# Cartel Pricing Dynamics in the Presence of an Antitrust Authority*

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**Abstract**

Price-fixing is characterized when firms are concerned about creating suspicions that a cartel has formed. Antitrust laws have a complex effect on pricing as they interact with the conditions determining the internal stability of the cartel. The qualitative properties of pricing dynamics are characterized and the impact of antitrust policy is explored.

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# 1 Introduction

As evidenced by recent cases in lysine, graphite electrodes, and vitamins, price-fixing remains a perennial problem. Though there is a voluminous theoretical literature on collusive pricing, an important dimension to price-fixing cartels has received little attention. In light of its illegality, a critical goal faced by a cartel is to avoid the appearance that there is a cartel. Firms want to raise prices but not suspicions that they are coordinating their behavior. If high prices or rapidly increasing prices or, more generally, anomalous price movements may make customers and the antitrust authorities suspicious that a cartel is operating, one would expect this to have implications for how the cartel prices. The objective of this paper is to explore these implications especially with respect to pricing dynamics. Also of interest is understanding the impact of antitrust policy.

In an earlier paper (Harrington, 2002), I explored this issue by characterizing the joint profit maximizing price path under the constraint of possible detection and antitrust penalties. Assuming that the probability of detection is sensitive to price changes, it was shown that the cartel gradually raises price with price converging to a steady-state level. Comparative statics on the steady-state price reveal that it is decreasing in the damage multiple and the probability of detection but is independent of the level of fixed fines. Furthermore, if fines are the only penalty, the cartel's steady-state price is the same as in the absence of antitrust laws, though fines do affect the path to the steady-state. Another intriguing result is that a more stringent standard for calculating damages increases the steady-state price.

That analysis presumed that the incentive compatibility constraints ensuring the internal stability of the cartel were not binding. In the current paper, these constraints are explicitly introduced and allowed to bind. The optimal cartel price path is characterized and three considerations come into play - a desire to set high prices to realize high profit, a desire to gradually raise price so as to make detection less likely, and a need to adjust price so as to maintain the internal stability of the cartel. After laying out the model in Section 2 and defining an optimal collusive price path in Section 3, its intertemporal properties are characterized in Section 4. Section 5 investigates the role of antitrust policy and, in particular, identifies some possible perverse effects from the prohibition of price-fixing.

## 2 Model

Consider an industry with $n$ symmetric firms. $\overline{\pi}(P_i, P_{-i})$ denotes firm $i$'s profit when its price is $P_i$ and all other firms charge a common price of $P_{-i}$. Define $\pi(P) \equiv \overline{\pi}(P, P)$ to be a firm's profit and $D(P)$ a firm's demand when every firm charges $P$. The space of feasible prices is $\Omega$ which is assumed to be a non-empty, compact, convex subset of $\Re_+$. An additional restriction will be placed on $\Omega$ later.

**A1** Either: i) $\overline{\pi}(P_i, P_{-i})$ is continuous in $P_i$ and $P_{-i}$, quasi-concave in $P_i$, and $\exists$ unique $\widehat{P}$ such that $P \gtreqqless \psi(P)$ as $P \lesseqqgtr \widehat{P}$ where $\psi(P_{-i}) \in \arg\max \overline{\pi}(P_i, P_{-i})$; or ii)

$$\overline{\pi}(P_i, P_{-i}) = \begin{cases} (P_i - c)\, nD(P_i) & \text{if } P_i < P_{-i} \\ (P_i - c)\, D(P_i) & \text{if } P_i = P_{-i} \\ 0 & \text{if } P_i > P_{-i} \end{cases}$$

and $D(c) > 0$.

Part (i) of A1 results in the stage game encompassing many differentiated products models while (ii) makes it inclusive of the Bertrand price game (homogeneous goods and constant marginal cost). Allowing for the latter is important for some existence results though the characterization results hold much more generally. $\widehat{P}$ will denote a symmetric Nash equilibrium price under either (i) or (ii) where, in the latter case, $\widehat{P} = c$. Let $\widehat{\pi} \equiv \pi\left(\widehat{P}\right)$ be the associated profit. As a convention, $\overline{\pi}(\psi(P), P) = (P - c)\, nD(P)$ under (ii). A2 defines the cartel profit function and the joint profit-maximizing price.

**A2** $\pi(P)$ is differentiable and quasi-concave in $P$, if $\pi(P) > 0$ then it is strictly quasi-concave in $P$, and $\exists P^m > \widehat{P}$ such that $\pi(P^m) > \pi(P)\, \forall P \neq P^m$.

Firms engage in this price game for an infinite number of periods. The setting is one of perfect monitoring so that firms' prices over the preceding $t-1$ periods are common knowledge in period $t$. In this paper, "detection" always refers to a third party, such as buyers, detecting the existence of a cartel. Assume a firm's payoff is the expected discounted sum of its income stream where the common discount factor is $\delta \in (0, 1)$.

If firms form a cartel, they meet to determine price. Assume these meetings, and any associated documentation, provides the "smoking gun" if an investigation is pursued.[1] The cartel is detected with some probability and incurs penalties in that event. Assume,

[1] Though it is assumed that suspicions lead to an investigation and conviction with probability one, all results go through if the probability of these events is required only to be positive.

for simplicity, that detection results in the discontinuance of collusion forever. Detection in period $t$ then generates a terminal payoff of $[\widehat{\pi}/(1-\delta)] - X^t - F$ where $X^t$ is a firm's damages and $F$ is any (fixed) fines (which may include the monetary equivalent of prison sentences). If not detected, collusion continues on to the next period.

Damages are assumed to evolve in the following manner:

$$X^t = \beta X^{t-1} + \gamma x\left(P^t\right) \text{ where } \beta \in [0,1) \text{ and } \gamma \geq 0.$$

As time progresses, damages incurred in previous periods become increasingly difficult to document and $1-\beta$ measures the rate of the deterioration of the evidence. $x(P^t)$ is the level of damages incurred by each firm in the current period where $\gamma$ is the damage multiple applied. While U.S. antitrust law specifies treble damages, $\gamma$ is often well less than three because of an out-of-court settlement.

**A3** $x : \Omega \rightarrow \Re_+$ is bounded and continuous and is non-decreasing over $\left[\widehat{P}, P^m\right]$.

Current U.S. antitrust practice is $x(P^t) = \left(P^t - \widehat{P}\right) D(P^t)$ where $\widehat{P}$ is referred to as the "but for price" and is the price that would have occurred but for collusion. By the boundedness of $x(\cdot)$, it follows that damages are bounded by $\overline{X} \equiv \gamma\overline{x}/(1-\beta)$ where $\overline{x} \geq x(P)\,\forall P \in \Omega$. We then have that $X^{t-1} \in \left[0, \overline{X}\right]$.

Detection of a cartel can occur from many sources; some of which are related to price - such as customer complaints - and some of which are unrelated to price - such as internal whistleblowers. Hay and Kelley (1974) find that detection was attributed to a complaint by a customer or a local, state, or federal agency in 13 of 49 price-fixing cases. In the recent graphite electrodes case, it was reported that the investigation began with a complaint from a steel manufacturer which is a purchaser of graphite electrodes (Levenstein and Suslow, 2001). High prices or price increases or simply anomalous price movements may cause customers to become suspicious and pursue legal action or share their suspicions with the antitrust authorities.[2] Though it isn't important for my model, I do imagine that buyers (in many price-fixing cases, they are industrial buyers) are the ones who may become suspicious about collusion.[3]

[2] The Nasdaq case is one in which truly anomalous pricing resulted in suspicions about collusion. It was scholars rather than market participants who observed that dealers avoided odd-eighth quotes and ultimately explained it as a form of collusive behavior (Christie and Schultz, 1994). The market-makers paid an out-of-court settlement of around $1 billion.

