Equivalence Scales Reconsidered♯

This version, March 2002.

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Paper presented at the MICRO seminar, Universitá Luigi Bocconi,
Milan, Italy, 14 January 2003.

Abstract

The paper proposes a new and normative approach for adjusting households' incomes in order to account for the heterogeneity of needs across income recipients when measuring inequality and welfare. We derive the implications for the structure of the adjustment method of two conditions concerned with the way the ranking of situations is modified by a change in the reference household type and by more equally distributed living standards across households. Our results suggest that concern for greater equality in living standards conflicts with the basic welfarist principle of symmetrical treatment of individuals which is at the core of the standard equivalence scale approach. *Journal of Economic Literature* Classification Number: D31, D63. *Keywords*: Heterogeneous Households, Equivalent Income Function, Size Adjusting Function, Equivalence Scales, Lorenz dominance.

1. Introduction and Overview

When households are identical in all other respects than income, a large consensus prevails among the profession to appeal to measures consistent with the Lorenz quasi-ordering when making judgements about inequality, welfare and poverty. The normative justification for this practice originates in the relationship between the Lorenz quasi-ordering and the condition of *equality preference*, according to which a rich to poor transfer of income is deemed socially desirable. Therefore a progressive transfer will always be recorded as a distributional improvement by any Lorenz consistent measures.

In practice however, normative evaluations typically involve the comparison of incomes pertaining to households which differ in many respects other than income. Among these factors, the size and the composition of a household are regarded as the main circumstances that affect its members' well-being. If one agrees that households' needs must be taken into account in the assessment of alternative situations, then one has to develop normative criteria

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♯ This paper forms part of the research programme of the TMR network **Living Standards, Inequality and Taxation** [Contract No. ERBFMRXCT 980248] of the European Communities whose financial support is gratefully acknowledged. We have benefited from useful remarks from our audiences at various seminars and conferences, and we are in particular indebted to Stephen Bazen and Nicolas Gravel for useful conversations and suggestions.

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that take full account of the households' incomes and circumstances. The construction of such measures requires assumptions about the relationship between households' incomes and needs on the one hand, and households' well-being on the other hand.

One way to do this would be to develop criteria that generalise the Lorenz quasi-ordering in this multidimensional framework. That was precisely the aim of Atkinson and Bourguignon (1987), who introduced the so-called *sequential Lorenz dominance* criterion for making comparisons of living standards across populations whose members differ in income and needs. However, despite its theoretical attractiveness this approach has met with only limited success in empirical work, where researchers seem to favour the conventional equivalence scale approach in order to take a family's circumstances into account[1]. Given an arbitrary household type – generally a single adult – the procedure consists in deriving the *household equivalent income* obtained by deflating the household's original income by a scale factor which reflects its needs, and then weighting the resulting figures by the number of persons in the household. The equivalent income is a cardinal measure of the household's well-being and represents that income which, if given to the single adult, will allow it to attain the same welfare level as a *typical member* of the original household. In the second stage one applies the conventional measures designed for the homogeneous case to the comparison of the weighted distributions of equivalent incomes.

This two-stage procedure is particularly convenient since it assumes that the determination and the evaluation of the adjusted distributions are distinct and independent issues. This implies that, when deciding which equivalence scale to choose, one does not have to worry about the criterion that will be used later on for comparing the adjusted distributions. On the other hand the way of accommodating household needs is given *a priori* and one may conceive of other possible ways of determining the equivalent income e.g., using absolute rather than relative scales (see Blackorby and Donaldson (1994)). This element of arbitrariness would not be too much of a problem if the results one obtains were shown to be robust to the choice of the adjustment method. However, even in the case where one views relative equivalence scales as the relevant method to adjust income for needs, there is strong evidence that the choice of these scale values can significantly affect the normative conclusions that are drawn (see e.g. Buhmann, Rainwater, Schmaus and Smeeding (1988)).

Even if there is a certain degree of disagreement regarding the appropriate values of the scale factors – or more generally the way incomes have to be adjusted – most of the profession would however accept this general procedure for passing normative judgements in a heterogeneous environment. Conforming with this practice, we henceforth will refer to living standards for making inequality and welfare comparisons, but we will avoid placing too much structure on the procedures used for accommodating differences in needs. Following Donald-

[1] Part of the explanation may originate in the fact that the implementation of sequential Lorenz dominance requires that the marginal distributions of household types be identical in the situations under comparison. Jenkins and Lambert (1993) have proposed an extension of the sequential Lorenz dominance criterion that permits comparisons of welfare to be made across populations of households with different distributions of needs.

son and Pendakur (1999), we introduce an *equivalent income function* in order to compare the living standards of households with differing needs. Given a reference household type, the equivalent income function specifies the transformations to be performed in order to convert households' incomes into equivalent incomes. Similarly, in order not to constrain the way family composition is assimilated, we introduce a *size adjusting function* which specifies the weight attached to each household's type. This results in a flexible *adjustment method* which encompasses most of the situations encountered in the literature. In a second stage, comparisons of situations are made by means of *multidimensional quasi-orderings* which consist in comparing the adjusted distributions by means of the standard Lorenz criteria used in the homogeneous case. This ensures that the conclusions obtained will hold for the largest variety of inequality indices and social welfare functions, as well as most poverty measures.

Having placed as few constraints as possible on the way welfare and inequality comparisons in a heterogeneous context are performed, our aim is to examine the implications for the structure of the adjustment method of two particular normative conditions. Our first condition requires that the result of the comparisons not be affected by the choice of the reference household type. For instance, if a distribution is ranked above another distribution when a single adult is taken as the reference type, then this should continue to be the case when a couple is substituted for a single adult. The application of this condition considerably restricts the class of admissible equivalent income functions, which reduces to *income-independent* [*relative* or *absolute*] *equivalence scales* depending on the Lorenz quasi-ordering used for comparing the adjusted distributions. However, this condition makes no particular recommendation concerning the way the equivalent incomes have to be weighted, and our results appear to be compatible to a large extent with the standard approach. Things change more substantially when we introduce our second condition according to which a transfer of income, that reduces the inequality of living standards between two households, must not decrease welfare or increase inequality. In this case the equivalent income function and the size adjusting function are no longer independent, and their respective forms are completely determined by the Lorenz quasi-orderings selected for comparing the adjusted distributions. In particular, when incomes are positive, the implied equivalence scales are independent of household income and the size adjusting function is proportional to the scale factors.

The paper is organized as follows. Section 2 sets out the background by considering the case of a homogeneous population and introduces the basic Lorenz quasi-orderings that will be used later on for comparing the adjusted distributions. The possibility that households differ in other respects than income is fully recognized in Section 3 where we propose an alternative approach for making welfare and inequality comparisons. We define in turn the adjustment method, which permits to transform income distributions for heterogeneous households into fictitious distributions for a homogeneous population, and the multidimensional quasi-orderings, which enable us to compare situations across heterogeneous populations. In Section 4 we investigate the implications for the adjustment method of the introduction of our two normative conditions and this is where our key results are presented. Section 5 concludes the paper summarizing our main conclusions and contrasting these with the standard approach, while Section 6 contains the proofs of our results.

2. Welfare and Inequality Measurement for a Homogeneous Population

We consider a *population* or *society* $S := \{1, 2, \ldots, n\}$ consisting of n ($n \geq 2$) homogeneous households, and we assume that incomes are drawn from an interval D which, depending on the context, will be equal to $\mathbb{R}$ or $\mathbb{R}_{++}$. It is convenient to frame the analysis in terms of weighted income distributions since we will make extensive use of weights when heterogeneous households are considered. A typical *income distribution* is a composite vector $(\mathbf{x} \mid \mathbf{w}) := (x_1, \ldots, x_n \mid w_1, \ldots, w_n)$, where $x_i \in D$ and $w_i > 0$ are respectively the *income* and the *weight* of household i, and we let $\mu(\mathbf{x} \mid \mathbf{w}) := \sum_{j=1}^{n} \big(w_j / \sum_{i=1}^{n} w_i\big) x_j$ represent the *weighted mean* of distribution $(\mathbf{x} \mid \mathbf{w})$. For notational convenience, we will call $\mathbf{x} := (x_1, \ldots, x_n)$ an *income profile* and $\mathbf{w} := (w_1, \ldots, w_n)$ a *weight profile*, and we denote as $\mathcal{Y}(D)$ the set of income distributions. Furthermore we let $(\mathbf{x}^{\mathbf{x}} \mid \mathbf{w}^{\mathbf{x}}) := (x_1^{\mathbf{x}}, \ldots, x_n^{\mathbf{x}} \mid w_1^{\mathbf{x}}, \ldots, w_n^{\mathbf{x}})$ stand for a non-decreasing re-arrangement of $(\mathbf{x} \mid \mathbf{w})$ defined by $\mathbf{x}^{\mathbf{x}} = \Pi \mathbf{x}$ and $\mathbf{w}^{\mathbf{x}} = \Pi \mathbf{w}$ where Π is a permutation matrix such that $x_1^{\mathbf{x}} \leq x_2^{\mathbf{x}} \leq \cdots \leq x_n^{\mathbf{x}}$. We denote as $F(\,\cdot\,; (\mathbf{x} \mid \mathbf{w}))$ the *cumulative distribution function* of $(\mathbf{x} \mid \mathbf{w}) \in \mathcal{Y}(D)$ defined by

$$F(z; (\mathbf{x} \mid \mathbf{w})) := \sum_{j=1}^{q(z;\mathbf{x})} \frac{w_j^{\mathbf{x}}}{\sum_{i=1}^{n} w_i^{\mathbf{x}}}, \ \forall\, z \in (-\infty, +\infty), \tag{2.1}$$

where $q(z; \mathbf{x}) := \#\{i \in \{1, 2, \ldots, n\} \mid x_i^{\mathbf{x}} \leq z\}$ is the number of households who receive in distribution $(\mathbf{x} \mid \mathbf{w})$ an income no greater than z.

