INSTANTANEOUS GRATIFICATION

By Christopher Harris and David Laibson*

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Abstract. We propose a tractable continuous-time model of hyperbolic discounting that can be used to study the behavior of liquidity-constrained consumers. We show that our dynamically *inconsistent* model shares the same value function as a related dynamically *consistent* optimization problem with a wealth contingent utility function. Using this partial equivalence, we can show both existence and *uniqueness* of a hyperbolic equilibrium. We also show that the equilibrium consumption function is continuous and monotonic in wealth. None of these properties apply generally to analogous discrete-time models of hyperbolic discounting. All of the pathological properties of discrete-time hyperbolic models are eliminated by our continuous-time model.

JEL classification: C6, C73, D91, E21.

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1. Introduction

Robert Strotz (1956) first suggested that discount rates are higher in the short run than in the long run. Almost every experimental study on time preference has supported his conjecture (Ainslie 1992). To capture this empirical regularity, Laibson (1997a) adopted a discrete-time discount function, $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots .\}$, which Phelps and Pollak (1968) had previously used to model intergenerational time preferences. With $\beta < 1$, this 'hyperbolic' discount function captures the gap between a high short-run discount rate and a low long-run rate. In the last several years, this discrete-time discount function has been used to model a wide range of behavior: e.g., saving, contracts, job search.[1]

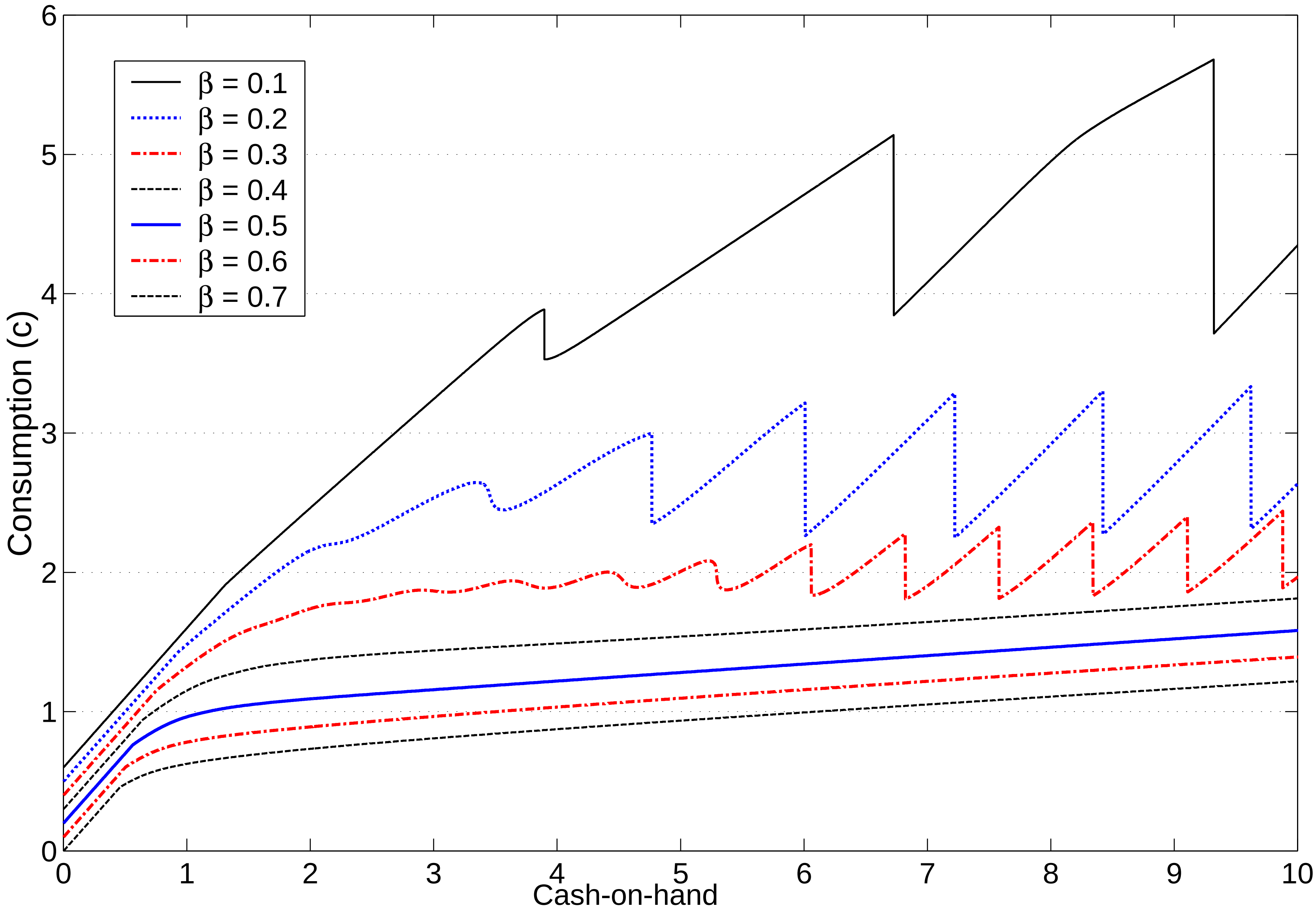
The hyperbolic discount function implies that current preferences are inconsistent with those held in the future. Beginning with the work of Strotz, such dynamic inconsistency has been analyzed by treating the individual as a sequence of independent selves whose choices are modelled as an intrapersonal game.

This game-theoretic framework has proved fruitful. A recurrent problem has, however, plagued most of these hyperbolic applications: strategic interaction among intrapersonal selves often generates counterfactual policy functions. Hyperbolic consumption functions need not be globally monotonic in wealth, and may even drop discontinuously at a countable number of points. Numerous authors, including Laibson (1997b), Morris and Postlewaite (1997), O'Donoghue and Rabin (1999a), Harris and Laibson (2001b), and Krusell and Smith (2000) have identified hyperbolic examples in which the consumption function has negatively sloped intervals or downward discontinuities. Figure 1 plots examples of such 'pathological' consumption functions.

Two solutions to this problem have been proposed. First, Harris and Laibson (2001b) point out that pathologies occur only when the model is calibrated in a

[1] For some examples, see O'Donoghue and Rabin (1999b), Angeletos, Laibson, Repetto, Tobacman and Weinberg (2001), and Della Vigna and Paserman (2000).

Figure 1: Consumption functions for $\beta \in \{0.1, 0.2, \ldots, 0.7\}$*



*These consumption functions are taken from discrete time simulations in Harris and Laibson (2001b). These simulations assume iid income, a risk-free asset, and CRRA. The short-run discount factor is $\beta\delta$. The long-run discount factor is δ=.95. The plotted consumption functions are shifted upward (in increments of .1) so they do not overlap.

limited region of the parameter space. When the hyperbolic model is calibrated with reasonable levels of noise (i.e. income volatility) and reasonable values for other preference and technology parameters, the pathologies typically vanish. However, Harris and Laibson (2001b) acknowledge that there *do* exist defensible calibrations for which the pathologies are still present (notably when the coefficient of relative risk aversion lies well below unity).

Second, O'Donoghue and Rabin (1999a) point out that pathologies arise only to the extent that consumers recognize that their own preferences are dynamically inconsistent. If consumers do not recognize this, then they will not have any incentive to act strategically vis-à-vis their own future selves. Hence, naive consumers who do not anticipate their own dynamic inconsistency will not exhibit pathologies. However, this solution requires that consumers be *completely* naive about their own future preferences. Any partial knowledge of future dynamic inconsistency reinstates the pathologies.

In the current paper we identify a solution to the pathology problem that is more robust than either of those cited above. First, we propose a continuous-time model of time discounting that captures the qualitative properties of the discrete-time hyperbolic model. This model distinguishes between the 'present' and the 'future'. The present is valued discretely more than the future, mirroring the one-time drop in valuation implied by the discrete-time quasi-hyperbolic discount function (Phelps and Pollak 1998, Laibson 1997) and its continuous-time generalizations (Barro 1999, Luttmer and Mariotti 2000). In addition, we assume that the transition from the present to the future is determined by a constant hazard rate. This simplifying assumption enables us to reduce our problem to a system of two differential equations that characterize present and future value functions.

Second, we show that our model has a limit case that is analytically tractable and psychologically relevant. This is the case in which the present is vanishingly short.

By focusing on this psychologically important limit case, we take the phrase "instantaneous gratification" literally. We analyze a model in which individuals prefer gratification in the present instant discretely more than consumption in the momentarily delayed future. This model is a useful benchmark that captures the essence of nearby models in which the present is short, but not precisely instantaneous.

Third, we show that the instantaneous-gratification model, which is dynamically *inconsistent*, shares the same value function as a related dynamically *consistent* optimization problem with a wealth-contingent utility function. Using this partial equivalence, we can show both existence and *uniqueness* of the hyperbolic equilibrium. However, our economy is not observationally equivalent to the related dynamically consistent optimization problem. The partial equivalence applies to the value functions but *not* to the policy functions.

We also show that the equilibrium consumption function of the hyperbolic problem is continuous and monotonic in wealth. The monotonicity property relies on the condition that the long-run discount rate is weakly greater than the interest rate. When this inequality is satisfied, all of the pathological properties of discrete-time hyperbolic models are eliminated by our continuous-time model.

Two other sets of authors have analyzed hyperbolic preferences in continuous time. Barro (1999) analyzes the choices of hyperbolic agents with constant relative risk aversion. He focuses on the general equilibrium implications of hyperbolic discounting and the ways in which hyperbolic economies may be observationally equivalent to exponential economies. Luttmer and Mariotti (2000) analyze the choices of agents with arbitrary discount functions, constant relative risk aversion, and stochastic asset returns. Luttmer-Mariotti generalize Barro's observational-equivalence result, but also identify particular endowment processes for which the hyperbolic model has interesting new asset-pricing implications (e.g., an elevated equity premium). Luttmer and Mariotti work with general discount functions and consider numerous special cases.

They have independently identified some properties of the particular case in which the present is vanishingly short. However, their findings do not overlap with ours.

Barro and Luttmer-Mariotti both restrict their analysis to *linear* policy rules. The existence of a linear equilibrium depends on special preference assumptions (constant relative risk aversion) and market assumptions (complete markets enabling sales of future labor income). We do not make restrictive assumptions of this kind: we work with a broad class of preferences; and we introduce the constraint that consumers may not borrow against future labor income. We pursue these generalizations for greater realism. Our problem does not admit a linear equilibrium. We have to contend with the pathologies that arise in our general setting, but do not arise under the Barro/Luttmer-Mariotti assumptions in either discrete or continuous time.

Our results also differ from Barro and Luttmer-Mariotti in that we are able to prove uniqueness of Markov equilibrium in the class of *all* policy rules. This is a desirable and unexpected result, since the hyperbolic model is a dynamic game, and can therefore generate non-uniqueness. For example, Krusell and Smith (2000) have shown that hyperbolic Markov equilibria are *not* unique in a deterministic discrete-time setting. In the current paper, we provide two uniqueness results. First, we prove uniqueness in a class of continuous-time models with stochastic asset returns. Second, we propose a refinement that uses the unique equilibrium in the stochastic setting to select a sensible unique equilibrium in the deterministic setting. This refinement takes the natural approach of selecting the limiting equilibrium obtained as the noise in the asset returns vanishes.

The rest of the paper formalizes these claims. In Section 2 we present our general continuous-time model and formulate some of the properties of this model. In Section 3 we describe an important limit case of our model. We call this limit case the instantaneous-gratification model. In Section 4 we show that the instantaneous-gratification model has the same *value function* as a particular dynamically consistent optimization problem. We call this latter problem the 'equivalent problem', but note that it is *not* observationally equivalent to the hyperbolic problem. The instantaneous-gratification model shares the same long-run discount rate as the equivalent problem, but the two problems have different instantaneous utility functions and different equilibrium policy functions.[2] In Section 5, we use our partial equivalence result to derive several important properties of the instantaneous-gratification problem, including equilibrium existence, equilibrium uniqueness, consumption-function continuity, and consumption-function monotonicity. In Section 6 we derive the deterministic version of the instantaneous-gratification model, and provide a complete analysis of the case of constant relative risk aversion. In Section 7 we formulate results that complement and generalize the results of Section 3. In Section 8 we conclude.

2. A Continuous-Time Consumption Model

Our modelling framework incorporates liquidity constraints, an important qualitative feature of consumers' planning problems (cf. Deaton 1991, Carroll 1992, 1997).

2.1. Dynamics. Time, t, is indexed by the real numbers. At a point in time, the consumer has wealth $x \in [0, +\infty)$. The consumer receives a continuous flow of labor income $y \in (0, +\infty)$.

Since wealth is a stock variable, if $x > 0$, the consumer may choose any instantaneous consumption level $c \in (0, +\infty)$. If $x = 0$, she may choose any instantaneous consumption level $c \in (0, y]$; i.e., when wealth is zero, instantaneous consumption must lie weakly below instantaneous income. In particular, the consumer may never borrow.

Whatever the consumer does not consume is invested in an asset, the returns

[2] By contrast, see Barro (1999), Laibson (1996), and Luttmer and Mariotti (2000) for the special case — log utility and no liquidity constraints — in which observational equivalence of the policy functions *does* hold.

on which are distributed normally with mean μdt and variance $\sigma^2 dt$, where $\mu \in (-\infty, +\infty)$ and $\sigma \in (0, +\infty)$. The change in her wealth at time t is therefore

$$dx = (\mu x + y - c)\, dt + \sigma x dz,$$

where z is a standard Wiener process.

We could easily generalize this framework by adding a stochastic source of labor income. For example, stochastic increments of labor income could follow a Poisson arrival process. We do not pursue this generalization, since it would not qualitatively change the analysis that follows.

2.2. The Consumer. The consumer is modeled as a sequence of autonomous selves. Each self controls consumption in the 'present' and cares about but does not directly control consumption in the 'future.'

In the standard discrete-time formulation of quasi-hyperbolic preferences, the present consists only of the current (single) period. The future consists of all periods after the current period, and a period $n \geq 1$ steps into the future is discounted with the overall discount factor $\beta\delta^n$ (Phelps and Pollak 1968, Laibson 1997).

This model can be extended to continuous-time and generalized in two ways. First, the present can last for any duration $T \in (0, \infty)$. Second, the duration of the present, T, can be random. We assume that self i is "born" at time t_i. Self i retains control of the consumption decision from date t_i to date $t_{i+1} = t_i + T_i$, where T_i is distributed exponentially with parameter $\lambda \in [0, +\infty)$. Hence, λ represents the arrival rate of transitions from the present to the future. At time t_{i+1}, self $i+1$ takes control of the consumption decision and this new self retains control of consumption until the next transition date, $t_{i+2} = t_{i+1} + T_{i+1}$.

