Optimal Lending Contracts and Firms' Survival with Moral Hazard *

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Abstract

There is widespread evidence supporting the conjecture that borrowing constraints have important implications for firm growth and survival. In this paper we model a multi-period borrowing/lending relationship with asymmetric information. We show that borrowing constraints emerge as a feature of the optimal long-term lending contract, and that such constraints relax as the value of the borrower's claim to future cash-flows increases. We also show that the optimal contract has interesting implications for firm dynamics. In agreement with the empirical evidence, as age and size increase, the sensitivity of investment declines, and firm survival increases.

Key words. Optimal Contract, Borrowing Constraints, Moral Hazard, Survival.

JEL Codes: D82, G32, L14.

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1 Introduction

It has long been emphasized[1]that for a large part of the cross-section of firms, investment responds to innovations in the cash-flow process. Interestingly, it has also been established that the sensitivity of capital expenditures to variations in cash flows is much greater for small and young firms. Such sensitivity may be due to a simple selection process, as in the models of Jovanovic [17] and Alti [2]. In either paper, entrepreneurs start out not knowing the productivity of their firms and learn about it by observing the cash-flows realizations. An alternative explanation, investigated in a number of recent empirical studies,[2]is that the investment cash-flow sensitivity is due to financing constraints. In this paper we consider an infinitely repeated borrowing and lending relationship, where asymmetric information with respect to the borrower's (firm's) cash-flows induces moral hazard. We show that borrowing constraints emerge as a feature of the optimal long-term lending contract, and that such constraints relax as the value of the borrower's claim to future cash-flows increases. The optimal contract predicts that, in agreement with the empirical evidence,[3] the survival rate increases, and the investment cash-flow sensitivity decreases, with age and size.

The model is as follows. At time zero the borrower (entrepreneur) has a project that requires a fixed initial investment. The initial investment is financed by a lender (bank), with unlimited resources. Once the project is in operation, an advancement of working capital is required each period, which is also provided by the bank. Revenues increase with the amount of capital advanced. At any point in time the project can be discontinued. Both, the borrower and the lender, are risk neutral and discount future flows at the same rate. Furthermore, every period there is a positive, constant probability that revenues are lost. We assume that the revenues' realization is private information for the firm; the lender does not have the possibility to verify whether revenues are strictly positive or zero.

To our knowledge, Fazzari, Hubbard and Petersen [11] are the first to use the

[1]See Fazzari, Hubbard, and Petersen [11], for example

[2]See the survey later in this section.

[3]See, for example, Evans [10], Hall [16], and Dunne, Roberts, and Samuelson [8].

cross-sectional variation in the correlation between investment and cash-flows to investigate whether the same correlation is as a signal of financing constraints. They estimate reduced-form investment equations across groups of firms classified by their dividend behavior. Their results indicate a substantially greater sensitivity of investment to cash flows for those firms that plow back most of their earnings. It has been rightly pointed out, however, that a positive investment-cash flow sensitivity can be interpreted as a signal of borrowing constraints only if we can adequately control for the marginal value of investment. A natural way to do so is to include Tobin's Q in the regression equation. The problem is that if average Q (the measure actually observed) differs from marginal Q, cash-flows could be positively correlated with investment just because they provide information on the marginal value of investment that is not already reflected in the control.

Gilchrist and Himmelberg [14] address this issue. In their study, they use estimates from a set of VAR forecasting equations in order to construct a proxy for the expected discounted stream of marginal return to investment. Most importantly, cash flow is included as one of the fundamentals in the VAR equations, so that any additional sensitivity of investment to cash flow can be interpreted as evidence of capital market imperfections. Their results are essentially consistent with the findings of Fazzari, Hubbard, and Petersen.

Gertler and Gilchrist [13] consider the response of manufacturing firms of different size to innovations in monetary policy. They find that small firms account for a disproportionate share of the decline in production that follows a tightening of monetary policy.

Whited [22] applies a methodology already used in consumption studies in order to study the effects of borrowing constraints on household expenditures. This methodology exploits the observation that the investment Euler equation of the standard neoclassical model should be violated for firms that face financing constraints. Whited partitions the firms in his sample in two sets depending on several measures of financial distress and finds that the unconstrained Euler equation fails to hold for the a priori constrained firms, while it performs much better for the remaining firms.

There is an extensive literature on optimal financing contracts with moral hazard. In borrowing/lending relationships, moral hazard can have either one of two sources: limited commitment (risk of repudiation) or asymmetric information. To our knowledge, Eaton and Gersowitz [9] were the first to model the case of limited commitment. A more recent contribution in this line of research is Albuquerque and Hopenhayn [1], which in some sense can be considered companion to this paper, given the similarity of the two environments. Atkeson [3] studies the effects of both limited commitment and asymmetric information in his analysis of international lending.

Gertler [12] studies the optimal contract between a lender and a borrower in a three-period production economy with asymmetric information. Even though our structure and Gertler's look very much alike, they differ in at least two fundamental respects. First, Gertler focuses on the ability of his model to generate large output fluctuation through the magnification of relatively small productivity shocks, while our interest is mainly in the predictions for industry dynamics. Second, while Gertler considers a finite-horizon model, our choice is to analyze the infinite-horizon case. The recursive structure of our problem allows us a much sharper characterization of the optimal contract and of its implications.

The studies that are closest to ours are working papers by Quadrini [18] and DeMarzo and Fishman [7]. Both characterize the optimal lending contract in environments characterized by asymmetric information. Quadrini, in particular, focuses on the implications of the contract for firm dynamics. DeMarzo and Fishman instead, are interested mainly in the implementation of the long-term arrangement by means of simple contracts.

Finally, we want to acknowledge that, as it is the case for all papers that study infinitely repeated relationships with moral hazard, our own paper owes a great deal to the early work of Radner [19], Rogerson [20], Green [15], and Spear and Srivastava [21].

The remainder of this paper is organized as follows. The model is introduced in Section 2. In Section 3 we characterize the main properties of the optimal contract. The implications for firm survival are described in Section 4. In Section 5 we provide necessary and sufficient conditions for the existence of the optimal contract and we relate the value attributed by the contract to the entrepreneur, to his initial capital. Section 6 concludes.

2 The Model.

Time is discrete and the time horizon is infinite. At time zero the entrepreneur has a project which requires a fixed initial investment $I_0 > 0$ and a per-period investment of working capital. Let k_t be the amount of working capital invested in the project - its scale - in period t. The project is successful with probability p, in which case the entrepreneur collects revenues $R(k_t)$. If the project fails, revenues are zero. We assume that the function R is continuous, uniformly bounded from above, and strictly concave. At the beginning of every period the project can be liquidated. The liquidation generates a scrap value S.

We assume that the lender cannot observe the revenue outcome. In other words, such outcome is *private information* for the entrepreneur.

The entrepreneur's net worth is given by $M < I_0$. Therefore, to undertake the project, he requires a lender (bank) to finance part of the initial setup cost and the project investments in every period. We assume that in every period the entrepreneur is liable for payments to the lender only to the extent of current revenues. Therefore the firm is restricted at all times to a nonnegative cash flow. [4]

Both the borrower and the lender are risk neutral, discount flows using the same discount factor $\delta \in (0,1)$, and are able to commit to a long term contract.

We model the relation between the bank and the entrepreneur as a messaging game. At time 0 the bank makes a take-it-or-leave-it offer to the entrepreneur. The offer consists of a contract whose terms can be contingent on all public information. Let θ be a Bernoulli random variable, with $\theta \in \Theta \equiv \{H, L\}$ and $prob\{\theta = H\} = p$. Revenues are positive (and equal to $R(k)$) when $\theta = H$ and identically zero when $\theta = L$. We invoke the Revelation Principle to reduce the message space to the set Θ. Thus a reporting strategy for the entrepreneur is given by $\hat{\boldsymbol{\theta}} = \{\hat{\theta}_t(\theta^t)\}_{t=1}^{\infty}$, where $\theta^t = (\theta_1, ..., \theta_t)$.

[4] This assumption can be easily relaxed to a lower bound.