Rather than focusing on a particular index, we are interested in deriving results which prove to be valid for a large set of value judgements. A typical *social judgement* is represented by a reflexive and transitive binary relation $\geq_J$ on the set of distributions $\mathcal{Y}(D)$ and we denote respectively as $\sim_J$ and $>_J$ its symmetric and asymmetric components defined in the usual way. We assume throughout that the social judgement $\geq_J$ satisfies the following:

Distributional Invariance [DI]: For all $(\mathbf{x} \mid \mathbf{u}), (\mathbf{y} \mid \mathbf{v}) \in \mathcal{Y}(D)$, we have $(\mathbf{x} \mid \mathbf{u}) \sim_J (\mathbf{y} \mid \mathbf{v})$, whenever $F(z; (\mathbf{x} \mid \mathbf{u})) = F(z; (\mathbf{x} \mid \mathbf{v}))$, for all $z \in (-\infty, +\infty)$.

This condition is actually fulfilled by most of the normative criteria used for comparing income distributions. In addition the social judgement is assumed to express a concern for equality in the sense that rich to poor transfers of income always result in a socially higher ranked state. More precisely we will say that distribution $(\mathbf{x} \mid \mathbf{w})$ is obtained from distribution $(\mathbf{y} \mid \mathbf{w})$ by a *progressive transfer* if there exists $\Delta > 0$ and two households $i, j \in S$ such that (i) $x_k = y_k$, for all $k \neq i, j$, (ii) $x_i = y_i + (\Delta / w_i)$, (iii) $x_j = y_j - (\Delta / w_j)$, and the positions of the income recipients on the income scale have not been modified i.e., $y_g \leq y_k$ implies

$x_g \leq x_k$, for all $g, k \in S$ $(g \neq k)$. Then we impose the social judgement $\geq_J$ to fulfil the following principle:

Transfer Principle [TP]: For all $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$, we have $(\mathbf{x} \mid \mathbf{w}) \geq_J (\mathbf{y} \mid \mathbf{w})$, whenever $(\mathbf{x} \mid \mathbf{w})$ is obtained from $(\mathbf{y} \mid \mathbf{w})$ by a *progressive transfer*[2].

All Lorenz consistent social judgements verify the two properties above and in this paper we will focus on such quasi-orderings. We let $F^{-1}(\,\cdot\,; (\mathbf{x} \mid \mathbf{w}))$ represent the *inverse cumulative distribution function* – equivalently the *quantile function* – of $(\mathbf{x} \mid \mathbf{w})$ obtained by letting $F^{-1}(0; (\mathbf{x} \mid \mathbf{w})) := x_1^{\mathbf{x}}$ and

$$F^{-1}(p; (\mathbf{x} \mid \mathbf{w})) := Inf\big\{z \in (-\infty, +\infty) \,\big|\, F(z; (\mathbf{x} \mid \mathbf{w})) \geq p\big\},\ \forall\, p \in (0, 1] \tag{2.2}$$

(see Gastwirth (1971)). The *generalized Lorenz curve* of distribution $(\mathbf{x} \mid \mathbf{w})$ – denoted as $L(p; (\mathbf{x} \mid \mathbf{w}))$ – is then defined by

$$GL(p; (\mathbf{x} \mid \mathbf{w})) := \int_0^p F^{-1}(q; (\mathbf{x} \mid \mathbf{w}))\, dq,\ \forall\, p \in [0, 1]. \tag{2.3}$$

We are now in a position to introduce the Lorenz quasi-ordering that constitutes the basis of the criteria used in the paper in order to make inequality and welfare comparisons.

Definition 2.1: Given two income distributions $(\mathbf{x} \mid \mathbf{u}), (\mathbf{y} \mid \mathbf{v}) \in \mathcal{Y}(D)$, we will say that $(\mathbf{x} \mid \mathbf{u})$ *Lorenz dominates* $(\mathbf{y} \mid \mathbf{v})$, which we write $(\mathbf{x} \mid \mathbf{u}) \geq_L (\mathbf{y} \mid \mathbf{v})$, if and only if $GL(p; (\mathbf{x} \mid \mathbf{u})) \geq GL(p; (\mathbf{y} \mid \mathbf{v}))$, for all $p \in (0, 1)$ and $GL(1; (\mathbf{x} \mid \mathbf{u})) = GL(1; (\mathbf{y} \mid \mathbf{v}))$.

In this paper we focus on three social judgements consistent with the Lorenz quasi-ordering. The first criterion introduced by Shorrocks (1983) expresses a concern for efficiency in the sense that, other things equal, social welfare cannot decrease when the income of at least one household increases.

Definition 2.2: Given two income distributions $(\mathbf{x} \mid \mathbf{u}), (\mathbf{y} \mid \mathbf{v}) \in \mathcal{Y}(D)$, we will say that $(\mathbf{x} \mid \mathbf{u})$ *generalised Lorenz dominates* $(\mathbf{y} \mid \mathbf{v})$, which we write $(\mathbf{x} \mid \mathbf{u}) \geq_{GL} (\mathbf{y} \mid \mathbf{v})$, if and only if $GL(p; (\mathbf{x} \mid \mathbf{u})) \geq GL(p; (\mathbf{y} \mid \mathbf{v}))$, for all $p \in (0, 1]$.

Our next two social judgements rule out any efficiency considerations and are basically concerned with the way income is distributed among households. Formally this is achieved by normalizing in an appropriate way the distributions under comparison. Given the distribution $(\mathbf{x} \mid \mathbf{w}) \in \mathcal{Y}(D)$ where $D = \mathbb{R}_{++}$, we indicate by $\hat{\mathbf{x}} := (\hat{x}_1, \ldots, \hat{x}_n)$ the *reduced income profile* obtained from $\mathbf{x}$ by letting $\hat{x}_i := x_i / \mu(\mathbf{x} \mid \mathbf{w})$, for all $i \in S$. We denote as $RL(p; (\mathbf{x} \mid \mathbf{w}))$ the *relative Lorenz curve* of distribution $(\mathbf{x} \mid \mathbf{w}) \in \mathcal{Y}(D)$ defined by $RL(p; (\mathbf{x} \mid \mathbf{w})) := GL(p; (\hat{\mathbf{x}} \mid \mathbf{w}))$, for all $p \in [0, 1]$.

[2] The requirement in the statement of the *Transfer Principle* that the distributions involved have the same weight profile is not restrictive (see Ebert and Moyes (2002)).

DEFINITION 2.3: Given two income distributions $(\mathbf{x}\,|\,\mathbf{u}), (\mathbf{y}\,|\,\mathbf{v}) \in \mathcal{Y}(D)$ with $D = \mathbb{R}_{++}$, we will say that $(\mathbf{x}\,|\,\mathbf{u})$ *relative Lorenz dominates* $(\mathbf{y}\,|\,\mathbf{v})$, which we write $(\mathbf{x}\,|\,\mathbf{u}) \geq_{RL} (\mathbf{y}\,|\,\mathbf{v})$, if and only if $RL(p; (\mathbf{x}\,|\,\mathbf{u})) \geq RL(p; (\mathbf{y}\,|\,\mathbf{v}))$, for all $p \in (0, 1)$.

Our final inequality quasi-ordering – inspired by Kolm (1976) – pays attention to the distribution of the average income shortfalls from mean income. Given the distribution $(\mathbf{x}\,|\,\mathbf{w}) \in \mathcal{Y}(D)$, we let $\tilde{\mathbf{x}} := (\tilde{x}_1, \ldots, \tilde{x}_n)$ represent the *centered income profile* obtained from $\mathbf{x}$ by letting $\tilde{x}_i := x_i - \mu(\mathbf{x}\,|\,\mathbf{w})$, for all $i \in S$. The *absolute Lorenz curve* of distribution $(\mathbf{x}\,|\,\mathbf{w}) \in \mathcal{Y}(D)$ – denoted as $AL(p; (\mathbf{x}\,|\,\mathbf{w}))$ – is defined by $AL(p; (\mathbf{x}\,|\,\mathbf{w})) := GL(p; (\tilde{\mathbf{x}}\,|\,\mathbf{w}))$, for all $p \in [0, 1]$. It specifies for every poorest fraction of the population the per capita amount of income needed in order to provide these households with the mean income.

DEFINITION 2.4: Given two income distributions $(\mathbf{x}\,|\,\mathbf{u}), (\mathbf{y}\,|\,\mathbf{v}) \in \mathcal{Y}(D)$ with $D = \mathbb{R}$, we will say that $(\mathbf{x}\,|\,\mathbf{u})$ *absolute Lorenz dominates* $(\mathbf{y}\,|\,\mathbf{v})$, which we write $(\mathbf{x}\,|\,\mathbf{u}) \geq_{AL} (\mathbf{y}\,|\,\mathbf{v})$, if and only if $AL(p; (\mathbf{x}\,|\,\mathbf{u})) \geq AL(p; (\mathbf{y}\,|\,\mathbf{v}))$, for all $p \in (0, 1)$.

Clearly $(\mathbf{x}\,|\,\mathbf{u}) \geq_{L} (\mathbf{y}\,|\,\mathbf{v})$ implies that $(\mathbf{x}\,|\,\mathbf{u}) \geq_{J} (\mathbf{y}\,|\,\mathbf{v})$, for all $J \in \{GL, RL, AL\}$. Although we focus on the three Lorenz quasi-orderings defined above, we emphasize that our results can be easily extended to other related quasi-orderings such as the intermediate Lorenz criteria inspired by Bossert and Pfingsten (1990) or the poverty dominance criteria studied by Foster and Shorrocks (1988).

