

Inefficiency in Legislative Policy-Making: A Dynamic Analysis*

Abstract

This paper develops an infinite horizon model of public spending and taxation in which policy decisions are determined by legislative bargaining. The policy space incorporates both productive and distributive public spending and distortionary taxation. The productive spending is investing in a public good that benefits all citizens (e.g., national defense or air quality) and the distributive spending is district-specific transfers (e.g., pork barrel spending). Investment in the public good creates a dynamic linkage across policy-making periods. The analysis explores the dynamics of legislative policy choices, focusing on the efficiency of the steady state level of taxation and allocation of tax revenues. The model sheds new light on the efficiency of legislative policy-making and has a number of novel positive implications.

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1 Introduction

It has long been argued that legislatures in which representatives are elected by geographically defined districts will make inefficient decisions. According to conventional wisdom, legislators will try to benefit their constituents at the expense of the general community through pork barrel spending and other distributive policies (e.g., cotton or tobacco subsidies). This leads to both excessive spending and a misallocation of government revenues between distributive policies and important national public goods.

Despite this widely held view, formal political theory tells us little about the dynamics of legislative policy choices. Will legislative policy-making result in a long run size of government that is too large? What will be the time path of investment in national public goods and the long run levels of these goods? What features of the environment determine the magnitude of the distortions arising from legislative policy-making? This paper analyzes these questions in a novel infinite horizon model of legislative policy-making. The model is of an economy in which policy choices are made by a legislature comprised of representatives elected by single-member, geographically defined districts. In each period, the legislature chooses the level of a distortionary income tax and decides how to allocate tax revenues between investment in a public good that benefits all citizens (national defense or air quality) and district-specific unproductive transfers (pork barrel spending). Thus the model incorporates both productive and distributive public spending and distortionary taxation. The dynamic linkage across periods is created by the public good which is the state variable. Legislative policy-making in each period is modelled using the legislative bargaining approach of Baron and Ferejohn (1989).

The results of the model provide a rigorous formal underpinning for the conventional wisdom described above by showing conditions under which the steady state size of government (as measured by the tax rate) is too large and the level of public goods is too low. However, we also show that this conventional view needs qualification. When the economy's taxable capacity is small relative to its public good needs, legislative decisions will actually be efficient in the long run, despite the fact that legislators can benefit their districts via distributive policies. Moreover, the nature of the inefficiency emerging from legislative choice could be quite different from that which the conventional wisdom assumes. In particular, legislators could hold back on public good spending recognizing that creating too large a stock of these goods could lead future legislators to

start engaging in pork barrel spending. This means that the overall size of government could be below optimal and revenues could be allocated to their most productive uses - namely, maintaining public good levels.

The model also generates a number of novel positive implications. First, it suggests that the size of the legislative coalitions passing budgets may decline over time as a country builds up its stock of public goods. Second, the model suggests that societies in which citizens have a more elastic labor supply will enjoy better quality government. A higher elasticity of labor supply, reduces pork barrel spending but not the long run levels of public goods. Finally, the model suggests that the quality of government as measured by the proportion of revenues devoted to distributive policies is inversely correlated with the productivity of the private sector. The model yields these clean comparative statics results because we fully characterize the set of equilibria and, in particular, the conditions for a unique equilibrium.

In studying the efficiency of politically determined policy choices, our paper contributes to the literature on the theory of political failure.¹ A number of works have explored the efficiency of legislative decision-making from a static perspective. In a well-known paper, Weingast, Shepsle and Johnsen (1981) argue that distributive policy-making will lead to excessive government spending. However, they do not model the process of passing legislation, assuming instead that legislative policy-making is governed by a “norm of universalism”.² In a legislative bargaining model in which proposals need to be approved by a qualified majority, Baron (1991) shows that legislators may propose projects whose aggregate benefits are less than their costs, when these benefits can be targeted to particular districts. Related models of legislative bargaining are elaborated by Persson and Tabellini (2000) and Austen-Smith and Banks (2005). An interesting dynamic analysis of the problem is provided by LeBlanc, Snyder and Tripathi (2000) who argue that majority-rule legislatures will under-invest in public goods.³ They make their argument in the context of a finite

¹ This literature seeks to develop an understanding of the performance of political institutions in allocating public resources that matches our understanding of the performance of markets in the allocation of private resources. It includes the papers by Acemoglu (2004), Acemoglu and Robinson (2001), Besley and Coate (1998), Coate and Morris (1995), (1999), Lizzeri and Persico (2001), Persson and Svensson (1989), Tabellini and Alesina (1990), and Wittman (1989).

² Under this norm, each legislator unilaterally decides on the level of spending he would like on projects in his own district and the aggregate level of taxation is determined by the need to balance the budget. Distributive policy-making then becomes a pure common pool problem.

³ Velasco (1999) develops an analysis of the accumulation of public debt that models public decision-making as a dynamic common pool problem. The key assumption of this approach is that (as in the static model of Weingast, Shepsle and Johnsen (1981)) legislators can all choose the amount of spending they want for their constituents.

horizon model in which in each period a legislature allocates a pool of exogenously given resources between targeted transfers and a public investment that serves to increase the amount of revenue available in the next period. Like us, they employ the bargaining approach to model legislative policy-making. Our model differs from theirs in that it is infinite horizon, taxation is distortionary and investment in public goods yields benefits for more than one period. These features explain our more nuanced set of conclusions concerning the efficiency of legislative policy-making.

In solving a dynamic political bargaining model, our paper also contributes to the literature on the legislative bargaining approach.⁴ While most papers in this literature focus on the choices made for a single policy period, a few have explored dynamic decision making. Baron (1996), Baron and Herron (2003), and Kalandrakis (2004) all study dynamic models in which a legislature makes policy choices in each period.⁵ In these models, the dynamic linkage across periods is created by the assumption that today's policy choice determines tomorrow's default outcome should the legislature not come to agreement. This creates complex interactions as today's policy-makers choose policy, taking into account the implications of shifting next period's status quo.⁶ However, in these models the policy choices in each period are purely distributive so there are no implications for efficiency except in so far as citizens are risk averse.⁷ Our analysis also differs from these papers in that the dynamic linkage is created by the accumulation of the stock of public goods.

Finally, our paper contributes to a small literature trying to develop infinite horizon political economy models of policy-making that incorporate rational, forward-looking decision makers. It Thus, there is no voting on spending bills and hence no need for legislators to build coalitions to pass them.

⁴ This literature seeks to understand how legislatures choose policies in multi-dimensional policy-making environments where the median voter theorem cannot be applied and party attachments are weak. The approach stems from the seminal work of Baron and Ferejohn (1989) and includes the papers by Banks and Duggan (2000), Eraslan (2002) and Eraslan and Merlo (2002). An alternative "demand bargaining" approach to the problem is developed by Morelli (1999).

⁵ A recent paper related to this literature is Gomes and Jehiel (2004). They present a general model of infinitely repeated coalitional bargaining that can be interpreted as legislative bargaining. Their environment, however, differs from the rest of the literature because they assume that there are no restrictions in monetary transfers among players, so the proposer can extract all the expected continuation values from the other players with no limit. This assumption does not seem to fit with the realities of legislative bargaining where pork transfers are restricted to be positive and is key to their results. Moreover they study results for the asymptotic case when $\delta \rightarrow 1$. With respect to our model, they do not study the growth path of the public good or the type of inefficiencies that prevail when this condition for efficiency is not satisfied.

⁶ Epple and Riordan (1987) present a related analysis in which legislators take turns in being the proposer as opposed to the proposer being randomly drawn each period. They also consider non-stationary equilibria.

⁷ Baron (1996) studies a one-dimensional policy space, Baron and Herron (2003) a two-dimensional policy space, and Kalandrakis (2004) a divide-the-dollar game.

has been well recognized in the political economy literature that many interesting issues arise from recognizing the dynamic linkage of policies across periods. Such linkages either arise *directly* as with public investment or debt, or *indirectly* because today's policy choices impact citizens' private investment decisions. However, extending standard static models to understand fully dynamic policy-making has proved difficult, even in the case of one-dimensional policy environments. Accordingly, most dynamic analyses employ two period models. While these have yielded many useful insights, they do not permit the development of predictions about the dynamics of policy choices or steady state policy levels and this has spawned the recent research effort on infinite horizon models. Krusell and Rios-Rull (1999) embed a negative income tax system into the neo-classical growth model and assume that the rate of taxation is determined by the median voter in each period. They are able to solve their model numerically and use it to make predictions concerning the long run size of government. Using a simpler underlying economic model, Hassler, Rodriguez Mora, Storesletten and Zilibotti (2003) develop an overlapping generations model of the welfare state where in each period the level of welfare benefits is determined by majority voting. They are able to provide analytical solutions of their model. Hassler, Krusell, Storesletten and Zilibotti (2004) extend this approach to a richer economic environment in which the welfare state provides an insurance role.⁸ These models differ from ours in that the policy space is one-dimensional and the dynamic linkage occurs by impacting private investment decisions. Closer to our paper is Azzimonti Renzo (2005) who studies policy-making in an infinite horizon model in which in each period the winning political party allocates revenue between targeted group-specific public goods and a public infrastructure good that serves to make the economy more productive in the future. The major difference is that the winning political party is a policy dictator, and therefore there is no need to build legislative coalitions as in our model.⁹

The organization of the remainder of the paper is as follows. Section 2 describes the model. Section 3 creates a benchmark for comparison by solving for the efficient solution. The heart of the paper is Section 4 which solves for equilibrium policy choices. Section 5 develops the model's implications for the efficiency of legislative decision-making and points out some of its more interesting positive implications. A brief conclusion is offered in Section 6. An Appendix

⁸ Hassler, Storesletten and Zilibotti (2003) develop a related model where tax revenues are used to finance public goods.

⁹ In addition, Azzimonti Renzo is primarily interested in understanding the implications of asymmetries in the popularity of political parties for the time path of government spending and investment.

contains proofs of some of the more technical results.

2 The model

A continuum of infinitely lived citizens live in n identical districts indexed by $i = 1, \dots, n$. The size of the population in each district is normalized to be one. There are three goods - a public good g ; consumption z ; and labor l . The consumption good is produced from labor according to the technology $z = wl$. The public good can be produced from the consumption good according to the technology $g = z/p$. The public good is an investment good in the sense that units produced in the current period yield benefits in future periods.¹⁰ Specifically, if the level of the public good at time t is g_t and I_t new units are produced in period t , then the level in period $t + 1$ is

$$g_{t+1} = (1 - d)g_t + I_t.$$

The parameter d is the depreciation rate of the public good.

Each citizen's per period utility function is

$$z + Ag^\alpha - \frac{l^{(1+\frac{1}{\varepsilon})}}{\varepsilon + 1}.$$

where $\alpha \in (0, 1)$. The parameter A measures the relative importance of the public good to the citizens. Citizens discount future per period utilities at rate δ .

The assumptions on technology imply that the price of the public good is p and the wage rate is w . Moreover, the quasi-linear utility specification implies that the interest rate is $\rho = 1/\delta - 1$. At this interest rate, citizens have no incentive to save or borrow. At wage rate w , each citizen will work an amount $l^*(w) = (\varepsilon w)^\varepsilon$, so that ε is the elasticity of labor supply. The associated per period indirect utility function is given by

$$u(w, g; A) = \frac{\varepsilon^\varepsilon w^{\varepsilon+1}}{\varepsilon + 1} + Ag^\alpha.$$

Public decisions are made by a legislature consisting of representatives from each of the n districts. One citizen from each district is selected to be that district's representative. Since all citizens are the same, the identity of the representative is immaterial and hence the selection process can be ignored. The legislature meets at the beginning of each period. These meetings

¹⁰ Many public goods have this feature. Important examples are national defense activities, such as building tanks and training troops; scientific knowledge, such as cancer research; and environmental clean ups.

take only an insignificant amount of time, and representatives undertake private sector work in the rest of the period just like everybody else. The affirmative votes of $q < n$ representatives are required to enact any legislation. The only way the legislature can raise funds is via a proportional tax on labor income.¹¹ Tax revenues can be used to finance investment in the public good, but can also be diverted to finance targeted district-specific transfers, which are interpreted as (non-distortionary) pork barrel spending.¹²

To describe how legislative decision-making works, suppose the legislature is meeting at the beginning of a period in which the current level of the public good is g . One of the legislators is randomly selected to make the first policy proposal, with each representative having an equal chance of being recognized. A proposal is described by an $n + 2$ -tuple $\{r, s_1, \dots, s_n, x\}$, where r is the income tax rate; s_i is the proposed transfer to district i 's residents; and x is the proposed new level of the public good. The tax revenues raised under the proposal are given by $R(r) = nrwl^*(w(1 - r))$ and the proposal must satisfy the budget constraint that $\sum_i s_i \leq B(r, x; g)$ where $B(r, x; g)$ denotes the difference between tax revenues and investment spending; i.e.,

$$B(r, x; g) = R(r) - p[x - (1 - d)g].$$

The set of constraints is completed by the non-negativity constraints that $s_i \geq 0$ for each district i (which rules out financing public investment via district specific lump sum taxes).¹³

If the proposal is accepted by q legislators, then the plan is implemented and the legislature adjourns until the beginning of the next period. At that time, the legislature meets again with the only difference being that the initial level of public capital is x . If, on the other hand, the first proposal is not accepted, another legislator is chosen to make a proposal. There are $T \geq 2$ such proposal rounds, each of which takes a negligible amount of time. If the process continues until proposal round T , and the proposal made at that stage is rejected, then a legislator is appointed to choose a default policy. The legislator is required to choose a policy that treats districts uniformly, meaning that if he chooses transfers, he must choose a uniform transfer that goes to all districts.¹⁴

¹¹ For simplicity and to focus on the legislative investment decision, we abstract from the possibility of financing expenditures with public debt. In the conclusion, we discuss a way in which the model may be adapted to explore the implications of government borrowing.

¹² The district-specific transfers could be either direct grants to particular localities or earmarks for specific public projects that the districts would undertake anyway. In the latter case, the earmarks would be non-distortionary and equivalent to a direct transfer.

¹³ For analytical convenience, we do not impose the constraint that investment is non-negative; i.e., $x \geq (1 - d)g$. Thus, we are assuming, effectively, that investment is reversible.

3 The social planner's solution

To create a normative benchmark with which to compare the political equilibrium, we begin by describing the policies that would be chosen by a social planner whose objective was to maximize aggregate utility. In a period in which the current level of the public good is g , the planner's problem is to choose a new level of the public good x , a vector of transfers (s_1, \dots, s_n) , and a tax rate r to solve the problem

$$\begin{aligned} & \max nu(w(1-r), g) + \sum_i s_i + \delta V(x) \\ \text{s.t. } & s_i \geq 0 \text{ for all } i \text{ and } \sum_i s_i \leq B(r, x; g), \end{aligned}$$

where $V(x)$ denotes the planner's value function.

This problem can be simplified by observing that if $B(r, x; g)$ were positive, the planner would want to use all the available surplus revenues to finance transfers and hence $\sum_i s_i = B(r, x; g)$. Moreover, aggregate utility is independent of how the planner allocates this surplus across the districts because citizens' utilities are linear in consumption. Thus, we can eliminate the choice variables (s_1, \dots, s_n) and reformulate the problem as that of choosing a new level of the public good x and a tax rate r to solve

$$\begin{aligned} & \max nu(w(1-r), g) + B(r, x; g) + \delta V(x) \\ \text{s.t. } & B(r, x; g) \geq 0. \end{aligned}$$

The problem in this form is fairly standard. The social planner's value function must satisfy the functional equation

$$V(g) = \max_{\{r, x\}} \{nu(w(1-r), g) + B(r, x; g) + \delta V(x) : B(r, x; g) \geq 0\},$$

and familiar arguments can be applied to show that it exists and is differentiable, increasing and strictly concave. From this, the properties of the optimal policy may readily be deduced.

¹⁴ This assumption guarantees that if the legislature is unable to agree on a policy proposal, the default outcome will be efficient in the sense of maximizing the legislators' average utility, and so independent from the identity of the last proposer. The assumption helps keep the model tractable. Nonetheless, it is important to note that when the number of proposal rounds T is large, the particular default policy that is chosen has only a small effect on equilibrium payoffs, which vanishes as $T \rightarrow \infty$.