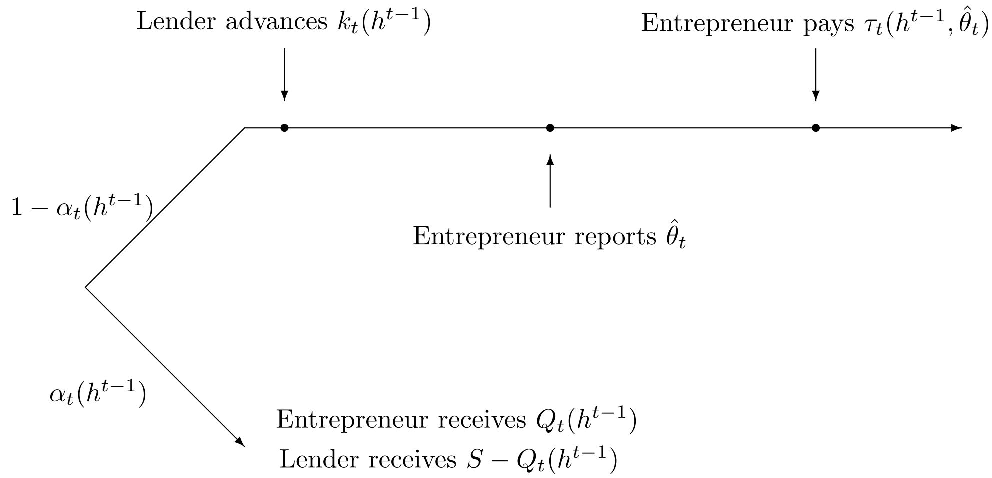

Figure 1: The Timing.

We say that a sequence of realizations θ^t induces an history $h^t = (\hat{\theta}_1, ..., \hat{\theta}_t)$. A long-term contract is denoted by $\boldsymbol{\sigma} = \{\alpha_t(h^{t-1}), Q_t(h^{t-1}), k_t(h^{t-1}), \tau_t(h^t)\}$. That is, the contract specifies a contingent policy of liquidation probabilities α_t and transfers Q_t from the lender to the entrepreneur (conditional on liquidation) and capital advancements k_t and transfers τ_t from the entrepreneur to the lender (conditional on survival).

The timing is the same at every information node (i.e. after every history h^t) and is described in Figure 1. At the beginning of the period, the lender has the chance of liquidating the project. The contract dictates that he will do so with probability $\alpha_t(h^{t-1})$. In case liquidation occurs, the entrepreneur is compensated with a value $Q_t(h^{t-1})$, while the lender receives $S - Q_t(h^{t-1})$. In the alternative case, in which the project is not liquidated, the lender provides the entrepreneur with capital $k_t(h^{t-1})$. Thereafter, the entrepreneur observes the revenue realization and makes a report to the lender. The lender will require a transfer $\tau_t(h^t)$, where $h^t = (h^{t-1}, \hat{\theta}_t)$.

At every time t, conditional on success, the entrepreneur will receive a net cash-flow $R(k_t) - \tau_t$. We assume, without loss of generality, that these resources are fully consumed by the entrepreneur (i.e. not reinvested in the business venture).[5] As a consequence, the nonnegativity constraint on cash-flows of the firm requires that $\tau_t \leq R(k_t)$, implying that when the project fails no payments are made to the bank.

[5]Alternatively, we might allow the entrepreneur to save and assume that the bank can observe the return on his wealth and monitor the size of the project.

We are now in the position to provide our definition of feasible contract.

Definition 1 *A contract* $\boldsymbol{\sigma}$ *is feasible if* $\forall\, t \geq 1$ *and* $\forall\, h^{t-1} \in \Theta^t$
(i) $\alpha_t(h^{t-1}) \in [0,1]$,
(ii) $Q_t(h^{t-1}) \geq 0$,
(iii) $\tau_t(h^{t-1}, H) \leq R(k_t(h^{t-1}))$,
(iv) $\tau_t(h^{t-1}, L) \leq 0$.

After every history h^{t-1}, a pair $(\hat{\boldsymbol{\theta}}, \boldsymbol{\sigma})$ implies expected discounted cash flows for the entrepreneur and the lender. Denote such values as $V_t(\hat{\boldsymbol{\theta}}, \boldsymbol{\sigma}, h^{t-1})$ and $B_t(\hat{\boldsymbol{\theta}}, \boldsymbol{\sigma}, h^{t-1})$, respectively.

Definition 2 *A contract* $\boldsymbol{\sigma}$ *is incentive compatible if* $\forall\, \hat{\boldsymbol{\theta}}$.

$$V_1(\boldsymbol{\theta}, \boldsymbol{\sigma}, h^0) \geq V_1(\hat{\boldsymbol{\theta}}, \boldsymbol{\sigma}, h^0)$$

Now define the set $\mathcal{V} \equiv \{V \mid \exists\, \boldsymbol{\sigma} \;\; s.t.\; (feas),\; (ic)\; and\; V_1\left[\boldsymbol{\sigma}, \boldsymbol{\theta}, h^0\right] = V\}$. $\mathcal{V}$ is the set of equity values that can be generated by feasible and incentive compatible contracts. Then, if V is the expected discounted cash flow that the lender offers to the entrepreneur, the bank will choose the contract that delivers to him the value *sup* $\mathcal{B}(V)$, where $\mathcal{B}\;(V) \equiv \{B \mid \exists\, \boldsymbol{\sigma} \;\; s.t.\; V_1\left[\boldsymbol{\sigma}, \boldsymbol{\theta}, h^0\right] = V$ and $B_1\left[\boldsymbol{\sigma}, \boldsymbol{\theta}, h^0\right] = B\}$.

2.1 A Benchmark: Contracts under Symmetric Information.

Before venturing into the characterization of the optimal contract with private information, we find it useful to study the case of symmetric information. This is, the scenario in which the lender observes the revenue realizations and thus can condition the terms of the contract on them. Since the two agents discount cash flows at the same rate, the optimal contract will maximize the total expected discounted profits for the match. In turn, this result is achieved by having the lender provide the entrepreneur with the unconstrained efficient amount of capital in every period. The properties we have assumed for the revenue functions R are clearly sufficient for existence and uniqueness of the solution to the static profit maximization problem. Thus the unconstrained efficient capital advancement is given by

$k^* \equiv \arg\max_k pR(k) - k$ and the per-period total surplus is $\pi^* \equiv \max_k pR(k) - k$. Then the total surplus generated by the optimal contract under symmetric information is given by $\widetilde{W} \equiv \frac{\pi^*}{1-\delta}$. In order to avoid trivial results in the analysis that follows, we assume that $\widetilde{W} > S$.

All subdivisions of the surplus $\widetilde{W}$ among the lender and entrepreneur are feasible, provided that a nonnegative value is assigned to the entrepreneur. It is enough to choose the appropriate stationary transfer policy $\tau(i),\ i = H, L$.

2.2 The Contract with Private Information: a Recursive Representation.

Following Green [15] we now recast the lender's problem in recursive form, adopting the entrepreneur's entitlement V as state variable.[6] It is important to notice that, the entrepreneur being risk-neutral, the state variable V is in effect the expected discounted value of the cash-flows that will accrue to the entrepreneur himself. Therefore, it can be thought of as the value of the firm's equity. We denote the value function of the problem as $W(V)$. $W(V)$ is the expected discounted sum of net revenues for the match when the entitlement of the borrower is V. In finance jargon, W denotes the total value of the firm. The residual $B \equiv W - V$ is the value of debt. At the beginning of every period the lender decides whether to liquidate the project, obtaining the value S, or keep it in operation. We start by defining the value of letting the entrepreneur continuing. The value that the contract promises to the entrepreneur is given by the expected current net cash-flow plus the discounted value of the stream of future cash-flows. Therefore the optimal choices of a pair (k, τ) and continuation values (V^H, V^L) must satisfy the following consistency requirement (promise-keeping constraint):

$$V = p\,(R(k) - \tau) + \delta\left[pV^H + (1-p)\,V^L\right]. \tag{1}$$

[6] Green [15] and Spear and Srivastava [21] were the first to show that, under mild boundedness conditions, there exists a recursive formulation for the maximization problem faced by the principal in models of repeated moral hazard. Such conditions hold in our case. We decide to omit the proof of equivalence between the sequence problem and the recursive problem in our case, because it consists of the mere application of the techniques used by Atkenson and Lucas [4], among the others.