3. WELFARE AND INEQUALITY MEASUREMENT FOR HETEROGENEOUS POPULATIONS

From now on we suppose that the population S is composed of heterogeneous households and that each household is distinguished by two attributes: *income* and *type*. We assume that there exists a given and finite number of types H $(2 \leq H \leq n)$ and we let $\mathbb{H} := \{1, 2, \ldots, H\}$ represent the set of possible types. The household's type $m \in \mathbb{H}$ may be best interpreted as an index of neediness which is increasing with *family size*[3]. A *situation* is a partitioned vector $(\mathbf{x}; \mathbf{m}) := (x_1, \ldots, x_n; m_1, \ldots, m_n)$, where $x_i \in D$ and $m_i \in \mathbb{H}$ are respectively the income and the type of household i, and we let $\mathcal{Z}(D)$ represent the set of situations.

In order to make comparisons of living standards across households, we assume a *household utility function* $U : D \times \mathbb{H} \to \mathbb{R}$ so that $U(y; m)$ represents the *utility* or *well-being* of a typical member of a household with income y and type m. The household utility function is supposed to be (i) continuous and increasing in y, (ii) non-increasing in m, and such that

[3]We associate for convenience the household's type with family size but insist that the framework might be extended by replacing $m \in \mathbb{H}$ with a vector of household characteristics comprising the number of adults, the number and age of children, the health status of family members, for instance, in addition to family size. All of our results apply to this less restrictive setting provided that general agreement can be reached regarding the ranking of needs or types on the basis of these vectors of characteristics.

(iii) $U(D;h) = U(D;m)$, for all $h, m \in \mathbb{H}$, and we denote as $\mathbf{U}$ the set of such functions[4]. For our analysis it is crucial that utility levels – not necessarily utility differences – can be compared across households so that the utility function U is defined up to an increasing transformation. To say things differently what is important is the indifference map in the income-need space. Indicating by r the reference household type, the *equivalent income function* $E : \mathbb{H} \times D \times \mathbb{H} \to D$ is defined by

$$U\left(E(r;(y;m));r\right) = U(y;m),\ \forall\, y \in D,\ \forall\, r, m \in \mathbb{H}, \tag{3.1}$$

which upon inverting yields $E(r;(y;m)) = U^{-1}\left(U(y;m);r\right)$. In other words, $E(r;(y;m))$ represents the *equivalent income* of a type-m household with income y, i.e., the amount of income needed by a type-r household in order to achieve the same living standard as a household of type m with income y. Two ordinally equivalent utility functions define the same equivalent income function: if $U^* = \psi \circ U^\circ$, for some ψ increasing, then $E^*(r;(y;m)) = E^\circ(r;(y;m))$, for all $y \in D$ and all $r, m \in \mathbb{H}$. The next conditions follow from the definition of the equivalent income function and the properties of the household utility function.

Identity [ID]: $E(r;(y;r)) = y,\ \forall\, r \in \mathbb{H},\ \forall\, y \in D$.

Income Monotonicity [IM]: $E(r;(y;m))$ is continuous and increasing in y, $\forall\, r, m \in \mathbb{H}$.

Type Monotonicity [TM]: $E(r;(y;m))$ is non-increasing in m, $\forall\, r, m \in \mathbb{H},\ \forall\, y \in D$.

Path Independence [PI]: $E(r;(y;m)) = E(r;(E(h;(y;m));h)), \forall\, r, h, m \in \mathbb{H}, \forall\, y \in D$.

Fixed Domain [FD]: $E(r;(D;m)) = D,\ \forall\, r, m \in \mathbb{H}$.

According to *Identity*, the equivalent income is equal to the household's original income when the household's type is identical to the chosen reference type. *Income Monotonicity* and *Type Monotonicity* require that the equivalent income is increasing with income and non-increasing with needs respectively. *Type Monotonicity* is a direct consequence of the assumption that the utility functions of the different types cannot intersect and are nested in the sense that the needier the household the lower the graph of its utility function. It implicitly supposes that we are able to order household types according to the attainable welfare the household can reach given its income. *Path Independence* ensures that the order in which the transformations are performed when converting the income of a type-m household into the equivalent income for the reference type r does not matter. The four conditions above are rather innocuous and impose apparently little structure on the equivalent income function. The remaining condition – *Fixed Domain* – might seem more restrictive as it requires that the income interval D is mapped onto itself irrespective of the type of the household. It follows that the equivalent income function defines an onto mapping on the interval D for

[4] It is helpful to assume the existence of an *ethical planner* who is in charge of evaluating households' circumstances relying on the principle of extended sympathy.

all types. When $D = \mathbb{R}_{++}$, an implication of this condition is that differences in needs have little impact on the living standards when incomes are arbitrarily small.

For future reference, we let $\mathbb{E}$ represent the set of admissible equivalent income functions that fulfil the five conditions above. It is always possible to derive equivalence scales by imposing a particular functional form for the equivalent income function. Given the equivalent income function $E \in \mathbb{E}$, the *relative equivalence scale* R solves $E(r;(y;m)) = y/R(r;(y;m))$ so that $R(r;(y;m)) = y/E(r;(y;m))$, for all $y \in D = \mathbb{R}_{++}$ and all $r, m \in \mathbb{H}$. Similarly we obtain the *absolute equivalence scale* A by letting $E(r;(y;m)) = y - A(r;(y;m))$ so that $A(r;(y;m)) = y - E(r;(y;m))$, for all $y \in D = \mathbb{R}$ and all $r, m \in \mathbb{H}$. It must be emphasized that neither definition implies that the equivalence scale's values are independent of income. However, because income-independent equivalence scales will play an important part in our results, we find it convenient for later reference to indicate by $K(r;m)$ the *income-independent relative equivalence scale*, which solves $R(r;(u;m)) = R(r;(v;m))$, for all $u, v \in D$ and all $r, m \in \mathbb{H}$. Similarly we denote as $L(r;m)$ the *income-independent absolute equivalence scale*, which solves $A(r;(u;m)) = A(r;(v;m))$, for all $u, v \in D$ and all $r, m \in \mathbb{H}$.

In order to make the definition of the adjustment process complete we have to determine how equivalent incomes have to be weighted. In order to allow for the greatest generality we will assume that the weight attached to each household's equivalent income depends on its type and on the reference household type. More precisely we define a *size adjusting function* $w : \mathbb{H} \times \mathbb{H} \to \mathbb{R}_{++}$ such that $w(r;m)$ is the weight attributed to a type-m household when the reference type is r and we denote as $\mathbb{W}$ the set of size adjusting functions. The reference household type, the equivalent income function and the size adjusting function fully describe the *adjustment method*, which is represented by the triple $(r;(E;w)) \in \mathbb{A} := \mathbb{H} \times \mathbb{E} \times \mathbb{W}$. Given the adjustment method $(r;(E;w)) \in \mathbb{A}$, we assign to every situation $(\mathbf{x};\mathbf{m}) \in \mathcal{Z}(D)$ the *adjusted distribution* $(E(r;(\mathbf{x};\mathbf{m})) \mid w(r;\mathbf{m}))$ where $E(r;(\mathbf{x};\mathbf{m})) := (E(r;(x_1;m_1)), \ldots, E(r;(x_n;m_n)))$ and $w(r;\mathbf{m}) := (w(r;m_1), \ldots, w(r;m_n))$ represent the *equivalent income profile* and the *weight profile* respectively.

The current practice when making comparisons of income distributions across and within heterogeneous populations consists of comparing the adjusted distributions by means of Lorenz consistent unidimensional quasi-orderings or indices. Restricting attention to the three social judgements we introduced in Section 2, we consider three families of *multidimensional quasi-orderings* for making comparisons of situations.

Definition 3.1: Given two situations $(\mathbf{x}^*;\mathbf{m}^*), (\mathbf{x}^\circ;\mathbf{m}^\circ) \in \mathcal{Z}(D)$, we will say that $(\mathbf{x}^*;\mathbf{m}^*)$ *dominates* $(\mathbf{x}^\circ;\mathbf{m}^\circ)$ *for* $(r;(E;w)) \in \mathbb{A}$ *and* $J \in \{GL, RL, AL\}$, which we write $(\mathbf{x}^*;\mathbf{m}^*) \succeq [r, E, w, J]\ (\mathbf{x}^\circ;\mathbf{m}^\circ)$, if and only if

$$(E(r;(\mathbf{x}^*;\mathbf{m}^*)) \mid w(r;\mathbf{m}^*)) \geq_J (E(r;(\mathbf{x}^\circ;\mathbf{m}^\circ)) \mid w(r;\mathbf{m}^\circ)). \tag{3.2}$$

We denote as $\succ[r, E, w, J]$ and $\sim[r, E, w, J]$ the asymmetric and symmetric components of $\succeq[r, E, w, J]$ defined by substituting respectively $>_J$ and $\sim_J$ for $\geq_J$ in (3.2). The multidimensional quasi-ordering $\succeq[r, E, w, J]$ is flexible enough to encompass most of the different approaches adopted in the literature.

4. Consistent Multidimensional Welfare and Inequality Comparisons

We have introduced a flexible method for comparing situations which, assuming that a particular social judgement has been agreed upon, depends on the equivalent income function, the size adjusting function and the reference type. Our definition imposes very few constraints on the multidimensional quasi-ordering and, as a result, some measures may appear disputable from an ethical point of view. In order to rule out undesirable elements, we will impose our multidimensional quasi-orderings to satisfy two apparently reasonable normative conditions. These requirements are concerned with the way the ranking of situations is modified by a change in the reference household type and by more equally distributed utilities across households, respectively.

4.1. *Independence with Respect to the Household's Reference Type*

We confine attention to the situation where the size adjusting function is given and is such that the weights vary proportionally with the reference type so that $w(r; m) = \lambda(r; h)w(h; m)$, for all $r, h, m \in \mathbb{H}$ [5]. Since there is no particular ethical reason in our model for selecting one household type rather than another as the reference, it would appear reasonable to require that the normative conclusions implied by the multidimensional quasi-ordering do not depend on the chosen reference type. Actually, our definition of a multidimensional quasi-ordering does not exclude the possibility of the results of the comparison being modified by a change in the choice of reference household type, as is shown in the example below.