In this continuous-time framework, the set of selves is countable. One new self is associated with each transition date $t \in \{t_0 = 0, t_1, t_2, ...\}$. For self t_i the present lasts from time t_i to the stochastic transition time $t_{i+1} = t_i + T_i$. For self t_i the future begins at time $t_{i+1} = t_i + T_i$ and lasts forever.

We assume that self i values the future discretely less than the present. Specifically, self i's preferences are given by

$$\mathrm{E}_{t_i}\left[\int_{t_i}^{t_i+T_i} e^{-\gamma(s-t_i)}U\left(c\left(s\right)\right)ds + \alpha\int_{t_i+T_i}^{+\infty} e^{-\gamma(s-t_i)}U\left(c\left(s\right)\right)ds\right], \qquad (1)$$

where $\gamma \in (0, +\infty)$, $\alpha \in (0, 1]$, and $U : (0, +\infty) \to \mathbb{R}$. Because the transition date $t_i + T_i$ is stochastic, self i has a stochastic discount function,

$$D_\lambda(t_i, s) = \left\{ \begin{array}{lll} e^{-\gamma(s-t_i)} & \text{if} & s \in [t_i, t_i + T_i) \\ \alpha e^{-\gamma(s-t_i)} & \text{if} & s \in [t_i + T_i, +\infty) \end{array} \right\}.$$

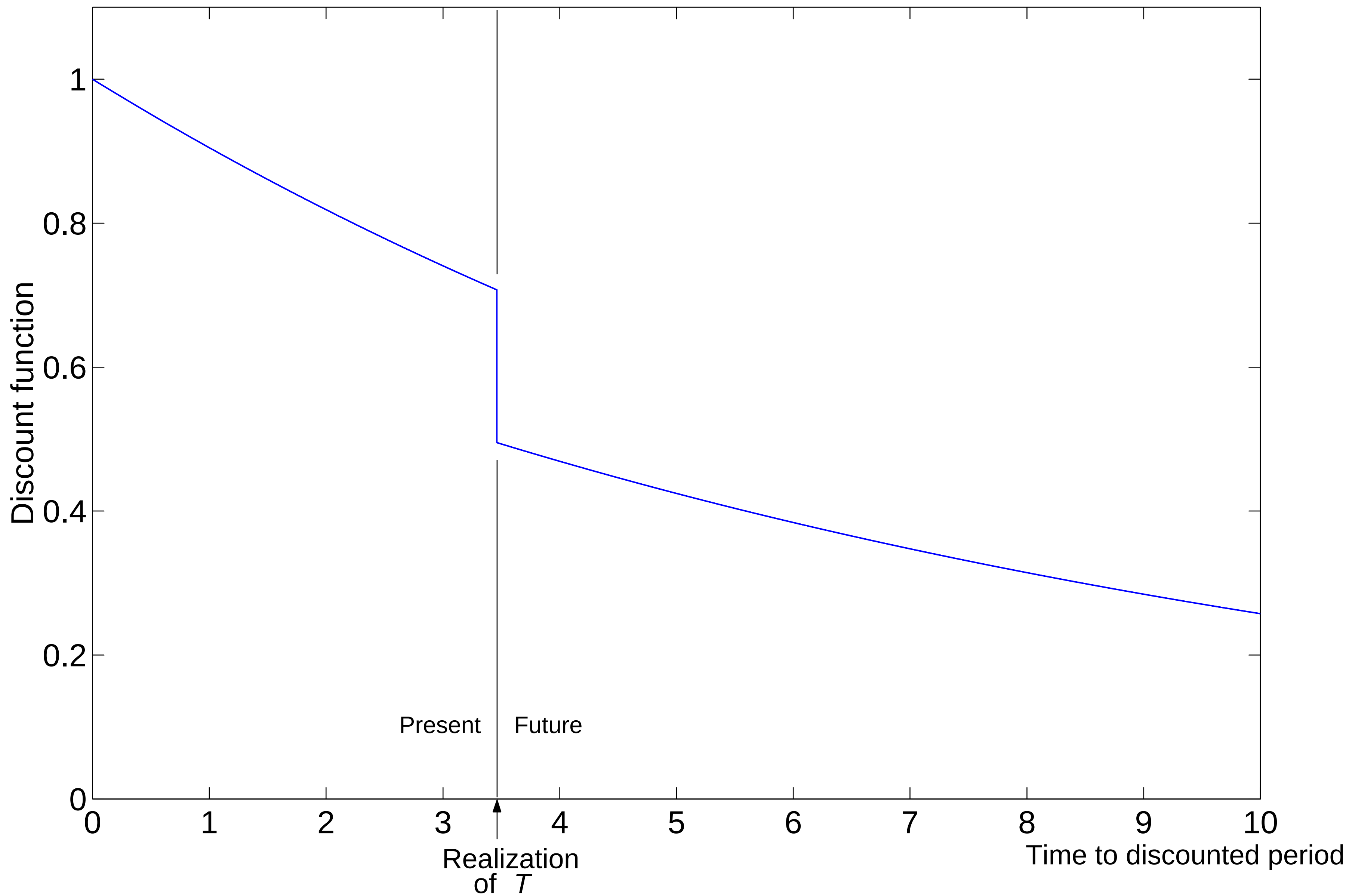
$D_\lambda(t_i, s)$ decays exponentially at rate γ up to time $t_i + T_i$, drops discontinuously at $t_i + T_i$ to a fraction α of its level just prior to $t_i + T_i$, and decays exponentially at rate γ thereafter. Hence, self t_i discounts all flows in the 'future' — i.e., flows that come after time $t_i + T_i$ — with an extra factor of α. This continuous-time formalization is close to some of the deterministic discount functions used in Barro (1999) and Luttmer and Mariotti (2000). However, we assume that the duration of the present, T_i, is stochastic. Figure 2 plots a single realization of this discount function, with $t_i = 0$ and $T_i = 3.4$.

When $\lambda = 0$ our discount function reduces to the standard exponential discount function, namely

$$D_0(t_i, s) = e^{-\gamma(s-t_i)} \text{ for all } s \in [t_i, +\infty).$$

As $\lambda \to \infty$ the discount function converges to a deterministic jump function with a

Figure 2: Realization of discount function (α=0.7, γ=0.1)



jump at $s = t_i$, namely

$$D_\infty(t_i, s) = \left\{ \begin{array}{lll} 1 & \text{if} & s = t_i \\ \alpha e^{-\gamma(s-t_i)} & \text{if} & s \in (t_i, +\infty) \end{array} \right\}.$$

Letting λ go to infinity captures the special case in which the present is vanishingly short. We shall return to this case below.

Figure 3 plots the expected value of the discount function for a set of λ values: $\lambda \in \{0, 0.1, 1, 10, \infty\}$. Analytically, the expected value is given by,

$$\mathrm{E}_t D_\lambda(t_i, s) = e^{-\lambda(s-t_i)} e^{-\gamma(s-t_i)} + (1 - e^{-\lambda(s-t_i)}) \alpha e^{-\gamma(s-t_i)}.$$

2.3. Exponential Assumptions. We shall need the following assumptions for the analysis of the model with finite λ:

E1 U is three times continuously differentiable on $(0, +\infty)$;

E2 $U'(c) > 0$ for all $c \in (0, +\infty)$;

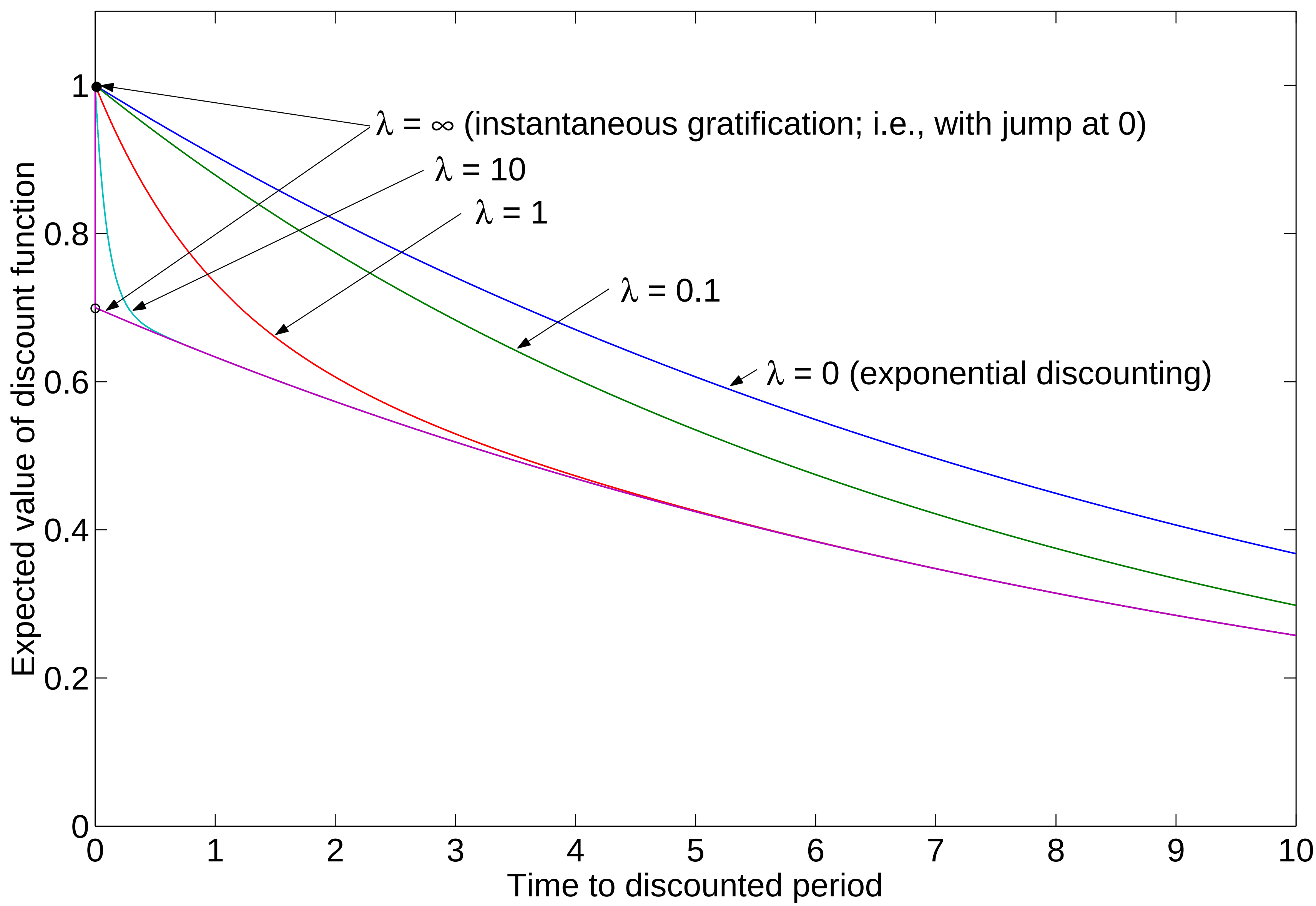
E3 there exist $0 < \underline{\rho} \leq \overline{\rho} < +\infty$ such that $\underline{\rho} \leq \frac{-cU''(c)}{U'(c)} \leq \overline{\rho}$ for all $c \in (0, +\infty)$;

E4 $\gamma > \max_{\rho \in [\underline{\rho}, \overline{\rho}]} (1 - \rho) \left(\mu - \frac{1}{2} \rho \sigma^2 \right)$.

Assumptions E1-E3 can be summarized by saying that the consumer has bounded relative risk aversion, or BRRA for short. Assumption E4 is the natural integrability condition for an exponential consumer with BRRA preferences: it ensures that the expected payoff of such a consumer is well defined.

Assumptions E1-E4 can be dramatically simplified if U has constant relative risk aversion ρ. In this case E1-E3 reduce to $\rho > 0$, and E4 reduces to $\gamma > (1 - \rho) \left(\mu - \frac{1}{2} \rho \sigma^2 \right)$.

Figure 3: Expected value of discount function for $\lambda \in \{0, 0.1, 1, 10, \infty\}$



2.4. Equilibrium. We confine attention to the set of perfect equilibria in stationary Markov strategies. More precisely, we focus on perfect equilibria in regular consumption functions.

Definition 1. *A consumption function* $C : [0, +\infty) \to (0, +\infty)$ *is* ***regular*** *iff* $C(0) \in (0, y]$ *and* C *is Lipschitz continuous.*

In other words: consumption must take place out of labor income when $x = 0$; and there exists $L \in [0, +\infty)$ such that, for all $x_1, x_2 \in [0, +\infty)$, $|C(x_1) - C(x_2)| \leq L|x_1 - x_2|$.

Now suppose that C is a regular consumption function. Then, for all $x \in [0, +\infty)$, we may find the timepath X_x^C of wealth starting at x by solving the equation

$$\begin{aligned} dX_x^C(t) &= \left(\mu X_x^C(t) + y - C\left(X_x^C(t)\right)\right) dt + \sigma X_x^C(t)\, dz(t), \\ X_x^C(0) &= x. \end{aligned}$$

We define the continuation-value function V by the formula

$$V(x) = \mathrm{E}\left[\int_0^{+\infty} e^{-\gamma t} U\left(C\left(X_x^C(t)\right)\right) dt\right].$$

The continuation-value function V discounts utility flows exponentially, with discount rate γ. We define the current-value function W by the formula

$$W(x) = \mathrm{E}\left[\int_0^{T} e^{-\gamma t} U\left(C\left(X_x^C(t)\right)\right) dt + \alpha e^{-\gamma T} V\left(X_x^C(T)\right)\right], \tag{2}$$

where T represents the next stochastic transition date. In the definition of W, the continuation-value function V is discounted by two multiplicative terms: the present-future discount factor α, and the standard exponential expression $e^{-\gamma T}$. The α factor reflects the one-time discounting that arises during a transition between the "present" and the "future." The two terms in the integral of equation (2) are directly comparable to the two terms in equation (1), which describes the actor's preferences.

Using this notation, we can define equilibrium as follows.

