Model misspecification, the equilibrium natural rate and the equity premium

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Abstract

This paper analyses the importance of risk and uncertainty premia in determining the equilibrium natural rate of interest. Within a simple general equilibrium model of a monetary economy, we derive real and nominal interest rates and the equity premium in a consistent manner, and we show that the effects of risk and uncertainty on all these variables are interconnected. Realistic equity premia are generated assuming that agents use robust investment rules to deal with the possibility of model misspecification.

Our results show that natural rates are affected by sizable uncertainty premia of two sorts: the first is associated with ambiguity of the reference model of technology growth; the second is related to uncertainty on the future course of monetary policy. Using similar parameter values, the equilibrium natural rate is typically calibrated to be much smaller than in the deterministic steady state of SDGE models.

Keywords: natural rate of interest, equity premium puzzle, risk-free rate puzzle, robust control, model misspecification.

JEL Classification: E43, G11

1.3

1 Introduction

The equilibrium real interest rate plays an important, if sometimes implicit, role in monetary policy. In the context of the literature on simple interest rate rules spawned from Taylor93, the equilibrium real rate is a crucial determinant of the level around which the nominal interest rate fluctuates at business cycle frequencies. FuhrerMoore95, for example, point out that a rule prescribing changes in the level of the policy interest rate in response to inflation "would require knowledge of the equilibrium value of [the nominal interest rate], which implicitly requires knowledge of the equilibrium real rate of interest. If the monetary authority were to use the level rule with an incorrect estimate of the equilibrium real rate, inflation would never reach its target" (p. 1063).

The equilibrium real interest rate is closely related to the Wicksellian natural rate of interest, defined in Woodford as "the equilibrium real rate of return in case of fully flexible prices." In DSGE models with nominal rigidities, where typically shocks have persistent effects, the equilibrium real interest rate will pin down the steady-state value of the natural rate, and the short-term fluctuations of the natural rate around its long term value will

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potentially play an important role in optimal monetary policy prescriptions. Since the focus of DSGE models is on business cycle dynamics, the equilibrium value of the natural rate is normally recovered in the deterministic steady-state of the economy. The result is that it corresponds simply to the inverse of the time discount parameter of the representative household of the model.

This result is due to many simplifying assumptions, amongst which one can list: on the one hand, the zero rates of growth of per-capita consumption and population in equilibrium, and the absence of finite life effects; on the other hand, the absence of steady state effects of risk, due to the focus on the deterministic steady state. One could defend the zero population growth and the absence of finite life effects assumptions on empirical grounds: at least in developed countries, the first is often approximately zero and the impact of finite life effects on the equilibrium interest rate is very small in models á la Blanchard85. The assumption of absence of steady state risk premia, on the other hand, is sometimes justified on the grounds that, for reasonable values of the risk aversion parameter, such premia are very small in general equilibrium models.

The latter conclusion is, however, at odds with evidence from finance data. Standard models are known to generate the risk-free rate puzzle–see Weil89–that is an equilibrium value of the risk-free rate which is too high compared to the data. Since the risk-free rate and the equilibrium natural real rate are equivalent notions, the latter is also misspecified if the risk-free rate is. Conversely, if risk effects can be shown to play an important role on the risk-free rate, they should also be acknowledged in the natural rate. A realistic general equilibrium model of the equilibrium interest rate should be consistent with the finance evidence on the risk free rate.

This paper attempts to build such a model exploiting the link between the natural rate of monetary economics and the risk-free rate of financial economics. More precisely, we focus on a simple economy with white noise shocks and no endogenous persistence, where there is no difference between the short and long term values of the natural rate. In this simplified economy, we study the properties of the nonlinear stochastic equilibrium, rather than those of the deterministic steady-state (and we also allow for trend consumption growth). We can therefore analyse the importance of risk effects in determining the natural interest rate. The natural rate of our model will correspond to the *equilibrium* natural rate of models with nominal rigidities and where shocks have persistent effects.

In order to help generate realistic results on the equity premium in a general equilibrium framework, we build on recent developments in economics and finance emphasising the role of model uncertainty – see e.g., AndersonHansenSargent, Maenhout on the ability of model uncertainty premia to solve the equity premium puzzle. More specifically, agents are supposed to base their behaviour on robust portfolio rules in a simple general equilibrium monetary model. The resulting model-uncertainty premia affect, in a consistent manner, the natural rate and the equity premium.

All results are obtained analytically, in closed form. The paper shows, first, that in a stochastic environment one can define two natural real rates, depending on whether they are based on the real expected return on nominal bonds, or the real expected return on risk-free bonds. We will refer to these as the natural nominal rate – which is therefore defined as a *real* rate – and natural real rate, respectively. The key difference is that the first, unlike the second, is affected by inflation premia, which are here due to both risk and model uncertainty. Since the natural nominal rate is the equilibrium real interest rate which should enter Taylor-type rules, the model suggests the existence of a possible link between the degree of aggressiveness of the rule and the level around which the nominal interest rate is stabilised. Different degrees of aggressiveness would in fact modify the volatility of inflation, with consequences on the inflation premium and, in turn, the equilibrium interest rate of the Taylor rule.

Second, we show that both natural rates can be heavily affected by uncertainty premia of two sorts: the first is associated with ambiguity of the reference model of technology growth; the second is related to uncertainty on the monetary policy rule. An impact on the natural rate of monetary policy uncertainty is somewhat surprising, especially since money is superneutral in the model. We show that it is due to the fact that, in an uncertain environment, ambiguity of monetary policy tends to increase the fuzziness of the dynamics of technology growth. Loosely speaking, uncertainty over equilibrium inflation can increase the overall level of uncertainty in the economy and thus affect the premium required on real returns.

The presence of inflation and nominal variables in the model also allows us to show that the equity premium puzzle is probably deeper than typically thought, because average real rates – often used to calibrate the risk-free rate – actually include an inflation premium. In general equilibrium models, the inflation premium tends to be correlated with the equity premium, because permanent technology shocks (which generate the equity premium) are also deflationary (or inflationary depending on their sign). In a general equilibrium model, therefore, the difference between the return on equity and the average real rate – which is the exact theoretical equivalent of the "empirical" measure of the equity premium used first by MehraPrescott – is much harder to boost than the difference between the return on equity and the risk-free rate.

The model can be used to study the plausible movements of the natural rate in response to events such as an increase of monetary policy uncertainty, or an increase in households' confidence in a new economy-type scenario for future productivity growth. We argue that the first scenario could characterise US economic developments in the seventies, while the second one could be seen as a way to rationalise investors' beliefs in the nineties. Under these interpretations, we show that our model can explan the prolonged period of negative real rates in the seventies, and the findings in Blanchard93 of a link between the equity premium and the Great Inflation of those years. We also show that the new economy-type scenario would tend to increase the natural rate.

Finally, to investigate the quantitative implications of the model, we perform a tentative calibration exercise. This exercise confirms that uncertainty premia effects are likely to be quantitatively important determinants of the natural rates. Using similar parameter values, the equilibrium natural rate is typically calibrated to be much smaller than in the deterministic steady state of SDGE models.

The paper is organised as follows. Section 2 provides a motivation for and outlines the main ingredients of our model. The equity premium and the risk-free rate are derived in Section 3, that also illustrates the differential impacts on prices of monetary policy and technological uncertainty. Section 4 analyses the determinants of the nominal interest rate and inflation. It also derives the natural nominal rate and the *relative* equity premium, where the latter is defined as the difference between the return on equity and the ex-ante real, rather than risk-free, rate. The results of two simple comparative static exercises and a welfare analysis are presented in Section 5, while the calibration is described in Section 6. Section 7 concludes. All technical details are consigned to the appendix.

2 Robust consumption in a monetary economy

Since our aim is to derive the natural rate of interest in a stochastic equilibrium, we wish to construct a model which retains three key features.

First, we want to adopt a general equilibrium framework, to be able to relate the natural rate of interest to taste and other exogenous parameters. We also wish to use a monetary model, to be able to relate the natural rate of interest, a "real variable", to the nominal interest rate. We introduce money in the model through the money-in-the-utility-function formulation. As a result, inflation and monetary policy will also appear explicitly in the model, albeit in a stylised fashion. More specifically, monetary policy will be implemented according to a simple stochastic money growth rule. While unsuitable to describe actual policy moves over short term horizons, a stochastic money growth rule is sufficient to capture a key long run determinant of the nominal interest rate, namely the trend rate of growth of money. This assumption implies that any short-term policy reaction to economic developments is subsumed in the properties of shocks to the rate of growth of money. The correlation of money growth shocks with technology shocks, in particular, could be viewed as the sign of a more frequent, if not systematic, response of monetary policy to the latter shocks. For example, a positive correlation may be attributed to the willingness of the central bank to neutralise the effects of technology shocks on inflation, so that a negative (i.e. inflationary) technology shock would often be met by a monetary contraction.

Second, we want the model to be able to deliver plausible values for the natural rate from a quantitative viewpoint. Given the close link between the natural rate and the equity premium, this implies that the model should also generate plausible values for the equity premium. To achieve this goal, we assume that agents adopt robust rules when making consumption and portfolio decisions, in order to take into account the possibility of model misspecification. Robust control has been advocated by HansenSargent02, and Anderson-HansenSargent and Maenhout have argued that it can help to generate plausible values for the equity premium in a general equilibrium model. We adopt a generalisation of the robust control approach due to UppalWang. The generalisation allows one to model separately uncertainty over different state variables of the model. Thus, we assume that agents are uncertain about the trend rate of growth of an endogenous stochastic endowment–which we refer to as technology growth–and about the trend rate of growth of money. We will show that both sources of uncertainty, which are possibly "correlated", play an important role in equilibrium.

Finally, we wish to keep the model tractable and focus on the basic long term trends in monetary policy and technology. We therefore abstract from nominal rigidities and also adopt the assumption that all shocks are white noise. Serially correlated shocks would probably improve the ability of the model to track down short-term variations in the natural rate of interest, but they would not alter its fundamental long-term determinants. As a result, there is no difference in the model between the short-run and the long-run natural rates. For simplicity, we will therefore omit the "long-run" qualifier in most of the discussion below.

2.1 A monetary economy

The basic ingredients of the model are relatively standard. The model is in continuous time and related to those used by Stulz86 and RebeloXie99. Excluding money, it can be seen as a continuous time equivalent of the lucas78 model.

The representative household chooses an optimal consumption/investment plan, which requires selecting a consumption rate and the portfolio shares of a number of assets in order to maximise the intertemporal utility function

$$\int_0^\infty e^{-\delta t} \frac{u\left(c_t, m_t\right)^{1-\gamma}}{1-\gamma} dt \qquad \gamma \neq 1 \tag{1}$$

where $\delta > 0$ is the rate of time preference, $\gamma > 0$ is the coefficient of relative risk aversion, c_t is consumption and m_t represents real money balances (henceforth, we drop all time subscripts to simplify the notation). Current utility, in turn, takes the CES form

$$u\left(c, m\right) = \left(\alpha^{\frac{1}{\psi}} c^{\frac{\psi-1}{\psi}} + \left(1-\alpha\right)^{\frac{1}{\psi}} m^{\frac{\psi-1}{\psi}}\right)^{\frac{\psi}{\psi-1}}$$

where $0 < \alpha \leqslant 1$ is a constant and ψ is the elasticity of intra-temporal substitution between consumption and real money balances.[1] The investment opportunity set is kept simple, but it is sufficient to analyse the equilibrium variables of interest. Thus, the household can invest its wealth in: claims to the future endowment stream, i.e. equity yielding a risky return; real balances; a real bond, in zero net supply, which yields a safe real interest rate; a nominal bond, also in zero net supply, which yields a nominal interest rate. Since money is injected in the economy through stochastic nominal transfers, we also introduce an additional asset which ensures the household from future transfer shocks and thus completes the markets.

If we denote the representative household's real wealth by w, this can be invested in: equity s, yielding a stochastic instantaneous return $\mu_s dt + \sigma_s dz_s$, where dz_s is a standard Brownian motion; a nominal bond B yielding a safe instantaneous nominal return Rdt; a real bond b yielding a safe instantaneous real return rdt; nominal money M yielding a zero nominal return; an asset representing the expected present discounted value of future money transfers, x.[2]

If P denotes the price of the consumption good, real wealth can be written as

$$w = s + \frac{M}{P} + \frac{B}{P} + b + x \tag{2}$$

or, in terms of investment shares ω_i in asset i ($\omega_i \equiv \frac{i}{w}$ and $\sum_i \omega_i = 1$), $1 = \omega_s + \omega_B + \omega_b + \omega_x + \omega_M$.

All the asset returns will be determined in equilibrium, together with the inflation rate. The only exogenous processes driving our model economy are the endowment process and the money growth process. To maintain tractability, both processes are assumed to be geometric Brownian motions. Specifically, the endowment y is injected as

$$\frac{dy}{y} = \mu_y \; dt + \sigma_y \; dz_y \tag{3}$$

and nominal money is printed according to the rule

$$\frac{dM}{M} = \mu_M \; dt + \sigma_M \; dz_M \tag{4}$$

where dz_y and dz_M are standard, possibly correlated, Brownian motion processes (with correlation coefficient ρ_{My}), and μ_y, μ_M, σ_y and σ_M are constant, exogenous parameters.