Example 4.1: Let $\mathbb{H} = \{1, 2\}$ and consider a population $S := \{1, 2\}$ such that household 1 is of type 1 and household 2 of type 2 so that $\mathbf{m} = (1, 2)$. The weighting function is given by $w(r; m) := m$, for all $r, m \in \mathbb{H}$, and the equivalent income function E is defined by $E(1; (y; 1)) = E(2; (y; 2)) = y$, $E(2; (y; 1)) = E^{-1}(1; (y; 2))$,

$$E(1; (y; 2)) = \begin{cases} \ln(1 + y), & \text{for } y \leq 2.513, \\ y/2, & \text{for } 2.513 < y, \end{cases} \tag{4.1}$$

for all $y \in D = \mathbb{R}_{++}$. Consider the following income profiles: $\mathbf{x}^1 := (0.40, 0.82)$, $\mathbf{y}^1 := (1.20, 0.22)$; $\mathbf{x}^2 := (1.15, 0.54)$, $\mathbf{y}^2 := (0.60, 0.22)$; and $\mathbf{x}^3 := (1.15, 0.91)$, $\mathbf{y}^3 := (0.80, 0.22)$. Letting $\Delta J(r; p; g) := J(p; (E(r; (\mathbf{x}^g \,|\, \mathbf{m})); \mathbf{m})) - J(p; (E(r; (\mathbf{y}^g \,|\, \mathbf{m})); \mathbf{m}))$, for $r = 1, 2$, $p \in [0, 1]$, $g = 1, 2, 3$ and $J \in \{GL, RL, AL\}$, application of our different criteria gives the results summarized in Table 4.1.

[5] This admits as a particular case the standard approach where weights are set equal to the number of persons in the household i.e., $\lambda(r; m) = 1$ and $w(r; m) = m$, for all $r, m \in \mathbb{H}$.

Table 4.1

	$\Delta GL(r;p;1)$		$\Delta RL(r;p;2)$		$\Delta AL(r;p;3)$	
p	$r=1$	$r=2$	$r=1$	$r=2$	$r=1$	$r=2$
0	.000	.000	.000	.000	.000	.000
1/3	.067	.090	.015	−.007	.011	−.027
2/3	.200	.290	.030	−.015	.022	−.054
1	.000	−.209	.000	.000	.000	.000

It is possible to reverse dominance or turn it into non-comparability for any of the social judgements we have considered by suitably choosing the reference type. The choice of the reference household type therefore plays a crucial role in the heterogeneous context, when the normative conclusions are based on the comparisons of adjusted distributions. In order to avoid the kind of situations illustrated in Example 4.1, we introduce the following condition:

Reference Independence [RI]: Let $(r;(E;w)) \in \mathbb{A}$ and $J \in \{GL, RL, AL\}$ be given. Then, for all $(\mathbf{x};\mathbf{m}), (\mathbf{y};\mathbf{m}) \in \mathcal{Z}(D)$ such that $E(r;(\mathbf{x};\mathbf{m}))$ and $E(r;(\mathbf{y};\mathbf{m}))$ are non-decreasingly arranged:

(4.2) $$\forall\, r, h \in \mathbb{H}\ (r \neq h) : (\mathbf{x};\mathbf{m}) \succeq [r, E, w, J]\, (\mathbf{y};\mathbf{m}) \Longrightarrow (\mathbf{x};\mathbf{m}) \succeq [h, E, w, J]\, (\mathbf{y};\mathbf{m}).$$

This condition requires that, other things equal, the ranking of situations does not depend on the particular chosen reference type when adjusting incomes for needs[6]. It is a particularly weak condition since (4.2) is requested to hold in the very specific cases where the distributions of equivalent incomes are similarly arranged and the distributions of household characteristics in both situations are identical. The next result draws out the implications of *Reference Independence* for the equivalent income function.

Proposition 4.1: *Let $(s;(E;w)) \in \mathbb{A}$ with $w(s;m) = \lambda(s;h)w(h;m)$, for all $h, m \in \mathbb{H}$, and $J \in \{GL, RL, AL\}$. Then the multidimensional quasi-ordering $\succeq[s, E, w, J]$ verifies Reference Independence if and only if, for all $y \in D$ and all $r, m \in \mathbb{H}$:*

(4.3.a) *$E(r;(y;m)) = y/K(r;m)$, whenever $J = RL, GL$ with $D = \mathbb{R}_{++}$,*

(4.3.b) *$E(r;(y;m)) = y - L(r;m)$, whenever $J = AL, GL$ with $D = \mathbb{R}$,*

where $K(r;m)$ and $L(r;m)$ are non-decreasing in m and such that $K(r;r) = 1$ and $L(r;r) = 0$, for all $r \in \mathbb{H}$, and $K(r;m) = K(r;h)K(h;m) > 0$ and $L(r;m) = L(r;h) + L(h;m)$, for all $r, h, m \in \mathbb{H}$.

Although it would appear to be a weak condition, *Reference Independence* narrows down considerably the set of admissible equivalent income functions, ruling out in particular income-

[6] Conditions of independence with respect to a chosen base reference are very common in measurement economics.

dependent equivalence scales. As far as relative Lorenz dominance is concerned, adjusting incomes by means of scale factors is the appropriate technique for making comparisons of income distributions across heterogeneous populations. If one appeals to the absolute Lorenz criterion for comparing the distributions of equivalent incomes, then income-independent absolute equivalence scales are the only possible technique for accommodating differences in needs. In the case of welfare comparisons, the equivalent income function is determined by the definition of the income range: absolute scales obtain when no lower bounds are imposed on household income while relative scales are the only possibility when incomes are restricted to be positive. A direct consequence is that the way incomes are adjusted in order to take needs into account is conditional upon the Lorenz quasi-ordering one employs when comparing the distributions of equivalent incomes: the equivalent income function and the social judgement cannot be chosen independently.

On the other hand, Proposition 4.1 does not impose any restriction on the way family size has to be integrated and one is allowed to choose any system of weights provided they depend only on the household types. In particular the standard approach satisfies *Reference Independence* as long as (i) the inequality measures are consistent with the relative Lorenz quasi-ordering, and (ii) the welfare measures are consistent with the generalised Lorenz quasi-ordering and incomes be positive. Although our approach is different, it must be emphasized that the restrictions we have obtained are to a large extent compatible with the standard practice.

4.2. *A Concern for More Equal Distributions of Well-Being*

The requirement that the ranking of the social states does not depend on the household type chosen as the reference may be considered too strong a requirement[7]. We will henceforth consider that a general consensus prevails regarding the choice of the reference type, which will be assumed to be fixed throughout.

In the homogeneous case there is a large consensus among the profession to consider that a progressive transfer always results in a social improvement. In the heterogeneous case the relationship between income and living standards is more complicated, and a transfer of income from a richer household to a poorer one does not necessarily entail a reduction in the inequality of living standards. This originates in the fact that needs interact with income in the determination of the household's well-being, so that the differences in needs may offset the positive impact on living standards of a transfer of income. As suggested by Ebert (1995), a possible extension of the *Transfer Principle* to the heterogeneous case would involve the equivalent incomes of the households taking part in the transfer. More precisely, given the adjustment method $(r;(E;w)) \in \mathbb{A}$ and two situations $(\mathbf{x};\mathbf{m}), (\mathbf{y};\mathbf{m}) \in \mathcal{Z}(D)$, we will say that situation $(\mathbf{x};\mathbf{m})$ is obtained from situation $(\mathbf{y};\mathbf{m})$ by means of an (E,r)*-progressive*

[7] For instance, anyone who adheres to the principle of welfarism will certainly select the single adult as the natural reference household type.

TABLE 4.2

p	$\Delta GL(p;\vartheta)$ $\vartheta = b$	$\Delta GL(p;\vartheta)$ $\vartheta = 2$	$\Delta RL(p;\vartheta)$ $\vartheta = b$	$\Delta RL(p;\vartheta)$ $\vartheta = 2$	$\Delta AL(p;\vartheta)$ $\vartheta = b$	$\Delta AL(p;\vartheta)$ $\vartheta = 2$
0	.000	.000	.000	.000	.000	.000
$1/(2+\vartheta)$	.142	.125	.041	.037	.142	.135
$(1+\vartheta)/(2+\vartheta)$	.000	$-.041$	.000	$-.003$	.000	$-.010$
1	.000	$-.041$	.000	.000	.000	.000

transfer if there exists $\Delta > 0$ and two households $i, j \in S$ with $m_i \neq m_j$ such that

(4.4.a) $\quad x_k = y_k,$ for all $k \neq i, j;$

(4.4.b) $\quad x_i = y_i + \Delta$ and $x_j = y_j - \Delta;$

(4.4.c) $\quad E(r;(y_i;m_i)) < E(r;(x_i;m_i)) \leq E(r;(x_j;m_j)) < E(r;(y_j;m_j));$

and the positions of all households on the equivalent income scale are not affected i.e.,

$$E(r;(y_1;m_1)) \leq \cdots \leq E(r;(y_n;m_n)) \text{ and } E(r;(x_1;m_1)) \leq \cdots \leq E(r;(x_n;m_n)), \tag{4.5}$$

assuming that households are labelled in such a way that equivalent incomes are non-decreasingly arranged.

An (E, r)-progressive transfer makes the living standards of the households involved closer and may therefore be considered a move in the direction of greater equality. Since by definition our multidimensional quasi-orderings incorporate a concern for equality in terms of equivalent incomes, one expects that an (E, r)-progressive transfer will always be recorded as an improvement. However this is not the case as the following example demonstrates.

EXAMPLE 4.2: Let $\mathbb{H} = \{1, 2\}$ and consider a population $S := \{1, 2, 3\}$ such that households 1 and 3 are of type 1 and household 2 is of type 2 so that $\mathbf{m} = (1, 2, 1)$. We choose the single adult as the reference type which will be kept fixed throughout the example. The equivalent income function E is defined by $E(1;(y;1)) = y$ and $E(1;(y;2)) = y/b$, for all $y \in D = \mathbb{R}_{++}$, where $b = 1.5$ is the scaling factor for type 2-household. The size adjusting function is given by $w(1;1) := 1$ and $w(1;2) := \vartheta > 0$. Consider the income profiles $\mathbf{y} := (2, 6, 4)$ and $\mathbf{x} := (2.5, 5.5, 4)$. The corresponding distributions of equivalent incomes are $E(1;(\mathbf{y};\mathbf{m})) = (2, 4, 4)$ and $E(1;(\mathbf{x};\mathbf{m})) = (2.5, 3.66, 4)$, so that $(\mathbf{x};\mathbf{m})$ follows from $(\mathbf{y};\mathbf{m})$ by means of an $(E, 1)$-progressive transfer. The weights depend on the value of the parameter ϑ and we use $\mathbf{w}(\vartheta) := w(1;\mathbf{m}) = (1, \vartheta, 1)$ to indicate the vector of weights. Letting $\Delta J(p;\vartheta) := J(p;(E(1;(\mathbf{x};\mathbf{m})) \mid \mathbf{w}(\vartheta))) - J(p;(E(1;(\mathbf{y};\mathbf{m})) \mid \mathbf{w}(\vartheta)))$, for $p \in [0, 1]$, $\vartheta \in \{b, 2\}$ and $J \in \{GL, RL, AL\}$, we obtain the results depicted in Table 4.2.

This example shows that adjusting household size by the number of persons can result in unexpected conclusions: although one situation appears to be more desirable than the other, none of our multidimensional quasi-orderings actually confirms this intuition. The generalised, relative and absolute Lorenz curves of the adjusted distributions cross, which leaves room for conflicting opinions: given any social judgement among those considered here, it is always possible to find two measures consistent with it that lead to opposite conclusions[8]. But Example 4.2 also indicates that it is possible to find a size adjusting function such that the $(E,1)$-progressive transfer has the desired impact on the ranking of situations. Although it is premature to generalise, it is interesting to note that this happens precisely in the case where the weight given to the couple is set equal to the scale factor employed for computing its equivalent income. In order to avoid the kind of situation depicted in the example above, we impose the following condition on our multidimensional quasi-orderings.

Between-Type Transfer Principle [BTP]: Let $(r;(E;w)) \in \mathbb{A}$, $J \in \{GL, RL, AL\}$ be given, and consider any $(\mathbf{x};\mathbf{m}), (\mathbf{y};\mathbf{m}) \in \mathcal{Z}(D)$. Then, $(\mathbf{x};\mathbf{m}) \succeq [r,E,w,J]\,(\mathbf{y};\mathbf{m})$, whenever $(\mathbf{x};\mathbf{m})$ is obtained from $(\mathbf{y};\mathbf{m})$ by means of an (E,r)-progressive transfer.

This condition says that, if a household with a higher utility gives income to a household with a lower utility in such a way that it is still better-off after the transfer, then the resulting situation constitutes a social improvement. The *Between-Type Transfer Principle* is not as innocuous as it might look like at first sight, and it may conflict with other ethical principles. For instance one may perfectly argue that it is not obvious that $(\mathbf{x};\mathbf{m})$ improves upon $(\mathbf{y};\mathbf{m})$ in Example 4.2 since the two persons in the couple lose 0.34 units of equivalent income each while only one person gains 0.50 units of equivalent income. Actually it is a direct consequence of Example 4.2 that anyone who subscribes to the *utilitarian rule* cannot accept the *Between-Type Transfer Principle*. Indeed since the generalised Lorenz curves of the distributions of equivalent incomes weighted by the households' sizes intersect, it is always possible to find a household utility function $U \in \mathbf{U}$ such that $V(\cdot) := U(\cdot;1)$ is increasing and concave, and $W_V(\mathbf{x}\,|\,\mathbf{m}) := m_1 V(x_1) + m_2 V(x_2\,/\,b) < m_1 V(y_1) + m_2 V(y_2\,/\,b) =: W_V(\mathbf{y}\,|\,\mathbf{m})$[9]. On the other hand there exist rules – the *maximin* and the *leximin* for instance – that give equal weight to individuals and that verify the *Between-Type Transfer Principle*. The next result indicates the implications for the adjustment method of requiring that less unequally distributed living standards be recorded as a social improvement by any of our multidimensional quasi-orderings.

Proposition 4.2: *Let $(s;(E;w)) \in \mathbb{A}$ and $J \in \{GL, RL, AL\}$. Then, the multidimensional quasi-ordering $\succeq[s,E,w,J]$ verifies the Between-Type Transfer Principle if and only if, for all*

[8] Example 4.2 is a generalisation of Glewwe (1991), who uses the Theil index of inequality in order to show that an (E,r)-progressive transfer may imply an increase in inequality in terms of equivalent incomes.
[9] Choosing for instance the utility function $V(y) = y^{(1-\eta)}/(1-\eta)$ with constant relative inequality aversion $\eta \in [0,\infty)$, this happens precisely when $\eta < 0.5372$.

$y \in D$ and all $r, m \in \mathbb{H}$:

(4.6.a) *$E(r;(y;m)) = y/K(r;m)$ and $w(r;m) = \lambda(r)K(r;m)$, whenever $J = RL, GL$ with $D = \mathbb{R}_{++}$,*

(4.6.b) *$E(r;(y;m)) = y - L(r;m)$ and $w(r;m) = \lambda(r)$, whenever $J = AL, GL$ with $D = \mathbb{R}$,*

for some $\lambda(r) > 0$, where $K(r;m)$ and $L(r;m)$ are non-decreasing in m and such that $K(r;r) = 1$ and $L(r;r) = 0$, for all $r \in \mathbb{H}$, and $K(r;m) = K(r;h)K(h;m) > 0$ and $L(r;m) = L(r;h) + L(h;m)$, for all $r, h, m \in \mathbb{H}$.

As it could have been be anticipated from our discussion of Example 4.2, the *Between-Type Transfer Principle* completely determines the structure of the adjustment method. Not only do we obtain the same equivalent income functions as those implied by *Reference Independence*, but also we establish the precise way in which family size has to be incorporated. For relative inequality comparisons of adjusted distributions to be normatively significant, these two conditions imply that differences in needs must be accommodated by means of income-independent relative scales and the corresponding weights have to be proportional to these scale factors. The same adjustment method is obtained for welfare comparisons based on the generalised Lorenz criterion when incomes are restricted to be positive. Proposition 4.2 thus provides arguments in favour of the weighting procedure proposed by Ebert (1999) and Pyatt (1990) when income-independent equivalence scales are employed. If absolute Lorenz dominance is substituted for relative Lorenz dominance or if we are interested in welfare comparisons for unbounded incomes, then the same weight has to be given to all equivalent incomes irrespective of the household type and income-independent absolute scales are required for making the proper adjustments for family needs[10]. Choosing methods different from those identified in Proposition 4.2 runs the risk that inequality increases and welfare decreases as a result of an (E, r)-progressive transfer. We do not claim that such adjustment methods are inadequate but rather that they rely on principles different from the ones we have adopted here.

So far we have restricted our attention to pairs of situations that differ with respect to households' incomes but not to households' needs in order to derive our results. Our reader might therefore be under the impression that the adjustment methods – and more generally the multidimensional quasi-orderings – we have obtained only permit comparisons of situations *within the same heterogeneous population* or *across populations with the same distribution of needs* [see the proofs of Propositions 4.1 and 4.2]. Actually, it is a straightforward but tedious exercise to show that the adjustment methods we have characterized can be used for making comparisons of living standards in the general case where the marginal distributions of needs differ across populations.

[10]Incidently we note that the resulting weights are defined up to a proportional factor, which depends on the reference type, a restriction that was assumed in Proposition 4.1.

5. Discussion and Relationship with the Literature

By a way of a conclusion we would like to discuss our results in comparison with the standard method of making comparisons of inequality and welfare when the income receiving units differ with respect to needs. Firstly, whichever condition – *Reference Independence* or *Between-Type Transfer Principle* – we impose on the multidimensional quasi-ordering, the equivalent income function we derive implies that the equivalence scales are income-independent. Experimental studies as well as empirical work however tend to find that scale values are sensitive to households' incomes (see e.g. Kapteyn, Kooreman and Willemse (1988), Donaldson and Pendakur (2002)). Secondly, the *Between-Type Transfer Principle* implies in addition that the way household composition has to be incorporated is not independent of the chosen equivalent income function, and therefore the implicit equivalent scale.

5.1. *Equivalence Scales and Income-Independence*

While they place constraints on the *structure* of the adjustment method, our results leave open the choice of the precise *values* of the corresponding equivalence scales. Different procedures, such as expert opinions, interviews with consumers or econometric studies, are used for determining the values of the equivalence scales (see e.g. Coulter, Cowell and Jenkins (1992) for a review). We are particularly interested here by the implications of our results for the econometric determination of equivalence scales. So far we have taken a *normative approach* and assumed the existence of an ethical observer, who computes a household's well-being on the basis of its income and type. There is no requirement that the resulting household utility function be related to the household's effective behaviour. This must be contrasted with the *positive approach* where equivalence scales – and more generally equivalent income functions – are derived from the observation of households' consumption patterns. Estimation of equivalence scales requires coherence between the household's preferences and the household utility function, which results in severe restrictions being imposed on the structure of preferences (see Blundell and Lewbel (1991), Blackorby and Donaldson (1993)). From a practical point of view, it is of importance to know to which extent our results are compatible with these restrictions. Consider the case where we are interested in making welfare or relative inequality comparisons and where household income is positive. Propositions 4.1 and 4.2 would recommend the use of income-independent relative equivalence scales for adjusting incomes for needs. Substituting the resulting equivalent income function into the household [indirect] utility function, we obtain

$$U(y;m) = U\left(y/K(r;m);r\right) = \widehat{U}^r\left(y/K(r;m)\right),\ \forall\, y \in \mathbb{R}_{++},\ \forall\, r,m \in \mathbb{H}, \tag{5.1}$$

where $\widehat{U}^r$ is the [indirect] utility function of the reference household type[11]. Condition (5.1) is actually nothing else than *relative equivalence scale exactness* (Blackorby and Donaldson (1993)) or *independence of base level* (Blundell and Lewbel (1991)). Similarly *absolute*

[11] We omit the price system in the indirect utility function by assuming that prices are constant throughout.

equivalence scale exactness (Blackorby and Donaldson (1994)) would obtain in the case of welfare and absolute inequality comparisons with unbounded incomes. This means that there are cases where the equivalent income functions estimated from consumption patterns are compatible with our results. For instance, *relative equivalence scale exactness* ensures that comparisons of adjusted distributions of positive incomes by means of the generalised Lorenz criterion satisfy *Reference Independence*. Of course, this no longer happens if equivalence scale exactness is replaced by a weaker condition. The conditions of *relative* and *absolute generalised equivalence scale exactness* introduced by Donaldson and Pendakur (1999, 2002) allow the scales to depend on household income, something which is confirmed by their empirical results. One might therefore be tempted to reject *Reference Independence* and the *Between-Type Transfer Principle* in order to reconcile the theory with the data. We claim that this attitude is rather extreme as it is not obvious that these two conditions are responsible for the difficulty.

Considering first Proposition 4.1, we must distinguish different cases depending on the social judgement one uses for comparing the adjusted distributions. Appealing to the relative Lorenz quasi-ordering for making comparisons of adjusted distributions leaves no room for income-dependent scales. However, if comparisons of adjusted distributions of positive incomes are made by means of the generalised or absolute Lorenz quasi-orderings, then a necessary and sufficient condition for *Reference Independence* to be satisfied is that:

$$E(r;(y;m)) = -L(r;m) + K(r;m)y,\ \forall\, y \in D,\ \forall\, r, m \in \mathbb{H} \tag{5.2}$$

[see the proof of Proposition 4.1]. Such an equivalent income function does not rule out income-dependent equivalence scales[12]. Actually, it is the fact that we also impose *Type Monotonicity*, and it is this in conjunction with the income domain which gives rise to the income-independence of the equivalence scales. Turning now to our second result, we note that the main implication of the *Between-Type Transfer Principle* is that the adjustment method satisfies the following condition:

$$w(r;m)E(r;(y;m)) = \chi(r;m) + \lambda(r)y,\ \forall\, y \in D,\ \forall\, r, m \in \mathbb{H}, \tag{5.3}$$

where χ and λ are arbitrary functions [see the proof of Proposition 4.2]. Actually, condition (5.3) is also sufficient for the multidimensional quasi-orderings to satisfy the *Between-Type Transfer Principle*. Once again, it is *Type Monotonicity* which, when combined with the income domain, implies *equivalence scale exactness*[13].

The discussion above clearly indicates that, by dropping *Type Monotonicity*, it is possible to satisfy *Reference Independence* and/or the *Between-Type Transfer Principle*, and to have

[12] The corresponding income-dependent relative and absolute equivalence scales are given by $R(r;(y;m)) = K(r;m)y/(y - L(r;m))$ and $A(r;(y;m)) = ((K(r;m) - 1)y + L(r;m))/K(r;m)$ respectively (compare with Donaldson and Pendakur (1999)).

[13] It is easily verified that the equivalent income function considered by Donaldson and Pendakur (1999) verifies condition (5.3) under the further requirement that family size is incorporated in a suitable way.

income-dependent equivalence scales at the same time, at least in certain cases. However, from a normative point of view it is difficult to dispense with the condition of *Type Monotonicity*, unless the ordering of household types on the basis of neediness makes no particular sense. Consider the case where there is no ambiguity regarding the ranking in terms of well-being of two types of households: a couple composed of two adults and a single adult. Discarding *Type Monotonicity* means that we admit the possibility that, for some income levels, the couple is better-off than the single adult while the opposite occurs for other income values.

5.2. *The Conflict Between Equality Preference and Welfarism*

Although this does not play any role in the practical determination of equivalence scales, a second main difference with the way equivalence scales are currently applied is that our results recommend weighting family size by a factor that depends on the chosen equivalence scale. In contrast the standard approach consists of weighting the households' equivalent incomes by the number of persons in the household. This adjustment procedure would be the natural choice if one adheres to the principle of *welfarism* according to which (i) the individuals constitute the appropriate units for social evaluation, and (ii) the only relevant information for making normative assessment consists of the utilities enjoyed by the individuals irrespective of their other circumstances. This means that the individuals' utilities – and therefore the equivalent incomes that are derived from them – capture all the features that are considered as affecting individuals' well-being. Other non-utility information, such as the names of the individuals or the type of household to which they belong, should not be taken into account. Individuals have to be treated symmetrically and their utilities have to be given equal weight in the social evaluation whatever their family circumstances. This is related to the interpretation of the household utility function, which is typically viewed as the *utility per head* in the household[14]. Then the equivalent income is the income which, if given to the reference household type, guarantees that its representative member utility will be equal to the representative member utility in the original household.

The standard approach and the model developed in this paper appear to rely on different and non-compatible ethical principles. It is clear that the *Between-Type Transfer Principle* rules out *utilitarianism* as a relevant social judgement in our model. On the other hand there exist welfarist rules such as the maximin and the leximin for instance that are consistent with the former condition. But it is precisely because the ranking of situations they generate does not depend on the *particular* system of weights one chooses that the maximin and the leximin verify the *Between-Type Transfer Principle*. Actually Proposition 4.2 suggests that there is an irreducible conflict between the principle of *greater equality* captured by the *Between-Type Transfer Principle* and the principle of *symmetrical treatment of individuals* inherent in welfarism. The choice between these two principles is a matter of ethical preference which

[14] Blackorby and Donaldson (1993) have proposed a more elaborate model, where $U(y;m)$ is interpreted as the *representative member utility*, i.e., the utility that has to be given to all members of a household of type m with income y in order to reach the same household's well-being, assuming that the household distributes resources optimally among its members.

cannot be resolved exclusively on theoretical grounds, and as such falls beyond the scope of this paper.

6. Proofs of the Results

Given a transformation $f \in \mathcal{F}(D) := \{f : D \to D \text{ continuous and increasing}\}$ and a distribution $(\mathbf{x} \mid \mathbf{w}) \in \mathcal{Y}(D)$, we denote as $(f(\mathbf{x}) \mid \mathbf{w})$ the transformed distribution where $f(\mathbf{x}) := (f(x_1), \ldots, f(x_n))$. The three following lemmata (see Ebert and Moyes (2002) for the proofs) will be useful when proving Proposition 4.1.

Lemma 6.1: *Let $D = \mathbb{R}$ or $D = \mathbb{R}_{++}$ and $f \in \mathcal{F}(D)$. Then the two following statements are equivalent:*

(a) For all $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$: $(\mathbf{x} \mid \mathbf{w}) \geq_{GL} (\mathbf{y} \mid \mathbf{w}) \Longrightarrow (f(\mathbf{x}) \mid \mathbf{w}) \geq_{GL} (f(\mathbf{y}) \mid \mathbf{w})$.

(b) f is concave.

Lemma 6.2: *Let $D = \mathbb{R}_{++}$ and $f \in \mathcal{F}(D)$. Then the two following statements are equivalent:*

(a) For all $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$: $(\mathbf{x} \mid \mathbf{w}) \geq_{RL} (\mathbf{y} \mid \mathbf{w}) \Longrightarrow (f(\mathbf{x}) \mid \mathbf{w}) \geq_{RL} (f(\mathbf{y}) \mid \mathbf{w})$.

(b) $f(y) = \beta y$, for all $y \in D$ $(\beta > 0)$.

Lemma 6.3: *Let $D = \mathbb{R}$ and $f \in \mathcal{F}(D)$. Then the two following statements are equivalent:*

(a) For all $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$: $(\mathbf{x} \mid \mathbf{w}) \geq_{AL} (\mathbf{y} \mid \mathbf{w}) \Longrightarrow (f(\mathbf{x}) \mid \mathbf{w}) \geq_{AL} (f(\mathbf{y}) \mid \mathbf{w})$.

(b) $f(y) = \alpha + \beta y$, for all $y \in D$ $(\alpha \in \mathbb{R}, \beta > 0)$.