Definition 2. *A regular consumption function C is a* ***regular equilibrium*** *iff:*

1. *For all regular consumption functions $\widetilde{C}$ and all $x \in [0, +\infty)$, we have*

$$W(x) \geq \mathrm{E}\left[\int_0^T e^{-\gamma t} U\left(\widetilde{C}\left(X_x^{\widetilde{C}}(t)\right)\right) dt + \alpha e^{-\gamma T} V\left(X_x^{\widetilde{C}}(T)\right)\right].$$

2. *For all $x \in [0, +\infty)$, we have $V(x) \geq \frac{U(y)}{\gamma}$.*

The first condition in our definition of equilibrium reflects our assumption that the current self maintains control of the consumption decision for the duration of the present — i.e., until the next stochastic transition date T periods in the future. Since any single self controls consumption for a time interval with a strictly positive measure, the consumption strategy of the current self has a mathematically meaningful impact on the discounted present value of total utility.[3]

The second condition in our definition of equilibrium requires that equilibrium continuation-payoff functions must be bounded below by the payoff function associated with the myopic policy "always consume deterministic labor income y". This requirement rules out equilibria supported by policy functions that generate expected utility of $-\infty$. Such infinitely bad policy functions can in general be equilibria since no single self has an incentive to deviate.

[3] *Alternatively*, we could have assumed that there exists a *continuum* of selves. In this formulation each self, t, controls consumption for only an instant, dt. In addition, each self has its own stochastic present, from t to $t + T_t$. Self t applies the discount factor α to all utility flows after date $t + T_t$. This continuum-self model yields identical results to the countable-self model as long as we close the continuum-self model with the heuristic equilibrium condition $U'(c) = W'(x)$: marginal utility of consumption equals the marginal value of wealth.

2.5. Characterization of Equilibrium. In this subsection we formulate the Bellman system that characterizes regular equilibria. We also formulate an existence theorem for such equilibria. To conserve space, we state these motivating theorems without providing formal proofs. Instead, we provide the basic intuitions behind the results.

For all $\phi \in (0, +\infty)$, put

$$f(\phi) = \operatorname*{argmax}_{c \in (0,+\infty)} U(c) - c\phi \text{ and } f_0(\phi) = \operatorname*{argmax}_{c \in (0,y]} U(c) - c\phi.$$

Then:

Definition 3. *The finite-λ Bellman system is*

$$\frac{1}{2}\sigma^2 x^2 W'' + (\mu x + y - C) W' - \gamma W - \lambda (W - \alpha V) + U(C) = 0, \quad (3)$$

$$\frac{1}{2}\sigma^2 x^2 V'' + (\mu x + y - C) V' - \gamma V + U(C) = 0, \quad (4)$$

$$C = f(W') \quad (5)$$

when $x > 0$, *and*

$$(y - C) W' - \gamma W - \lambda (W - \alpha V) + U(C) = 0, \quad (6)$$

$$(y - C) V' - \gamma V + U(C) = 0, \quad (7)$$

$$C = f_0(W') \quad (8)$$

when $x = 0$.

Equation (3) can be understood in the usual way by applying Ito's Lemma. Intuitively, $\frac{1}{2}\sigma^2 x^2 W''$ represents the expected value of instantaneous changes in W arising from Brownian volatility in the returns process; $(\mu x + y - C) W'$ represents the expected value of instantaneous changes in W arising from expected changes in wealth; $-\gamma W$ represents the expected value of instantaneous changes in W arising from exponential discounting at rate γ; $-\lambda(W - \alpha V)$ represents the expected value of instantaneous changes in W arising from the stochastic arrival (with hazard rate λ) of a transition between the "present" with value W and the "future" with continuation value αV; $U(C)$ represents the instantaneous value of the consumption flow. Equation (4) can be understood in the same way. The only difference is that there is no longer a transition effect, since V describes continuation payoffs *after* the future has arrived. From the perspective of each self, the future only arrives once. Equations (6) and (7) are analogous to equations (3) and (4). The only difference is that they apply to the special case $x = 0$.

Equations (5) and (8) express the fact that consumption is chosen optimally. When consumption is not constrained, equations (5) and (8) imply that

$$U'(C(x)) = W'(x).$$

Intuitively, the marginal utility of a unit of consumption is equal to the marginal value of a unit of wealth.

We then have the following characterization theorem.

Theorem 4. *Suppose that Assumptions (E1-E4) are satisfied. Then a regular consumption function C is a regular equilibrium iff: W and V are continuously differentiable on $[0, +\infty)$ and twice continuously differentiable on $(0, +\infty)$; $V(x)$ is bounded below by $\frac{U(y)}{\gamma}$; and (W, V, C) satisfies the finite-λ Bellman system.* ■

We also have the following existence theorem.

Theorem 5. *Suppose that Assumptions (E1-E4) are satisfied. Then there exists a regular equilibrium.* ■

This theorem can be motivated as follows. The structure of the problem implies that, for any Borel measurable strategies, the value functions W and V must be bounded. The smooth noise associated with the asset returns implies that W'' and V'' must be bounded in the interior $(x > 0)$. Optimization implies that $U'(C(x)) = W'(x)$, so $U''(C(x))C'(x) = W''(x)$. Hence, provided that consumption is not too large, boundedness of W'' implies boundedness of C'. In other words, C is Lipschitz continuous. Finally, special arguments are required at the boundary $(x = 0)$, where the smooth noise vanishes since the diffusion process is scaled by x.

Remark 6. *It can in fact be shown that all equilibria are regular. We have not formulated a theorem along these lines, because doing so would involve a major digression. Among other things, we would have to explain what is meant by a solution to the dynamics on the interior of* $(0, +\infty)$ *when the consumption function is only Borel measurable; and we would have to explain how to normalize the dynamics when the consumption function is not linearly bounded. Moreover the apparatus that we would have to develop along the way would ultimately turn out to be redundant.*

3. The Instantaneous-Gratification Model

The continuous-time consumption model presented in the last subsection has an immediate advantage over its discrete-time analogue: there exists an equilibrium consumption function C that is continuous everywhere on $[0, +\infty)$. Indeed, it is possible to show that for this model all equilibrium consumption functions are continuous everywhere on $[0, +\infty)$. However, the principal pathology of the discrete-time hyperbolic consumption model remains: there may be intervals on which $C' < 0$.

Fortunately, we need not be interested in the general case of the continuous-time consumption model. The urge for "instantaneous gratification" suggests that the present – i.e., the interval from t to $t + T$ during which consumption is particularly highly valued – is very short. These observations lead us to consider the limiting case in which $\lambda \to +\infty$, and hence the present becomes vanishingly short. Recall that λ is the arrival rate of transitions from the present to the future. We refer to the limiting case, $\lambda = +\infty$, as the instantaneous-gratification model since in this limit case the highly rewarding present becomes instantaneously short. Before proceeding, it may be helpful to emphasize that we will study the limit of the equilibria as $\lambda \to +\infty$. In this way we are led to a definition of the equilibrium of the limit problem.

3.1. Hyperbolic Assumptions. We shall also need the following further assumptions, which are specific to the hyperbolic context:

H1 there exist $-\infty < \underline{\pi} \leq \overline{\pi} < +\infty$ such that $\underline{\pi} \leq \frac{-cU'''(c)}{U''(c)} \leq \overline{\pi}$ for all $c \in (0, +\infty)$;

H2 $\alpha + \underline{\rho} - 1 > 0$;

H3 $(2 - \alpha)\,\underline{\rho} - (1 - \alpha)\,\overline{\pi} > 0$;

H4 $\gamma > \max_{\rho \in \left[\frac{\overline{\rho}(\alpha+\underline{\rho}-1)}{(2-\alpha)\overline{\rho}-(1-\alpha)\underline{\pi}}, \frac{\underline{\rho}(\alpha+\overline{\rho}-1)}{(2-\alpha)\underline{\rho}-(1-\alpha)\overline{\pi}}\right]} (1-\rho)\left(\mu - \frac{1}{2}\rho\sigma^2\right)$.

Assumption H1 requires that the coefficient of relative prudence is bounded; Assumptions H2 and H3 ensure that equilibrium in the instantaneous-gratification model is equivalent to maximization in an equivalent problem; and Assumption H4 is the integrability condition for the equivalent problem.

Assumptions H1-H4 can be dramatically simplified if U has a constant coefficient of relative risk aversion ρ. In this case H1-H3 reduce to $\alpha + \rho - 1 > 0$, and H4 reduces to E4. In practice, calibrated models will usually satisfy this inequality: empirical estimates of the coefficient of relative risk aversion typically lie between 1 and 5 (i.e. $1 < \rho < 5$), and the short-run discount factor is typically thought to be at least 0.5 (i.e. $\alpha > 0.5$).[4] Hence, we can expect calibrated versions of the model to satisfy the

[4]See Laibson et al (1998) and Ainslie (1992).

inequality $\alpha + \rho - 1 > 0$. However, for completeness, we discuss the case $\alpha + \rho - 1 \leq 0$ in Section 7.

3.2. The Bellman System of the Instantaneous-Gratification Model. In this section, we characterize the limiting equilibrium obtained as $\lambda \to +\infty$. We first define the instantaneous-gratification Bellman system.

Definition 7. *The instantaneous-gratification Bellman system is*

$$\overline{W} = \alpha \overline{V}, \tag{9}$$

$$\frac{1}{2}\sigma^2 x^2 \overline{V}'' + \left(\mu x + y - \overline{C}\right) \overline{V}' - \gamma \overline{V} + U\left(\overline{C}\right) = 0, \tag{10}$$

$$\overline{C} = f\left(\overline{W}'\right) \tag{11}$$

when $x > 0$*; and*

$$\overline{W} = \alpha \overline{V}, \tag{12}$$

$$\left(y - \overline{C}\right) \overline{V}' - \gamma \overline{V} + U\left(\overline{C}\right) = 0, \tag{13}$$

$$\overline{C} = f_0\left(\overline{W}'\right) \tag{14}$$

when $x = 0$.

It is easy to motivate this definition of the instantaneous-gratification Bellman system: if we divide equations (3) and (6) of the finite-λ Bellman system through by λ and let $\lambda \to +\infty$, then we obtain equations (9) and (12) of the instantaneous-gratification Bellman system; and equations (10), (11), (13) and (14) of the instantaneous-gratification Bellman system are identical to equations (4), (5), (7) and (8) of the finite-λ Bellman system.

Equations (9) and (12) reflect the fact that as $\lambda \to +\infty$ the discount function drops essentially immediately to a fraction α of its initial value, and that the current-value function $\overline{W}$ is therefore α times the continuation-value function $\overline{V}$. Here $\overline{V}$ represents the *exponentially* discounted value of the stream of equilibrium utility flows. From the perspective of the current self, all of those future utility flows are additionally discounted by factor α, so $\overline{W} = \alpha\overline{V}$. Equations (10) and (13) can be understood in the usual way by applying Ito's Lemma. Equations (5) and (8) express the fact that consumption is chosen optimally. When consumption is not constrained, they imply that $U'\left(\overline{C}(x)\right) = \overline{W}'(x)$. Moreover equations (9) and (12) imply that $\overline{W}'(x) = \alpha\overline{V}'(x)$. Hence,

$$U'\left(\overline{C}(x)\right) = \overline{W}'(x) = \alpha\overline{V}'(x).$$

Intuitively, the marginal utility of a unit of consumption is equal to the marginal value of a unit of wealth, which is equal to $\alpha\overline{V}'(x)$.

Equations (9 - 14) are similar — although not identical — to the system of equations that would arise if a consumer were dynamically consistent. To simplify comparison, assume that the consumer is always in the interior of her state space and her choice space. Then the Bellman system that would apply to a completely exponential agent (i.e., with $\alpha = 1$), is given by

$$\frac{1}{2}\sigma^2 x^2 \overline{V}_E'' + \left(\mu x + y - \overline{C}_E\right)\overline{V}_E' - \gamma\overline{V}_E + U\left(\overline{C}_E\right) = 0, \tag{15}$$

$$U'\left(\overline{C}_E(x)\right) = \overline{V}_E'(x). \tag{16}$$

Compare these equations to the analogous equations for the instantaneous-gratification model (again assuming that the consumer is always in the interior of her state and choices spaces):

$$\frac{1}{2}\sigma^2 x^2 \overline{V}'' + \left(\mu x + y - \overline{C}\right)\overline{V}' - \gamma\overline{V} + U\left(\,\overline{C}\,\right) = 0, \tag{17}$$

$$U'\left(\overline{C}(x)\right) = \alpha\overline{V}'(x). \tag{18}$$

The only difference between Bellman system (15-16) and Bellman system (17-18) is the α term that multiplies $\overline{V}'$ in the last equation. Since $\alpha < 1$ and the utility function is concave, this difference drives up consumption in the instantaneous-gratification model relative to consumption in the exponential model. Naturally, this increase in consumption also drives a wedge between the value functions, $\overline{V}_E$ and $\overline{V}$, which characterize the two problems.

Formal motivation for the definition of the instantaneous-gratification Bellman system can be obtained as follows. Using equations (9) and (12), we may substitute for $\overline{W}$ in equations (11) and (14). Similarly, using equations (11) and (14), we may substitute for $\overline{C}$ in equations (10) and (13). In this way, we obtain the system:

$$\frac{1}{2}\sigma^2 x^2 \overline{V}'' + \left(\mu x + y - f\left(\alpha\overline{V}'\right)\right)\overline{V}' - \gamma\overline{V} + U\Big(f\left(\alpha\overline{V}'\right)\Big) = 0 \tag{19}$$

when $x > 0$; and

$$\left(y - f_0\left(\alpha\overline{V}'\right)\right)\overline{V}' - \gamma\overline{V} + U\Big(f_0\left(\alpha\overline{V}'\right)\Big) = 0 \tag{20}$$

when $x = 0$. We then have the following theorem, the proof of which is omitted to conserve space.

Theorem 8. *Suppose that Assumptions E1-E4 and H1-H4 are satisfied. Then, as* $\lambda \to +\infty$, V *converges uniformly on compact subsets of* $[0, +\infty)$ *to a limit function*

$\overline{V}$ *which is the unique viscosity solution*[5] *of the system (19-20).* ■

The central difficulty in proving Theorem 8 relates to the fact that, while C is continuous on the whole of $[0,+\infty)$, its limit $\overline{C}$ may have an upward jump at 0. The strategy for circumventing this difficulty can be explained as follows. Suppose, for the sake of discussion, that:

1. There exists $K \in [0,+\infty)$ such that, for all $\lambda \in [0,+\infty)$, $C(x) \leq K(1+x)$.
2. There exist functions $\overline{W}, \overline{V} : [0,+\infty) \to \mathbb{R}$ such that:
   (a) $\overline{W}$ and $\overline{V}$ are twice continuously differentiable on $[0,+\infty)$;
   (b) $W \to \overline{W}$ and $V \to \overline{V}$ uniformly on compact subsets of $[0,+\infty)$ as $\lambda \to +\infty$;
   (c) $W' \to \overline{W}'$, $W'' \to \overline{W}''$, $V' \to \overline{V}'$ and $V'' \to \overline{V}''$ uniformly on compact subsets of $(0,+\infty)$ as $\lambda \to +\infty$.

(We emphasize that, while we assume that the bound K on C is independent of λ, we do not assume that the Lipschitz constant L of C is independent of λ. By the same token, while we assume that W and V converge uniformly on compact subsets of the closed interval $[0,+\infty)$, we only assume that W', W'', V' and V'' converge uniformly on compact subsets of the open interval $(0,+\infty)$. In this way, we allow for the possibility that $\overline{C}$ may have an upward jump at 0.)

We may then reason as follows. Put $Z = W - \alpha V$. Multiplying equations (4) and (7) by α and subtracting them from equations (3) and (6), we obtain

$$\frac{1}{2}\sigma^2 x^2 Z'' + (\mu x + y - C) Z' - (\gamma + \lambda) Z + (1-\alpha) U(C) = 0$$

[5] *See Crandall et al (1992) for a "user's guide" to viscosity solutions.*

when $x \geq 0$. In other words, Z is the expected present discounted value of the flow of utility $(1-\alpha)\,U(C)$ up to time T_t. Hence $Z \to 0$ as $\lambda \to +\infty$, and $\overline{W} = \alpha\overline{V}$. That is, equations (9) and (12) hold.

Next, put $\overline{C} = f\left(\overline{W}'\right)$ for $x > 0$ and $\overline{C} = f_0\left(\overline{W}'\right)$ for $x = 0$. In other words, choose $\overline{C}$ in such a way as to ensure that equations (11) and (14) hold. Then, passing to the limit in equation (4), we obtain

$$\frac{1}{2}\sigma^2 x^2 \overline{V}'' + \left(\mu x + y - \overline{C}\right)\overline{V}' - \gamma\overline{V} + U\left(\overline{C}\right) = 0 \tag{21}$$

when $x > 0$. That is, equation (10) holds. There are then two cases to consider.

In the first case, $\overline{W}'(0) \geq U'(y)$. In this case, $\overline{C}(0) = \overline{C}(0+) \leq y$. We may therefore pass to the limit in equation (21) to obtain

$$\left(y - \overline{C}\right)\overline{V}' - \gamma\overline{V} + U\left(\overline{C}\right) = 0$$

when $x = 0$. In other words, equation (13) holds in this case.

In the second case, $\overline{W}'(0) < U'(y)$. In this case, $\overline{C}(0) = y < \overline{C}(0+)$. In particular, there is an upward jump in $\overline{C}$ at 0. We therefore proceed as follows. Let x^0 be the time-path for assets starting at 0. Then we have

$$dx^0(t) = \left(\mu x^0(t) + y - C\left(x^0(t)\right)\right)dt + \sigma x^0(t)\,dz(t),$$
$$x^0(0) = 0$$

and

$$V(0) = \mathrm{E}\left[\int e^{-\gamma t} U\left(C\left(x^0(t)\right)\right)dt\right].$$

Let $\overline{x}^0$ be any limit point of x^0. Since $\overline{C}(0+) > y$, $\overline{x}^0$ must remain trapped at 0 forever. We can therefore find a timepath of mixed consumptions $\kappa : [0, +\infty) \to \mathcal{P}([0, K])$ such that

$$d\overline{x}^0(t) = y - \int c d\kappa (c \mid t) = 0$$

for all $t \geq 0$ and

$$\overline{V}(0) = \mathrm{E}\left[\int e^{-\gamma t} \left(\int U(c)\, d\kappa(c \mid t)\right) dt\right].$$

Hence

$$\frac{U(y)}{\gamma} \leq \overline{V}(0) \leq \mathrm{E}\left[\int e^{-\gamma t} U\left(\int c d\kappa (c \mid t)\right) dt\right] = \mathrm{E}\left[\int e^{-\gamma t} U(y)\, dt\right] = \frac{U(y)}{\gamma}.$$

In particular, $\overline{V}(0) = \frac{U(y)}{\gamma}$. Since $\overline{C}(0) = y$, it follows that equation (13) holds in this case too.

Remark 9. *This derivation of the boundary condition for the Bellman system of the instantaneous-gratification model highlights the importance of the requirement that* $V \geq \frac{U(y)}{\gamma}$ *in the definition of equilibrium.*

4. The Equivalence Result

In the present section we show that, under appropriate assumptions, the value function $\overline{V}$ of the hyperbolic consumer in the instantaneous-gratification model is also the value function of an exponential consumer with an appropriately chosen utility function.

This result can be motivated by comparing the Bellman system with $\lambda = 0$ with the Bellman system of the instantaneous-gratification model. For all $\phi \in (0, +\infty)$, put

$$h(\phi) = U(f(\phi)) - f(\phi)\,\phi \text{ and } h_0(\phi) = \max_{c\in(0,y]} U(f_0(\phi)) - f_0(\phi)\,\phi.$$

Then, putting $\lambda = 0$ in the Bellman system with finite λ, we obtain:

$$\frac{1}{2}\sigma^2 x^2 W'' + (\mu x + y)\,W' - \gamma W + h(W') = 0 \tag{22}$$

when $x > 0$; and

$$yW' - \gamma W + h_0(W') = 0 \tag{23}$$

when $x = 0$. Secondly, for all $\phi \in (0, +\infty)$, put

$$\widehat{h}(\phi) = U(f(\alpha\phi)) - f(\alpha\phi)\,\phi \text{ and } \widehat{h}_0(\phi) = U(f_0(\alpha\phi)) - f_0(\alpha\phi)\,\phi.$$

Then the Bellman system of the instantaneous-gratification model can be written:

$$\frac{1}{2}\sigma^2 x^2 \overline{V}'' + (\mu x + y)\,\overline{V}' - \gamma \overline{V} + \widehat{h}\left(\overline{V}'\right) = 0 \tag{24}$$

when $x > 0$; and

$$y\overline{V}' - \gamma \overline{V} + \widehat{h}_0\left(\overline{V}'\right) = 0 \tag{25}$$

when $x = 0$.

Now, provided that $\widehat{h}$ is decreasing and convex (which is guaranteed by Assumptions H2 and H3), we can find a utility function $\widehat{U}$ such that

$$\widehat{h}(\phi) = \max_{c\in(0,+\infty)} \widehat{U}(c) - c\phi.$$

Similarly, provided that $\widehat{h}_0$ is decreasing and convex (which is again guaranteed by Assumptions H2 and H3), we can find a utility function $\widehat{U}_0$ such that

$$\widehat{h}_0(\phi) = \max_{c\in(0,y]} \widehat{U}_0(c) - c\phi.$$

However, unlike h and h_0, $\widehat{h}$ and $\widehat{h}_0$ are not generated by the same utility function. On the contrary, $\widehat{U}_0$ dominates $\widehat{U}$. In particular, we have $\widehat{U}_0(y) > \widehat{U}(y)$. Hence, in order to obtain the desired equivalence, the utility function of the exponential consumer must be made to depend on her wealth as her consumption. Specifically, she must use the utility function $\widehat{U}$ when $x > 0$ and the utility function $\widehat{U}_0$ when $x = 0$.

4.1. The Utility Function in the Interior. Recall that $\widehat{h}(\phi) = U(f(\alpha\phi)) - f(\alpha\phi)\phi$ for all $\phi \in (0,+\infty)$, and put

$$\rho(c) = \frac{-cU''(c)}{U'(c)}, \quad \pi(c) = \frac{-cU'''(c)}{U''(c)}, \quad \theta(c) = \frac{\pi(c)}{\rho(c)}$$

for all $c \in (0,+\infty)$. Then:

Lemma 10. *Suppose that Assumptions E1-E4 and H1-H3 hold. Then:*

1. $\widehat{h}'(\phi) < 0$ *for all* $\phi \in (0,+\infty)$;
2. $\widehat{h}''(\phi) > 0$ *for all* $\phi \in (0,+\infty)$;
3. *there exist* $0 < \underline{\rho}_{\widehat{h}} \leq \overline{\rho}_{\widehat{h}} < +\infty$ *such that* $\underline{\rho}_{\widehat{h}} \leq \frac{-\phi\widehat{h}''(\phi)}{\widehat{h}'(\phi)} \leq \overline{\rho}_{\widehat{h}}$ *for all* $\phi \in (0,+\infty)$.

Proof. Note first that

$$\begin{aligned}\widehat{h}(\phi) &= U(f(\alpha\phi)) - f(\alpha\phi)\phi \\ &= U(f(\alpha\phi)) - \alpha f(\alpha\phi)\phi - (1-\alpha) f(\alpha\phi)\phi \\ &= h(\alpha\phi) - (1-\alpha) f(\alpha\phi)\phi.\end{aligned}$$

Hence

$$\begin{aligned}\widehat{h}'(\phi) &= \alpha h'(\alpha\phi) - (1-\alpha) f(\alpha\phi) - (1-\alpha)\alpha f'(\alpha\phi)\phi \\ &= -\alpha f(\alpha\phi) - (1-\alpha) f(\alpha\phi) - (1-\alpha)\alpha f'(\alpha\phi)\phi \\ &= -f(\alpha\phi) - (1-\alpha) f'(\alpha\phi)\alpha\phi \\ &= -f(\alpha\phi)\left(1 + (1-\alpha)\frac{f'(\alpha\phi)\alpha\phi}{f(\alpha\phi)}\right) \\ &= -f(\alpha\phi)\left(1 + (1-\alpha)\frac{U'(f(\alpha\phi))}{f(\alpha\phi)U''(f(\alpha\phi))}\right) \\ &= -f(\alpha\phi)\left(1 - \frac{1-\alpha}{\rho(f(\alpha\phi))}\right) \\ &= \frac{-f(\alpha\phi)}{\rho(f(\alpha\phi))}(\alpha + \rho(f(\alpha\phi)) - 1).\end{aligned}$$

Part 1 of the lemma therefore follows from Assumption H2.

Second, as shown above, we have

$$\widehat{h}'(\phi) = -f(\alpha\phi) - (1-\alpha) f'(\alpha\phi)\alpha\phi.$$

Hence

$$\begin{aligned}
\widehat{h}''(\phi) &= -\alpha f'(\alpha\phi) - (1-\alpha) f'(\alpha\phi)\alpha - (1-\alpha)\alpha f''(\alpha\phi)\alpha\phi \\
&= -\alpha f'(\alpha\phi)\left(1 + (1-\alpha)\left(1 + \frac{f''(\alpha\phi)\alpha\phi}{f'(\alpha\phi)}\right)\right) \\
&= \frac{-\alpha}{U''(f(\alpha\phi))}\left(1 + (1-\alpha)\left(1 - \frac{U'''(f(\alpha\phi))U'(f(\alpha\phi))}{U''(f(\alpha\phi))^2}\right)\right) \\
&= \frac{-\alpha}{U''(f(\alpha\phi))}(1 + (1-\alpha)(1-\theta(f(\alpha\phi)))) \\
&= \frac{-\alpha}{U''(f(\alpha\phi))}((2-\alpha) - (1-\alpha)\theta(f(\alpha\phi))).
\end{aligned}$$

Part 2 of the lemma therefore follows from Assumption H3.

Third, using the final expressions obtained above for $\widehat{h}'(\phi)$ and $\widehat{h}''(\phi)$, we have

$$\frac{-\phi\widehat{h}''(\phi)}{\widehat{h}'(\phi)} = \frac{(2-\alpha) - (1-\alpha)\theta(f(\alpha\phi))}{\alpha + \rho(f(\alpha\phi)) - 1}.$$

Hence

$$\underline{\rho}_{\widehat{h}} \leq \frac{-\phi\widehat{h}''(\phi)}{\widehat{h}'(\phi)} \leq \overline{\rho}_{\widehat{h}},$$

where

$$\underline{\rho}_{\widehat{h}} = \frac{(2-\alpha)\underline{\rho} - (1-\alpha)\overline{\pi}}{\underline{\rho}(\alpha + \overline{\rho} - 1)} \text{ and } \overline{\rho}_{\widehat{h}} = \frac{(2-\alpha)\overline{\rho} - (1-\alpha)\underline{\pi}}{\overline{\rho}_U\left(\alpha + \underline{\rho} - 1\right)}.$$

This establishes part 3 of the lemma. ■

In view of Lemma 10, we may apply Fenchel's Theorem to conclude that, if we define the function $\widehat{U} : (0, +\infty) \to \mathbb{R}$ by the formula

$$\widehat{U}(\widehat{c}) = \min_{\phi \in (0,+\infty)} \widehat{h}(\phi) + \widehat{c}\phi,$$

then

$$\widehat{h}(\phi) = \max_{\widehat{c} \in (0,+\infty)} \widehat{U}(\widehat{c}) - \phi \widehat{c}$$

for all $\phi \in (0, +\infty)$.

Theorem 11. *Suppose that Assumptions E1-E4 and H1-H3 hold. Then:*

1. $\widehat{U}$ *is twice continuously differentiable;*
2. $\widehat{U}'(\widehat{c}) > 0$ *for all* $\widehat{c} \in (0, +\infty)$*;*
3. *there exist* $0 < \underline{\rho}_{\widehat{U}} \leq \overline{\rho}_{\widehat{U}} < +\infty$ *such that* $\underline{\rho}_{\widehat{U}} \leq \frac{-\widehat{c}\widehat{U}''(\widehat{c})}{\widehat{U}'(\widehat{c})} \leq \overline{\rho}_{\widehat{U}}$ *for all* $\widehat{c} \in (0, +\infty)$.

Proof. Put

$$\widehat{g}(\widehat{c}) = \operatorname{argmin}_{\phi \in (0,+\infty)} \widehat{h}(\phi) + \widehat{c}\phi.$$

Then

$$\widehat{U}'(\widehat{c}) = \widehat{g}(\widehat{c}), \quad \widehat{U}''(\widehat{c}) = \frac{-1}{\widehat{h}''(\widehat{g}(\widehat{c}))}$$

and

$$\frac{-\widehat{c}\widehat{U}''(\widehat{c})}{\widehat{U}'(\widehat{c})} = \frac{\widehat{h}'(\widehat{g}(\widehat{c}))}{-\widehat{g}(\widehat{c})\,\widehat{h}''(\widehat{g}(\widehat{c}))}.$$

In particular, we may put

$$\underline{\rho}_{\widehat{U}} = \frac{1}{\overline{\rho}_{\widehat{h}}} \text{ and } \overline{\rho}_{\widehat{U}} = \frac{1}{\underline{\rho}_{\widehat{h}}}.$$

This completes the proof of the theorem. ■

4.2. The Utility Function at the Boundary. Recall that $\widehat{h}_0(\phi) = U(f_0(\alpha\phi)) - f_0(\alpha\phi)\phi$ for all $\phi \in (0, +\infty)$. Then:

Lemma 12. *Suppose that Assumptions E1-E4 and H1-H3 hold. Then*

$$\widehat{h}_0(\phi) = \left\{ \begin{array}{cc} U(y) - \phi y & \text{if } 0 < \phi \leq \frac{U'(y)}{\alpha} \\ \widehat{h}(\phi) & \text{if } \frac{U'(y)}{\alpha} < \phi < +\infty \end{array} \right\}.$$

Moreover $\widehat{h}_0'\left(\frac{U'(y)}{\alpha}-\right) \leq \widehat{h}_0'\left(\frac{U'(y)}{\alpha}+\right)$. *In particular,* $\widehat{h}_0$ *is strictly decreasing and convex.*

Proof. The first statement is immediate from the definition of $\widehat{h}_0$. It implies that

$$\begin{aligned} \widehat{h}_0'\left(\frac{U'(y)}{\alpha}+\right) &= \widehat{h}'\left(\frac{U'(y)}{\alpha}\right) = \frac{-f(U'(y))}{\rho(f(U'(y)))}(\alpha + \rho(f(U'(y))) - 1) \\ &= -y\left(\frac{\alpha + \rho(y) - 1}{\rho(y)}\right) \geq -y = \widehat{h}_0'\left(\frac{U'(y)}{\alpha}-\right). \end{aligned}$$

This completes the proof of the lemma. ■

In view of Lemma 10, we may apply Fenchel's Theorem to conclude that, if we define the function $\widehat{U}_0 : (0, y] \to \mathbb{R}$ by the formula

$$\widehat{U}_0(\widehat{c}) = \min_{\phi \in (0, +\infty)} \widehat{h}_0(\phi) + \widehat{c}\phi,$$

then

$$\widehat{h}_0(\phi) = \max_{\widehat{c} \in (0, y]} \widehat{U}_0(\widehat{c}) - \phi\widehat{c}.$$

Now put

$$\psi = \frac{\alpha + \rho(y) - 1}{\rho(y)}.$$

Then:

Theorem 13. *Suppose that Assumptions E1-E4 and H1-H3 hold. Then*

$$\widehat{U}_0(\widehat{c}) = \left\{ \begin{array}{cc} \widehat{U}(\widehat{c}) & \textit{if } 0 < \widehat{c} < \psi y \\ \widehat{U}(\psi y) + (\widehat{c} - \psi y)\widehat{U}'(\psi y) & \textit{if } \psi y \leq \widehat{c} \leq y \end{array} \right\}.$$

Moreover $\widehat{U}_0(y) = U(y)$*.*

Proof. We have

$$\widehat{h}_0(\phi) = \left\{ \begin{array}{ll} U(y) - \phi y & \text{if } 0 < \phi \leq \frac{U'(y)}{\alpha} \\ \widehat{h}(\phi) & \text{if } \frac{U'(y)}{\alpha} < \phi < +\infty \end{array} \right\}$$

and

$$\widehat{h}_0'(\phi) \left\{ \begin{array}{ll} = -y & \text{if } 0 < \phi < \frac{U'(y)}{\alpha} \\ \in [-y, -\psi y] & \text{if } \phi = \frac{U'(y)}{\alpha} \\ = \widehat{h}'(\phi) & \text{if } \frac{U'(y)}{\alpha} < \phi < +\infty \end{array} \right\}.$$

Hence

$$\min_{\phi \in (0,+\infty)} \widehat{h}_0(\phi) + \widehat{c}\phi = \left\{ \begin{array}{ll} \min_{\phi \in (0,+\infty)} \widehat{h}(\phi) + \widehat{c}\phi & \text{if } 0 < \widehat{c} < \psi y \\ \widehat{h}\left(\frac{U'(y)}{\alpha}\right) + \widehat{c}\frac{U'(y)}{\alpha} & \text{if } \psi y \leq \widehat{c} \leq y \\ -\infty & \text{if } y < \widehat{c} < +\infty \end{array} \right\}.$$

Moreover

$$\min_{\phi\in(0,+\infty)} \widehat{h}(\phi)+\widehat{c}\phi=\widehat{U}(\widehat{c})$$

and

$$\begin{aligned}\widehat{h}\left(\frac{U'(y)}{\alpha}\right)+\widehat{c}\frac{U'(y)}{\alpha} &= \widehat{h}\left(\frac{U'(y)}{\alpha}\right)+\psi y\frac{U'(y)}{\alpha}+(\widehat{c}-\psi y)\frac{U'(y)}{\alpha} \\ &= \widehat{U}(\psi y)+(\widehat{c}-\psi y)\widehat{U}'(\psi y).\end{aligned}$$

Finally,

$$\widehat{h}\left(\frac{U'(y)}{\alpha}\right)+y\frac{U'(y)}{\alpha}=U(y)-y\frac{U'(y)}{\alpha}+y\frac{U'(y)}{\alpha}=U(y).$$

This completes the proof of the theorem. ■

4.3. The Equivalent Consumption Problem. The analysis of Sections 4.1 and 4.2 shows that $\overline{V}$ is the value function for the consumption problem of a consumer whose wealth evolves according to the same dynamics as in the original problem, but whose preferences are given by

$$\mathrm{E}_t\left[\int_t^{+\infty} e^{-\gamma(s-t)}\left(\chi_{\{\widehat{x}(s)=0\}}\widehat{U}_0(\widehat{c}(s))+\chi_{\{\widehat{x}(s)>0\}}\widehat{U}(\widehat{c}(s))\right)ds\right].$$

In other words, the equivalent consumer uses a standard discount function that decays exponentially at rate γ, but uses a non-standard utility function that depends on her wealth.

Remark 14. *We denote consumption and wealth in the equivalent problem by $\widehat{c}$ and $\widehat{x}$ in order to emphasize the fact that the equivalent consumer makes different consumption choices from the original hyperbolic consumer. In other words, the* *equivalent problem is not observationally equivalent to the original problem.*

Figure 4 plots an example of $\widehat{U}$ and $\widehat{U}_0$ for the special case in which U has a constant coefficient of relative risk aversion $\rho \neq 1$. For this special case we have the closed-form solutions

$$\widehat{U}\left(\widehat{c}\right) = \frac{\psi^{\rho}}{\alpha} U\left(\widehat{c}\right)$$

and

$$\widehat{U}_0\left(\widehat{c}\right) = \left\{ \begin{array}{cc} \widehat{U}\left(\widehat{c}\right) & \text{if } 0 < \widehat{c} < \psi y \\ \widehat{U}\left(\psi y\right) + \frac{1}{\alpha}\left(\widehat{c} - \psi y\right) & \text{if } \psi y \leq \widehat{c} \leq y \end{array} \right\},$$

where

$$\psi = \frac{\alpha + \rho - 1}{\rho}.$$

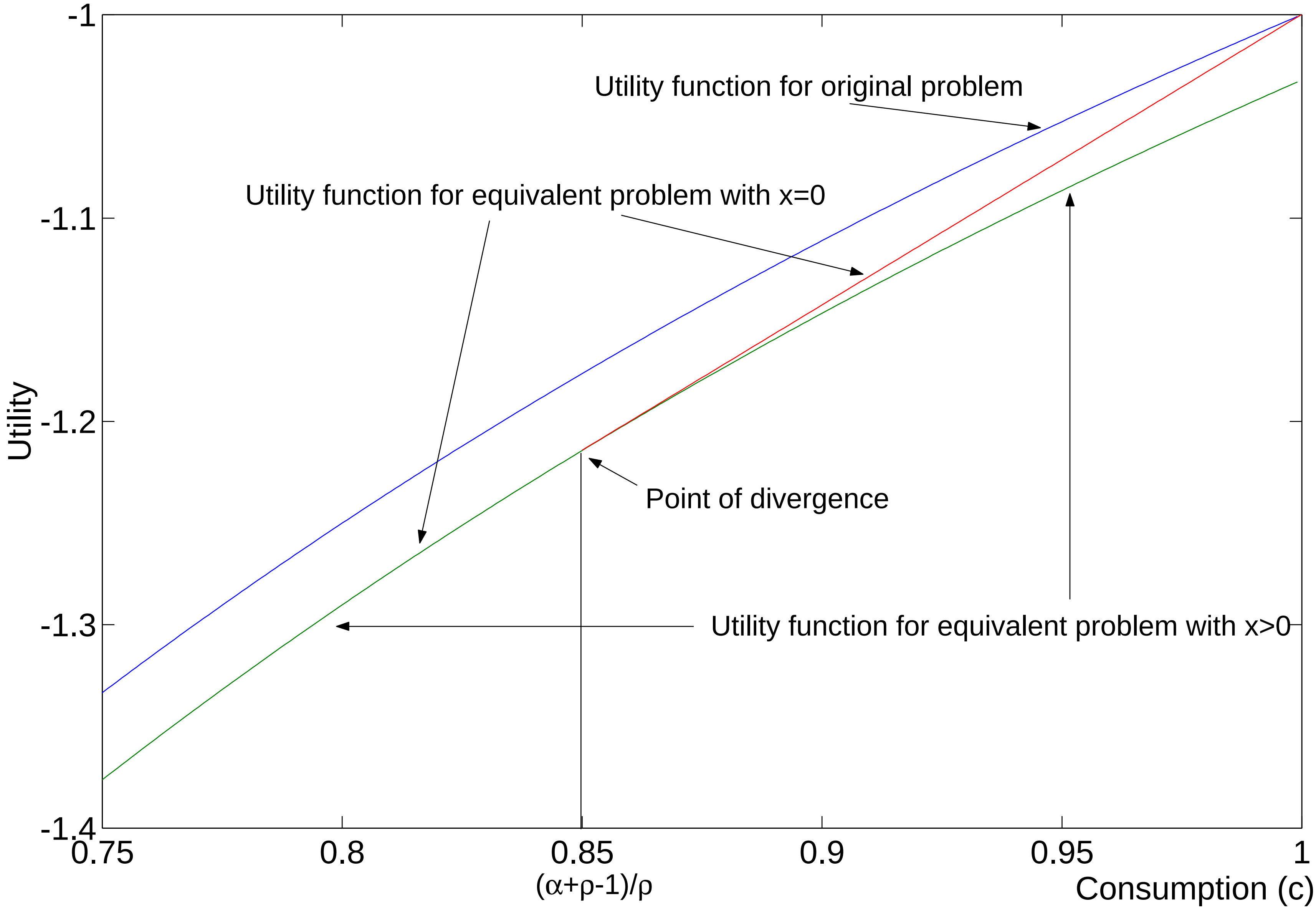
5. Some Features of the Instantaneous-Gratification Model

In the present section, we exploit the partial equivalence result of Section 4 to investigate the instantaneous-gratification model. We establish the existence and uniqueness of equilibrium, the continuity of the consumption function in the interior of the wealth space, a sufficient condition for the monotonicity of the consumption function, a generalized Euler equation governing the evolution of the marginal utility of consumption and a corresponding equation governing the evolution of consumption itself. Assumptions E1-E4 and H1-H4 will be in force throughout the section.

5.1. Existence and Uniqueness of Equilibrium.

Theorem 15. *The Bellman system of the instantaneous-gratification model has a unique solution* $\left(\overline{V}, \overline{C}\right)$.

Figure 4: Utility functions for equivalent problem (α=.7, ρ=2)



Proof. The equivalence result of Section 4 shows that $\left(\overline{V}, \overline{C}\right)$ solves the Bellman system of the instantaneous-gratification model iff $\overline{V}$ solves the Bellman equation of the equivalent problem, $\overline{C} = f\left(\alpha \overline{V}'\right)$ when $x > 0$ and $\overline{C} = f_0\left(\alpha \overline{V}'\right)$ when $x = 0$. Moreover standard considerations show that the Bellman equation of the equivalent problem possesses a unique solution. ■

5.2. Continuity of the Consumption Function.

Theorem 16. *We have:*

1. $\overline{C}$ *is continuous when* $x > 0$*;*
2. *there exists* $\mu_{crit} \in (-\infty, +\infty)$ *such that* $\overline{C}(0) < \overline{C}(0+)$ *for all* $\mu < \mu_{crit}$ *and* $\overline{C}(0) = \overline{C}(0+)$ *for all* $\mu \geq \mu_{crit}$*.*

Proof. Note first that $\overline{C}$ is continuous in the interior because $\overline{C} = f\left(\overline{V}'\right)$ there. Secondly, $\overline{C}(0) = f_0\left(\alpha \overline{V}'(0)\right) = y \wedge f\left(\alpha \overline{V}'(0)\right) \leq f\left(\alpha \overline{V}'(0)\right) = \overline{C}(0+)$. Hence $\overline{C}(0) \leq \overline{C}(0+)$, with equality iff $\overline{V}'(0) \geq \frac{U'(y)}{\alpha}$. Thirdly, let $\widetilde{V}$ be the value function of the restricted version of the equivalent consumption problem in which the consumer has utility function $\widehat{U}$ instead of $\widehat{U}_0$ when her wealth is 0. It can be shown that $\overline{V}'(0) \geq \frac{U'(y)}{\alpha}$ iff $\widetilde{V}(0) \geq \frac{U(y)}{\gamma}$. Moreover: $\widetilde{V}(0)$ is strictly increasing in μ; $\widetilde{V}(0) = \frac{\widehat{U}(y)}{\gamma} < \frac{U(y)}{\gamma}$ for all μ sufficiently small; and $\widetilde{V}(0) \to +\infty$ as $\mu \to +\infty$. ■

Remark 17. *In the case* $\mu < \mu_{crit}$*, the consumer dissaves when her wealth is low, spends all her wealth in finite time, and experiences a discontinuous drop in consumption when her wealth runs out. In the case* $\mu > \mu_{crit}$*, asset returns are high enough to induce the consumer to save when her wealth is low.*

5.3. Monotonicity of the Consumption Function.

Theorem 18. *Suppose that* $\gamma \geq \mu$*. Then* $\overline{C}' > 0$ *when* $x > 0$*.*

Proof. Note first that $\overline{C} = f\left(\overline{V}'\right)$ in the interior. Hence $\overline{C}$ is continuously differentiable there, and $\overline{C}' = f'\left(\overline{V}'\right)\overline{V}''$. Hence $\overline{C}' > 0$ iff $\overline{V}'' < 0$. Secondly, differentiating equation (24) with respect to x, we obtain

$$\frac{1}{2}\sigma^2 x^2 \overline{V}''' + (\mu x + y)\overline{V}'' - \gamma \overline{V}' + \sigma^2 x \overline{V}'' + \mu \overline{V}' + \widehat{h}'\left(\overline{V}'\right)\overline{V}'' = 0$$

or

$$\overline{V}''' = \frac{2}{\sigma^2 x^2}\left((\gamma - \mu)\overline{V}' - \left(\left(\mu + \sigma^2\right)x + y + \widehat{h}'\left(\overline{V}'\right)\right)\overline{V}''\right).$$

In particular, if $\overline{V}'' = 0$, then

$$\overline{V}''' = \frac{2}{\sigma^2 x^2}(\gamma - \mu)\overline{V}' \geq 0.$$

Hence, if there exists $x_1 \in (0, +\infty)$ such that $\overline{V}''(x_1) \geq 0$, then $\overline{V}'' \geq 0$ on $(x_1, +\infty)$. Thirdly, if there exists $x_1 \in (0, +\infty)$ such that $\overline{V}''(x_1) \geq 0$ on $(x_1, +\infty)$, then $\overline{V}$ grows at least linearly; and this contradicts the assumption that $\rho \geq \underline{\rho} > 0$. Overall, then, we must have $\overline{V}'' < 0$ on $(0, +\infty)$. ■

Remark 19. *Theorem 25 below shows that, if $\gamma < \mu$, then a drop in consumption may occur.*

5.4. The Generalized Euler Equation. Since $U'\left(\overline{C}\right)$ may have a discontinuity at 0, we cannot use Itô's Lemma to study its dynamics. We can, however, use Itô's Lemma to study the dynamics of $M = \overline{V}'$. These dynamics are very closely related to those of $U'\left(\overline{C}\right)$. Indeed, we have $U'\left(\overline{C}\right) = \alpha M$ for $x > 0$. Moreover:

1. if $\overline{C}(0+) = \overline{C}(0)$, then the dynamics of M are identical to those of $U'\left(\overline{C}\right)$;

2. if $\overline{C}(0+) > \overline{C}(0)$ and $x(0) \in (0, +\infty)$, then the dynamics of M are identical to those of $U'(\overline{C})$ on the interval $(0, \tau)$, where τ is the first time that x hits 0; and

3. if $\overline{C}(0+) > \overline{C}(0)$ and $x(0) = 0$, then the dynamics of M are identical to those of $U'(\overline{C})$, in the sense that both are trivial.

The two dynamics only differ if $\overline{C}(0+) > \overline{C}(0)$ and $x(0) \in (0, +\infty)$, in which case $U'(\overline{C})$ jumps up at τ.

Theorem 20. *We have:*

$$\frac{dM}{M} = \left(\gamma - \mu + \sigma^2 \rho(\overline{C}) \frac{x\overline{C}'}{\overline{C}} + (1-\alpha)\overline{C}'\right) dt - \sigma\rho(\overline{C}) \frac{x\overline{C}'}{\overline{C}} dz \tag{26}$$

if either $x > 0$ or $x = 0$ and $\overline{C}(0+) = \overline{C}(0)$; and

$$\frac{dM}{M} = 0$$

if $x = 0$ and $\overline{C}(0+) > \overline{C}(0)$.

This theorem gives an exact expression for the rate of growth of M. The equation includes deterministic terms (i.e. the terms which include dt) and a stochastic term (i.e. the final term, which includes dz). The stochastic term captures the negative effect that positive wealth shocks have on marginal utility.

The term γdt implies that marginal utility rises more quickly the higher the long-run discount rate γ. The term $-\mu dt$ implies that marginal utility rises more slowly the higher the rate of return μ. The term $\sigma^2\rho(\overline{C})\frac{x\overline{C}'}{\overline{C}}dt$ captures two separate effects. First, asset income uncertainty σ^2 affects the savings decision. Second, since marginal utility is non-linear in consumption, asset income uncertainty affects the average value of future marginal utility. The net impact of these two effects is always positive. The term $(1-\alpha)\overline{C}'dt$ captures the effect of hyperbolic discounting. Naturally, when $\alpha = 1$, this effect vanishes and the model coincides with the standard exponential discounting case.

Proof. We begin by applying Itô's Lemma to M to obtain

$$dM = \left(\frac{1}{2}\sigma^2 x^2 M'' + \left(\mu x + y - \overline{C}\right) M'\right) dt + \sigma x M' dz. \tag{27}$$

Next, we put $\widetilde{C} = f(\alpha M)$. Then, differentiating equation (10) with respect to x, we have

$$\frac{1}{2}\sigma^2 x^2 M'' + \left(\mu x + y - \widetilde{C}\right) M' - \gamma M + \sigma^2 x M' + \mu M - \widetilde{C}' M + U'\left(\widetilde{C}\right)\widetilde{C}' = 0$$

when $x > 0$. Moreover this equality extends by continuity to the case $x = 0$. Hence

$$\begin{aligned}
\frac{1}{2}\sigma^2 x^2 M'' + \left(\mu x + y - \overline{C}\right) M' &= \frac{1}{2}\sigma^2 x^2 M'' + \left(\mu x + y - \widetilde{C}\right) M' + \left(\widetilde{C} - \overline{C}\right) M' \\
&= \gamma M - \sigma^2 x M' - \mu M + \widetilde{C}' M - U'\left(\widetilde{C}\right)\widetilde{C}' + \left(\widetilde{C} - \overline{C}\right) M' \\
&= \gamma M - \sigma^2 x M' - \mu M + \widetilde{C}' M - \alpha M \widetilde{C}' + \left(\widetilde{C} - \overline{C}\right) M' \\
&= \left(\gamma - \mu + (1-\alpha)\widetilde{C}'\right) M - \left(\sigma^2 x - \left(\widetilde{C} - \overline{C}\right)\right) M'
\end{aligned}$$

and

$$\frac{dM}{M} = \left(\gamma - \mu + (1-\alpha)\widetilde{C}' - \sigma^2 x \frac{M'}{M} + \left(\widetilde{C} - \overline{C}\right)\frac{M'}{M}\right) dt + \sigma x \frac{M'}{M} dz.$$

Next,

$$\frac{M'}{M} = \frac{U''\left(\widetilde{C}\right)\widetilde{C}'}{U'\left(\widetilde{C}\right)} = \frac{\widetilde{C}U''\left(\widetilde{C}\right)}{U'\left(\widetilde{C}\right)}\frac{\widetilde{C}'}{\widetilde{C}} = -\rho\left(\widetilde{C}\right)\frac{\widetilde{C}'}{\widetilde{C}}.$$

Hence

$$\frac{dM}{M} = \left(\gamma - \mu + (1-\alpha)\,\widetilde{C}' + \sigma^2\rho\left(\widetilde{C}\right)\frac{x\widetilde{C}'}{\widetilde{C}} - \left(\widetilde{C} - \overline{C}\right)\rho\left(\widetilde{C}\right)\frac{\widetilde{C}'}{\widetilde{C}}\right)dt$$
$$-\sigma\rho\left(\widetilde{C}\right)\frac{x\widetilde{C}'}{\widetilde{C}}dz. \tag{28}$$

In particular, we have the first statement of the Theorem. As for the second statement, note that if $x = 0$ and $\overline{C}\,(0+) > \overline{C}\,(0)$ then $\overline{C}\,(0) = y$ and therefore it follows directly from equation (27) that $dM = 0$. In particular, we have

$$\left(\widetilde{C} - \overline{C}\right)\rho\left(\widetilde{C}\right)\frac{\widetilde{C}'}{\widetilde{C}} = \gamma - \mu + (1-\alpha)\,\widetilde{C}'.$$

I.e. the correction term in equation (28) exactly cancels the other terms. ■

5.5. The Dynamics of Consumption. Since $\overline{C}$ may have a discontinuity at 0, we cannot use Itô's Lemma to study its dynamics. We can, however, use Itô's Lemma to study the dynamics of $\widetilde{C} = f\,(\alpha M)$. Just as the dynamics of M were very closely related to those of $U'\big(\overline{C}\,\big)$, so the dynamics of $\widetilde{C}$ are very closely related to those of $\overline{C}$. Indeed, we have $\widetilde{C} = \overline{C}$ for $x > 0$. Moreover:

1. if $\overline{C}\,(0+) = \overline{C}\,(0)$, then the dynamics of $\widetilde{C}$ are identical to those of $\overline{C}$;
2. if $\overline{C}\,(0+) > \overline{C}\,(0)$ and $x\,(0) \in (0, +\infty)$, then the dynamics of $\widetilde{C}$ are identical to those of $\overline{C}$ on the interval $(0, \tau)$, where τ is the first time that x hits 0; and

3. if $\overline{C}(0+) > \overline{C}(0)$ and $x(0) \in 0$, then the dynamics of $\widetilde{C}$ are identical to those of $\overline{C}$, in the sense that both are trivial.

The two dynamics only differ if $\overline{C}(0+) > \overline{C}(0)$ and $x(0) \in (0, +\infty)$, in which case $\overline{C}$ jumps down at τ.

For all $c \in (0, +\infty)$, put $\pi(c) = -\frac{cU'''(c)}{U''(c)}$. Then:

Theorem 21. *We have:*

$$\frac{d\widetilde{C}}{\widetilde{C}} = -\left(\frac{\gamma - \mu}{\rho\left(\widetilde{C}\right)} + \sigma^2 \frac{x\widetilde{C}'}{\widetilde{C}} + \frac{(1-\alpha)\widetilde{C}'}{\rho\left(\widetilde{C}\right)} - \frac{1}{2}\sigma^2 \pi\left(\widetilde{C}\right)\left(\frac{x\widetilde{C}'}{\widetilde{C}}\right)^2\right) dt + \sigma \frac{x\widetilde{C}'}{\widetilde{C}} dz \quad (29)$$

if either $x > 0$ or $x = 0$ and $\overline{C}(0+) = \overline{C}(0)$; and

$$\frac{d\widetilde{C}}{\widetilde{C}} = 0$$

if $x = 0$ and $\overline{C}(0+) > \overline{C}(0)$.

Equation (29), which describes the evolution of consumption, compares closely to equation (26), which describes the evolution of marginal utility. To underscore the similarities, begin with equation (26), replace $\overline{C}$ with $\widetilde{C}$, and then divide by $\rho\left(\widetilde{C}\right)$. There are only two differences between the resulting equation and equation (29): a series of sign reversals; and the appearance of the new deterministic term $-\frac{1}{2}\sigma^2 \pi\left(\widetilde{C}\right)\left(\frac{x\widetilde{C}'}{\widetilde{C}}\right)^2 dt$. The sign reversals reflect the inverse relationship between consumption and marginal utility. The new deterministic term reflects the effects of prudence. The sign of the prudence effect depends on the sign of U''': when $U''' < 0$, the prudence term raises the growth rate of consumption. (Cf. Kimball (1990).)

Proof. We have

$$\frac{d\widetilde{C}}{\widetilde{C}} = \frac{1}{\widetilde{C}}\left(f'(\alpha M)\,\alpha dM + \frac{1}{2}f''(\alpha M)\,\alpha^2 (dM)^2\right)$$

(applying Itô's Lemma to $\widetilde{C} = f(\alpha M)$)

$$= \frac{1}{\widetilde{C}}\left(f'(\alpha M)\,\alpha M\frac{dM}{M} + \frac{1}{2}f''(\alpha M)\,\alpha^2 M^2\left(\frac{dM}{M}\right)^2\right)$$

(collecting terms)

$$= \frac{U'\left(\widetilde{C}\right)}{\widetilde{C}U''\left(\widetilde{C}\right)}\frac{dM}{M} - \frac{1}{2}\frac{U'''\left(\widetilde{C}\right)U'\left(\widetilde{C}\right)^2}{\widetilde{C}U''\left(\widetilde{C}\right)^3}\left(\frac{dM}{M}\right)^2$$

(because U' is the inverse of f)

$$= -\frac{1}{\rho\left(\widetilde{C}\right)}\frac{dM}{M} + \frac{1}{2}\frac{\pi\left(\widetilde{C}\right)}{\rho\left(\widetilde{C}\right)^2}\left(\frac{dM}{M}\right)^2$$

(by definition of $\rho\left(\widetilde{C}\right)$ and $\pi\left(\widetilde{C}\right)$)

$$= -\frac{1}{\rho\left(\widetilde{C}\right)}\left(\left(\gamma - \mu + (1-\alpha)\,\widetilde{C}' + \sigma^2\rho\left(\widetilde{C}\right)\frac{x\widetilde{C}'}{\widetilde{C}} - \left(\widetilde{C} - \overline{C}\right)\rho\left(\widetilde{C}\right)\frac{\widetilde{C}'}{\widetilde{C}}\right)dt\right.$$
$$\left. - \sigma\rho\left(\widetilde{C}\right)\frac{x\widetilde{C}'}{\widetilde{C}}dz\right) + \frac{1}{2}\frac{\pi\left(\widetilde{C}\right)}{\rho\left(\widetilde{C}\right)^2}\left(\sigma\rho\left(\widetilde{C}\right)\frac{x\widetilde{C}'}{\widetilde{C}}\right)^2 dt$$

(substituting for $\frac{dM}{M}$ from equation (28))

$$= -\left(\frac{\gamma - \mu + (1-\alpha)\widetilde{C}'}{\rho\left(\widetilde{C}\right)} + \sigma^2 \frac{x\widetilde{C}'}{\widetilde{C}} - \frac{1}{2}\pi\left(\widetilde{C}\right)\left(\sigma\frac{x\widetilde{C}'}{\widetilde{C}}\right)^2 - \left(\widetilde{C} - \overline{C}\right)\frac{\widetilde{C}'}{\widetilde{C}}\right)dt$$
$$+\sigma\frac{x\widetilde{C}'}{\widetilde{C}}dz$$

(simplifying). ■

6. The Deterministic Case

In the present section, we again exploit the equivalence result of Section 4 to take the limit of the instantaneous-gratification model obtained as $\sigma \to 0+$. Recall that σ represents the standard deviation of asset returns. We show that, by viewing the deterministic instantaneous-gratification model as a limiting case of the stochastic instantaneous-gratification model, we are able to pinpoint a unique equilibrium value function for the deterministic model. Furthermore, the Bellman system for the deterministic value function turns out to be particularly tractable. For one thing, while the Bellman system for the stochastic value function is a *second*-order non-autonomous ordinary differential equation, the Bellman system for the deterministic value function is a *first*-order non-autonomous ordinary differential equation. For another, in the case in which U has constant relative risk aversion, the Bellman system for the deterministic value function possesses a symmetry that allows us to transform it into a first-order *autonomous* ordinary differential equation. We are therefore able to provide a complete analysis of equilibrium in this case. In particular, we obtain an example that shows that the condition $\gamma \geq \mu$ used in our proof of monotonicity of the consumption function is necessary, at least in the deterministic case. Assumptions E1-E4 and H1-H4 will be in force throughout the section.

6.1. Derivation of the Deterministic Model. The following theorem describes the sense in which the deterministic instantaneous-gratification model is the limit of the stochastic instantaneous-gratification model. The proof of the theorem follows standard lines and is omitted to conserve space.[6]

Theorem 22. *We have:*

1. *there is a continuous function* $\overline{V}_0 : [0, +\infty) \to \mathbb{R}$ *such that* $\overline{V} \to \overline{V}_0$ *uniformly on compact subsets of* $[0, +\infty)$ *as* $\sigma \to 0+$*;*
2. $\overline{V}_0$ *is the unique viscosity solution of*

$$(\mu x + y) \overline{V}_0' - \gamma \overline{V}_0 + \widehat{h} \left(\overline{V}_0' \right) = 0 \tag{30}$$

when $x > 0$ *and*

$$y \overline{V}_0' - \gamma \overline{V}_0 + \widehat{h}_0 \left(\overline{V}_0' \right) = 0 \tag{31}$$

when $x = 0$. ■

In particular, we obtain an equilibrium-refinement result for the deterministic model. By letting $\sigma \to 0+$ we select a sensible equilibrium for the deterministic model ($\sigma = 0$). Krusell and Smith (2000) have shown that hyperbolic Markov equilibria are *not* unique in a deterministic discrete-time setting. Our refinement provides a natural method for selecting among these equilibria.

6.2. The Case of Constant Relative Risk Aversion. In this section we adopt the following parametric assumptions:

[6] See Crandall et al (1992) for a "user's guide" to viscosity solutions.

P1 ρ is constant;

P2 $\mu > 0$.

Under these assumptions, we can transform the non-autonomous system (30-31) into an autonomous system.

Lemma 23. *Suppose that assumptions P1-P2 hold, and that $\rho \neq 1$. Put*

$$l = \log(\mu x + y) \text{ and } v(l) = \frac{\overline{V}_0\left(\frac{\exp(l)-y}{\mu}\right)}{(1-\rho)\,U(\exp(l))}.$$

Then $\overline{V}_0$ satisfies equations (30-31) iff v satisfies

$$\mu\left((1-\rho)\,v + v'\right) - \gamma v + \widehat{h}\left(\mu\left((1-\rho)\,v + v'\right)\right) = 0 \tag{32}$$

when $l > \log(y)$ and

$$\max\left\{\frac{1}{\alpha}, \mu\left((1-\rho)\,v + v'\right)\right\} - \gamma v + \widehat{h}\left(\max\left\{\frac{1}{\alpha}, \mu\left((1-\rho)\,v + v'\right)\right\}\right) = 0 \tag{33}$$

when $l = \log(y)$.

Proof. We proceed in three steps. First, put

$$v_0(x) = \frac{\overline{V}_0(x)}{(1-\rho)\,U(\mu x + y)}.$$

Then (30) holds iff

$$0 = (\mu x + y)(1-\rho)\left(\mu U' v_0 + U v_0'\right) - \gamma(1-\rho)\,U v_0 + \widehat{h}\left((1-\rho)\left(\mu U' v_0 + U v_0'\right)\right)$$

(because $\overline{V}_0' = (1-\rho)\left(\mu U' v_0 + U v_0'\right)$, and where we have suppressed the dependence of U and v_0 on $\mu x + y$ and x respectively)

$$\Leftrightarrow 0 = (\mu x + y)\left(\frac{\mu U' v_0}{U} + v_0'\right) - \gamma v_0 + \frac{\widehat{h}\left((1-\rho)\left(\mu U' v_0 + U v_0'\right)\right)}{(1-\rho)\,U}$$

(dividing through by $(1-\rho)\,U$)

$$\Leftrightarrow 0 = \left((1-\rho)\,\mu v_0 + (\mu x + y)\,v_0'\right) - \gamma v_0 + \widehat{h}\left(\frac{(1-\rho)\left(\mu U' v_0 + U v_0'\right)}{\left((1-\rho)\,U\right)^{\frac{\rho}{\rho-1}}}\right)$$

(because $(\mu x + y)\,U' = (1-\rho)\,U$ and $\widehat{h}$ is homogeneous of degree $1 - \frac{1}{\rho}$)

$$\Leftrightarrow 0 = \left((1-\rho)\,\mu v_0 + (\mu x + y)\,v_0'\right) - \gamma v_0 + \widehat{h}\left((1-\rho)\,\mu v_0 + (\mu x + y)\,v_0'\right) \tag{34}$$

(because $\left((1-\rho)\,U\right)^{\frac{\rho}{\rho-1}} = U'$). Secondly, put

$$v\,(l) = v_0\left(\frac{\exp\,(l) - y}{\mu}\right).$$

Then (34) holds iff

$$0 = \left((1-\rho)\,\mu v + \mu v'\right) - \gamma v + \widehat{h}\left((1-\rho)\,\mu v + \mu v'\right)$$

(because $v_0'\,(x) = \frac{\mu v'(\log(\mu x + y))}{(\mu x + y)}$). Thirdly, note that

$$\overline{V}_0' \geq \frac{U'\,(y)}{\alpha} \Leftrightarrow \left((1-\rho)\,\mu v + \mu v'\right) \geq \frac{1}{\alpha}.$$

The same chain of reasoning therefore shows that (31) holds iff

$$0 = \max\left\{\frac{1}{\alpha}, (1-\rho)\,\mu v + \mu v'\right\} - \gamma v + \widehat{h}\left(\max\left\{\frac{1}{\alpha}, (1-\rho)\,\mu v + \mu v'\right\}\right). \blacksquare$$

Lemma 24. *Suppose that assumptions P1-P2 hold, and that* $\rho = 1$. *Put*

$$l = \log(\mu x + y) \text{ and } v(l) = \overline{V}_0\left(\frac{\exp(l) - y}{\mu}\right) - \frac{U(\exp(l))}{\gamma}.$$

Then $\overline{V}_0$ *satisfies equations (30-31) iff* v *satisfies*

$$\mu\left(\frac{1}{\gamma} + v'\right) - \gamma v + \widehat{h}\left(\mu\left(\frac{1}{\gamma} + v'\right)\right) = 0 \tag{35}$$

when $l > \log(y)$ *and*

$$\max\left\{\frac{1}{\alpha}, \mu\left(\frac{1}{\gamma} + v'\right)\right\} - \gamma v + \widehat{h}\left(\max\left\{\frac{1}{\alpha}, \mu\left(\frac{1}{\gamma} + v'\right)\right\}\right) = 0 \tag{36}$$

when $l = \log(y)$.

Proof. We proceed in three steps. First, put

$$v_0(x) = \overline{V}_0(x) - \frac{U(\mu x + y)}{\gamma}.$$

Then (30) holds iff

$$(\mu x + y)\overline{V}_0' - \gamma \overline{V}_0 + \widehat{h}\left(\overline{V}_0'\right) = 0$$

$$0 = (\mu x + y)\left(\frac{\mu U'}{\gamma} + v_0'\right) - \gamma\left(\frac{U}{\gamma} + v_0\right) + \widehat{h}\left(\frac{\mu U'}{\gamma} + v_0'\right)$$

(because $\overline{V}_0' = \frac{\mu U'}{\gamma} + v_0'$, and where we have suppressed the dependence of U and v_0 on $\mu x + y$ and x respectively)

$$\Leftrightarrow 0 = \left(\frac{\mu}{\gamma} + (\mu x + y)\, v_0'\right) - \gamma\left(\frac{U}{\gamma} + v_0\right) + \widehat{h}\left(\frac{\frac{\mu}{\gamma} + (\mu x + y)\, v_0'}{(\mu x + y)}\right)$$

(because $(\mu x + y)\, U' = 1$)

$$\Leftrightarrow 0 = \left(\frac{\mu}{\gamma} + (\mu x + y)\, v_0'\right) - \gamma v_0 + \widehat{h}\left(\frac{\mu}{\gamma} + (\mu x + y)\, v_0'\right) \tag{37}$$

(because $\widehat{h}\,(\phi) = -\log(\alpha\phi) - \frac{1}{\alpha}$ and $U = \log(\mu x + y)$). Secondly, put

$$v\,(l) = v_0\left(\frac{\exp(l) - y}{\mu}\right).$$

Then (37) holds iff

$$0 = \left(\frac{\mu}{\gamma} + \mu v'\right) - \gamma v + \widehat{h}\left(\frac{\mu}{\gamma} + \mu v'\right)$$

(because $v_0'\,(x) = \frac{\mu v'(\log(\mu x + y))}{(\mu x + y)}$). Thirdly, note that

$$\overline{V}_0' \geq \frac{U'\,(y)}{\alpha} \Leftrightarrow \left(\frac{\mu}{\gamma} + \mu v'\right) \geq \frac{1}{\alpha}.$$

The same chain of reasoning therefore shows that (31) holds iff

$$0 = \max\left\{\frac{1}{\alpha}, \frac{\mu}{\gamma} + \mu v'\right\} - \gamma v + \widehat{h}\left(\max\left\{\frac{1}{\alpha}, \frac{\mu}{\gamma} + \mu v'\right\}\right). \blacksquare$$

Theorem 25. *Suppose that assumptions P1-P2 hold. Then:*

1. *If $\mu \leq \gamma < +\infty$, then $\overline{C}' > 0$ on $(0, +\infty)$.*

2. If $\alpha\mu < \gamma < \mu$, then there exists $x_1 \in (0, +\infty)$ such that:

(a) $\overline{C}' > 0$ on $(0, x_1)$;

(b) $\overline{C}' > 0$ and $\overline{C}'' = 0$ on $(x_1, +\infty)$; and

(c) $\overline{C}(x_1+) < \overline{C}(x_1-)$.

3. If $\gamma \leq \alpha\mu$, then $\overline{C}' > 0$ and $\overline{C}'' = 0$ on $(0, +\infty)$.

Moreover, if $\gamma > \alpha\mu$, then $\overline{C}(x) > \mu x + y$ for all $x \in [0, +\infty)$. In particular, there is always a unique solution to the wealth dynamics.

The three cases of Theorem 25 are illustrated in Figures 5, 6 and 7. These figures are effectively phase portraits: although equations (32) and (35) are first-order autonomous ordinary differential equations, jumps can occur in v', and it is therefore helpful to include v' in the portrait. It is possible for v' to jump up but not down. Intuitively speaking, this is because v is the upper envelope of smooth functions, and can therefore have convex kinks but not concave kinks.

The first element of the portraits is the graph of the Hamilton-Jacobi function H, which consists of the locus of points (v', v) satisfying equation (30). The second element of the portraits is the horizontal line $v = a$. This corresponds to the value that the consumer obtains if she has wealth 0 and consumes y forever. The third element is the vertical line at $v' = 0$. The intersection of this line with the graph of H yields the steady state $(0, v(\infty))$, where $v(\infty)$ corresponds to the value that the consumer obtains is she has very large wealth and consumes out of this wealth at a constant rate forever. The fourth element is the horizontal line $v = v(\infty)$.

There are three possible positions for the steady state: between the left-hand intersection (L, a) of $v = a$ with the graph of H and the minimum of H (Figure 5); between the minimum of H and the right-hand intersection (R, a) of $v = a$ with the

Figure 5: Plot of H as a function of v’ ($\rho = 1/2$, $\mu = 0.05$, $\alpha = 0.7$, $\gamma = 0.07$)

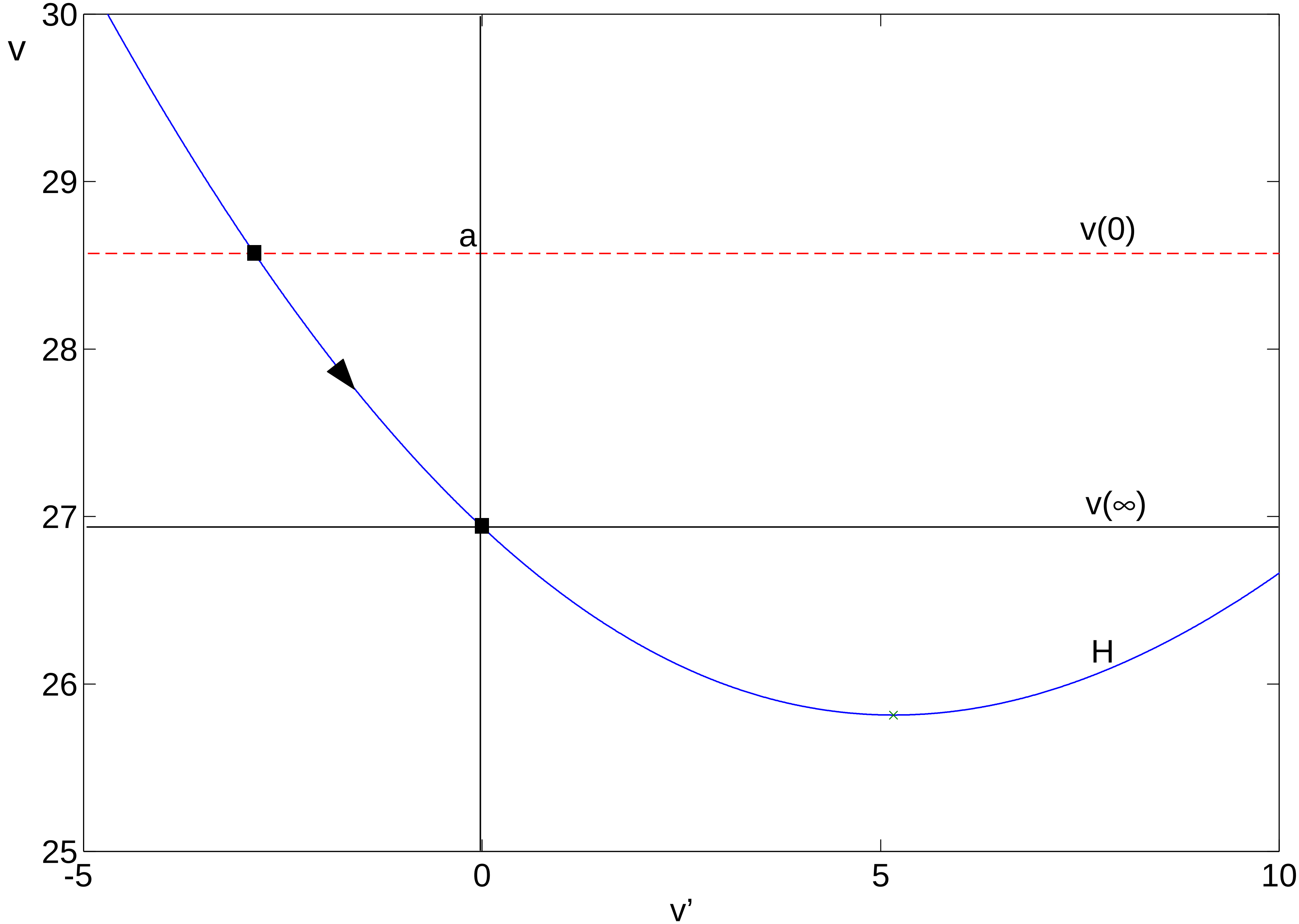


Figure 6: Plot of H as a function of v, ($\rho = 1/2$, $\mu = 0.05$, $\alpha = 0.7$, $\gamma = 0.04$)

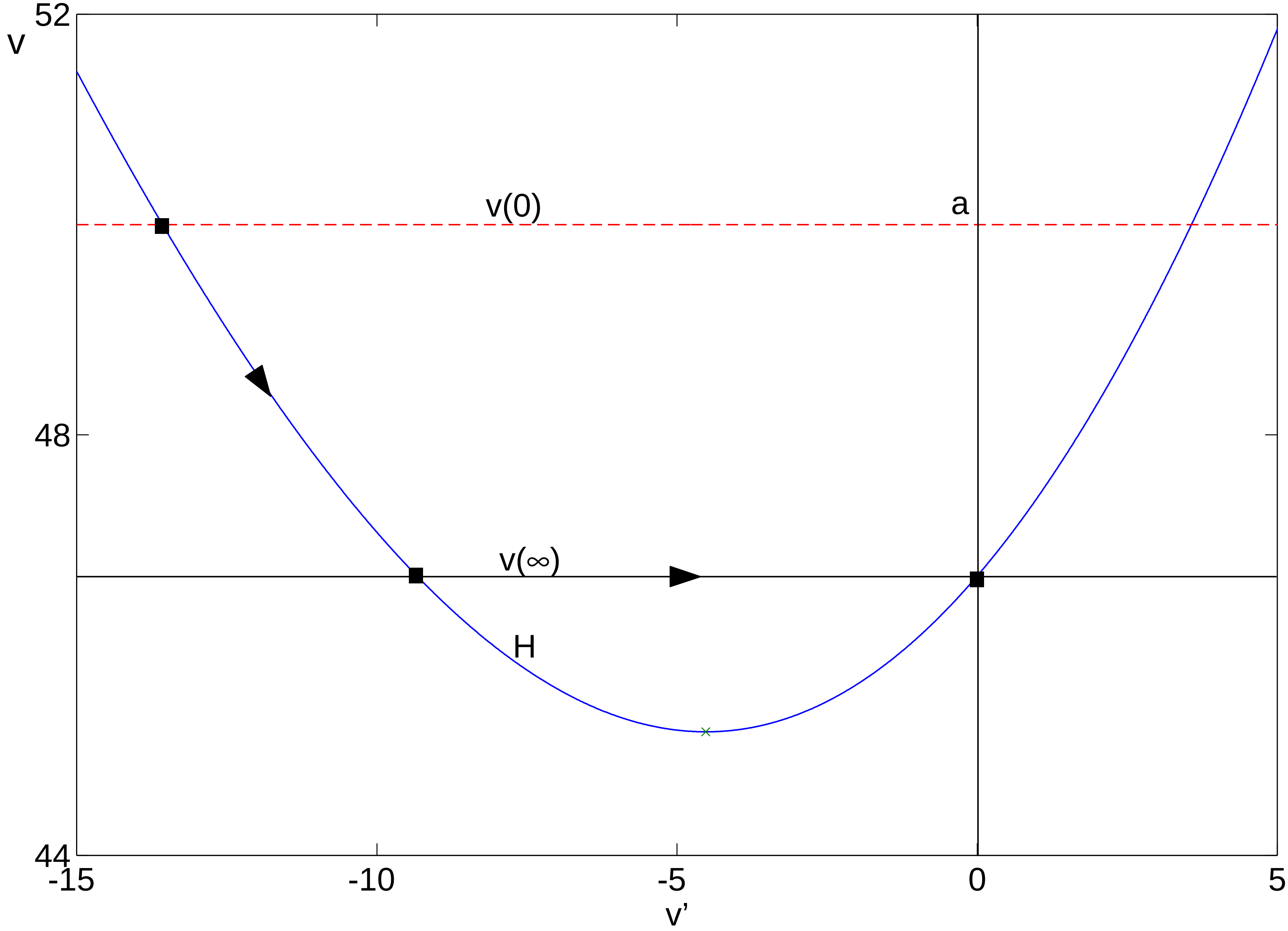
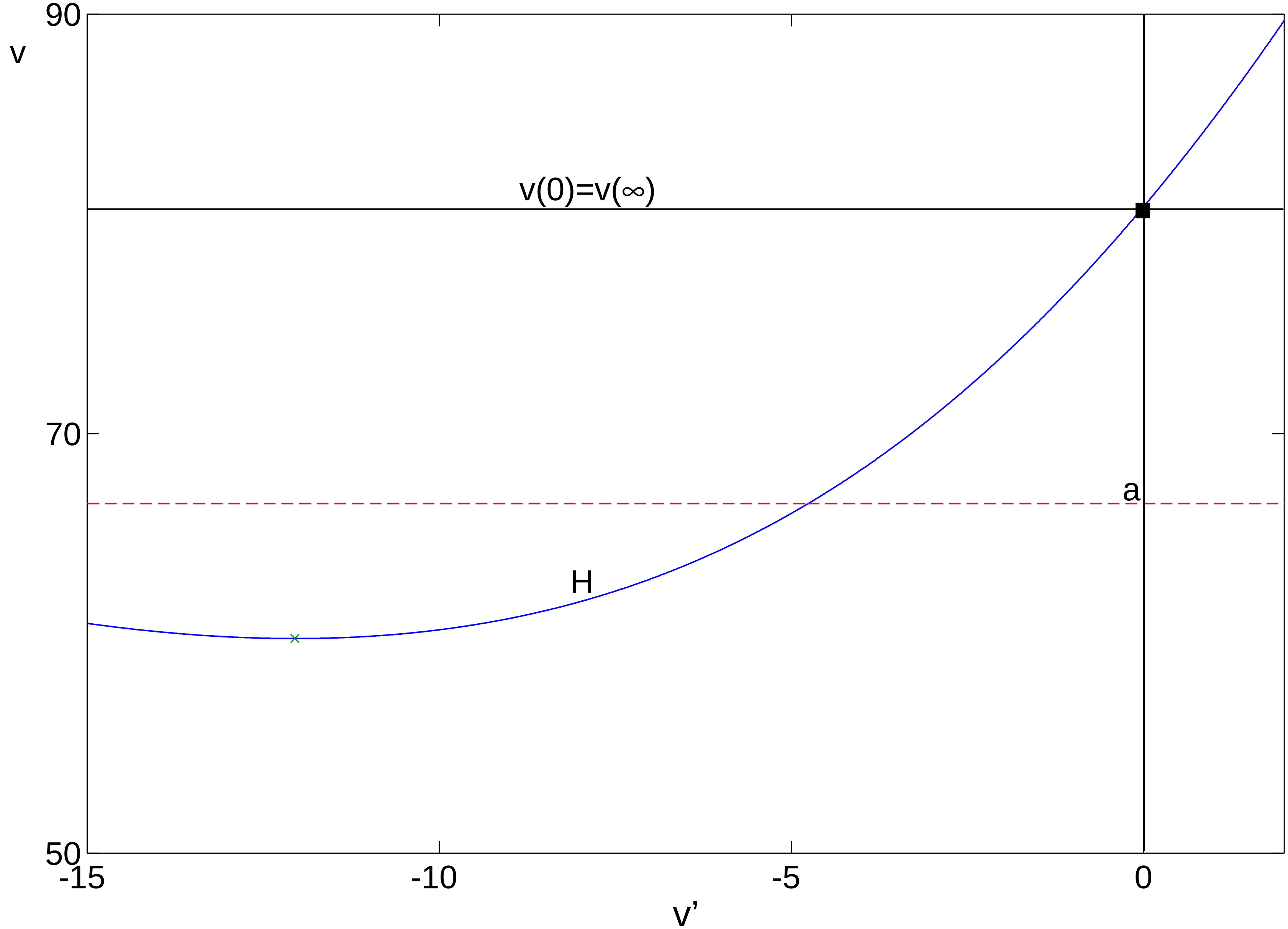


Figure 7: Plot of H as a function of v', ($\rho = 1/2$, $\mu = 0.05$, $\alpha = 0.7$, $\gamma = 0.03$)



graph of H (Figure 6); and to the right of the right-hand intersection of $v = a$ with the graph of H (Figure 7). In Figure 5, the phase path begins at (L, a), and falls smoothly to $(0, v(\infty))$. In Figure 6, the phase path again begins at (L, a), but now it falls smoothly to the level $v = v(\infty)$, at which point it jumps to $(0, v(\infty))$. In Figure 7, the phase path begins at (R, a) and remains there.

Proof. Put

$$a = \left\{ \begin{array}{ll} \frac{1}{(1-\rho)\gamma} & \text{if } \rho \neq 1 \\ 0 & \text{if } \rho = 1 \end{array} \right\}.$$

Then, in view of Lemmas 23 and 24, there exists a smooth function $H : (-\infty, +\infty) \to \mathbb{R}$ such that:

1. $v = H(v')$;
2. $H'' > 0$;
3. $H(\phi) \to +\infty$ as $\phi \to \pm\infty$;
4. $\min H < a$; and
5. if $H'(0) \leq 0$ then $H(0) < a$.

Moreover, because v is a viscosity solution of the equation $v = H(v')$, any switches that occur between the two values of v' consistent with any given v must be from the lower to the higher value of v'. There are therefore three possibilities:

1. If $H'(0) \leq 0$, then: $v(0) = a$; $v' < 0$ on $(0, +\infty)$; and v asymptotes to $H(0)$.
2. If $H'(0) > 0$ and $H(0) < a$, then there exists $x_1 \in (0, +\infty)$ such that: $v(0) = a$; $v' < 0$ on $(0, x_1)$; and $v = H(0)$ on $[x_1, +\infty)$. In particular, v' jumps from the lower to the higher of the two values in $H^{-1}(H(0))$ at x_1.

3. If $H(0) \geq a$, then: $v = H(0)$ on $[0, +\infty)$.

Moreover, it can be shown that

$$H'(0) = -\frac{\gamma - \mu}{\gamma - \mu(1-\rho)}, \quad \max H^{-1}(H(a)) = \frac{\gamma - \alpha\mu}{\alpha\mu\gamma}.$$

Finally, it can be shown that $\overline{C}(x) > \mu x + y$ iff $v'(l) < \frac{\gamma - \alpha\mu}{\alpha\mu\gamma}$. Hence $\overline{C}(x) > \mu x + y$ for all $x \in [0, +\infty)$ iff $H(0) < a$. In particular, while $\overline{C}$ fails to be unique at x_1 in the second of our three cases, there is nonetheless a unique solution to the dynamics even in that case. ■

7. The Bellman System of the Instantaneous-Gratification Model Revisited

Theorem 8 covers the case in which U has constant relative risk aversion $\rho > 1 - \alpha$. It is also possible to prove a satisfactory limit theorem that covers the case in which U has constant relative risk aversion $\rho < 1 - \alpha$. In order to formulate such a theorem, we introduce the following assumptions, which complement Assumptions H2 and H3:

H2′ $\alpha + \overline{\rho} - 1 < 0$;

H3′ $(2 - \alpha)\overline{\rho} - (1 - \alpha)\underline{\pi} < 0$.

The theorem, the proof of which is omitted, is then as follows.

Theorem 26. *Suppose that Assumptions E1-E4, H1, H2′, H3′ and H4 hold. Then* $W \to \frac{\alpha U(y)}{\gamma}$ *and* $V \to \frac{U(y)}{\gamma}$ *uniformly on compact subsets of* $[0, +\infty)$ *as* $\lambda \to +\infty$. ■

These limiting value functions reflect the fact that as $\lambda \to +\infty$ the consumption rate also goes to $+\infty$. The infinite consumption rate arises because when $\rho < 1 - \alpha$, the utility function is not sufficiently bowed to dampen the feedback effects that arise in hyperbolic models. Indeed, the feedback effects drive consumption to infinity: e.g., "If the next self is going to consume at a high rate, then I should consume at an even higher rate, etc..." An infinite consumption rate implies that wealth immediately collapses to 0. So total utility is just the flow utility of consuming labor income each period. Hence, $W \to \frac{\alpha U(y)}{\gamma}$ and $V \to \frac{U(y)}{\gamma}$. Note that the instantaneous consumption boom has no impact on total utility because the consumption boom is only of duration dt and generates diminishing returns in utility.

Comparing Assumptions H2 and H3 with Assumptions H2′ and H3′, it is clear that there is a knife-edge case in between, namely the case in which U has constant relative risk aversion $\rho = 1 - \alpha$. We have not analyzed this case. However, we would expect it to resemble the case $\rho < 1 - \alpha$ covered by Theorem 26.

Finally, note that Theorem 8 continues to hold when Assumptions H2 and H3 are replaced by the following, significantly weaker, assumptions:

H2″ $\alpha + \liminf_{c \to +\infty} \rho(c) - 1 > 0$;

H3″ $(2 - \alpha) \liminf_{c \to +\infty} \rho(c) - (1 - \alpha) \limsup_{c \to +\infty} \pi(c) > 0$.

Assumptions H2″ and H3″ ensure that $\widehat{h}$ is decreasing and convex near 0. This is enough to ensure that consumption remains bounded as $\lambda \to +\infty$. These assumptions are, however, consistent with $\widehat{h}$ being increasing or concave away from 0. In other words, for some BRRA utility functions, the instantaneous-gratification problem is not value-function equivalent to any exponential consumption problem.

8. Conclusions

We have described a continuous-time model of hyperbolic discounting. Our model allows for a general class of preferences, includes liquidity constraints, and places no restrictions on equilibrium policy functions. The model is also psychologically relevant. We take the phrase "instantaneous gratification" literally. We analyze a model in which individuals prefer gratification in the present instant discretely more than consumption in the momentarily delayed future. In this simple setting, equilibrium is unique and the consumption function is continuous. When the long-run discount rate weakly exceeds the interest rate, the consumption function is also monotonic. All of the pathologies that characterize discrete-time hyperbolic models vanish.

9. References

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