To understand the optimal policy, note first that the revenue maximizing tax rate is given by $r = 1/(1 + \varepsilon)$ and hence the maximum level of revenue that can be raised during the period is $R(1/(1 + \varepsilon))$. The levels of the public good x that the planner can implement with an initial level g are therefore given by the interval $[0, (1 - d)g + R(1/(1 + \varepsilon))/p]$. If the planner would like to invest in the public good (i.e., $x \geq (1 - d)g$) he will choose an income tax rate just sufficient to finance this investment. Raising surplus revenues to finance pork barrel spending will never be optimal because taxation is distortionary. Thus, the tax rate is $r = r(x, g)$ where the function $r(x, g)$ is implicitly defined by the equality $B(r, x; g) = 0$. On the other hand, if the planner would like to disinvest in the public good (i.e., $x < (1 - d)g$) he will redistribute the proceeds through pork rather than through an earnings subsidy. This means that the tax rate will equal 0 (as opposed to being negative) and $B(r, x; g) > 0$.

Given the properties of the value function, there will exist some critical level of the public good \hat{g} such that for all $g \leq \hat{g}$ the planner will want to invest and for all $g > \hat{g}$ he will want to disinvest. Accordingly, the optimal policy functions must be such that

$$x^o(g) = \begin{cases} \arg \max\{nu(w(1 - r(x, g)), g) + \delta V(x)\} & \text{if } g \leq \hat{g} \\ (1 - d)\hat{g} & \text{if } g > \hat{g} \end{cases}$$

and

$$r^o(g) = \begin{cases} r(x^o(g), g) & \text{if } g \leq \hat{g} \\ 0 & \text{if } g > \hat{g} \end{cases}.$$

Moreover, on the interval $[0, \hat{g}]$, the optimal public good level $x^o(g)$ must satisfy the first order condition:

$$\delta V'(x) = nw \frac{\partial u(w(1 - r(x, g)), g)}{\partial w} \frac{\partial r(x, g)}{\partial x}.$$

After computing the derivatives on the right hand side, this can be rewritten as

$$\delta V'(x) = \left[\frac{1 - r(x, g)}{1 - r(x, g)(1 + \varepsilon)} \right] \cdot p.$$

This is the *Euler equation* for the planner's problem. The term on the left hand side is the *social marginal benefit* of the public good and the term on the right is the *social marginal cost*. The social marginal cost is the product of the price of the public good p and the *marginal cost of public funds*. The latter always exceeds 1 and is higher the more elastic is the supply of labor and the

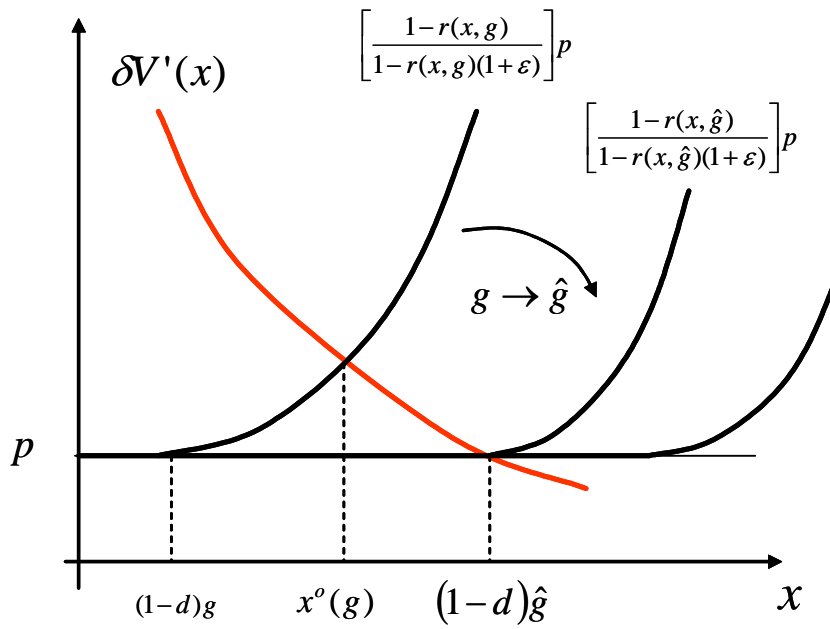


Fig 1.a

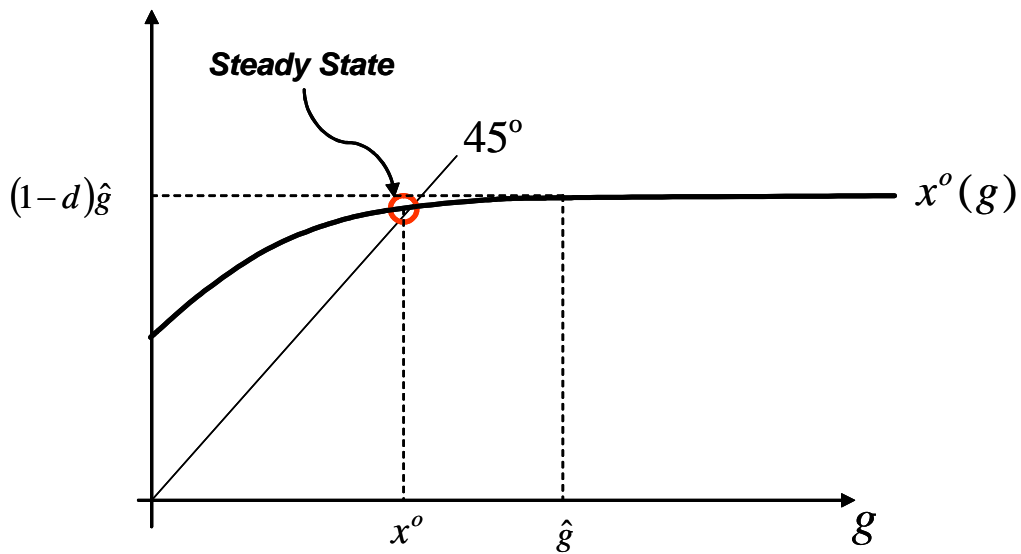


Fig 1.b

Figure 1: The planner's Problem

greater is the tax rate.¹⁵

The planner's solution converges to a unique steady state (r^o, x^o) . To compute this, we first need to find an expression for the social marginal benefit of the public good. We have that

$$V(x) = \begin{cases} \max_z \{nu(w(1 - r(z, x)), x) + \delta V(z)\} & \text{if } x \leq \hat{g} \\ nu(w, x) + p(1 - d)(x - \hat{g}) + \delta V((1 - d)\hat{g}) & \text{if } x > \hat{g} \end{cases}$$

Using the *Envelope Theorem* and computing the derivative $\partial r/\partial x$, we have that

$$V'(x) = \begin{cases} nA\alpha x^{\alpha-1} + \frac{1-r(x^o(x),x)}{1-r(x^o(x),x)(1+\varepsilon)}p(1-d) & \text{if } x \leq \hat{g} \\ nA\alpha x^{\alpha-1} + p(1-d) & \text{if } x > \hat{g} \end{cases}$$

To understand this note that there are two future benefits of investing more in the public good. First, public good consumption is higher in the next period. Second, less investment will be necessary next period or, if $x > \hat{g}$, more disinvestment will be possible. The first term in the expression measures the former effect and the second term the latter. The size of the latter effect depends on whether x is larger or smaller than \hat{g} . If less investment is necessary next period, this will mean a lower tax rate. If more disinvestment is possible, this will mean a larger transfer. The value of a tax rate reduction is greater than an equally costly transfer increase because taxation is distortionary. While the expressions differ, however, it is important to note that the social marginal benefit of the public good is continuous at $x = \hat{g}$ because $r(x^o(\hat{g}), \hat{g}) = 0$.

At a steady state, we have that $x^o = x^o(x^o)$ which must mean that $x^o < \hat{g}$. Substituting in the expression for the social marginal benefit of the public good into the Euler equation, it follows that at a steady state

$$\delta[nA\alpha(x^o)^{\alpha-1} + \frac{1 - r(x^o, x^o)}{1 - r(x^o, x^o)(1 + \varepsilon)}p(1 - d)] = \frac{1 - r(x^o, x^o)}{1 - r(x^o, x^o)(1 + \varepsilon)} \cdot p. \quad (1)$$

It is easy to verify that this equation has a unique solution and this is the planner's steady state level of the public good. The steady state tax rate is given by $r^o = r(x^o, x^o)$.

Although the analysis of the planner's problem is relatively standard, it is useful to have a graphical representation of the solution. This will set the stage for the more complicated political equilibrium. Figure 1.a shows the Euler equation. The decreasing function is the social marginal

¹⁵ The "marginal cost of public funds" represents the social cost of raising an additional \$1 of tax revenue. The difference between the marginal cost of public funds and \$1 is a measure of the distortions taxation is creating.

benefit of the public good $\delta V'(x)$. The increasing functions are the social marginal costs evaluated at different initial public capital levels. Note that these curves are always above p since $r(x, g) \geq 0$. The intersection of these two loci gives us the planner's investment choice $x^o(g)$, which is represented in Figure 1.b. This curve is increasing on $[0, \hat{g}]$ and constant thereafter. It can be shown to have a slope less than 1 and hence intersects the 45° line once. This intersection identifies the steady state level x^o . It is apparent from the Figure that from any initial level the equilibrium level of the public good converges monotonically to the steady state x^o . The tax rate can be shown to be monotonically decreasing if, as seems natural, the economy starts out with a public good level lower than the steady state level.

4 Political equilibrium

We look for stationary equilibria in which any representative selected to propose at proposal round $\tau \in \{1, \dots, T\}$ of the meeting at some time t uses a proposal strategy that depends only on the current level of the public good. We focus on symmetric equilibria in which each representative use the same proposal strategy and treats the other representatives anonymously. Such equilibria are characterized by a collection of functions: $\{r_\tau(g), s_\tau(g), x_\tau(g)\}_{\tau=1}^T$. Here $r_\tau(g)$ is the income tax rate that is proposed at round τ when the initial level of the public good is g and $x_\tau(g)$ is the new level of public good. The proposer also offers a transfer of $s_\tau(g)$ to the districts of $q - 1$ randomly selected representatives where q is the size of a minimum winning coalition.¹⁶ Any remaining tax revenues are used to provide pork for his own district. As standard in the literature on legislative bargaining, we assume that legislators do not use weakly dominated strategies when voting for a policy proposal: so they vote in favor if and only if their utility under the proposal is at least as large as their continuation value in the event it is rejected. We focus, without loss of generality, on equilibria in which at each round τ proposals are immediately accepted by at least q legislators, so that on the equilibrium path, no meeting lasts more than one proposal round.¹⁷

Accordingly, the policies that are actually implemented in equilibrium are described by $\{r_1(g)$,

¹⁶ In our model, while the proposer is free to offer transfers to more than $q - 1$ representatives, there is no loss in generality in considering only minimal winning coalitions since in no stationary equilibrium would the proposer find it optimal to offer transfers to more than $q - 1$ representatives.

¹⁷ Since there is no discounting within bargaining stages, there are equilibria in which, in correspondence to some state, a proposal may be rejected at some stage τ and accepted at a later stage $\tau' \leq T$. Without loss of generality, however, we can ignore these equilibria, since they would be payoff equivalent to equilibria with immediate agreement.

$s_1(g), x_1(g)\}$.

To be more precise, $\{r_\tau(g), s_\tau(g), x_\tau(g)\}_{\tau=1}^T$ is an *equilibrium* if at each proposal round τ and all public good levels $g \geq 0$, the equilibrium proposal maximizes the proposer's payoff subject to the incentive constraint of getting the required number of affirmative votes and the feasibility constraint that transfers be non-negative. Formally, $(r_\tau(g), s_\tau(g), x_\tau(g))$ must solve the problem

$$\max_{(r,s,x)} u(w(1-r), g) + B(r, x; g) - (q-1)s + \delta v_1(x)$$

subject to

$$u(w(1-r), g) + s + \delta v_1(x) \geq v_{\tau+1}(g),$$

$$B(r, x; g) \geq (q-1)s, \text{ and } s \geq 0,$$

where $v_1(x)$ is the legislators' round one value function (which describes the expected future payoff of a legislator at the beginning of a period in which the current level of public good is x) and $v_{\tau+1}(g)$ is the expected future payoff of a legislator in the out-of-equilibrium event that the proposal at round τ is rejected. The first constraint is the incentive constraint and the latter two are the feasibility constraints.

The legislators' round one value function is defined recursively by

$$v_1(g) = u(w(1-r_1(g)), g) + \frac{B(r_1(g), x_1(g); g)}{n} + \delta v_1(x_1(g)). \quad (2)$$

To understand this recall that a legislator is chosen to propose in round one with probability $1/n$. If chosen to propose, he obtains a payoff in that period of

$$u(w(1-r_1(g)), g) + B(r_1(g), x_1(g); g) - (q-1)s_1(g).$$

If he is not chosen to propose, but is included in the minimum winning coalition, he obtains a payoff of $u(w(1-r_1(g)), g) + s_1(g)$ and if he is not included he obtains a payoff of just $u(w(1-r_1(g)), g)$. The probability that he will be included in the minimum winning coalition, conditional on not being chosen to propose, is $(q-1)/(n-1)$. Taking expectations, the pork barrel transfers $s_1(g)$ cancel and the period payoff is as described in (2).

For all proposal rounds $\tau = 1, \dots, T-1$ the expected future payoff of a legislator if the round τ proposal is rejected is

$$v_{\tau+1}(g) = u(w(1-r_{\tau+1}(g)), g) + \frac{B(r_{\tau+1}(g), x_{\tau+1}(g); g)}{n} + \delta v_1(x_{\tau+1}(g)).$$

This reflects the assumption that the round $\tau + 1$ proposal will be accepted. Recall that if the round T proposal is rejected, the assumption is that a legislator is appointed to choose policy and that he must choose a uniform policy which, if it involves transfers, must make the same transfer to all districts. Thus,

$$v_{T+1}(g) = \max_{(r,x)} \left\{ u(w(1-r), g) + \frac{B(r, x; g)}{n} + \delta v_1(x) : B(r, x; g) \geq 0 \right\}.$$

An equilibrium $\{r_\tau(g), s_\tau(g), x_\tau(g)\}_{\tau=1}^T$ is said to be *concave* if the associated round one legislators' value function $v_1(g)$ is strictly concave. We will restrict attention to concave equilibria in what follows and henceforth when we refer to an equilibrium it should be understood to be concave.¹⁸ Note also that economy-wide aggregate utility in an equilibrium is given by $nv_1(g)$. This follows from the fact that each district has a population of size 1 and citizens obtain the same payoffs as their representatives.

Our analysis of political equilibrium is divided into three parts. We begin by assuming that an equilibrium exists and characterize what the equilibrium policy proposals must look like. We then show that there are three possible types of equilibrium, each of which has distinct dynamics and a different steady state. Finally, we will establish the conditions under which each of these three types of equilibrium exists. Throughout the analysis, we will maintain the following assumption:

Assumption 1:

$$\frac{R\left(\frac{1-q/n}{1+\varepsilon-q/n}\right)}{p} < \left(\frac{\delta q A \alpha}{p(1-\frac{q}{n}\delta(1-d))}\right)^{\frac{1}{1-\alpha}}.$$

Basically, this requires that the tax base of the economy be not so large that a minimum winning coalition of legislators could accumulate their desired level of the public good in a single period. The role this assumption plays will become apparent shortly.

4.1 The equilibrium policy proposals

The basic structure of the equilibrium policy proposals is easily understood. To get support for his proposal, the proposer must obtain the votes of $q - 1$ other representatives. Accordingly, given that utility is transferable, he is effectively making decisions to maximize the utility of q legislators. The optimal policy will depend on the state variable g . If the stock of the public good

¹⁸ As we will prove, a concave equilibrium always exists in our model. Although we could not find an example of a non-concave equilibrium, we have not formally proven that such an equilibrium can not exist.

is sufficiently low, then even though the proposer is only taking into account the well-being of q legislators, he will still not want to divert resources to transfers. Transfers require *either* reduced public investment *or* increased taxation and when the stock of the public good is low, the marginal benefit of public investment and the marginal cost of taxation are both too high to make transfers attractive. The proposer will therefore choose the tax rate - public investment pair that maximizes collective utility and the outcome will be *as if* he is maximizing the utility of the legislature as a whole. Once the stock of the public good becomes sufficiently large, then the opportunity cost of transfers is lessened and the collective utility of the q legislators will be maximized by diverting some resources to pork. Accordingly, the proposer will propose pork for the districts associated with his minimum winning coalition.

In any equilibrium, therefore, there will exist a threshold level of the public good that divides the state space into two ranges. In the lower range, in every proposal round, the proposer will propose a *unanimous coalition solution* in which the policy proposal maximizes aggregate legislator utility implying that no revenues are devoted to pork. These proposals will be supported by the entire legislature. In the upper range, in every proposal round the proposer chooses a *minimum winning coalition solution* in which the proposer provides pork for his own district and those of a minimum winning coalition of representatives. The tax rate-investment pair maximizes the aggregate utility of q legislators, given that they appropriate all the surplus revenues. The transfer paid out to coalition members is just sufficient to make them in favor of accepting the proposal. Thus, only those legislators whose districts receive pork vote for these proposals.