Proof of Proposition 4.1: Because sufficiency is obvious, we only have to prove that RI implies (4.3.a) and (4.3.b). To simplify notation, we let $\mathbf{u} := (w(r; m_1), \ldots, w(r; m_n))$ and $\mathbf{v} := (w(h; m_1), \ldots, w(h; m_n))$. Given $J \in \{GL, RL, AL\}$, we have to show that, if

$$(E(r; (\mathbf{x}; \mathbf{m})) \mid \mathbf{u}) \geq_J (E(r; (\mathbf{y}; \mathbf{m})) \mid \mathbf{u}) \Longrightarrow (E(h; (\mathbf{x}; \mathbf{m})) \mid \mathbf{v}) \geq_J (E(h; (\mathbf{y}; \mathbf{m})) \mid \mathbf{v}), \tag{6.1}$$

for all $(\mathbf{x}; \mathbf{m}), (\mathbf{y}; \mathbf{m}) \in \mathcal{Z}(D)$ and all $r, h \in \mathbb{H}$, then $E(r; (\,\cdot\,; h))$ must verify (4.3.a) and (4.3.b). Since by assumption $w(r; m) = \lambda(r; h) w(h; m)$, for all $r, h, m \in \mathbb{H}$, we have $\mathbf{v} = \lambda(r; h)\mathbf{u}$, so that distributions $(E(h; (\mathbf{x}; \mathbf{m}) \mid \mathbf{v})$ and $(E(h; (\mathbf{x}; \mathbf{m}) \mid \mathbf{u})$ [similarly $(E(h; (\mathbf{y}; \mathbf{m}) \mid \mathbf{v})$ and $(E(h; (\mathbf{y}; \mathbf{m}) \mid \mathbf{u})$] have the same cumulative distribution function. Furthermore by PI we also have

$$E(h; (y; m)) = E(h; (E(r; (y; m)); r)), \ \forall\, y \in D, \ \forall\, r, h, m \in \mathbb{H}. \tag{6.2}$$

Letting $\mathbf{r} := (r, \ldots, r)$, $\mathbf{s}^* = (s_1^*, \ldots, s_n^*)$, $\mathbf{s}^\circ = (s_1^\circ, \ldots, s_n^\circ)$, $s_i^* = E(r; (x_i; m_i))$ and $s_i^\circ = E(r; (y_i; m_i))$, for all $i = 1, 2, \ldots, n$, and appealing to DI, we deduce that statement (6.1) is equivalent to

$$(\mathbf{s}^* \mid \mathbf{u}) \geq_J (\mathbf{s}^\circ \mid \mathbf{u}) \Longrightarrow (E(h; (\mathbf{s}^*; \mathbf{r})) \mid \mathbf{u}) \geq_J (E(h; (\mathbf{s}^\circ; \mathbf{r})) \mid \mathbf{u}), \tag{6.3}$$

for all $(\mathbf{s}^* \mid \mathbf{u}), (\mathbf{s}^\circ \mid \mathbf{u}) \in \mathcal{Y}(D)$. We examine successively the cases where the social judgement is captured by the generalised, relative and absolute Lorenz quasi-orderings.

Case 1: $J = GL$. Appealing to Lemma 6.1, we know that a necessary condition for (6.3) to hold is that the transformation $f(\cdot) := E(h; (\cdot; r))$ is concave. Interchanging the indices r and h in (6.1) and using Lemma 6.1 again, we deduce that $E(r; (\cdot; h)) = E^{-1}(h; (\cdot; r))$ must be concave. Therefore, we conclude that $E(r; (\cdot; h))$ is affine i.e.,

$$E(r; (y; h)) = \alpha(r; h) + \beta(r; h)y, \ \forall\, y \in D, \ \forall\, r, h \in \mathbb{H}, \tag{6.4}$$

for some functions $\alpha(\cdot; \cdot)$ and $\beta(\cdot; \cdot)$. If $D = \mathbb{R}$, then TM implies that $\beta(r, h) = 1$, for all $r, h \in \mathbb{H}$ $(r \neq h)$. Indeed, suppose that $\beta(r, h) > 1$ for some $r \neq h$. Then it is possible to find $y^\circ, y^* \in D$ with $y^\circ < y^*$ such that

$$E\left(r; (y^\circ; h)\right) = \alpha(r, h) + \beta(r, h)y^\circ < y^\circ = E\left(r; (y^\circ; r)\right), \text{ and} \tag{6.5.a}$$

$$E\left(r; (y^*; h)\right) = \alpha(r, h) + \beta(r, h)y^* > y^* = E\left(r; (y^*; r)\right), \tag{6.5.b}$$

which contradicts TM. Similarly, if $\beta(r, h) < 1$ for some $r \neq h$, then one can find $y^\circ, y^* \in D$ with $y^\circ > y^*$ such that (6.5.a) and (6.5.b) hold. Therefore, we conclude that

$$E(r; (y; h)) = \alpha(r, h) + y, \ \forall\, y \in D, \ \forall\, r, h \in \mathbb{H}. \tag{6.6}$$

If $D = \mathbb{R}_{++}$, then TM implies that $\alpha(r, h) = 0$, for all $r, h \in \mathbb{H}$ $(r \neq h)$. Indeed, if $\alpha(r, h) > 0$ for some $h \neq r$, then we obtain $E(r; (D; h)) > Inf D$, for all $r, h \in \mathbb{H}$, when $y \to Inf D$, which contradicts TM, and we therefore conclude that

$$E(r; (y; h)) = \beta(r, h)y, \ \forall\, y \in D, \ \forall\, r, h \in \mathbb{H}. \tag{6.7}$$

Case 2: $J = RL$ with $D = \mathbb{R}_{++}$. We deduce from Lemma 6.2 that (6.3) will hold provided that if and only if $E(h; (\cdot; r))$ is proportional so is its inverse. Thus $E(r; (y; h)) = \beta(r; h)y$, for all $y \in D$ and all $r, h \in \mathbb{H}$, and TM can be satisfied.

Case 3: $J = AL$ with $D = \mathbb{R}$. Using similar reasoning, we deduce from Lemma 6.3 that $E(h; (\cdot; r))$ is defined by (6.4). Invoking TM as in Case 1 above, we finally obtain $\beta(r; h) = 1$ so that $E(r; (y; h)) = \alpha(r; h) + y$, for all $y \in D$ and all $r, h \in \mathbb{H}$.

Defining $L(r; m) := -\alpha(r; m)$ and $K(r; m) := \beta(r; m)^{-1}$, and appealing to ID and FD, we obtain finally $L(r; r) = 0$ and $K(r; r) = 1$, for all $r \in \mathbb{H}$, $L(r; m) = L(r; h) + L(h; m)$ and $K(r; m) = K(r; h)K(h; m) > 0$, for all $r, h, m \in \mathbb{H}$, respectively. □

Before we proceed to the proof of Proposition 4.2, we find it convenient to introduce the following technical result we state without proof[15].

[15] The interested reader is referred to Ebert and Moyes (2001) which provides the details of the proof.

LEMMA 6.4: *Given $D \subseteq \mathbb{R}$ and $r, h, m \in \mathbb{H}$, consider the function $\psi : \mathbb{H} \times D \times \mathbb{H} \to D$. Then, the solution to*

$$\psi(r;(u+\Delta;h)) + \psi(r;(v-\Delta;m)) = \psi(r;(u;h)) + \psi(r;(v;m)), \tag{6.8}$$

for all $u, v \in D$ and all $\Delta > 0$ such that $v - \Delta \in D$, is given by

$$\psi(r;(y;m)) = \chi(r;m) + \lambda(r)y, \ \forall\, y \in D, \ \forall\, r, m \in \mathbb{H}, \tag{6.9}$$

for some functions $\chi : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ and $\lambda : \mathbb{H} \to \mathbb{R}$.

PROOF OF PROPOSITION 4.2: Because sufficiency is obvious, we only have to prove that BTP implies (4.6.a) and (4.6.b), which is achieved in three steps.

STEP 1: Given the equivalent income function $E \in \mathbb{E}$ and the reference type $r \in \mathbb{H}$, we first indicate a way of constructing two situations $(\mathbf{x};\mathbf{m})$ and $(\mathbf{y};\mathbf{m})$ such that the former results from the latter by means of an (E,r)-progressive transfer. To this end, we let $g \in \mathbb{H}$ be an arbitrary type and we consider the two following cases.

CASE 1: $g < r$. Then, we have $E(r;(y;r)) = y \leq E(r;(y;g))$, for all $y \in D$, by ID and TM. We claim that, given any $v_g \in D$, it is always possible to find $u_g, t_g \in D$ with $u_g < v_g < t_g$ and $\Delta > 0$ such that:

$$E\left(r;(v_g;g)\right) < E\left(r;(v_g+\Delta;g)\right) \leq E\left(r;(t_g-\Delta;r)\right) < E\left(r;(t_g;r)\right); \text{ and} \tag{6.10.a}$$

$$E\left(r;(u_g-\Delta;r)\right) < E\left(r;(u_g;r)\right) \leq E\left(r;(v_g;g)\right) < E\left(r;(v_g+\Delta;g)\right). \tag{6.10.b}$$

Condition (6.10.a) follows from the fact that the income domain D is unbounded from above. Since the income domain D is open, given any $v \in D$, it is always possible to find $u < v$ such that $E(r;(u;r)) \leq E(r;(v;g))$, from which condition (6.10.b) follows. We then define situations $(\mathbf{x}^1;\mathbf{m}^1)$, $(\mathbf{y}^1;\mathbf{m}^1)$, $(\mathbf{x}^2;\mathbf{m}^2)$ and $(\mathbf{y}^2;\mathbf{m}^2)$ as indicated in Table 6.1.

TABLE 6.1

i	:	1	$\cdots$	$n-3$	$n-2$	$n-1$	n
m_i^1	:	g	$\cdots$	g	g	r	r
y_i^1	:	v_g	$\cdots$	v_g	v_g	t_g	t_g
x_i^1	:	v_g	$\cdots$	v_g	$v_g+\Delta$	$t_g-\Delta$	t_g
m_i^2	:	r	$\cdots$	r	r	g	g
y_i^2	:	$u_g-\Delta$	$\cdots$	$u_g-\Delta$	$u_g-\Delta$	$v_g+\Delta$	$v_g+\Delta$
x_i^2	:	$u_g-\Delta$	$\cdots$	$u_g-\Delta$	u_g	v_g	$v_g+\Delta$

Case 2: $r \leq g$. Then, we have $E(r;(y;r)) = y \geq E(r;(y;g))$, for all $y \in D$, by ID and TM. We claim that, given any $v_g \in D$, it is always possible to find $u_g, t_g \in D$ with $u_g < v_g < t_g$ and $\Delta > 0$ such that:

(6.11.a) $\quad E\left(r;(v_g - \Delta; g)\right) < E\left(r;(v_g; g)\right) \leq E\left(r;(t_g; r)\right) < E\left(r;(t_g + \Delta; r)\right);$ and

(6.11.b) $\quad E\left(r;(u_g; r)\right) < E\left(r;(u_g + \Delta; r)\right) \leq E\left(r;(v_g - \Delta; g)\right) < E\left(r;(v_g; g)\right).$

Condition (6.11.a) follows from the fact that the income domains $D = \mathbb{R}$ and $D = \mathbb{R}_{++}$ are unbounded from above. Similarly, it is always possible to find incomes such that (6.11.b) holds invoking the unboundness from below of $D = \mathbb{R}$. We introduce the situations $(\mathbf{x}^3; \mathbf{m}^3)$, $(\mathbf{y}^3; \mathbf{m}^3)$, $(\mathbf{x}^4; \mathbf{m}^4)$ and $(\mathbf{y}^4; \mathbf{m}^4)$ represented in Table 6.2.