Note that monetary policy does not react systematically to economic develpments (it is "passive", in the traditional sense used by Friedman). Nevertheless, some interaction between monetary policy and real activity does take place through the covariance between monetary and technology shocks. The covariance between real and monetary shocks will play a crucial role in shaping premia attached to monetary policy uncertainty.

Given the exogenous processes, we conjecture that, in equilibrium, all shocks will be lognormally distributed, so that all asset prices can be characterised as geometric Brownian motion processes with constant drift $\mu_j dt$ and constant standard deviation σ_j, for each asset j.

The aforementioned ingredients of the model are standard. The main aspect in which we depart from previous analyses is to account for model misspecification.

[1] The elasticity of intra-temporal substitution does not play any significant role in the model. More precisely, all our propositions would remain unchanged if we selected the simpler Cobb-Douglas form $u(c, m) = c^{\alpha} m^{(1-\alpha)}$, where $\psi = 1$. We adopt the more general CES formulation to be able to investigate the case of $\gamma < 1$ in the numerical simulations without having to assume at the same time that consumption and money are substitutes (i.e. that the cross derivative $u_{cm} < 0$).

[2] Throughout the paper, real variables are denoted in lower-case letters; nominal variables in capital letters.

2.2 Model misspecification

Unfortunately, the simple model outlined above is well-known to be ill-suited to account for the observed behaviour of economic agents towards risk when solved under rational expectations. In order to improve its empirical performance, we follow the suggestion of the recent literature on robust control and use a different assumption on individual's behaviour. Specifically, we assume that the households are uncertain about the exact distributions of the rate of growth of money and the rate of growth of technology. They therefore use the available model as a benchmark and try to devise consumption and investment strategies which can yield "not too undesirable" outcomes if the model does turn out to be misspecified.

Robust rules have been advocated by HansenSargent01 and HansenSargent02. The fundamental premise of the Hansen and Sargent proposal is to point out that models are only approximations of reality. A real world investor would realise that his or her model of the world is potentially misspecified. The investor would then presumably try to take model uncertainty into account when making decisions. Rather than striving to learn the true model over time, the robust decision-maker would simply concentrate on devising decision rules which can help to avoid large mistakes. The available model would therefore only be used as a reference point, to be compared to potential alternatives. The idea is that the differences between alternative models and the reference model are difficult to detect empirically, but consequential for the decision maker.

AndersonHansenSargent have applied this approach to portfolio choice, showing that it gives rise to "model uncertainty premia" alongside standard risk premia. Further characteristics of robust portfolio choice have been derived by Maenhout. In both these papers, model uncertainty is described by a single parameter, which reflects the overall level of model uncertainty in the economy. Building on this approach, UppalWang have shown how to deal with differential degrees of model misspecification for the different state variables of a model. Specifically, they deal with a framework in which the marginal probability laws of each state variable are known with some degree of imprecision, a degree which is possibly different for different state variables.

The general set-up of the UppalWang approach to portfolio choice is summarised in the appendix. For our model, it boils down to assuming that the stochastic processes for the endowment and money growth processes are not known with full confidence. As written above, the processes represent the status quo knowledge of the representative household and they are defined under a certain probability measure P, the "reference probability" or "reference model". If P were known with full confidence, there would be no uncertainty, or ambiguity, as to the reference model. Since households are not fully confident on P, they entertain the possibility that different models may be true, where the different models are indexed by probabilities Q^{ξ} induced by perturbations ξ's of the state vector. In continuous time, Q^{ξ} is then defined by $dQ^{\xi} = \xi dP$ and under the Q^{ξ} measure the law of motion followed by the state vector is simply modified by a drift-adjustment term.

Intuitively, households take as a benchmark a model where the trend rates of technology and money growth are $\mu_y dt$ and $\mu_M dt$, respectively. When taking decisions, however, they entertain the possibility that the actual drifts will be different from the benchmark values, consistently with the idea that it is difficult to estimate exactly the drift of the processes followed by the state variables. The magnitude of the deviations from the benchmark taken into account will depend on a matrix Φ (with elements $\phi_M, \phi_y, \phi_{My} \geqslant 0$, where the subscripts y and M refer to the technology and money growth process, respectively) that indexes the household's confidence on the joint law of motion of the state vector.

As in the case with a single source of model misspecification, when the elements in Φ are very large, the household is extremely confident on P and it will not bother taking alternative models into account. Consumption and portfolio choices then collapse to the standard expected utility case. At the other extreme, when the elements in Φ are very small, the household is so uncertain on the reference model that it will act based on a worst-case scenario. In all intermediate cases, the household balances his concern about model misspecification and the knowneldge that it has about the economy, as represented by P. The UppalWang approach allows us to analyse independently different levels of confidence in the specification of technology growth and the monetary policy rule. For example, if the household is extremely uncertain about the marginal distribution of the endowment process, it will entertain the possibility of large perturbations of its law of motion and the ϕ_y element of the Φ matrix will be close to zero. If, on the contrary, the household is very confident in the marginal distribution of the endowment process, ϕ_y will be large. Similar considerations apply to the marginal distribution of the money supply process, as indexed by ϕ_M. Finally, given that the money supply and technology growth processes are possibly correlated, the term ϕ_{My} will index the household's confidence in the joint distribution of the two processes.

In the appendix, we show that, applying the UppalWang set-up to this problem, the representative household must solve the following problem

$$
\begin{aligned}
0 \quad = \quad & \sup_{c,\omega} \inf_{v_y, v_M} \left\{ \frac{\left(\alpha^{\frac{1}{\psi}} c^{\frac{\psi-1}{\psi}} + (1-\alpha)^{\frac{1}{\psi}} m^{\frac{\psi-1}{\psi}}\right)^{\frac{\psi(1-\gamma)}{\psi-1}}}{1-\gamma} - \delta V + V_w w \mu_w + \frac{1}{2} V_{ww} w^2 \sigma_w^2 \right. \\
& + V_w w \left(v_y y \sigma_{wy} + v_M M \sigma_{Mw}\right) + \frac{1}{2} \psi\left(V\right) w \left(v_y^2 y^2 \sigma_y^2 \left(2\phi_y + \phi_{My}\right)\right. \\
& \left.\left. + v_M^2 M^2 \sigma_M^2 \left(\phi_{My} + 2\phi_M\right) + 2 v_M v_y M y \sigma_{My} \phi_{My}\right)\right\}
\end{aligned} \tag{5}
$$

where ω is the vector of portfolio weights $\omega = (\omega_s, \omega_M, \omega_B, \omega_x, \omega_b)$, v_y, v_M and v_{yM} are perturbations of the state vector, and the drift and standard deviation of wealth, μ_w and σ_w, are known endogenous functions of the other state variables. Finally, the term $\psi\left(V\right)$ is the function which converts the penalty for taking into account the information provided by a certain perturbation of the state vector into units of utility. The particular functional form of $\psi\left(\cdot\right)$ is chosen for analytical convenience. We follow Maenhout and UppalWang and choose $\psi\left(V\right)$ to be proportional to the value function V as $\psi\left(V\right) = (1-\gamma)\, V$.

Note that the first four terms within curly brackets of equation (**??**) are those which appear in the standard planning problem. The fifth term reflects the risk adjustment of wealth which arises when the hypothesis of model misspecification is entertained. Following UppalWang, we will use this term – which depends on misspecification of both the money and technology processes – to evaluate how pessimistic is the worst alternative scenario considered by the household. Finally, the last term within curly brackets of equation (**??**) represents the penalty function used to evaluate the alternative models.

The first order conditions of the household's optimisation problem include equations which express the demand for each available asset. These equations can be rewritten as asset pricing equations, which determine the return on any risky asset as the sum of the safe return, r, plus a premium:

$$
\mu_i = r + \gamma\, \rho_{iw}\, \sigma_i\, \sigma_w - v_y\, y\, \rho_{iy}\, \sigma_i\, \sigma_y - v_M\, M\, \rho_{iM}\, \sigma_i\, \sigma_M \tag{6}
$$

where μ_i is the expected real return on the asset, σ_i its instantaneous standard deviation and ρ_{ij} denotes its instantaneous correlation with process j ($j = w, y, M$).

Note, first, that in the absence of model misspecification, i.e. when the v_y and v_M perturbations are set to zero, equation (**??**) collapses to the standard equation derived in merton71 for the case of lognormal asset prices. The excess return on asset i must include a component which is proportional to the covariance between shocks to the asset price and shocks to real wealth. If the covariance is positive, the asset is risky and must pay a premium. If the covariance is negative, the asset provides a hedge against adverse fluctuations of real wealth: it therefore trades at a discount.

Accounting for model misspecification produces an additional wegde between the risky and riskless returns. We show in the appendix that $v_y, v_M < 0$. Hence, as already noted by UppalWang, model misspecification creates a sort of "hedging demand motive" in the demand for risky assets, which is of a similar form to that emphasised by merton73. Equation (**??**) shows that the hedging demand motive can also be viewed from the perspective of pricing. A "model uncertainty premium", in the words of AndersonHansenSargent, crops up in the asset pricing equation alongside the standard risk premium. The model uncertainty premium is the compensation required by households interested in taking robust decisions to hold assets whose return is uncertain. Since misspecification pertains, by construction, to the evolution of technology and/or the rate of growth of money, the premium crops up when shocks to the return on asset i are correlated with technology and/or money supply shocks.

3 Asset pricing in a monetary equilibrium

3.1 A robust equilibrium asset pricing equation

Definition 1 *The robust equilibium is characterised by consumption and money demand rules, portfolio shares and prices such that: (i) the households solve equation (**??**) subject to the boundary condition* $\lim_{t\to\infty} E\left[\exp\left(-\delta t\right)V\right] = 0$ *; (ii) markets clear continuously, i.e.* $c = y$, $m^d = M^s/P$, $\omega_B = \omega_x = \omega_b = 0$, $\omega_s + \omega_M + \omega_x = 1$.

Substituting out the perturbations v_y and v_M in in equation (**??**), we obtain the asset pricing equation

$$\mu_i = r + \gamma\, \rho_{iw}\, \sigma_i\, \sigma_w + \frac{1}{\Phi_y}\, \rho_{iy}\, \sigma_i\, \sigma_y + \frac{1}{\Phi_M}\, \rho_{My}\, \sigma_y\, \rho_{iM}\, \sigma_i \tag{7}$$

where $\Phi_y \equiv \frac{\phi_M \phi_{My} + 2\phi_M \phi_y + \phi_y \phi_{My} + \frac{1}{2}\phi_{My}^2\left(1-\rho_{My}^2\right)}{\phi_M + \frac{1}{2}\phi_{My}\left(1-\rho_{My}^2\right)}$ and $\Phi_M \equiv \frac{\phi_M \phi_{My} + 2\phi_M \phi_y + \phi_y \phi_{My} + \frac{1}{2}\phi_{My}^2\left(1-\rho_{My}^2\right)}{\phi_y}$. Note that $\Phi_y, \Phi_M \geq 0$ because all ϕ's are positive by construction and $0 \leq \rho_{My} \leq 1$. Note also that, under the assumption of a unique source of model misspecification, Φ_y and Φ_M simplify to $\Phi_y = \phi$ and $\Phi_M \to \infty$.

In equilibrium, both components of the model-uncertainty premium work in the sense of increasing the equilibrium expected return on a risky asset when the asset price covaries positively with the process which is potentially misspecified. This is entirely symmetric to the standard risk premium, which is positive for assets whose price covaries positively with real wealth. Unlike the standard risk premium, however, the factor of proportionality (the "price of risk") is given by a convolution of the parameters which capture model misspecification.

Specifically, both Φ_y and Φ_M are increasing in ϕ_y and ϕ_M, respectively. Thus, quite intuitively, the premium attached to technological uncertainty is increasing in the amount of ambiguity of the reference model of technology growth. Similarly, the premium attached to monetary policy uncertainty is decreasing in the level of confidence in the reference model of the monetary policy rule.

The two terms Φ_M and Φ_y, however, are not affected only by ϕ_M and ϕ_y, respectively: all ϕ_i parameters contribute to shape them. More precisely, Φ_y is decreasing in ϕ_M and Φ_M is decreasing in ϕ_y. In other words, an increase in confidence in the reference model of technology growth determines an increase in the uncertainty premium attached to monetary uncertainty (over and above the aforementioned fall in the premium attached to technological uncertainty). The intuition is that, for given ϕ_M and ϕ_{My}, an increase in ϕ_y means that the existing uncertainty over the correlation between technology and monetary shocks can be attributed to a larger proportion to uncertainty on the marginal distribution of monetary shocks. In this respect, the ambiguity of the reference model of the monetary policy rule increases, thus making Φ_M smaller. This *relative* effect disappears when there are no links between technological and monetary uncertainty, i.e. when $\phi_{My} = 0$: in this special case, Φ_y and Φ_M are independent of ϕ_M and ϕ_y, respectively. A similar intuition explains why Φ_y and Φ_M are also increasing in ϕ_{My}. Such an increase implies that the two shocks are easier to distinguish, so that a smaller uncertainty premium becomes necessary. Finally, as a consistency check, note that $\lim_{\phi_M \to \infty} \Phi_M = \infty$ and $\lim_{\phi_y \to \infty} \Phi_y = \infty$.