This discussion motivates:

Proposition 1: *Let $\{r_\tau(g), s_\tau(g), x_\tau(g)\}_{\tau=1}^T$ be an equilibrium with associated value function $v_1(g)$. Then there exists $g^*(v_1) > 0$ such that: (i) if $g \in [0, g^*(v_1)]$*

$$(r_\tau(g), s_\tau(g), x_\tau(g)) = (r(x^*(g; v_1), g), 0, x^*(g; v_1)) \text{ for all } \tau = 1, \dots, T,$$

where $r(x, g)$ is the tax rate function from the analysis of the planner's problem and

$$x^*(g; v_1) = \arg \max_x \{u(w(1 - r(x, g)), g) + \delta v_1(x)\};$$

and (ii) if $g \in (g^*(v_1), \infty)$

$$(r_\tau(g), x_\tau(g)) = (r^*, x^*(v_1)) \text{ for all } \tau = 1, \dots, T,$$

where

$$(r^*, x^*(v_1)) = \arg \max_{(r, x)} q[u(w(1-r), g) + \delta v_1(x)] + B(r, x; g),$$

and

$$s_\tau(g) = \begin{cases} \frac{B(r^*, x^*(v_1); g)}{n} & \text{for all } \tau = 1, \dots, T-1 \\ v_{T+1}(g) - u(w(1-r^*), g) - \delta v_1(x^*(v_1)) & \text{for } \tau = T \end{cases}$$

The threshold level of the public good is $g^*(v_1)$ and the Proposition tells us that the tax rate - public investment pair proposed in each proposal round will maximize aggregate legislator utility when $g \leq g^*(v_1)$ and the utility of q legislators when $g > g^*(v_1)$. It is straightforward to verify that

$$r^* = \frac{1 - q/n}{1 + \varepsilon - q/n}$$

and that

$$x^*(v_1) = \arg \max_x \{\delta q v_1(x) - px\}.$$

Thus, both the tax rate and level of the public good proposed in the minimum winning coalition range are independent of the existing level of the public good and, furthermore, the tax rate is independent of the value function. Another useful fact is that at $g = g^*(v_1)$, the unanimous and minimum winning coalition solutions coincide so that

$$(r(x^*(g^*(v_1); v_1), g^*(v_1)), 0, x^*(g^*(v_1); v_1)) = (r^*, 0, x^*(v_1)). \quad (3)$$

This implies that $B(x^*(v_1), r^*; g^*(v_1)) = 0$, an observation that permits the computation of the threshold value $g^*(v_1)$ once $x^*(v_1)$ is known.

Proposition 1 provides us with the basic picture of what the equilibrium proposals look like. The next step is to obtain more information about the function $x^*(g; v_1)$ and the public good level $x^*(v_1)$. Since these obviously depend upon the nature of the equilibrium value function, we must investigate this first. Proposition 1 allows us to write

$$v_1(x) = \begin{cases} \max_y \{u(w(1-r(y, x)), x) + \delta v_1(y)\} & \text{if } x \leq g^*(v_1) \\ u(w(1-r^*), x) + \frac{B(r^*, x^*(v_1); x)}{n} + \delta v_1(x^*(v_1)) & \text{if } x > g^*(v_1) \end{cases}.$$

As we will see, this value function is differentiable everywhere except at $g = g^*(v_1)$.¹⁹ Thus, by the *Envelope Theorem* we have that:

$$v_1'(x) = \begin{cases} A\alpha x^{\alpha-1} + \left(\frac{1-r(x^*(x;v_1),x)}{1-r(x^*(x;v_1),x)(1+\varepsilon)}\right)\left(\frac{p(1-d)}{n}\right) & \text{if } x < g^*(v_1) \\ A\alpha x^{\alpha-1} + \frac{p(1-d)}{n} & \text{if } x > g^*(v_1) \end{cases}.$$

The equilibrium value function has a kink at $g^*(v_1)$ in the sense that the left hand derivative $\lim_{x \nearrow g^*(v_1)} v_1'(x)$ exceeds the right hand derivative $\lim_{x \searrow g^*(v_1)} v_1'(x)$. This reflects the fact that the value of an additional unit of public good is lower in the minimum winning coalition range. As in the planner's problem, there are two sources of future benefits from investing more in the public good. First, public good consumption is higher in the next period. Second, less investment will be necessary next period. The benefits from the first effect are the same whether or not the legislature is choosing to make transfers in the next period, but the benefits of the second effect do depend on this choice. In the unanimous coalition case, the lower investment translates into a lower tax rate. With minimum winning coalitions, it means higher transfers. But the value of a tax rate reduction is greater than the value of an equally expensive transfer increase because taxes are distortionary and hence the kink.

We can now use the expression for the slope of the value function to characterize the function $x^*(g; v_1)$. If $v_1(x)$ is differentiable at the solution, then $x^*(g; v_1)$ satisfies the first order condition

$$\delta v_1'(x) = \left(\frac{1-r(x,g)}{1-r(x,g)(1+\varepsilon)}\right)\left(\frac{p}{n}\right).$$

This first order condition reflects the fact that any increase in public good investment must be financed by an increase in taxes. Thus, the increase is worthwhile if and only if the per capita benefit of an additional unit of public good $\delta v_1'(x)$ exceeds the per capita tax cost which is $\left(\frac{1-r}{1-r(1+\varepsilon)}\right)\left(\frac{p}{n}\right)$. The latter is the product of the per capita cost of the public good and the marginal cost of public funds.

There are three cases to be considered. The first is that at $g^*(v_1)$ the discounted right hand derivative of the value function exceeds the per capita tax cost. In this case, $\delta v_1'(x)$ must equal the per capita tax cost in the minimum winning coalition range. Accordingly, $x^*(g; v_1)$ exceeds

¹⁹ As it will be clear from the proof of Proposition 4, this follows by standard arguments (see Stokey, Lucas and Prescott [1989]).

$g^*(v_1)$ and satisfies the first order condition

$$\delta[A\alpha x^{\alpha-1} + \frac{p(1-d)}{n}] = (\frac{1-r(x,g)}{1-r(x,g)(1+\varepsilon)})(\frac{p}{n}). \quad (4)$$

The second case is that at $g^*(v_1)$ the left hand derivative of the value function is less than the tax cost. In this case, $\delta v_1'(x)$ must equal the tax cost in the unanimous coalition range. Accordingly, $x^*(g; v_1)$ is smaller than $g^*(v_1)$ and satisfies the first order condition

$$\delta[A\alpha x^{\alpha-1} + (\frac{1-r(x^*(x; v_1), x)}{1-r(x^*(x; v_1), x)(1+\varepsilon)})(\frac{p(1-d)}{n})] = (\frac{1-r(x,g)}{1-r(x,g)(1+\varepsilon)})(\frac{p}{n}). \quad (5)$$

The third case is that the tax cost lies between the left and right hand derivatives of the value function at $g^*(v_1)$. In this case, we must have that $x^*(g; v_1) = g^*(v_1)$.

Turning to $x^*(v_1)$, if $v_1(x)$ is differentiable at the solution, then $x^*(v_1)$ satisfies the first order condition

$$\delta v_1'(x) = \frac{p}{q}.$$

This first order condition reflects the fact that any increase in public good investment simply reduces the amount of transfers received by the q representatives in the minimum winning coalition. Thus, the increase is worthwhile if and only if the per capita benefit of an additional unit of public good $\delta v_1'(x)$ exceeds the per capita transfer cost to coalition members which is p/q .

There are again three cases. The first is that at $g^*(v_1)$ the discounted right hand derivative of the value function exceeds the transfer cost p/q . In this case, $\delta v_1'(x)$ must equal p/q in the minimum winning coalition range. Accordingly, $x^*(v_1)$ exceeds $g^*(v_1)$ and satisfies the first order condition

$$\delta[A\alpha x^{\alpha-1} + \frac{p(1-d)}{n}] = \frac{p}{q}. \quad (6)$$

The second possibility is that the left hand derivative at $g^*(v_1)$ is less than p/q . In this case, $\delta v_1'(x)$ must equal the transfer cost in the unanimity range. Accordingly, $x^*(v_1)$ must be smaller than $g^*(v_1)$ and must satisfy the first order condition

$$\delta[A\alpha x^{\alpha-1} + (\frac{1-r(x^*(x; v_1), x)}{1-r(x^*(x; v_1), x)(1+\varepsilon)})(\frac{p(1-d)}{n})] = \frac{p}{q}. \quad (7)$$

The third possibility is that p/q lies between the left and right hand derivatives of the value function at $g^*(v_1)$. In which case we must have that $x^*(v_1) = g^*(v_1)$.

4.2 The three types of equilibrium

We can now bring the information just obtained about the function $x^*(g; v_1)$ and the public good level $x^*(v_1)$ together with Proposition 1 to get a picture of how equilibrium plays out. There are three possibilities to be considered. In a *Type I equilibrium*, $x^*(v_1)$ exceeds $g^*(v_1)$, so that if the current level of the public good is in the minimum winning coalition range, the equilibrium investment decision will be such as to keep it there. In a *Type II equilibrium*, $x^*(v_1)$ is less than $g^*(v_1)$, so that in the minimum winning coalition range, the equilibrium investment decision will be such as to put the public good level back in the interior of the unanimity range. In a *Type III equilibrium*, $x^*(v_1)$ is equal to $g^*(v_1)$ so that the equilibrium investment decision will be such as to put the public good level back at the boundary of the two ranges. We consider each in turn.

4.2.1 Type I equilibrium

In this case, $x^*(v_1)$ must satisfy the first order condition (6). Solving this implies that

$$x^*(v_1) = \left(\frac{\delta q A \alpha}{p(1 - \frac{q}{n} \delta(1-d))} \right)^{\frac{1}{1-\alpha}}.$$

Furthermore, since $B(x^*(v_1), r^*; g^*(v_1)) = 0$, this implies that

$$g^*(v_1) = \frac{p x^*(v_1) - R(r^*)}{p(1-d)}.$$

This gives us closed form solutions for $x^*(v_1)$ and $g^*(v_1)$. For this equilibrium to exist, the parameters of the model must be such as to imply that $x^*(v_1)$ is indeed greater than $g^*(v_1)$ and we will give a condition for this below.

It remains to describe the behavior of $x^*(g; v_1)$. From (3) we know that at $g = g^*(v_1)$ it must be the case that $x^*(g; v_1) = x^*(v_1)$. This implies that the marginal cost curve for the unanimity case $(\frac{1-r(x, g^*(v_1))}{1-r(x, g^*(v_1))(1+\varepsilon)}) (\frac{p}{n})$ will equal p/q at x equal to $x^*(v_1)$. Now define the public good levels \hat{g} and g' from the equations

$$\delta[A\alpha g^*(v_1)^{\alpha-1} + \frac{p(1-d)}{n}] = \left(\frac{1-r(g^*(v_1), \hat{g})}{1-r(g^*(v_1), \hat{g})(1+\varepsilon)} \right) \left(\frac{p}{n} \right)$$

and

$$\delta[A\alpha g^*(v_1)^{\alpha-1} + \left(\frac{1-r(x^*(v_1), g^*(v_1))}{1-r(x^*(v_1), g^*(v_1))(1+\varepsilon)} \right) \left(\frac{p(1-d)}{n} \right)] = \left(\frac{1-r(g^*(v_1), g')}{1-r(g^*(v_1), g')(1+\varepsilon)} \right) \left(\frac{p}{n} \right).$$

Observe that $g' < \hat{g} < g^*(v_1)$.

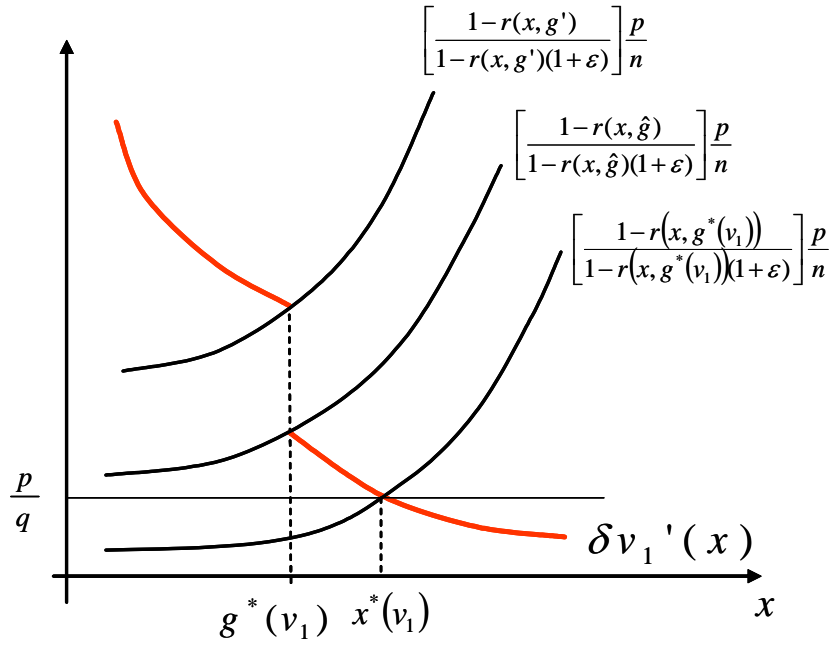


Fig 2.a

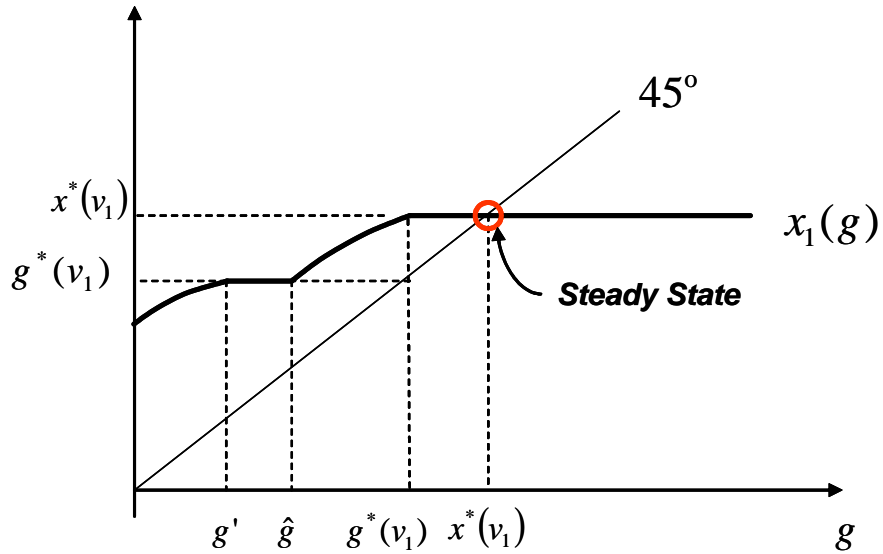


Fig 2.b

Figure 2: Type I equilibrium: $x^*(v_1) > g^*(v_1)$

On the interval $[\widehat{g}, g^*(v_1)]$, $x^*(g; v_1)$ is implicitly defined by the first order condition (4). It is increasing in g and exceeds $g^*(v_1)$. Effectively in this part of the unanimity range, the public good is sufficiently valuable that the aggregate utility of the legislators is maximized by choosing a level of investment that will induce pork to be distributed in the next period. On the interval $[g', \widehat{g}]$, $x^*(g; v_1)$ is constant and equal to $g^*(v_1)$. Here the proposer is deterred from investing more than $g^*(v_1)$ because it will result in a minimum winning coalition solution in the next period. Finally, on the interval $[0, g']$, $x^*(g; v_1)$ is implicitly defined by the first order condition (5) and is less than $g^*(v_1)$. In this range, the proposer chooses the level of investment that maximizes aggregate legislator utility knowing that in the next period the proposer will do the same.

The situation is illustrated in Figure 2. In panel (a) the vertical axis measures units of consumption and the horizontal axis measures units of x . The downward sloping line with the discontinuity at $g^*(v_1)$ is $\delta v_1'(x)$ and hence represents the per capita benefit of an additional unit of the public good. The horizontal line p/q represents the per capita marginal cost in the minimum winning coalition range. The upward sloping convex lines $(\frac{1-r(x,g)}{1-r(x,g)(1+\varepsilon)})(\frac{p}{n})$ represent the per capita marginal cost of the public good in the unanimity range. Lower initial levels of the public good shift these lines to the left.

