Noise, Information and the Favorite-Longshot Bias*

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Abstract

According to the favorite-longshot bias, bets with short odds (i.e., favorites) offer higher expected payoff than those with long odds (i.e., longshots). This paper proposes an explanation for the bias based on a model of equilibrium betting in parimutuel markets. The bias (or its reverse) arises because privately informed bettors place bets without knowing the final distribution of bets. The direction and extent of the bias depend on the amount of private information relative to noise that is present in the market.

Keywords: Parimutuel betting, favorite-longshot bias, private information, noise, lotteries.

JEL Classification: D82 (Asymmetric and Private Information), D83 (Search; Learning; Information and Knowledge), D84 (Expectations; Speculations).

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1 Introduction

Betting markets provide a natural environment for testing theories of decision making under uncertainty and price formation. The uncertainty about the value of the assets traded in betting markets is resolved unambiguously and the outcome is publicly observed. In addition, in most cases it is reasonable to presume that the realized outcomes are exogenous with respect to market prices. In regular financial markets instead, the intrinsic value of assets is observed only in the long run and can be affected by market prices.

In horse-racing tracks and lottery games throughout the world, the most common procedure adopted is parimutuel betting.[1] According to the parimutuel system, the holders of winning tickets divide the total amount of money bet on all outcomes in proportion to their bets, net of taxes and expenses. Parimutuel payoff *odds* thus depend on the distribution of bets and are determined by the gambling public itself. Because these odds are not skewed by price-setting suppliers (such as bookmakers), parimutuel betting markets are particularly well suited to test market efficiency.[2]

The most widely documented empirical regularity observed in horse-betting markets is the *favorite-longshot bias*: horses with "short" odds (i.e., favorites) tend to win more frequently than indicated by their odds, while horses with "long" odds (i.e., longshots) win less frequently.[3] This implies that the expected returns on longshots is much lower than on favorites. This finding is puzzling, because the simple logic of arbitrage suggests that expected returns should be equalized across horses. To further add to the puzzle, note that for parimutuel *lottery* games, such as Lotto, a *reverse* favorite-longshot bias always results: the expected payoff is lower on numbers that attract a higher-than-proportional amount of bets.[4]

The goal of this paper is to formulate a simple theoretical model of parimutuel betting

[1] Recently, the parimutuel structure was adopted in markets for contingent claims on various economic indices. As Economides and Lange (2005) explain, these markets allow traders to hedge risks related to variables such as U.S. nonfarm payroll employment and European harmonized indices of consumer prices.

[2] Given the mutual nature of parimutuel markets, bettors are simultaneously present on both sides of the market. Essentially, the demand for a bet on an outcome generates the supply on all the other outcomes. See e.g. Levitt (2004) on supply-driven distortions in sport betting markets not using the parimutuel system.

[3] For surveys of the extensive literature we refer to Thaler and Ziemba (1988), Hausch and Ziemba (1995), Sauer (1998), and Jullien and Salanié (2002).

[4] See Forrest, Gulley and Simmons (2000) for a test of rational expectations in lotteries. The reverse favorite-longshot bias is also present in some betting markets (see e.g. Busche and Hall, 1988, Vaughan Williams and Paton, 1998, and our discussion below).

that provides an informational explanation for the occurrence of the favorite-longshot bias and its reverse. In the model, a finite number of bettors decide simultaneously whether, and on which of two outcomes, to bet. Each bettor's payoff has two components. First, there is a "common value" component, equal to the expected monetary payoff of the bet based on the final odds.[5] Second, bettors derive a private utility from gambling (see e.g., Conlisk, 1993). For simplicity, we set this "recreational value" to be the same for all bettors.[6]

Our model allows bettors to have heterogeneous beliefs based on the observation of a private signal. We aim to characterize the effect of private information on market outcomes. While private information is clearly absent in lottery games, there is evidence it is present in horse betting (see Section 6). In a limit case relevant for lottery games, the signal contains no information.

We model betting as a *simultaneous*-move game. This is a realistic description of lottery games, in which the numbers picked by participants are not made public before the draw. For horse-race betting, the distribution of bets (or, equivalently, the provisional odds given the cumulative bets placed) are displayed over time on the tote board and updated at regular intervals until post time, when betting is closed. However, a large proportion of bets is placed in the very last seconds before post time (see National Thoroughbred Racing Association, 2004).[7] Thus, we focus on the last-minute simultaneous betting game.

We show that the favorite-longshot bias depends on the amount of information relative to noise that is present in equilibrium. Suppose first that the bettors' signals are completely uninformative about the outcome, as is the case in lottery games. Market odds then will vary randomly, mostly due to the noise contained in the signal. Since all numbers are equally likely to be drawn, and the jackpot is shared among the lucky few who pick the winning number, the expected payoff is automatically lower for numbers that attract more than their fair share of bets. Borrowing the terminology used in horse betting, lottery

[5]By assuming risk neutrality, we depart from a large part of the betting literature since Weitzman (1965) in which bettors are risk loving. See Section 6 for a discussion of alternative models presented in the literature.

[6]In the case of parimutuel derivative markets mentioned in footnote 1 this private value is given by the value of hedging against other pre-existing risks correlated with the outcome on which betting takes place.

[7]Ottaviani and Sørensen (2004) endogenize the timing of bets in a dynamic model. They show that small privately informed bettors have an incentive to wait till post time, and thus end up betting simultaneously.

outcomes with short market odds yield lower expected returns than outcomes with long market odds. More generally, when signals contain little information, or there is aggregate uncertainty about the final distribution of bets due to noise, our model predicts a *reverse favorite-longshot bias* — as is observed in parimutuel lotteries.

Now, as bettors have more private information, in the realized bets there is less noise and more of this information. However, the final odds do not directly reflect the information revealed by the bets. When a large number of bets are placed on an outcome, it means that many bettors privately believed that this outcome was more likely. Conversely, the occurrence of long odds (i.e., few bets) reveals unfavorable evidence. We show that if there are many bettors and/or they have sufficient private information, the favorites (or longshots) are more (or less) likely to win than indicated by the realized market odds. This results in the *favorite-longshot bias*, as observed in horse-betting markets. In this simple observation lies the main contribution of this paper.

In the literature, Shin (1991 and 1992) has formulated an information-based explanation of the favorite-longshot bias in the context of *fixed-odds* betting. While our informational assumptions are similar to Shin's, we focus on parimutuel betting. The logic of the result is different in the two markets, as explained in detail by Ottaviani and Sørensen (2005).[8]

In the context of parimutuel markets, Potters and Wit (1996) derived the favorite-longshot bias as a deviation from a rational expectations equilibrium. In Potters and Wit's model, when privately informed bettors are given the opportunity to adjust their bets at the final market odds, it is assumed that they ignore the information contained in the bets.[9] Instead, in the Bayes-Nash equilibrium of our model, bettors understand that bets are informative, but are unable to adjust their bets to incorporate this information because they do not observe the final market odds.

Ours is the first paper that studies the favorite-longshot bias in an equilibrium model of parimutuel betting with private information. The only other paper in the literature

[8] As shown by Ottaviani and Sørensen (2005), the parimutuel payoff structure has a built-in insurance against adverse selection, that is not present in fixed-odds markets. In parimutuel markets, an increase in the number of informed bettors drives market odds to be more extreme and so reduces the favorite-longshot bias. In fixed-odds markets instead, an increase in the fraction of informed bettors strengthens the favorite-longshot, because adverse selection is worsened by the increased presence of informed bettors.

[9] A similar approach is to let bettors have heterogeneous beliefs, rather than private information (e.g., Eisenberg and Gale, 1959, Theorem 2 of Ali, 1977, Chadha and Quandt, 1996). Then there is no information content in the bets.

that has studied equilibria in parimutuel markets with private information is Koessler and Ziegelmeyer (2002). They considered bettors with binary signals. By assuming instead that bettors' beliefs are continuously distributed (as in other auction theoretic models of price formation), our analysis is greatly simplified. In the context of our model, we can then endogenize the bettors' participation decision, derive comparative statics results for changes in the track take, and develop implications for the favorite-longshot bias.