For given coefficient Φ_y, technological uncertainty is represented by a scenario in which technology growth, hence the rate of growth of consumption, turnes out to be lower than the level predicted by the reference model. In order to insure themselves against such a scenario, households would like to hold assets which yield a high return when technology growth is surprisingly low. These are assets whose price correlation with technology shocks, ρ_{iy}, is negative. Thus, assets that covary positively with technology growth would have to pay a model uncertainty premium.

The reason why monetary uncertainty also matters for real asset returns is slightly more convoluted. Since the household only cares about real returns, a misspecification of the money growth process is not directly relevant. Indeed, equation (**??**) shows that the standard deviation of monetary shocks has no effect on the model uncertainty premium. However, since the joint distribution of monetary and technology shocks is uncertain, misspecification of the money growth and technology growth processes may be mixed up. Hence, if the correlation between money and technology growth is positive in the reference model, the possibility that trend money growth is lower than in the reference model is "bad news," in the sense that it may imply that technology growth is also lower than in the reference model. Assets whose prices covary positively with monetary shocks are therefore subject to uncertainty risk, in the sense that they yield a low return exactly when consumption growth may also be low if the model is misspecified.

Note that the monetary uncertainty premium is increasing in the correlation ρ_{My}. If $\rho_{My} < 0$, assets whose price is positively correlated with monetary shocks will trade at a discount, because they provide insurance against the risk of model misspecification. Only when $\rho_{My} = 0$ does the model uncertainty premium related to the money growth process vanish. When monetary and technology shocks are uncorrelated in the reference model, there is in fact no reason for households to believe that misspecification of money growth may be a signal of misspecification of the model of technology growth. Even if the $\rho_{My} = 0$ belief were held with very low confidence, monetary shocks would continue to be inconsequential for real returns. Nevertheless, uncertainty on the monetary policy rule would continue to affect optimal portfolio shares also in this special case. The asset pricing equation would simplify to $\mu_i = r + \gamma\,\rho_{iw}\,\sigma_i\,\sigma_w + \frac{1}{\phi_{My}+2\,\phi_y}\,\rho_{iy}\,\sigma_i\,\sigma_y$ and the degree of confidence in the monetary policy rule would affect optimal portfolio shares through the uncertainty premium on technology shocks. More precisely, the smaller ϕ_{My}, i.e. the larger the uncertainty on the actual correlation between monetary and technology shocks, the larger the risk that the reference model may be overestimating the rate of technology growth and, in turn, the larger the uncertainty premium required because of potential misspecification of the technology process.

To gain further insights into equation (**??**), we consider a few special cases. The first is characterised by the highest level of ambiguity on the correlation between money growth and technology shocks, that is the case $\phi_{My} = 0$. It follows that the ambiguity parameters simplify to $\Phi_y|_{\phi_{My}\to 0} = 2\phi_y$ and $\Phi_M|_{\phi_{My}\to 0} = 2\phi_M$, so that $\mu_i = r + \gamma\,\rho_{iw}\,\sigma_i\,\sigma_w + \frac{1}{2\phi_y}\,\rho_{iy}\,\sigma_i\,\sigma_y + \frac{1}{2\phi_M}\,\rho_{My}\,\sigma_y\,\rho_{iM}\,\sigma_i$. In this case the two uncertainty premia are directly proportional to the degree of ambiguity existing on the evolution of the corresponding exogenous variable. Specifically, the uncertainty premium related to the evolution of technology is larger the higher the ambiguity of the technology process – that is, the lower ϕ_y. Similar considerations apply for the component related to uncertainty on money growth. Note that $2\,\phi_y < \Phi_y$ and $2\,\phi_M < \Phi_M$, so the result is an unambiguous increase in both uncertainty premia.

Another interesting special case occurs when there is full confidence in the marginal distribution of technology growth, so that $\phi_y \to \infty$. It follows that $\mu_i = r + \gamma\,\rho_{iw}\,\sigma_i\,\sigma_w + \frac{1}{2\,\phi_M + \phi_{My}}\,\rho_{My}\,\sigma_y\,\rho_{iM}\,\sigma_i$ .Under this scenario, there is still the possibility that the correlation between monetary and technology shocks implied by the reference model is misspecified. As a result, the joint distribution of monetary and technology shocks remains ambiguous, and an uncertainty premium continues to be required by households even if there is no uncertainty on the marginal distribution of technology growth.

Monetary uncertainty only becomes totally irrelevant in the case, already analysed by Maenhout, of a unique source of model misspecification. This leads to $\mu_i = r + \gamma\,\rho_{iw}\,\sigma_i\,\sigma_w + \frac{1}{\phi}\,\rho_{iy}\,\sigma_i\,\sigma_y$, where the separate premium required for uncertainty over the evolution of money growth disappears.

3.2 Misspecification and the equity premium

Using equation (**??**) on equity and solving out for the exogenous variables, we can finally obtain

Proposition 1 *The equity premium, $EP \equiv \mu_S - r$, is given by*

$$EP = \left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\sigma_y^2 \tag{8}$$

The equity premium is increasing in the uncertainty over the distribution of both technology and monetary shocks.

Note that this result encompasses both the case of a single source of model misspecification, when the equity premium collapses to $EP = \left(\gamma + \frac{1}{\phi}\right)\sigma_y^2$, and the case of a standard rational expecations equilibrium, where $EP = \gamma\sigma_y^2$. For future reference, note also that in the special case with maximum uncertainty about the correlation between the shocks, i.e. $\phi_{My} = 0$, the equity premium simplifies to $EP|_{\phi_{My}=0} = \left(\gamma + \frac{1}{2}\frac{1}{\phi_y} + \frac{1}{2}\frac{\rho_{My}^2}{\phi_M}\right)\sigma_y^2$.

The case of a single source of misspecification illustrates the observational equivalence results noted in AndersonHansenSargent and Maenhout that the ambiguity parameter is not distinguishable from risk aversion. A high ambiguity of the reference model could potentially explain equity premia of any size. The level of ϕ necessary to generate the observed equity premium may imply an unrealistic degree of pessimism–see Maenhout.

Equation (**??**) includes a richer description of the level of ambiguity. Specifically, the equity premium tends to be larger than in the rational expectations solution also because of the aforementioned impact on the asset pricing equation of uncertainty over nominal money growth. The reason is that, in equilibrium, equity prices covary perfectly with consumption growth. Equity is therefore a prime example of an asset which tends to yield a surprisingly low return if the reference model of monetary policy is misspecified and correlated with consumption growth.

Proposition **??** highlights how changes in households' confidence in the monetary policy rule, which imply changes in confidence in the equilibrium inflation level consistent with the reference model, can have an impact on the equity premium. More specifically, an increase in uncertainty over trend money growth, hence over equilibrium inflation, will boost the equity premium, as $\partial EP/\partial\phi_M = -2\left(\rho_{My}^2/\Phi_M^2\right)\sigma_y^2 < 0$. Note also that the size of this effect is larger when technological uncertainty is greater, as $\partial^2 EP/\partial\phi_M\partial\phi_y = -4\left(1/\Phi_y\right)\left(\rho_{My}^2/\Phi_M^2\right)\left(\phi_{My}/\phi_y\right)\sigma_y^2 < 0$. These results are consistent with the idea that the equity premium may change when monetary institutions change, to the extent that the change affects households' level of uncertainty over the policy rule. The recent stronger focus on price stability by most central banks should, for example, have resulted in a reductiong of the level of ambiguity over long term money growth, thus a fall in Φ_M and in the relative equity premium.

These results can be seen as consistent with the evidence presented in Blanchard93 on historical movements in the US equity premium. Amongst other findings, Blanchard93 argues that movements in the equity premium around a long term trend are correlated with movements in inflation. The correlation is especially clear in the seventies, when the sharp increase in inflation is associated with a high equity premium, and in the eighties, when a low premium is associated with the fall in inflation. Blanchard93 also argues that "identifying the reason why inflation affects the premium is even more [difficul]" (p. 105). One possibile explanation mentioned in the paper is related to the possibility that investors suffer from money illusion, as argued by ModiglianiCohn. Our alternative explanation in terms of changes in the level of monetary policy uncertainty is also consistent with the postulates of rational decision making.

3.3 Misspecification and the risk-free rate puzzle

Once the model is solved for the expected endogenous return on equity, the asset pricing equation for equity can be inverted to solve for the risk-free rate. This yields the

Proposition 2 *In the robust equilibrium, the risk-free interest rate is*

$$r = \delta + \gamma\mu_y - (1+\gamma)\left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2} \tag{9}$$

The (real) risk-free rate is decreasing in the level of ambiguity of the reference model and, in particular, in the level of uncertainty over trend money growth.

This proposition shows that the impact of model misspecification on the risk-free interest rate is always negative. Specifically, a fall in households' confidence on the accuracy of the reference model has the same effect as an increase of "technological risk," σ_y. These imply a less predictable future evolution of the endowment, thus an increase in precautionary savings and a fall in the risk-free rate.

To compare this result to the standard model, it is useful to rewrite the equilibrium risk-free rate as

$$r = \delta + \gamma\mu_{\widehat{y}} - \gamma^2\frac{\sigma_y^2}{2} - (1+\gamma)\left(\frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2} \tag{9b}$$

where $\mu_{\widehat{y}}$ is defined as $\mu_{\widehat{y}} \equiv \frac{E[d\ln y]}{dt}$ and we have used the equivalence $d\ln y = \frac{dy}{y} - \frac{\sigma_y^2}{2}dt$. In other words, the above expression would have been obtained if we had assumed at the outset that technology shocks followed the process $d\ln y = \mu_{\widehat{y}}dt + \sigma_y dz_y$, rather than adopting assumption (**??**). This slight change facilitates the comparison with some of the literature on the equity premium puzzle. As a consistency check, it can be noted that in the absence of model misspecification, i.e. when $\Phi_y, \Phi_M \to \infty$, the model yields the conventional result $r = \delta + \gamma\,\mu_{\widehat{y}} - \gamma^2\,\frac{\sigma_y^2}{2}$ – see e.g. equation (15) in Campbell02.

In the conventional model, a similar downward pressure on the risk-free rate – which is necessary to address the Weil89 risk-free rate puzzle – can be exercised by increasing the coefficient of relative risk aversion, γ. Such pressure, however, also implies a fall in the elasticity of intertemporal substitution $(1/\gamma)$. Households will be less willing to shift present consumption into the future, which will tend to increase the equilibrium risk-free rate. The first order effect of an increase in the risk aversion coefficient would be to *increase* the risk-free rate (via the term $\gamma\mu_{\widehat{y}}$). As emphasised by Campbell02, a low risk-free rate in the conventional model can only be obtained as a knife-edge solution when γ is sufficiently high and the negative term $-\gamma^2\frac{\sigma_y^2}{2}$ predominates.

A general equilibrium model where agents follow robust control policies can solve the risk-free puzzle. A unique source of misspecification is, in principle, sufficient. In that case, equation (**??**) becomes $r|_{\Phi_y=\phi,\Phi_M=\infty} = \delta + \gamma\left(\mu_{\widehat{y}} - \frac{1}{\phi}\frac{\sigma_y^2}{2} - \left(\gamma + \frac{1}{\phi}\right)\frac{\sigma_y^2}{2}\right)$, where the risk-free rate remains strictly decreasing in the degree of model misspecification. A low risk-free rate can thus be obtained by increasing sufficiently the degree of model misspecification. Maenhout has shown that, in this case, the solution of the risk-free rate puzzle via the model misspecification hypothesis is analogous to the one which would be obtained using recursive utility preferences á la epstein-zin89, weil90 – or, in continuous time, duffie-epstein92a and svensson89.[3] Recursive utility allows one to modify the risk aversion parameter without affecting intertemporal substitutability.

This paper highlights that the *real* risk-free rate can also be affected by uncertainty over a *nominal* variable, namely trend money growth. This result appears most clearly if one assumes that there is full confidence in the benchmark model's description of the marginal distribution of technology shocks, so that $\phi_y \to \infty$ and the risk-free rate becomes $r|_{\phi_y\to\infty} = \delta + \gamma\,\mu_{\widehat{y}} - \gamma^2\frac{\sigma_y^2}{2} - (1+\gamma)\frac{\rho_{My}^2}{2\phi_M+\phi_{My}}\frac{\sigma_y^2}{2}$. Once again, the correlation between monetary and technology shocks, ρ_{My}, plays a key role. In spite of the fact that the marginal distribution of the technology shocks is known with full confidence, the joint distribution of monetary and technology shocks remains ambiguous. Hence, if trend money supply growth turned out to be misspecified and actually lower than in the reference model, the rate of growth of the endowment may also be lower when $\rho_{My} > 0$. This would call for an increase of precautionary savings and cause a fall in the risk-free rate. Similarly, a lower trend money supply growth than in the reference model may imply a high rate of growth of the endowment when $\rho_{My} < 0$. The present discounted value of future endowments, however, would also be negatively correlated with money growth, so there would be scope for reducing the savings rate. The net effect would also be that of a fall in the risk-free rate.

Such increase in precautionary savings only fails to be called for when the correlation between nominal and real shocks is zero in the reference model. When this happens, the distribution of the endowment is believed to be independent of that of money. There would therefore be no need to increase precautionary savings.

Note, finally, that in the special case in which there is maximum uncertainty about the correlation between the shocks, i.e. $\phi_{My} = 0$, the risk-free rate (**??**) simplifies to $r|_{\phi_{My}=0} = \delta + \gamma\mu_y - (1+\gamma)\left(\gamma + \frac{1}{2}\frac{1}{\phi_y} + \frac{1}{2}\frac{\rho_{My}^2}{\phi_M}\right)\frac{\sigma_y^2}{2}$.

4 Misspecification, the natural rate of interest and the relative equity premium

4.1 A risk- and uncertainty-adjusted Fisher equation

We are now ready to derive the equilibrium nominal variables of the model. In order to highlight the effect of inflation on the nominal interest rate, we are going to assume that the central bank steers the money supply in order to achieve an inflation objective π^* in

[3] AndersonHansenSargent prove that, when model uncertainty is described by a single parameter, investors' preferences are observationally equivalent to recursive utility. UppalWang, however, show that the equivalence does not carry through in the case of multiple sources of model uncertainty.

expected terms.[4]

We could specify such target directly in terms of the rate of growth of prices P. For bond holders, however, the price level matters as a deflator of the price of nominal bonds, whose real value is B/P. Given the nonlinearity of the model, an inflation target specified in terms of the rate of growth of P would imply that some convexity terms would play a role on the equilibrium interest rate. In order to highlight the "pure" influence of risk and uncertainty aversion on the equilibrium interest rate, we therefore specify the inflation target in terms of the price deflator $1/P$, i.e. $E[\pi^*] \equiv -(1/P)\,E[d(1/P)]/dt$. It follows that

Proposition 3 *In the robust equilibrium, the trend rate of growth of money which supports the exogenous inflation target* $E[\pi^*]$ *is* $\mu_M^* = E[\pi^*] + \mu_y + \sigma_M^2 - \rho_{My}\,\sigma_M\,\sigma_y$. *Given this equation, the nominal interest rate can be written as*

$$R = r + E[\pi^*] + BP \tag{10}$$

*where r is the risk-free rate defined in equation (**??**) and the inflation risk-plus-uncertainty premium BP is given by*

$$BP = \left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\sigma_y^2 - \left(\gamma + \frac{1}{\Phi_y} + \frac{1}{\Phi_M}\right)\rho_{My}\,\sigma_M\,\sigma_y \tag{11}$$

The definition of the expected inflation target in the Proposition 3 is a stochastic version of the quantity equation. As in the nonstochastic case, it states that trend money growth has to account for trend inflation and trend real growth. When the stochastic nature of the model is taken into account, money growth also needs to be higher the higher the variance of monetary shocks and the lower their correlation with technology shocks. These terms appear because, by construction, a higher variance of the rate of growth of prices translates into a larger rate of growth of $1/P$. In order to keep the rate of growth of $1/P$ constant, trend money growth must increase.

Given the exogenous inflation objective, the equilibrium nominal interest rate can be derived using the general asset pricing equation (**??**). Equation (**??**) shows that it obeys an inflation risk- and uncertainty-adjusted Fisher equation. Since the real return on nominal bonds is negatively correlated with inflation, the inflation premium will be positive when inflationary shocks are negatively correlated with, respectively, shocks to real wealth, to money and to technology growth.

In this model, surprise inflation arises for two reasons: because of negative technology shocks and because of positive money growth shocks. The former are clearly undesirable, because they are perfectly correlated with (they are actually identical to) shocks to real wealth. Thus, they command positive risk and uncertainty premia, represented by the component $\left(\gamma + 1/\Phi_y + \rho_{My}^2/\Phi_M\right)\sigma_y^2$. As to inflationary episodes due to shocks to money growth, these only command a premium insofar as they are negatively correlated with technology shocks–that is, insofar as they tend to be accompanied by negative shocks to wealth. If instead shocks to money growth are positively correlated with technology shocks, then the surprisingly low real return on nominal bonds due to an unexpected monetary expansion tends to be compensated by a positive endowment shock. When $\rho_{My} > 0$, nominal bonds

[4] Monetary and technological shocks make it impossible to attain the inflation objective at all times. Inflation volatility is typically non-zero in the model. It disappears, so that inflation becomes deterministic, only if $\rho_{My} = 1$ and $\sigma_M = \sigma_y$. This can be appreciated from the expression for equilibrium inflation

$$\frac{dP}{P} = \left(\mu_M - \mu_y + \sigma_y^2 - \rho_{My}\sigma_M\sigma_y\right)dt + \sigma_M dz_M - \sigma_y dz_y$$

are a hedge against negative shocks to wealth and therefore pay a negative premium. This is the component $-(\gamma + 1/\Phi_y + 1/\Phi_M)\,\rho_{My}\sigma_M\sigma_y$.[5]

All in all, the overall sign of the bond premium is ambiguous, as it depends on whether the premium required because of real shocks is larger or smaller than the discount accepted because of the possibility of monetary shocks. The bond premium can in fact be split into two components: $(\gamma + 1/\Phi_y)\left(\sigma_y^2 - \rho_{My}\sigma_M\sigma_y\right)$ and $(1/\Phi_M)\,(1/\sigma_M)\,\rho_{My}\sigma_y\left(\rho_{My}\sigma_M\sigma_y - \sigma_M^2\right)$. In empirical calibrations where the standard deviation of technology shocks is approximated by that of consumption growth, the covariance between consumption and monetary shocks is positive but typically smaller than the variance of either consumption or money growth. Hence, the first component of the bond premium tends to be positive, while the second one is often negative.

Finally, note that the size of the bond premium relative to the equity premium depends on the sign of the covariance between money and technology shocks. When this is positive, as is often the case in the empirical evidence, the equity premium is larger–which appears to be intuitively appealing. Ultimately, the sign and size of the bond premium must be determined empirically.

4.2 The natural rate(s) of interest

So far, the only "real interest rate" discussed in the model has been the risk-free rate. The model, however, allows one to analyse explicitly another definition of real interest rate, namely the ex-ante real return on nominal bonds. The latter is simply the nominal rate deflated by the rate of inflation expected over the same time horizon. Unlike the risk-free rate it is observable, if some measure of inflation expectations is available.

Both these real interest rates are related to Wicksell's natural rate of interest which, following Woodford, can be defined as "the equilibrium real rate of return in case of fully flexible prices". In the deterministic steady-state of macroeconomic applications, this notion is typically found to correspond to the inverse of the time discount parameter of the representative household of the model. Since in our stochastic equilibrium, we can define two real interest rates consistent with stable inflation, we will refer to them as natural real rate–or expected real return on the risk-free bond–and natural nominal rate–or expected real return on the nominal bond, respectively.

Definition 2 *The natural nominal rate is defined as the equilibrium real return on the nominal bond, or* $\overline{r} \equiv R - E[\pi^*]$*. The natural real rate is defined as the equilibrium real return on the safe bond, which is equal to the the risk-free. It follows that* $\overline{r} = r + BP$*.*

While being related, the two natural rates are quite different in terms of their determinants.

The natural real rate is independent of monetary developments, namely the variance of monetary shocks. This is a useful property, since it implies that, in a more realistic model with serially shocks and nominal rigidities, the natural real rate could be possibly used as a policy indicator – as suggested by Woodford. Note however that, when households apply robust control as in this model, the natural real rate will still be affected by monetary policy uncertainty and by the correlation between real and monetary shocks (for the reasons discussed regarding the risk-free rate). In this respect, the natural real rate is not entirely insulated from monetary developments.

A more important shortcoming of the natural real rate as a policy indicator is that it would never, if not by chance, be observed in the data – not even in the long run. Even if

[5] Note that the bond premium discussed here is different from the term premium typically analysed in term structure models. The latter is normally a relative premium of long-term over short-term bonds. Here, the bond premium is an absolute premium paid by (very) short-term bonds over and above the risk-free rate.

no shocks occurred in the economy for protracted periods of time, inflation premia would continue to affect the equilibrium return on nominal bonds, thus driving a wedge between the natural real and nominal rates. In this simple model, equilibrium expected returns and premia are constant, so the wedge between the two natural rates is not too consequential. In a more realistic model where inflation premia moved with the inflation cycle, however, the wedge may play an important role in reducing the usefulness of the natural real rate as a policy indicator.

The natural nominal rate is obviously not subject to the aforementioned shortcoming. This is in fact the level around which interest rates would be moved if monetary policy were set following an interest rate rule.

As a policy indicator, however, even the natural nominal rate is not free from drawbacks. Specifically, it is affected by the standard deviation of monetary shocks, through the inflation premium. Consequently, the equilibrium natural nominal rate is not independent of the monetary policy rule followed by the central bank. Under a Taylor-type rule, for example, the equilibrium value of the policy interest rate would be affected by the parameters defining the short-term features of the rule. Different rule tolerating various degrees of short-term inflation volatility would be consistent with different equilibrium values of the natural nominal rate.

To conclude this section, note that the difference between the two natural rates vanishes in a non-stochastic economy (or if households are risk-neutral). In this special case, all rates collapse to the sum of the rate of time preference plus the average (and, in this model, marginal) rate of growth of productivity adjusted for intertemporal substitutability: $r = \delta + \gamma\mu_y$. Only when the steady-state rate of growth of productivity is zero, we recover the standard result of SDGE models that the equilibrium natural rate coincides with the rate of time preference: $r = \delta$ (see e.g. CEE03).

4.3 The relative equity premium

In standard calibrations of the equity premium puzzle–starting from MehraPrescott–the risk-free rate is approximated by the average over a long period of the ex-ante real rate–the natural nominal rate according to the definition above. Once we acknowledge the conceptual role of bond premia in driving a wedge between the risk-free rate and the natural nominal rate, the standard calibration procedure is no longer valid.

We therefore define a new measure of the equity premium which is consistent with the standard calibration scheme. Specifically, we define the "observed" equity premium as the difference between the expected (average) return on equity μ_s and the nominal natural rate $\overline{r}$. This is a *relative* premium, as it describes the size of the equity premium over and above the premium on nominal bonds. From equation (**??**), the relative equity premium can be obtained immediately, since $REP \equiv \mu_s - \overline{r} = \mu_s - r - (\overline{r} - r) = EP - BP$. It is therefore possible to establish the following

Proposition 4 *The "relative equity premium", defined as the expected real return paid by equity over the expected real return on nominal bonds, is given by*

$$REP = \left(\gamma + \frac{1}{\Phi_y} + \frac{1}{\Phi_M}\right) \rho_{My}\,\sigma_M\,\sigma_y \tag{12}$$

¿From the conceptual viewpoint, the difference between the equity premium proper and the relative equity premium is stark. Most notably, monetary shocks matter for the relative equity premium, while they are irrelevant for the equity premium proper. The relative premium is increasing in the volatility (i.e standard deviation) of money growth shocks, when the correlation ρ_{My} is positive, and it could be negative when $\rho_{My} < 0$. The equity premium proper is instead always positive and it is not directly affected by monetary developments.

¿From a quantitative viewpoint, equation (**??**) deepens the equity premium puzzle. First, the covariance between shocks to money and consumption growth – the variables often used in calibrations of the equity premium – is often smaller than the variance of consumption growth. As will become apparent in Section 6, this implies that a larger risk-aversion parameter is necessary to generate the ex-post averages of the equity premium in a standard rational expectations model. In a standard rational expectations equilibrium, the equity premium and the relative equity premium would in fact be given by $EP = \gamma\sigma_y^2$ and $REP = \gamma\rho_{My}\sigma_M\sigma_y$, respectively.

A second way in which equation (**??**) deepens the equity premium puzzle is by showing that the same (technological) factors that produce the equity premium in a simple rational expectations model will also tend to generate an inflation premium on nominal bonds. In this model, the latter component of the inflation premium (the first in eqution (**??**)) is exactly identical to the equity premium: thus, the two components cancel each other out when the relative equity premium is constructed as the difference between the equity premium and the bond premium. Such a stark result is, obviously, model dependent. In more general models, however, one would also expect high equity premia to be associated with high inflation premia, when both premia are linked to shocks that depress consumption and boost inflation at the same time. This is typically the case for permanent technological shocks – ultimately because of the simple arithmetics of the quantity equation. It should also be the case, however, for cost-push shocks in neo-Keynesian models, since they also induce a negative correlation between consumption and inflation – see e.g. ClaridaGaliGertler.

Incidentally, in our model the monetary policy rule that generates a relative equity premium equal to the equity premium proper must be such that $BP = 0$, i.e. that $\sigma_M = \sigma_y$ and $\rho_{My} = 1$. In words, monetary shocks must have the same size as technological shocks and, in addition, they must be such as to perfectly offset the inflationary or deflationary pressures of technological shocks.

Note, finally, that both the equity premium and the relative equity premium are affected by monetary policy uncertainty. Both premia would therefore be influenced by a change in the policy uncertainty premium at times of high and volatile inflation.

5 Comparative statics and welfare

In this section, we present two simple comparative statics exercises, where the model is used to gauge the likely impact on the natural rate of two realistic events. The first application shows how the model can be used to predict the impacts on the relative equity premium and on the natural rate of an increase in uncertainty over the future course of monetary policy. The second exercise analyses the consequences on the natural rate of a "new economy" scenario. We conclude the section with the welfare analysis of the model.

5.1 Monetary uncertainty and a higher risk premium in the seventies

We have already discussed above the ability of the model to generate an increase in the equity premium proper during a period of high inflation, which is consistent with the evidence presented in Blanchard93. We analyse here how the same phenomenon affects the relative equity premium and the natural rates of interest.

In the conventional model used by MehraPrescott, an increase in the relative equity premium at times of high inflation can only be explained if the standard deviation of money also increases with inflation. This channel, however, is likely to play a very minor quantitative role, for reasonable degrees of risk aversion.

Our model can potentially account for much larger increases in the relative equity premium at time of high inflation, if the latter also causes a fall in households' confidence in the future course of monetary policy – that is, a fall in ϕ_M. Using equilibrium conditions, we can establish that, for realistic values of ρ_{My}, σ_y and σ_M (that is, if $\rho_{My} > 0$ and $\rho_{My}\sigma_y < \sigma_M$), $\partial REP/\partial\phi_M < 0$, $\partial r/\partial\phi_M > 0$ and $\partial\overline{r}/\partial\phi_M > 0$ (see the appendix). Hence, an increase in monetary policy uncertainty, i.e. a fall in ϕ_M, leads unambiguously to an increases in the relative equity premium. The size of the increase can be large. For example, when uncertainty about the joint distribution of shocks is maxium, i.e. $\phi_{My} = 0$, the derivative of the relative equity premium simplifies to $\partial REP/\partial\phi_M = -\left(\frac{1}{2}/\phi_M^2\right)\rho_{My}\,\sigma_M\,\sigma_y$, which can become infinitely large as ϕ_M becomes smaller and smaller.

Parallel to the increase in the relative equity premium, an increase in monetary policy uncertainty also determines a fall in the natural rates. This result is consistent with the prolonged fall of ex-ante real rates during the seventies. Such fall was particularly pronounced for short-term rates – that reached negative levels – but it was also observed at longer maturities (see Figure 1).

5.2 A new economy scenario

Some recent papers (see the next section) have claimed that the equity premium may have shrunk in recent years, possibly as a consequence of the "new economy". It is therefore interesting to investigate whether a fall in the relative premium because of new economy "effects" can be justified in a general equilibrium model. A related point, which is of interest for monetary policy, concerns the consequences of the new economy on the natural rate of interest. Permanent changes in the latter would require changes in the policy interest rate to maintain the policy stance unchanged.

A key issue is to understand what exactly should be interpreted as "new economy". One ingredient should certainly be an increase in the expected rate of productivity growth. We also assume here that another ingredient is a change in the households' confidence in their knowledge of the new, higher rate of growth of productivity. The new economy will therefore be captured through an increase in μ_y and a change in ϕ_y.

In the appendix we show that, for realistic parameter values, $\partial REP/\partial\phi_y < 0$, $\partial r/\partial\phi_y > 0$ and $\partial\overline{r}/\partial\phi_y > 0$.

The relative equity premium would obviously remain unchanged in the standard model, since it is not affected by first moments. In our model, the premium falls if the new economy scenario also involves an increased confidence in the trend rate of growth of productivity suggested by the reference model. The relative premium increases instead, if the new economy implies a lower confidence in the reference model.

Changes in investors' confidence in the reference model also affect the natural rates. These tend to increase if the new economy implies an increase in investors' confidence in the future rate of productivity growth, and to fall otherwise. The sensitivity of the natural rates to changes in ϕ_y can be much larger than the sensitivity to changes in μ_y. As a result, the uncertainty premium effect is likely to be predominant in shaping the direction of the change in the natural rates.

5.3 Welfare

To complete the analysis of the model, this section evaluates its welfare implications. Given the model ingredients, it is not surprising that the Friedman rule emerges as the optimal monetary policy.

Proposition 5 *The optimised level of the maximum value function $V(w)$ is*

$$V = w^{1-\gamma}\left((1-\alpha)^{(1-\alpha)}\,\alpha^{\alpha}\right)^{1-\gamma} R^{-(1-\gamma)\,(1-\alpha)}\frac{\left(\frac{c}{s}\right)^{1-\gamma}}{1-\gamma} \tag{13}$$

where R is as defined in equation (??) and $\frac{c}{s} = \delta - (1-\gamma)\left(\mu_y - \left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}\right)$. Welfare is maximised when the rate of growth of money is set to yield $R = 0$, i.e. when expected inflation is

$$E\pi^{Fried} = -\delta - \gamma\mu_y - (1-\gamma)\left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2} + \left(\gamma + \frac{1}{\Phi_y} + \frac{1}{\Phi_M}\right)\rho_{My}\,\sigma_M\,\sigma_y \tag{14}$$

The optimum value of the function $V(\cdot)$ is decreasing in the nominal interest rate, which increases the opportunity cost of holding real balances. It follows that monetary policy can maximise utility by lowering the nominal interest rate as much as possible. Given the zero lower bound on the nominal interest rate, optimal policy will be the Friedman rule prescribing a rate of growth of money such that the nominal interest rate is zero.

Under certainty, the Friedman rule yields an equilibrium rate of deflation equal to minus the rate of time preference. The explicit presence of risk in the equilibrium of the model drives a wedge between the inflation rate compatible with the Friedman rule and the natural real rate, r. Under the Friedman rule, inflation is equal to minus the *ex-ante* real interest rate and it is therefore affected by the bond risk premium.

If such premium is positive, the real rate will be larger than under certainty equivalence and the Friedman rule will imply a higher rate of deflation than under certainty, the more so the higher the level of ambiguity over the money and/or technology processes. If, however, the bond premium is negative, the Friedman rule could possibly be associated with zero or even positive inflation rates.

6 A simple numerical calibration

This section evaluates the quantitative implications of the model for the two natural rates and the equity premium. The calibration is mainly based on US data, for which long data series are available. For comparison purposes, we also present some tentative results related to the euro area.

The standard preference parameters are calibrated using values consistent with the macroeconomic literature. More specifically, for time preference parameter, δ, we use two values consistent with the DSGE literature, 3% and 4% – see e.g. CEE03 and SmetsWouters03. For the coefficient of relative risk aversion, γ, the simulations are based on the values of 0.5 and 6. The first value could be seen as a lower bound, which is consistent with a very small degree of risk aversion, but implies an elasticity of intertemporal substitution which is possibly too large. The second value is on the high side, though not as large as the largest considered by MehraPrescott ($\gamma = 10$).

The parameters which determine how extreme is the worst case scenario considered by investors are less easy to pin down. Their values are therefore selected so as to yield, if possible, the "observed" value of the natural nominal rate and the equity premium. We then investigate the implications of the simulations for the natural real rate and the other premia. We also check how pessimistic is the worst case scenario in terms of the implied rate of growth of consumption. As shown in Section ??, the worst case scenario is such that the rate of growth of consumption is adjusted downwards by the amount $v_y\sigma_{yw}y + v_M\sigma_{Mw}M$. Plugging in the equilibrium values of v_y and v_M, the adjustment turns out to be given by $-\left(1/\Phi_y + \rho_{My}^2/\Phi_M\right)\sigma_y^2$, so that the worst case scenario for growth becomes $WCG =$

$\mu_y - \left(1/\Phi_y + \rho^2_{My}/\Phi_M\right)\sigma^2_y$.[6] This quantity will be monitored in the simulations below to gauge whether our assumptions on the ambiguity parameters are plausible.

For the US, the moments of consumption and money growth (i.e. μ_y, σ_y, σ_M and ρ_{My}) are calibrated from the series of quarterly consumption and yield data collected by Campbell99 and from data on money (M1) available from the Federal Reserve Board.[7] Since the latter dataset is only available from the beginning of 1959, the sample period used in the calibration is 1959Q1-1999Q4. For the euro area, we use the data constructed for the ECB's area wide model – see FaganHenryMestre – complemented by ECB data for money aggregates.

In our US sample, the average premium on equity is broadly consistent with the conventional wisdom. The average excess return on equity is in fact 5.2 percent, while the ex-ante real interest rate is equal to 1.5 percent. A recent stream of research from Blanchard93 to FamaFrench02, however, has argued that the post-WWII premium could be much lower than previously thought. The largest estimate of the equity premium from 1951 to 2000 in FamaFrench02, for example, is only 4.3 percent (and it could be as low as 2.6). In our calibration, we use the sample value as our main benchmark. The lower value suggested by FamaFrench02 is used to gauge some suggestive evidence on possible recent evolution of the natural rates.

6.1 The baseline calibration

As a benchmark for comparison, Table 1 calibrates the model for the cases of no ambiguity ($\phi_y = \phi_{My} = \phi_M = \infty$) and a single source of ambiguity ($\Phi_y = \phi$, $\Phi_M = \infty$). The table reports the equilibrium values of the natural rates, the relative equity premium, the equity premium proper, the bond premium and, finally, the worst case scenario for consumption growth, all in percentage points.

For each of the two values of δ taken into account, the first row shows the value of γ necessary to generate the observed average real rate of 1.5 percent. The implied coefficient of risk aversion is implausibly high, which reiterates the evidence of a risk-free rate puzzle. As argued in section **??**, looking at the relative equity premium deepens the equity premium puzzle. The exceedingly large risk aversion parameter necessary to match the average real rate does generate a large equity premium proper, but is only capable of producing a relative premium of 2 percent.

The other rows in the table are based on values of γ of 0.5 or 6 and selecting ϕ so as to match the observed ex-ante real rate. The results show that, consistently with Maenhout, this model is capable of generating a high equity premium, but only at the cost of a possibly overly pessimistic assumption on the worst case scenario. When the risk aversion parameter is equal to 6, the worst case scenario is such that steady-state consumption growth is approximately -3 percent. For the lower risk aversion parameter, however, the worst case scenario jumps to -22 percent, certainly an implausible assumption.

The main calibration results are shown in Table 2, which is based on the $\phi_{My} = 0$ assumption throughout. This assumption reflects the hypothesis that ambiguity over the correlation between the two stochastic variables of the model is extremely high – the correlation coefficient between the two shocks is in fact very difficult to estimate precisely in-sample. In the first half of the table, our baseline calibration, the other parameters, ϕ_M and ϕ_y, are chosen to jointy match the sample mean of the ex-ante real rate and the relative equity premium. In the second half of the table, we perform comparative static exercises

[6] When the investor has no confidence in the reference model and chooses a min-max approach ($\phi_y = \phi_{My} = \phi_M = 0$), the risk adjustment term on wealth becomes infinitely large.

[7] The data are from Table 1, Money stock measures H.6, available from http://www.federalreserve.gov/releases.

aimed to capture, respectively, a new-economy type scenario and a "monetary instability" scenario.

The baseline calibration confirms that the model can indeed generate the observed relative equity premium, though only assuming a relatively high degree of pessimism of the representative household. The worst-case scenario for consumption growth entertained by the household is never higher than -2 percent. Less pessimistic assumptions are only possible for much larger values of γ, which however also yield unrealistically high values for the bond premium.

The better performance of the model with respect to the case of a single source of model misspecification arises from the contribution of monetary uncertainty. This helps to boost the relative equity premium without producing large effects on the degree of pessimism in consumption growth (the effect on the latter is in fact weighed by the squared correlation between money and output shocks). The calibrations also produce an estimated inflation premium on nominal bonds that, with one exception, varies between -50 and 30 basis points. These are not unrealistic values. The ratio between the sample standard deviations of the ex-ante real rate and equity prices is approximately 1/10, which compares to a similar or lower ratio for the bond and relative equity premia.

When it is negative, the bond premium helps to explain the equity premium puzzle since, by construction, it is subtracted from the equity premium proper to obtain the relative equity premium. The bond premium also drives a wedge between natural real and nominal rates, which can be positive or negative depending on the sign of the premium. For given natural nominal rate of 1.5 percent, the natural real rate is estimated to be between 1.1 and 2.0 percent, depending on parameter values (and dismissing the unrealistic configuration $\delta = 3$, $\gamma = 0.5$). This is much lower than the values of 3 or 4 percent implied by the non-stochastic steady-state of the model (for the case of zero equilibrium consumption growth).

6.2 Comparative static exercises

In the second third of Table 2, we focus on the $\gamma = 6$ case and analyse the effects of a 20 percent increase in ϕ_y and a 20 percent fall in ϕ_M, respectively. These two comparative static exercises aim to capture, respectively, a new-economy type scenario and a "monetary instability" scenario.

For the new economy scenario, we abstract from changes in the expected rate of growth of technology and simply focus on the assumption that new economy means increased confidence in the rate of technology growth of the reference model. The table shows that this assumption leads to a quite large increase in the natural rates. The natural real rate jumps up by between 2.5 and 2.8 percentage points and the natural nominal rate by just over 2 percentage points. The bond and equity premia fall because of the increased confidence of households in the reference model of the economy.

The monetary instability scenario is presented next, at the end of Table 2. In this case, the natural real rate falls by between 0.2 and 0.6 percentage points, while the natural nominal rate falls by a whole percentage point. The difference is accounted for by a 70 to 80 basis points fall in the bond premium, which turns negative and provides the main contribution to the increase of the relative equity premium.

Table 3 and 4 present some suggestive evidence on the recent evolution of the natural rates. More specifically, Table 3 focuses on the post-1984 period, which could be seen as coincident with a regained monetary stability in the United States. It then derives model-consistent natural real rate and premia based on the sample values of the moments of the exogenous variables, the ex-post natural nominal rate and the FamaFrench02 estimates pf the post-war equity premium. For comparative purposes, Table 4 replicates the same exercise for the euro area based on the 1997-2001 sample and on the same value of the equity premium as in the US.

Table 3 appears to confirm that the post-1984 data are consistent with a period of increased confidence in the monetary policy rule. The value of ϕ_M does in fact increase by approximately 150 percent. At the same time, the model interprets the data as consistent with a fall of confidence, though smaller in percentage terms, in the reference model of technology growth. The net effect is a sizable increase in both the natural nominal rate (imposed in the calibration) and the natural real rate. The latter appears to hover around the 3 percent level.

The euro area data appear to be consistent with a similar level of ambiguity of the reference model of technology growth, but much larger confidence in monetary stability. Almost identical values of the natural nominal rate and the equity premium are consistent with a much higher value of the ϕ_M parameter. Consequently, the natural real rate appears to be lower than in the US and approximately equal to 2.5 percent. The bond premium also turns out to be closer to zero. The euro area results should obviously be interpreted with special caution, given the particularly short sample on which they are based.

7 Conclusions

Two definitions of the equilibrium natural rate of interest are available in a stochastic general equilibrium model. While the difference between the two rates, due to inflation premia, can be small from an empirical viewpoint, it is consequential if the natural rate of interest is to be used for policy analysis.

¿From a more general viewpoint, risk and uncertainty premia appear to be important to reconcile the long term natural rate generated by general equilibrium models with the evidence on the equity premium and risk-free rate puzzles. Both the equity premium and the natural rates are related to households' confidence in the monetary policy rule followed by the central bank and perceptions of technology growth.

Table 1: Risk premia and real rates under a single source of model misspecification

United States
(in percentage points)

δ	γ	$\phi \times 100$	$\overline{r}$	r	REP	EP	BP	WCG
3	564	∞	1.5	−1.6	2.0	5.1	3.0	2.5
3	0.5	0.096	1.5	−9.8	7.5	18.8	11.3	−16.2
3	6.0	0.315	1.5	−2.0	2.3	5.8	3.5	−3.2
4	564	∞	1.5	−1.6	2.0	5.1	3.1	2.5
4	0.5	0.071	1.5	−13.8	10.2	25.4	15.3	−22.9
4	6.0	0.297	1.5	−2.2	2.4	6.1	3.7	−3.5

Source: Campbell (2002) and Federal Reserve Board. Sample: 1959Q1-1999Q4.
Note: under the assumption of a unique source of misspecification ϕ; σ_M, σ_y and ρ_{My} are sample moments of real consumption growth and money growth M1 (2.36%, 0.95% and 0.16, respectively); ϕ is chosen to match the sample mean of the ex-ante real interest rate (1.46%).

Table 2: Risk premia and real rates under multiple sources of model misspecification

United States
(in percentage points)

δ	γ	$\phi_y \times 100$	$\phi_M \times 100$	$\overline{r}$	r	REP	EP	BP	WCG
3	0.5	0.048	0.127	1.5	−2.9	5.2	9.5	4.3	−7.0
3	6.0	0.103	0.053	1.5	2.0	5.2	4.6	−0.5	−2.0
4	0.5	0.085	0.059	1.5	1.1	5.2	5.5	0.3	−3.0
4	6.0	0.095	0.055	1.5	1.6	5.2	5.0	−0.2	−2.4
3	6.0	0.124	0.053	3.6	4.6	4.9	3.9	−1.0	−1.3
4	6.0	0.114	0.055	3.8	4.4	4.9	4.2	−0.6	−1.6
3	6.0	0.103	0.042	0.5	1.8	6.0	4.7	−1.3	−2.1
4	6.0	0.095	0.044	0.5	1.4	6.0	5.1	−0.9	−2.5

Source: Campbell (2002) and Federal Reserve Board. Sample: 1959Q1-1999Q4.
Note: under the assumption $\phi_{My} = 0$; σ_M, σ_y and ρ_{My} are sample moments of real consumption growth and money growth M1 (2.56%, 0.95% and 0.36, respectively); in the $\gamma = 1$ case, ϕ_y and ϕ_M are chosen to match the sample mean of the ex-ante real rate (1.46%) and to yield the largest possible relative equity premium; in the $\gamma = 4$ case, ϕ_y and ϕ_M are chosen to match the sample means of the ex-ante real rate and of the relative equity premium (5.23%). When only one of these two parameters is free, the other is chosen to to match the sample means of the ex-ante real rate.

Table 3: Recent low-equity-premium scenario

(a) United States
(in percentage points)

δ	γ	$\phi_y \times 100$	$\phi_M \times 100$	$\overline{r}$	r	REP	EP	BP	WCG
3	6.0	0.085	0.134	2.5	3.2	4.3	3.6	−0.7	−1.4
4	6.0	0.076	0.175	2.5	2.8	4.3	4.0	−0.3	−1.8

Source: Campbell (2002) and Federal Reserve Board. Sample: 1984Q1-1998Q4.
Note: under the assumption $\phi_{My} = 0$; σ_y, σ_M and ρ_{My} sample moments of real consumption growth and money growth M1; ϕ_M and ϕ_y are chosen to match the average ex-ante real rate and the largest estimate of the equity premium in Fama and French (2002).

(b) Euro area
(in percentage points)

δ	γ	$\phi_y \times 100$	$\phi_M \times 100$	$\overline{r}$	r	REP	EP	BP	WCG
3	6.0	0.079	0.583	2.4	2.7	4.3	4.0	−0.3	−1.7
4	6.0	0.071	3.570	2.4	2.3	4.3	4.4	0.1	−2.1

Source: Fagan, Henry and Mestre (2001) ECB. Sample: 1997Q1-2001Q4.
Note: under the assumption $\phi_{My} = 0$; σ_M, σ_y and ρ_{My} are sample moments of real consumption growth and money growth M1 (1.47%, 2.29% and 0.3269, respectively, where the latter is estimated over the period 1980-2001); ϕ_M and ϕ_y are chosen to match the average ex-ante real interest rate and the same equity premium as in the US.

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Source: Campbell (1999).
Note: the short rate is the T-bill rate; the long rate is the 10-year treasury yield. The two series are deflated using expected inflation (over the respective relevant horizons) as obtained from a quarterly VAR including the two nominal rates, the inflation rate and the rate of growth of consumption, and estimated over the 1952Q1-1998Q4 period.

A Appendix

A.1 The Uppal and Wang (2002) approach

The general set-up of the robust approach to portfolio choice can be summarised as follows. The status quo knowledge of an investor about the uncertainty in the economy is described by a certain probability measure P, the "reference probability" or "reference model". If P, which is typically the result of some estimation, were estimated with full confidence, there would be no uncertainty, or ambiguity, as to the reference model. However, possibly due to insufficient data, the estimation of P is not fully confident. The investor therefore entertains the possibility that different models may be true, where the different models are indexed by probabilities Q^w given by $dQ^w = w\left(X_{t+1}\right) dP$ where X_t is a state variable and $w\left(x\right)$ is a density function. The investor also has an index that tells him, given the available information, how each alternative model compares to the reference model. AndersonHansen-Sargent, Maenhout and UppalWang use relative entropy, defined as the expected value of the log-likelihood ratio between the two models, i.e. the log of the Radon-Nykodym derivative above: $\phi L\left(w\right) = \phi E_t^w \left[\ln w\left(x_{t+1}\right)\right]$. Note that the conditional expectation is evaluated with respect to the density of the alternative model. The likelihood ratio is adjusted for the level of ambiguity of knowledge of the reference model, which is represented by the nonnegative parameter ϕ.

In order to use his or her available information about potential model misspecification, as summarised by $\phi L\left(w\right)$, the investor employs a certainty equivalent function to evaluate his future utility. In discrete time,

$$V_t = u\left(c_t\right) + \beta \inf_w \left[\phi L\left(w\right) + E_t^w \left[V_{t+1}\right]\right] \tag{15}$$

where $\beta > 0$ is the time discount parameter. This formulation implies that the investor considers whether to use an alternative model to evaluate his future utility. The term $\phi L\left(w\right)$ is a penalty for rejecting the reference model P and using Q^w instead. The magnitude of the penalty depends on the level of ambiguity in the reference model. When $\phi \approx \infty$, the investor is extremely confident on P and he will not enterntain alternatives. The scheme then collapses to standard expected utility. If, at the other extreme, $\phi \approx 0$, the investor is so uncertain that he will act based on a worst-case scenario, ie $V_t = u\left(c_t\right) + \beta \inf V_{t+1}$. In all intermediate cases, the investor balances his concern about model misspecification and the knowneldge that he has about the economy, as represented by P.

In order to deal with different levels of ambiguity for the different state variables belonging to the vector $\mathbf{X}_t = \left(X_{1t}, ..., X_{nt}\right)$, UppalWang introduce an extension of the model above. The function used to evaluate current and future utility becomes

$$V_t = u\left(c_t\right) + \beta \inf_{\xi, \xi_{ij}} \left\{ \frac{\psi\left(V_t\right)}{2} \sum_{i,j=1}^{n} \phi_{ij} L\left(\xi_{ij}\right) + E_t^{\xi}\left[V_{t+1}\right] \right\}, \tag{16}$$

where $\phi_{ij} = \phi_{ji} \geq 0$. The ξ's are perturbations of the state vector that induce alternative models Q^{ξ}. Each ξ_{ij} represents a source of information about the joint distribution of X_{it} and X_{jt}. The term $\psi\left(V_t\right) \phi_{ij} L\left(\xi_{ij}\right)/2$ is the penalty for taking into account that source of information, a penalty constructed as the product of the "likelihood ratio", $\phi_{ij} L\left(\xi_{ij}\right)$, by the term $\psi\left(V_t\right)$ which converts the penalty to units of utility (consistently with the units of $E^{\xi}\left[V_{t+1}\right]$). The particular functional form of $\psi\left(\cdot\right)$ is chosen for analytical convenience.

In continuous time, the model is specified as follows.

The state vector $\mathbf{X}_t$ follows the process $d\mathbf{X}_t = \mu_X dt + \sigma_X d\mathbf{z}_t$, where $\mathbf{z}_t$ is an n-dimensional Brownian motion and the drift and diffusion terms are possibly functions of the state and of time $\mu_X = \mu_X\left(\mathbf{X}_t, t\right)$, $\sigma_X = \sigma_X\left(\mathbf{X}_t, t\right)$. The perturbations ξ_{ij} and ξ are specified as perturbations of the distribution of the state vector such that

$$d\xi_t = -\xi_t \left[\sum_{i=1}^{n} a_{it} \right]' dz_t \quad (17)$$

$$d\xi_{ijt} = \xi_{ijt} \left[a_{it} + a_{jt} \right]' dz_t \quad (18)$$

where $a_{it}^T = (0, ...0, v_{it}, 0, ..., 0)\, \sigma_X$ and $\xi_0 = \xi_{ij0} = 1$. Q^ξ is then defined by $dQ^\xi = \xi dP$, so that, under the Q^ξ measure, $d\mathbf{z}_t^\xi = d\mathbf{z}_t + \sum_{i=1}^n a_{it} dt$ is a Brownian motion and the state vector is modified by a drift-adjustment term: $d\mathbf{X}_t = (\mu_X - \sigma_X \sum_{i=1}^n a_{it})\, dt + \sigma_X (\mathbf{X}_t, t)\, d\mathbf{z}_t^\xi$.

Given the form of a_{it} and the ξ_t and ξ_{ijt}, the continuous time version of the maximum value function is

$$0 = \inf_{v_i} \left\{ u(c) - \delta V + V_t + \mu_X V_X + \frac{1}{2} tr \left(V_{XX} \sigma_X \sigma_X' \right) + \mathbf{v}' \sigma_X \sigma_X' V_X + \frac{\psi(V)}{2} \mathbf{v}' \Phi \mathbf{v} \right\} \quad (19)$$

where $\delta > 0$ is the rate of time preference, $\mathbf{v} = (v_1, ..., v_n)'$ and Φ is the $n \times n$ matrix

$$\Phi = \begin{pmatrix} \left(\phi_{11} + \sum_{j=1}^n \phi_{1j}\right) \sigma_1^2 & \phi_{12}\sigma_{12} & ... & \phi_{1n}\sigma_{1n} \\ \phi_{12}\sigma_{12} & \left(\phi_{22} + \sum_{j=1}^n \phi_{2j}\right) \sigma_2^2 & ... & \phi_{2n}\sigma_{2n} \\ ... & ... & ... & ... \\ \phi_{1n}\sigma_{1n} & \phi_{2n}\sigma_{2n} & ... & \left(\phi_{nn} + \sum_{j=1}^n \phi_{nj}\right) \sigma_n^2 \end{pmatrix} \quad (20)$$

which indexes the investor's confidence on the law of motion of the state vector.

The two last terms in the Hamilton-Jacobi-Bellman equation are those induced by ambiguity. The term $\mathbf{v}'\sigma_X \sigma_X' V_X$ is induced by the change of measure from P to Q^w: by Girsanov's theorem, this is entirely captured by a simple change of drift. This outcome of the way of modelling uncertainty is therefore entirely consistent with the idea that it is difficult to estimate the drift of the processes followed by the state variables. The last term, $\psi(V)\, \mathbf{v}'\Phi\mathbf{v}/2$, is the penalty function.

A.2 The model in the main text

For the model in the main text, the state vector is represented by the endowment and money growth processes, reported below for convenience

$$\begin{aligned} \frac{dy}{y} &= \mu_y\, dt + \sigma_y\, dz_y \\ \frac{dM}{M} &= \mu_M\, dt + \sigma_M\, dz_M \end{aligned}$$

We assume that households are not entirely sure on the above specification of endowment and money growth. These equations are only used as approximations. They represent the reference model, which households are going to compare to possible alternative when taking decisions.

In terms of the UppalWang set-up, the model in the text corresponds to the $n = 2$ case, with two exogenous state variables, y and M. It follows that $\mathbf{v} = (v_y, v_M)$ and $\sigma_X = (\sigma_y y, \sigma_M M)'$, so that

$$\begin{aligned} \sigma_X \sigma_X' &= \begin{pmatrix} \sigma_y y \\ \sigma_M M \end{pmatrix} \begin{pmatrix} \sigma_y y & \sigma_M M \end{pmatrix} \\ &= \begin{pmatrix} \sigma_y^2 y^2 & \sigma_{My} M y \\ \sigma_{My} M y & \sigma_M^2 M^2 \end{pmatrix} \end{aligned}$$

and

$$\Phi = \begin{pmatrix} (2\phi_y + \phi_{My})\,\sigma_y^2 y^2 & \phi_{My}\sigma_{My}My \\ \phi_{My}\sigma_{My}My & (\phi_{My} + 2\phi_M)\,\sigma_M^2 M^2 \end{pmatrix}$$

Note that in the special case of a single source of model misspecification, Φ simplifies to

$$\Phi = \phi \cdot \begin{pmatrix} \sigma_y^2 y^2 & \sigma_{My}My \\ \sigma_{My}My & \sigma_M^2 M^2 \end{pmatrix}$$

For convenience, note that the five assets available in the economy have returns defined as follows

$$\begin{aligned}
\frac{ds}{s} &= \left(\mu_s - \frac{y}{s}\right)\, dt + \sigma_s\, dz_s \\
\frac{dM}{M} &= \left(-\mu_P + \sigma_P^2\right)\, dt - \sigma_P\, dz_P \\
\frac{dB}{B} &= \left(R - \mu_P + \sigma_P^2\right)\, dt - \sigma_P\, dz_P \\
\frac{db}{b} &= r\, dt \\
\frac{dx}{x} &= \left(\mu_x - \frac{\tau}{x}\right)\, dt + \sigma_x\, dz_x
\end{aligned}$$

Note that: the returns on equity, s, and on the present discounted value of future transfers, x, include a dividend yield as well as a capital gain; the returns on money and the nominal bond are zero and Rdt, respectively, in nominal terms, and negatively affected by inflation in real terms. In the equations above, R, r, the ratios $\frac{y}{s}$ and $\frac{\tau}{x}$ and all μ_i's and σ_i's have to be determined in equilibrium.

It follows that the maximum value function of the optimisation problem of the representative household can be written as a function $V = V(w_t, k_t, M_t, t)$ and it will have to solve

$$\begin{aligned}
0 &= \sup_{c,\omega}\inf_{v_y,v_M} \{u(c) - \delta V + V_t + V_w\mu_w w + k\mu_y V_y + M\mu_M V_M \\
&\quad + \frac{1}{2}V_{ww}\sigma_w^2 w^2 + V_{yw}\sigma_{wy}wy + V_{Mw}\sigma_{Mw}Mw + \frac{1}{2}V_{yy}\sigma_y^2 y^2 \\
&\quad + V_{My}\sigma_{My}My + \frac{1}{2}V_{MM}\sigma_M^2 M^2 + V_w w\left(v_y\sigma_{yw}y + v_M\sigma_{Mw}M\right) \\
&\quad + V_y y\left(v_y\sigma_y^2 y + v_M\sigma_{My}M\right) + V_M M\left(v_y\sigma_{My}y + v_M\sigma_M^2 M\right) \\
&\quad + \frac{1}{2}\psi(V)\,\big(v_y^2\sigma_y^2 y^2(2\phi_y + \phi_{My}) \\
&\quad + v_M^2\sigma_M^2 M^2(\phi_{My} + 2\phi_M) + 2v_M v_y\sigma_{My}yM\phi_{My}\big)\}
\end{aligned}$$

where $\omega = (\omega_s, \omega_M, \omega_B, \omega_x, \omega_b)$ are the porfolio shares in the 5 assets available in the economy and μ_w and σ_w are shorthand notation for the drift and standard deviation of the wealth process

$$\begin{aligned}
\frac{dw}{w} &= \left[(\omega_M + \omega_B)\left(R - \mu_P + \sigma_P^2 - r\right) + \omega_s(\mu_s - r) + \omega_x(\mu_x - r) - \omega_M R + r - \frac{c}{w}\right] dt \\
&\quad - (\omega_M + \omega_B)\,\sigma_P dz_P + \omega_s\sigma_s dz_s + \omega_x\sigma_x dz_x
\end{aligned}$$

so that σ_w^2 is the variance of the rate of growth of wealth and σ_{iw} the covariance between shocks to wealth and to process i. Note that the above equation incorporates the constraint $\omega_b = 1 - \omega_s - \omega_M - \omega_B - \omega_x$.

Given the utility function postulated above and that the money supply and production processes are lognormal, we can guess that V will be independent of the state variables M

and y and of time. If all derivatives of the maximum value function with respect to these arguments are set to zero, the Hamilton-Jacobi-Bellman equation simplifies to

$$\begin{aligned} 0 &= \sup_{c,\omega} \inf_{v_y, v_M} \left\{ \frac{\left(\alpha^{\frac{1}{\psi}} c^{\frac{\psi-1}{\psi}} + (1-\alpha)^{\frac{1}{\psi}} m^{\frac{\psi-1}{\psi}} \right)^{\frac{\psi(1-\gamma)}{\psi-1}}}{1-\gamma} - \delta V + V_w w \mu_w + \frac{1}{2} V_{ww} w^2 \sigma_w^2 \right. \\ & +V_w w \left(v_y y \sigma_{wy} + v_M M \sigma_{Mw} \right) + \frac{1}{2} \psi(V) \left(v_y^2 y^2 \sigma_y^2 \left(2\phi_y + \phi_{My} \right) \right. \\ & \left. \left. + v_M^2 M^2 \sigma_M^2 \left(\phi_{My} + 2\phi_M \right) + 2 v_M v_y y M \sigma_{My} \phi_{My} \right) \right\} \end{aligned}$$

Using the conjecture $V(w) = \kappa \frac{w^{1-\gamma}}{1-\gamma}$ and adopting the functional form $\psi(V) = (1-\gamma) V$, the first order conditions of the problem can be written as

$$\begin{aligned} \frac{\partial}{\partial c} &: \alpha^\gamma c_t^{-\gamma} R^{\frac{\psi(\gamma\psi-1)}{\psi-1}} \left(R(1-\alpha) + R^\psi \alpha \right)^{\frac{1-\gamma\psi}{\psi-1}} = w^{-\gamma}\kappa \\ \frac{\partial}{\partial \omega_M} &: m = c \frac{1-\alpha}{\alpha} R^{-\psi} \\ \frac{\partial}{\partial \omega_B} &: R - \mu_P + \sigma_P^2 - r = -\gamma\sigma_{Pw} + v_y y \sigma_{Py} + v_M M \sigma_{MP} \\ \frac{\partial}{\partial \omega_s} &: \mu_s - r = \gamma\sigma_{sw} - v_y y \sigma_{sy} - v_M M \sigma_{Ms} \\ \frac{\partial}{\partial \omega_x} &: \mu_x - r = \gamma\sigma_{wx} - v_y y \sigma_{xy} - v_M M \sigma_{Mx} \\ \frac{\partial}{\partial v_y} &: \gamma\sigma_{wy} + v_y \sigma_y^2 y \left(2\phi_y + \phi_{My} \right) + v_M \sigma_{My} M \phi_{My} = 0 \\ \frac{\partial}{\partial v_M} &: \gamma\sigma_{Mw} + v_M \sigma_M^2 M \left(\phi_{My} + 2\phi_M \right) + v_y \sigma_{My} y \phi_{My} = 0. \end{aligned}$$

Postulate now that all covariances will have constant correlation coefficients, so that $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ with ρ_{ij} being the correlation coefficient and σ_i and σ_j denoting standard deviations (all constants).

The 3rd to 5th first order conditions are asset pricing equations. For a generic asset i, one would obtain $\mu_i - r = \gamma\rho_{iw}\sigma_i\sigma_w - v_y y \rho_{iy}\sigma_i\sigma_y - v_M M \rho_{iM}\sigma_i\sigma_M$, which is equation (**??**) in the text.

It is then possible to solve the last two equations for v_M and v_y to obtain

$$\begin{aligned} v_M &= \frac{\gamma}{M} \frac{\rho_{wy}\sigma_w\sigma_y\rho_{My}\sigma_M\sigma_y\phi_{My} - \rho_{Mw}\sigma_M\sigma_w\sigma_y^2 \left(2\phi_y + \phi_{My} \right)}{\sigma_M^2 \left(\phi_{My} + 2\phi_M \right) \sigma_y^2 \left(2\phi_y + \phi_{My} \right) - \rho_{My}^2 \sigma_M^2 \sigma_y^2 \phi_{My}^2} \\ v_y &= \frac{\gamma}{y} \frac{\rho_{wy}\sigma_w\sigma_y\sigma_M^2 \left(2\phi_M + \phi_{My} \right) - \rho_{My}\sigma_M\sigma_y\rho_{Mw}\sigma_M\sigma_w\phi_{My}}{\rho_{My}^2 \sigma_M^2 \sigma_y^2 \phi_{My}^2 - \sigma_M^2 \left(\phi_{My} + 2\phi_M \right) \sigma_y^2 \left(2\phi_y + \phi_{My} \right)} \end{aligned}$$

These can be used in the asset pricing equations to find, for the generic asset i,

$$\begin{aligned} \mu_i &= r + \gamma\rho_{iw}\sigma_i\sigma_w + \gamma \frac{\rho_{wy}\sigma_w\sigma_y\sigma_M^2 \left(2\phi_M + \phi_{My} \right) - \rho_{My}\sigma_M\sigma_y\rho_{Mw}\sigma_M\sigma_w\phi_{My}}{\rho_{My}^2 \sigma_M^2 \sigma_y^2 \phi_{My}^2 - \sigma_M^2 \left(\phi_{My} + 2\phi_M \right) \sigma_y^2 \left(2\phi_y + \phi_{My} \right)} \rho_{iy}\sigma_i\sigma_y \\ & + \gamma \frac{\rho_{wy}\sigma_w\sigma_y\rho_{My}\sigma_M\sigma_y\phi_{My} - \rho_{Mw}\sigma_M\sigma_w\sigma_y^2 \left(2\phi_y + \phi_{My} \right)}{\sigma_M^2 \left(\phi_{My} + 2\phi_M \right) \sigma_y^2 \left(2\phi_y + \phi_{My} \right) - \rho_{My}^2 \sigma_M^2 \sigma_y^2 \phi_{My}^2} \rho_{My}\sigma_y\rho_{iM}\sigma_i \end{aligned}$$

which can be written as equation (**??**) in the text.

Now solve the macroeconomic side of the model. Recall that, in equilibrium, $c = y$ and that the total return on equity includes both a capital gain and a dividend, i.e. $ds/s = (\mu_s - c/s)\,dt + \sigma_s dz_s$. The same holds true for the sum of real money holdings and the x asset, which can be defined as $\overline{x} \equiv x + m$, which yields the expected return μ_x plus the opportunity cost of holding money, equal to the foregone interest Rm, so that $d\overline{x}/\overline{x} = (\mu_x - Rm/\overline{x})\,dt + \sigma_x dz_x$.

Conjecture that the evolution of real wealth is independent of monetary factors. Since nominal and real bonds are in zero net supply, equilibrium wealth is given by $w = s + \overline{x}$. It follows that $dw/w = ds/s = d\overline{x}/\overline{x}$. Matching drift and diffusion terms, $(\mu_s - c/s) = (\mu_x - Rm/x)$ and $\sigma_s dz_s = \sigma_x dz_x = \sigma_w dz_w$. Using the latter equality in the asset pricing equations for equity and the x asset, it follows that $\mu_s = \mu_x$, so that $c/s = Rm/\overline{x}$ and, rearranging terms, $\overline{x}/s = Rm/c = R^{1-\psi}(1-\alpha)/\alpha$. From the equilibrium composition of real wealth, it finally follows that $w = \left(1 + R^{1-\psi}(1-\alpha)/\alpha\right)s$. This can be used to substitute out w in the first order conditions and to obtain, amongts other results,

$$c = R^{-\frac{\psi(1-\gamma)}{\gamma(1-\psi)}} \left(R(1-\alpha) + R^{\psi}\alpha\right)^{-\frac{1-\gamma}{\gamma(1-\psi)}} \kappa^{-\frac{1}{\gamma}} s \tag{21}$$

so that

$$\mu_w = \mu_s - R^{-\frac{\psi(1-\gamma)}{\gamma(1-\psi)}} \left(R(1-\alpha) + R^{\psi}\alpha\right)^{-\frac{1-\gamma}{\gamma(1-\psi)}} \kappa^{-\frac{1}{\gamma}} \tag{22}$$

The values of v_M, v_y, μ_w, and the second moments of wealth can be substituted back in the Bellman equation to verify that the conjectured form of the maximum value function does solve the equation for a constant value of κ. This value can be substituted in equations (**??**) and (**??**) so as to express them as functions of exogenous variables and first and second moments of the evolution of equity, s. Finally, noting that c is proportional to s, so that $dc/c = ds/s$, but also $dc/c = dy/y$ because of the equilibrium condition $c = y$, it follows that $ds/s = dy/y$. Hence $\sigma_s = \sigma_y$, $\rho_{sy} = 1$, $\rho_{Ms} = \rho_{My}$ and equation (**??**) must equal μ_y, which implies

$$\mu_S = \delta + \gamma\mu_y + (1-\gamma)\left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}$$

Using this value in the asset pricing equation for equity and solving for r one finds

$$r = \delta + \gamma\mu_y - (1+\gamma)\left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}$$

that is Proposition **??**. It follows that the equity premium is simply $EP = \left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\sigma_y^2$, as in Proposition **??**.

For the inflation rate, note that $Rm/c = R^{1-\psi}(1-\alpha)/\alpha$ implies that, applying Ito's lemma,

$$\frac{dP}{P} = \left(\mu_M - \mu_y + \sigma_y^2 - \rho_{My}\sigma_M\sigma_y\right)dt + \sigma_M dz_M - \sigma_y dz_y \tag{23}$$

and

$$\frac{d\frac{1}{P}}{\frac{1}{P}} = -\left(\mu_M - \mu_y - \sigma_M^2 + \rho_{My}\sigma_M\sigma_y\right)dt - \sigma_M dz_M + \sigma_y dz_y$$

Defining the inflation target as minus the drift of the inverse of the price level, $E[\pi^*] = \mu_M - \mu_y - \sigma_M^2 + \rho_{My}\sigma_M\sigma_y$, it follows that this can be attained by an appropriate choice of the average rate of growth of money: $\mu_M^* = E[\pi^*] + \mu_y + \sigma_M^2 - \rho_{My}\,\sigma_M\,\sigma_y$. Using equation (**??**) in the asset pricing equation for nominal bonds, it is finally possible to obtain the nominal interest rate

$$
\begin{aligned}
R &= r + E\left[\pi^*\right] - \gamma\, \rho_{Pw}\, \sigma_P\, \sigma_w - \frac{1}{\Phi_y}\, \rho_{Py}\, \sigma_P\, \sigma_y - \frac{1}{\Phi_M}\, \rho_{My}\, \sigma_y\, \rho_{MP}\, \sigma_P \\
&= r + E\left[\pi^*\right] + \left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right) \sigma_y^2 - \left(+\frac{1}{\Phi_y} + \frac{1}{\Phi_M}\right) \rho_{My}\, \sigma_M\, \sigma_y \\
&= E\left[\pi^*\right] + \delta + \gamma\mu_y - (\gamma - 1)\left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right) \frac{\sigma_y^2}{2} - \left(\gamma + \frac{1}{\Phi_y} + \frac{1}{\Phi_M}\right) \rho_{My}\, \sigma_M\, \sigma_y
\end{aligned}
$$

Using the equilibrium relationships, one can also find

$$
\begin{aligned}
v_y y &= -\frac{2\,\phi_M + \phi_{My}\left(1 - \rho_{My}^2\right)}{2\,\phi_y\, \left(\phi_{My} + 2\,\phi_M\right) + \phi_{My}\, \left(\left(1 - \rho_{My}^2\right)\, \phi_{My} + 2\,\phi_M\right)} \\
v_M M &= -\frac{2\,\rho_{My}\,\phi_y}{2\,\phi_y\, \left(\phi_{My} + 2\,\phi_M\right) + \phi_{My}\, \left(\left(1 - \rho_{My}^2\right)\, \phi_{My} + 2\,\phi_M\right)} \frac{\sigma_y}{\sigma_M}
\end{aligned}
$$

and

$$
\kappa^{\frac{1}{\gamma}} = R^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}} \left(1-\alpha\right)^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} \alpha^{-\alpha\left(1-\frac{1}{\gamma}\right)} \frac{1}{\delta - (1-\gamma)\left(\mu_y - \left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}\right)}
$$

The maximum value function is therefore given by

$$
V = w^{1-\gamma}\left(\frac{1-\alpha}{R}\right)^{(1-\gamma)\,(1-\alpha)} \alpha^{\alpha\,(1-\gamma)} \frac{\left(\delta - (1-\gamma)\left(\mu_y - \left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}\right)\right)^{1-\gamma}}{1-\gamma}
$$

where

$$
\begin{aligned}
\frac{\partial V}{\partial \mu_M} &= \frac{\partial V}{\partial R}\frac{\partial R}{\partial \mu_M} \\
&= -w^{1-\gamma}\left(\frac{c}{s}\right)^{1-\gamma}\left(\frac{1-\alpha}{R}\right)^{1+(1-\gamma)(1-\alpha)} \alpha^{\alpha\,(1-\gamma)} < 0.
\end{aligned}
$$

A.2.1 Feasibility

Feasibility of the optimal plan implies that optimum consumption is $c\,(t) \geq 0$. In order to ensure that this is satisfied, it is necessary to assume that

$$
\delta \geq (1-\gamma)\left(\mu_y - \left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}\right)
$$

which simply states that the rate at which future consumption is discounted to the present must be larger than the risk and uncertainty adjusted rate of growth of the endowment.

A.2.2 Concavity of the maximum

Given the form of the maximum value function, $V = \frac{w^{1-\gamma}}{1-\gamma}\kappa$, concavity of the maximum implies $\frac{\partial^2 V}{\partial w^2} < 0$ (see Merton p. 105). Since $\frac{\partial^2 V}{\partial w^2} = -\gamma w^{-1-\gamma}\kappa$, concavity of the maximum requires $\kappa > 0$. In equilibrium

$$
\kappa = \left(\frac{1-\alpha}{R}\right)^{(1-\alpha)(1-\gamma)} \alpha^{\alpha(1-\gamma)} \frac{1}{\delta - (1-\gamma)\left(\mu_y - \left(\gamma + \frac{1}{\Phi_y} + \frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}\right)}
$$

which is always positive provided the feasibility condition is satisfied.

A.2.3 Transversality condition

Consider now the transversality condition $\lim_{t\to\infty} E\left[\exp\left(-\delta t\right)V\right]=0$. This requires that

$$\lim_{t\to\infty} E\left[e^{-\delta t}\frac{w^{1-\gamma}\alpha^{\alpha\,(1-\gamma)}\left(\frac{1-\alpha}{R}\right)^{(1-\gamma)\,(1-\alpha)}}{1-\gamma}\left(\delta-(1-\gamma)\left(\mu_y-\left(\gamma+\frac{1}{\Phi_y}+\frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}\right)\right)^{1-\gamma}\right]=0$$

where everything is constant apart from w. The limit can therefore be rewritten as

$$\frac{\alpha^{\alpha\,(1-\gamma)}\left(\frac{1-\alpha}{R}\right)^{(1-\gamma)\,(1-\alpha)}\left(\delta-(1-\gamma)\left(\mu_y-\left(\gamma+\frac{1}{\Phi_y}+\frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}\right)\right)^{1-\gamma}}{1-\gamma}\lim_{t\to\infty} E\left[e^{-\delta t}w^{1-\gamma}\right]$$

so that, in general, the transversality condition is satisfied when $\lim_{t\to\infty} E\left[e^{-\delta t}w^{1-\gamma}\right]=0$.

Given the equilibrium rate of growth of wealth, we obtain that this limit tends to 0 as t tends to infinity provided that

$$\delta-(1-\gamma)\left(\mu_y-\left(\gamma+\frac{1}{\Phi_y}+\frac{\rho_{My}^2}{\Phi_M}\right)\frac{\sigma_y^2}{2}\right)>\left(1-\frac{1}{\gamma}\right)\left(1-\left(\gamma+\frac{1}{\Phi_y}+\frac{\rho_{My}^2}{\Phi_M}\right)\right)\frac{\sigma_y^2}{2}$$

Now notice that the left hand side is always positive, by virtue of the feasibility condition. A sufficient condition for the trasversality condition to be satisfied is therefore that the right had side of the equation is negative. When $\gamma>1$, thus $\frac{\gamma-1}{\gamma}>0$, it follows that $1-\left(\gamma+\frac{1}{\Phi_y}+\frac{\rho_{My}^2}{\Phi_M}\right)<0$ and and the trasversality condition is always satisfied. When $\gamma<1$, $\frac{\gamma-1}{\gamma}<0$ and the sign of $1-\left(\gamma+\frac{1}{\Phi_y}+\frac{\rho_{My}^2}{\Phi_M}\right)$ is likely to be negative for high levels of ambiguity in the reference model, so the sufficient condition may not be satisfied.

A.2.4 A few derivatives

$$\begin{aligned}
\frac{\partial REP}{\partial\phi_M} &= -2\frac{1}{\Phi_M}\left(\frac{1}{\Phi_y}+\frac{1}{\Phi_M}\left(1-\frac{\phi_M}{\phi_y}\right)\right)\rho_{My}\,\sigma_M\,\sigma_c<0 \qquad \rho_{My}>0\\
\frac{\partial r}{\partial\phi_M} &= (1+\gamma)\,\frac{1}{\Phi_M^2}\rho_{My}^2\sigma_c^2>0\\
\frac{\partial\overline{r}}{\partial\phi_M} &= \frac{1}{\Phi_M}\left\{(\gamma-1)\,\frac{1}{\Phi_M}\rho_{My}^2\sigma_c^2+2\left(\frac{1}{\Phi_y}+\frac{1}{\Phi_M}\left(1-\frac{\phi_M}{\phi_y}\right)\right)\rho_{My}\,\sigma_M\,\sigma_c\right\}>0 \qquad \rho_{My}>0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial REP}{\partial\phi_y} &= -2\,\frac{\phi_M}{\phi_y}\,\frac{1}{\Phi_y}\,\frac{1}{\Phi_M}\,\rho_{My}\,\sigma_M\,\sigma_c<0 \qquad \rho_{My}>0\\
\frac{\partial REP}{\partial\mu_y} &= 0\\
\frac{\partial r}{\partial\phi_y} &= (1+\gamma)\,\frac{1}{\Phi_y^2}\,\sigma_c^2>0\\
\frac{\partial r}{\partial\mu_y} &= \gamma>0\\
\frac{\partial\overline{r}}{\partial\phi_y} &= \frac{1}{\Phi_y}\left((\gamma-1)\,\frac{1}{\Phi_y}\,\sigma_c^2+2\,\frac{\phi_M}{\phi_y}\,\frac{1}{\Phi_M}\,\rho_{My}\,\sigma_M\,\sigma_c\right)>0\\
\frac{\partial\overline{r}}{\partial\mu_y} &= \gamma>0
\end{aligned}$$