The minimum winning coalition public good level is determined by the intersection of the marginal benefit curve and the curve p/q . Given that $x^*(v_1)$ exceeds $g^*(v_1)$, the intersection must occur to the right of $g^*(v_1)$ as illustrated. The function $x^*(g; v_1)$ is determined by the intersection of the marginal benefit and marginal tax cost curve $(\frac{1-r(x,g)}{1-r(x,g)(1+\varepsilon)})(\frac{p}{n})$. At $g = g^*(v_1)$, this intersection occurs at $x^*(v_1)$. As we lower g the marginal cost curve shifts leftward and we trace out successively lower levels of x . In the interval $[g', \widehat{g}]$, lowering g has no impact on the level of the public good that is chosen. On the interval $[0, g']$, the marginal cost curve intersects the upper branch of the marginal benefit curve and $x^*(g; v_1)$ once again decreases as we lower the initial level of the public good.

Panel (b) of Figure 2 uses this information about $x^*(g; v_1)$ and $x^*(v_1)$ to graph the equilibrium level of x as a function of the state variable g . This curve intersects the 45° line at $x^*(v_1)$. It is apparent from this Figure that from any initial condition the equilibrium level of the public good converges monotonically to the steady state $x^*(v_1)$. The steady state tax rate is r^* . In the steady state, the proposer raises more revenue than necessary to maintain the public good level at $x^*(v_1)$ and this revenue is used to finance pork. Thus we have:

Proposition 2: Let $\{r_\tau(g), s_\tau(g), x_\tau(g)\}_{\tau=1}^T$ be a Type I equilibrium. Then, the equilibrium level of the public good and tax rate converge monotonically to the steady state $(r^*, (\frac{\delta q A \alpha}{p(1-\frac{q}{n}\delta(1-d))})^{\frac{1}{1-\alpha}})$. In this steady state, in each period the districts of a minimum winning coalition of representatives receive pork.

4.2.2 Type II equilibrium

If $x^*(v_1)$ is less than $g^*(v_1)$, it must be the case that $x^*(v_1)$ satisfies the first order condition (7). This is not a closed form solution for $x^*(v_1)$, since it depends upon $x^*(x^*(v_1), v_1)$. Turning to $x^*(g; v_1)$, we know from (3) that at $g = g^*(v_1)$ it must be the case that $x^*(g; v_1) = x^*(v_1)$. On the interval $[0, g^*(v_1)]$, $x^*(g; v_1)$ is implicitly defined by the first order condition (5). It is increasing over this range and thus less than $g^*(v_1)$. Accordingly, the proposer chooses the optimal level of investment knowing that in the next period the proposer will also be maximizing aggregate legislator utility.

The situation is illustrated in Figure 3. The interpretation of the various curves in panel (a) are as for Figure 2(a). Given that $x^*(v_1)$ is less than $g^*(v_1)$, the intersection of the marginal benefit curve and the curve p/q must occur to the left of $g^*(v_1)$ as illustrated. The function $x^*(g; v_1)$ is determined by the intersection of the marginal benefit curve and the marginal cost curve $(\frac{1-r(x,g)}{1-r(x,g)(1+\varepsilon)})(\frac{p}{n})$. At $g = g^*(v_1)$, this intersection occurs at $x^*(v_1)$. As we lower g the marginal cost curve shifts leftward and we trace out successively lower levels of x .

Panel (b) of Figure 3 uses this information about $x^*(g; v_1)$ and $x^*(v_1)$ to graph the equilibrium level of x as a function of the initial public good level g . This curve intersects the 45° line at \hat{x} . It is apparent from this Figure that from any initial level of the public good the equilibrium converges monotonically to the steady state \hat{x} . The steady state tax rate is $\hat{r} = r(\hat{x}, \hat{x})$. The steady state (\hat{r}, \hat{x}) involves no pork and satisfies:

$$\delta[A\alpha\hat{x}^{\alpha-1} + (\frac{1-\hat{r}}{1-\hat{r}(1+\varepsilon)})(\frac{p(1-d)}{n})] = (\frac{1-\hat{r}}{1-\hat{r}(1+\varepsilon)})(\frac{p}{n}).$$

It follows from (1) that (\hat{r}, \hat{x}) must equal the planner's steady state. This yields:

Proposition 3: Let $\{r_\tau(g), s_\tau(g), x_\tau(g)\}_{\tau=1}^T$ be a Type II equilibrium. Then, the equilibrium level of the public good and tax rate converge monotonically to the planner's steady state (r^o, x^o) .

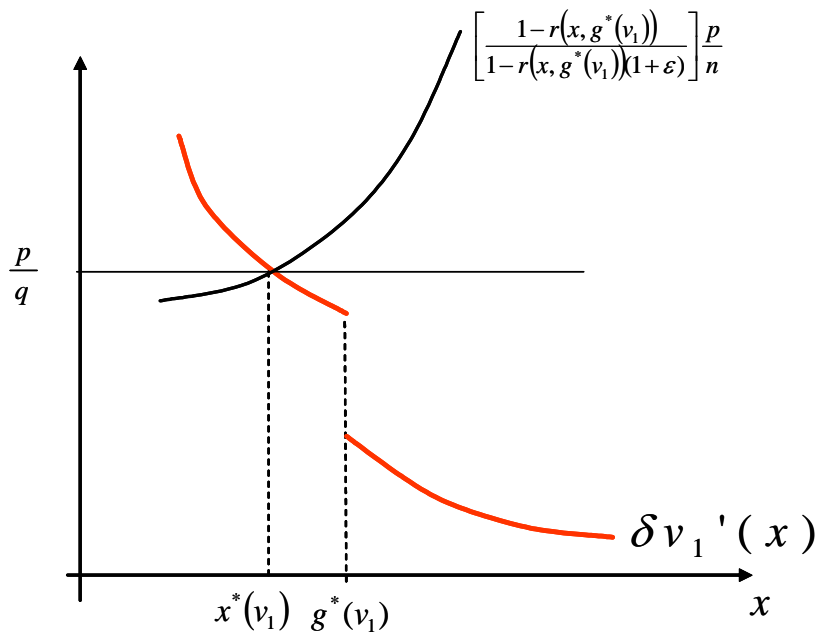


Fig 3.a

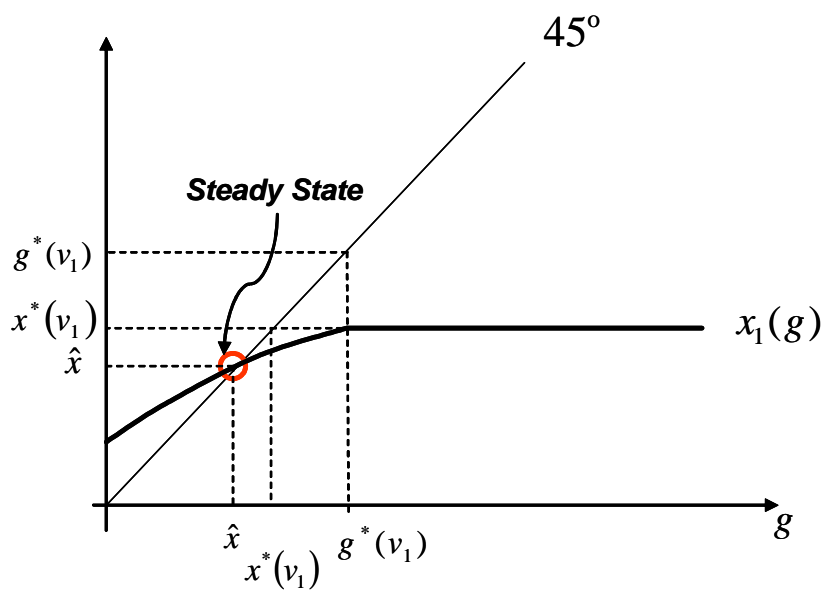


Fig 3.b

Figure 3: Type II equilibrium: $x^*(v_1) < g^*(v_1)$

4.2.3 Type III equilibrium

If $x^*(v_1)$ is equal to $g^*(v_1)$, it must be the case that

$$\lim_{x \nearrow g^*(v_1)} \delta v_1'(x) \geq \frac{p}{q} \geq \lim_{x \searrow g^*(v_1)} \delta v_1'(x).$$

Moreover, since $B(x^*(v_1), r^*; g^*(v_1)) = 0$, we can solve the budget equation to obtain

$$x^*(v_1) = g^*(v_1) = \frac{R(r^*)}{pd}.$$

Turning to $x^*(g; v_1)$, we know that at $g = g^*(v_1)$ it must be the case that $x^*(g; v_1) = g^*(v_1)$.

Now define the public good level g' from the equation

$$\delta[A\alpha g^*(v_1)^{\alpha-1} + (\frac{1 - r(g^*(v_1), g^*(v_1))}{1 - r(g^*(v_1), g^*(v_1))(1 + \varepsilon)})(\frac{p(1-d)}{n})] = (\frac{1 - r(g^*(v_1), g')}{1 - r(g^*(v_1), g')(1 + \varepsilon)})(\frac{p}{n}).$$

It is the case that $g' < g^*(v_1)$. On the interval $[g', g^*(v_1)]$, $x^*(g; v_1)$ is constant and equal to $g^*(v_1)$. Here the proposer is deterred from investing more than $g^*(v_1)$ because it will result in a minimum winning coalition solution being selected in the next period. On the interval $[0, g']$, $x^*(g; v_1)$ is implicitly defined by the first order condition (5) and is less than $g^*(v_1)$. In this range, the proposer chooses the optimal level of investment knowing that in the next period the proposer will also be maximizing aggregate legislator utility.

The situation is illustrated in Figure 4. Given that $x^*(v_1)$ equals $g^*(v_1)$, the curve p/q must intersect the marginal benefit curve at the point of discontinuity as illustrated in panel (a). The marginal cost curve for the unanimity case $(\frac{1 - r(x, g^*(v_1))}{1 - r(x, g^*(v_1))(1 + \varepsilon)})(\frac{p}{n})$ will equal p/q at x equal to $g^*(v_1)$. As we lower g , the marginal cost curve shifts leftward but on the interval $[g', g^*(v_1)]$ this has no impact on the level of the public good that is chosen. On the interval $[0, g']$, the marginal cost curve intersects the upper branch of the marginal benefit curve and $x^*(g; v_1)$ decreases as the initial level of the public good is lowered.

Panel (b) of Figure 4 graphs the equilibrium level of x as a function of the initial public good level g . This curve intersects the 45° line at $g^*(v_1)$. It is apparent from this Figure that from any initial condition the equilibrium converges monotonically to the steady state $g^*(v_1)$. The steady state tax rate is r^* which also equals $r(g^*(v_1), g^*(v_1))$. This steady state involves no pork, but is not the planner's steady state. Thus we have:

Proposition 4: *Let $\{r_\tau(g), s_\tau(g), x_\tau(g)\}_{\tau=1}^T$ be a Type III equilibrium. Then, the equilibrium levels of the public good and tax rate converge monotonically to the steady state $(r^*, \frac{R(r^*)}{pd})$. In this steady state, no districts receive pork.*

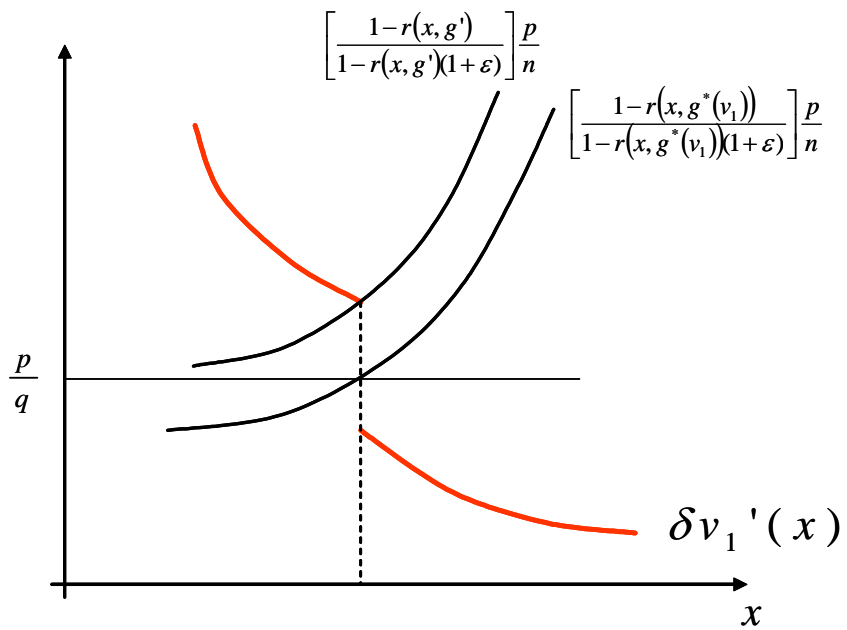


Fig 4.a

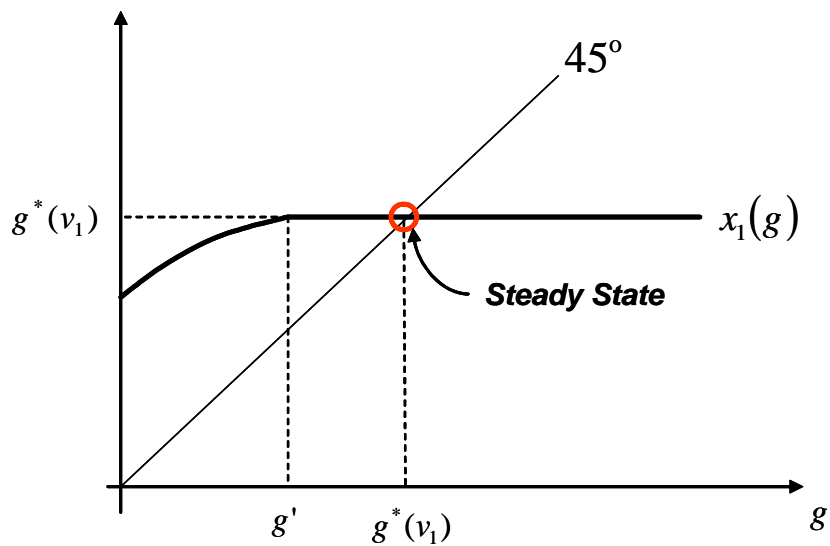


Fig 4.b

Figure 4: Type III equilibrium: $x^*(v_1) = g^*(v_1)$

4.3 Existence, uniqueness and multiplicity of equilibria

The foregoing analysis has provided a complete characterization of equilibrium. It has shown that there are three possible types of equilibrium and has described the dynamics of the equilibrium policy proposals in each case. Moreover, it solved for the steady state levels of public goods and taxes in each case. It remains to establish the conditions under which each type of equilibrium exists. There are two main questions of interest. First, does an equilibrium always exist and, second, is it possible that two or more types of equilibria can co-exist?

Recall that A parameterizes the value of the public good to the citizens. Define \bar{A} to be the value of A that makes the marginal benefit of the public good in the minimum winning coalition range equal to p/q at $R(r^*)/pd$; that is,

$$\delta[\bar{A}\alpha(\frac{R(r^*)}{pd})^{\alpha-1} + \frac{p(1-d)}{n}] = \frac{p}{q}. \quad (8)$$

Similarly, let \underline{A} be the value of A that makes the marginal benefit of the public good in the unanimity range equal to p/q at $R(r^*)/pd$; that is,

$$\delta[\underline{A}\alpha(\frac{R(r^*)}{pd})^{\alpha-1} + (\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})] = \frac{p}{q}. \quad (9)$$

Notice that \underline{A} must be less than \bar{A} since, holding constant preferences, the marginal benefit of the public good is higher in the unanimity range. Then we have the following result:

Proposition 5: *(i) If $A \in (0, \underline{A})$ there is a unique equilibrium and this equilibrium is a Type I equilibrium. (ii) If $A > \bar{A}$ there is a unique equilibrium and this equilibrium is a Type II equilibrium. (iii) If $A \in [\underline{A}, \bar{A}]$ there are three equilibria, one of each type.*

Thus, there always exists an equilibrium and, for a range of the parameter space, all of the three types of equilibria discussed above co-exist.²⁰ Multiple equilibria arise because there are complementarities between the public good decisions of different proposers in different periods. Consider a proposer deciding whether to use p units of tax revenue to purchase an additional unit of the public good or to finance pork for the districts of a minimum winning coalition of representatives. The gain from the latter strategy is p/q . As already noted, the gain from the

²⁰ It should also be noted that for a range of the parameter space $((0, \underline{A}) \cup (\bar{A}, \infty))$ the equilibrium is unique (at least in the class of symmetric Markov equilibria with concave value functions). Previous research in the legislative bargaining literature has provided uniqueness results for Baron and Ferejohn's (1989) static model (Eraslan (2002)). Uniqueness is perhaps more surprising in a dynamic model where the bargaining process may lead to inefficient outcomes because of the complementarities discussed above.

former strategy can be decomposed into two parts. One part is deterministic and directly affects the proposer’s future utility: investing in an additional unit of public good generates a benefit of $A\alpha g^{\alpha-1}$ in the following period, independently of the behavior of other representatives. But the other part depends on the strategy used by representatives in the future. If the representatives in the future are “virtuous,” they respond to the reduced need to invest in the public good by imposing a lower tax rate. If they are not virtuous, they respond by increasing pork barrel spending. Virtuous behavior increases the incentives to invest in the public good and, in this way, is self-reinforcing.

5 Implications of the model

The results of the previous section provide a complete picture of what political equilibrium looks like. In this section, we draw out some of the implications of the model. We first discuss what it tells us about the efficiency of legislative decision-making and then turn to some of its positive implications.

5.1 The efficiency of political equilibrium

To understand the model’s implications concerning efficiency, we will focus on a comparison of the equilibrium and planner’s steady states. We will refer to the steady state levels of the public good and tax rate in the planner’s solution as “the efficient levels”. This is motivated by the fact that the planner’s solution is the unique Pareto efficient policy sequence in the set of policy sequences that provide all citizens with the same expected payoff. Since all citizens have the same expected payoff in political equilibrium, divergencies between equilibrium and planner’s steady states represent Pareto inefficiencies and thereby constitute “political failures” in the sense defined by Besley and Coate (1998).

We begin by understanding how the equilibrium steady state differs from the planner’s in the three types of equilibrium. For the Type I equilibrium we can show:

Lemma 1: *Let $\{r_\tau(g), s_\tau(g), x_\tau(g)\}_{\tau=1}^T$ be a Type I equilibrium. Then, the steady state public good level is below the efficient level. Moreover, the steady state tax rate is above the efficient level if $A \in (0, \underline{A})$ and below it if $A \in (\underline{A}, \bar{A}]$.*

Thus, public goods are always under-provided in this type of equilibrium, but the overall size of government as measured by the tax rate may be below or above the efficient level. When

$A \in (0, \underline{A})$ this type of equilibrium reflects well the conventional wisdom concerning legislative decision-making. Namely, that it results in government being too large in the sense that taxes are higher than the efficient level and, in addition, that these tax revenues are misallocated, with public goods being under-provided and pork being over-provided. When $A \in (\underline{A}, \bar{A}]$ (and we are in a Type I equilibrium) the picture differs from the conventional one because government is below its efficient scale, although revenues are still misallocated to pork.

We know from Proposition 2, that for the Type II equilibrium, the equilibrium and efficient steady states coincide. This just leaves the Type III equilibrium, for which we can establish:

Lemma 2: *Let $\{r_\tau(g), s_\tau(g), x_\tau(g)\}_{\tau=1}^T$ be a Type III equilibrium. Then, the steady state public good level is below the efficient level.*

Since there is no pork in the Type III equilibrium, this result implies that the overall size of government is below optimal. The nature of the inefficiency therefore differs completely from that suggested by the conventional wisdom. Government is not only too small, but available tax revenues are allocated efficiently. The legislature is not willing to choose a public good level higher than $g^*(v_1)$ because they know that the public good level will go back down to $g^*(v_1)$ the next period, so there will be only one period of additional public good consumption. Moreover, the extra revenues that will be saved by not having to purchase as much public good in the next period will just be spent on transfers rather than reducing taxes. Spending on transfers is less efficient in an ex-ante sense than reducing taxes because of the deadweight cost of taxation.

Combining the results of Lemmas 1 and 2 and Propositions 2-5, the efficiency implications of the model can be summarized as follows:

Proposition 6: *If $A < \underline{A}$ then the steady state equilibrium tax rate is too high and the steady state equilibrium level of public goods is too low. If $A > \bar{A}$, then the steady state equilibrium is efficient. If $A \in [\underline{A}, \bar{A}]$ then the steady state equilibrium could be efficient or inefficient. If it is inefficient, there are two possibilities. In the first, the tax rate will be too low but revenues will be allocated efficiently. In the second, the tax rate will also be too low and some revenues will be used to finance pork.*

What are the basic reasons for the inefficiencies in policy-making that can arise in the model? Four factors are crucial in the sense that, without them, the equilibrium policy choices would be Pareto efficient. The first is *majoritarian decision making*. With unanimity (i.e., $q = n$), it

is the case that $\underline{A} = \overline{A} = 0$ which implies that the political equilibrium is always efficient.²¹ Majoritarian decision making allows legislative coalitions to benefit their districts at the expense of non-coalition members. The second is the availability of *distributive policies*; i.e., policies whose benefits can be targeted narrowly and whose costs are financed centrally. If there were no such policies, there would be no way for majoritarian coalitions to transfer wealth to themselves. The third factor is *political uncertainty* arising from the random allocation of proposal power. If the proposer was constant over time, the political equilibrium would certainly differ from the planner's solution, but it would be Pareto efficient. In particular, it would not be possible to make the citizens in the proposer's district better off. The final factor is *lack of commitment*. The inefficiencies in both the Type I and III equilibria could be resolved by Coasian bargaining between present and future legislators. However, such Coasian bargains are not possible because the identity of future legislators is not clear and even if it were, future promises would not be credible.

In interpreting Proposition 6, it is important not to over-simplify the conclusions concerning the role of the public good taste parameter A . It may seem unsurprising that when A is high enough, the political equilibrium is efficient. After all, the higher is the benefit of the public good, the higher is the incentive to invest in it, independently of the decision making process. In our model, however, the consequences of A for efficiency are indirect and more subtle than this. In a static model, the marginal benefit of the public good necessarily depends only on the exogenous parameters such as A : a high enough level of A would imply that the marginal benefit of g is higher than the marginal benefit of pork transfers, and therefore that all tax revenues would be used for g . But in our dynamic model, the marginal benefit of investing in the public good is endogenous because it depends on the level accumulated in previous periods. As A becomes higher, the long run level of the public good increases as more accumulation takes place: this compensates for the higher level of A by reducing the marginal benefit of investment. For this reason, a high absolute value of A is not sufficient to guarantee the equilibrium level of the public good is efficient. For example, as the depreciation rate d converges to 0, both \underline{A} and \overline{A} converge to infinity (see the proof of Proposition 7). Thus, the equilibrium is inefficient when d is very small, irrespective of the level of A . This reflects the fact that when d is very small, if the equilibrium were efficient, there would be almost no investment in g and therefore close to zero taxes in the steady state. But

²¹ Indeed, generally there will exist some critical $\tilde{q} < n$ such that if $q \geq \tilde{q}$, legislative policy-making is efficient.

this cannot happen because when taxes are close to zero the marginal deadweight cost of taxation is close to zero, so it would always be optimal to raise taxes to fund pork barrel transfers. In the steady state, therefore, we have an efficient equilibrium only when the long run level of investment is high enough to imply that the (endogenous) deadweight cost of taxation is higher than the marginal benefit of pork transfers. This suggests that we should expect efficient equilibria not only when the depreciation of public goods is high, but also when there is sustained technological progress that favors high long run levels of public good investments.

Under what circumstances is the political equilibrium more likely to be inefficient? Proposition 6 suggests that a feel for this can be obtained by studying how the critical public good preference thresholds \underline{A} and \bar{A} depend upon the underlying parameters. Ceteris paribus, factors that serve to raise \underline{A} and \bar{A} make inefficiency more likely. The relevant parameters of interest are the discount rate δ , the price of the public good p , the rate of depreciation d and the size of the majority required to pass legislation q . Also of interest is the elasticity of labor supply ε . When analyzing this, however, one must recognize that the laissez-faire national income at wage w is $nw(\varepsilon w)^\varepsilon$ and hence raising the elasticity of labor supply ε increases the size of the tax base. To circumvent this, we consider *compensated* changes in the elasticity; i.e., changes in ε compensated by changes in w that leave the size of the tax base constant. We now have:

Proposition 7: *An increase in the price of the public good p or the economy's wage rate w induces an increase in \underline{A} and \bar{A} , while an increase in either the discount rate δ or the required majority q induces a decrease in \underline{A} and \bar{A} . A compensated increase in the elasticity of labor supply ε also induces a decrease in \underline{A} and \bar{A} .*

Thus, political equilibrium is more likely to be inefficient when citizens are more impatient, public goods are more expensive and the private sector is more productive. On the other hand, equilibrium is more likely to be efficient when super-majorities are required to pass tax and spending bills and when the elasticity of labor supply is high. The latter result is particularly interesting and reflects the logic that when tax revenues are more costly to raise, they will be less likely to be squandered on pork.

The proposition does not speak to the impact of an increase in the depreciation rate because it is ambiguous. While \underline{A} and \bar{A} are decreasing in d for sufficiently small d , the derivatives may switch sign at larger d . However, there are two interesting facts to note about the depreciation

rate. First, as noted above, as d converges to 0, both \underline{A} and \overline{A} converge to infinity. Thus, the equilibrium *must* be inefficient when d is very small and the inefficiency takes the conventional form in which the steady state tax rate is too high and revenues are misallocated to pork. Second, when $d = 1$, $\underline{A} = \overline{A}$ and the third type of equilibrium cannot arise. This reflects the fact that there is no difference between the marginal benefit of investment in the unanimity and minimum winning coalition ranges when depreciation is 100%. In particular, investing more in the public good leads to no tax or transfer changes in the next period, because none of the extra investment carries over to the next period. Moreover, Assumption 1 implies that when $d = 1$, $A > \overline{A}$, meaning that the equilibrium must be efficient.²²

5.2 Some positive implications

The first interesting positive implication of the model concerns the *dynamics of legislative coalitions*. Consider the case in which $A < \underline{A}$ so that the steady state involves under-provision of the public good and over-taxation. In the steady state, budgets will be approved only by a minimum winning coalition of representatives. However, assuming that the economy started out with a sufficiently low level of the public good, in the early phases of governance as the public good was accumulated, the legislature would not have engaged in distributive policy-making and budgets would have been approved unanimously. Thus, the model suggests that we might observe a decline over time in both the quality of government and the degree of consensus in the legislature. Relatedly, starting from a steady state in which budgets are passed by minimum winning coalitions, if the value of the public good increases very dramatically (for example, as result of a new military threat), we would expect to see a shift to unanimous coalitions as the legislature ramps up investment in the public good.

A second interesting implication concerns the *elasticity of labor supply*. Consider two societies that differ only in the elasticity of their citizens' labor supply (ε_0 and ε_1) and their wage levels (w_0 and w_1). Suppose the wage levels across the two communities are such as to make the size of the laissez-faire national income the same (i.e., $w_0(\varepsilon_0 w_0)^{\varepsilon_0}$ equals $w_1(\varepsilon_1 w_1)^{\varepsilon_1}$). Finally, suppose that in both societies, the steady state involves under-provision of the public good and excessive

²² In assuming that public investment increases the revenue available for redistribution in the next period, Leblanc, Snyder and Tripathi (2000) effectively assume that $d = 1$. Moreover, their assumption that the pool of available revenue the legislature has available is always sufficient to finance the surplus maximizing level of investment (p.29) is the exact opposite of Assumption 1. This explains why their model has a unique inefficient equilibrium.

taxation (i.e., $A < \min\{\underline{A}_0, \underline{A}_1\}$). Then, the steady state level of public goods will be the same in both societies, but the tax rate will be lower in the society with a higher elasticity of labor supply. Thus, the citizens in the society with the low elasticity would be better off with the public good - tax rate pair that arises in the society with the high elasticity.²³ In this sense, societies where citizens have a high elasticity of labor supply will have better quality governments.

Finally, consider the implications of an increase in the *productivity of the private sector* as measured by the wage rate w . Assuming that the steady state involves pork, an increase in w has no impact on the size of government as measured by the tax rate nor on the steady state level of the public good. However, because tax revenues are higher, the amount of pork is increased and the quality of government is lower. Compare this with the implications of an increase in the *productivity of public spending* as measured by A . Again, assuming that the steady state involves pork, a small increase in A has no impact on the size of government but does increase the quality of government by raising the level of the public good. Thus, increases in private productivity worsen the quality of government, while increases in public productivity improve it.

6 Conclusion

This paper has presented an infinite horizon model of public spending and taxation in which policy decisions are made by a legislature consisting of representatives from geographically defined districts. In each period, policies are determined by legislative bargaining. The model incorporates both productive and distributive public spending and distortionary taxation. The dynamic linkage across policy-making periods is created by the fact that the productive spending has long run benefits. Despite the fact that the policy space is quite rich, the model is tractable. Key to this tractability is that, distributive policies notwithstanding, all citizens are ex ante identical at the beginning of each period so equilibrium can be summarized by a single value function. This permits a particularly clean analysis of the welfare properties of equilibrium.

The welfare analysis sheds new light on the efficiency of politically determined policy choices. First, it provides a rigorous formal underpinning for the conventional wisdom that legislatures in which representatives are elected by geographically defined districts will produce a long run size of government that is too large and long run levels of national public goods that are too low.

²³ The public good - tax rate pair in the high elasticity society will necessarily be feasible for the low elasticity society, because the tax rate will yield more tax revenues in the low elasticity society.

Proposition 7 provides conditions under which this is the unique outcome. Roughly speaking, these conditions require that the taxable capacity of the economy is large enough to easily meet the needs of maintaining an adequate level of public goods.

However, the analysis also shows that the conventional wisdom needs qualification. When the economy's taxable capacity is small relative to its public good needs, legislative decisions will actually be efficient in the long run. Legislators will not choose to redistribute to their districts when maintaining public good levels requires a level of taxation that creates significant distortions in the economy. Moreover, the direction of the distortions emerging from legislative choice could be the opposite from that suggested by the conventional wisdom. Specifically, legislators can hold back on public good spending in the belief that accumulating too large a stock of these goods will lead future legislators to start engaging in pork barrel spending. This behavior yields an equilibrium with no distributive policy-making and an overall size of government that is below optimal.

The model also yields some interesting positive implications. First, it suggests that when the need for public goods is acute, then the legislature will approve its budgets by unanimity, but as the need for public investment is saturated, the political equilibrium will shift to a regime of minimal winning coalitions. Thus, the size of the legislative coalitions passing budgets may decline over time as a country builds up its stock of public goods. Second, the model suggests that societies in which citizens have a more elastic labor supply will enjoy better quality governments. A higher elasticity of labor supply, reduces distributive policies but not the long run levels of public goods. Finally, the model suggests that the quality of government as measured by the proportion of revenues devoted to distributive policies is inversely correlated with the productivity of the private sector.

There are many ways in which the model might usefully be developed.²⁴ Perhaps the most important task is to incorporate government borrowing. To maintain tractability, one could simplify the model by assuming that the public good was not durable but allow the legislature to finance spending by a combination of taxation and issuing public debt. Debt would then form the dynamic linkage across periods. In this context, it would be natural to assume that the per period value of public goods was stochastic (reflecting, for example, wars or terrorist threats) and study

²⁴ It might also be interesting to incorporate into the framework the alternative "demand bargaining" approach to legislative policy-making suggested by Morelli (1999).

how debt, spending and taxation responded to shocks. This would facilitate a political economy investigation of the normative theory of debt and taxation suggested by Barro (1979).

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7 Appendix

Proof of Proposition 1: To prove the result, we will need some additional notation. For any strictly concave function $v(x)$ consider the problem for all $\mu \in [0, \infty)$

$$\begin{aligned} \max_{\{r,x\}} \mu[u(w(1-r), g) + \delta v(x)] + B(r, x; g) \\ \text{s.t. } B(r, x; g) \geq 0 \end{aligned} \tag{10}$$

Interpreting $v(x)$ as the expected payoff with x units of the public good, the problem is to maximize the aggregate utility of μ legislators under the assumption that any revenue that is not used for investment in the public good is used to finance pork in these legislators' districts. Under the assumption that v is strictly concave, there is a unique solution to this problem given by $r(g; \mu, v)$ and $x(g; \mu, v)$.

Note the following facts about this problem. First, for μ sufficiently small, the solution will involve a positive budget surplus (i.e., $B(r, x; g) > 0$). Second, for μ sufficiently large, the optimal tax rate will be such that all tax revenues are used to finance investment in the public good and hence $B(r, x; g) = 0$. Third, if it is the case that for some $\tilde{\mu}$ it is optimal to select a tax rate-public good pair such that all revenues are used for investment (i.e., $B(r, x; g) = 0$), then this must also be optimal for all $\mu > \tilde{\mu}$.

Now define $\mu(g; v)$ to be the size of the smallest group of legislators who would choose to devote all revenues to investment. Formally,

$$\mu(g; v) = \min\{\mu \in [0, \infty) : B(r(g; \mu, v), x(g; \mu, v); g) = 0\}.$$

Then all groups of legislators of size less than $\mu(g; v)$ would devote some revenues to pork and all larger groups would devote all revenues to investment. It should be noted that $\mu(g; v)$ exists and is unique for all g . We now have:

Lemma A.1: *Let $\{r_\tau(g), s_\tau(g), x_\tau(g)\}_{\tau=1}^T$ be an equilibrium with associated payoff function $v_1(g)$. (i) If $\mu(g; v_1) \leq q$, then*

$$(r_\tau(g), s_\tau(g), x_\tau(g)) = (r(g; n, v_1), 0, x(g; n, v_1)) \text{ for all } \tau = 1, \dots, T.$$

(ii) If $\mu(g; v_1) > q$, then

$$(r_\tau(g), x_\tau(g)) = (r(g; q, v_1), x(g; q, v_1)) \text{ for all } \tau = 1, \dots, T,$$

and

$$s_\tau(g) = \begin{cases} \frac{B(r(g;q,v_1),x(g;q,v_1);g)}{n} & \text{for all } \tau = 1, \dots, T-1 \\ v_{T+1}(g) - u(w(1-r(g;q,v_1)),g) - \delta v_1(x(g;q,v_1)) & \text{for } \tau = T \end{cases}$$

Proof of Lemma A.1: We begin by considering the problem of the proposer in the final round T . From the discussion in the text, we know that $(r_T(g), s_T(g), x_T(g))$ must solve the round T proposer's problem

$$\max_{(r,s,x)} [u(w(1-r),g) + B(r,x;g) - (q-1)s + \delta v_1(x)]$$

subject to

$$u(w(1-r),g) + s + \delta v_1(x) \geq v_{T+1}(g),$$

$$B(r,x;g) \geq (q-1)s, \quad \text{and } s \geq 0,$$

where

$$v_{T+1}(g) = \max_{(r,x)} \{u(w(1-r),g) + \frac{B(r,x;g)}{n} + \delta v_1(x) : B(r,x;g) \geq 0\}.$$

We now establish:

Claim A.1: Let (r_T, s_T, x_T) solve the round T proposer's problem. Then (r_T, x_T) solves problem (10) with $\mu = q$ and

$$s_T = v_{T+1}(g) - \delta v_1(x_T) - u(w(1-r_T),g).$$

Proof of Claim A.1: It is easy to see that

$$s_T = v_{T+1}(g) - \delta v_1(x_T) - u(w(1-r_T),g).$$

For if this were not the case it would follow from the definition of $v_{T+1}(g)$ that either the incentive constraint is violated or $s_T > 0$ and we could create a preferred proposal by just reducing s_T . It follows that we can write the proposer's payoff as

$$q [u(w(1-r_T),g) + \delta v_1(x_T)] + B(r_T, x_T; g).$$

Now suppose that (r_T, x_T) does not solve problem (10) with $\mu = q$. Let (r', x') solve problem (10) with $\mu = q$ and

$$s' = v_{T+1}(g) - \delta v_1(x') - u(w(1-r'),g).$$

Then, the proposer's payoff under the proposal (r', s', x') is

$$q [u(w(1-r'),g) + \delta v_1(x')] + B(r', x'; g).$$

By construction, the incentive constraint is satisfied and, by definition of $v_{T+1}(g)$, $s' \geq 0$. Note also that

$$\begin{aligned}
B(r', x'; g') - (q-1)s' &= (q-1)[u(w(1-r'), g') + \delta v_1(x')] + B(r', x'; g') - (q-1)v_{T+1}(g) \\
&= q[u(w(1-r'), g') + \delta v_1(x')] + B(r', x'; g') \\
&\quad - [u(w(1-r'), g') + \delta v_1(x') + (q-1)v_{T+1}(g)] \\
&\geq q[u(w(1-r'), g') + \delta v_1(x')] + B(r', x'; g') - qv_{T+1}(g),
\end{aligned}$$

where the last inequality follows by definition of $v_{T+1}(g)$. The difference on the right hand side must be non-negative. To see this, note that

$$v_{T+1}(g) = u(w(1-r^*), g) + \delta v_1(x^*) + B(r^*, x^*; g)/n$$

for some (r^*, x^*) such that $B(r^*, x^*; g) \geq 0$ and hence

$$\begin{aligned}
q[u(w(1-r'), g') + \delta v_1(x')] + B(r', x'; g') &\geq q[u(w(1-r^*), g) + \delta v_1(x^*)] + B(r^*, x^*; g) \\
&\geq qv_{T+1}(g).
\end{aligned}$$

Thus, (r', s', x') is feasible for the proposer's problem and yields a higher payoff than (r_T, s_T, x_T) - a contradiction. This completes the proof of Claim A.1. ■

It follows from Claim A.1 that if $\mu(g; v_1) \leq q$, then $(r_T(g), s_T(g), x_T(g))$ equals $(r(g; n, v_1), 0, x(g; n, v_1))$, while if $\mu(g; v_1) > q$, then $(r_T(g), s_T(g), x_T(g))$ equals

$$(r(g; q, v_1), v_{T+1}(g) - u(w(1-r(g; q, v_1)), g) - \delta v_1(x(g; q, v_1)), x(g; q, v_1)).$$

Now consider the round $T-1$ proposer's problem

$$\max_{(r, s, x)} [u(w(1-r), g) + B(r, x; g) - (q-1)s + \delta v_1(x)]$$

subject to

$$u(w(1-r), g) + s + \delta v_1(x) \geq v_T(g),$$

$$B(r, x; g) \geq (q-1)s, \quad \text{and} \quad s \geq 0.$$

If $\mu(g; v_1) \leq q$ then we know that

$$v_T(g) = u(w(1-r(g; n, v_1)), g) + \delta v_1(x(g; n, v_1)) = v_{T+1}(g)$$

so applying the exact same logic as above implies that the solution to the round $T - 1$ proposer's problem is $(r(g; n, v_1), 0, x(g; n, v_1))$. Repeated application of the same logic implies that the solution to the proposer's problem is $(r(g; n, v_1), 0, x(g; n, v_1))$ in all earlier proposal rounds.

On the other hand, if $\mu(g; v_1) > q$ then we know that

$$v_T(g) = u(w(1 - r(g; q, v_1)), g) + \delta v_1(x(g; q, v_1)) + \frac{B(r(g; q, v_1), x(g; q, v_1); g)}{n}.$$

So we need to show that the solution to the round $T - 1$ proposer's problem with this level of reservation utility is

$$(r(g; q, v_1), \frac{B(r(g; q, v_1), x(g; q, v_1); g)}{n}, x(g; q, v_1))$$

Let $(r_{T-1}, s_{T-1}, x_{T-1})$ denote the solution. It is straightforward to show the desired result if $s_{T-1} > 0$, so we need only rule out the possibility that $s_{T-1} = 0$. If $s_{T-1} = 0$, it must be the case that $B(r_{T-1}, x_{T-1}; g) > 0$ and that (r_{T-1}, x_{T-1}) solves the problem

$$\begin{aligned} & \max u(w(1 - r), g) + \delta v_1(x) + B(r, x; g) \\ & \text{s.t. } u(w(1 - r), g) + \delta v_1(x) \geq v_T(g) \end{aligned}$$

Now consider the proposal

$$(r', s', x') = (r(g; q, v_1), \frac{B(r(g; q, v_1), x(g; q, v_1); g)}{n}, x(g; q, v_1))$$

The payoff to the proposer under the policy (r', s', x') is

$$q[u(w(1 - r'), g) + \delta v_1(x')] + B(r', x'; g) - (q - 1)v_T(g) \tag{11}$$

But we know that $u(w(1 - r_{T-1}), g) + \delta v_1(x_{T-1}) \geq v_T(g)$, and hence a lower bound of (11) is:

$$q[u(w(1 - r'), g) + \delta v_1(x')] + B(r', x'; g) - (q - 1)[u(w(1 - r_{T-1}), g) + \delta v_1(x_{T-1})].$$

The payoff to the proposer under the optimal policy $(r_{T-1}, 0, x_{T-1})$ is given by:

$$u(w(1 - r_{T-1}), g) + \delta v_1(x_{T-1}) + B(r_{T-1}, x_{T-1}; g).$$

It must be the case that

$$\begin{aligned} & u(w(1 - r_{T-1}), g) + \delta v_1(x_{T-1}) + B(r_{T-1}, x_{T-1}; g) \\ & > q[u(w(1 - r'), g) + \delta v_1(x')] + B(r', x'; g) - (q - 1)[u(w(1 - r_{T-1}), g) + \delta v_1(x_{T-1})] \end{aligned}$$

which implies that

$$\begin{aligned} & q[u(w(1 - r_{T-1}), g) + \delta v_1(x_{T-1})] + B(r_{T-1}, x_{T-1}; g) \\ & > q[u(w(1 - r'), g) + \delta v_1(x')] + B(r', x'; g). \end{aligned}$$

This contradicts the fact that $(r', x') = (r(g; q, v_1), x(g; q, v_1))$.

Repeated application of the same logic implies that the solution to the proposer's problem is

$$(r(g; q, v_1), \frac{B(r(g; q, v_1), x(g; q, v_1); g)}{n}, x(g; q, v_1)).$$

in all earlier proposal rounds. This completes the proof of Lemma A.1. ■

Lemma A.1 tells us what equilibrium proposals must look like. The next step is to develop expressions for $(r(g; n, v_1), x(g; n, v_1))$ and $(r(g; q, v_1), x(g; q, v_1))$. If $\mu(g; v_1) \leq q$, then it must be the case that $B(r(g; n, v_1), x(g; n, v_1); g) = 0$. It follows that we can write

$$(r(g; n, v_1), x(g; n, v_1)) = (r(x^*(g; v_1), g), x^*(g; v_1)),$$

where $r(x, g)$ is the tax rate function from the analysis of the planner's problem, and

$$x^*(g; v_1) = \arg \max_x \{u(w(1 - r(x, g)), g) + \delta v_1(x)\}.$$

If $\mu(g; v_1) > q$, then $B(r(g; q, v_1), x(g; q, v_1); g) > 0$ and hence $(r(g; q, v_1), x(g; q, v_1))$ must solve the unconstrained problem

$$\max_{(r, x)} q[u(w(1 - r), g) + \delta v_1(x)] + B(r, x; g). \quad (12)$$

Notice that the solutions to this problem are independent of g and, moreover, the tax rate is independent of the value function. Thus, we may write the solutions as $(r^*, x^*(v_1))$.

To complete the proof of the Proposition, it only remains to show that there exists a unique $g^*(v_1) > 0$ such that $\mu(g; v_1) \leq q$ for all $g \leq g^*(v_1)$ and $\mu(g; v_1) > q$ for all $g > g^*(v_1)$. We begin with the following useful observation.

Claim A.2: For any strictly concave function $v(x)$, $g \geq 0$ and $\mu \in [0, \infty)$, let $(\hat{r}(g; \mu, v), \hat{x}(g; \mu, v))$ be the solution of the problem

$$\max_{\{r, x\}} \mu[u(w(1 - r), g) + \delta v(x)] + B(r, x; g) \quad (13)$$

and define $\hat{\mu}(g; v)$ as

$$\widehat{\mu}(g; v) = \min\{\mu \in (0, \infty) : B(\widehat{r}(g; \mu, v), \widehat{x}(g; \mu, v); g) = 0\}.$$

Then, it is the case that $\widehat{\mu}(g; v) = \mu(g; v)$.

Proof of Claim A.2: Assume first that $\widehat{\mu}(g; v) < \mu(g; v)$, then for any $\mu \in (\widehat{\mu}(g; v), \mu(g; v))$ the unconstrained solution $(\widehat{r}(g; \mu, v), \widehat{x}(g; \mu, v))$ would violate the budget constraint:

$$B(\widehat{r}(g; \mu, v), \widehat{x}(g; \mu, v); g) < 0$$

(note that $B(\widehat{r}(g; \mu, v), \widehat{x}(g; \mu, v); g)$ is strictly decreasing in μ since $\widehat{r}(g; \mu, v)$ is strictly decreasing and $\widehat{x}(g; \mu, v)$ is non decreasing in μ). This implies that the constrained solution $(r(g; \mu, v), x(g; \mu, v))$ to problem (10) satisfies the budget constraint with equality: i.e.,

$$B(r(g; \mu, v), x(g; \mu, v); g) = 0.$$

This implies that $\mu(g; v) \leq \mu$, a contradiction.

Assume now that $\widehat{\mu}(g; v) > \mu(g; v)$, then for any $\mu \in (\mu(g; v), \widehat{\mu}(g; v))$, since $\mu < \widehat{\mu}(g; v)$, it must be $B(\widehat{r}(g; \mu, v), \widehat{x}(g; \mu, v); g) > 0$. In this case the solution of the unconstrained problem is the same as the constrained solution $r(g; \mu, v), x(g; \mu, v)$ (since the constraint is not binding): so $B(r(g; \mu, v), x(g; \mu, v); g) > 0$. However, since $\mu > \mu(g; v)$ at the the solution of the constrained problem, the constraint must be satisfied as equality, $B(r(g; \mu, v), x(g; \mu, v); g) = 0$, a contradiction. This completes the proof of Claim A.2. ■

We can now show that $\mu(\cdot; v_1)$ is an increasing and continuous function. For monotonicity, let $g' > g$, $\mu' = \mu(g'; v_1)$ and $\mu = \mu(g; v_1)$. We need to show that $\mu' > \mu$. Suppose, to the contrary, that $\mu' \leq \mu$. By Claim A.2, we know that $(r(g; \mu, v_1), x(g; \mu, v_1))$ solves the problem

$$\max_{(r, x)} \mu[u(w(1-r), g) + \delta v_1(x)] + B(r, x; g),$$

while $(r(g'; \mu', v_1), x(g'; \mu', v_1))$ solves the problem

$$\max_{(r, x)} \mu'[u(w(1-r), g') + \delta v_1(x)] + B(r, x; g').$$

It can easily verified that $r(g'; \mu', v_1) \geq r(g; \mu, v_1)$ and $x(g'; \mu', v_1) \leq x(g; \mu, v_1)$. Thus, since $g' > g$

$$B(r(g'; \mu', v_1), x(g'; \mu', v_1); g') > B(r(g; \mu, v_1), x(g; \mu, v_1); g) = 0$$

which contradicts the definition of μ' .

For continuity, let $g \geq 0$ and consider a sequence $g_n \rightarrow g$. Letting $\mu_n = \mu(g_n; v_1)$, we need to show that $\mu_n \rightarrow \mu(g; v_1)$. Note first that for any g there is an upperbound M such that $\mu(g_n; v_1) \in [0, M]$ (at least for n large enough): so we can assume, without loss of generality, that the limit $\mu^\infty = \lim_{n \rightarrow \infty} \mu_n$ exists. Since $B(r, x; g)$ is continuous in all its arguments and since $r(g; \mu, v_1)$ and $x(g; \mu, v_1)$ are continuous in g and μ by the *Theorem of the Maximum*, we have that

$$\lim_{n \rightarrow \infty} B(r(g_n; \mu_n; v_1), x(g_n; \mu_n; v_1), v_1); g_n) = B(r(g; \mu^\infty, v_1), x(g; \mu^\infty, v_1); g).$$

Moreover, since $B(r(g_n; \mu_n; v_1), x(g_n; \mu_n; v_1), v_1); g_n) = 0$ for all g_n we have that

$$B(r(g; \mu^\infty, v_1), x(g; \mu^\infty, v_1); g) = 0.$$

Clearly, it can not be that $\mu^\infty < \mu(g; v_1)$, because this would violate the definition of $\mu(g; v_1)$. Assume then that $\mu^\infty > \mu(g; v_1)$. In this case, we must have that $x(g; \mu^\infty, v_1) \geq x(g; \mu(g; v_1), v_1)$ and $r(g; \mu^\infty, v_1) < r(g; \mu(g; v_1), v_1)$, but this would imply

$$B(r(g; \mu^\infty, v_1), x(g; \mu^\infty, v_1), v_1); g) < B(r(g; \mu(g; v_1), v_1), x(g; \mu(g; v_1), v_1); g) = 0$$

which is a contradiction.

The final step is to show that $\mu(0; v_1) < q$ while for g large enough $\mu(g; v_1) > q$. The latter is obvious, and the former is implied by Assumption 1. To see this, suppose to the contrary, that $\mu(0; v_1) \geq q$. Then it would follow that $\mu(g; v_1) > q$ for all $g > 0$. This would imply that for all $g > 0$

$$(r_1(g), s_1(g), x_1(g)) = (r^*, \frac{B(r^*, x^*(v_1); g)}{n}, x^*(v_1)),$$

and hence that

$$v_1(g) = u(w(1 - r^*), g) + \frac{B(r^*, x^*(v_1); g)}{n} + \delta v_1(x^*(v_1))$$

This in turn implies that

$$v_1'(g) = A\alpha g^{\alpha-1} + \frac{p(1-d)}{n}$$

and hence that

$$x^*(v_1) = \left(\frac{\delta q A \alpha}{p(1 - \frac{q}{n} \delta (1-d))} \right)^{\frac{1}{1-\alpha}}.$$

But then, since $\mu(0; v_1) \geq q$ it must be the case that

$$B(\widehat{r}(0; q, v_1), \widehat{x}(0; q, v_1); 0) = B(r^*, x^*(v_1); 0) \geq 0$$

or, equivalently,

$$\frac{R(r^*)}{p} \geq \left(\frac{\delta q A \alpha}{p(1 - \frac{q}{n} \delta(1-d))} \right)^{\frac{1}{1-\alpha}}$$

which violates Assumption 1 (since $r^* = (1 - q/n)/(1 + \varepsilon - q/n)$).

It now follows that there exists a unique $g^*(v_1) > 0$ such that $\mu(g^*(v_1); v_1) = q$. Because $\mu(\cdot; v_1)$ is increasing, this $g^*(v_1)$ will have the property that for all $g \leq g^*(v_1)$, $\mu(g; v_1) \leq q$ and for all $g > g^*(v_1)$, $\mu(g; v_1) > q$. Thus, the proof of the Proposition is complete. *QED*

Proof of Proposition 5: The proof will proceed in two parts. First, we develop necessary and sufficient conditions for the existence of an equilibrium of each of the three types. Then we analyze when these conditions will be satisfied, relating them to \underline{A} and \bar{A} .

Existence of a Type I equilibrium

From the analysis preceding Proposition 2 in this case we have that

$$x^*(v_1) = \left(\frac{\delta q A \alpha}{p(1 - \frac{q}{n} \delta(1-d))} \right)^{\frac{1}{1-\alpha}} \equiv x^*$$

and that

$$g^*(v_1) = \frac{px^* - R(r^*)}{p(1-d)} \equiv g^*.$$

It then follows that $x^* > g^*$ if and only if

$$\frac{R(r^*)}{pd} > \left[\frac{\delta q A \alpha}{p(1 - \frac{q}{n} \delta(1-d))} \right]^{\frac{1}{1-\alpha}}. \quad (14)$$

This inequality is therefore a necessary condition for the existence of a Type I equilibrium.

In such an equilibrium, if $g > g^*$ then the legislature will choose the public good level x^* and tax rate r^* that period and every period thereafter. It must therefore be the case that the value function $v_1(g)$ satisfies:

$$v_1(g) = \begin{cases} \max_x \{u(w(1-r(x,g)), g) + \delta v_1(x)\} & g \leq g^* \\ u(w(1-r^*), g) + \frac{B(r^*, x^*; g)}{n} + \frac{\delta}{1-\delta} (u(w(1-r^*), x^*) + \frac{B(r^*, x^*; x^*)}{n}) & g > g^* \end{cases} \quad (15)$$

The question is whether there exists a strictly concave value function that satisfies this relationship when inequality (14) is satisfied. If so, there will exist an associated equilibrium $\{(r_\tau(g), s_\tau(g), x_\tau(g))\}_{\tau=1}^T$ in which if $g \in [0, g^*]$

$$(r_\tau(g), s_\tau(g), x_\tau(g)) = (r(x^*(g; v_1), g), 0, x^*(g; v_1)) \text{ for all } \tau = 1, \dots, T,$$

and if $g \in (g^*, \infty)$

$$(r_\tau(g), x_\tau(g)) = (r^*, x^*) \text{ for all } \tau = 1, \dots, T,$$

and

$$s_\tau(g) = \begin{cases} \frac{B(r^*, x^*; g)}{n} & \text{for all } \tau = 1, \dots, T-1 \\ v_{T+1}(g) - u(w(1-r^*), g) - \delta v_1(x^*) & \text{for } \tau = T \end{cases}$$

The proof of the following Lemma establishes not only that there does exist such a function but also that it is unique.

Lemma A.2: *There exists a Type I equilibrium if and only if inequality (14) is satisfied. Moreover, there is a unique such equilibrium.*

Proof of Lemma A.2: Let $\bar{g} > x^*$ be an arbitrarily large but bounded scalar. We will first restrict the range of public good levels to $[0, \bar{g}]$ and prove the existence of a unique $v_1(g)$ that satisfies (15) when this assumption is satisfied. Then we will extend the solution for $g > \bar{g}$. Define for $g \in [g^*, \bar{g}]$ the function

$$\bar{v}(g) = u(w(1-r^*), g) + \frac{B(r^*, x^*; g)}{n} + \frac{\delta}{1-\delta} (u(w(1-r^*), x^*) + \frac{B(r^*, x^*; x^*)}{n})$$

This function $\bar{v}(g)$ is continuous, bounded and strictly concave on $[g^*, \bar{g}]$. Then let F denote the set of continuous, bounded, weakly concave functions $v : \mathfrak{R}_+ \rightarrow \mathfrak{R}$ such that $v(g) = \bar{v}(g)$ for all $g \in [g^*, \bar{g}]$. This set is non empty, closed, bounded, and convex. Finally, define the functional Ψ on F as follows:

$$\Psi(v)(g) = \begin{cases} \max_x \{u(w(1-r(x, g)), g) + \delta v(x)\} & g \in [0, g^*] \\ u(w(1-r^*), g) + \frac{B(r^*, x^*; g)}{n} + \delta \bar{v}(x^*) & g \in (g^*, \bar{g}] \end{cases}. \quad (16)$$

For a given expected continuation value function v at time $t+1$, $\Psi(v)$ provides the expected value function of a legislator at time t in an equilibrium with $x^*(v_1) > g^*(v_1)$.

We will now prove that there exists a unique $v_1 \in F$ such that $v_1 = \Psi(v_1)$. The first step is to show that Ψ maps F into itself; i.e., that $\Psi(v) \in F$. It is immediate that $\Psi(v)(g) = \bar{v}(g)$ for all $g \in [g^*, \bar{g}]$, and that $\Psi(v)$ is bounded on $[0, \bar{g}]$. However, we need to prove that $\Psi(v)$ is continuous and (strictly) concave.

Continuity. The function $\Psi(v)$ is continuous on $[0, g^*)$ by the *Theorem of the Maximum*, and on $(g^*, \bar{g}]$ by definition. We just need to show that it is continuous at $g = g^*$. Since $B(r^*, x^*; g^*) =$

0, we have that

$$\begin{aligned}\lim_{g \searrow g^*} \Psi(v)(g) &= u(w(1-r^*), g^*) + \frac{B(r^*, x^*; g^*)}{n} + \delta \bar{v}(x^*) \\ &= u(w(1-r^*), g^*) + \delta \bar{v}(x^*).\end{aligned}$$

Next note that $(r^*, x^*) = (r(x^*(g^*; v), g^*), x^*(g^*; v))$. To see this, suppose the converse. Then, since $B(r^*, x^*; g^*) = 0$, it must be that:

$$u(w(1-r(x^*(g^*; v), g^*)), g^*) + \delta v(x^*(g^*; v)) > u(w(1-r^*), g^*) + \delta v(x^*),$$

which implies that

$$\begin{aligned}& q[u(w(1-r(x^*(g^*; v), g^*)), g^*) + \delta v(x^*(g^*; v))] + B(r(x^*(g^*; v), g^*), x^*(g^*; v); g^*) \\ & > q[u(w(1-r^*), g^*) + \delta v(x^*)] + B(r^*, x^*; g^*).\end{aligned}$$

But this is a contradiction since (r^*, x^*) solves the problem

$$\max_{(r,x)} q[u(w(1-r), g) + \delta v(x)] + B(r, x; g).$$

To see the latter, note that given that $x^* > g^*$, we know that $v(x^*) = \bar{v}(x^*)$ and, by construction, $\delta q \bar{v}'(x^*) = p$. It follows, therefore, that

$$\begin{aligned}\lim_{g \nearrow g^*} \Psi(v)(g) &= u(w(1-r(x^*(g^*; v), g^*)), g^*) + \delta v(x^*(g^*; v)) \\ &= u(w(1-r^*), g^*) + \delta \bar{v}(x^*) = \lim_{g \searrow g^*} \Psi(v)(g).\end{aligned}$$

Strict Concavity. We proceed in three steps.

Step 1. $\Psi(v)$ is strictly concave on $[0, g^*]$. In this case the budget constraint is binding and the value function is:

$$\Psi(v)(g) = \max_{(r,x)} \left\{ \begin{array}{l} u(w(1-r), g) + \frac{B(r,x;g)}{n} + \delta v(x) \\ B(r, x; g) \geq 0 \end{array} \right\}. \quad (17)$$

Take two points g_1 and g_2 with $0 \leq g_1 < g_2 \leq g^*$, and a scalar $\phi \in [0, 1]$. Define r_i and x_i to be the optimal policies with public good level g_i $i = 1, 2$. Let $g_\phi = \phi g_1 + (1 - \phi) g_2$, $r_\phi = \phi r_1 + (1 - \phi) r_2$, and $x_\phi = \phi x_1 + (1 - \phi) x_2$. Since $v(x)$ is concave, the function $u(w(1-r), g) + B(r, x; g)/n + \delta v(x)$

is concave in (r, x, g) . Thus, we have that

$$\begin{aligned}\phi\Psi(v)(g_1) + (1 - \phi)\Psi(v)(g_2) &= \phi\left[u(w(1 - r_1), g_1) + \frac{B(r_1, x_1; g_1)}{n} + \delta v(x_1)\right] \\ &\quad + (1 - \phi)\left[u(w(1 - r_2), g_2) + \frac{B(r_2, x_2; g_2)}{n} + \delta v(x_2)\right] \\ &< u(w(1 - r_\phi), g_\phi) + \frac{B(r_\phi, x_\phi; g_\phi)}{n} + \delta v(x_\phi).\end{aligned}$$

Since $B(r, x; g)$ is concave in (r, x, g) , we have that $B(r_\phi, x_\phi; g_\phi) \geq 0$, so that

$$\begin{aligned}u(w(1 - r_\phi), g_\phi) + \frac{B(r_\phi, x_\phi; g_\phi)}{n} + \delta v(x_\phi) &\leq \max_{(r, x)} \left\{ \begin{array}{l} u(w(1 - r), g_\phi) + \frac{B(r, x; g_\phi)}{n} + \delta v(x) \\ B(r, x; g_\phi) \geq 0 \end{array} \right\} \\ &= \Psi(v)(g_\phi).\end{aligned}$$

Therefore $\phi\Psi(v)(g_1) + (1 - \phi)\Psi(v)(g_2) < \Psi(v)(g_\phi)$ as required.

Step 2. $\Psi(v)$ is strictly concave on $(g^*, \bar{g}]$. This is immediate from the definition of $\Psi(v)(g)$.

Step 3. $\Psi(v)$ is strictly concave on $[0, \bar{g}]$. Let g_1 and g_2 be such that $0 \leq g_1 \leq g^* < g_2 \leq \bar{g}$. We have two possible cases. First it may be that $g_\phi \leq g^*$. For all $g \in [0, \bar{g}]$, let $(r'(g; v), x'(g; v))$ be the solution to the problem

$$\begin{aligned}\max_{(r, x)} u(w(1 - r), g) + \frac{B(r, x; g)}{n} + \delta v(x) \\ B(r, x; g) \geq 0\end{aligned},$$

and let

$$\Xi(v)(g) = u(w(1 - r'(g; v)), g) + \frac{B(r'(g; v), x'(g; v); g)}{n} + \delta v(x'(g; v)).$$

We have that $\Xi(v)(g) \geq \Psi(v)(g)$ for all $g \in [0, \bar{g}]$. Indeed, the two functionals are equivalent for $g \in [0, g^*]$ but, if $g > g^*$, $\Psi(v)(g)$ is less than $\Xi(v)(g)$. Therefore we have:

$$\begin{aligned}\phi\Psi(v)(g_1) + (1 - \phi)\Psi(v)(g_2) &\leq \phi\Xi(v)(g_1) + (1 - \phi)\Xi(v)(g_2) \\ &< u(w(1 - r'_\phi), g_\phi) + \frac{B(r'_\phi, x'_\phi; g_\phi)}{n} + \delta v(x'_\phi)\end{aligned}$$

where $r'_\phi = \phi r'(g_1; v) + (1 - \phi)r'(g_2; v)$ and $x'_\phi = \phi x'(g_1; v) + (1 - \phi)x'(g_2; v)$. Moreover, $B(r'_\phi, x'_\phi; g_\phi) \geq$

0, implying:

$$\begin{aligned}
\Psi(v)(g_\phi) &= \max_{(r,x)} \left\{ \begin{array}{l} u(w(1-r), g_\phi) + B(r, x; g_\phi)/n + \delta v(x) \\ B(r, x; g_\phi) \geq 0 \end{array} \right\} \\
&\geq u(w(1-r'_\phi), g_\phi) + \frac{B(r'_\phi, x'_\phi; g_\phi)}{n} + \delta v(x'_\phi) \\
&> \phi \Psi(v)(g_1) + (1-\phi) \Psi(v)(g_2).
\end{aligned}$$

The second case arises when $g_\phi > g^*$. Let $\psi \in [0, 1]$ be such that $g^* = \psi g_1 + (1-\psi) g_2$; by the previous step, we have that $\Psi(v)(g^*) > \psi \Psi(v)(g_1) + (1-\psi) \Psi(v)(g_2)$ (since obviously $g^* \in [0, g^*]$). Take now a scalar $\eta \in [0, 1]$ such that $\eta g^* + (1-\eta) g_2 = g_\phi$. Since $\Psi(v)$ is strictly concave and continuous in $g \geq g^*$, it must be that $\Psi(v)(g_\phi) > \eta \Psi(v)(g^*) + (1-\eta) \Psi(v)(g_2)$. Therefore we have:

$$\begin{aligned}
\Psi(v)(g_\phi) &> \eta \Psi(v)(g^*) + (1-\eta) \Psi(v)(g_2) > \eta \psi \Psi(v)(g_1) + (1-\eta \psi) \Psi(v)(g_2) \\
&= \phi \Psi(v)(g_1) + (1-\phi) \Psi(v)(g_2)
\end{aligned}$$

where the second inequality follows from $\Psi(v)(g^*) > \psi \Psi(v)(g_1) + (1-\psi) \Psi(v)(g_2)$, and the last equality follows from the definitions of η and ψ : $\phi g_1 + (1-\phi) g_2 = g_\phi = \eta g^* + (1-\eta) g_2 = \eta \psi g_1 + (1-\eta \psi) g_2$, implying $\phi = \eta \psi$.

Given that $\Psi(v) \in F$, to prove existence and uniqueness of a fixpoint of Ψ in F , it is sufficient to prove that $\Psi(\cdot)$ is a contraction in F . Let $\omega_1, \omega_2 \in F$ be such that $\omega_1(g) \leq \omega_2(g)$ for all $g \in [0, \bar{g}]$. Define $x_{\omega_i}(g)$ as a solution of $\max_x \{u(w(1-r(x, g)), g) + \delta \omega_i(x)\} \forall i = 1, 2$. For $g \in [0, g^*]$, we have:

$$\begin{aligned}
\Psi(\omega_2)(g) &= \max_x \{u(w(1-r(x, g)), g) + \delta \omega_2(x)\} \geq u(w(1-r(x_{\omega_1}(g), g)), g) + \delta \omega_2(x_{\omega_1}(g)) \\
&\geq u(w(1-r(x_{\omega_1}(g), g)), g) + \delta \omega_1(x_{\omega_1}(g)) \\
&= \Psi(\omega_1)(g)
\end{aligned}$$

and, by definition, $\Psi(\omega_2)(g) = \Psi(\omega_1)(g)$ for $g \in (g^*, \bar{g}]$. So $\Psi(\cdot)$ satisfies Blackwell's monotonicity condition (cf. Blackwell (1965)). Let a be a weakly positive scalar, then for any $g \in [0, g^*]$ and $v \in F$ we have:

$$\Psi(v+a)(g) = \max_x \{u(w(1-r(x, g)), g) + \delta v(x)\} + \delta a = \Psi(v)(g) + \delta a$$

and $\Psi(v+a)(g) = \Psi(v)(g)$ if $g \in (g^*, \bar{g}]$. Since $\delta \in (0, 1)$, we conclude that Blackwell's discounting condition is satisfied as well (cf. Blackwell (1965)). It follows that Blackwell's sufficient conditions

are satisfied and, by Theorem 5 in Blackwell (1965), $\Psi(\cdot)$ is a contraction with modulus δ . From all these properties, it follows that there exists a unique continuous, bounded, strictly concave value function v_1 that satisfies (15).

To see that the equilibrium value function can be extended for $g > \bar{g}$, note that we can define $v(g) = \bar{v}(g)$ for $g > \bar{g}$. The resulting value function is continuous, concave and continues to be a fixpoint of (16). This completes the proof of Lemma A.2. ■

Existence of a Type II equilibrium

In this case, Proposition 3 tells us that the equilibrium converges monotonically to the planner's steady state. Thus, it must be the case that for all $g \leq g^*(v_1)$, $v_1(g) = V(g)/n$ where $V(g)$ is the planner's value function. This means that $x^*(v_1) = x^{**}$ where

$$\delta \frac{V'(x^{**})}{n} = \frac{p}{q}.$$

This in turn implies that

$$g^*(v_1) = \frac{px^{**} - R(r^*)}{p(1-d)} \equiv g^{**}.$$

It turns out that $x^{**} < g^{**}$ if and only if the marginal cost of public funds at the planner's steady state exceeds the ratio n/q ; that is,

$$\left(\frac{1 - r(x^o, x^o)}{1 - r(x^o, x^o)(1 + \varepsilon)} \right) > \frac{n}{q}. \quad (18)$$

To see this, note from the Euler equation for the planner's problem, that at the planner's steady state

$$\delta V'(x^o) = p \left(\frac{1 - r(x^o, x^o)}{1 - r(x^o, x^o)(1 + \varepsilon)} \right).$$

Thus, if the condition is satisfied then it must be the case that $V'(x^o) > V'(x^{**})$ which by the concavity of the planner's value function implies that $x^{**} > x^o$. But since the condition implies that $r^o = r(x^o, x^o) > r^*$, this means that

$$x^o = \frac{px^o - R(r^o)}{p(1-d)} < \frac{px^{**} - R(r^*)}{p(1-d)} \equiv g^{**}$$

From Figure 3(b) it is clear that this implies that $x^o(g^{**}) = x^{**} < g^{**}$.

For $g > g^{**}$, we know that the legislature selects the public good level x^{**} which puts proposals back into the no-pork region in one period. Accordingly, it must be the case that

$$v_1(g) = \begin{cases} \frac{V(g)}{n} & g \leq g^{**} \\ u(w(1-r^*), g) + \frac{1}{n}B(r^*, x^{**}; g) + \delta \frac{V(x^{**})}{n} & g > g^{**} \end{cases}$$

It is now straightforward to show that this is indeed an equilibrium value function and is strictly concave. The associated equilibrium policy proposals $\{(r_\tau(g), s_\tau(g), x_\tau(g))\}_{\tau=1}^T$ are such that if $g \in [0, g^{**}]$

$$(r_\tau(g), s_\tau(g), x_\tau(g)) = (r(x^*(g; v_1), g), 0, x^*(g; v_1)) \text{ for all } \tau = 1, \dots, T,$$

and if $g \in (g^{**}, \infty)$

$$(r_\tau(g), x_\tau(g)) = (r^*, x^{**}) \text{ for all } \tau = 1, \dots, T,$$

and

$$s_\tau(g) = \begin{cases} \frac{B(r^*, x^{**}; g)}{n} & \text{for all } \tau = 1, \dots, T-1 \\ v_{T+1}(g) - u(w(1-r^*), g) - \delta v_1(x^{**}) & \text{for } \tau = T \end{cases}.$$

Thus we have:

Lemma A.3: *There exists a Type II equilibrium if and only if inequality (18) is satisfied. Moreover, there is a unique such equilibrium.*

Existence of a Type III equilibrium

In this case, we know that

$$x^*(v_1) = g^*(v_1) = \frac{R(r^*)}{pd} \equiv \tilde{x}.$$

Further, we know that it must be the case that

$$\delta[A\alpha\tilde{x}^{\alpha-1} + \frac{p(1-d)}{n}] \leq \frac{p}{q} \leq \delta[A\alpha\tilde{x}^{\alpha-1} + (\frac{1-r^*}{1-r^*(1+\varepsilon)}) (\frac{p(1-d)}{n})]. \quad (19)$$

In such an equilibrium, if $g > \tilde{x}$ then the legislature will choose the public good level \tilde{x} and tax rate r^* that period and every period thereafter. It must therefore be the case that the value function $v_1(g)$ satisfies

$$v_1(g) = \begin{cases} \max_x \{u(w(1-r(x, g)), g) + \delta v_1(x)\} & g \leq \tilde{x} \\ u(w(1-r^*), g) + \frac{B(r^*, \tilde{x}; g)}{n} + \frac{\delta}{1-\delta} u(w(1-r^*), \tilde{x}) & g > \tilde{x} \end{cases}. \quad (20)$$

The question is whether there exists a strictly concave value function which satisfies this relationship when inequalities (19) are satisfied. If so, there will exist an associated equilibrium $\{(r_\tau(g), s_\tau(g), x_\tau(g))\}_{\tau=1}^T$ in which if $g \in [0, \tilde{x}]$

$$(r_\tau(g), s_\tau(g), x_\tau(g)) = (r(x^*(g; v_1), g), 0, x^*(g; v_1)) \text{ for all } \tau = 1, \dots, T,$$

and if $g \in (\tilde{x}, \infty)$

$$(r_\tau(g), x_\tau(g)) = (r^*, \tilde{x}) \text{ for all } \tau = 1, \dots, T,$$

and

$$s_\tau(g) = \begin{cases} \frac{B(r^*, \tilde{x}; g)}{n} & \text{for all } \tau = 1, \dots, T-1 \\ v_{T+1}(g) - u(w(1-r^*), g) - \delta v_1(\tilde{x}) & \text{for } \tau = T \end{cases}$$

The following Lemma shows that the answer is yes.

Lemma A.4: *There exists a Type III equilibrium if and only if inequality (19) is satisfied. Moreover, there is a unique such equilibrium.*

Proof of Lemma A.4: The proof is similar to the proof of Lemma A.2. Let $\bar{g} > \tilde{x}$ be an arbitrarily large but bounded scalar. We will first restrict the range of public good levels to $[0, \bar{g}]$ and prove the existence of a unique $v_1(g)$ that satisfies (20) when this assumption is satisfied, then we will extend the solution for $g > \bar{g}$. Define for $g \in [\tilde{x}, \bar{g}]$ the function

$$\tilde{v}(g) = u(w(1-r^*), g) + \frac{B(r^*, \tilde{x}; g)}{n} + \frac{\delta}{1-\delta} u(w(1-r^*), \tilde{x}).$$

Then let \tilde{F} denote the set of continuous, bounded, weakly concave functions $v : \mathfrak{R}_+ \rightarrow \mathfrak{R}$ such that $v(g) = \tilde{v}(g)$ for all $g \in [\tilde{x}, \bar{g}]$. Finally, define the functional $\tilde{\Psi}$ on \tilde{F} as follows:

$$\tilde{\Psi}(v)(g) = \begin{cases} \max_x \{u(w(1-r(x, g)), g) + \delta v(x)\} & g \in [0, \tilde{x}] \\ u(w(1-r^*), g) + \frac{1}{n} B(r^*, \tilde{x}; g) + \delta \tilde{v}(\tilde{x}) & g \in (\tilde{x}, \bar{g}] \end{cases}.$$

It can be shown that $\tilde{\Psi}(v) \in \tilde{F}$ and, further, that $\tilde{\Psi}(v)$ is strictly concave. It can also be shown that $\tilde{\Psi}(v)$ is a contraction mapping which implies that there exists a unique function v_1 such that $v_1 = \tilde{\Psi}(v_1)$. This function is strictly concave and satisfies (20). As in the case with $x^*(v_1) > g^*(v_1)$, the value function can be extended in (\bar{g}, ∞) by defining $v(g) = \tilde{v}(g)$ for $g > \bar{g}$. The resulting value function is continuous, concave and continues to be a fixpoint of (20). This completes the proof of Lemma A.4. ■

When are the conditions satisfied?

Define \bar{A} to be the value of A that would be such as to make the discounted marginal benefit of the public good in the minimum winning coalition range equal to p/q at the public good level $R(r^*)/pd$; that is,

$$\delta \left[\bar{A} \alpha \left(\frac{R(r^*)}{pd} \right)^{\alpha-1} + \frac{p(1-d)}{n} \right] = \frac{p}{q}.$$

Similarly, let \underline{A} be the value of A that would be such as to make the discounted marginal benefit of the public good in the unanimity range equal to p/q at the public good level \tilde{x} ; that is,

$$\delta[\underline{A}\alpha\left(\frac{R(r^*)}{pd}\right)^{\alpha-1} + \left(\frac{1-r^*}{1-r^*(1+\varepsilon)}\right)\left(\frac{p(1-d)}{n}\right)] = \frac{p}{q}.$$

Notice that \underline{A} must be less than \bar{A} since, holding constant public good preferences, the value of an additional unit is higher in the unanimity range.

Now we have the following convenient result.

Lemma A.5: (i) Condition (14) is satisfied if and only if $A \in (0, \bar{A})$. (ii) Condition (18) is satisfied if and only if $A > \underline{A}$. (iii) Condition (19) is satisfied if and only if $A \in [\underline{A}, \bar{A}]$.

Proof of Lemma A.5: (i) Let

$$x^*(A) = \left(\frac{\delta q A \alpha}{p(1 - \frac{q}{n}\delta(1-d))}\right)^{\frac{1}{1-\alpha}}$$

Then, we know that condition (14) is satisfied if and only if $x^*(A) < \frac{R(r^*)}{pd}$ or equivalently if only if $A \in (0, \hat{A})$ where

$$x^*(\hat{A}) = \frac{R(r^*)}{pd}.$$

But note that

$$\begin{aligned} x^*(\hat{A}) &= \frac{R(r^*)}{pd} \Leftrightarrow \left(\frac{\delta q \hat{A} \alpha}{p(1 - \frac{q}{n}\delta(1-d))}\right)^{\frac{1}{1-\alpha}} = \frac{R(r^*)}{pd} \\ &\Leftrightarrow \delta[\hat{A}\alpha\left(\frac{R(r^*)}{pd}\right)^{\alpha-1} + \frac{p(1-d)}{n}] = \frac{p}{q} \end{aligned}$$

which implies that $\hat{A} = \bar{A}$.

(ii) Condition (18) is that

$$\left(\frac{1-r(x^o, x^o)}{1-r(x^o, x^o)(1+\varepsilon)}\right) > \frac{n}{q}.$$

Let $x^o(A)$ denote the planner's steady state public good level with public good preference parameter A and $r^o(A) = r(x^o(A), x^o(A))$ the associated tax rate. Clearly, $x^o(A)$ and $r^o(A)$ are increasing in A . Letting \tilde{A} be such that

$$\left(\frac{1-r^o(\tilde{A})}{1-r^o(\tilde{A})(1+\varepsilon)}\right) = \frac{n}{q},$$

it is clear that condition (18) is satisfied if and only if $A > \tilde{A}$. But note that $r^o(\tilde{A}) = r^*$ and that

$$\delta n \tilde{A} \alpha x^o(\tilde{A})^{\alpha-1} = p[1 - \delta(1-d)] \left[\frac{1-r^*}{1-r^*(1+\varepsilon)} \right]$$

or, equivalently,

$$\delta[\underline{A}\alpha x^o(\tilde{A})^{\alpha-1} + (\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})] = \frac{p}{q}.$$

Furthermore, $B(x^o(\tilde{A}), r^*; x^o(\tilde{A})) = 0$ which implies that

$$x^o(\tilde{A}) = \frac{R(r^*)}{pd}$$

and hence that $\tilde{A} = \underline{A}$ as required.

(iii) This is immediate.

This completes the proof of Lemma A.5. ■

The proposition now follows by combining Lemmas A.2 - A.5. *QED*

Proof of Lemma 1: We know from Proposition 2 that the equilibrium steady state is

$$(r^*, (\frac{\delta q A \alpha}{p(1-\frac{q}{n}\delta(1-d))})^{\frac{1}{1-\alpha}}).$$

The planner's steady state is given by:

$$\delta[nA\alpha(x^o)^{\alpha-1} + (\frac{1-r^o}{1-r^o(1+\varepsilon)})p(1-d)] = \frac{1-r^o}{1-r^o(1+\varepsilon)} \cdot p.$$

and

$$R(r^o) = pdx^o$$

which means that

$$\delta[A\alpha(\frac{R(r^o)}{pd})^{\alpha-1} + (\frac{1-r^o}{1-r^o(1+\varepsilon)})\frac{p(1-d)}{n}] = \frac{1-r^o}{1-r^o(1+\varepsilon)} \cdot \frac{p}{n}$$

The equilibrium steady state satisfies

$$\delta[A\alpha x^{\alpha-1} + \frac{p(1-d)}{n}] = \frac{p}{q}.$$

We know that

$$\delta[\underline{A}\alpha(\frac{R(r^*)}{pd})^{\alpha-1} + (\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})] = \frac{p}{q} = \frac{1-r^*}{1-r^*(1+\varepsilon)} \cdot \frac{p}{n}.$$

Thus, if $A < \underline{A}$ we have that

$$\delta[A\alpha(\frac{R(r^*)}{pd})^{\alpha-1} + (\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})] < \frac{1-r^*}{1-r^*(1+\varepsilon)} \cdot \frac{p}{n}$$

This implies that $r^* > r^o$ - the equilibrium tax rate is higher than the planner's tax rate. On the other hand, if $A \in (\underline{A}, \bar{A}]$ we have that

$$\delta[A\alpha(\frac{R(r^*)}{pd})^{\alpha-1} + (\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})] > \frac{1-r^*}{1-r^*(1+\varepsilon)} \cdot \frac{p}{n}$$

which implies that $r^* < r^o$.

What about the level of public goods? The level of public goods at the equilibrium satisfies

$$\delta[A\alpha x^{\alpha-1} + \frac{p(1-d)}{n}] = \frac{1-r^*}{1-r^*(1+\varepsilon)} \cdot \frac{p}{n} > \frac{1-r^o}{1-r^o(1+\varepsilon)} \cdot \frac{p}{n}$$

Thus, it is clear that the level of public goods is below that at the planner's solution, because the marginal cost is higher and the marginal benefit is lower. *QED*

Proof of Lemma 2: It suffices to show that $r^* < r^o$. But we know from the proof of Lemma 1 that this follows if $A \in (\underline{A}, \bar{A}]$. But we also know that when this type of equilibrium exists it must be the case that $A \in [\underline{A}, \bar{A}]$. *QED*

Proof of Proposition 7: Solving (8) and (9) for \underline{A} and \bar{A} yields

$$\underline{A} = \frac{p(1-\delta(1-d))(\frac{R(r^*)}{pd})^{1-\alpha}}{q\delta\alpha}$$

and

$$\bar{A} = \frac{p(1-\frac{q}{n}\delta(1-d))(\frac{R(r^*)}{pd})^{1-\alpha}}{q\delta\alpha}.$$

Moreover, we have that

$$R(r^*) = nr^*(1-r^*)^\varepsilon w^{\varepsilon+1} \varepsilon^\varepsilon = \frac{(n-q)\varepsilon^{2\varepsilon} w^{\varepsilon+1}}{(1+\varepsilon-q/n)^{\varepsilon+1}}.$$

From these expressions it is clear that \underline{A} and \bar{A} are decreasing in δ and q and increasing in p and w . For the claims about the impact of increasing the elasticity of labor supply ε , define the function $w(\varepsilon)$ from the equality

$$nw^{\varepsilon+1} \varepsilon^\varepsilon = K,$$

for some constant K . Then let $\tilde{R}(\varepsilon)$ be the function that equals $R(r^*)$ when the elasticity is ε and the wage is $w(\varepsilon)$; that is,

$$\tilde{R}(\varepsilon) = \frac{(n-q)\varepsilon^{2\varepsilon} w(\varepsilon)^{\varepsilon+1}}{(1+\varepsilon-q/n)^{\varepsilon+1}} = \frac{(1-q/n)\varepsilon^\varepsilon}{(1+\varepsilon-q/n)^{\varepsilon+1}}.$$

We need to show that $\tilde{R}(\varepsilon)$ is decreasing in ε . Taking logs, we have that

$$\begin{aligned}\ln R(\varepsilon) &= \ln(1 - q/n)\varepsilon^\varepsilon - \ln(1 + \varepsilon - q/n)^{\varepsilon+1} \\ &= \ln(1 - q/n) + \varepsilon \ln \varepsilon - (\varepsilon + 1) \ln(1 + \varepsilon - q/n)\end{aligned}$$

Thus,

$$\frac{d \ln R(\varepsilon)}{d\varepsilon} = \ln \varepsilon - \ln(1 + \varepsilon - q/n) + 1 - \left(\frac{\varepsilon + 1}{\varepsilon + 1 - q/n}\right) < 0$$

which implies the result. *QED*