The paper proceeds as follows. Section 2 formulates the model. Section 3 analyzes how the recreational value of betting affects the nature of the equilibrium. Section 4 characterizes the equilibrium when the recreational value is so large that no bettors abstain. In this case, the favorite-longshot bias arises when the amount of information relative to noise is high. Section 5 considers the equilibrium when some bettors rationally choose to abstain. We show that a lower recreational value from betting leads to reduced participation by poorly informed bettors and to a greater favorite-longshot bias. Section 6 compares our theory with alternative explanations for the favorite-longshot bias and discusses the empirical evidence in light of our results. Section 7 concludes.

2 Model

Bets can be placed on the realization of a binary random variable. The two possible outcomes are denoted by $x = -1, 1$. In the case of a horse race, there are two horses and the outcome is the identity of the winning horse. For a lottery, the outcome is one number drawn from two. The case of two outcomes is the simplest with which to illustrate the logic of our information-based explanation of the favorite-longshot bias, but our results apply to more realistic settings with more than two outcomes.[10]

There are N bettors with a common *prior belief*, $q = \Pr(x = 1)$, possibly formed after the observation of a common signal. In addition, each bettor $i \in \{1, ..., N\}$ is privately endowed with *signal* s_i. After receiving the signal, each of the N bettors decides whether to abstain or to bet on either outcome $x = -1$ or $x = 1$. Bettors are ex-ante identical and differ only by the realization of their private signal.

Bettors are assumed to be risk neutral and to maximize the expected monetary return, plus a fixed recreational utility value from betting, denoted by u. This recreational value

[10] See Section 7 for an informal discussion on the effect of an increase in the number of outcomes on the level of noise.

generates a demand for betting. Without the addition of this value, there would be no betting in equilibrium under risk neutrality, as predicted by the no trade theorem (Milgrom and Stokey, 1982).[11]

All money that is bet on the two outcomes is placed in a common pool, from which a fraction, $\tau \in [0, 1)$, is subtracted for taxes and other expenses. The remaining money is returned to those who placed bets on the winning outcome, x. We assume that no payment is returned to the bettors if no bets were placed on the winning outcome.[12] Let b_x denote the total amount bet on outcome x. If x is the winner, then every unit bet on outcome x receives the monetary payoff $(1 - \tau)(b_x + b_{-x})/b_x$.

The private signals observed by the bettors are assumed to be identically and independently distributed conditional on state x. Since there are only two states, the likelihood ratio $f(s|x = 1)/f(s|x = -1)$ is strictly increasing in s. Upon observation of signal s, the prior belief q is updated according to Bayes' rule into the *posterior belief*, $p = \Pr(x = 1|s)$. Given the one-to-one mapping between the signal realization and the posterior belief, it is convenient to place assumptions on the conditional distributions of the posterior belief (see Appendix A of Smith and Sørensen, 2000).

The posterior belief, p, is assumed to be distributed according to the continuous cumulative distribution G with density g on $[0, 1]$.[13] By the law of iterated expectations, the prior must satisfy $q = E[p] = \int_0^1 pg(p)\, dp$. The conditional densities of the posterior belief can be derived as $g(p|x = 1) = pg(p)/q$ and $g(p|x = -1) = (1 - p)g(p)/(1 - q)$. Note that these relations are necessary, since Bayes' rule yields $p = qg(p|1)/g(p)$ and $1 - p = (1 - q)g(p| - 1)/g(p)$. We have $g(p|1)/g(p| - 1) = (p/(1 - p))((1 - q)/q)$, reflecting that high beliefs in outcome 1 are more frequent when outcome 1 is true. This monotonicity of the likelihood ratio implies that $G(p|x = 1)$ is higher than $G(p|x = -1)$

[11] Instead of giving to each bettor a constant recreational utility of betting, we could have equivalently introduced two separate populations of bettors: outsiders (motivated by recreation) and insiders (motivated by expected monetary return). Following this alternative approach, Ottaviani and Sørensen (2004) obtain a closed-form solution for the equilibrium resulting when there are is an *infinite* number of informed bettors.

Since noise disappears when the number of bettors tends to infinity, it is essential for the purpose of our analysis to study the general case with a finite number of bettors. The model with a representative bettor presented in this paper has the advantage of leading to a closed-form solution of the equilibrium with a *finite* number of bettors.

[12] Our results hold with alternative rules on how the pool is split when no one bets on the winning horse. For example, the pool could be divided equally among all losers.

[13] In the presence of discontinuities in the posterior belief distribution, the only symmetric equilibria might involve mixed strategies. Our results can be extended to allow for these discontinuities.

in the first-order stochastic dominance (FOSD) order, i.e. $G(p|1) - G(p|-1) < 0$ for all p such that $0 < G(p) < 1$.

In the course of the analysis, we impose additional restrictions on the distribution of the posterior beliefs. The posterior belief distribution is said to be *symmetric* if $G(p|1) = 1 - G(1-p|-1)$ for all $p \in [0,1]$, i.e., if the chance of posterior p conditional on state $x = 1$ is equal to the chance of posterior $1-p$ conditional on state $x = -1$. The signal distribution is said to be *unbounded* if $0 < G(p) < 1$ for all $p \in (0,1)$, so that there is strictly positive probability that the posterior beliefs are close to certainty.

Example. Our results are conveniently illustrated by the class of conditional signal densities $f(s|x=1) = (1+a)s^a$ and $f(s|x=-1) = (1+a)(1-s)^a$ for $s \in [0,1]$, with corresponding distribution functions $F(s|x=1) = s^{a+1}$ and $F(s|x=-1) = 1-(1-s)^{a+1}$, parametrized by the quality of information, $a > 0$. The posterior odds ratio is

$$\frac{p}{1-p} = \frac{q}{1-q}\frac{f(s|1)}{f(s|-1)} = \frac{q}{1-q}\left(\frac{s}{1-s}\right)^a. \tag{1}$$

In this class of examples, posterior beliefs are clearly unbounded and symmetric. The signal becomes uninformative when $a \to 0$ and perfectly informative when $a \to \infty$.

3 Equilibrium

After observing their private signals, bettors simultaneously decide which of three actions to take: abstain, bet on -1, or bet on 1. This amounts to employing a strategy that maps every signal into one of the three actions. In equilibrium, every bettor correctly conjectures the strategies employed by the opponents, and plays the best response to this conjecture. We restrict our attention to symmetric Bayes-Nash equilibria, in which all bettors use the same strategy.[14] Below, we characterize the symmetric equilibrium and show that it is unique.

Consider the problem of a bettor with posterior belief p. The payoff from abstention is 0. The expected payoff of a unit bet on outcome $y \in \{-1,1\}$ is $U(y|p) = pW(y|1) + (1-p)W(y|-1) - 1 + u$, where $W(y|x)$ is the expected payoff of a bet on outcome y conditional on state x being realized. Note that $U(1|p) = pW(1|1) - 1 + u$ since

[14] When N is not too large, there may also be asymmetric equilibria, as noted by Koessler and Ziegelmeyer (2002).

$W(y|-y) = 0$, because a bet on outcome y pays out nothing in state $-y$. Likewise, $U(-1|p) = (1-p)W(-1|-1) - 1 + u$.

Clearly $\partial U(1|p)/\partial p = W(1|1) > 0$ and $\partial U(-1|p)/\partial p = -W(-1|-1) < 0$. The best response of each individual bettor therefore has a cutoff characterization. There exist threshold posterior beliefs, $\hat{p}_{-1}, \hat{p}_1 \in [0,1]$ with $\hat{p}_{-1} \leq \hat{p}_1$, such that if $p < \hat{p}_{-1}$ it is optimal to bet on $x = -1$; if $p \in [\hat{p}_{-1}, \hat{p}_1)$, it is optimal to abstain; and if $p \geq \hat{p}_1$ it is optimal to bet on $x = 1$. (With continuous beliefs, the tie-breaking rule is clearly inessential.)