By the unimprovability principle, the incentive compatibility principle stated in the previous section holds if and only if the following condition holds:

$$R(k) - \tau + \delta V^H \geq R(k) + \delta V^L.$$

This condition is known in the literature as temporary incentive-compatibility constraint. We rewrite it as

$$\tau \leq \delta \left(V^H - V^L\right). \tag{2}$$

Finally, the limited liability constraint requires that

$$\tau \leq R\left(k\right). \tag{3}$$

A continuation value V^i $(i = H, L)$ must be supported by a feasible continuation contract. Any positive continuation value $V^i \geq 0$ is feasible, since it can be obtained by giving the entrepreneur no capital and a transfer equal to V^i, and then liquidating the project. Any continuation value $V^i < 0$ is not feasible, for it would violate the limited liability constraint (3) in some future period. Hence, a value V can be supported by a feasible contract if, and only if, $V \geq 0$. The optimal value of the program (P1) defines the value of continuing the project, denoted as $\widehat{W}\left(V\right)$.

$$\begin{aligned} \widehat{W}\left(V\right) = \max_{k,\tau,V^H,V^L} & \; pR\left(k\right) - k + \delta\left[pW\left(V^H\right) + \left(1-p\right)W(V^L)\right] \\ & \text{subject to (1), (2), (3),} \\ & \text{and } V^H, V^L \geq 0. \end{aligned} \tag{P1}$$

We now turn to the liquidation decision. Allowing for randomizations over the liquidation decision is equivalent to assuming that at the beginning of every period the lender offers a lottery to the borrower. The firm is liquidated with probability α, in which case the borrower receives Q, and it is kept in operation with probability $1-\alpha$. In the latter case, the borrower receives V_c, where c is mnemonic for *continuation*.[7]

Then the function $W\left(V\right)$ solves the following functional equation:

[7] In Section 3 we will show that randomizing on the liquidation choice is actually welfare-improving.

$$W(V) = \max_{\alpha \in [0,1], Q, V_c} \alpha S + (1-\alpha) \widehat{W}(V_c)$$
$$\text{subject to } \alpha Q + (1-\alpha) V_c = V, \tag{P2}$$
$$\text{and } Q, V_c \geq 0.$$

We now turn to the characterization of the value function. Existence and uniqueness of $W(V)$ follow from standard dynamic programming arguments. In order to prove monotonicity and concavity we just need to verify that the contraction mapping defined by the programs (P1) and (P2) maps increasing and concave functions into increasing and concave functions. The proofs of Propositions 1 and 2 show that if the function $W(V)$ in problem (P1) satisfies these properties, so does $\widehat{W}(V)$. Figure 3 below shows that the mapping defined by the problem (P2) preserves the two properties. Also, notice that $\widehat{W}(0) = \delta S$. In fact, given the non-negativity constraints on the continuation values, the only way to award a value of 0 to the borrower is to set $k = \tau = 0$ and liquidate with probability 1 in the following period.

It is easy to see that if the borrower's entitlement reaches the level $\widetilde{V} \equiv \frac{pR(k^*)}{1-\delta}$, then the choices $k = k^*$, $\tau = 0$, and $V^H = V^L = \widetilde{V}$ are feasible and are surplus-maximizing. This implies that whenever the threshold $\widetilde{V}$ is reached, no further constraints on capital will be necessary. The value entitlement is so high that the firm is given every period the optimal capital advancement k^* with no need of repayment. As a consequence, we have that $W(\widetilde{V}) = \widetilde{W}$.

The results reached so far are formally stated in the next two propositions.[8]

Proposition 1 *The value function is strictly increasing up to* $\widetilde{V}$ *and constant at the level* $\widetilde{W}$ *thereafter.*

Proposition 2 *The value function is concave.*

3 Properties of the Optimal Lending Contract.

In this section we characterize analytically the properties of the optimal lending contract. We begin by discussing the optimal liquidation policy and we then char-

[8]The proofs which are not included in the main body of the paper can be found in the Appendix.

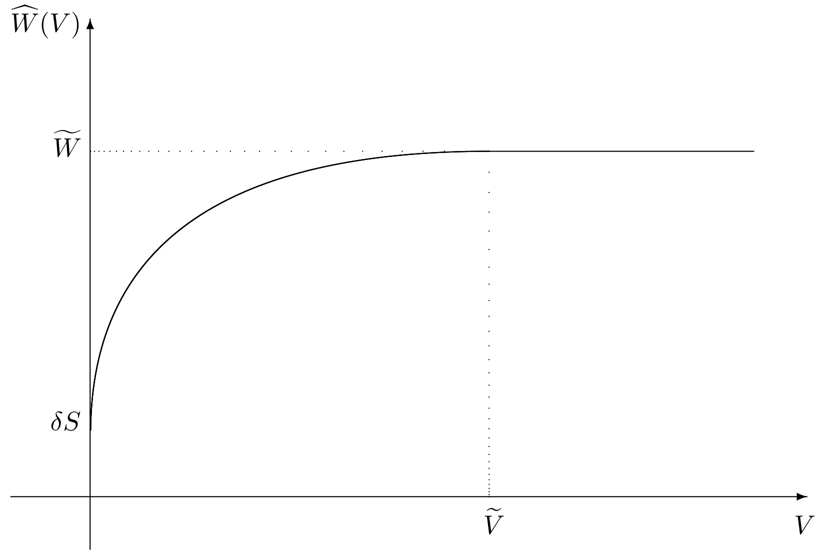

Figure 2: The Function $\widehat{W}(V)$.

acterize the policy functions (k, τ, V^H, V^l). The results derived here will be used in Section 4 in order to determine predictions for firm survival.

3.1 Liquidation.

Given the discussion carried out above, it is clear that there exists a range of values for the entrepreneur's entitlement V, such that liquidation of the project is the surplus-maximizing decision. Figure 3 shows that randomizing over such decision yields a Pareto-improvement over deterministic liquidation. Since the objective function in (P2) does not depend on the value Q and the function $\widehat{W}(V)$ is increasing, it is clearly always optimal to set $Q = 0$ and $(1-\alpha)V_c = V$. The continuation value V_c that maximizes the surplus is given by V_r, where V_r is such that $\widehat{W}(V_r)$ lies on the tangent departing from S. Proposition 3 and Corollary 1 summarize these findings.

Proposition 3 *There exists a value V_r, $0 < V_r < \widetilde{V}$ such that $W(V)$ is linear over $[0, V_r]$.*

Corollary 1 *The probability of liquidation is given by*

$$\alpha(V) = \begin{cases} \frac{(V_r - V)}{V_r} & \text{if } V \in [0, V_r], \\ 0 & \text{if } V > V_r. \end{cases} \tag{4}$$

Notice that for $V \leq V_r$the probability of liquidation is decreasing linearly in the entrepreneur's value entitlement.

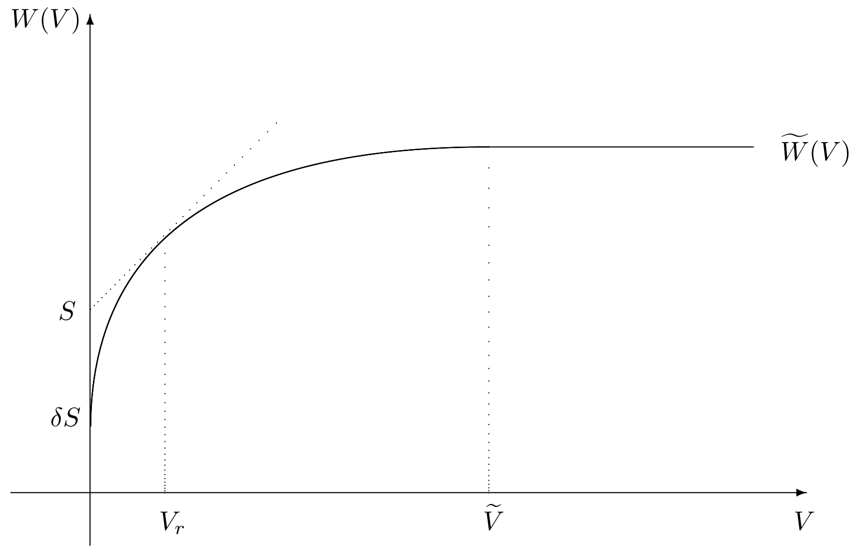

Figure 3: The Liquidation Decision.

3.2 Optimal Pay-back Policy.

Proposition 4 states that, prior to achieving the unconstrained optimum $\widetilde{V}$, it is optimal to set $\tau = R(k)$. That is, all cash-flows produced by the project accrue to the lender and the entrepreneur receives nothing. Once the value V reaches the threshold $\widetilde{V}$, the optimal pay-back policy becomes indeterminate, because the surplus does not depend on the time pattern of the transfer τ. In particular, among the optimal pay-back policies is the one that requires the entrepreneur to transfer all the cash flows to the lender until his value reaches $\widetilde{V}$, and to pay nothing thereafter.

The economic intuition behind these findings is straightforward. At every date t, the lender needs to decide how to deliver on his promise V_t. He can reward the entrepreneur with cash (i.e. allowe him to retain part of the current revenues) or promise him an higher future value. Given that both agents are risk-neutral and discount future cash-flows at the same rate, the optimal pay-back schedule is the one that allows the equity value V to reach the threshold $\widetilde{V}$ in the shortest time possible. In fact, as we have already argued, when $V = \widetilde{V}$, it will possible for the lender to advance the unconstrained efficient capital forever after.

Proposition 4 *When $V^H(V) < \tilde{V}$, a necessary condition for the optimal contract is that $\tau = R(k)$. On the other hand, when $V^H(V) \geq \tilde{V}$, any value τ such that $pR(k) + \delta V^L \leq V$ is consistent with the optimal contract.*

Proof. For the sake of contradiction, assume there exists V such that $W\left(V^H\right)$ is strictly increasing and $\tau < R(k)$. Then it is possible to increase both τ and V^H in such a way that the constraints are still satisfied, but the surplus of the match has increased strictly. ■

Proposition 4 implies that, for the purpose of characterizing the remaining policy functions, there is no loss of generality in making the following assumption.

Assumption 1 *From now on we assume that $\tau = R(k)$ for every V.*

We will see shortly that this hypothesis facilitates greatly our analysis.

3.3 The Dynamics of The Entrepreneur's Value V.

Assumption 1 allows us to rewrite the problem (P1) as follows.

$$\widehat{W}(V) = \max_{k,\tau,V^H,V^L} pR(k) - k + \delta\left[pW\left(V^H\right) + (1-p)W(V^L)\right]$$

$$\text{subject to } \delta\left[pV^H + (1-p)V^L\right] = V \tag{5}$$

$$R(k) \leq \delta\left(V^H - V^L\right) \tag{6}$$

$$\text{and } V^H, V^L \geq 0.$$

Proposition 2 (concavity) implies that the value function is differentiable almost everywhere. In turn, this means that in all but a countable number of entitlements V, the following conditions are necessary for the optimal contract:

$$R^{'}(k) = \frac{1}{p-\mu} \tag{7}$$

$$W^{'}\left(V^H\right) = \left(\lambda - \frac{\mu}{p}\right) \tag{8}$$

$$W^{'}\left(V^L\right) = \lambda + \frac{\mu}{1-p} \tag{9}$$

$$\lambda, \mu \geq 0$$

, where λ and μ are the multipliers attached to constraints (5) and (6) respectively. Using the envelope condition along with conditions (8) and (9), we also have that

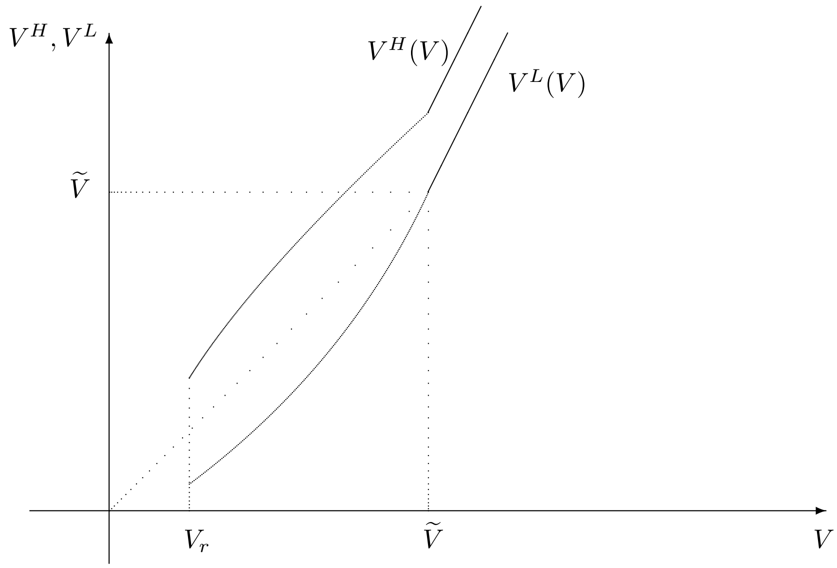

Figure 4: The Dynamics of the Entrepreneur's Entitlement.

$$\lambda = W'(V) = pW'\left(V^H\right) + (1-p)\,W'\left(V^L\right). \tag{10}$$

Figure 4 shows the qualitative behavior of the functions $V^H(V)$ and $V^L(V)$.[9] We already know that, for $V < V_r$, randomizing over the liquidation decision emerges as an optimal strategy. Proposition 5 states that for $V \geq V_r$ the continuation values are nondecreasing functions of the value entitlement. Notice that as a consequence of Proposition 4, for $V^H(V) > \tilde{V}$ the shape of the function $V^H(V)$ depends on the particular choice of pay-back policy $\tau(V)$. Consistently with Assumption 1, we choose to represent the case in which $\tau(V) = R[k(V)]$ for every V.

Proposition 5 *The following conditions hold in the optimal contract:*

1. *(i) $V < \tilde{V}$ implies $V^L(V) < V < V^H(V)$; (ii) $V \geq \tilde{V}$ implies $V^L(V) \leq V \leq V^H(V)$.*

2. *the policy functions $V^H(V)$ and $V^L(V)$ are nondecreasing in V.*

Up to this point we have dedicated a sizeable part of our discussion to the liquidation decision and to the recommendations of the contract in the case in which V reaches the threshold $\widetilde{V}$. However, we have not discussed whether and under which conditions either one of the two events actually occur. We are now in the

[9]Lemma 4 shows that the policies V^H and V^L are single-valued and continuous.

position to fill this gap. Proposition 6 characterizes the limiting behavior of the stochastic process for the equity value V_t. It turns out that either the firm exits (it is liquidated) or it reaches the unconstrained optimum.

Lemma 1 *The optimal contract implies that V_t is a submartingale.*

Proof. Using (1) and (3) we obtain that $V \leq pV^1 + (1-p)\,V^L$. ■

Proposition 6 *The firm either exits or reaches the unconstrained optimum almost surely. That is, the limit distribution of firm's value exists and is degenerate, with two absorbing points.*

Proof. Given Lemma 1, the convergence of V_t follows easily from a standard Submartingale Convergence Theorem like Theorem 7.4.3 in Ash (1972). Since $V^H(V)$ is continuous and increasing for $V \leq \widetilde{V}$ and $V^H\left(\widetilde{V}\right) > \widetilde{V}$, it follows that a finite sequence of positive revenue realizations is sufficient for a firm with entitlement V to reach the state $\widetilde{V}$. Since any such sequence occurs with positive probability, a firm which does not exit reaches the unconstrained state with probability measure one. On the other hand, $V^L(V)$ is also increasing, continuous and uniformly bounded from below, implying that $\lim_{V \to V_r} V^L(V) < V_r$. Thus a sequence of negative shocks is sufficient to lead to the randomization area a firm starting from any $V < \widetilde{V}$. In other words, a firm that does not reach $\widetilde{V}$ will exit almost surely. ■

Figure 6 shows a few sample paths for the equity value V. While some firms end up being liquidated, others reach the unconstrained state $\widetilde{V}$. We find it remarkable that a simple model with i.i.d. shocks is able to generate such a level of heterogeneity in the evolution of the stochastic process V_t.

3.4 The Optimal Capital Advancement Policy.

In Section 1 we have briefly summarized a series of empirical papers that support the conjecture that the observed investment-cash flow sensitivity is actually due to existence of borrowing constraints. In subsection 2.1 we have shown that, under symmetric information, it would always be optimal for the lender to provide the entrepreneur with the unconstrained efficient amount of capital. However, this ceases to be the case when the lender cannot observe the firm's cash flows. Proposition 7 proves that borrowing constraints are indeed a feature of the optimal long-term contract under asymmetric information.

Proposition 7 *i) $V < \tilde{V}$ implies $k(V) < k^*$;(ii) $V \geq \tilde{V}$ implies $k(V) = k^*$.*

Using equations (7), (8) and (9) we obtain that a necessary condition for the optimal contract is that

$$(1-p)[W'(V^L) - W'(V^H)] = 1 - \frac{1}{pR'(k)}. \tag{11}$$

Condition (11) shows that the optimal capital advancement k satisfies $k < k^*$ if and only if $W'(V^L) > W'(V^H)$. In light of Proposition 5, strict concavity of the value function $W(V)$ would be sufficient to prove Proposition 7. Even without this property, however, we are able to show that the result holds.