Table 6.2

i	:	1	$\cdots$	$n-3$	$n-2$	$n-1$	n
m_i^3	:	r	$\cdots$	r	r	g	g
y_i^3	:	u_g	$\cdots$	u_g	u_g	v_g	v_g
x_i^3	:	u_g	$\cdots$	u_g	$u_g + \Delta$	$v_g - \Delta$	v_g
m_i^4	:	g	$\cdots$	g	g	r	r
y_i^4	:	$v_g - \Delta$	$\cdots$	$v_g - \Delta$	$v_g - \Delta$	$t_g + \Delta$	$t_g + \Delta$
x_i^4	:	$v_g - \Delta$	$\cdots$	$v_g - \Delta$	v_g	t_g	$t_g + \Delta$

We note that by construction, situation $(\mathbf{x}^s; \mathbf{m}^s)$ is obtained from situation $(\mathbf{y}^s; \mathbf{m}^s)$ by means of an (E, r)-progressive transfer, for $s = 1, 2, 3, 4$, and that this holds for all $v_g \in D$ and all $\Delta > 0$ sufficiently small.

Step 2: Let $\psi(r;(y;m)) := w(r;m)E(r;(y;m))$, for all $y \in D$ and all $r, m \in \mathbb{H}$. We show that, if the multidimensional quasi-ordering $\succeq [r, E, w, J]$ satisfies BTP, then, given any $r, h, m \in \mathbb{H}$, it must be the case that

(6.12) $$\psi(r;(u+\Delta;h)) + \psi(r;(v-\Delta;m)) = \psi(r;(u;h)) + \psi(r;(v;m)),$$

for all $u, v \in D$ and all $\Delta > 0$ such that $v - \Delta \in D$. We consider successively three cases and examine in each case the implication of BTP when $J \in \{GL, RL, AL\}$. To simplify notation, given any $(\mathbf{x}; \mathbf{m})$ and $(\mathbf{y}; \mathbf{m})$, we find it convenient to let $\mathbf{x}^* := E(r;(\mathbf{x};\mathbf{m}))$, $\mathbf{y}^* := E(r;(\mathbf{y};\mathbf{m}))$ and $\mathbf{w}^* := (w_1^*, \ldots, w_n^*)$ where $w_i^* = w\left(r; m_i\right)$, for all $i \in S$.

Case 1: $r \leq h < m$. Let $g \in \{h, m\}$ and choose first $(\mathbf{x}; \mathbf{m}) = (\mathbf{x}^3; \mathbf{m}^3)$ and $(\mathbf{y}; \mathbf{m}) = (\mathbf{y}^3; \mathbf{m}^3)$. A necessary condition for $(\mathbf{x}^* \mid \mathbf{w}^*) \geq_{GL} (\mathbf{y}^* \mid \mathbf{w}^*)$ is that

(6.13) $$\sum_{j=1}^{k} \left(\frac{w_j^*}{\sum_{i=1}^{n} w_i^*}\right) x_j^* \geq \sum_{j=1}^{k} \left(\frac{w_j^*}{\sum_{i=1}^{n} w_i^*}\right) y_j^*,$$

for all $k = 1, 2, \ldots, n$, which implies that

$$\psi(r; (u_g + \Delta; r)) + \psi(r; (v_g - \Delta; g)) \geq \psi(r; (u_g; r)) + \psi(r; (v_g; g)). \tag{6.14}$$

Now for $(\mathbf{x}^* \mid \mathbf{w}^*) \geq_{RL} (\mathbf{y}^* \mid \mathbf{w}^*)$, it is necessary that

$$\sum_{j=1}^{k} \left(\frac{w_j^*}{\sum_{i=1}^n w_i^*} \right) \frac{x_j^*}{\mu\left(\mathbf{x}^* \mid \mathbf{w}^*\right)} \geq \sum_{j=1}^{k} \left(\frac{w_j^*}{\sum_{i=1}^n w_i^*} \right) \frac{y_j^*}{\mu\left(\mathbf{y}^* \mid \mathbf{w}^*\right)}, \tag{6.15}$$

for all $k = 1, 2, \ldots, n-1$. In particular, for $k = n-1$, (6.15) actually reduces to (6.14). Similarly a necessary condition for $(\mathbf{x}^* \mid \mathbf{w}^*) \geq_{AL} (\mathbf{y}^* \mid \mathbf{w}^*)$ is that

$$\sum_{j=1}^{k} \left(\frac{w_j^*}{\sum_{i=1}^n w_i^*} \right) \left[x_j^* - \mu\left(\mathbf{x}^* \mid \mathbf{w}^*\right)\right] \geq \sum_{j=1}^{k} \left(\frac{w_j^*}{\sum_{i=1}^n w_i^*} \right) \left[y_j^* - \mu\left(\mathbf{y}^* \mid \mathbf{w}^*\right)\right], \tag{6.16}$$

for all $k = 1, 2, \ldots, n-1$, which again implies (6.14). Choosing next $(\mathbf{x}; \mathbf{m}) = (\mathbf{x}^4; \mathbf{m}^4)$ and $(\mathbf{y}; \mathbf{m}) = (\mathbf{y}^4; \mathbf{m}^4)$, we establish using a similar reasoning that

$$\psi(r; (v_g - \Delta; g)) + \psi(r; (t_g + \Delta; r)) \leq \psi(r; (v_g; g)) + \psi(r; (t_g; r)). \tag{6.17}$$

Using condition ID, and combining inequalities (6.14) and (6.17), we obtain

$$\psi(r; (v_g; g)) + \psi(r; (v_g - \Delta; g)) = w(r; r)\Delta, \ \forall\, g \in \{h, m\}, \tag{6.18}$$

and we conclude that (6.12) holds, for all $u = v_h$, $v = v_m$, and all $\Delta > 0$ sufficiently small.

CASE 2: $h < m \leq r$. One proves along a similar argument that (6.12) holds, for all $v_h, v_m \in D$ and all $\Delta > 0$ sufficiently small, choosing successively $(\mathbf{x}; \mathbf{m}) = (\mathbf{x}^s; \mathbf{m}^s)$ and $(\mathbf{y}; \mathbf{m}) = (\mathbf{y}^s; \mathbf{m}^s)$ with $s = 1, 2$, for $g \in \{h, m\}$.

CASE 3: $h < r \leq m$. Repeated application of the above argument gives (6.12) choosing (i) $(\mathbf{x}; \mathbf{m}) = (\mathbf{x}^s; \mathbf{m}^s)$ and $(\mathbf{y}; \mathbf{m}) = (\mathbf{y}^s; \mathbf{m}^s)$ with $s = 1, 2$, whenever $g = h$, and (ii) $(\mathbf{x}; \mathbf{m}) = (\mathbf{x}^s; \mathbf{m}^s)$ and $(\mathbf{y}; \mathbf{m}) = (\mathbf{y}^s; \mathbf{m}^s)$ with $s = 3, 4$, whenever $g = m$.

STEP 3: We have shown that, if the multidimensional quasi-ordering $\succeq [r, E, w, J]$ satisfies BTP, then condition (6.12) must hold. Invoking Lemma 6.4, we conclude that

$$w(r; m)E(r; (y; m)) =: \psi(r; (y; m)) = \chi(r; m) + \lambda(r)y, \ \forall\, y \in D, \ \forall\, r, m \in \mathbb{H}. \tag{6.19}$$

Letting $\alpha(r; m) := \chi(r; m)/w(r; m)$ and $\beta(r; m) := \lambda(r)/w(r; m)$, (6.19) can be equivalently rewritten as

$$E(r; (y; m)) = \alpha(r; m) + \beta(r; m)y, \ \forall\, y \in D, \ \forall\, r, m \in \mathbb{H}. \tag{6.20}$$

Case 1: $J = GL, AL$ with $D = \mathbb{R}$. Appealing to similar arguments as in the proof of Proposition 4.1, one can show that TM implies that $\beta(r;m) = 1$, for all $r, m \in \mathbb{H}$ $(r \neq m)$. Therefore, we conclude that:

(6.21.a) $$E(r;(y;m)) = \alpha(r;m) + y, \ \forall\, y \in D, \ \forall\, r, m \in \mathbb{H}, \text{ and}$$

(6.21.b) $$w(r;m) = \lambda(r), \ \forall\, r, m \in \mathbb{H}.$$

Case 2: $J = GL, RL$ with $D = \mathbb{R}_{++}$. Using the same technique as in the proof of Proposition 4.1, one verifies that $\alpha(r;m) = 0$, for all $r, m \in \mathbb{H}$ $(r \neq m)$, and we therefore conclude that

(6.22.a) $$E(r;(y;m)) = \beta(r;m)y, \ \forall\, y \in D, \ \forall\, r, m \in \mathbb{H}, \text{ and}$$

(6.22.b) $$w(r;m) = \lambda(r)/\beta(r;m), \ \forall\, r, m \in \mathbb{H}.$$

Finally, the proof is made complete by defining $L(r;m) := -\alpha(r;m)$ and $K(r;m) := \beta(r;m)^{-1}$, and appealing to ID and FD. □

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