Suppose that all opponents adopt the cutoff strategy defined by the thresholds $\hat{p}_{-1}$ and $\hat{p}_1$. Because the others' signals and corresponding bets are uncertain, the conditional winning payoff is also random. For instance, $G(\hat{p}_{-1}|1)$ is the probability that any opponent bets on outcome -1 in state 1. We have:

Lemma 1 *When all opponents use the thresholds $\hat{p}_{-1} \leq \hat{p}_1$, then the expected return on a successful bet is given by*

$$W(1|1) = \begin{cases} (1-\tau)\frac{1-G(\hat{p}_1|1)+G(\hat{p}_{-1}|1)\left[1-G(\hat{p}_1|1)^{N-1}\right]}{1-G(\hat{p}_1|1)} & \text{if } G(\hat{p}_1|1) < 1, \\ (1-\tau)\left[1+(N-1)G(\hat{p}_{-1}|1)\right] & \text{if } G(\hat{p}_1|1) = 1, \end{cases} \quad (2)$$

$$W(-1|-1) = \begin{cases} (1-\tau)\frac{G(\hat{p}_{-1}|-1)+[1-G(\hat{p}_1|-1)]\left[1-[1-G(\hat{p}_{-1}|-1)]^{N-1}\right]}{G(\hat{p}_{-1}|-1)} & \text{if } G(\hat{p}_{-1}|-1)>0, \\ (1-\tau)\left[1+(N-1)\left[1-G(\hat{p}_1|-1)\right]\right] & \text{if } G(\hat{p}_{-1}|-1)=0. \end{cases} \quad (3)$$

Proof. See the Appendix. □

Depending on the model's parameters, the symmetric equilibrium can take one of three forms. First, all bettors participate in betting. Second, there is a non-empty set of intermediate beliefs at which bettors refrain from betting. Third, no bets are placed regardless of the posterior belief, i.e., $\hat{p}_{-1} = 0$ and $\hat{p}_1 = 1$. The equilibrium regime crucially depends on the level of the recreational value, u:[15]

Proposition 1 *Assume that the posterior belief is unbounded. There exists a uniquely defined critical value $u^* \in (\tau, 1)$ such that in any symmetric equilibrium:*

1. *if $u \geq u^*$, all bettors bet actively,*

[15] Proposition 1 can be extended to the case in which the posterior belief is not unbounded, i.e. $G(p) = 0$ or $G(p) = 1$ for some interior $p \in (0,1)$. With bounded beliefs, let $\bar{p} = \max\left\{1 - \max_{G(p)=0} p, \min_{G(p)=1} p\right\}$ denote the most extreme belief that any bettor attaches to any of the two outcomes. Let $\bar{u} = 1 - \bar{p}(1-\tau)$. Then $u^* > \bar{u} \geq \tau$ and cases 1 and 3 of Proposition 1 hold as stated. The claim in case 2 is valid under the more stringent condition that $u \in (\bar{u}, u^*)$. In case $u \in (\tau, \bar{u}]$, the proof establishes the weaker claim that there exists *an* equilibrium in which every bettor abstains.

2. if $u \in (\tau, u^*)$, some, but not all, bettors abstain,

3. if $u \leq \tau$, all bettors abstain.

Proof. See the Appendix. □

When the recreational value is high enough or, equivalently, the takeout rate is low enough, even bettors without strong posterior beliefs bet on one of the two horse — this is the "no abstention" regime analyzed in Section 4. With low recreational value or high takeout rate, bettors without strong private beliefs prefer not to place any bet — the "abstention" regime analyzed in Section 5. Additional increases in the takeout further reduce participation, until when the market completely breaks down according to the logic of the no trade theorem.[16]

4 No Abstention

Consider the first regime identified in Proposition 1. Since the overall amount of information present in the market is constant, this case allows us to isolate the informational drivers of the favorite-longshot bias.

In this case, the symmetric equilibrium strategy is characterized by the single threshold $\hat{p} \equiv \hat{p}_{-1} = \hat{p}_1 \in [0, 1]$ such that for $p < \hat{p}$ it is optimal to bet on $x = -1$, and for $p \geq \hat{p}$ it is optimal to bet on $x = 1$.

Proposition 2 *Suppose that $u \geq u^*$. With N bettors, there exists a unique symmetric equilibrium in which all bettors with $p \geq \hat{p}^{(N)}$ bet on outcome 1, and all bettors with $p < \hat{p}^{(N)}$ bet on outcome -1, where $\hat{p}^{(N)} \in (0, 1)$. As the number of bettors N tends to infinity, the cutoff belief $\hat{p}^{(N)}$ tends to the unique solution to*

$$\frac{p}{1-p} = \frac{1 - G(p|1)}{G(p|-1)}. \tag{4}$$

Proof. See the Appendix. □

To understand the simple intuition for the equilibrium condition (4), consider the expected payoff achieved by a bettor with posterior p who bets on outcome 1. If winning, the bettor shares the pool with all those who also picked 1. In the limit case with a large

[16] Bettors with beliefs $p = 1$ and $p = 0$ always participate, but these beliefs have probability zero.

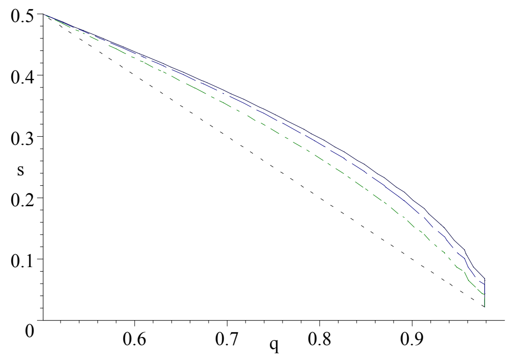

Figure 1: The equilibrium cutoff signal $\hat{s}_N$ in the linear signal example is plotted against the prior belief $q \in [1/2, 1]$ for $N = 1, 2, 3, 4, 100$, in a progressively thicker shade. The downward sloping diagonal ($\hat{s}_1 = q$) corresponds to the optimal rule ($\hat{p}_1 = 1/2$) for a single bettor. As the number of players increases, the cutoff signal increases and converges to the limit $\hat{s}(q)$.

number of players, the law of large numbers implies that there is no uncertainty in the conditional distribution of the opponents. Since all bettors use the same cutoff strategy $\hat{p}$, the fraction of bettors who picked 1 in state $x = 1$ is $1 - G(\hat{p}|1)$. The expected payoff from a bet on outcome 1 is then $p/(1 - G(\hat{p}|1))$. Similarly, the expected payoff from -1 is $(1-p)/G(\hat{p}|-1)$. The payoff of an indifferent bettor satisfies equation (4). The proof of the Proposition also provides a closed-form solution for the symmetric equilibrium resulting with a finite number of bettors.

Example. To illustrate how the equilibrium depends on the number of bettors, N, consider the example introduced at the end of Section 2, for the special case with $a = 1$. This corresponds to a binary signal with uniformly distributed precision. Through the translation of signals into posterior beliefs (1), the cutoff posterior belief defining the equilibrium corresponds to a cutoff private signal $\hat{s}^{(N)}$. By the equilibrium condition (10) reported in the Appendix, the cutoff private signal is

$$\frac{q}{1-q} = \frac{1-\hat{s}^{(N)}}{\hat{s}^{(N)}} \frac{1-F\left(\hat{s}^{(N)}|1\right)}{F\left(\hat{s}^{(N)}|-1\right)} \frac{1-\left(1-F\left(\hat{s}^{(N)}|-1\right)\right)^N}{1-\left(F\left(\hat{s}^{(N)}|1\right)\right)^N}.$$

Figure 1 plots the equilibrium signal cutoff, $\hat{s}^{(N)}(q)$, for different values of N, as a function of the prior $q > 1/2$. A single bettor ($N = 1$) optimally bets on the horse that is more likely to win according to the posterior belief — in this sense, betting is "truthful" in the absence of competition among bettors. With $N > 1$ bettors instead, equilibrium betting is biased in favor of the ex-ante less likely outcome, $y = -1$. To see this, note the cutoff signal of the bettor indifferent between a bet on 1 and a bet on -1 satisfies $\hat{s}^{(N)}(q) > 1 - q$, corresponding to a posterior belief $\hat{p}^{(N)}(q) > 1/2$. Thus, bettors with beliefs in the interval $\left[1/2, \hat{p}^{(N)}(q)\right]$ bet on outcome $y = -1$, even though the truthful strategy prescribes to bet on outcome $y = 1$.

Intuitively, this is an implication of the winner's curse. Conditional on state x, the opponents are more likely to receive information in favor of state x, and thus to bet on outcome x. This induces a positive correlation between the true state and the number of bets placed on it, thereby depressing the return on the winner. Since outcome $x = 1$ is ex-ante more likely to occur, it is also more likely to suffer from this unfavorable adjustment in payoff created by the parimutuel scheme. This creates a contrarian incentive to bet on the ex-ante less likely outcome, $y = -1$. When $N > 1$, truthful betting for the outcome which is most likely to occur according to the posterior belief is only an equilibrium if the two outcomes are ex-ante equally likely, $q = 1/2$.[17]

4.1 Favorite-Longshot Bias

The symmetric equilibrium strategy with N bettors has cutoff posterior belief $\hat{p}^{(N)}$. Since signals are random, bets on a given outcome follow a binomial distribution. For any $k = 0, \ldots, N$,

$$\Pr\left(k \text{ bet } 1 \text{ and } N-k \text{ bet } -1|x\right) = \binom{N}{k}\left[1 - G\left(\hat{p}^{(N)}|x\right)\right]^{k} G\left(\hat{p}^{(N)}|x\right)^{N-k}.$$

Since higher private beliefs are more frequent when the state is higher, $G\left(\hat{p}^{(N)}|-1\right) > G\left(\hat{p}^{(N)}|1\right)$, the distribution of bets is FOSD higher when $x = 1$ than when $x = -1$.

When k bets are placed on outcome 1, a bet on outcome 1 pays out $1+\rho$ when winning, where $\rho = [N(1-\tau) - k]/k$ is the *market odds* ratio for outcome 1. These market odds are such that the total payments to winners are $k(1+\rho) = (1-\tau)N$, equal to the total amount bet, net of the track take. This verifies that in the parimutuel system the budget is

[17] We conclude that truthful betting (for the most likely outcome according to the posterior belief) is not an equilibrium in parimutuel betting with asymmetric information, other than in the degenerate game with only one player and in the special case with symmetric prior. Furthermore, the equilibrium in the limit game with an infinite number of players is not truthful. Note the contrast with the positive results on truthtelling obtained by Cabrales, Calvó-Armengol and Jackson (2003) for insurance mutual schemes that use proportional payment/reimbursement rules with a parimutuel structure.

automatically balanced regardless of the state. The implied *market probability* for outcome 1 is $\pi = k/N = (1-\tau)/(1+\rho)$, equal to the fraction of money bet on this outcome.

We are now ready for the key step of our analysis. A rational observer of the final bets distribution would instead update to the *posterior odds* ratio by using Bayes' rule,

$$\beta = \frac{1-q}{q}\frac{\Pr(k \text{ bet } 1|-1)}{\Pr(k \text{ bet } 1|1)} = \frac{1-q}{q}\left(\frac{1-G\left(\hat{p}^{(N)}|-1\right)}{1-G\left(\hat{p}^{(N)}|1\right)}\right)^{k}\left(\frac{G\left(\hat{p}^{(N)}|-1\right)}{G\left(\hat{p}^{(N)}|1\right)}\right)^{N-k}.$$

In general, this posterior odds ratio is different from the market odds ratio. The posterior probability, $\Pr(x=1|k \text{ bet } 1)$, is derived from the posterior odds as $1/(1+\beta)$. When exactly k bets are placed on 1, the empirical frequency of state 1 should be approximately equal to $1/(1+\beta)$ by the law of large numbers. The posterior odds incorporate the information revealed in the betting distribution and adjust for noise, and thus are the correct estimators of the empirical odds.

The favorite-longshot bias identified in the horse-betting data suggests that the difference between the market odds ratio and the posterior odds ratio is systematic: when the market odds ratio, ρ, is large, it is smaller than the corresponding posterior odds ratio, β. Thus, a longshot is less likely to win than is suggested by the market odds.

Our model allows us to uncover a systematic relation between posterior and market odds depending on the interplay between the amount of noise and information contained in the bettors' signal. To appreciate the role played by *noise*, note that market odds can range from zero to infinity, depending on the realization of the signals. For example, if most bettors happen to draw a low signal, then the market odds of outcome 1 will be very long. If the signals contain little information, then the posterior odds are close to the prior odds even if the market odds are extreme. In this case, deviations of the market odds from the prior odds are largely due to the randomness contained in the signal, so that the reverse of the favorite-longshot bias is present (i.e., the market odds are more extreme than the posterior odds).

As the number of bettors increases, the realized market odds contain more and more *information*, so that the posterior odds are more and more extreme for any market odds different from 1. We can therefore establish:

Proposition 3 *Let* $\pi^* \in (0,1)$ *be defined by*

$$\pi^* = \frac{\log\frac{1-G(\hat{p}|1)}{1-G(\hat{p}|-1)}}{\log\frac{1-G(\hat{p}|1)}{1-G(\hat{p}|-1)} + \log\frac{G(\hat{p}|-1)}{G(\hat{p}|1)}} \quad (5)$$

where $\hat{p}$ is the unique solution to the limit equilibrium condition (4). Take as given any market implied probability $\pi \in (0,1)$ for outcome $x=1$. As the number of bettors tends to infinity, a longshot's market probability $\pi < \pi^*$ (respectively a favorite's $\pi > \pi^*$) is strictly greater (respectively smaller) than the associated posterior probability $1/(1+\beta)$.

Proof. Let $\pi < \pi^*$ be given. The desired inequality is

$$\frac{1-\pi}{\pi} < \frac{1-q}{q}\left(\frac{1-G\left(\hat{p}^{(N)}|-1\right)}{1-G\left(\hat{p}^{(N)}|1\right)}\right)^k \left(\frac{G\left(\hat{p}^{(N)}|-1\right)}{G\left(\hat{p}^{(N)}|1\right)}\right)^{N-k}.$$

Take the natural logarithm, use $k/N=\pi$, and re-arrange to arrive at the inequality

$$\frac{1}{N}\log\frac{(1-\pi)\,q}{\pi\,(1-q)} < \pi\log\frac{1-G\left(\hat{p}^{(N)}|-1\right)}{1-G\left(\hat{p}^{(N)}|1\right)} + (1-\pi)\log\frac{G\left(\hat{p}^{(N)}|-1\right)}{G\left(\hat{p}^{(N)}|1\right)}.$$

The left-hand side tends to zero as N tends to infinity. The right-hand side tends to a positive limit, precisely since $\pi < \pi^*$. □

Long (short) market odds are shorter (longer) than the posterior odds, in accordance with the favorite-longshot bias. The turning point, π^*, is a function of how much more informative is the observation that the private belief exceeds $\hat{p}$ as compared to the observation that it falls short of $\hat{p}$. By definition (5), the Bayesian inference on x, based on the observation of k bettors with beliefs below $\hat{p}$ and $N-k$ bettors with beliefs above $\hat{p}$, is exactly neutral when $k/N=\pi^*$.

In Proposition 3, the realized market probability $\pi \in (0,1)$ is held constant as the number of bettors N tends to infinity. Since the probability distribution of π is affected by changes in N, it is natural to wonder whether this probability distribution of realized market probabilities can become so extreme that it is irrelevant to look at fixed non-extreme realizations. This is not the case. By the strong law of large numbers, in the limiting case as N goes to infinity, outcome 1 receives the fraction $1-G(\hat{p}|x)$ of total bets. The noise vanishes and the market probability π tends almost surely to the limit $G(\hat{p}|x)$ in state x. Thus, the observation of the market bets eventually reveals the true outcome, so that the posterior odds ratio becomes more extreme (either diverging to infinity or converging to zero). In the limit, there is probability one that the realized market odds are less extreme than the posterior odds. This fact supports the favorite-longshot bias as the theoretical prediction of our simple model for the case with many informed bettors.

In the special case with symmetric prior $q = 1/2$ and symmetric signal distribution (implying that $G(1/2|1) = 1 - G(1/2|-1)$), the symmetric equilibrium has $\hat{p}^{(N)} = \hat{p} = 1/2$ for all N. The turning point then is $\pi^* = 1/2$. In this simplified case, we can further illuminate the fact that the favorite-longshot bias arises when the realized bets contain more information than noise.

Proposition 4 *Assume $q = 1/2$ and that the belief distribution is symmetric. Take as given any market implied probability $\pi \in (0,1)$ for outcome $x = 1$. If either the signal informativeness $G(1/2|-1)/G(1/2|1)$ or the number of bettors, N, are sufficiently large, then a longshot's market probability $\pi < 1/2$ (respectively, a favorite's $\pi > 1/2$) is strictly greater (respectively, smaller) than the associated posterior probability $1/(1+\beta)$.*

Proof. Let $\pi < 1/2$ be given. The desired inequality is

$$\frac{1-\pi}{\pi} < \left(\frac{1-G(1/2|-1)}{1-G(1/2|1)}\right)^k \left(\frac{G(1/2|-1)}{G(1/2|1)}\right)^{N-k} = \left(\frac{G(1/2|-1)}{G(1/2|1)}\right)^{N-2k}.$$

Take the natural logarithm, use $k/N = \pi$, and re-arrange to arrive at the inequality

$$\frac{1}{1-2\pi} \log \frac{1-\pi}{\pi} < N \log \frac{G(1/2|-1)}{G(1/2|1)}. \tag{6}$$

Since $\pi < 1/2$ and $G(1/2|-1) > G(1/2|1)$, all terms are positive. The right-hand side of (6) tends to infinity when the informativeness ratio $G(1/2|-1)/G(1/2|1)$, or the number of bettors N, tends to infinity. The inequality is reversed if $\pi > 1/2$. □

As illustrated by inequality (6), the favorite-longshot bias arises when there are many bettors (i.e., N is large) or when all bettors are well informed (i.e., $G(1/2|-1)/G(1/2|1)$ is large). Since the left-hand side of (6) is a strictly increasing function of $\pi < 1/2$, it is harder to satisfy the inequality at market probabilities closer to zero. This is natural, because the bettors must reveal more information through their bets in order for the posterior probability to become very extreme. At the easiest point, $\pi = 1/2$, the condition is $2 < N \log[G(1/2|-1)/G(1/2|1)]$. Note that the favorite-longshot bias can result even in the presence of a single bettor, provided this bettor is sufficiently well informed.

The more bettors there are, the greater is the return for any fixed π, because of the greater favorite-longshot bias. This does not take into account the fact that the number of bettors also affects the probability distribution over π. The law of large numbers implies

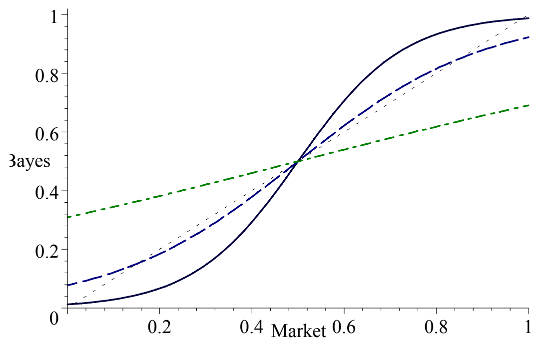

Figure 2: In the example with fixed number $N = 4$ of insiders, the inferred Bayesian probability is plotted against the market implied probability $\pi \in [0, 1]$ when the chance of betting correctly is $\eta_1 = 11/20, 13/20, 3/4$, represented in progressively thicker shade. The diagonal corresponds to the absence of a favorite-longshot bias.

that a greater number of bettors will generate a less random π. Ottaviani and Sørensen (2004) analyze this limit by considering a continuum of informed bettors. In the limit, the implied market probability in state x is deterministic and fully reveals the state. In a related comparative statics exercise, Ottaviani and Sørensen (2004) establish that the realized market odds are more extreme when there are more informed bettors, thus resulting in a less pronounced favorite-longshot bias.

Example. Consider the example introduced at the end of Section 2 and assume that the prior is fair, $q = 1/2$. Symmetry of the prior and symmetry of the signal distributions in this example, $F(s|1) = 1 - F(1 - s| - 1)$, imply that the conditional distributions of the posterior are also symmetric, $G(p|1) = 1 - G(1 - p| - 1)$. As noted before Proposition 4, the equilibrium threshold belief is then $\hat{p}^{(N)} = \hat{p} = 1/2$, corresponding to the signal threshold $\hat{s}^{(N)} = \hat{s} = 1/2$.

First, we illustrate how the extent of the favorite-longshot bias depends on the ratio of information to noise, as described in Proposition 4. Note that the greater is a, the more informative is the signal about x in the sense of Blackwell. An increase in a yields a mean-preserving spread of the signal distribution $F(s)$, and also serves, through (1), to make every signal result in a posterior belief further away from the prior $q = 1/2$. Conditional on state x, individual bets are independent and correct with chance $\eta_1 \equiv 1 - F(1/2|1) = 1 - 2^{-1-a}$. There is a one-to-one mapping between $a \in (0, \infty)$ and $\eta_1 \in (1/2, 1)$. The

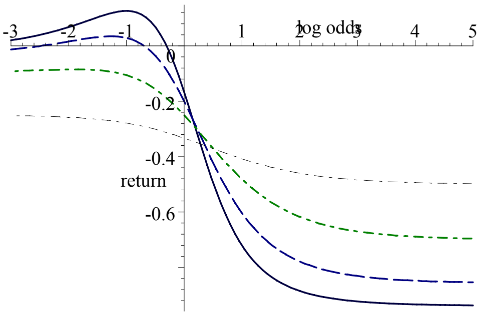

Figure 3: The expected return on an extra bet on outcome 1 in the signal example with $a = 1$ is plotted against the natural logarithm of the market odds ratio. The prior is always set at $q = 1/2$. The curves correspond to $N = 1, 2, 3, 4$, drawn in a progressively thicker shade.

market probability π for outcome 1 is distributed with density $\binom{N}{N\pi}\eta_1^{N\pi}(1-\eta_1)^{N(1-\pi)}$ conditional on state 1. This market probability π is associated with posterior probability

$$\beta = \frac{\eta_1^{N\pi}(1-\eta_1)^{N(1-\pi)}}{\eta_1^{N\pi}(1-\eta_1)^{N(1-\pi)} + \eta_1^{N(1-\pi)}(1-\eta_1)^{N\pi}}.$$

Figure 2 shows how the favorite-longshot bias arises when the signal is sufficiently informative. When the signal is very noisy ($\eta_1 = 11/20$) there is a reverse favorite-longshot bias. As the signal informativeness rises above a critical value, the favorite-longshot bias arises in an ever larger region around $\pi = 1/2$. In the limit, as the signal becomes perfectly revealing, the curve for the posterior probability becomes the step function that rises from 0 to 1 at $\pi = 1/2$.

Second, we illustrate how the favorite-longshot bias depends on the number of bettors N. This could be done in a plot very similar to Figure 2. To compare our theoretical results to the empirical findings, we depict in Figure 3 the expected payoff to an extra bet on outcome 1 as it varies with the market odds. Here $\tau = 0$, $a = 1$ and $\eta_1 = 3/4$, so the implied posterior probability is $3^{2k}/\left(3^N + 3^{2k}\right)$ when k bettors have bet on outcome 1. The expected payoff from an extra bet on outcome 1 is $\left[(N+1)\,3^{2k}\right]/\left[\left(3^N + 3^{2k}\right)(k+1)\right] - 1$. The curves generated in this stylized example have similar features to the one reported in Thaler and Ziemba's (1988) Figure 1, plotting the empirical expected return for horses with different market odds.

4.2 Extension: Common Error

We now extend the model to account for the possibility that there is some residual uncertainty. For instance, even after pooling the private information of a large number of bettors, the actual outcome of the race is still uncertain. We show that our main result also holds when there is a common error component in the information held by the bettors.

We modify the model by assuming that the outcome is z, while the state x is a noisy binary signal of the outcome z. The private signal is informative about x, but, conditional on x, its distribution is independent of z. In the limit with infinite N, the symmetric equilibrium features the favorite-longshot bias:

Proposition 5 *Assume that the belief distribution is symmetric and that the state x is a symmetric binary signal of the outcome z, with $\Pr(x = 1|z = 1) = \Pr(x = -1|z = -1) \equiv \sigma > 1/2$. In the limit as $N \to \infty$, in the symmetric equilibrium the market odds ratio is less extreme than the associated posterior odds ratio.*

Proof. See the Appendix. □

5 Abstention

We now turn to the second case of Proposition 1, in which bettors with intermediate private beliefs abstain. As the recreational value of betting decreases or, equivalently, the level of the track take increases, bettors with weak signals prefer not to participate because their expected loss from betting is not compensated by the recreational value. As a result, participation decreases and, along with it, the overall amount of information that is present in the market is also reduced. However, we establish below that abstention increases the amount of information relative to noise that is contained in the equilibrium bets. Thus, abstention strengthens the favorite-longshot bias.

If $u \in (\tau, u^*)$, the indifference conditions that determine the equilibrium are

$$\hat{p}_1 W(1|1) = 1 - u, \tag{7}$$

$$(1 - \hat{p}_{-1}) W(-1| - 1) = 1 - u. \tag{8}$$

Proposition 6 *When $u \in (\tau, u^*)$, there exists a unique symmetric equilibrium, defined by the unique solution $0 < \hat{p}_{-1} < \hat{p}_1 < 1$ to equations (7) and (8). When u increases or τ*

decreases, participation by bettors increases: $\hat{p}_{-1}$ *rises and* $\hat{p}_1$ *falls. In the limit as* $u \to u^*$*, we have* $\hat{p}_1 - \hat{p}_{-1} \to 0$*. In the limit as* $u \to \tau$*, we have* $\hat{p}_{-1} \to 0$ *and* $\hat{p}_1 \to 1$*.*

Proof. See the Appendix. □

Proposition 6 establishes that the proportion of active bettors increases with the recreational value of betting. The willingness to bet increases in $u \in (\tau, u^*)$ continuously and monotonically, from full abstention to no abstention.

Intuitively, a greater u results in a reduction in the favorite-longshot bias. When u is greater, the abstention region is smaller, as less informed bettors join the pool of bettors. The bets that are placed then contain *relatively* more noise and less information. According to the logic of Proposition 4, less informative bets reduce the favorite-longshot bias.

Proposition 7 *Assume that the belief distribution is symmetric and unbounded. Take as given any bet realization with total amounts* $b_1, b_{-1} > 0$ *placed on the two outcomes. If* $u \in (\tau, u^*)$ *is sufficiently close to* τ*, a longshot's market probability* $\pi = b_1/(b_1 + b_{-1}) < 1/2$ *(respectively a favorite's* $\pi > 1/2$*) is strictly greater (respectively smaller) than the associated posterior probability* $1/(1+\beta)$*.*

Proof. See the Appendix. □

Note that if there is a reverse favorite-longshot bias at u^* (as shown possible in Section 4.1), it will persist as u falls below the critical value u^*. This follows from the fact that the thresholds change continuously. According to Proposition 7, the reverse bias is overturned as u approaches τ.

Example. We now illustrate our findings for the signal distribution presented above with $a = 1$ and symmetric prior $q = 1/2$. For this symmetric example, we have $W(1|1) = W(-1|-1)$ and the solution to equations (7) and (8) satisfies $\hat{p}_1 = 1 - \hat{p}_{-1}$. Either of these equations yields

$$\frac{1-u}{1-\tau} = \hat{p}_1 \frac{1 - G(\hat{p}_1|1) + G(\hat{p}_{-1}|1)\left[1 - G(\hat{p}_1|1)^{N-1}\right]}{1 - G(\hat{p}_1|1)} = \hat{p}_1 \frac{2 - (1-\hat{p}_1)\,\hat{p}_1^{2(N-1)}}{1+\hat{p}_1}.$$

The critical $u^{*(N)}$ corresponds to the cutoff belief $\hat{p}_1 = 1/2$ at which $\left(1 - u^{*(N)}\right)/(1-\tau) = \left(2 - 2^{1-2N}\right)/3$. Note that as $N \to \infty$, we obtain the limit equation $(1-u)/(1-\tau) =$

$2\hat{p}_1 / (1 + \hat{p}_1)$, solved by $\hat{p}_1 = (1 - u) / (1 + u - 2\tau)$. For any N, the solution for $\hat{p}_1$ rises from 1/2 to 1 as $(1 - u) / (1 - \tau)$ increases from $\left(1 - u^{*(N)}\right) / (1 - \tau)$ to 1 (and so u decreases from $u^{*(N)}$ to τ). Moreover, abstention becomes more attractive as the number of bettors increases.

6 Related Literature and Empirical Implications

This paper has explored a new model of parimutuel betting with private information and used it to develop an information-based explanation for the favorite-longshot bias. In this section, we present the alternative explanations for this phenomenon that have been proposed in the literature, and then confront the implications of the different theories with the existing empirical evidence.

The most notable alternative theories are the following:

- Griffith (1949) suggested that the bias might be due to the tendency of individual decision makers to *overestimate* small probability events.
- Weitzman (1965) hypothesized that individual bettors are *risk loving*, and thus are willing to accept a lower expected payoff when betting on riskier longshots.[18]
- Isaacs (1953) noted that an informed *monopolist bettor* who can place multiple bets would not set the expected return of its marginal bet to zero, since this would destroy the return on the inframarginal bets.
- Hurley and McDonough (1995) and Terrell and Farmer (1996) showed that the favorite-longshot bias would result also when there are many competing bettors because the amount of *arbitrage* is *limited* by the track take.

While each of these explanations for the favorite-longshot bias has merit, the information-based explanation developed in this paper has a number of advantages.

First, our theory builds on the realistic assumption that the differences in beliefs among bettors are generated by *private information* (see e.g., Crafts 1985). Only by modeling

[18] See also Rosett (1965), Quandt (1976) and Ali (1977) on the risk-loving explanation. According to Golec and Tamarkin (1998), the bias is compatible with preferences for skewness rather than risk. Jullien and Salanié (2000) use data from fixed-odds markets to argue in favor of non-expected utility models.

explicitly the informational determinants of beliefs, we can address the natural question of information aggregation.

Second, our information-based theory offers a parsimonious explanation for the favorite-longshot bias and its *reverse*. None of the alternative theories can account for the reverse favorite-longshot bias. Our theory also predicts that the favorite-longshot bias is lower or reversed when the number of bettors is relatively low (relative to the number of states), as verified empirically by Sobel and Raines (2003) and Gramm and Owens (2005).[19]

Third, theories based on private information are able to explain the occurrence of the favorite-longshot bias in both *fixed-odds* and parimutuel markets, as well as the lower level of bias observed in parimutuel markets, as documented by Bruce and Johnson (2001) among others. Intuitively, in the parimutuel system the payoff conditional on outcomes that attracted few bets is high because the parimutuel odds are adjusted automatically to balance the budget. This favorable effect on the conditional payoff counteracts the unfavorable updating of the probability that this outcome occurs. See Ottaviani and Sørensen (2005) for a detailed comparison of the outcomes of parimutuel and competitive fixed-odds betting markets within a unified model.

Fourth, our theory is compatible with the *reduced* favorite-longshot bias in "exotic" bets (such as exacta bets, which specify the winner and the runner up in a race), as observed by Asch and Quandt (1988). As also stressed by Snowberg and Wolfers (2005), risk-loving explanations do not seem compatible with arbitrage across exacta and win pools. Asch and Quandt concluded in favor of private information because the payoffs on winners tend to be more depressed in the exacta than in the win pool.[20]

Finally, our theory is compatible with the fact that *late bets* tend to contain more information about the horses' finishing order than earlier bets, as observed by Asch, Malkiel and Quandt (1982). Ottaviani and Sørensen (2004) have demonstrated that late informed betting results in equilibrium when (many small) bettors are allowed to optimally time their bets.

[19] The preponderance of noise might also account for some of Metrick's (1996) findings in NCAA basketball tournament betting pools.

[20] They observed that the market probability recovered from the win pool overestimates market probability on the exacta pool, by a much larger margin for winning than for losing horses.

7 Conclusion

We have built a simple model of parimutuel betting with private information. The sign and extent of the favorite-longshot bias depend on the amount of information relative to noise present in the market. When there is little private information, posterior odds are close to prior odds, even with extreme market odds. In this case, deviations of market odds from prior odds are mostly due to the noise contained in the signal. Given the parimutuel payoff structure, this introduces a systematic bias in expected returns. The market odds tend to be more extreme than the posterior odds, resulting in a reverse favorite-longshot bias.

As the number of bettors increases, the realized market odds contain more information and less noise. For any fixed market odds, the posterior odds then are more extreme, increasing the extent of the favorite-longshot bias. Note that the favorite-longshot bias always arises with a large number of bettors, provided that they have *some* private information. This is confirmed by Ottaviani and Sørensen (2004) in a model with a continuum of privately informed bettors. In that setting, there is no noise, so that the favorite-longshot bias always results.

Our theory can be tested by exploiting the variation across betting environments. The amount of private information tends to vary consistently depending on the nature and prominence of the underlying event. Similarly, the amount of noise present depends on the number of outcomes, as well as on the observability of past bets. For example, relative to the number of bettors, the number of outcomes is much higher in lotteries than in horse races. This means that there is a sizeable amount of noise in lotteries and exotic bets, despite the large number of tickets sold.

We have argued that the information-based explanation of the favorite-longshot bias is promising and appears to be broadly consistent with the available evidence. Additional empirical work is necessary to compare the performance of this theory with the alternatives.

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Appendix: Proofs

Proof of Lemma 1. Let $\eta_{-1} = G(\hat{p}_{-1}|1)$, $\eta_0 = G(\hat{p}_1|1) - G(\hat{p}_{-1}|1)$, and $\eta_1 = 1 - G(\hat{p}_1|1)$ denote the chances for any opponent to take the respective action in state 1. Likewise, let $\zeta_{-1} = G(\hat{p}_{-1}|-1)$, $\zeta_0 = G(\hat{p}_1|-1) - G(\hat{p}_{-1}|-1)$, and $\zeta_1 = 1 - G(\hat{p}_1|-1)$.

We derive $W(1|1)$. Clearly, $W(-1|-1)$ can be derived in the same way. By definition,

$$W(1|1) = (1-\tau)\sum_{\hat{b}_1=0}^{N-1}\sum_{\hat{b}_{-1}=0}^{N-1-\hat{b}_1}\frac{\hat{b}_{-1}+\hat{b}_1+1}{\hat{b}_1+1}p\left(\hat{b}_{-1},\hat{b}_1|1\right),$$

where

$$p\left(\hat{b}_{-1},\hat{b}_1|1\right) = \frac{(N-1)!}{\hat{b}_{-1}!\hat{b}_1!\left(N-1-\hat{b}_{-1}-\hat{b}_1\right)!}\eta_{-1}^{\hat{b}_{-1}}\eta_1^{\hat{b}_1}\eta_0^{N-1-\hat{b}_{-1}-\hat{b}_1}$$

is the probability that the number of bets by the opponents on outcome -1 is equal to $\hat{b}_{-1}$ and the number of bets on outcome 1 is $\hat{b}_1$.

In case $\eta_1 = 0$, the distribution degenerates to be binomial with chance η_{-1} for bets on outcome -1. Then we can reduce $W(1|1) = (1-\tau)\sum_{\hat{b}_{-1}=0}^{N-1-\hat{b}_1}\left(\hat{b}_{-1}+1\right)p\left(\hat{b}_{-1}|1\right) = (1-\tau)\left((N-1)\eta_{-1}+1\right)$ since $(N-1)\eta_{-1}$ is the expected number of bets on outcome -1.

For the remainder, assume $\eta_1 > 0$. Let $n = \hat{b}_{-1}+\hat{b}_1$ denote the realized number of bets by the opponents. Clearly, n is binomially distributed with parameter $\eta_{-1}+\eta_1$. Moreover, conditional on n, $\hat{b}_1$ is binomially distributed with (updated) chance $\eta_1/\left(\eta_{-1}+\eta_1\right)$. Notice that given the number n of betting opponents, $W(1|1)$ is equal to

$$(1-\tau)\sum_{k=0}^{n}\frac{n+1}{k+1}\binom{n}{k}\left(\frac{\eta_1}{\eta_{-1}+\eta_1}\right)^k\left(\frac{\eta_{-1}}{\eta_{-1}+\eta_1}\right)^{n-k} = (1-\tau)\frac{1-\left(\frac{\eta_{-1}}{\eta_{-1}+\eta_1}\right)^{n+1}}{\left(\frac{\eta_1}{\eta_{-1}+\eta_1}\right)},$$

where the simplification follows from Newton's binomial,

$$\sum_{K=0}^{N}\binom{N}{K}\eta^K\zeta^{N-K} = (\eta+\zeta)^N. \tag{9}$$

Since n is binomially distributed, the expected payoff $W(1|1)$ is

$$\begin{aligned}
&(1-\tau)\sum_{n=0}^{N-1}\frac{1-\left(\frac{\eta_{-1}}{\eta_{-1}+\eta_1}\right)^{n+1}}{\left(\frac{\eta_1}{\eta_{-1}+\eta_1}\right)}\binom{N-1}{n}\left(\eta_{-1}+\eta_1\right)^n\eta_0^{N-1-n}\\
=\;&(1-\tau)\frac{\eta_{-1}+\eta_1}{\eta_1}-(1-\tau)\frac{\eta_{-1}}{\eta_1}\sum_{n=0}^{N-1}\binom{N-1}{n}\eta_{-1}^n\eta_0^{N-1-n}\\
=\;&(1-\tau)\frac{\eta_{-1}+\eta_1-\eta_{-1}\left(\eta_{-1}+\eta_0\right)^{N-1}}{\eta_1},
\end{aligned}$$

where we used (9) again. □

Proof of Proposition 1. First, suppose that $u \leq \tau$. Any active bettor places a unit bet, which is immediately reduced to $1-\tau$ before being placed in the pool of money to be returned to winners. By the logic of the no trade theorem, it is therefore not possible for all active bettors to expect a return in excess of $1-\tau$. But anyone who expects a return no greater than $1-\tau \leq 1-u$, is better off keeping the unit bet and forgoing the recreational payoff. Some active bettors are therefore better off abstaining, which implies that there can be no active bettors at all.

When everyone is betting, the symmetric equilibrium is determined by the common threshold value $\hat{p} \equiv \hat{p}_{-1} = \hat{p}_1$. A bettor with belief $\hat{p}$ is indifferent among betting on the two outcomes, and so satisfies $\hat{p}W(1|1) = (1-\hat{p})W(-1|-1)$. It will follow from Proposition 2 that this equation has a unique solution for $\hat{p}$. Now, the critical value u^* is defined by $u^* = 1 - \hat{p}W(1|1)$, which is precisely the value at which the worst-off active bettor is indifferent between betting and abstaining.

Finally, suppose that $u > \tau$, and yet no one is betting. An individual with posterior belief p sufficiently close to 1 will then gain from deviating to betting on outcome 1, since the expected utility from so doing is arbitrarily close to $(1-\tau) - 1 + u > 0$. □

Proof of Proposition 2. We look for a symmetric equilibrium, in which each bettor adopts the same cutoff $\hat{p}$. Consider the optimal response of one individual to all other bettors using $\hat{p}$. From Lemma 1, we see that $0 < 1-\tau \leq W(1|1), W(-1|-1) \leq (1-\tau)(N-1)$. Now, let $\check{p} \in (0,1)$ be the unique solution to the indifference condition

$$\check{p}W(1|1) = (1-\check{p})W(-1|-1).$$

If the individual has belief $\check{p}$, he is indifferent between betting on either of the two horses, $U(1|\check{p}) = U(-1|\check{p})$. Symmetric equilibrium requires $\check{p} = \hat{p}$. Note that the assumption $u \geq u^*$ guarantees that betting on one of the outcomes is better than abstention.

If $G(\hat{p}) = 1$, Lemma 1 reduces the indifference condition to $\check{p}(N-1) = (1-\check{p})$ or $\check{p} = 1/N$. Thus, if $G(1/N) = 1$, then $\hat{p}^{(N)} = 1/N$ defines the symmetric equilibrium, and all bettors always bet on outcome -1. Likewise, when $G(1-1/N) = 0$, then $\hat{p}^{(N)} = 1-1/N$ defines the equilibrium where everyone always bets on outcome 1. If $G(\hat{p}) \in (0,1)$, expressions (2) and (3) reduce to

$$W(1|1) = (1-\tau)\frac{1-[G(\hat{p}|1)]^N}{1-G(\hat{p}|1)},$$

and

$$W(-1|-1) = (1-\tau)\frac{1-[1-G(\hat{p}|-1)]^N}{G(\hat{p}|-1)}.$$

At a symmetric equilibrium $\check{p} = \hat{p}$, and the indifference condition becomes

$$\frac{\hat{p}}{1-\hat{p}} = \frac{1-G(\hat{p}|1)}{G(\hat{p}|-1)}\frac{1-[1-G(\hat{p}|-1)]^N}{1-G(\hat{p}|1)^N}. \tag{10}$$

The existence of a unique solution $p^{(N)}$ to this equation follows from the fact that the left-hand side of (10) is strictly increasing, ranging from zero to infinity as $\hat{p}$ ranges over $(0,1)$, while the positive right-hand side of (10) is weakly decreasing in $\hat{p}$ since $W(1|1) = (1-\tau)\sum_{k=0}^{N-1}[G(\hat{p}|1)]^k/N$ is increasing in $\hat{p}$ and similarly $W(-1|-1)$ is decreasing in $\hat{p}$. Since $1/(N-1) \leq W(1|1)/W(-1|-1) \leq N-1$, the solution is in the range $[1/N, 1-1/N]$. We conclude that this cutoff $\hat{p}^{(N)}$ defines the unique symmetric equilibrium when $G(1-1/N) > 0$ and $G(1/N) < 1$.

Finally, let $\hat{p}$ be the unique solution to the limit equation (4) and let an arbitrary $\varepsilon > 0$ be given. Notice that for N sufficiently large, $G(1-1/N) > 0$ and $G(1/N) < 1$, so the solution $\hat{p}^{(N)}$ is defined by (10). By monotonicity, at $\hat{p}+\varepsilon$ the left-hand side of (4) exceeds the right-hand side. By pointwise convergence of the right-hand side of (10) to the right-hand side of (4), for sufficiently large N, the left-hand side of (10) exceeds the right-hand side at $\hat{p}+\varepsilon$. Likewise, for sufficiently large N, the right-hand side of (10) exceeds the left-hand side at $\hat{p}-\varepsilon$. It follows that $\hat{p}^{(N)} \in (\hat{p}-\varepsilon, \hat{p}+\varepsilon)$ when N is sufficiently large.□

Proof of Proposition 5. With a signal realization s that induces private belief p we now have $\Pr(x = z = 1|s) = \Pr(z = 1|x = 1)\Pr(x = 1|s) = \sigma p$. If all bettors use symmetric strategies, we have $W(y = 1|x = z = 1) = W(y = -1|x = z = -1) \equiv \chi$ and $W(y = 1|x = 1, z = -1) = W(y = -1|x = 1, z = -1) \equiv \psi$. The expected payoff from a bet on outcome 1 is $U(y = 1|p) = \sigma p\chi + (1-\sigma)(1-p)\psi$ while $U(y = -1|p) = \sigma(1-p)\chi + (1-\sigma)p\psi$, so that $U(y = 1|p) - U(y = -1|p)$ is weakly increasing in p if and only if $\sigma\chi \geq (1-\sigma)\psi$.

We now show that $\sigma\chi \geq (1-\sigma)\psi$. If instead $\sigma\chi < (1-\sigma)\psi$, a symmetric equilibrium would necessarily have a cutoff at $1/2$, but since those with $p > 1/2$ would bet on outcome

-1, there would be more bets on this outcome when $x = 1$, and so $\chi > \psi$. Since also $\sigma > 1 - \sigma$, this is incompatible with $\sigma\chi < (1 - \sigma)\psi$, a contradiction.

If $\sigma\chi \geq (1 - \sigma)\psi$ holds with equality, then $\sigma > 1 - \sigma$ implies $\chi < \psi$. If the inequality is strict, then the unique equilibrium has a cutoff at 1/2, and since more people bet on outcome 1 when $x = 1$, we get again $\chi < \psi$. In the limit with infinite N, there is no uncertainty about how much is bet on outcome y given x. Since the remaining amount is bet on $y = -1$, we obtain the relation $1 = 1/W(y = z|x = z) + 1/W(y = z|x \neq z) = 1/\chi + 1/\psi$. Since $W(y = 1|x = z = 1) = \chi < \psi$, outcome 1 is the favorite when $x = 1$. Having observed that outcome 1 has odds $W(y = 1|x = z = 1)$, one can infer that $x = 1$, and so the posterior probability for outcome $z = 1$ is $\sigma = \Pr(z = 1|x = 1)$. The expected return on the favorite, $\sigma\chi$, is then immediately weakly greater than the expected return on the longshot, $(1 - \sigma)\psi$, by the inequality $\sigma\chi \geq (1 - \sigma)\psi$. In addition, the favorite-longshot bias carries over since the favorite has greater posterior odds than market odds, i.e. $\sigma \geq 1/\chi$. This inequality is true, since $1 = 1/\chi + 1/\psi$ can be solved for $\psi = \chi/(\chi - 1)$ and $\sigma\chi \geq (1 - \sigma)\psi$ then reduces to $\sigma \geq 1/\chi$. □

Proof of Proposition 6. We are considering the case $0 < \hat{p}_{-1} < \hat{p}_1 < 1$, so optimality of the threshold rule implies (7) and (8). Using the notation from the proof of Lemma 1, rewrite (7) as

$$\eta_{-1} = \frac{\frac{\eta_1}{\hat{p}_1}\frac{1-u}{1-\tau} - \eta_1}{1 - (1 - \eta_1)^{N-1}} \equiv H(\hat{p}_1). \tag{11}$$

With $\eta_1 = 1 - G(\hat{p}_1|1)$, the right-hand side H is a continuous function of $\hat{p}_1 \in (0, 1)$. It tends to infinity as $\hat{p}_1$ tends to 0, and tends to $(\tau - u)/[(1 - \tau)(N - 1)] < 0$ as $\hat{p}_1$ tends to 1. Moreover, it can be checked that H is a decreasing function whenever it is positive. It follows that for every $\hat{p}_{-1} \in [0, 1]$ there is a unique solution $\hat{p}_1 \in (0, 1)$ to equation (7). Thus equation (7) defines a continuous curve in the space $(\hat{p}_{-1}, \hat{p}_1) \in [0, 1]^2$. Using the implicit function theorem, it can be verified that this curve is downward sloping. Likewise, for every $\hat{p}_1 \in [0, 1]$ there is a unique solution $\hat{p}_{-1} \in (0, 1)$ to (8), and the set of solutions defines a downward sloping curve. The solution is unique since curve (8) is steeper than curve (7) wherever the two curves cross. This is a consequence of the monotonicity of $g(p|1)/g(p|-1)$ mentioned in Section 2.

For the comparative statics result, notice that an increase in u will imply a lower value of $H(\hat{p}_1)$ for every given $\hat{p}_1$. The curve defined by (7) therefore shifts down: for every given $\hat{p}_{-1}$, the resulting $\hat{p}_1$ is lower than before. Likewise, the curve defined by (8) shifts up: for every given $\hat{p}_1$, the resulting $\hat{p}_{-1}$ is larger than before. Since the (8) is steeper, the intersection of the two curves must move in the direction where $\hat{p}_{-1}$ rises and $\hat{p}_1$ falls. $\square$

Proof of Proposition 7. Under the symmetry assumptions, we have $\hat{p}_{-1} = 1 - \hat{p}_1$. Using notation from the proof of Lemma 1, this implies $\eta_1 = \zeta_{-1} > \eta_{-1} = \zeta_1$ and $\eta_0 = \zeta_0$.

Suppose that the realized bet amounts are $b_{-1}, b_1 > 0$. The market implied probability for outcome 1 is $\pi = b_1/ (b_1 + b_{-1})$. The bet distribution in state 1 is

$$p(b_{-1}, b_1|1) = \frac{N!}{b_{-1}!b_1!(N - b_{-1} - b_1)!}\eta_{-1}^{b_{-1}}\eta_1^{b_1}\eta_0^{N-b_{-1}-b_1},$$

and analogously in state -1. The posterior odds ratio is $\beta = p(\hat{b}_{-1}, \hat{b}_1| - 1)/p(\hat{b}_{-1}, \hat{b}_1|1) = (\zeta_1/\eta_1)^{b_1-b_{-1}}$. If $\pi < 1/2$, or $b_{-1} > b_1$, the desired inequality of the favorite-longshot bias is $(1 - \pi)/\pi < \beta$, i.e., $[\pi/[b_1(1 - 2\pi)]] \log[(1 - \pi)/\pi] < \log(\eta_1/\zeta_1)$. This inequality holds at the fixed realization b_{-1}, b_1, once the ratio η_1/ζ_1 is sufficiently large. But

$$\frac{\eta_1}{\zeta_1} = \frac{1 - G(\hat{p}_1|1)}{1 - G(\hat{p}_1| - 1)} = \frac{\int_{\hat{p}_1}^1 g(p|1)\, dp}{\int_{\hat{p}_1}^1 g(p| - 1)\, dp} = \frac{\int_{\hat{p}_1}^1 \frac{p}{1-p} g(p| - 1)\, dp}{\int_{\hat{p}_1}^1 g(p| - 1)\, dp} > \frac{\hat{p}_1}{1 - \hat{p}_1}$$

and the last expression tends to infinity as u tends to τ, according to Proposition 6. $\square$