[3] "As a general rule, the [Antitrust] Division follows leads generated by disgruntled employees, unhappy customers, or witnesses from ongoing investigations. As such, it is very much a reactive agency with

To capture these ideas in a tractable manner, I specify an exogenous probability of detection function which depends on the current and previous periods' price vectors. $\phi\left(\underline{P}^t, \underline{P}^{t-1}\right)$ is the probability of detection when the cartel is active where $\underline{P}^t \equiv (P_1^t, \ldots, P_n^t)$.[4] As a notational convention, the vector of prices will be replaced with a scalar when firms charge a common price. This specification can capture how high prices and big price changes can create suspicions among buyers that firms may not be competing.[5] The impact of the properties of this detection technology on the joint profit-maximizing price path was explored in Harrington (2002). There it was found that cartel pricing dynamics are empirically plausible when detection is driven by price changes rather than price levels. As a result, in this paper I will largely focus on when detection is sensitive to price changes. A4 specifies that the probability of detection is minimized when prices don't change and is weakly higher with respect to price increases. These seem plausible in the context of a stationary environment so that buyers do not expect to see much in terms of price fluctuations.

**A4** $\phi : \Omega^{2n} \rightarrow [0,1]$ is continuous and: i) $\phi(P^o, P^o) \leq \phi(P', P^o)$ and $\phi(P^o, P^o) \leq \phi(P^o, P')$, $\forall P', P^o \in \Omega$; ii) if $\underline{P}'' \geq \underline{P}' \geq \underline{P}^o$ (component-wide) then $\phi\left(\underline{P}'', \underline{P}^o\right) \geq \phi\left(\underline{P}', \underline{P}^o\right)$.

To derive various properties of the cartel price path, additional restrictions will later be placed on $\phi$. For purposes of generality, I have sought to impose the minimal restrictions for a particular property to be true; hence, the form of those restrictions will vary with the result. For the reader who prefers to have one unifying set of assumptions, it can be shown that all of the various assumptions made on $\phi$ in this paper hold for the following two classes of functions.[6] For the first class, suppose the probability of detection function is additively separable in the individual price changes:

$$\phi\left(\underline{P}^t, \underline{P}^{t-1}\right) = \sum_{j=1}^{n} \omega_j\left(\underline{P}^t\right) \widetilde{\phi}\left(P_j^t - P_j^{t-1}\right),$$

respect to the search for criminal antitrust violations. ... Customers, especially federal, state, and local procurement agencies, play a role in identifying suspicious pricing, bid, or shipment patterns." [McAnney, 1991, pp. 529, 530]

[4] In much of the analysis, it is unnecessary to specify the exact form of detection when the cartel collapses. At this point, it is sufficient to suppose that detection can occur during the post-cartel phase but it need not be as likely as when the cartel was active.

[5] While customers are implicitly assumed to be forgetful in that their likelihood of becoming suspicious depends only on recent prices, the inclusion of a more comprehensive price history would significantly complicate the analysis, by greatly expanding the state space, without any apparent gain in insight.

[6] The proof is available on request.

where $\omega_j : \Omega \to [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Assume $\widetilde{\phi} : \Omega \to [0,1]$ is differentiable, $\widetilde{\phi}'(\varepsilon) \geq (\leq) 0$ when $\varepsilon \geq (\leq) 0$, and $\widetilde{\phi}''(\varepsilon) \geq 0$ when $\varepsilon \geq 0$. Thus, when the price change is negative, the probability of detection is non-increasing in the price change and, when the price change is positive, it is a weakly convex non-decreasing function of the price change. The second class has detection depend on movements in a summary statistic of firms' prices. Suppose

$$\phi\left(\underline{P}^t, \underline{P}^{t-1}\right) = \widetilde{\phi}\left(f\left(\underline{P}^t\right) - f\left(\underline{P}^{t-1}\right)\right)$$

where $f : \Omega^n \to \Omega$ and: i) $f(P, \ldots, P) = P$; and ii) if $P^o \leq P$ then $f(P, \ldots, P^o, \ldots, P) \leq P$. $\widetilde{\phi}$ has the properties specified above. Examples for $f$ include the average price - either unweighted or weighted by market share - and the median price.

This modelling of detection warrants further discussion since it does not model those agents who might engage in detection. The first point to make concerns tractability. With two distinct sources of structural dynamics - detection and antitrust penalties - in addition to the usual (repeated game-style) behavioral dynamics, this model is rich enough to provide new insight into cartel pricing dynamics even with a reduced form modelling of the detection process and its complexity already pushes the boundaries of formal analysis. It would seem prudent to understand the workings of this model before moving on to the much more difficult problem of endogenizing the probability of detection function. Tractability issues aside, there is another motivation which makes the analysis of intrinsic interest. The objective of this paper is to develop insight and testable hypotheses about cartel pricing. A good model of the detection process is then one that is a plausible description of how cartel members *perceive* the detection process. To my knowledge, there is little evidence from past cases that cartels hold a sophisticated view of buyers (which is implied if one were to model buyers as strategic agents and derive an equilibrium). It strikes me as quite reasonable that firms might simply postulate that higher prices or bigger price changes result in a greater likelihood of creating suspicions without having derived that relationship from first principles about buyers. Thus, even if this modelling of the detection process is wrong, the resulting statements about cartel pricing may be right if that model is a reasonable representation of firms' perceptions.

In period 1, firms have the choice of forming a cartel, and risking detection and penalties, or earning non-collusive profit of $\widehat{\pi}$. If they choose the former, they can, at any time, choose to discontinue colluding. However, a finitely-lived cartel will cause collusion to unravel so that, in equilibrium, firms either collude forever or not at all (subject to the cartel being exogenously terminated because of detection). Firms are then not allowed to form and dissolve a cartel more than once. While the possibility of temporarily shutting down the cartel is not unreasonable (firms may want to "lay low" for a bit of time), the analysis is complicated enough without allowing for such. Exploration of that strategic option is left for future research.

**Related Work** Though no previous work allows for the rich set of dynamics of this model, there have been papers which take account of detection considerations in the context of cartel pricing. Block, Nold, and Sidak (1981) offer a static model in which the probability of detection is increasing in the price level. Spiller (1986), Salant (1987), and Baker (1988) allowed buyers to adjust their purchases under the anticipation that they may be able to collect multiple damages if sellers were shown to have been colluding. Also within a static setting, Besanko and Spulber (1989, 1990), LaCasse (1995), Polo (1997), Souam (2001), and Schinkel and Tuinstra (2002) explore a context in which firms have private information, which influences whether or not they collude, and either the government or buyers must decide whether to pursue costly legal action. Three papers consider a dynamic setting. Cyrenne (1999) modifies Green and Porter (1984) by assuming that a price war, and the ensuing raising of price after the war, results in detection for sure and with it a fixed fine. Spagnolo (2000) and Motta and Polo (2003) consider the effects of leniency programs on the incentives to collude in a repeated game of perfect monitoring.[7] Though considering collusive behavior in a dynamic setting with antitrust laws, these papers exclude the sources of dynamics that are the foci of the current analysis; specifically, that the probability of detection and penalties are sensitive to firms' pricing behavior. It is that sensitivity that will generate predictions about cartel pricing dynamics.

## 3 Optimal Symmetric Subgame Perfect Equilibrium

The cartel's problem is to choose an infinite price path so as to maximize the expected sum of discounted income subject to the price path being incentive compatible (IC). In determining the set of IC price paths, the assumption is made that deviation from the collusive path results in the cartel being dissolved and firms behaving according to a Markov Perfect Equilibrium (MPE).

Suppose a firm deviates and the cartel collapses. Since cartel meetings are no longer taking place, the damage variable simply depreciates at the exogenous rate of $1-\beta$: $X^t =$

[7] Rey (2001) offers a nice review of some of this work along with other theoretical analyses pertinent to optimal antitrust policy.

$\beta X^{t-1}$.[8] This is still a dynamic problem, however, in that price movements can create suspicions and, while firms are no longer colluding, an investigation could reveal evidence of past collusion. The state variables at $t$ are the vector of lagged prices, $\underline{P}^{t-1}$, and (common) damages, $X^{t-1}$. A MPE is then defined by a stationary policy function which maps $\Omega^n \times \Re_+$ into $\Omega$. Let $V_i^{mpe}\left(\underline{P}^{t-1}, X^{t-1}\right)$ denote firm $i$'s payoff at a MPE. When there is a symmetric state and a symmetric MPE, the payoff is denoted $V^{mpe}\left(P^{t-1}, X^{t-1}\right)$.

For the characterization of cartel pricing, it is not necessary to characterize a MPE; it being sufficient that the MPE payoff satisfy the following condition:

$$\frac{\widehat{\pi}}{1-\delta} \geq V_i^{mpe}\left(\underline{P}^{t-1}, X^{t-1}\right) \geq \frac{\widehat{\pi}}{1-\delta} - \beta X^{t-1} - F, \forall \left(\underline{P}^{t-1}, X^{t-1}\right) \in \Omega^n \times \left[0, \overline{X}\right]; \quad (1)$$

that is, a MPE results in a payoff weakly lower than the static Nash equilibrium payoff but weakly higher than the static Nash equilibrium payoff less the cost of incurring the penalties for sure. The issue then is under what conditions does a MPE exist which satisfies (1). Note that it holds if the post-cartel price path is sufficiently close to pricing at $\widehat{P}$ and the probability of incurring penalties during the post-cartel phase is sufficiently great. For example, (1) holds when infinite repetition of the static Nash equilibrium is a MPE; a sufficient condition for which is that the stage game is the Bertrand price game.[9] In the ensuing analysis, (1) is assumed in some cases and in others occurs for free; being implied by other assumptions. Though this property need not always hold (for an example where it doesn't, see Harrington 2003), it is useful to limit our attention to when it does so as to be able to provide a coherent set of results. Let me emphasize that I could have done away with (1) by simply focusing on the Bertrand price game. The route I have taken is more general as, by assuming a MPE payoff satisfies (1), it includes the Bertrand price game as a special case.

It is natural to assume that, at the start of the cartel, damages are zero and firms are charging the non-collusive price: $\left(P^0, X^0\right) = \left(\widehat{P}, 0\right)$. While many of the ensuing results are robust to these initial conditions, they will be assumed throughout the paper so as to simplify some of the proofs. Before providing the conditions defining the cartel solution, the assumption is made that damages are assessed only in those periods for which the cartel has been functioning properly and, more specifically, are not assessed when a firm

[8] Implicit in this specification is that damages stop accumulating once the cartel is dismantled. Though this is a useful approximation, if the post-cartel price exceeds $\widehat{P}$, it is because of past collusion so one could argue that additional damages should be assessed. Whether they are, in practice, is another matter.

[9] Another sufficient condition is that the probability of detection (when the cartel is inactive) is independent of an individual firm's price when that price is different from a common price charged by other firms. The proofs are available on request.

deviates from the cartel price. Thus, when a firm considers cheating on the agreement, it assumes the act of deviation negates damages for that period.[10]

As the focus is on symmetric collusive solutions, it is sufficient to define the state variables as $\left(P^{t-1}, X^{t-1}\right) \in \Omega \times \left[0, \overline{X}\right]$. The firms' problem is either to not form a cartel - and price at $\widehat{P}$ in every period with each firm receiving a payoff of $\widehat{\pi}/(1-\delta)$ - or form a cartel and choose a price path so as to:

$$\max_{\{P^t\}_{t=1}^{\infty} \in \Gamma} \sum_{t=1}^{\infty} \delta^{t-1} \Pi_{j=1}^{t-1} \left[1-\phi\left(P^j, P^{j-1}\right)\right] \pi\left(P^t\right) \tag{2}$$
$$+\sum_{t=1}^{\infty} \delta^t \phi\left(P^t, P^{t-1}\right) \Pi_{j=1}^{t-1} \left[1-\phi\left(P^j, P^{j-1}\right)\right] \left[\left(\widehat{\pi}/(1-\delta)\right) - \sum_{j=1}^{t} \beta^{t-j} \gamma x\left(P^j\right) - F\right]$$

where

$$\begin{aligned}
\Gamma \quad &\equiv \quad \{\left\{P^t\right\}_{t=1}^{\infty} \in \Omega^{\infty} : \sum_{\tau=t}^{\infty} \delta^{t-\tau} \Pi_{j=t}^{\tau-1} \left[1-\phi\left(P^j, P^{j-1}\right)\right] \pi\left(P^{\tau}\right) \\
&\quad + \sum_{\tau=t}^{\infty} \delta^{\tau-t+1} \phi\left(P^{\tau}, P^{\tau-1}\right) \Pi_{j=t}^{\tau-1} \left[1-\phi\left(P^j, P^{j-1}\right)\right] \left[\left(\widehat{\pi}/(1-\delta)\right) - \sum_{j=1}^{\tau} \beta^{\tau-j} \gamma x\left(P^j\right) - F\right] \\
&\geq \quad \max_{P_i} \overline{\pi}\left(P_i, P^t\right) + \delta \phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right) \left[\left(\widehat{\pi}/(1-\delta)\right) - \sum_{j=1}^{t-1} \beta^{t-j} \gamma x\left(P^j\right) - F\right] \\
&\quad + \delta \left[1-\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\right] V_i^{mpe}((P^t, \ldots, P_i, \ldots, P^t), \sum_{j=1}^{t-1} \beta^{t-j} \gamma x\left(P^j\right)), \\
\forall t \quad &\geq \quad 1\}
\end{aligned}$$

In (2), $\Pi_{j=1}^{t-1} \left[1-\phi\left(P^j, P^{j-1}\right)\right]$ is the probability that the cartel has not been detected as of period $t$. $\Gamma$ is the set of price paths that satisfy the incentive compatibility constraints (ICCs). A solution to (2) is referred to as an Optimal Symmetric Subgame Perfect Equilibrium (OSSPE) price path.

As I do not have a general proof of existence for a pure-strategy MPE, it is necessary to assume A5 so as to provide a general proof of the existence of an OSSPE price path.[11] Recall that if the stage game is the Bertrand price game then infinite repetition of the

[10] In practice, it is not clear when damages are no longer assessed and this assumption is probably as good as any other. Furthermore, it has a nice property which is useful for both analytical and numerical work. If damages were assigned in the period that a firm deviated then, entering the post-deviation phase, firms would have different levels of damages and this would expand the state space. Let me add that all results have been derived when it is instead assumed that damages are assessed in the period of deviation based on the price that the cartel set (which also serves to preserve common damages).

[11] To my knowledge, there is no general existence theorem for Markov Perfect Equilibrium, even in mixed strategies, when the state space is uncountable; see Fudenberg and Tirole (1991).

static Nash equilibrium is a MPE. This satisfies both the existence and continuity specified in A5.

**A5** $\forall \left(\underline{P}^{t-1}, X^{t-1}\right) \in \Omega^n \times \left[0, \overline{X}\right]$, $\exists$ a Markov Perfect Equilibrium and, furthermore, $\exists$ a continuous function $V_i^{mpe} : \Omega^n \times \left[0, \overline{X}\right] \rightarrow \Re$ such that $V_i^{mpe}\left(\underline{P}^{t-1}, X^{t-1}\right)$ is the payoff associated with a Markov Perfect Equilibrium.

Define the firms' choice set as $\{No\ Cartel\} \cup \Gamma$ where it is understood that choosing an element from $\Gamma$ implies forming a cartel while choosing *No Cartel* implies all firms price at $\widehat{P}$ in all periods. An OSSPE price path is a selection from $\{No\ Cartel\} \cup \Gamma$ that maximizes each firm's payoff. All proofs are in the Appendix.

**Theorem 1** *If A1-A5 hold then an OSSPE price path exists.*

$V\left(P^{t-1}, X^{t-1}\right)$ will denote the payoff that is associated with an OSSPE path. When something is stated to be a property of an OSSPE path, it is meant to refer to an OSSPE path that involves cartel formation.

In order to simplify the proofs, the assumption is made from hereon that $\Omega = [0, P^m]$ so that the cartel does not set price above the simple monopoly price. While I don't believe this assumption is essential for any result, I cannot dismiss the possibility that an OSSPE path would have price exceed the simple monopoly price in some periods. I will later elaborate on this point and will note in the proofs where this assumption is used. However, I also conjecture that the most relevant part of the parameter space is where an OSSPE path lies below $P^m$.[12]

# 4 Dynamic Properties of the Collusive Price Path

When ICCs are not binding, an OSSPE price path is non-decreasing over time as the cartel gradually raises price to reduce the probability of detection while achieving higher profit (Harrington, 2002). When instead cartel stability is a concern, the analysis is more subtle. Critical is how these ICCs evolve over time, in response to the state variables, and whether collusion is becoming more or less difficult. Our approach to this problem has three steps. First, a numerical analysis is conducted so as to identify what types of price

[12] If all of this creates doubt for the reader, a sufficient condition for this assumption to be made without loss of generality is for demand to be perfectly (or sufficiently) inelastic up to some maximal price, $P^m$, and zero thereafter. Prices in excess of $P^m$ then generate zero demand and can be shown never to be part of an OSSPE path.

paths may occur. Second, an analytical characterization of the price path is conducted for special cases of the model in order to provide some intuition for the numerical results. Third, the analytical and numerical results are pulled together to draw some general conclusions about the properties of the collusive price path.

### 4.1 Numerical Analysis

Consider an industry with symmetrically differentiated products. Using a common demand system (Vives, 1999), if all firm demands are non-negative then firm $i$'s demand is

$$\begin{aligned} D\left(P_i, P_{-i}\right) &= \left(\frac{a}{b+(n-1)e}\right) - \left(\frac{b+(n-2)e}{(b+(n-1)e)(b-e)}\right) P_i \\ &\quad + \left(\frac{e(n-1)}{(b+(n-1)e)(b-e)}\right) P_{-i}, \end{aligned}$$

where $a > 0$ and $b > e > 0$.[13] The firm cost function is $C(q) = cq$, where $a > c \geq 0$. The static Nash equilibrium price is

$$\widehat{P} = \frac{(a+c)(b-e)+(n-1)ec}{2(b+(n-2)e)-(n-1)e},$$

and the joint profit-maximizing price is $(a+c)/2$. Assume the damage function is $x(P) = \left(P - \widehat{P}\right) D(P,P)$ and suppose detection depends on movement in the average transaction price,[14]

$$f(P_1, \ldots, P_n) = \sum_{i=1}^{n} \left(\frac{D(P_i, P_{-i})}{\sum_{j=1}^{n} D(P_j, P_{-j})}\right) P_i.$$

Letting $f^t \equiv f\left(P_1^t, \ldots, P_n^t\right)$, the probability of detection when the cartel is active is specified to be

$$\phi\left(f^t, f^{t-1}\right) = \begin{cases} \min\left\{\alpha_0 + \alpha_1^u \left(f^t - f^{t-1}\right)^2, 1\right\} & \text{if } f^t \geq f^{t-1} \\ \\ \min\left\{\alpha_0 + \alpha_1^d \left(f^t - f^{t-1}\right)^2, 1\right\} & \text{if } f^t < f^{t-1} \end{cases}$$

and when the cartel is inactive it is the same though with a zero constant. I then allow for an asymmetric response to price increases and price decreases and consider parameter

[13] This demand system is derived from the following utility function:

$$U(q_1, \ldots, q_n) = a\sum_{i=1}^{n} q_i - \left(\frac{1}{2}\right)\left(b\sum_{i=1}^{n} q_i^2 + e\sum_{i=1}^{n}\sum_{j\neq i} q_i q_j\right)$$

where $q_i$ is the amount consumed of firm $i$'s product.

[14] Numerically, it is useful to have a summary statistic of the lagged price vector so as to limit the dimensionality of the state space.

values such that $0 \leq \alpha_1^d \leq \alpha_1^u$. When the cartel is active, $\alpha_0$ captures sources of detection that are independent of price movements. Note that the average transaction price from the preceding period rather than the previous period's price is a state variable though, when firms set a common price, the two are identical. For numerical analysis, the state space is $\Delta^*$ which is a discretized version of $\Delta \equiv \left[\widehat{P}, P^m\right] \times [0, \gamma x(P^m)/(1-\beta)]$. For all of the numerical results reported here, it is assumed that $\Delta^*$ is $30 \times 30$ and thus has 900 states.

The numerical method involves two stages: i) solving for a MPE for the post-deviation game; and ii) solving the cartel's problem. A statute of limitations is imposed so that antitrust penalties can only be levied if detection occurs within $T$ periods of the cartel's dissolution. This allows a MPE to be solved through backward induction. Let $W^\tau(f', X')$ denote a firm's MPE payoff in the $\tau^{th}$ period after a deviation given a lagged average transaction price of $f'$ and damages of $X'$. As $W^{T+1}(f, X) = \widehat{\pi}/(1-\delta)$, the post-deviation period $T$ symmetric equilibrium price is defined by:

$$\begin{aligned}\widetilde{P}^T \quad \in \quad & \arg\max_{P_i} \overline{\pi}\left(P_i, \widetilde{P}^T\right) + \delta\phi\left(f\left(\widetilde{P}^T, \ldots, P_i, \ldots, \widetilde{P}^T\right), f'\right)\left[(\widehat{\pi}/(1-\delta)) - \beta X' - F\right] \\ & + \delta\left[1 - \phi\left(f\left(\widetilde{P}^T, \ldots, P_i, \ldots, \widetilde{P}^T\right), f'\right)\right](\widehat{\pi}/(1-\delta)).\end{aligned}$$

Using the first-order condition, it is possible to derive a closed form solution for $\widetilde{P}^T$ with which one can derive $W^T(f', X')$. This is done for each $(f', X') \in \Delta^*$. Using a Chebychev polynomial to interpolate, the evaluation of $W^T(f', X')$ is extended to $\Delta$. Interpolation involves 20 basis functions and an equal number of interpolation nodes. The $T - 1^{st}$ post-deviation equilibrium price is defined by:

$$\begin{aligned}\widetilde{P}^{T-1} \quad \in \quad & \arg\max_{P_i} \overline{\pi}\left(P_i, \widetilde{P}^{T-1}\right) + \delta\phi\left(f\left(\widetilde{P}^{T-1}, \ldots, P_i, \ldots, \widetilde{P}^{T-1}\right), f'\right)\left[(\widehat{\pi}/(1-\delta)) - \beta X' - F\right] \\ & + \delta\left[1 - \phi\left(f\left(\widetilde{P}^{T-1}, \ldots, P_i, \ldots, \widetilde{P}^{T-1}\right), f'\right)\right] W^T\left(f\left(\widetilde{P}^{T-1}, \ldots, P_i, \ldots, \widetilde{P}^{T-1}\right), \beta X'\right).\end{aligned}$$

As the first-order condition does not have a closed-form solution, I solve it using the bisection method starting with bounds of $\widehat{P}$ and $f'$ (as one can show that the MPE price cannot be higher than $f'$). Note that this method requires not only a good approximation of the post-deviation value function but also its derivative. To make sure the approximation is a good one, I compare the solution with that derived using exhaustive search of the price space (which does not rely on approximating the derivative), for several parameter configurations. The two solutions are very close. Solving for the symmetric equilibrium price for all states in $\Delta^*$, interpolation is used again to derive $W^{T-1}(f', X') \; \forall (f', X') \in \Delta$. Iterating this process ultimately leads to the derivation of $W^1(f', X')$ which is the same as $V^{mpe}(f', X')$.

Given the MPE payoff function, the remaining problem is a single-agent constrained dynamic programming problem:

$$\begin{aligned} V\left(P^{t-1}, X^{t-1}\right) &= \max_{P \in \Omega^*} \pi(P) + \delta\phi\left(P, P^{t-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \beta X^{t-1} - \gamma x(P) - F\right] \\ &\quad + \delta\left[1 - \phi\left(P, P^{t-1}\right)\right] V\left(P, \beta X^{t-1} + \gamma x(P)\right) \end{aligned} \tag{3}$$

subject to

$$\begin{aligned} &\pi(P) + \delta\phi\left(P, P^{t-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \beta X^{t-1} - \gamma x(P) - F\right] \\ &\quad + \delta\left[1 - \phi\left(P, P^{t-1}\right)\right] V\left(P, \beta X^{t-1} + \gamma x(P)\right) \quad \geq \\ &\max_{P_i \in \Omega^*} \overline{\pi}(P_i, P) + \delta\phi\left(f(P, \ldots, P_i, \ldots, P), P^{t-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \beta X^{t-1} - F\right] \\ &+ \delta\left[1 - \phi\left(f(P, \ldots, P_i, \ldots, P), P^{t-1}\right)\right] V^{mpe}\left(f(P, \ldots, P_i, \ldots, P), \beta X^{t-1}\right). \end{aligned} \tag{4}$$

$\Omega^*$ is a discretized version of $\Omega$ and contains 100 equidistant prices from $\left[\widehat{P}, P^m\right]$. (3)-(4) is solved through function iteration with a discretized state space of $30 \times 30$. The value function is approximated by a linear spline with 30 basis functions and an equal number of interpolation nodes.[15] An initial value function is specified, for which the above constrained optimization problem was solved.[16] This produces new values for each state in $\Delta^*$. Interpolation using a linear spline then produces a new value function defined on $\Delta$. This process is iterated until convergence is achieved.[17] For purposes of comparison, the same process is run on the unconstrained dynamic programming problem as defined by (3).

The output produced is: i) the MPE value and policy functions; ii) the constrained dynamic programming value and policy functions and the price, damage, and value paths when $\left(P^0, X^0\right) = \left(\widehat{P}, 0\right)$; iii) the MPE price path when the state equals its steady-state values for the solution in (ii); and iv) the unconstrained dynamic programming value and policy functions and the price, damage, and value paths when $\left(P^0, X^0\right) = \left(\widehat{P}, 0\right)$.

[15] As this numerical method does not require approximation of the derivative of the value function, I use the linear spline rather than the Chebychev polynomial.

[16] The initial coefficients for the linear spline are set at 10,000; resulting in the initial value function well exceeding the present value of the unconstrained joint profit maximum. Thus, convergence occurs from above. This is important since if the initial value function is set too low, it could converge to $V^{mpe}(\cdot)$ or there may not exist any price which satisfies (4). Note that this operator on the value function is not assured of being a contraction mapping.

[17] The convergence criterion is that the norm of the difference of the coefficient vectors between iterations is less than $5 \times 10^{-10}$.

There are 13 parameters: demand and cost parameters - $a, b, e, c$; detection parameters - $\alpha_0, \alpha_1^u, \alpha_1^d$; penalty parameters - $\beta, \gamma, F, T$; discount factor, $\delta$; and number of firms, $n$. The benchmark parameter configuration is:

$$\begin{aligned} a &= 100, b = 2, e = 1, c = 0, n = 4, \delta = .5, \beta = .75, \gamma = 1 \\ \alpha_0 &= .025, \alpha_1^u = 16/\left(P^m - \widehat{P}\right)^2, \alpha_1^d = .8/\left(P^m - \widehat{P}\right)^2, F = 0, T = 8. \end{aligned}$$

Note that if $\alpha_0 = 0$ and $\alpha_1^u = \omega/\left(P^m - \widehat{P}\right)^2$ then a price increase of $\left(P^m - \widehat{P}\right)/\sqrt{\omega}$ results in detection for sure. Throughout the analysis, $a, e, c, F$, and $T$ do not vary. The model was run for 32 parameter configurations; 26 of which involved cartel formation.[18] These configurations involved various modifications to the benchmark configuration and included the following values:

$$\begin{aligned} \delta &\in \{.3, .4, .5, .6, .9\}, \beta \in \{.5, .75, .9\}, b \in \{2, 3\}, n \in \{2, 4, 6, 8\} \\ \alpha_1^u &\in \left\{1/\left(P^m - \widehat{P}\right)^2, 16/\left(P^m - \widehat{P}\right)^2, 32/\left(P^m - \widehat{P}\right)^2\right\} \\ \alpha_1^d &\in \left\{0, .8/\left(P^m - \widehat{P}\right)^2\right\}, \alpha_0 \in \{.025, .05\}, \gamma \in \{1, 2\}. \end{aligned}$$

For the benchmark case, Figure 1 shows the value and policy functions for both the cartel problem and the non-collusive solution. When the initial state is the non-collusive price with zero damages, the resulting time paths are shown in Figure 2. The ICCs bind as the cartel raises price to 37 which is below the unconstrained steady-state price of 46. Also shown is the MPE price path starting at the price and damages associated with the cartel steady-state. Note that the MPE price doesn't immediately fall to the non-collusive level of 20 as firms mediate their price drops so as to make detection less likely.

Surveying the results for all of the parameter configurations, two qualitatively distinct cartel price paths emerge. First, the cartel price path is monotonically increasing, as represented in Figure 2. The cartel gradually raises price - so as to avoid detection - and price achieves some steady-state level which is typically below the monopoly price because it isn't worth it for the cartel to risk detection by further raising price or it isn't feasible for the cartel to do so. This monotonicity of price - which is proven when ICCs do not bind in Harrington (2002) - can then still occur when ICCs bind. Second, and more interestingly, the cartel price path initially increases and then declines; approaching its steady-state level from above. A representative example is shown in Figure 3 where price

[18] A typical case took 3-5 hours on a Dell Workstation PWS 350 with a 1.8 GHz Intel Xeon processor. When $\beta$ and/or $\delta$ are close to one, it can take much longer.

rises from 20 to over 45 during the first ten periods and then the cartel gives up about 10% of its price increase as it falls to its steady-state level.[19]

To understand these numerical findings, I next analytically derive properties of an OSSPE price path for special cases of the model. In particular, I consider each of the two dynamics - detection and penalties - in isolation. In Section 4.2, penalties are fixed but the probability of detection remains endogenous. As in the case when ICCs do not bind, an OSSPE price path is shown to be increasing over time. In Section 4.3, I allow penalties to evolve but fix the probability of detection. After price is raised in the first period, an OSSPE path is declining thereafter. In Section 4.4, general conclusions are drawn from the numerical and analytical results and comparative dynamics are performed.

## 4.2 Pricing Dynamics with Endogenous Detection

Assume there are only fines: $\gamma = 0$ and $F > 0$. The lone state variable for the cartel is lagged price and, in the event of a deviation, the vector of lagged prices. Though penalties are fixed, the probability of detection is sensitive to how the cartel prices, as specified in A4. Further structure is required to establish our main result.

**B1** If $P' \geq P$ and $P' > P^o$ then $\phi(P', P) - \phi((P', \ldots, P^o, \ldots, P'), P)$ is non-increasing in $P$.

To interpret B1, suppose that the lagged price is $P$ and the cartel is to raise it to $P'$. If an individual firm considers deviating to a price of $P^o$, $\phi(P', P) - \phi((P', \ldots, P^o, \ldots, P'), P)$ is the associated difference in the probability of detection between colluding and deviating. B1 says that if the cartel is raising price by a greater amount then this differential in the probability is greater. Section 2 described a class of probability of detection functions whereby B1 holds. The presumed property for a MPE payoff is stated as B2.

**B2** $\widehat{\pi}/(1-\delta) \geq V_i^{mpe}(\underline{P}) \geq (\widehat{\pi}/(1-\delta)) - F, \forall \underline{P} \in \Omega^n$.

Theorem 2 shows that when penalties are fixed and only detection is sensitive to the price path, the cartel price path is non-decreasing over time.

**Theorem 2** *Assume A1-A2, A4-A5, B1-B2, and $\gamma = 0$. If $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is an OSSPE path then $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is non-decreasing over time.*

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[19] The modifications to the benchmark case are: $\beta = .9$, $\alpha_0 = .05$, $\alpha_1^u = 2/\left(P^m - \widehat{P}\right)^2$, $\alpha_1^d = 0$, and $\delta = .6$.

When the cartel is unconstrained by concerns about stability (that is, the ICCs are not binding), the optimal price path is non-decreasing over time. Since bigger price movements are more likely to trigger suspicions about a cartel having been formed, the cartel gradually raises price so as to balance profit and the probability of detection. Thus, if the price path is decreasing when ICCs bind, it is because incentive compatibility requires it. The issue then is under what circumstances does the cartel find itself charging a price that it can't sustain. In the proof of Theorem 2, it is established that if it is IC to raise price to some level then it is IC to keep price at that level. Therefore, it is never necessary to reduce price in order to maintain the stability of the cartel which implies that the price path is non-decreasing over time.

In understanding the role of B1 in proving Theorem 2, consider the ICC associated with the cartel currently pricing at $P''$. If a firm deviates and prices at $P^o < P''$, it changes its current profit by an amount $\overline{\pi}(P^o, P'') - \pi(P'')$ and alters the future payoff, in the event the cartel is not detected, by $V_i^{mpe}(P'', \ldots, P^o, \ldots, P'') - V(P'')$. Those components to the ICC are the same regardless of whether the cartel is raising price to $P''$ or keeping it there. What differs is how cheating influences the current likelihood of detection. Suppose the cartel is raising price from $P'$ to $P''$. If a firm goes along with that price increase, detection occurs with probability $\phi(P'', P')$ while if a firm deviates by pricing at $P^o$ then the probability of detection is $\phi((P'', \ldots, P^o, \ldots, P''), P')$ so that cheating changes the probability of detection by $\phi(P'', P') - \phi((P'', \ldots, P^o, \ldots, P''), P')$. If instead the cartel is maintaining price at $P''$ then cheating alters the probability of detection by $\phi(P'', P'') - \phi((P'', \ldots, P^o, \ldots, P''), P'')$. B1 ensures us that

$$\phi(P'', P') - \phi((P'', \ldots, P^o, \ldots, P''), P') \geq \phi(P'', P'') - \phi((P'', \ldots, P^o, \ldots, P''), P'')$$

so that cheating has a more favorable effect on the likelihood of detection when the cartel is raising price than when it is keeping price constant. Given the other components of the ICCs are identical, if it is IC to raise price to $P''$ then it is IC to keep price at $P''$.

This can be stated more intuitively. We have that the probability of detection is greater when price changes are more significant. If a cartel keeps price constant then cheating - with the associated drop in price - can not make detection any less likely. In contrast, if the cartel is raising price then cheating - by not raising price as much - can reduce the extent of price fluctuations and thereby make detection less likely. Thus, if a firm found it unprofitable to cheat when the cartel raised price to some level, it isn't then profitable when the cartel is keeping price at that level. In brief, concerns about detection makes cheating less profitable, *ceteris paribus*, when the cartel is keeping prices stable than when it is raising price. It follows that price need never be lowered in order to maintain the stability of the cartel and, therefore, the price path is non-decreasing over time. In conclusion, when the dynamics are solely due to how the price path influences the likelihood of detection, concerns about cartel stability do not alter the qualitative properties of the optimal cartel price path - it is increasing just as when ICCs do not bind.

### 4.3 Pricing Dynamics with Endogenous Penalties

In proving Theorem 2, it was crucial that penalties were fixed; for if penalties evolve then the ICCs could change so that it may not be IC to keep price constant. To consider the dynamics emanating from the endogeneity of penalties, suppose detection is independent of prices - being exclusively driven by such factors as internal whistleblowers - and $\gamma > 0$ so that penalties are sensitive to the prices set.

**C1** $\exists \phi^o \in (0,1)$ such that $\phi\left(\underline{P}', \underline{P}^o\right) = \phi^o \ \forall \underline{P}', \underline{P}^o \in \Omega^n$.

It will be useful to explicitly specify the likelihood of detection after the cartel has collapsed. Let $\rho(\tau)$ denote the probability of detection $\tau$ periods after the last cartel meeting (which was in the period during which a firm cheated). As specified in C2, detection is less likely in at least some periods when the cartel is inactive than when it is active.

**C2** $\rho(0) = \phi^o$, $\rho(\tau) \leq \phi^o \ \forall \tau$, and $\rho(\tau) < \phi^o$ for some $\tau$.

As the probability of detection is fixed, the problem simplifies considerably. First, it is straightforward to show that the unique MPE is the infinite repetition of the static Nash equilibrium.[20] Second, the optimal deviation price is that which maximizes current profit, $\psi(P^t)$. Since the probability of detection is fixed and the price at which a firm deviates doesn't influence penalties (where recall it is assumed that damages are not assessed when the cartel is not functioning), a deviating firm's price only affects current profit. Using these properties, the cartel's problem can be stated as:

$$\max_{\{P^t\}_{t=1}^{\infty} \in \Omega^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} (1-\phi^o)^{t-1} \left[\pi\left(P^t\right) - \theta^c \gamma x\left(P^t\right)\right] + \kappa^c \left[(\widehat{\pi}/(1-\delta)) - F\right] \tag{5}$$

subject to:

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t} (1-\phi^o)^{\tau-t} \left[\pi\left(P^{\tau}\right) - \theta^c \gamma x\left(P^{\tau}\right)\right] + \kappa^c \left[(\widehat{\pi}/(1-\delta)) - F\right] - \theta^c \beta X^{t-1} \geq \overline{\pi}\left(\psi\left(P^t\right); P^t\right) + \delta\left(\widehat{\pi}/(1-\delta)\right) - \theta^d \beta X^{t-1} - \kappa^d F, \ \forall t \geq 1,$$

[20] The proof is available on request.

where $\theta^c \equiv \delta\phi^o / [1-\delta\beta(1-\phi^o)]$, $\theta^d \equiv \sum_{\tau=0}^{\infty} \delta (\delta\beta)^{\tau} \Pi_{h=0}^{\tau-1} (1-\rho(h)) \rho(\tau)$, $\kappa^c \equiv [\delta\phi^o / (1-\delta(1-\phi^o))]$, and $\kappa^d \equiv \sum_{\tau=0}^{\infty} \delta^{\tau+1} \Pi_{h=0}^{\tau-1} (1-\rho(h)) \rho(\tau)$. $\pi(P^t) - \theta^c \gamma x(P^t)$ represents the net income from collusion in period $t$. A firm receives profit of $\pi(P^t)$ by colluding but incurs a liability in the form of $\theta^c \gamma x(P^t)$ which is the expected present value of damages.[21] This expression is multiplied by $(1-\phi^o)^{t-1}$ which is the probability that the cartel has not yet been detected. Turning to the ICCs, $\theta^c$ $\left(\theta^d\right)$ and $\kappa^c$ $\left(\kappa^d\right)$ measure the marginal effect of damages and fines, respectively, on the collusive (punishment) payoff. It follows from C1-C2 that $\theta^d < \theta^c$ and $\kappa^d < \kappa^c$; a key implication of which is that if, starting from period $t$, some price path is IC given $X^{t-1} = X'$ then it is also IC if $X^{t-1} < X'$ as the collusive payoff is decreasing with respect to damages at a faster rate then the deviation payoff.

The next assumption says that the difference between the maximal current profit and the collusive profit is increasing in the collusive price. It'll imply that if a price path is IC then so is a price path which is identical except that the period $t$ price is lower.

**C3** $\overline{\pi}(\psi(P), P) - \pi(P)$ is increasing in $P$ $\forall P \geq \widehat{P}$.

In proving the results of this section, it will be useful to pose the cartel's problem as choosing a level of damages rather than price. As this approach requires that $x(\cdot)$ be one-to-one, C4 strengthens A3 by assuming the damage function is strictly monotonic over the relevant domain.

**C4** $x(\cdot)$ is differentiable and non-decreasing, $x\left(\widehat{P}\right) = 0$, and $x$ is strictly increasing over $\left[\widehat{P}, P^m\right)$.

Defining $\xi(x)$ as the price that generates current damage penalties of $d$, it is implicitly defined by: $d = \gamma x(\xi(d))$. $\xi$ is well-defined $\forall d \in \left[\gamma x\left(\widehat{P}\right), \gamma x(P^m)\right]$.

**C5** $\pi(\xi(d))$ is concave in $d$ $\forall d \in \left[\gamma x\left(\widehat{P}\right), \gamma x(P^m)\right]$.

It can be shown that C5 holds when demand is weakly concave, marginal cost is constant, damages take the standard form, and the but for price weakly exceeds the competitive price.[22] Note that C4 is also implied by these conditions.

The next result shows that damages are non-decreasing over time.

[21] More specifically, the expected present value of damages is $\gamma x(P^t) \sum_{\tau=0}^{\infty} \delta [\delta\beta(1-\phi^o)]^{\tau} \phi^o$ where $(1-\phi^o)^{\tau} \phi^o$ is the probability of detection in $\tau$ periods and $\beta^{\tau} \gamma x(P^t)$ is the value of damages at that time.

[22] The proof is available on request.

**Lemma 3** *Assume A1-A2 and C1-C5. If* $\left\{\overline{X}^t\right\}_{t=1}^{\infty}$ *is consistent with an OSSPE then* $\left\{\overline{X}^t\right\}_{t=1}^{\infty}$ *is non-decreasing.*

C6 imposes quasi-concavity of net income - profit less the expected present value of damages. Sufficient conditions for C6 are strict concavity of the profit and damage functions.

**C6** $\exists P^+ \in \left(\widehat{P}, P^m\right]$ such that $\pi'(P) - \theta^c \gamma x'(P) \gtreqless 0$ as $P \lesseqgtr P^+ \; \forall P \in \left[\widehat{P}, P^m\right]$.

Theorem 4 shows that though the cartel raises price in the first period, it (weakly) decreases price thereafter. Recall that it is assumed the probability of detection is fixed but penalties are sensitive to the price path.[23]

**Theorem 4** *Assume A1-A2 and C1-C6. If* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is an OSSPE price path then* $\overline{P}^1 > P^0$ *and it is non-increasing* $\forall t \geq 1$.

The logic behind the proof and the result is as follows. As the probability of detection is independent of lagged prices, all dynamics come from the evolution of damages. Since detection is more likely when the cartel is active, the collusive payoff is more sensitive than the deviation payoff to damages. Given that damages grow on the cartel price path (Lemma 3), the collusive payoff is then declining at a faster rate over time than is the deviation payoff. This tightens ICCs and, in order to ensure they are satisfied, the cartel may need to lower price (by C3). Note that though price is falling over time, its decline is sufficiently moderate so that damages rise.

It is easy to argue that, when ICCs are binding, the price path is strictly decreasing in some periods. Suppose, contrary to the claim, that the price path never decreases. Then, by Theorem 4, it is constant starting with period 1 and let $P'$ be this constant price. With a constant price, $X^t$ is strictly increasing and converging to $\gamma x(P') / (1-\beta)$. Suppose the ICC at $t'$ is binding so that the collusive payoff equals the payoff to cheating. Given, by supposition, the cartel price is $P'$ in both $t'$ and $t'+1$ periods, the ICC at $t'+1$ is identical to that at $t'$ except that inherited damages are higher at $t'+1$. Given that the collusive payoff declines faster with respect to damages than the payoff to cheating, if the two payoffs are equal at $t'$ then, since $X^{t'-1} < X^{t'}$, the collusive payoff is strictly less than the payoff to cheating at $t'+1$ which violates incentive compatibility. This contradiction means that the original supposition that the price path is constant is false. Combined with Theorem 4, the price path is then decreasing in some periods.

[23] It is worth noting that, when the ICCs do not bind, the cartel raises price in the first period and keeps it fixed thereafter when the probability of detection is fixed.

### 4.4 Discussion and Comparative Dynamics

Let us now pull together the various pieces of this section. To begin, if the ICCs are not binding, the cartel price path rises over time (Harrington, 2002). When ICCs bind, the central issue is whether the cartel will, at some point, be forced to lower price so as to maintain cartel stability. Both the sensitivity of detection to price movements and the sensitivity of penalties to price levels are pertinent to this issue. Focusing on the former dynamic, Theorem 2 showed that raising price made cartel stability easier so that there is never a need to lower price. More specifically, if a firm did not find it optimal to cheat when other firms were raising price then it is not optimal for them to cheat when other firms are keeping price constant. Thus, higher prices are easier to sustain as lagged price is higher. In contrast, the evolution of penalties can have the opposite effect - collusion may be more difficult as firms collude longer. As penalties grow, cartel members become increasingly concerned with the prospects of detection. If detection is less likely when collusion stops, there is an added incentive for a firm to cheat. With rising penalties as firms collude longer, the cartel must lower price so as to counterbalance this increased desire to deviate. The extent to which rising penalties make the cartel less stable then depends on whether cheating - with the ensuing collapse of the cartel - makes detection less likely. If the probability of detection is sufficiently insensitive to the price decline that would ensue in the post-cartel periods then a firm reduces the probability of paying antitrust penalties by cheating and causing the cartel to dissolve. In that case, this dynamic forces price down. However, if instead a post-cartel price war is likely to trigger detection, rising penalties serve to stabilize the cartel. Firms increasingly prefer to maintain relatively stable cartel prices than to risk detection by inducing a price war. Thus, when detection is sensitive to price declines, these two dynamics reinforce themselves to result in a rising price path.

Given this discussion, the numerical price paths originally derived are easy to understand. When the probability of detection is sufficiently sensitive to price increases, the cartel will gradually raise price for reasons that are clear. If, in addition, detection is sufficiently sensitive to price decreases then collusion will become easier over time which allows further price hikes; so the price path is always increasing. Collusion is becoming easier because penalties are growing - so avoiding detection is increasingly important - and the best way to avoid detection is to maintain moderately rising prices rather than experience a price war. When instead detection is fairly insensitive to price decreases then the price path, after initially rising, will eventually fall. The growing penalties make cheating increasingly attractive - as it brings collusion to an end and reduces the chances of having to pay these penalties - and the cartel must lower price as a result. Thus, the second dynamic eventually comes to dominate the pricing dynamics.

Comparative dynamics are performed and reported in Figures 4 and 5. The benchmark case is explored under various discount factors, $\delta \in \{.3, .4, .6, .9\}$, and number of firms, $n \in \{2, 4, 6, 8\}$. In the absence of antitrust policy, the standard result is that more patient firms result in higher cartel prices. The result here is different. As $\delta$ is raised, the price path initially shifts down though in the long-run prices are higher. This reflects two countervailing effects of $\delta$. First, there is the standard effect that more patient firms are less inclined to cheat and this loosens up ICCs and allows for a higher collusive price. This effect is what is causing the cartel to price higher in the long-run. Second, a cartel that raises price faster earns higher current profit but lowers its future payoff because detection is more likely and damages are larger. Thus, a cartel comprised of more patient firms will raise price slower.

Turning to the impact of market structure, increasing the number of firms has the usual effect of lowering the price that a cartel charges. (Note that the initial price for the cartel, $\widehat{P}$, changes with $n$.) What is interesting, however, is that having more firms results in a shorter transition path. For example, compare a duopoly with the case of four firms. The duopoly raises price from 33 to 45 and takes more than 30 periods to enact this 12 unit price hike. A cartel with four firms raises price from 20 to 37 and this 17 unit price increase is achieved in only 13 periods. More generally, the average price increase during the transition phase is monotonically increasing in the number of firms. Given that a cartel with more firms is starting at a lower price, the increase in profit from a given price hike is greater which makes the cartel more willing to run the risk of detection. The prediction is then made that a cartel with more firms will raise price faster and the transition phase will be shorter.

# 5 Possible Perverse Effects of Antitrust Laws

Having identified some properties of cartel pricing dynamics, the next step is to explore the impact of antitrust laws on the level of cartel prices. Of course, the primary goal of antitrust laws is to deter cartel formation altogether. In practice, the considerable length of time before cartels are detected[24] combined with the weak penalties[25] suggests that few cartels are discouraged from forming. However, even if a cartel is formed, one hopes that antitrust laws will induce the cartel to price lower to reduce the risk of detection and penalties in the event of detection. Furthermore, if the cartel price path is shifted down then clearly these laws reduce the profitability of forming a cartel - the cartel is induced to price lower and there is the possibility of penalties - and thus makes it less likely a cartel will form. If, however, antitrust laws induce the cartel to price higher than it is problematic as to whether these laws are even desirable.

To address the impact of antitrust laws on the cartel price path, the first task is to define the benchmark collusive price in the absence of antitrust laws. If detection considerations are removed then the model becomes a classical repeated game. In that the unique MPE for that game is infinite repetition of the static Nash equilibrium and given that we use MPE for the punishment in the game with antitrust laws, it is appropriate for the benchmark price to be the highest price supportable by a grim trigger strategy, which I denote $\widetilde{P}$.

**A6** $\widetilde{P}$ exists and is unique where if

$$\pi(P)/(1-\delta) \geq \overline{\pi}(\psi(P), P) + \delta(\widehat{\pi}/(1-\delta)) \ \forall P \in \left[\widehat{P}, P^m\right]$$

then $\widetilde{P} = P^m$ and otherwise $\widetilde{P} \in \left[\widehat{P}, P^m\right)$ and is defined by

$$\pi(P)/(1-\delta) \gtreqqless \overline{\pi}(\psi(P), P) + \delta(\widehat{\pi}/(1-\delta)) \text{ as } P \lesseqqgtr \widetilde{P}, \forall P \in \left[\widehat{P}, P^m\right].$$

It is not difficult to identify assumptions whereby antitrust laws result in lower prices in all periods. Assuming the probability of detection is fixed will suffice.[26] It is more interesting to consider when antitrust laws can have the perverse effect of raising the prices that the cartel sets. To make for a clean result, let us consider the extreme case when detection depends only on price movements. This is captured by assuming the baseline probability of detection, which is that associated with the price vector not changing, is zero.

**D1** $\phi : \Omega^{2n} \rightarrow \Re_+$ is continuously differentiable.

**D2** $\phi(P, P) = 0 \ \forall P \in \Omega$.

[24] Bryant and Eckard (1991) find that the mean and median duration of 184 (discovered) cartels was 7.27 and 5.80 years, respectively. Furthermore, 22% of the cartels lasted more than ten years.

[25] Lande (1993) persuasively argues that, in practice, penalties are on the order of single damages.

[26] A proof is available on request.

**D3** If $P'' \geq P'$ and $P'' \geq P^o$ then

$$\phi\left(P'', P'\right) + \phi\left(\left(P'', \ldots, P^o, \ldots, P''\right), P''\right) \geq \phi\left(\left(P'', \ldots, P^o, \ldots, P''\right), P'\right).$$

I believe results are robust to minor variations in D2 and this will be discussed later. