Consensus consumer and intertemporal asset pricing with heterogeneous beliefs*

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Abstract

The aim of the paper is to analyze the impact of heterogeneous beliefs in an otherwise standard competitive complete markets economy. We first show the existence of a consensus belief. The construction of a consensus *probability* belief, as well as a consensus consumer are shown to be also valid modulo a predictable (in discrete time) or finite variation (in continuous time) aggregation bias. We then use our construction of a consensus consumer to analyze the impact of beliefs heterogeneity on the CCAPM, on the expression of the risk free rate, on the assets price and volatility.

1. Introduction

The representative agent approach introduced by Negishi (1960) and developed by Rubinstein (1974), Breeden and Litzenberger (1978), Constantinides (1982) has become a significant cornerstone of theoretical and applied macroeconomics and has been the basis for many developments in finance. Among these developments, the Capital Asset Pricing Model (CAPM, Sharpe 1964 and Lintner 1965) and the Consumption based CAPM (CCAPM, Ingersoll, 1987, Huang and Litzenberger, 1988, Duffie, 1996) play an important role. Given their empirical tractability, these models have generated extensive empirical tests and subsequent theoretical extensions. However, as mentioned by Williams (1977), "difficulties remain, significant among which is the restrictive assumption of homogeneous expectations". It has been since repeatedly argued that the diversity of investors forecasts (due possibly but not exclusively to differences of information and/or of priors) is an important part of any proper understanding of the workings of asset markets (Lintner (1969), Rubinstein (1975, 1976), Gonedes (1976), Miller (1977), Williams (1977), Jarrow (1980), Mayshar (1981, 1983), Cragg and Malkiel (1982), Varian (1985, 1989), Abel (1989), Harris-Raviv (1993), Detemple and Murthy (1994), Basak (2000), Gallmeyer (2000), Welch (2000), Hollifield-Gallmeyer (2002), Calvet et al. (2002), Diether et al. (2002)).

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The aim of the present paper is to analyze the consequences of facing the issue of heterogeneous subjective beliefs in an otherwise standard Arrow-Debreu equilibrium economy. More precisely, we start from a given equilibrium with heterogeneous beliefs in an otherwise standard model and we investigate the following issues. 1) Is it possible to define consensus beliefs, i.e. "those beliefs which, if held by all individuals (...) would generate the same equilibrium prices as in the actual heterogeneous economy." (Rubinstein (1975) 2) Is it still possible in such a context to define a representative agent (or consensus consumer). 3) Is it possible "to construct sharing rules which indicate how consumption or portfolio choices of a particular consumer deviate from the per capita choices of the consensus consumer" (Rubinstein, 1976) or more precisely, is it possible to relate diversity of individual portfolios to heterogeneity of individual beliefs. 4) Can the introduction of heterogeneous beliefs help us to better understand the risk premium puzzle (Mehra-Prescott, 1985, see Kocherlakota, 1996 for a survey) or more generally, what is the impact of beliefs heterogeneity on the risk premium (or market price of risk). 5) Can the introduction of heterogeneous beliefs help us to better understand the risk free rate puzzle (Weil, 1989) or more generally, what is the impact of beliefs heterogeneity on the risk free rate. 6) Can the introduction of heterogeneous beliefs lead to higher prices volatility or more generally, what is the impact of beliefs heterogeneity on the assets price and volatility.

Since we want to focus on the impact of the presence of heterogeneous beliefs "per se", we consider these issues in the context of an otherwise standard model and in particular we do not suppose that there are short sale constraints (except in the additional remarks). We shall deal with a continuous time as well as with a discrete time framework.

The paper is organized as follows. We present in Section 2 a method to aggregate in an intertemporal framework heterogeneous individual subjective beliefs into a single consensus belief. Given an observed equilibrium with heterogeneous probabilities, we look for a consensus belief, which, if held by all investors would lead to an equivalent equilibrium, in the sense that it would leave invariant certain quantities to be defined. We first define in the spirit of Calvet et al. (2002) an "equivalent equilibrium of the first kind" where all investors would share the consensus belief, by 1) the equivalent equilibrium generates the same valuation of assets by the market (the same equilibrium market prices), 2) every investor is indifferent at the margin between investing one additional unit of income in the observed equilibrium (using his own subjective belief) and in the equivalent equilibrium (using the aggregate consensus belief). We prove the existence of such an equivalent equilibrium modulo income transfers. We then define an "equivalent equilibrium of the second kind" where all investors would share the consensus belief, by only imposing the first condition, i.e., the equilibrium market prices remain the same, and by prohibitting income transfers and we prove the existence of such an equilibrium. From there, we construct, as in the standard setting, a consensus consumer, i.e. a consumer endowed with the market portfolio and the consensus belief, who generates the same equilibrium prices as in the original equilibrium. We point out two essential effects of the introduction of heterogeneity in the investors beliefs on the equilibrium (state) price (density). There is first a change of probability effect, the consensus probability being a weighted average of the individual subjective probabilities and second, a (possibly negative) discount effect. As a consequence of the presence of this discount effect, the heterogeneous beliefs setting cannot in general be simply reduced to an homogeneous beliefs setting, with an average belief. We end Section 2 by showing for the class of linear risk tolerance utility functions how the consensus probability and the discount process can be explicitly obtained and we characterize the situations where the discount rate is positive (resp. negative).

For this class of utility functions (which includes logarithmic, exponential as well as power utility functions), it clearly appears that the consensus probability is directly related to the average belief and that the discount process is directly related to the dispersion of individual beliefs. In Section 3, we analyze the impact of these two effects on the state price density, the market price of risk, the market volatility and the risk-free rate. In particular, we derive an adjusted CCAPM formula. We prove that only the change of probability effect has an impact on the market price of risk. We find that the CCAPM formula under heterogeneous beliefs is given by the CCAPM formula in an economy where all investors would share the same probability belief namely the consensus belief obtained through the aggregation procedure. The impact of the introduction of heterogeneous beliefs on the market price of risk is then very clear: it leads to an increase (resp. decrease) of the market price of risk of a given asset (with respect to the homogeneous setting) if and only if the consensus probability is pessimistic (resp. optimistic), where pessimistic is meant in the sense that the instantaneous rate of return of the considered asset (under this probability) is lower that under the objective initial probability. Our results are consistent with those of Abel (2000), Cechetti et al. (2000), Hansen et al. (1999), or Anderson et al. (2000), which introduce distorded beliefs associated to cautious/pessimistic individual behavior. In our framework, the optimism/pessimism is relevant at the aggregate level, and we provide conditions on the individual subjective probabilities that lead to a pessimistic (resp. optimistic) consensus belief. For instance, in the case of linear risk tolerance utility functions, we prove that the consensus probability is pessimistic if there is no systematic bias on the belief upon the total wealth drift (or more generally, if the systematic bias is nonpositive), and if the optimistic (resp. pessimistic) investors have a risk tolerance lower (resp. higher) than the average.

Contrarily to the market price of risk, both the change of probability effect and the aggregation bias (in the form of a discount process) have an impact on the risk free rate. The impact of the change of probability effect from the initial probability to the consensus probability contributes to a lowering of the risk free rate if and only if the consensus probability is pessimistic. The impact of the aggregation bias contributes to an increase (resp. decrease) of the risk free rate when the "discount rate" is nonnegative, which has a clear interpretation: a nonnegative "discount rate" means that future consumption is less important for the representative agent, and leads to a higher equilibrium interest rate. Since we can for linear risk tolerance utility functions characterize the situations for which the discount rate is nonnegative or nonpositive, we are able, for this class of utility functions, to determine the impact of beliefs heterogeneity on the risk free rate.

We also analyze the impact of beliefs heterogeneity on the assets price and volatility. We find that the introduction of heterogeneity may lead to lower asset prices and higher volatility than in the standard setting. Related work includes Gallmeyer (2000), Hollifield-Gallmeyer (2002), Brennan-Xia (2001).

The results obtained in a continuous time framework are extended to a discrete time framework in Section 4. Section 5 is devoted to some additional and concluding remarks. In particular, it is shown that the beliefs heterogeneity induces a distortion of the risk sharing rule and that for each agent, this distortion is monotone in individual beliefs deviations from the consensus probability.

All the proofs are in Appendix A.

2. Consensus belief, consensus consumer

In the classical representative agent approach, all investors are taken as having the same subjective beliefs, the same utility functions and the same opportunity sets. In this section, we analyze to which extent this approach can be extended to heterogeneous subjective beliefs. More precisely, we start from a given equilibrium with heterogeneous beliefs in an otherwise standard model, and we explore to which extent it is possible 1) to define a consensus belief, i.e. "a belief, which, if held by all individuals (...) would generate the same equilibrium prices as in the actual heterogeneous economy" (Rubinstein, 1975) and 2) to define a representative agent (or a consensus consumer).

The model is standard, except that we allow the agents to have distinct subjective probabilities. The framework can be either discrete or continuous. However, in this section we will focus on the continuous time setting (see Section 4 for the discrete time setting). We fix a finite time horizon T on which we are going to treat our problem. We consider a filtrated probability space $\left(\Omega, (F_t)_{t\in[0,T]}, P\right)$, where the filtration $(F_t)_{t\in[0,T]}$ satisfies the usual conditions. Each investor indexed by $i = 1, \cdots, N$ solves a standard dynamic utility maximization problem. He has a current income at date t denoted by $e_t^{*^i}$ and a von Neumann-Morgenstern utility function for consumption of the form $E^{Q^i}\left[\int_0^T u_i(t, c_t)dt\right]$, where Q^i is a probability measure equivalent to P which corresponds to the subjective belief of individual i. If we denote by $\left(M_t^i\right)_{t\in[0,T]}$ the positive density process of Q^i with respect to P, then the utility function can be rewritten as $E^P\left[\int_0^T M_t^i u_i(t, c_t)dt\right]$.

We make the following classical assumptions.

Assumption

- for all $t \in \mathbf{T}$, $u_i(t, \cdot) : [k_i, \infty) \to \mathbb{R} \cup \{-\infty\}$ is of class C^1 on (k_i, ∞), strictly increasing and strictly concave[1],
- $u_i(., c)$ and $u_i'(t, .) = \frac{\partial u_i}{\partial c}(t, .)$ are continuous on $[0, T]$,
- for $i = 1, \cdots, N$, $P \otimes dt\left\{e^i > k_i\right\} > 0$ and $P \otimes dt\left\{e^i \geq k_i\right\} = 1$,
- there exists $\varepsilon > 0$ such that $e^* > \sum_{i=1}^N k_i + \varepsilon$, $P \otimes dt$ a.s.,
- $E^P\left[\int_0^T e_t^* dt\right] < \infty$,
- the density process M^i is uniformly bounded for $i = 1, \cdots, N$.

The first two conditions are classical regularity conditions. If we interpret k_i as a minimum subsistence level, the third condition can be interpreted as a survival condition for each agent[2].

[1] Note that we could easily generalize all the following results to the case where k_i is a function of t.

[2] In fact, this condition can be weakened. For instance, if the equilibrium price is known, we only need to impose that

$$E^P\left[\int_0^T q_t^* e_t^i dt\right] > E^P\left[\int_0^T q_t^* k_i dt\right],$$

which permits, through trade, to reach allocations $\left(y^{*^i}\right) > k_i$ $P \otimes dt$ a.s.

The fourth condition is a survival condition for the whole economy. The last two conditions are technical ones and are directly linked to the choice of L^1 as a consumption space for the agents[3].

We do not specify the utility functions u_i, although we shall focus on the classical cases of linear risk tolerance utility functions (which include logarithmic, power as well as exponential utility functions). We take the different subjective probabilities as given. As in Varian (1985,1989), Abel (1989) or Harris-Raviv (1993), they reflect difference of opinion among the agents rather than difference of information; indeed, "we assume that traders receive common information, but differ in the way they interpret this information" (Harris-Raviv ,1993). They might come from a Bayesian updating of the investors predictive distribution over the uncertain returns on risky securities as in e.g. Williams (1977), Detemple-Murthy (1994), Zapatero (1998), Gallmeyer (2000), Basak (2000), Hollifield-Gallmeyer (2002), but we do not make such an assumption; we only impose that the subjective probabilities be equivalent to the initial one. In continuous time, this mainly leads to assuming that investors differ only in their opinion about means and agree about variances. This hypothesis is reasonable, since as underlined by Hollifield-Gallmeyer (2002), "since aggregate consumption is observed continuously, all investors can perfectly estimate its volatility by computing the output process quadratic variation". Moreover, in the discrete time setting, Abel (1989) proves that "variations in variances can be ignored because such variations can be easily handled in the representative agent framework. Prices in such an economy are exactly the same as in an economy in which all investors have identical variances (given by some equally weighted average on the individual variances)". Notice that the above mentioned models with learning are not "more endogeneous" since the investor updating rule and the corresponding probabilities can be determined separately from his optimization problem (see e.g. Genotte, 1986).

In the remainder of the paper, an admissible consumption plan for agent i is an adapted $[k_i, \infty)$-valued process y^i such that $E^P\left[\int_0^T \left|y_t^i\right| dt\right] < \infty$. We recall that an equilibrium relatively to the beliefs $\left(M^i\right)$ and the income processes $\left(e^i\right)$ is defined by a positive, uniformly bounded price process q^* and a family of optimal admissible consumption plans $\left(y^{*^i}\right)$ such that markets clear, i.e.

$$\left\{\begin{array}{c} y^{*^i} = y^i(q^*, M^i, e^i) \\ \sum_{i=1}^N y^{*^i} = \sum_{i=1}^N e^i \equiv e^* \end{array}\right.$$

where

$$y^i(q, M, e) \equiv \arg \max_{E^P\left[\int_0^T q_t\left(y_t^i - e_t\right)dt\right] \leq 0} E^P\left[\int_0^T M_t u_i(c_t) dt\right].$$

We start from an equilibrium $\left(q^*, \left(y^{*^i}\right)\right)$ relatively to the beliefs $\left(M^i\right)$ and the income processes e^i. Such an equilibrium, when it exists, can be characterized by the first order necessary

[3] This pair of conditions can easily be replaced by

- $E^P\left[\int_0^T \left|e_t^i\right|^p dt\right] < \infty$
- $E^P\left[\int_0^T \left|M_t^i\right|^q dt\right] < \infty$

where p and q are such that $\frac{1}{p} + \frac{1}{q} = 1$.

conditions for individual optimality and the market clearing condition. These conditions can be written as follows

$$\left\{\begin{array}{ll} M_t^i u_i'(t, y^{*^i}) \leq \lambda_i q_t^*, & \text{on } \left\{y^{*^i} = k_i\right\} \\ M_t^i u_i'(t, y^{*^i}) = \lambda_i q_t^*, & \text{on } \left\{y^{*^i} > k_i\right\} \\ E^P\left[\int_0^T q_t^*\left(y_t^{*^i} - e_t^i\right) dt\right] = 0 & \\ \sum_{i=1}^N y^{*^i} = e^* & \end{array}\right. \tag{2.1}$$

for some set of positive Lagrange multipliers (λ_i).

In the next, we will say that $(q^*, (y^{*^i}))$ is an interior equilibrium relatively to the beliefs $\left(M^i\right)$ and the income processes $\left(e^i\right)$ if $y^{*^i} > k_i$, $P \otimes dt$ a.s. for $i = 1, \cdots, N$. Note that under the following additional condition

$$u_i'(t, k_i) = \infty \text{ for } t \in [0, T] \text{ and } i = 1, \cdots, N,$$

all the equilibria are interior ones.

Our first aim is to find an "equivalent equilibrium" in which the heterogeneous subjective beliefs would be aggregated into a common belief M. Following the approach of Calvet et al. (2002), we shall define an "equivalent equilibrium of the first kind" by two requirements. First, the "equivalent equilibrium" should generate the same equilibrium price process q^* as in the original equilibrium with heterogeneous beliefs, so that every asset gets the same valuation in both equilibria. Second, every investor should be indifferent at the margin between investing one additional unit of income in the original equilibrium with heterogeneous beliefs and in the "equivalent equilibrium", so that each asset gets the same marginal valuation by each investor (in terms of his marginal utility) in both equilibria. We shall see in Section 5 that this requirement is in fact equivalent to the condition that each investor's observed (or initial) demand be larger than (resp. equal to, less than) his demand in the "equivalent equilibrium" if and only if he attaches a subjective probability that is larger than (resp. equal to, less than) the aggregate common probability, which appears as a natural requirement for an aggregation procedure. The existence of such an "equivalent equilibrium of the first kind" is given by the following proposition.

Proposition 2.1. *Consider an interior equilibrium $(q^*, (y^{*^i}))$ relatively to the beliefs $\left(M^i\right)$ and the income processes $\left(e^i\right)$. There exists a unique positive and adapted process $(M_t)_{t\in[0,T]}$ with $M_0 = 1$, there exists a family of income processes $\left(\bar{e}^i\right)$ with $\sum_{i=1}^N \bar{e}^i = e^*$ and a unique family of individual consumption processes $(\bar{y}^i)$ such that $\left(q^*, (\bar{y}^i)\right)$ is an interior equilibrium relatively to the common belief M and the income processes $\left(\bar{e}^i\right)$ and such that individual marginal valuation remains the same, i.e.*

$$M_t^i u_i'(t, y^{*^i}) = M_t u_i'(t, \bar{y}^i), \qquad t \in [0, T], \, i = 1, \cdots, N.$$

This means that $\left(q^*, (\bar{y}^i)\right)$ is an equilibrium with income transfers relatively to the common belief M and the income processes $\left(e^{*^i}\right)$ such that individual marginal valuation are the same as in the original equilibrium with heterogeneous beliefs. In other words, we proved that modulo a feasible modification of the individual incomes (i.e. $\sum_{i=1}^N \bar{e}^i = \sum_{i=1}^N e^{*^i}$) the initial equilibrium price process remains an equilibrium price process in an homogeneous beliefs setting. The positive process M can then be interpreted as a consensus belief. In particular, if there is no heterogeneity, i.e. if all the investors have the same belief represented by $M^i = \tilde{M}$ for all i, we obtain $M = \tilde{M}$ and there is no transfer nor optimal allocations modification (i.e. $\bar{e}^i = e^i$ and $\bar{y}^i = y^{*^i}$ for all i). Our aggregation procedure satisfies then the so-called homogeneity requirement introduced by Rubinstein (1976): the aggregate belief is equal to the investors subjective probabilities when they happen to initially share the same belief.

As we said, the consensus belief that we obtained is such that the associated equilibrium price as well as the individual marginal valuation remain the same as in the heterogeneous framework. The individual marginal valuation invariance property is in fact equivalent to the invariance of the Lagrange multipliers. In other words, the "equivalent equilibrium of the first kind" is characterized by

$$\begin{cases} M_t u_i'(t, \bar{y}^i) = \lambda_i q_t^*, \qquad t \in [0,T] \\ E^P \left[\int_0^T q_t^* \left(\bar{y}_t^i - \bar{e}_t^i \right) dt \right] = 0 \\ \sum_{i=1}^N \bar{y}_t^i = e_t^* \end{cases}$$

where the $\lambda_i^{\prime}$s are the same as in the characterization of the initial equilibrium (Equation (2.1)). The cost we paid in order to maintain the Lagrange multipliers invariant has been to authorize income transfers between agents. Another way to construct an "equivalent equilibrium" and hence a consensus belief would be to prohibit transfers and to allow for individual marginal valuation modification. More precisely we have the following result, which permits to define the concept of "equivalent equilibrium of the second kind".

Proposition 2.2. *Consider an interior equilibrium* $(q^*, (y^{*^i}))$ *relatively to the beliefs* $\left(M^i\right)$ *and the income processes* $\left(e^i\right)$. *There exists a positive and adapted process* $(M_t)_{t\in[0,T]}$ *with* $M_0 = 1$ *and a family of individual consumption processes* $(\bar{y}^i)$, *such that* $\left(q^*, (\bar{y}^i)\right)$ *is an interior equilibrium relatively to the common belief* M *and the income processes* $\left(e^i\right)$.

We shall denote by $\left(\lambda_i'\right)$ the corresponding Lagrange multipliers. For both constructions, once the result on beliefs aggregation achieved, it is easy to construct as in the standard case, a representative agent, i.e. an expected utility maximizing aggregate investor, representing the economy in equilibrium. More precisely, we look for a single aggregate investor, endowed with the market portfolio, who, when maximizing his expected utility under the aggregate belief generates the same equilibrium prices as in the original equilibrium. The next proposition establishes the existence of such a representative agent.

As in the standard case, for $\alpha \in \left(\mathbb{R}_+^*\right)^N$, we introduce the function $u_\alpha(t,x) = \max_{\sum_{i=1}^N x_i \le x} \sum_{i=1}^N \frac{1}{\alpha_i} u_i(t, x_i)$.

Corollary 2.3. *Consider an interior equilibrium* $(q^*, (y^{*^i}))$ *relatively to the beliefs* $\left(M^i\right)$ *and the income processes* $\left(e^i\right)$. *There exists a consensus investor defined by the normalized von Neumann-Morgenstern utility function* u_λ *(resp.* $u_{\lambda'}$*) and the consensus belief* M *of Proposition 2.1 (resp. Proposition 2.2), in the sense that the portfolio* e^* *maximizes his expected utility* $E^P\left[\int_0^T M_t u(t, c_t) dt\right]$ *under the market budget constraint* $E^P \left[\int_0^T q_t^* \left(c_t - e_t^*\right) dt\right] \le 0$.

The construction of the representative agent is exactly the same as in the standard setting. As a consequence, all classical properties of the representative agent utility function remain valid in our setting (see e.g. Huang-Litzenberger, 1988). Among other properties, if all individual utility functions are state independent, then the aggregate utility function is also state independent, and if all individual utility functions exhibit linear risk tolerance, i.e. are such that $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i + \eta x$, then the aggregate utility function is also such that $-\frac{u'(t,x)}{u''(t,x)} = \bar{\theta} + \eta x$ where $\bar{\theta} = \sum_{i=1}^N \theta_i$.

Remark that our aggregation procedure applies to a framework where agents have common beliefs but possibly different state dependent utility functions of the following "separable" form

$$U_i(t,\omega,x) = v_i(t,\omega)\, u_i(t,x).$$

In that case[4], we obtain a representative agent utility function of the same form $U(t,\omega,x) = v(t,\omega)\, u(t,x)$, where ut is obtained from the u_i's as in the standard framework, and where v is an average of the v_i's.

Example 2.4 (A). *If the individual utility functions are of exponential type, i.e. if* $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i > 0$, *then*

$$\begin{aligned} u'(t,x) &= ae^{-x/\bar{\theta}} \\ M &= \prod_{i=1}^N \left(M^i\right)^{\theta_i/\bar{\theta}} \end{aligned}$$

for $a = e^{e_0^*/\bar{\theta}}$.

It is immediate by Hölder's Inequality that the consensus belief obtained in the previous example is not a martingale but a supermartingale; the consensus belief is not necessarily the density process of a given probability equivalent to P. In this example, the supermartingale property means that the beliefs heterogeneity induces a discount factor on the average utility at future dates. This effect will be analyzed in detail subsequently.

Example 2.5 (B). *If the individual utility functions are of power type, i.e. such that* $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i + \eta x$ *for* $\eta \neq 0$, *then*

$$\begin{aligned} u'(e) &= b(\bar{\theta} + \eta e)^{-\frac{1}{\eta}} \\ M &= \left[\sum_{i=1}^N \gamma_i (M_i)^\eta\right]^{\frac{1}{\eta}} \end{aligned}$$

for $b = (\bar{\theta} + \eta e_0)^{\frac{1}{\eta}}$ *and* $\gamma_i = \frac{\lambda_i^{-\eta}}{\sum_{j=1}^N \lambda_j^{-\eta}}$ *(we have* $\sum_{i=1}^N \gamma_i = 1$*).*

Notice that the consensus belief M *is then a supermartingale when* $\eta < 1$, *a martingale when* $\eta = 1$ *(logarithmic case), and a submartingale when* $\eta > 1$.

[4] Note that even if the v_i's are not martingales, our results still apply.

The consensus belief we constructed in Propositions (2.1) and (2.2) solves Rubinstein's (1976) weak aggregation problem. Indeed, prices are determined as if all the individual beliefs were described by M. Furthermore, if all the agents happen to share the same belief then the construction of M would lead to that common belief. We may ask whether the strong aggregation problem (with Rubinstein's (1976) terminology) is solved. In other words, is it possible to construct M and to obtain q^* as functions of the aggregate charasteristics (e.g. consumption) of the economy and not of the individual ones?

In the exponential case, we know that $M = \prod_{i=1}^N \left(M^i\right)^{\theta_i/\bar{\theta}}$, $u'(e^*) = a \exp -\frac{e^*}{\theta} = \exp -\frac{e^*-e_0^*}{\theta}$, so that the equilibrium price is given by $q^* = \prod_{i=1}^N \left(M^i\right)^{\theta_i/\bar{\theta}} \exp -\frac{e^*-e_0^*}{\theta}$, which solves the strong aggregation problem.

In the power utility functions case, the consensus belief is given by $M = \left[\sum_{i=1}^N \gamma_i (M_i)^\eta\right]^{\frac{1}{\eta}}$ and the equilibrium price process is given by $q^* = \left[\sum_{i=1}^N \gamma_i (M_i)^\eta\right]^{\frac{1}{\eta}} \left(\frac{\bar{\theta}+\eta e^*}{\bar{\theta}+\eta e_0^*}\right)^{-\frac{1}{\eta}}$ and both of them depend on the initial distribution of income through the γ_i's. Hence, the strong aggregation problem is not solved and M is a consensus characteristic but not a composite characteristic as defined by Rubinstein (1976).

However, as we shall see in the next section, the formulation $q_t^* = M_t u'(t, e_t^*)$ will enable us to compare the equilibrium under heterogeneous beliefs with the equilibrium in the standard setting.

It is interesting to notice in Examples (2.4) and (2.5) that for all utility functions in the classical class of linear risk tolerance utility functions, the consensus belief is obtained as a weighted average of the individual subjective beliefs. In the general case, since for all i, we have

$$\frac{1}{\lambda_i} M_t^i u_i'(t, y_t^{*^i}) = M u'(t, e_t^*)$$

we obtain

$$\begin{aligned} e_t^* &= \sum_{i=1}^N I_{u_i}\left(t, \lambda_i \frac{M_t}{M_t^i} u'(t, e^*)\right) \\ &= \sum_{i=1}^N I_{u_i}\left(t, \frac{M_t}{M_t^i} u_i'(t, \bar{y}_t^i)\right). \end{aligned}$$

It is clear then that we cannot have $M_t > M_t^i$ (resp. $M_t < M_t^i$), for all i, with a positive probability. Indeed, this would lead to $\sum_{i=1}^N I_{u_i}\left(t, \frac{M_t}{M_t^i} u_i'(t, \bar{y}_t^i)\right) < \sum_{i=1}^N \bar{y}_t^i = e_t^*$ (resp. $\sum_{i=1}^N I_{u_i}\left(t, \frac{M_t}{M_t^i} u_i'(t, \bar{y}_t^i)\right) > \sum_{i=1}^N \bar{y}_t^i = e_t^*$) with a positive probability which contradicts the equations above. Consequently, the consensus belief can still be considered as an average of the individual beliefs.

The process M represents a consensus belief, however, as seen above, except in the logarithmic case, it fails to be a martingale. Consequently, it can not be interpreted as the density process of a given probability measure. It is easy to see on the following example that it is not possible in general to recover the consensus belief as a martingale, as soon as we want the equilibrium price to remain the same and the optimal allocations in the equivalent equilibrium to be feasible, in the sense that they still add up to e^*.

Example 2.6. *Let us consider again the exponential utility functions case* $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i$*. Let us consider the "equivalent interior equilibrium" allocations* $(\bar{y}_t^i)$ *associated to the same equilibrium price* q^* *as in the initial equilibrium and to a common belief* M*. The first order conditions then lead to*

$$M_t u_i'(t, \bar{y}_t^i) = \varsigma_i M_t^i u_i'(t, y_t^{*^i})$$

where M *is the candidate consensus belief and where the* ς_i*'s are given positive multipliers. This leads to*

$$M_t = \left(\prod_{i=1}^{N} \varsigma_i \right)^{1/N} \left(\prod_{i=1}^{N} M_t^i \right)^{1/N}$$

and M *is a martingale only if all the* M^i*'s are equal.*

There is therefore in general no solution to the weak aggregation problem if we impose that M be a martingale, i.e. the density process of a given probability. This means that in the general case, there is a bias induced by the aggregation of the individual probabilities into a consensus probability. We shall see in the next propositions that the utility function of the representative agent is not an expectation of the future utility from consumption but an expectation of a discounted future utility from consumption. The (possibly negative) discount rate will be in average nonnegative (resp. zero, resp. nonpositive) when M is a supermartingale (resp. martingale, resp. submartingale).

In order to further specify our model, let us assume that $(F_t)_{t\in[0,T]}$ is the $P-$augmentation of the natural filtration generated by a Brownian motion W on (Ω, F, P), and that e^* and the M^i's satisfy the following stochastic differential equations[5]

$$\begin{aligned} de_t^* &= \alpha_t e_t^* dt + \beta_t e_t^* dW_t, \qquad \beta > 0 \\ dM_t^i &= \delta_t^i M_t^i dW_t, \qquad M_0^i = 1 \end{aligned}$$

If we further assume that the utility functions are of class $C^{1,3}$, the first order conditions for an interior equilibrium and Itô's Lemma give us that the equilibrium price q^*, the equilibrium allocations $\left(y^{*^i}\right)$ as well as M satisfy also stochastic differential equations of the form

$$\begin{aligned} dq_t^* &= \mu_{q^*}(t) q_t^* dt + \sigma_{q^*}(t) q_t^* dW_t \\ dy_t^{*^i} &= \mu_{y^*}^i(t) y_t^{*^i} dt + \sigma_{y^*}^i(t) y_t^{*^i} dW_t \\ dM_t &= \mu_M(t) M_t dt + \delta_M(t) M_t dW_t \end{aligned}$$

In the nexts, we will assume that δ_M satisfies the so-called Novikov's condition[6], i.e. $E\left[\exp\left(\int_0^T \delta_M^2 dt\right)\right] < \infty$.

[5] We assume that the coefficients of these SDE's satisfy the classical global Lipschitz and linear growth conditions that ensure existence and uniqueness of a strong solution (see Karatzas-Shreve (1988) for more details and weaker conditions).

[6] This condition ensures that $\int \delta_M dW$ is a martingale. It is easy to check that this condition is in particular satisfied if the total wealth is uniformly bounded and if, for $i = 1, \cdots, N$, we have $E\left[\exp\left(N \int \delta_i^2 dW\right)\right] < \infty$. This last condition is a little bit stronger than Novikov's condition for each δ_i, $i = 1, \cdots, N$.

Proposition 2.7. *Consider an interior equilibrium price process q^* relatively to the beliefs (M^i), and the income processes (e^i). There exists a positive martingale process $\bar{M}$ with $\bar{M}_0 = 1$, and a finite variation positive process B with $B_0 = 1$ such that, with the notations of the previous propositions*

$$\bar{M}_t B_t u'(t, e_t^*) = q_t^*.$$

These processes are given by

$$\begin{aligned} B_t &= \exp \int_0^t \mu_M(s) ds \\ \bar{M}_t &= \exp \left(\int_0^t \delta_M(s) dW_s - \frac{1}{2} \int_0^t \delta_M^2(s) ds \right). \end{aligned}$$

The process B measures the default of martingality of the consensus belief M and leads to a (possibly negative) discount of utility from future consumption through the "discount rate" $(-\mu_M)$. The adjustment process B measures then the aggregation bias induced by the heterogeneity of individual beliefs.

There are mainly two cases where there is no such adjustment effect, i.e. B is constant equal to 1:

- if all investors share the same belief. The consensus belief $\bar{M}$ is equal to that common belief,
- if all the utility functions are logarithmic. This property makes the case of logarithmic utility functions (which is often considered in the literature, see e.g. Rubinstein, 1976, Detemple-Murthy, 1974, Zapatero, 1998) very specific.

When B fails to be constant, it is a natural concern to determine whether it is greater or smaller than one, increasing or decreasing. This will permit to analyze the nature of this aggregation bias and its impact on the equilibrium state price density. For linear risk tolerance utility functions, the processes $\bar{M}$ and B can be explicitly computed. We shall denote the risk tolerance by $T_i(t) \equiv \frac{u_i'(t, y_t^{*i})}{u_i''(t, y_t^{*i})}$.

Proposition 2.8. *For linear risk tolerance utility functions, we have*

$$\begin{aligned} \delta_M &= \sum_{i=1}^N \kappa^i \delta^i \\ &= E^\kappa [\delta] \end{aligned}$$

and

$$\begin{aligned} \mu_M &= \frac{1}{2}(\eta - 1) \left[\sum_{i=1}^N \kappa^i \left(\delta^i\right)^2 - \left(\sum_{i=1}^N \kappa^i \delta^i \right)^2 \right] \\ &= \frac{1}{2}(\eta - 1) Var^\kappa [\delta] \end{aligned}$$

where $\kappa^i = \frac{T_i}{\sum_{i=1}^N T_i}$ *satisfy* $\sum_{i=1}^N \kappa^i = 1$ *and* E^κ *(resp.* Var^κ*) denote the expected value (resp. variance) of* δ *across agents with a weight* κ^i *for agent* i*.*

Notice that $\mu_M \leq 0$ *if and only if* $\eta \leq 1$ *(this encompasses the exponential case).*

For this class of utility functions, δ_M appears to be a risk-tolerance weighted average of the individual δ^i's and μ_M is proportional to the variance of the δ^i's with respect to the same weights. It appears then that the equilibrium price can be represented as an equilibrium price in an equivalent economy where the individual beliefs are replaced by a risk tolerance weighted average belief and where an additional effect is added in order to take the initial heterogeneity into account. This effect is measured by B or equivalently by μ_M which is directly related to the beliefs dispersion.

Furthermore, it is easy to see that B is nondecreasing, greater than 1 (resp. nonincreasing, lower than 1) if $\eta \geq 1$ (resp. $\eta \leq 1$).

Another way of interpreting the result of Proposition (2.7) is to introduce in the spirit of Calvet et al. (2002) an adjustment of the market portfolio. Indeed, these authors proved in a two dates framework where consumption takes place only at the final date that there exists an aggregate belief represented by a probability measure modulo a scalar adjustment of the total wealth. In a dynamic setting with consumption at all the possible dates, the scalar adjustment process has to be replaced by a dynamic adjustment proces. More precisely, we obtain the following

Proposition 2.9. *Consider an interior equilibrium price process* q^* *relatively to the beliefs* $\left(M^i\right)$ *and the income processes* $\left(e^i\right)$*. There exists a positive martingale process* $\left(\bar{M}_t\right)_{t\in[0,T]}$ *with* $M_0 = 1$*, and an adjusted market portfolio* $\bar{e}$ *such that with the notations of the previous propositions*

$$\bar{M}_t u'(t, \bar{e}_t) = q_t^*$$

where $\bar{e}_t = I_u(t, B_t u'(t, e_t^*))$ *is a finite variation transformation of the aggregate wealth of the economy*[7]*.*

As in Calvet et al. (2002), this aggregation procedure transforms individual utility functions into a utility function of the same type (and identical to the one obtained in the standard case of homogeneous beliefs), individual probabilities into a common probability and this aggregation is made possible by an adjustment of the market portfolio. However our adjustment does not correspond in general to Calvet et al. (2002) one. Indeed, their adjustment consists into the

[7] Besides, we obtain more explicitly, in the case of power utility functions (for $\bar{\theta} = 0$), $\bar{e} = B^{-\eta} e^*$, and for exponential utility functions $\bar{e} = e^* - \bar{\theta} \log B$. The adjusted market portfolio $\bar{e}$ satisfies then the following stochastic differential equation

$$d\bar{e}_t = \bar{e}_t \left(\alpha_t - \eta \mu_M(t)\right) dt + \beta_t \bar{e}_t dW_t$$

for power and logarithmic utility functions and

$$d\bar{e}_t = \left(\alpha_t \bar{e}_t - \bar{\theta} \mu_M(t)\right) dt + \beta_t \bar{e}_t dW_t$$

for exponential utility functions.

It clearly appears on these examples that the adjusted wealth has the same volatility as the initial one and has an adjusted drift. This adjustment can then be interpreted as a (possibly negative) discounted version of the initial wealth. It reflects the effect of the discount factor B. A positive (resp. negative) discount of future utility appears as equivalent to an increase (resp. reduction) of the economic growth.

multiplication of the initial wealth by a scalar (in their static setting). The analogous result in a dynamic setting would be the multiplication of the initial wealth by a predictable process. In our case, the adjustment is made in a predictable way but is not multiplicative. However, in their construction, one can not interpret the consensus belief as a predictably discounted consensus probability in an otherwise fixed framework (no wealth adjustment)..

Notice that we know (from the Examples) if the adjustment has to be made upwards or downwards in all the classical linear risk tolerance utility functions cases. Indeed, the adjustment direction depends on whether B is greater or smaller than one.

To summarize, we have pointed out through previous propositions two distinct effects of the introduction of some beliefs heterogeneity on the equilibrium price .

There is first a change of probability effect from P to the new common probability Q, whose density is given by $\bar{M}$. This aggregate probability can be seen (at least in the classical utility functions cases) as a weighted average of the individual subjective probabilities. The weights of this average are given by the individual risk tolerances exactly as in Rubinstein (1976) or Detemple-Murthy (1994) where the authors focused on logarithmic utility functions.

The second effect is represented by an "aggregation bias" of the market portfolio or of the equilibrium (state) price (density), which is of finite variation and takes the form of a discount factor. We are able, for linear risk tolerance utility functions, to determine if it is associated to a positive or negative discount rate. Indeed, the setting in which all agents would share the same homogeneous belief would lead to equilibrium prices of the form $\left(\bar{M}_t u'(t, e_t^*)\right)_{t\in[0,T]}$ whereas in our setting, equilibrium prices are given by $\left(\overline{M}_t B_t u'(t, e_t^*)\right)_{t\in[0,T]} = \left(\bar{M}_t u'(t, \bar{e}_t)\right)_{t\in[0,T]}$. Moreover, the adjustment process can be seen (at least in classical cases) as a measure of dispersion of individual beliefs.

We shall now analyze the impact of these two features on the equilibrium properties.

3. Asset pricing with heterogeneous beliefs

In this section, we use our construction of a representative consumer (Section 2) to study the impact of heterogeneity of beliefs on asset pricing. We first explore the impact on the equilibrium (state) price (density). We then turn to the impact on the CCAPM formula (or more precisely on the market price of risk (MPR)) and on the risk free rate. In particular, we wish to analyze if heterogeneous beliefs might contribute to both an increase in the MPR and a decrease in the risk free rate, thereby helping us to better understand the risk premium puzzle (Mehra-Prescott, 1985) as well as the risk free rate puzzle (Weil, 1989) -see Kocherlakota (1996) for a survey on these puzzles. We end this section by analyzing the impact on asset prices and volatility.

3.1. State price density

We have obtained in the setting with heterogeneous beliefs the following expression for the equilibrium (state) price (density) $q^* = Mu'(e^*) = \overline{M}Bu'(e^*)$ which we want to compare to the expression obtained in the standard setting, which is given by $q = u'(e^*)$. We shall consider as the standard setting an equilibrium under homogeneous beliefs (given by the objective probability P), and where the representative agent utility function is given by[8] u_λ with the same (λ_i) as in our heterogeneous beliefs setting. This is in particular the case when the standard setting equilibrium has the same Lagrange multipliers as in our framework, or when investors have linear risk tolerance utility functions, since in that case (see e.g. Huang-Litzenberger, 1988), the representative agent utility function does not depend upon the individual Lagrange multipliers (or initial allocations).

We recall that we denote by μ_M the drift of the consensus belief M (or indifferently of the adjustment process B, which means that $-\mu_M$ is the "discount rate") and by δ_M the volatility of the consensus belief M (or indifferently of the consensus probability density $\overline{M}$). We easily obtain the following result.

Proposition 3.1. *The drift and volatility of the equilibrium state price density q^* under heterogeneous beliefs are given by*

$$\begin{cases} \mu_{q^*} = \mu_q + \mu_M + \delta_M \sigma_q \\ \sigma_{q^*} = \sigma_q + \delta_M \end{cases}$$

where μ_q and σ_q denote the drift and volatility of the equilibrium state price density in the standard setting.

The adjustment process B plays no role on the volatility σ_{q^*} of the equilibrium state price density. The impact of heterogeneous beliefs on the volatility of the equilibrium state price density is given by δ_M, which corresponds to the change of probability effect from P to the consensus probability Q. Recall that in the case of linear risk tolerance utility functions, we have $\delta_M = \sum_{i=1}^N \kappa_i \delta_i$. The equilibrium state price density will be more (or less) volatile than in the standard case when $\delta_M \geq 0$ (resp. $\delta_M \leq 0$), which, as we shall see below, corresponds to an optimistic (resp. pessimistic) consensus probability.

The impact of heterogeneous beliefs on the drift μ_{q^*} of the equilibrium state price density is given by $\mu_M + \delta_M \sigma_q$, which corresponds to the effect of the adjustment process through μ_M (which in classical cases is essentially proportional to a variance of the different δ_i) and the effect of the covariance between $\frac{d\overline{M}}{\overline{M}}$ and $\frac{du'(e^*)}{u'(e)}$. The equilibrium state price density will have a higher drift (resp. lower drift) when, for instance, $\mu_M \geq 0$ and $\delta_M \leq 0$ (resp. $\mu_M \leq 0$ and $\delta_M \geq 0$). The first effect (of μ_M) is quite easy to explain in the light of our construction of a representative agent. Indeed, we have seen that the representative agent does not maximize the expected utility from future consumption but a discounted expectation of utility from future consumption. Furthermore, we proved, in the classical case of linear risk tolerance utility functions that the discount rate is positive if and only if $\eta \leq 1$. A positive discount rate corresponds then to a negative μ_M and a decrease of the drift of the state price density. This decrease reflects the fact that future consumption is less important for the representative agent than in the classical case where there is no discount. The nature of the second effect (of δ_M) can be analyzed, as mentioned above, in terms of pessimism/optimism of the consensus probability belief.

Since the expressions of the MPR and of the risk free rate are directly related to the drift and volatility of the equilibrium state price density, we shall now analyze the impact of the introduction of beliefs heterogeneity on these two quantities. Notice already that since the

[8] We recall that for $\alpha \in \left(\mathbb{R}_+^*\right)^N$, $u_\alpha(t,x) \equiv \max_{\sum_{i=1}^N x_i \leq x} \sum_{i=1}^N \frac{1}{\alpha_i} u_i(t, x_i)$.

MPR is related to σ_{q^*} $(=\sigma_q+\delta_M)$ and the risk free rate to μ_{q^*} $(=\mu_q+\mu_M+\delta_M\sigma_q)$ a positive effect (resp. negative) on both the MPR and the risk free rate (where a "positive effect" is meant with respect to the risk premium puzzle and the risk free rate puzzle and corresponds to a higher MPR and a lower risk free rate) is possible.

3.2. Adjusted CCAPM and Market Price of Risk

We suppose the existence of a riskless asset with price process S^0 such that $dS_t^0 = r_t^f S_t^0 dt$ and of a risky asset with "cum dividend" price process

$$dS_t = S_t\mu_R(t)\,dt + S_t\sigma_R(t)\,dW_t \qquad \sigma_R > 0.$$

Since q^* is a state price density, the price process S must be such that q^*S is a P-martingale so that as in the classical case (see, e.g. Duffie, 1996, or Huang-Litzenberger, 1988)

$$\mu_R - r^f = -\sigma_{q^*}\sigma_R. \tag{3.1}$$

Now, since $q_t^* = \overline{M}_t B_t u'(t, e_t^*)$, with B of finite variation, we have seen in Proposition (3.1) that

$$\sigma_{q^*} = \sigma_q + \delta_M. \tag{3.2}$$

We easily obtain the following expression for the market price of risk $\left(\frac{\mu_R - r^f}{\sigma_R}\right)$ in the heterogeneous beliefs setting.

Proposition 3.2. *The market price of risk under heterogeneous beliefs is given by*

$$\begin{aligned} MPR\,[heterogeneous] &= -\sigma_q - \delta_M && (3.3)\\ &= MPR\,[standard] - \delta_M \\ &= MPR\,[homogeneous\ under\ Q] && (3.4) \end{aligned}$$

The MPR under heterogeneous beliefs is greater (resp. lower) than in the standard setting if and only if $\delta_M \leq 0$ (resp. $\delta_M \geq 0$).

Equation (3.3) is the CCAPM formula under heterogeneous beliefs. This adjusted formula differs from the classical one only through the change of probability from P to the consensus probability Q and the MPR in the heterogeneous beliefs setting is given by the MPR in an economy where all investors would share the same belief Q. The adjustment process B plays no role.

The introduction of heterogeneity in the investors beliefs leads to higher market price of risk if and only if $\delta_M \leq 0$. In other words, the MPR under heterogeneous beliefs is greater than in the standard setting if and only if the consensus probability is pessimistic. Indeed, letting $W_t^Q = W_t - \int_0^t \delta_M(s)\,ds$, we know by Girsanov's Theorem that W_t^Q is a Q-Brownian motion, and the dynamics of S under Q is given by $dS_t = S_t\left[\mu_R(t) + \delta_M(t)\,\sigma_R(t)\right]dt + S_t\sigma_R(t)\,dW_t^Q$, hence a nonpositive δ_M decreases the representative agent instantaneous rate of return of the risky asset S. Under pessimism, the representative agent underestimates the rate of return of the risky asset, which leads to a higher equilibrium risk premium.

Our results are consistent with those of Abel (2000), Cechetti et al. (2000), Epstein and Wang (1994), Hansen et al. (1999), or Anderson et al. (2000), which introduce distorded beliefs associated to cautious/pessimistic individual behavior. Since we have obtained that the MPR in the heterogeneous beliefs setting is in fact given by the MPR under the homogeneous belief Q, we actually face the same issue as in e.g. Abel (2000), where investors have the same subjective probability, different from the initial objective probability. Abel (2000) shows in a discrete time setting, and for power utility functions, that "uniform pessimism" on the agents subjective probability leads to a higher risk premium.

Unlike in the setting of this paper, in our setting with heterogeneous beliefs, there is no need for all investors to be pessimistic, but a pessimism at the aggregate level is sufficient in order to ensure an increase in the market price of risk. More precisely, in the case of linear risk tolerance utility functions, there is no need for all δ_i to be nonpositive, it suffices that some average of the δ_i given by $\delta_M = \sum_{i=1}^N \kappa_i \delta_i$ be nonpositive. This is obviously the case when all investors have the same nonpositive δ_i, or when all agents have nonpositive, but possibly different δ_i (in particular, some investors might have the objective belief P). This can also be the case when some investors are always optimistic (i.e. such that $\delta_i \geq 0$) and some investors are always pessimistic (i.e. such that $\delta_i \leq 0$).

Remark 1. *For linear risk tolerance utility functions, the consensus belief is pessimistic if, for instance, there is no systematic error on the estimation of the total wealth drift, i.e. if* $\sum_{i=1}^N \delta_i = 0$*, or if the systematic error is nonpositive, i.e.* $\sum_{i=1}^N \delta_i \leq 0$*, and if the optimistic (resp. pessimistic) investors have a risk tolerance* $T_i(t) \equiv -\frac{u_i'(t, y_t^{*i})}{u_i''(t, y_t^{*i})}$ *lower (resp. higher) than the average.*

The same result holds true if $\sum_{i=1}^N \delta_i \leq 0$ *and if for all* i, $T_i \leq \frac{1}{N}\sum_{i=1}^N T_i$ *on* $\{\delta_i \geq 0\}$ *and* $T_i \geq \frac{1}{N}\sum_{i=1}^N T_i$ *on* $\{\delta_i \leq 0\}$*, which amounts to relate the notion of pessimism (optimism) to the states of the world.*

In all these situations, the MPR predicted by the adjusted CCAPM (3.3) is higher than the MPR predicted by the classical CCAPM. Opposite situations (associated to nonnegative δ_M) trivially lead to lower MPR.

3.3. Risk free rate

We have just seen that heterogeneity of beliefs leads to higher market price of risk (resp. lower) if and only if the consensus belief is pessimistic (resp. optimistic). We now analyze the expression of the risk free rate under heterogeneous beliefs, and in particular, we explore if the possible increase of the MPR (induced by a pessimistic aggregate probability) can be associated with a lowering of the risk free rate.

As seen in the previous subsection, since q^*S is a P-martingale, we easily get, as in the classical case (see Duffie, 1996, or Huang-Litzenberger, 1988)

$$r^f = -\mu_{q^*}. \tag{3.5}$$

Now, since $q^* = \overline{M}Bu'(e^*)$, with B of finite variation, we have seen in Proposition (3.1) that

$$\mu_{q^*} = \mu_q + \mu_M + \delta_M \sigma_q \tag{3.6}$$

hence the following expression of the risk free rate r^f in the heterogeneous beliefs setting.

Proposition 3.3. *The risk free rate under heterogeneous beliefs is given by*

$$\begin{aligned} r^f\,[heterogeneous] &= -\mu_q - \mu_M - \delta_M \sigma_q & (3.7) \\ &= r^f\,[standard] - \mu_M + \delta_M \left(-\frac{u''(e^*)}{u'(e^*)} \beta e^* \right) \\ &= r^f\,[homogeneous\ under\ Q] - \mu_M & (3.8) \end{aligned}$$

Contrarily to the MPR, both the change of probability effect and the aggregation bias have an impact on the risk free rate. The impact of the aggregation bias is represented by μ_M. If B is nondecreasing (resp. nonincreasing), then μ_M is nonnegative (resp. nonpositive) and contributes to a decrease (resp. increase) of the risk free rate. This effect has a clear interpretation; considering $(-\mu_M)$ as a discount rate, a nonpositive μ_M means that future consumption is less important for the representative agent, and leads to a higher equilibrium interest rate. We have seen that for power utility functions with $\eta \leq 1$, as well as for exponential utility functions, μ_M is nonpositive, so that the effect of the aggregation bias is towards an increase of the interest rate. For power utility functions with $\eta \geq 1$, we obtain a nonpositive discount rate $(-\mu_M)$. This is interesting in light of the risk free rate puzzle. As underlined by Weil (1989), "a value of β above 1 (which corresponds in our setting to a nonpositive discount rate) is a computer's solution of the risk free rate puzzle". Another interpretation of this effect is related to the decomposition of the heterogenous beliefs impact into a change of probability effect and a total wealth modification effect. Wen B is nondecreasing (resp. nonincreasing) this correspons to an increase (resp. decrease) of the economic growth and then to a decrease (resp. increase) of the risk-free rate.

The impact of the change of probability effect from P to the consensus probability Q is represented by the covariance between $\frac{dM}{M}$ and $\frac{du'(e^*)}{u'(e)}$, i.e. $\delta_M \left(-\frac{u''(e^*)}{u'(e^*)} \beta e^* \right)$ and contributes to a lowering of the risk free rate if and only if Q is pessimistic, i.e. $\delta_M \leq 0$. Under pessimism, the representative agent underestimates the consumption growth rate, which contributes to reducing the equilibrium risk free rate.

Combining both effects, we obtain that the impact of heterogeneity of investors beliefs is towards a lower (resp. higher) risk free rate if the aggregate probability is pessimistic (resp. optimistic) and B is nondecreasing (resp. nonincreasing). The effect may remain, for instance, downwards, if the aggregate probability is pessimistic, and if B is nonincreasing and such that $|\mu_M|$ is "small" which is associated in classical examples to a small dispersion of beliefs.

Example 3.4. *In the case of linear risk tolerance utility functions, the risk free rate formula can be written*

$$\begin{aligned} r^f\,[heterogeneous] - r^f\,[standard] &= -\mu_M + \delta_M \left(\sum_{i=1}^{N} \kappa_i \right) \beta e^* \\ &= \frac{1}{2}(\eta - 1) Var^{\kappa}[\delta] + \left(\frac{\beta e^*}{\overline{\theta} + \eta e^*} \right) E^{\kappa}[\delta] \end{aligned}$$

The first term is nonpositive (resp. nonnegative) if and only if $\eta \geq 1$ (resp. $\eta \leq 1$). In particular, it is nonnegative for exponential utility functions.

The second term is nonpositive (resp. nonnegative) if and only if the aggregate probability is pessimistic (resp. optimistic).

The formula derived for $\eta = 1$ is analogous to the one obtained in Zapatero (1998, Theorem 1), or in Basak (2000) or Detemple-Murthy (1994, Theorem 3.1) in their specific learning setting.

3.4. Asset price and volatility

As in the standard setting, we have for a risky asset with price process S and dividend yield process e^*,

$$q_t^* S_t = E_t \left[\int_t^T q_s^* e_s^* ds \right]$$

with $q_t^* = \overline{M}_t B_t u' (t, e_t^*)$ so that the asset price in an heterogeneous beliefs setting is given by

$$S_t \,[\text{heterogeneous}] = \frac{1}{u'(t, e_t^*)} E_t^Q \left[\int_t^T \frac{B_s}{B_t} u'(s, e_s^*) \, e_s^* ds \right]$$

which is to be compared to the following asset price formula in the standard setting

$$S_t \,[\text{standard}] = \frac{1}{u'(t, e_t^*)} E_t^P \left[\int_t^T u'(s, e_s^*) \, e_s^* ds \right].$$

As for the expression of the risk free rate, both effects of the change of probability and the "discount" aggregation bias are to be noticed.

If B is nonincreasing which is the case for linear risk tolerance utility functions when $\eta \leq 1$, the effect of the aggregation bias is towards a lowering of the asset price, and in that case,

$$S_t \,[\text{heterogeneous}] \leq \frac{1}{u'(t, e_t^*)} E_t^Q \left[\int_t^T u'(s, e_s^*) \, e_s^* ds \right]$$

where the quantity $\frac{1}{u'(t,e_t^*)} E_t^Q \left[\int_t^T \frac{B_s}{B_t} u'(s, e_s^*) \, e_s^* ds \right]$ corresponds to the asset price in a model where all investors share the same probability Q. The converse effect occurs for general power utility functions with $\eta \geq 1$. The interpretation here again is clear if we think of B as a discount factor.

As for the change of probability effect, we are led to compare $E_t^Q \left[\int_t^T u'(s, e_s^*) \, e_s^* ds \right]$ with $E_t^P \left[\int_t^T u'(s, e_s^*) \, e_s^* ds \right]$. The effect is towards a lowering of the asset price when $E_t^Q \left[u'(s, e_s^*) \, e_s^* \right] \leq E_t^P \left[u'(s, e_s^*) \, e_s^* \right]$ or $cov_t \left(\frac{\overline{M}_s}{\overline{M}_t}, u'(s, e_s^*) \, e_s^* \right) \leq 0$, which again is related to the pessimism/optimism of the consensus belief Q. Indeed, consider for intance the case of power utility functions, $u'(x) = (\eta x)^{-1/\eta}$, and suppose that for all t, α_t and β_t are deterministic. Then $E_t^Q \left[u'(s, e_s^*) \, e_s^* \right] = E_t^Q \left[(e_s^*)^{1-1/\eta} \right]$, and it is easy to see (See Appendix A, Proof of Inequality (3.9)) that if the consensus probability is pessimistic, i.e. if $\delta_M \leq 0$ and if $\eta \leq 1$, then

$$E_t^Q \left[(e_s^*)^{1-1/\eta} \right] \geq E_t^P \left[(e_s^*)^{1-1/\eta} \right]. \tag{3.9}$$

The same approach leads to the same result (resp. the opposite result, $E_t^Q\left[(e_s^*)^{1-1/\eta}\right] \leq E_t^P\left[(e_s^*)^{1-1/\eta}\right]$) when $\delta_M \geq 0$, and $\eta \geq 1$ (resp. $\delta_M \geq 0$ and $\eta \leq 1$, $\delta_M \leq 0$ and $\eta \geq 1$).

We now consider the impact of beliefs heterogeneity on the asset's volatility. We assume that for all t, $(\alpha_t, \beta_t) = (\alpha, \beta) \in \mathbb{R}^2$. Related work includes Gallmeyer (2000), Hollifield-Gallmeyer (2002), Brennan-Xia (2001). We know that

$$S_t = \frac{1}{q_t^*} E_t\left[\int_t^T q_s^* e_s^* ds\right]. \tag{3.10}$$

Letting $W_t^Q = W_t - \int_0^t \delta_M(s)\, ds$, we know by Girsanov's Theorem that W_t^Q is a Q-Brownian motion. Now, by the martingale representation Theorem, we have $dE_t^Q\left[\int_0^T B_s u'(s, e_s^*) e_s^* ds\right] = \Psi_t dW_t^Q$. By applying Itô's Lemma to Equation (3.10) and matching the diffusion coefficients with the dynamics of S given by $dS_t = S_t \mu_R(t)\, dt + S_t \sigma_R(t)\, dW_t$, we get (See Appendix A, Proof of Equation (3.11)),

$$\sigma_R(t) = \frac{\Psi_t}{E_t^Q\left[\int_t^T B_s u'(s, e_s^*) e_s^* ds\right]} - \frac{u''(t, e_t^*)}{u'(t, e_t^*)} \beta e_t^*. \tag{3.11}$$

In the case of power utility functions with $u'(x) = (\eta x)^{-1/\eta}$, we get

$$\sigma_R(t) = \beta + \frac{E_t^Q\left[\int_t^T B_s (e_s^*)^{-1/\eta} V_s ds\right]}{E_t^Q\left[\int_t^T B_s (e_s^*)^{1-1/\eta} ds\right]} \tag{3.12}$$

for $V_s = \int_t^s \mathcal{D}_t \mu_M(\nu) + \beta\left(1 - \frac{1}{\eta}\right) \mathcal{D}_t \delta_M(\nu)\, d\nu$ where $\mathcal{D}_t \mu_M$ and $\mathcal{D}_t \delta_M$ denote the Malliavin derivatives with respect to W^Q (See Appendix A, Proof of Equation (3.12)).

We deduce from Equation (3.12) that

1) In the standard setting, i.e., if all investors share the same belief, and if this belief is the objective initial probability P, then the volatility of the asset price is given by the volatility of the aggregate consumption process, i.e., $\sigma_R(t) = \beta$.

2) If all investors share the same "logarithmic type" utility function, i.e. if $\eta = 1$ in the setting of our linear risk tolerance utility functions, then we have seen that $\mu_M = 0$ (see Proposition 2.7), so that the volatility of the asset price is given by the volatility of the aggregate consumption process, i.e., $\sigma_R(t) = \beta$ whatever the dispersion of beliefs.

3) If all investors share the same beliefs Q, then $\mu_M = 0$, and $\sigma_R(t) > \beta$ (resp. $\sigma_R(t) < \beta$) if $\eta > 1$ and $\mathcal{D}_t \delta_M(\nu) \leq 0$ (resp. $\eta > 1$ and $\mathcal{D}_t \delta_M(\nu) \geq 0$) or $\eta < 1$ and $\mathcal{D}_t \delta_M(\nu) \geq 0$ (resp. $\eta < 1$ and $\mathcal{D}_t \delta_M(\nu) \leq 0$).

Consider for instance the case of Bayesian updating of investors beliefs, as in Detemple-Murthy (1994), Hollifield-Gallmeyer (2002), Gallmeyer (2000) or Ziegler (2000). Suppose that agents agree that the evolution of aggregate wealth is governed by $de_t^* = e_t^* \alpha dt + e_t^* \beta dW_t$, but that they do not know the true mean α and therefore estimate it from past data. Assuming that at date 0, agents view α as normally distributed with mean m_0 and variance $V_0 = E\left[(m_0 - \alpha)^2\right]$. From

Filtering Theory, we know that as new information arrives on e^*, agents update their estimate $\overline{\alpha}_t$ of the mean growth of aggregate wealth using

$$d\overline{\alpha}_t = \frac{\pi_t}{\beta}\left(dW_t + \frac{\alpha - \overline{\alpha}_t}{\beta}dt\right)$$

for

$$\pi_t = \left(\frac{1}{V_0} + \frac{1}{\beta^2 t}\right)^{-1}.$$

It is easy to see that this setting is equivalent to ours if we take $\delta_M(t) = \frac{\overline{\alpha}_t - \alpha}{\beta}$. We then obtain that for $\nu \geq t$, $\mathcal{D}_t\delta_M(\nu) = \frac{1}{\beta}\mathcal{D}_t\overline{\alpha}(\nu) = \frac{1}{\beta^2}\pi_t(\nu - t) \geq 0$.

This result helps us to understand the change of probability effect in a setting with learning, which contributes to an increase of assets volatility if and only if $\eta \leq 1$.

4) In the general case, we have $\sigma_R(t) > \beta$ when for instance $\eta < 1$, $\mathcal{D}_t\mu_M(\nu) \geq 0$ and $\mathcal{D}_t\delta_M(\nu) \geq 0$.

4. The discrete time setting

The framework is essentially the same as in Section 2, except that we now deal with a discrete time setting. We consider a filtrated probability space $\left(\Omega, (F_t)_{t\in\{0,...,T\}}, P\right)$. Each investor indexed by $i = 1, \cdots, N$ has a current income at date t denoted by $e_t^{*^i}$ and a von Neumann-Morgenstern utility function for consumption of the form $E^{Q^i}\left[\sum_{t=0}^T u_i(t, c_t)\right] = E^P\left[\sum_{t=0}^T M_t^i u_i(t, c_t)\right]$, where Q^i corresponds to the subjective belief of individual i and is a probability measure equivalent to P, with density process $\left(M_t^i\right)_{t\in\{0,...,T\}}$.

4.1. Consensus belief

We make the same classical assumptions on u_i as in Section 2. The definitions of an equilibrium and of an interior equilibrium remain the same, replacing $E^P\left[\int_0^T q_t^*\left(y_t^{*^i} - e_t^i\right)dt\right]$ by $E^P\left[\sum_{t=0}^T q_t^*\left(y_t^{*^i} - e_t^i\right)\right]$. We construct in the exact same way as in the continuous time setting a consensus belief and a consensus consumer. Indeed, it is immediate to verify that Proposition (2.1), Proposition (2.2) and Corollary (2.3) (and more generally, all that is done before Proposition (2.7), including the expression of the consensus belief and of the aggregate utility function obtained in the examples) remain valid as long as we replace $t \in [0, T]$ by $t \in \{0, ..., T\}$ and $E^P\left[\int_0^T x_t dt\right]$ by $E^P\left[\sum_{t=0}^T x_t\right]$ each time it is necessary. In particular, we obtain

Proposition 4.1. *Consider an interior equilibrium* $(q^*, (y^{*^i}))$ *relatively to the beliefs* $\left(M^i\right)$ *and the income processes* $\left(e^i\right)$*. There exists a consensus investor defined by the normalized von Neumann-Morgenstern utility function* u_λ *(resp.* $u_{\lambda'}$*) and the consensus belief* M *of Proposition 2.1 (resp. Proposition 2.2), in the sense that the portfolio* e^* *maximizes his expected utility* $E^P\left[\sum_{t=0}^T M_t u(c_t)\right]$ *under the market budget constraint* $E^P\left[\sum_{t=0}^T q_t^*(c_t - e_t^*)\right] \leq 0$.

Now, if we impose that M be a martingale, there is as previously an aggregation bias, which, in discrete time, is predictable.

Proposition 4.2. *Consider an interior equilibrium price process q^* relatively to the beliefs (M^i), and the income processes (e^i). There exists a positive martingale process $\bar{M}$ with $\bar{M}_0 = 1$, and a predictable positive process B with $B_0 = 1$ such that*

$$\bar{M}_t B_t u'(t, e_t^*) = q^*.$$

The process B is given by

$$B_t = B_{t-1} \frac{E_{t-1}[M_t]}{M_{t-1}}, \qquad B_0 = 1$$

and can be thought of as a discount factor.

The process B measures the default of martingality of the consensus belief M. If the investors utility functions are logarithmic, or if all investors share the same beliefs, then $B \equiv 1$. As in the continuous time setting, for linear risk tolerance utility functions, we are able to explicitly compute the process B and to determine whether it is increasing, decreasing, smaller or greater than one. The results are similar to the ones obtained in the continuous time setting.

Example 4.3. *For general power utility functions, the process B satisfies*

$$\frac{B_t}{B_{t-1}} = \frac{E_{t-1}\left[\left[\sum_{i=1}^N \gamma_i (M_t^i)^\eta\right]^{\frac{1}{\eta}}\right]}{\left[\sum_{i=1}^N \gamma_i (M_{t-1}^i)^\eta\right]^{\frac{1}{\eta}}}$$

so that by Minkovski's Lemma, B is nondecreasing, greater than 1 (resp. nonincreasing, lower than 1) if $\eta \geq 1$ (resp. $\eta \leq 1$).

In the exponential case, the process B satisfies

$$\frac{B_t}{B_{t-1}} = \frac{E_{t-1}\left[\prod_{i=1}^N \left(M_t^i\right)^{\theta_i/\bar{\theta}}\right]}{\prod_{i=1}^N \left(M_{t-1}^i\right)^{\theta_i/\bar{\theta}}}$$

so that by Hölder's Lemma, B is nondecreasing, greater than 1.

As in the continuous time setting, we have pointed out two distinct effects of the introduction of heterogeneous beliefs on the state price density. There is first a change of probability effect, from P to the consensus probability Q and the second effect is represented by a predictable aggregation bias B, which can be interpreted as a "discount effect". We shall now analyze the impact of these two features on the CCAPM and on the risk free rate.

4.2. Adjusted CCAPM and Risk Premium

We suppose the existence of a riskless asset with price process S^0 such that $S_0^0 = 1$ and $S_t^0 = \prod_{s=1}^t \left(1 + r_s^f\right)$ for some predictable risk free rate process r^f. We consider a risky asset with positive price process S and associated rate of return between date t and $(t+1)$ denoted by $R_{t+1} \equiv \frac{S_{t+1}}{S_t} - 1$. In such a context, since q^*S is a $P-$martingale, we obtain as in the classical case,

$$E_t^P \left[R_{t+1}\right] - r_{t+1}^f = -cov_t^P \left[\frac{q_{t+1}^*}{E_t^P \left[q_{t+1}^*\right]}, R_{t+1}\right]. \tag{4.1}$$

Now, since $q_t^* = \overline{M}_t B_t u'\left(t, e_t^*\right)$, with B predictable, Equation (4.1) can be written

$$\begin{aligned} E_t^P \left[R_{t+1}\right] - r_{t+1}^f &= -cov_t^P \left[\frac{\overline{M}_{t+1} B_{t+1} u'\left(t+1, e_{t+1}^*\right)}{E_t^P \left[\overline{M}_{t+1} B_{t+1} u'\left(t+1, e_{t+1}^*\right)\right]}, R_{t+1}\right] \\ &= -cov_t^P \left[\frac{\overline{M}_{t+1} u'\left(t+1, e_{t+1}^*\right)}{E_t^P \left[\overline{M}_{t+1} u'\left(t+1, e_{t+1}^*\right)\right]}, R_{t+1}\right] \end{aligned} \tag{4.2}$$

so that, as in the continuous time setting, the adjustment process B plays no role in the CCAPM formula with heterogeneous beliefs. This adjusted CCAPM formula differs from the following classical formula

$$E_t^P \left[R_{t+1}\right] - r_{t+1}^f = -cov_t^P \left[\frac{u'\left(t+1, e_{t+1}^*\right)}{E_t^P \left[u'\left(t+1, e_{t+1}^*\right)\right]}, R_{t+1}\right] \tag{4.3}$$

only through the change of probability from P to the consensus probability Q.

Comparing (4.2) and (4.3) implies that the introduction of heterogeneity in the beliefs leads to a higher risk premium if and only if

$$E_t^Q \left[\frac{u'\left(t+1, e_{t+1}^*\right) R_{t+1}}{E_t^Q \left[u'\left(t+1, e_{t+1}^*\right)\right]}\right] \leq E_t^P \left[\frac{u'\left(t+1, e_{t+1}^*\right) R_{t+1}}{E_t^P \left[u'\left(t+1, e_{t+1}^*\right)\right]}\right]$$

or equivalently if and only if

$$cov_t^{P_u} \left(\overline{M}_{t+1}, R_{t+1}\right) \leq 0 \tag{4.4}$$

where $\frac{dP_u}{dP}$ is given (up to a constant) by $u'\left(t+1, e_{t+1}^*\right)$. As in the continuous time setting, the introduction of heterogeneity in the beliefs leads to a higher risk premium when the aggregate probability exhibits some pessimism, in a sense to be defined. For instance, in the case of power utility functions $u'(x) = (\eta x)^{-1/\eta}$, if we assume, like in Abel (2000), that $\frac{e_{t+1}^*}{e_t^*}$ is i.id. and if the consensus belief happens to be uniformly pessimistic[9], which in the terminology of Abel (2000)

[9] More precisely, the subjective distribution $F^*(X)$ of X is characterized by uniform pessimism - with respect to the objective distribution $F(X)$ - if $F^*(X) = F(X \exp \Delta)$ for $\Delta > 0$. It is straightforward to show that under uniform pessimism $E^*[X^a] = \exp(-a\Delta) E[X^a]$ for any constant a.

means that the distribution of $\log \frac{e^*_{t+1}}{e^*_t}$ under Q is a leftward translation of the distribution of the same random variable under P, then

$$\begin{aligned}
E_t^Q\left[\frac{u'\left(t+1,e^*_{t+1}\right)e^*_{t+1}}{E_t^Q\left[u'\left(t+1,e^*_{t+1}\right)\right]}\right] &= E_t^Q\left[\frac{\left(\frac{e^*_{t+1}}{e^*_t}\right)^{1-1/\eta}}{E_t^Q\left[\left(\frac{e^*_{t+1}}{e^*_t}\right)^{-1/\eta}\right]}\right]\times e^*_t \\
&= E_t^P\left[\frac{\left(\frac{e^*_{t+1}}{e^*_t}\right)^{1-1/\eta}}{E_t^P\left[\left(\frac{e^*_{t+1}}{e^*_t}\right)^{-1/\eta}\right]}\right]\times \exp\left(-\Delta\right)e^*_t \\
&< E_t^P\left[\frac{u'\left(t+1,e^*_{t+1}\right)e^*_{t+1}}{E_t^P\left[u'\left(t+1,e^*_{t+1}\right)\right]}\right]
\end{aligned}$$

which leads to an increase in the risk premium.

More generally, let us define a pessimistic probability $Q \sim P$ by the fact that M_{t+1} (where $M_T = \frac{dQ}{dP}$) decreases with R_{t+1} conditionally to F_t for all t, which means that conditionally to date t information, the aggregate probability puts more weight on states of nature with lower returns at date $(t+1)$ (see Appendix B, Definition (5.4)). We prove (in Appendix B, Corollary 5.5) that if the aggregate probability is pessimistic, then the risk premium under heterogeneous beliefs is greater than in the standard setting.

Notice that as in the continuous time setting, there is no need for all investors to be pessimistic, but a pessimism at the aggregate level is sufficient in order to ensure an increase in the risk premium. As in the continuous time setting, we can for linear risk tolerance utility functions give conditions on the individual investors beliefs that lead to a pessimistic aggregate probability.

Example 4.4. *When* $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i > 0$, *if the objective probability is given by the (martingale process associated to the) geometric average of the different beliefs, i.e. if we assume that*

$$\frac{\prod_{i=1}^N \left(M^i_{t+1}\right)^{1/N}}{\prod_{i=1}^N \left(M^i_t\right)^{1/N}} = n^e_t \qquad t = 0,..,T,$$

for some F_t*-measurable random variable* n^e_t, *and if the pessimistic (resp. optimistic) agents have a risk tolerance* T_i *higher (resp. lower) than the average, then the risk premium in the heterogeneous beliefs setting is greater than in the standard setting (See Appendix B).*

When $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i + \eta x$ *for* $\eta \neq 0$, *the same result holds true, if the objective probability is given by the (martingale process associated to the)* $\eta-$ *average of the different beliefs, i.e. if we assume that*

$$\frac{\left[\sum_{i=1}^N \left(M^i_{t+1}\right)^{\eta}\right]^{1/\eta}}{\left[\sum_{i=1}^N \left(M^i_t\right)^{\eta}\right]^{1/\eta}} = n^p_t \qquad t = 0,..,T$$

for some F_t*-measurable random variable* n^p_t.

4.3. Risk free rate

We know that, as in the classical case, the risk free rate r^f is given by

$$1+r^f_{t+1}=\frac{q^*_t}{E^P_t\left[q^*_{t+1}\right]}.$$

Now, since $q^*_t=\overline{M}_t B_t u'(t,e^*_t)$, with B predictable, we obtain

$$\begin{aligned}1+r^f_{t+1}\,[\text{het.}] &= \left(\frac{B_t}{B_{t+1}}\right)\frac{\overline{M}_t u'(t,e^*_t)}{E^P_t\left[\overline{M}_{t+1}u'\left(t+1,e^*_{t+1}\right)\right]}\\ &= \left(\frac{B_t}{B_{t+1}}\right)\left[1+r^f_{t+1}\,[\text{homogeneous under } Q]\right]\end{aligned}$$

where r^f [homogeneous under Q] denotes the equilibrium risk free rate in a model where all investors share the same probability Q.

As in the continuous time setting, both the change of probability effect and the aggregation bias have an impact on the risk free rate. The impact of the aggregation bias leads to an decrease (resp. increase) of the interest rate if B is nondecreasing (resp. nonincreasing). For instance, in the case of power utility functions with $\eta\le 1$, or in the case of exponential utility functions, we know that the "discount factor" B is nonincreasing and therefore contributes to a higher risk free rate. Since a nonincreasing "discount factor" B corresponds to the fact that the representative agent "discounts" future consumption, it is natural to obtain a higher risk free rate. As above, the impact of the change of probability effect from P to the consensus probability Q depends on the sign of $E^Q_t\left[\frac{u'\left(t+1,e^*_{t+1}\right)}{u'(t,e^*_t)}\right]-E^P_t\left[\frac{u'\left(t+1,e^*_{t+1}\right)}{u'(t,e^*_t)}\right]$ or equivalently on the sign of $cov^P_t\left[\frac{\overline{M}_{t+1}}{\overline{M}_t},\frac{u'\left(t+1,e^*_{t+1}\right)}{u'(t,e^*_t)}\right]$, which is related to the optimism/pessimism of the consensus belief. Notice that in the case of power utility functions with $u'(x)=(\eta x)^{-1/\eta}$, if like in Abel (2000), $\frac{e^*_{t+1}}{e^*_t}$ is i.id., and if the consensus belief happens to be uniformly pessimistic (see the definition in Subsection 4.2), then

$$\begin{aligned}E^Q_t\left[\frac{u'\left(t+1,e^*_{t+1}\right)}{u'(t,e^*_t)}\right] &= E^Q_t\left[\left(\frac{e^*_{t+1}}{e^*_t}\right)^{-1/\eta}\right]\\ &= E^P_t\left[\left(\frac{e^*_{t+1}}{e^*_t}\right)^{-1/\eta}\right]\times\exp\left(\Delta/\eta\right)\\ &> E^P_t\left[\frac{u'\left(t+1,e^*_{t+1}\right)}{u'(t,e^*_t)}\right]\end{aligned}$$

which means that the change of probability effect contributes to a decrease of the risk free rate. More generally, with the definition of a pessimistic probability introduced in Subsection 4.2, we prove that a pessimistic consensus probability leads to a lower interest rate (See Appendix B).

5. Concluding and additional remarks

As underlined by Rubinstein (1976), one potential use of the aggregation procedure is to relate the heterogeneity of individual demands to the heterogeneity of individual beliefs. Is it possible with our aggregation procedure "to construct sharing rules which indicate how consumption or portfolio choices of a particular consumer deviate from the per capita choices of the consensus consumer" ? We adopt exactly the same approach as in Calvet et al. (2002).

It is well known that in an homogeneous beliefs setting, all the individual allocations are comonotonic (the so-called "risk sharing rule") and furthermore, in the case of linear risk tolerance utility functions, this risk sharing rule is linear, i.e. there exist constants a_i and b_i such that the optimal allocations satisfy $y^i = a_i + b_i e^*$. What is the impact of heterogeneity of beliefs on the risk sharing rule ?

We have in our setting, through Proposition (2.1),

$$M_t^i u_i' \left(t, y_t^{*^i}\right) = M_t u_i' \left(t, \overline{y}_t^i\right) = \lambda_i M_t u' \left(t, e_t^*\right)$$

hence we obtain

1) the optimal allocations $\overline{y}^i$ under the common belief M satisfy the classical risk sharing rule, since $\overline{y}^i = I_{u_i}\left(\lambda_i u'(e)\right)$. In the case of linear risk tolerance utility functions, the $\overline{y}^i$ satisfy the linear risk sharing rule.

2) the optimal allocations in the initial equilibrium y^{*^i} are such that

$$\begin{aligned} y^{*^i} &= \overline{y}^i + \left(y^{*^i} - \overline{y}^i\right) \\ &= \overline{y}^i + I_{u_i}\left(\frac{M}{M^i} u_i'\left(\overline{y}^i\right)\right) - \overline{y}^i \\ &= \overline{y}^i + I_{u_i}\left(\lambda_i \frac{M}{M^i} u'\left(e^*\right)\right) - I_{u_i}\left(\lambda_i u'\left(e^*\right)\right) \\ &= \overline{y}^i + \varphi^i(M) \end{aligned}$$

where $\varphi^i(M) \geq 0$ when $M^i \geq \overline{M}$, and $\varphi^i(M) \leq 0$ when $M^i \leq M$. The heterogeneity of the beliefs induces then a distortion of the risk sharing rule and for each agent this distortion is monotone in individual beliefs deviations from the aggregate probability. For instance, in the exponential case, we get that

$$y_t^{*^i} - \overline{y}_t^i = \theta_i \left(\log M_t^i - \log M_t\right).$$

If we want to compare with the classical risk sharing rule, we get in the case of power utility functions

$$y^{*^i} = \left[\left(a_i + \frac{\theta_i}{\eta}\right)\left(\frac{M}{M^i}\right)^{-\eta} - \frac{\theta_i}{\eta}\right] + b_i \left(\frac{M}{M^i}\right)^{-\eta} e^*$$

where $(a_i, b_i) \in \mathbb{R}^2$ are the coefficients of the linear risk sharing rule in the standard case. If $\theta_i = 0$, i.e. if $u'(x) = (\eta x)^{-1/\eta}$, then $a_i = 0$, and $y^{*^i} = b_i \left(\frac{M}{M^i}\right)^{-\eta} e^* = \beta_i e^*$ instead of $b_i e^*$ in the classical case, and the deviation with respect to the classical risk sharing rule is determined by $\frac{M^i}{M} = \frac{M^i}{\left[\sum_{i=1}^N \gamma_i (M^i)^\eta\right]^{1/\eta}}$.

Moreover, consider the following "comonotonicity property"

$$y_t^{*^i} \geq \overline{y}_t^i \text{ if and only if } M_t^i \geq \overline{M}_t$$

and

$$y_t^{*^i} \leq \overline{y}_t^i \text{ if and only if } M_t^i \leq \overline{M}_t$$

or in words, the condition that each investor's observed (or initial) demand be larger than (resp. equal to, less than) his demand in the "equivalent equilibrium" if and only if he attaches a subjective probability that is larger than (resp. equal to, less than) the aggregate common probability. This is a desirable property of the equivalent equilibrium, and it is in fact equivalent to the second condition imposed in the definition of an equilibrium of the first kind. This equivalence is easy to establish. We have just seen that the second invariance requirement implies the comonotonicity property. Conversely, by the first requirement, we must have for all $i = 1, ..., N$

$$\frac{M_t^i}{M_t} = \frac{\lambda_i}{\zeta_i} \frac{u_i'\left(t, \overline{y}_t^i\right)}{u_i'\left(t, y_t^{*^i}\right)},$$

where λ_i is as previously the Lagrange multiplier in the initial equilibrium, and ζ_i is the Lagrange multiplier in the "equivalent equilibrium". The second requirement is then equivalent to the condition that $\frac{\lambda_i}{\zeta_i} = 1$ for all $i = 1, ..., N$. Imposing directly the comonotonicity property, instead of the second requirement, leads to

$$\frac{M_t^i}{M_t} 1_{\left\{\frac{M_t^i}{M_t} \leq 1\right\}} \leq \frac{\lambda_i}{\zeta_i} \leq \frac{M_t^i}{M_t} 1_{\left\{\frac{M_t^i}{M_t} \geq 1\right\}}$$

and since $M_0^i = M_0 = 1$, $\frac{\lambda_i}{\zeta_i} = 1$ for all $i = 1, ..., N$.

Our aggregation procedure can also be used for the study of models with constraints. Indeed, let us consider a model where investors share common beliefs but are submitted to possible short sale constraints. The equilibrium prices and allocations in such a model are the same as in a model without short sale constraint, where the initially unconstrained investors' beliefs remain unchanged and where the initially constrained investors' beliefs are replaced by well chosen, more optimistic ones. The consensus consumer is then more optimistic (or more precisely, his/her consensus belief is more optimistic) and the impact on the MPR and risk free rate follows: the short sales constraints lead to a lower market price of risk and to a higher risk free rate.

The case of borrowing constraints can be analyzed similarly and leads to oppposite results. Indeed, the beliefs of the constrained agents have to be replaced by more pessimistic ones in order to maintain the equilibrium prices and allocations unchanged. This leads to a more pesssimistic consensus beliefs and contributes to a higher market price of risk and a lower risk free rate.

Appendix A

Proof of Proposition (2.1) Since q^* is an interior equilibrium price process relatively to the beliefs $\left(M^i\right)$, and the income processes e^i, we know that $\sum_{i=1}^N y^{*^i} = e^*$ and that there exist positive Lagrange multipliers (λ_i) such that for all i and for all t,

$$M_t^i u_i'\left(t, y_t^{*^i}\right) = \lambda_i q_t^*.$$

We consider the maximization problem

$$\left(\mathcal{P}^\lambda(t,\omega)\right) : \max_{\sum_{i=1}^N x_i \le e_t^*(\omega)} \lambda_i u_i(t, x_i).$$

Since all utility functions are increasing and concave and takes values in $\mathbb{R} \cup \{-\infty\}$, for each t and each ω this programm admits a solution $\left(y^{i,\lambda}(t,\omega)\right)$ and we have $\sum_{i=1}^N y^{i,\lambda}(t,\omega) = e^*(t,\omega)$. Furthermore, the process $\left(\frac{1}{\lambda_i} u_i'\left(t, y_t^{i,\lambda}\right)\right)_t$ is independent from i. We denote this process by $p^{(\lambda)}$. Letting $M^{(\lambda)} \equiv \frac{q^*}{p^{(\lambda)}}$, we then have for all i and for all t

$$M_t^{(\lambda)} u_i'\left(t, y_t^{i,(\lambda)}\right) = M_t^i u_i'\left(t, y_t^{*^i}\right).$$

The process M^λ is adapted and positive. Moreover, at date $t = 0$, we have for all i, $M_0^i = 1$, and $\sum_{i=1}^N \overline{y_0}^i = \sum_{i=1}^N y_0^{*^i} = e_0^*$, so that $M_0^\lambda = 1$.

As far as uniqueness is concerned, notice that any process y^i such that $\sum_{i=1}^N y^i = e^*$ and

$$M_t u_i'\left(t, y_t^i\right) = M_t^i u_i'\left(t, y_t^{*^i}\right)$$

for some positive process M is a solution of the maximization problem $\left(\mathcal{P}^\lambda\right)$. ■

Proof of Proposition (2.2) With the same notations as in the previous proof, we construct the following application

$$\begin{array}{cccc} \Phi & \Sigma & \to & T(\Sigma) \\ & (\alpha_i) & \to & E^P\left[\int_0^T q_t^*\left(y_t^{i,\alpha} - e_t^i\right) dt\right] \end{array}$$

where $\left(y^{i,\alpha}(t,\omega)\right)$ is the solution of $(\mathcal{P}^\alpha(t,\omega))$, where Σ is the simplex of $\mathbb{R}^N$, i.e. $\Sigma = \left\{x \in \mathbb{R}^N : \sum_{i=1}^N x_i = 1, x_i \ge 0, i = 1, \cdots, N\right\}$ and $T(\Sigma)$ is the tangent space to Σ i.e. $T(\Sigma) = \left\{x \in \mathbb{R}^N : \sum_{i=1}^N x_i = 0\right\}$.

The application Φ is well defined. Furthermore, $\alpha \to y^{i,\alpha}$ is continuous, $k_i \le y^{i,\alpha} \le e^* - \sum_{j \ne i} k_j$ and q^* is uniformly bounded. By the dominated convergence Theorem, the application Φ is then continuous. On the other hand, if for some i we have $\alpha_i = 0$ then $y^{i,\alpha} = 0$ and $\Phi_i(\alpha) < 0$. By the classical equilibrium for outward applications Theorem there exists then an interior zero for Φ. Let us denote by (β_i) this zero and by $(\bar{y}^i)$ the associated $\left(y^{i,\beta}\right)$. We have then

$$\begin{aligned} \sum_{i=1}^{N} \bar{y}^i &= e^* \\ \beta_i u_i'(t, \bar{y}^i) &= q \\ E^P \left[\int_0^T q_t^* \left(\bar{y}_t^i - e_t^i \right) dt \right] &= 0 \end{aligned}$$

where q is a given process. By construction, q and q^* are positive processes and it suffices to define then M by $M = \frac{q^*}{q} \times \frac{q_0}{q_0^*}$ and λ_i' by $\lambda_i' = \frac{1}{\beta_i} \frac{q_0}{q_0^*}$, $i = 1, \cdots, N$ in order to conclude. ■

Proof of Corollary (2.3) Similar to the proof of the analogous result in a standard setting.

Proof of Examples (2.4) and (2.5) Since, as seen in the proof of Corollary (2.3), the representative utility function u is given by

$$u_\lambda(t, x) = \max_{\sum_{i=1}^{N} x_i \le x} \sum_{i=1}^{N} \frac{1}{\lambda_i} u_i(t, x_i)$$

the expression of u_λ in the specific setting of linear risk tolerance utility functions is obtained as in the standard case (see e.g. Huang-Litzenberger, 1988).

The expression of M is obtained by using $M_t u_i'\left(t, \overline{y}_t^i\right) = M_t^i u_i'\left(t, y_t^{*^i}\right)$, as well as

$$\sum_{i=1}^{N} y^{*^i} = \sum_{i=1}^{N} \bar{y}^i = e^*.$$

Indeed, in the case of exponential utility functions, we have for all i,

$$M^i \exp\left(-\frac{y^{*^i}}{\theta_i}\right) = M \exp\left(-\frac{\overline{y}^i}{\theta_i}\right)$$

hence

$$\prod_{i=1}^{N} \left(M^i\right)^{\theta_i} \exp\left(-\sum_{i=1}^{N} y^{*^i}\right) = M^{\overline{\theta}} \exp\left(-\sum_{i=1}^{N} \overline{y}^i\right),$$

or equivalently

$$M = \prod_{i=1}^{N} \left(M^i\right)^{\frac{\theta_i}{\overline{\theta}}}.$$

In the case of power utility functions, we get for all i,

$$M^i \left(\theta_i + \eta y^{*^i}\right)^{-1/\eta} = M \left(\theta_i + \eta \overline{y}^i\right)^{-1/\eta} = \lambda_i M \left(\overline{\theta} + \eta e^*\right)^{-1/\eta} b$$

hence

$$\left(M^i\right)^\eta \lambda_i^{-\eta} = M^\eta \left(\overline{\theta} + \eta e^*\right)^{-1} b^\eta \left(\theta_i + \eta y^{*^i}\right)$$

and

$$M = \left[\sum_{i=1}^{N} \frac{\lambda_i^{-\eta}}{\sum_{i=1}^{N} \lambda_i^{-\eta}} \left(M^i\right)^{\eta} \right]^{1/\eta} .$$

■

Proof of Proposition (2.7) We know by Corollary (2.3) that there exists a consensus consumer with belief M and utility function u such that $M_t u'(t, e_t^*) = q_t^*$. Since $dM_t = M_t [\mu_M(t) dt + \delta_M(t) dW_t]$ with $M_0 = 1$, we have

$$M_t = \exp\left[\int_0^t \delta_M(u) dW_u + \int_0^t \left(\mu_M(u) - \frac{1}{2}\delta_M^2(u)\right) du\right],$$

hence with the notations of the Proposition, $M_t = B_t \overline{M}_t$. ■

Proof of Proposition (2.8) When $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i > 0$, then we know from Example (2.4) that $M_t = \prod_{i=1}^N \left(M_t^i\right)^{\theta_i/\bar{\theta}}$. Since $M_t^i = \exp\left[\int_0^t \delta^i(u) dW_u - \frac{1}{2}\int_0^t \left(\delta^i\right)^2(u) du\right]$, we have

$$\prod_{i=1}^{N} \left(M_t^i\right)^{\theta_i/\bar{\theta}} = \exp\left[\int_0^t \sum_{i=1}^{N} \frac{\theta_i}{\bar{\theta}} \delta^i(u) dW_u - \frac{1}{2}\int_0^t \sum_{i=1}^{N} \frac{\theta_i}{\bar{\theta}} \left(\delta^i\right)^2(u) du\right]$$

hence by Itô's Lemma,

$$\begin{cases} \delta_M = \sum_{i=1}^N \frac{\theta_i}{\bar{\theta}} \delta^i \\ \mu_M = -\frac{1}{2}\left[\sum_{i=1}^N \frac{\theta_i}{\bar{\theta}} \left(\delta^i\right)^2 - \left(\sum_{i=1}^N \frac{\theta_i}{\bar{\theta}} \delta^i\right)^2\right] \end{cases}$$

When $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i + \eta x$ for $\eta \neq 0$, the result is obtained in the same way, by applying Itô's Lemma to the expression of the consensus belief $M = \left[\sum_{i=1}^N \gamma_i (M^i)^{\eta}\right]^{\frac{1}{\eta}}$ obtained in Example (2.5). ■

Proof of Proposition (2.9) Immediate using Proposition (2.7). The results of the footnote are obtained through Itô's Lemma. ■

Proof of Proposition (3.1) Since $q^* = Mq$ with

$$\begin{aligned} dM_t &= M_t [\mu_M(t) dt + \delta_M(t) dW_t] \\ dq_t &= q_t \left[\mu_q(t) dt + \sigma_q(t) dW_t\right] \end{aligned}$$

we easily get, through Itô's Lemma,

$$dq_t^* = q_t^* \left[\left(\mu_q + \mu_M + \delta_M \sigma_q\right)(t) dt + \left(\sigma_q + \delta_M\right)(t) dW_t\right]$$

hence

$$\begin{cases} \mu_{q^*} = \mu_q + \mu_M + \delta_M \sigma_q \\ \sigma_{q^*} = \sigma_q + \delta_M \end{cases}$$

■

Proof of Proposition (3.2) Immediate, using Equations (3.1) and (3.2). ■

Proof of Remark (1) We know by Proposition (2.8) that for linear risk tolerance utility functions, letting $T_i \equiv -\frac{u_i'(t,y^{*i})}{u_i''(t,y^{*i})}$ denote the risk tolerance of agent i at date t, we have

$$\begin{aligned}\delta_M &= \sum_{i=1}^{N} \frac{T_i}{\sum_{i=1}^{N} T_i} \delta^i \\ &= \frac{1}{N} \sum_{i=1}^{N} \delta^i + \frac{1}{\sum_{i=1}^{N} T_i} \sum_{i=1}^{N} \left(T_i - \frac{1}{N} \sum_{i=1}^{N} T_i \right) \delta^i.\end{aligned}$$

It is then immediate that if $\sum_{i=1}^{N} \delta^i \le 0$, and if for all i , $T_i \le \frac{1}{N} \sum_{i=1}^{N} T_i$ on $\left\{\delta^i \ge 0\right\}$ and $T_i \ge \frac{1}{N} \sum_{i=1}^{N} T_i$ on $\left\{\delta^i \le 0\right\}$, then $\delta_M \le 0$. ■

Proof of Proposition (3.3) Immediate, using (3.5) and (3.6) . ■

Proof of Inequality (3.9) Since $de_t^* = e_t^* \left(\alpha_t dt + \beta_t dW_t\right)$, we have

$$e_t^* = e_0^* \exp\left[\int_0^t \left(\alpha_u - \frac{1}{2}\beta_u^2\right) du + \int_0^t \beta_u dW_u\right].$$

Letting $W_t^Q \equiv W_t - \int_0^t \delta_M(u)\, du$, we know, by Girsanov's Lemma, that W^Q is a Q-Brownian motion, and

$$\left(\frac{e_s^*}{e_t^*}\right)^{1-1/\eta} = \exp\left(1-1/\eta\right)\left[\int_t^s \left(\alpha_u - \frac{1}{2}\beta_u^2\right) du + \int_t^s \beta_u dW_u^Q\right] \exp\left(1-1/\eta\right) \int_t^s \beta_u \delta_M(u)\, du.$$

If $\delta_M \le 0$, and $\eta \le 1$, then $\exp\left(1-1/\eta\right) \int_t^s \beta_u \delta_M(u)\, du \ge 1$, hence,

$$\begin{aligned}E_t^Q\left[\left(e_s^*\right)^{1-1/\eta}\right] &= \left(e_t^*\right)^{1-1/\eta} E_t^Q\left[\left(\frac{e_s^*}{e_t^*}\right)^{1-1/\eta}\right] \\ &\ge \left(e_t^*\right)^{1-1/\eta} E_t^Q\left[\exp\left(1-1/\eta\right)\left[\int_t^s \left(\alpha_u - \frac{1}{2}\beta_u^2\right) du + \int_t^s \beta_u dW_u^Q\right]\right] \\ &\ge \left(e_t^*\right)^{1-1/\eta} E_t^P\left[\exp\left(1-1/\eta\right)\left[\int_t^s \left(\alpha_u - \frac{1}{2}\beta_u^2\right) du + \int_t^s \beta_u dW_u\right]\right] \\ &\ge E_t^P\left[\left(e_s^*\right)^{1-1/\eta}\right].\end{aligned}$$

The same approach leads to the same result (resp. the opposite result, $E_t^Q\left[\left(e_s^*\right)^{1-1/\eta}\right] \le E_t^P\left[\left(e_s^*\right)^{1-1/\eta}\right]$) when $\delta_M \ge 0$ and $\eta \ge 1$ (resp. $\delta_M \ge 0$ and $\eta \le 1$, $\delta_M \le 0$ and $\eta \ge 1$). ■

Proof of Equation (3.11) We have

$$\begin{aligned}S_t &= \frac{1}{q_t^*} E_t\left[\int_t^T q_s^* e_s^* ds\right] \\ &= \frac{1}{B_t u'\left(t, e_t^*\right)} E_t\left[\int_t^T \frac{\overline{M}_t}{\overline{M}_t} B_s u'\left(s, e_s^*\right) e_s^* ds\right] \qquad (5.1)\end{aligned}$$

$$= \frac{1}{B_t u'(t, e_t^*)} \left[E_t^Q \left[\int_0^T B_s u'(s, e_s^*) e_s^* ds \right] - \int_0^t B_s u'(s, e_s^*) e_s^* ds \right] \quad (5.2)$$

Since B is of finite variation, we get by Itô's Lemma that the volatility of $\frac{1}{B_t u'(t,e_t^*)}$ is given by $-\frac{u''(e_t^*)}{B_t[u'(e_t^*)]^2}\beta e_t^*$. Since $dE_t^Q \left[\int_0^T B_s u'(s, e_s^*) e_s^* ds\right] = \Psi_t dW_t^Q$, applying Itô's Lemma to Equation (5.2) and replacing $E_t \left[\int_t^T B_s u'(s, e_s^*) e_s^* ds\right]$ by $B_t u'(t, e_t^*) S_t$ leads to

$$\sigma_R(t) S_t = \frac{1}{B_t u'(t, e_t^*)} \Psi_t - \left(\frac{u''(t, e_t^*)}{B_t [u'(t, e_t^*)]^2} \beta e_t^* \right) (B_t u'(t, e_t^*) S_t)$$

hence

$$\sigma_R(t) = \frac{\Psi_t}{E_t^Q \left[\int_t^T B_s u'(s, e_s^*) e_s^* ds \right]} - \frac{u''(t, e_t^*)}{u'(t, e_t^*)} \beta e_t^*.$$

■

Proof of Equation (3.12) We know by the Clark-Ocone formula (see Nualart, 1995) that $\Psi_t = E_t^Q \left[\int_t^T \mathcal{D}_t (B_s u'(e_s^*) e_s^*) ds\right]$. Now, if $U_t \equiv B_t u'(e_t^*) e_t^*$, we obtain through Itô's Lemma that $dU_t = U_t [\mu_U(t) dt + \sigma_U(t) dW_t]$ with

$$\begin{cases} \mu_U = \mu_M + \delta_M \beta \left(1 - \frac{1}{\eta}\right) + \alpha \left(1 - \frac{1}{\eta}\right) + \frac{1}{2}\frac{\eta+1}{\eta^2}\beta^2 \\ \sigma_U = \beta \left(1 - \frac{1}{\eta}\right) \end{cases}.$$

Using the chain rule for Malliavin derivatives leads to

$$\mathcal{D}_t U_s = U_s \left[\beta \left(1 - \frac{1}{\eta}\right) + \int_t^s \mathcal{D}_t \mu_M(\nu) + \beta \left(1 - \frac{1}{\eta}\right) \mathcal{D}_t \delta_M(\nu) d\nu \right].$$

Replacing Ψ_t by $E_t^Q \left[\int_t^T \mathcal{D}_t U_s ds\right]$ and $\frac{u''(e_t^*)}{u'(e_t^*)}\beta e_t^*$ by $-\frac{\beta}{\eta}$ in Equation (3.11) gives us

$$\sigma_R(t) = \beta + \frac{E_t^Q \left[\int_t^T B_s u'(e_s^*) V_s ds\right]}{E_t^Q \left[\int_t^T B_s u'(e_s^*) e_s^* ds\right]}$$

for $V_s = \int_t^s \mathcal{D}_t \mu_M(\nu) + \beta \left(1 - \frac{1}{\eta}\right) \mathcal{D}_t \delta_M(\nu) d\nu$. ■

Appendix B

We start by recalling the definition of comonotonic random variables.

Definition 5.1. *Two random variables x and z are said to be comonotonic if both of them are nondecreasing measurable functions of a third one, i.e. there exists a random variable ξ and two nondecreasing functions f and g such that*

$$x = f(\xi) \text{ and } z = g(\xi).$$

It is easy to check that when two random variables x and z are comonotonic then $cov^Q(x,z) \geq 0$ with respect to any probability Q equivalent to P.

The following definition and lemma extend the comonotonicity concept and the associated covariance property.

Definition 5.2. *Two random variables x and z are said to be comonotonic conditionally to F_t if there exists a random variable ξ and two functions f and g on $\Omega \times \mathbb{R}$ such that*

$$x(\omega) = f(\omega, \xi(\omega)) \text{ and } z(\omega) = g(\omega, \xi(\omega)), \qquad P \text{ a.s.}$$

and such that f and g are F_t with respect to their first argument and nondecreasing with respect to their second argument more precisely $\omega \to f(\omega, .)$ and $\omega \to g(\omega, .)$ from Ω to the set of nondecreasing real valued functions endowed with the Borel structure associated with the Skorokhod topology are F_t-measurable.

Lemma 5.3. *For positive random variables x and z such that x and z are comonotonic conditionally to F_t, we have $cov_t^Q(x,z) \geq 0$ for all probability measure Q equivalent to P.*

Proof See Jouini-Napp (2003). ■

Definition 5.4. *In our context, we say that a probability P' on (Ω, F, P) equivalent to P, with density process (M_t') is pessimistic (with respect to R) if for all t, M_{t+1}' and $-R_{t+1}$ are comonotonic conditionally to F_t.*

Corollary 5.5. *If the aggregate probability $Q \sim P$ is pessimistic (with respect to R), then the risk premium (for R) under heterogeneous beliefs is greater than in the standard case.*

Proof We have seen that the introduction of heterogeneity in the beliefs leads to a higher risk premium if and only if

$$E_t^Q \left[\frac{u'\left(t+1, e_{t+1}^*\right) R_{t+1}}{E_t^Q \left[u'\left(t+1, e_{t+1}^*\right)\right]} \right] \leq E_t^P \left[\frac{u'\left(t+1, e_{t+1}^*\right) R_{t+1}}{E_t^P \left[u'\left(t+1, e_{t+1}^*\right)\right]} \right]$$

or equivalently if and only if $cov_t^{P_u}\left(\overline{M}_{t+1}, R_{t+1}\right) \leq 0$ where $\frac{dP_u}{dP}$ is given (up to a constant) by $u'\left(t+1, e_{t+1}^*\right)$ (see Equation 4.4). By definition, since Q is pessimistic with respect to R, the random variables $\overline{M}_{t+1}$ and $-R_{t+1}$ are comonotonic conditionally to F_t, and we deduce from the previous lemma that $cov_t^{\widetilde{P}}\left(\overline{M}_{t+1}, R_{t+1}\right) \leq 0$. ■

Proof of Example (4.4)

In the case of exponential utility functions, it is easy to see that under our assumptions, the random variable $\prod_{i=1}^N \left(M_{t+1}^i\right)^{\left(\theta_i - \frac{\overline{\theta}}{N}\right)/\overline{\theta}}$ is comonotonic with $-R_{t+1}$ conditionally to F_t. Corollary (5.5) concludes.

For power utility functions, let us remark that $\sum_{i=1}^N \gamma_i \left(M_{t+1}^i\right)^\eta = \sum_{i=1}^N \left(\gamma_i - \frac{1}{N}\right)\left(M_{t+1}^i\right)^\eta + \frac{1}{N}\sum_{i=1}^N \left(M_{t+1}^i\right)^\eta$. Under our assumptions, since $\frac{1}{N}\sum_{i=1}^N \left(M_{t+1}^i\right)^\eta$ is F_t-measurable, this expression is comonotonic with $-R_{t+1}$ conditionally to F_t and again, Corollary (5.5) concludes. ■

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