The economic intuition behind this finding is as follows. Lemma 3 in the appendix shows that for $V < \widetilde{V}$ the incentive compatibility constraint (2) binds. Therefore the level of capital advancement is tied to the spread in equity's values $(V^H - V^L)$. Jensen inequality implies that such spread is costly, so that $k < k^*$ unless the value function W is flat in the relevant portion. Figure 7 shows a few sample paths for the capital advancement. Paths terminate either because of liquidation or because the firm reached the optimal size k^*.

Given the results presented so far, the reader may expect the capital advancement to be strictly increasing in the value entitlement V. Proposition 8 shows that this is not the case. At a given value entitlement $V < \widetilde{V}$, any increase in k must be accompanied by an increase in the spread between promised values $(V^H - V^L)$. Condition (11) dictates that the optimal level of capital will equate the marginal benefit deriving from higher current surplus to the marginal cost consisting in lower future surplus. This implies that the capital advancement will be increasing in V if and only if this marginal cost is decreasing. We know that in a neighborhood of $\widetilde{V}$ this is actually the case, since the value function is globally concave, and linear for $V > \widetilde{V}$. However, this is not a global property. Just to the right of V_r, the marginal cost is increasing, since the value function is linear also over the randomization zone $[0, V_r]$.

Proposition 8 *There exist value entitlements V^* and V^{**} , with $V_r < V^* \leq V^{**} < \widetilde{V}$, such that:*

1. the policy function $k(V)$ is non-increasing for $V \in [V_r, V^*]$;
2. the policy function $k(V)$ is non-decreasing for $V \in \left[V^{**}, \widetilde{V}\right]$.

Proof. We have just seen that a necessary condition for the optimal contract is that $(1-p)\left(W'\left(V^L\right) - W'\left(V^H\right)\right) = 1 - \frac{1}{pR'(k)}$.

1) By Proposition 4, it follows that $V^L(V_r) < V_r$. By continuity of the function $V^L(\cdot)$, there exists $V > V_r$ such that $V^L(V) < V_r$. Combining this fact with monotonicity of $V^H(\cdot)$ and concavity of the value function yields the prediction that $\left(W'\left(V^L\right) - W'\left(V^1\right)\right)$ is non-decreasing and thus $k(\cdot)$ non-increasing at V_r.

2) We know that $V^H\left(\widetilde{V}\right) > \widetilde{V}$. By continuity of the function $V^H(\cdot)$, there exists $V < \widetilde{V}$ such that $V^H(V) > \widetilde{V}$. Combining this fact with monotonicity of $V^L(\cdot)$ and concavity of the value function yields the prediction that $\left(W'\left(V^L\right) - W'\left(V^H\right)\right)$ is non-increasing and thus $k(\cdot)$ non-decreasing at V. ■

All of our computer simulations show that the set of entitlement values V such that $k(V)$ is non-increasing is very small. Figure 8 shows the typical shape of the policy function for the capital advancement.

Given the available empirical evidence on the sensitivity of investment to cash-flows, it is of great interest to characterize how capital advancement reacts to the realizations of the revenue process in our model. Since by Proposition 5, $V^H(V) > V > V^L(V)$ for $V < \tilde{V}$, monotonicity of the policy function $k(V)$ would insure that, when $V < \tilde{V}$, the level of capital invested increases following a success (positive revenues) and decreases following a failure (no revenues). As we have just seen, however, the function $k(V)$ is not monotone. Thus we evaluate the performance of our model along this dimension by means of numerical experiments. It turns out that for most parameterizations, the result still holds. Figure 9 shows how the growth rate of capital responds to good and bad realizations of the revenue process. Success is followed by an increase in the capital invested, while failure triggers a decline. Also, notice how the magnitude of the response of investment to cash-flow innovations changes with V. Percentage changes in capital tend to be larger (in absolute value) for small values of V and show a tendency to decline as V increases. When V equals the threshold $\tilde{V}$, the level of capital invested is independent of cash-flows.

4 Firm's Survival.

As mentioned in the introduction, there is widespread agreement on the finding that failure rates decrease with firm's age and size. Moreover, conditional on age, a larger size implies a lower failure rate. Our model predicts that the conditional probability of survival increases with the value of the firm's equity V. The formal argument is given by Proposition 9, which states that the probability of a firm exiting before any given date, conditional on not having exited before, is weakly decreasing in the value of the firm's equity V, and it equals zero for $V \geq \widetilde{V}$.

Proposition 9 *Let T be the life length of the firm. Then the reliability function* $\mathrm{R}_V(t)$*, where* $\mathrm{R}_V(t) = \Pr(T > t \mid V)$*, is increasing in* V*.*

Proof. Let Ω_i consists of the events $\{\emptyset, \{L\}, \{H\}, \Omega_i\}$. Denote by $(\Omega_i, \mathcal{F}_i, \mu_i)$ the obvious probability space. Recall that θ_i is the random variable which equals H when revenues are strictly positive and equals L when revenues are zero. Then, for any $n \geq 1$, there exists a probability space $(\Omega_n, \mathcal{F}_n, \mu_n)$ such that $\Omega_n = \Pi_{i=1}^n \Omega_i$, $\mu_n = \Pi_{i=1}^n \mu_i$ and $\mathcal{F}_n$ is the Borel σ-algebra generated by Ω_n. Obviously the random sequence $\theta_n = \{\theta_i\}_{i=1}^n$ is measurable with respect to $\mathcal{F}_n$. For any measurable sequence e_n, denote by $W_{e_n}(V)$ the surplus attained by a match with starting entitlement V, after the sequence e_n has occurred. We say that the path e_n is terminated for V, if $T < n$ along that path. Finally and for every m, $m < n$, we say that $e_m \subset e_n$ if e_n and e_m coincide for $i \leq m$. Now consider any pair of entitlement values (V, V') such that $V < V'$. For every n, consider the set of measurable paths $\{e_n\}$ which are terminated for V. By monotonicity of the policy functions $V^H(V)$ and $V^L(V)$ it must be the case that those paths are not terminated yet for V', and $W_{e_n}\left(V'\right) > W_{e_n}(V)$. This fact insures that for any n, the set of measurable sequences which are terminated for the firm starting with V' has a lower probability measure than the ones which are terminated for the firm starting with V. ■

A corollary of Proposition 9 is that for every cohort of firms and for every value $V < \widetilde{V}$, the fraction of entrepreneurs having an equity value less than V is decreasing and goes asymptotically to zero. The mass of firms in the unconstrained state increases with time. In the limit, all surviving firms will receive the efficient amount of capital. This implies that the exit rate is decreasing with age and tends asymptotically to zero.

Corollary 2 *The exit rate decreases with firm age.*

It is important to notice that this result is due to the positive correlation between firm age and value V. If we condition on V, the relation between exit rate and age vanishes.

Proposition 8 shows that the optimal capital advancement policy is not increasing in the value V for every $V < \widetilde{V}$. However, as already argued in Subsection 3.4, our computer simulations show that for most parameterizations the set of equity values for which the function $k(V)$ is not monotone, is very small. To the extent that k is increasing in V, our model produces the further result that the failure rate is decreasing in size, both unconditionally and controlling for age.

5 Existence.

So far we have overlooked two important questions. Under which conditions does the optimal contract exist? Conditional on existence, how is the surplus (firm value) split among lender and borrower at the time in which the contract is signed (i.e. as of time $t = 0$)? In this section we address both questions.

Recall that the entrepreneur's net worth is given by $M < I_0$, where I_0 is the initial investment required by the project. Let V_0 denote the expected discounted cash flow that the contract assigns to the borrower at the time in which it is stipulated. Then

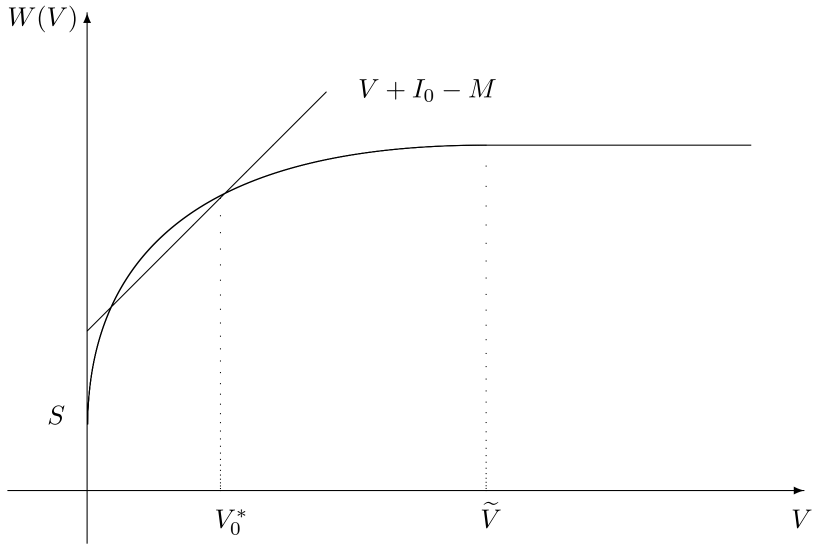

Figure 5: Existence.

the lender will be willing to provide financing to the entrepreneur, as long as V_0 satisfies

$$[W(V_0) - V_0] - [I_0 - M] \geq 0.$$

The term in the first square bracket is the portion of the expected discounted value of cash flows generated by the project, that accrues to the lender, i.e. the value of debt B_0. The term in the second square bracket is the portion of the initial investment that is financed by the lender. The entrepreneur will be willing to borrow as long as $V_0 > M$.

Thus the existence of the contract depends crucially on the difference $[I_0 - M]$. As it is clear from Figure 5, if such difference is large enough, a contract will not exist.

For those cases in which the contract exists, our assumptions are not sufficient to pin down the portion of surplus V_0 attributed to the borrower. Howerever, if we assume that the lending market is competitive, the value V_0 will be determined by the following simple problem:

$$\begin{aligned} &max\ V_0 \\ &s.t.\, [W\,(V_0) - V_0] - [I_0 - M] = 0. \end{aligned} \tag{12}$$

Differentiating the condition (12), we obtain that the optimal starting equity value V_0^* must satisfy

$$\frac{dV_0}{dM} = \frac{1}{1 - W'(V_0^*)}.$$

It follows that, as long as $W'(V_0^*) < 1$, the value of the entrepreneur V_0^* is increasing in the initial net worth M. It turns out that $W'(V_0^*) < 1$ is also a necessary condition for optimality. In fact, by Propositions 1 and 2, if there exists a value V_0 such that (12) holds and $W'(V_0) > 1$, then there must be a larger value V_0^* that satisfies (12) and $W'(V_0^*) < 1$. Thus we can state the following result.

Proposition 10 *The equity value V_0^* is strictly increasing in the net worth of the entrepreneur M.*

6 Conclusion.

In this paper we have modelled a multi-period borrowing/lending relationship where asymmetric information induces moral hazard. We have characterized the optimal lending contract and we have shown that financing constraints arise endogenously as a feature of such contract. We have also shown that, consistently with the empirical evidence, financing constraints relax, and the survival rate increases, with age and size.

There are obviously alternative ways of modelling borrowing constraints. At this stage, it is not clear which one deserves more merit. It is our opinion that empirical implications generated by different models can be used to discriminate among them. Albuquerque and Hopenhayn [1] analyzes with great generality a model of the borrower-lender relationship in which financing constraints arise as a result of the assumption of limited commitment on the side of the borrower. It is certainly of some interest to compare the predictions of the two models with respect to firm dynamics. The two environments differ in an important respect. In fact Albuquerque and Hopenhayn [1] predicts that, conditional on the state of nature, profits will be non-decreasing over time. In other words, every time in which the firm will transit in a state for which it had already transited in the past, it will receive a greater capital advancement than in that instance. Our model instead does not impose this restriction on firm dynamics.

In the introduction to this paper, we have recalled that economists still disagree on the causes of the observed correlation between investment and cash-flows. Once again, we believe that empirical implications of alternative models can be useful in order to discriminate among the possible explanations. In particular we notice that our model predicts that firm size is the real determinant of the cross-sectional variation in investment cash-flow sensitivity and survival rates. Such phenomena are correlated with age only because age is positively correlated with size. Selection models such as Jovanovic [17] and Alti [2] predict exactly the opposite. That is, age is the determinant of the variation in both investment cash-flow sensitivity and survival rates.

Finally, we suggest that our model has interesting applications in macroeconomics. Starting with Bernanke and Gertler [5], there has been a growing interest in understanding to what extent various forms of frictions in financial markets may generate and/or amplify macroeconomic fluctuations. However, the financial arrangements that are considered in this literature are never intertemporally optimal, meaning that contracts' provisions are not contingent on all public information. The only exception is recent work by Cooley, Quadrini, and Marimon [6], that embed the model by Albuquerque and Hopenhayn [1] in a general equilibrium framework. It would definitely be of interest to perform a similar exercise with the model developed in this paper, and then contrast the predictions generated by the two approaches.

A Appendix - Proofs and Lemmas

Proposition 1.

Proof. We first prove weak monotonicity. In order to do so we just need to prove that if the function $W(V)$ in the problem (P1) is non-decreasing, then so is $\widehat{W}(V)$. Thus assume the value function on the right-hand side of the operator is increasing. Consider any couple V, V' such that $V' > V$. Then it is feasible to choose $k\left(V'\right) = k(V)$, $\tau\left(V'\right) = \tau(V)$ and $V^H\left(V'\right) > V^H(V)$, $V^L\left(V'\right) > V^L(V)$ such that (1) is satisfied. It follows that $\widehat{W}\left(V'\right) \geq \widehat{W}(V)$. Now we prove that $V > \widetilde{V}$ implies $W(V) = \widetilde{W}$. Consider any V such that $V \geq \widetilde{V}$; then it must be that $W(V) \leq \widetilde{W}$ because the information-constrained optimal contract cannot attain a strictly higher surplus than the one attained by the contract with perfect information. Thus we just have to prove that $W\left(\widetilde{V}\right) = \widetilde{W}$. Notice that it is feasible to choose $V^H = V^L = \widetilde{V}$, $k = k^*$ and $\tau = 0$. Then it easy to verify that $\widetilde{W}$ satisfies the Bellman equation for $V = \widetilde{V}$. Finally we prove that $W(V) < \widetilde{W}$ implies $V < \widetilde{V}$. For the sake of contradiction, assume that there exists $V < \widetilde{V}$ such that $W(V) = \widetilde{W}$. In turn, this implies that $k = k^*$ and $W\left(V^H\right) = W\left(V^L\right) = \widetilde{W}$. By Lemma 2 below, we have that in this case $V^L(V) < V$ must hold. By repeating this argument, we eventually get that $W(V) = \widetilde{W}$ for every V. But this cannot be true because $W(0) = S$. Strict monotonicity over the range $\left[0, \widetilde{V}\right]$ follows as a corollary of Proposition 2 (concavity). To see why, assume by contradiction that there exists $V' < V < \widetilde{V}$ such that $W\left(V'\right) = W(V)$. Then by concavity of the value function it must be the case that $W\left(V'\right) = W(V) = \widetilde{W}$, which contradicts the result just proved. ■

Proposition 2.

Proof. The first thing to notice is that the feasible set as defined by constraints (1)-(3) is not convex. Thus we decide to introduce a change of variable. Let $x = R(k)$. Then we can rewrite the maximization problem as follows:

$$
\begin{aligned}
\widehat{W}(V) &= \max_{x,\tau,V^H,V^L} px - R^{-1}(x) + \delta\left[pW\left(V^H\right) + (1-p)W(V^L)\right], \\
&\quad \text{subject to } p(x-\tau) + \delta\left[pV^H + (1-p)V^L\right] = V, \\
&\quad \tau \le \delta\left(V^H - V^L\right), \\
&\quad \tau \le x, \\
&\quad \text{and } V^H, V^L \ge 0.
\end{aligned}
$$

Now assume that the value function $W(V)$ is concave. Then $\widehat{W}(V)$ is also concave. ■

Lemma 2 *Let $V < \widetilde{V}$ and $k = k^*$. Then it follows that $V^L(V) < V$.*

Proof. By (1), V^L is maximal when $\tau = R(k)$. By (2), we obtain that $R(k^*) \le \delta\left(V^H - V^L\right)$. Substituting in (1) we get that $pR(k^*) + \delta V^L \le V$ or $V^L \le \frac{V - pR(k^*)}{\delta}$. Since $\frac{\widetilde{V} - pR(k^*)}{\delta} = \widetilde{V}$, we have that $\frac{V - pR(k^*)}{\delta} < V$. Combining the two inequalities we get that $V^L < V$. ■

Lemma 3 *The IC constraint (2) is binding for every $V < \widetilde{V}$.*

Proof. By Proposition 7, $V < \tilde{V}$ implies that $k(V) < k^*$. For the sake of contradiction, assume there exists V such that $R(k) < \delta\left(V^H - V^L\right)$. Then by strict monotonicity of the revenue function it is possible to increase the surplus of the match strictly just by raising k. But this contradicts optimality. ■

Proof. We just need to show part ii). Since $W(V) = \widetilde{W} = \frac{pR(k^*) - k^*}{1-\delta}$ when $V \ge \tilde{V}$, it must be the case that $k(V) = k^*$. ■

Lemma 4 *The policies $V^H(V)$ and $V^L(V)$ are continuous and single-valued.*

Proof. Again, consider the problem as it is written above:

$$\widehat{W}(V) = \max_x px - R^{-1}(x) + \delta\left[pW\left(\frac{V + (1-p)x}{\delta}\right) + (1-p)W\left(\frac{V - px}{\delta}\right)\right]$$

It turns out that the function on the right-hand side is strictly concave in x, implying that $x(V)$ is single-valued. Then strict monotonicity of the revenue function and of the value function (on the range $\left[0, \widetilde{V}\right]$) imply that the policies $k(V)$,

$V^H(V)$ and $V^L(V)$ are also single-valued. Continuity follows by the Theorem of the Maximum, which implies that $V^H(V)$ and $V^L(V)$ are compact-valued and upper-hemicontinuous correspondences. ■

Proposition 5.

Proof. 1i) From above, we have that $V^H = \frac{V+(1-p)x}{\delta}$ $V^L = \frac{V-px}{\delta}$. This already implies $V^H > V$ and $V^H > V^L$. Then condition (10) implies that $V > V^L$. In fact $\lambda = W'\left(V^H\right) = W'\left(V^L\right)$ would imply $k = k^*$, which has been ruled out. Thus in solution it must be the case that $W'(V^1) < \lambda < W'\left(V^L\right)$.

1ii) Trivial.

2i) Assume there exist V, V' such that $V' > V$ and $V^L\left(V'\right) < V^L(V)$. By concavity of the value function and (9), $\mu\left(V'\right) \geq \mu(V)$ and thus $k\left(V'\right) \leq k(V)$. In turn, this implies that $\left(V^H - V^L\right)\left(V'\right) \leq \left(V^H - V^L\right)(V)$. Since $V^H\left(V'\right) \geq V^H(V)$ by (8), it must be the case that $V^L\left(V'\right) \geq V^L(V)$, contradicting the absurd assumption.

2ii) The proof of the monotonicity of $V^H(\cdot)$ is similar. Assume there exist V, V' such that $V' > V$ and $V^H\left(V'\right) < V^H(V)$. Then by concavity of the value function and (8), $\mu\left(V'\right) \leq \mu(V)$. On the other hand, since $V^L(\cdot)$ is non-decreasing, $\left(V^H - V^L\right)\left(V'\right) < \left(V^H - V^L\right)(V)$ and $k\left(V'\right) < k(V)$ as well. Finally, by (7) $\mu\left(V'\right) > \mu(V)$, leading to a contradiction. ■

Proposition 7.

Proof. i) By contradiction, assume that $k(V) = k^*$. If $W'\left(V^H\right) < W'\left(V^L\right)$, the surplus can be raised by decreasing V^H and increasing V^L. If $W'\left(V^H\right) = W'\left(V^L\right)$ then the value function is linear over the range $\left(V^L, V^H\right)$ and thus $pW\left(V^H\right) + (1-p)W\left(V^L\right) = W\left(\frac{V}{\delta}\right)$. On the other hand, $W(V) < \widetilde{W}$ implies $pW\left(V^H\right) + (1-p)W\left(V^L\right) < W(V)$. Thus $pW\left(V^H\right) + (1-p)W\left(V^L\right) = W\left(\frac{V}{\delta}\right) > W(V) > pW\left(V^H\right) + (1-p)W\left(V^L\right)$ which is a contradiction.ii) Notice that for $V \geq \tilde{V}$, $W(V) = \widetilde{W} \equiv \frac{pR(k^*)-k^*}{1-\delta}$. The only way such surplus can be achieved is by setting $k(V) = k^*$. ■

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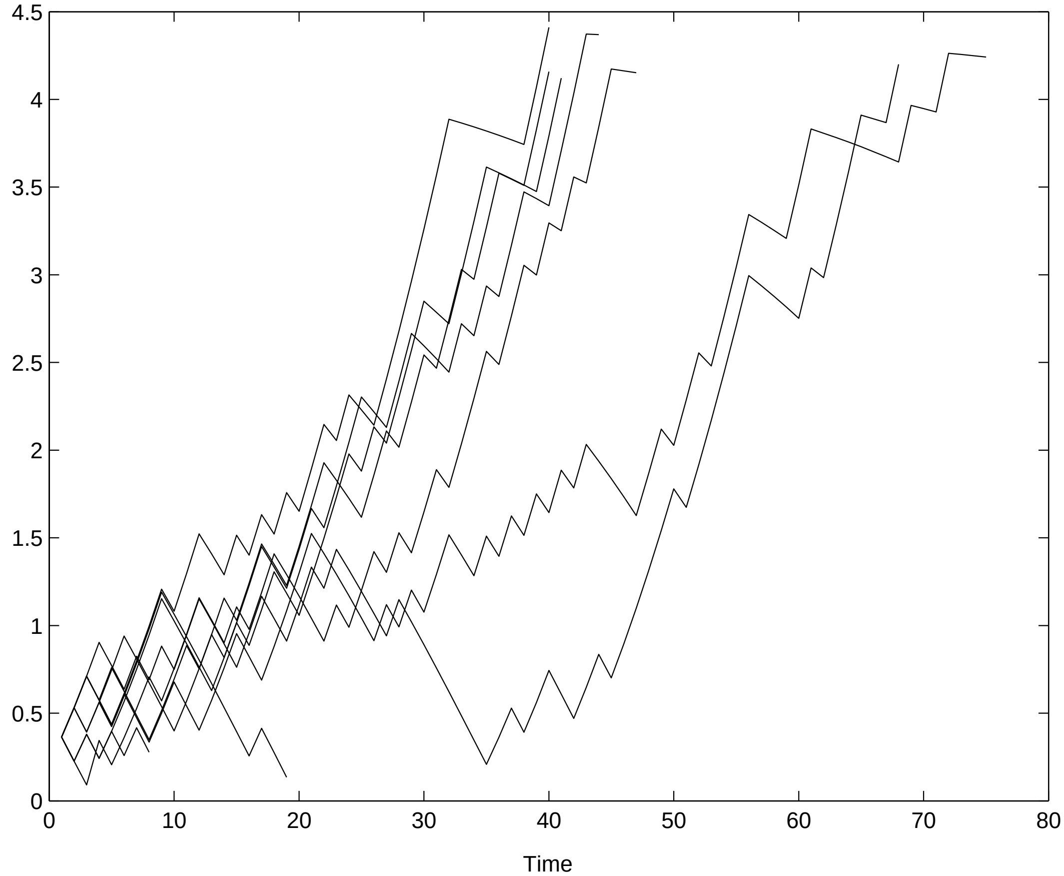

Figure 6: Sample Paths for the Equity Value V.

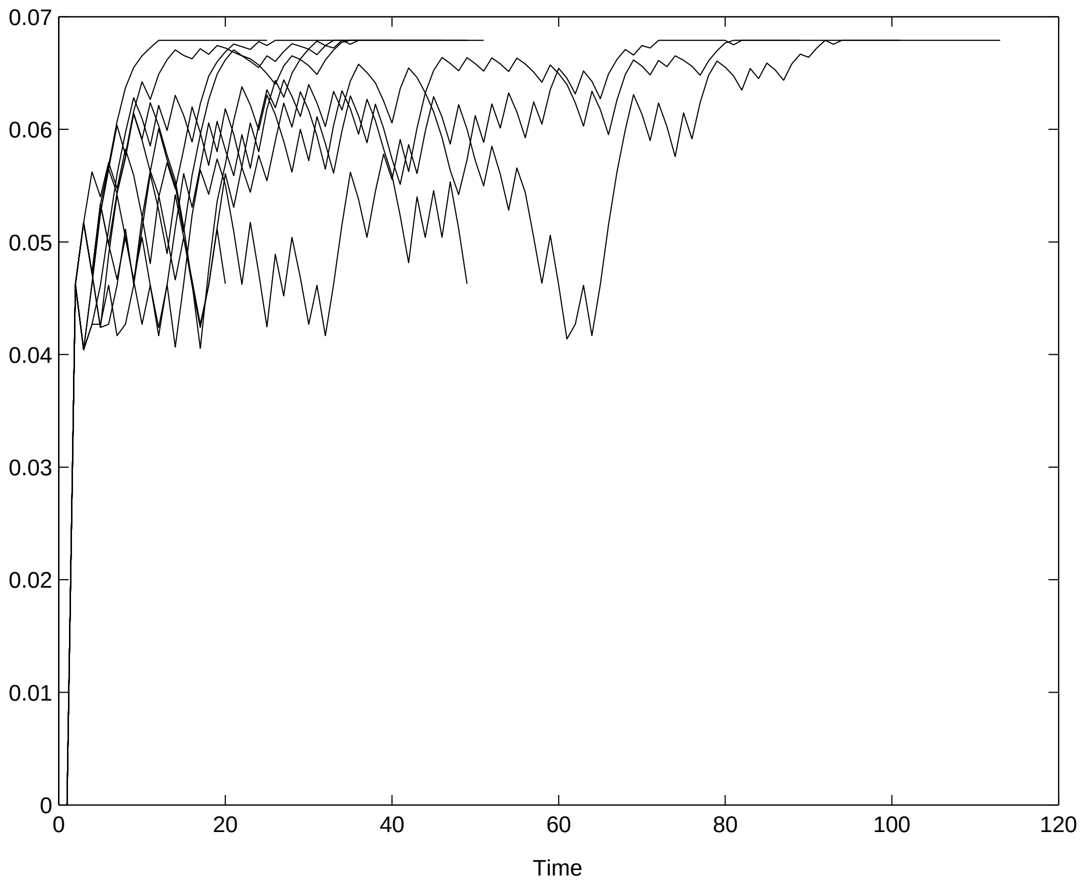

Figure 7: Sample Paths for the Capital Advancement k.

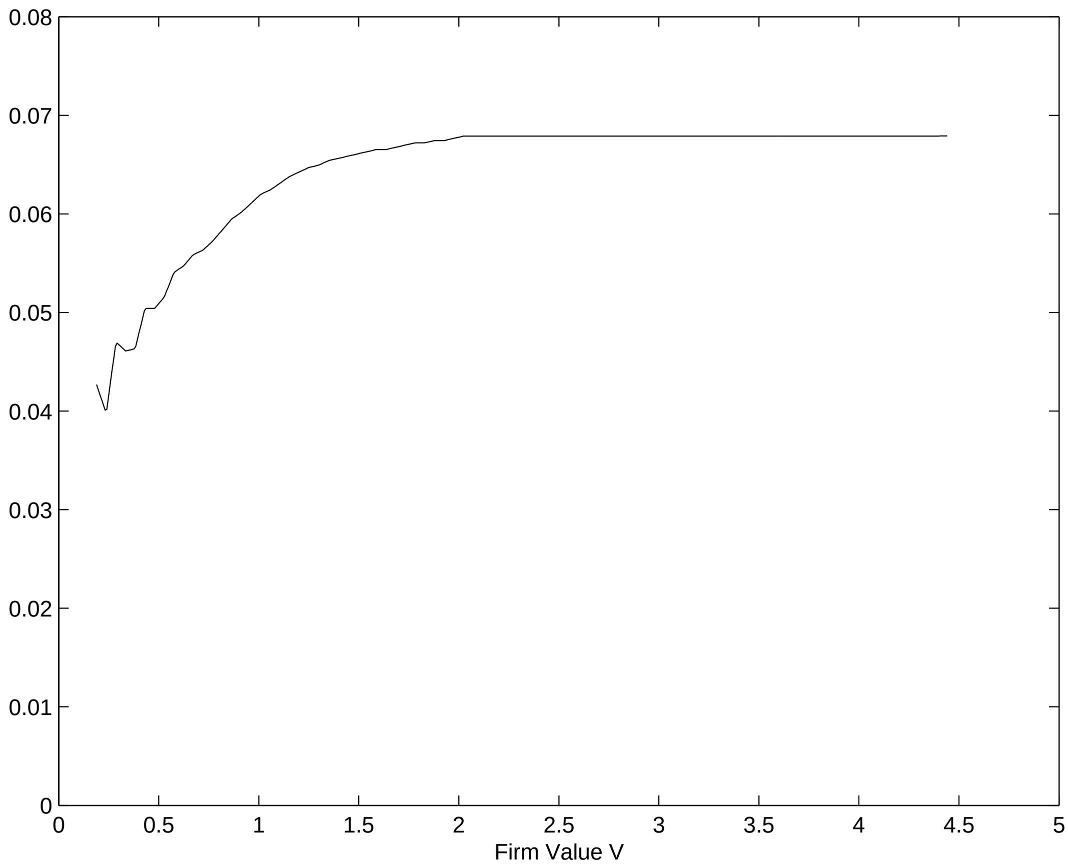

Figure 8: The Policy Function for the Capital Advancement.

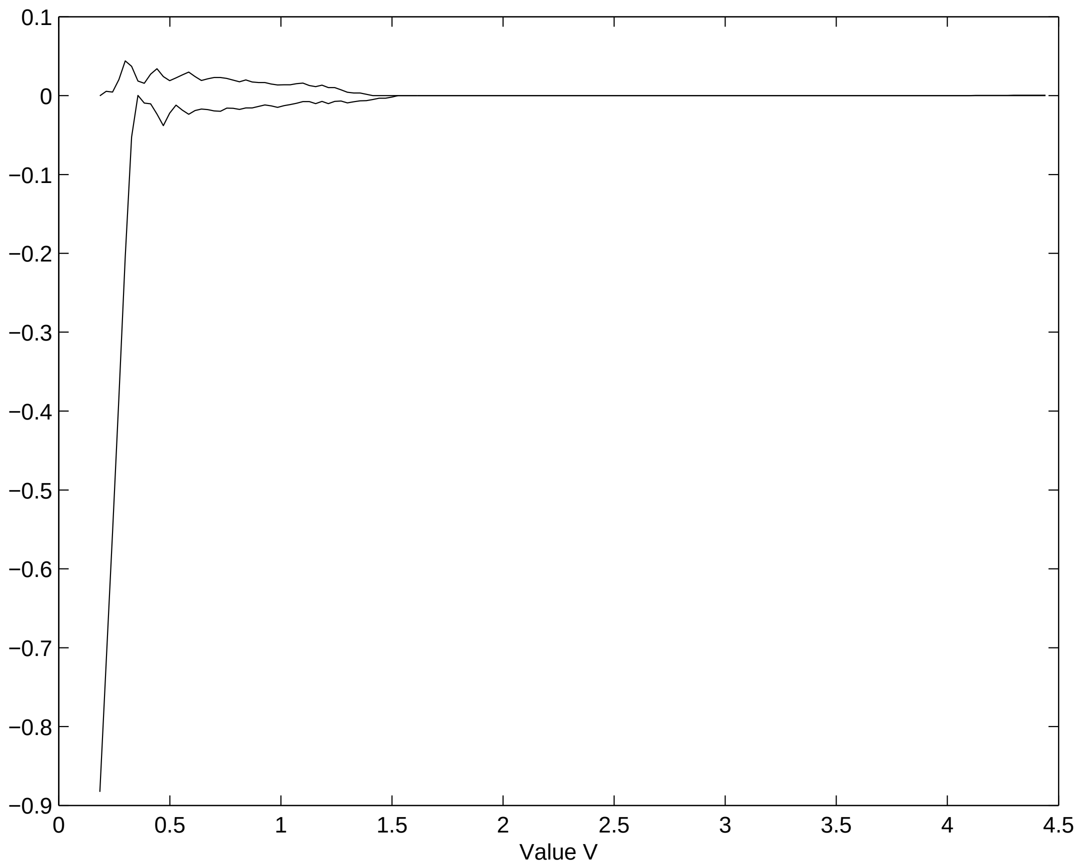

Figure 9: Growth rates of capital conditional on revenue realization.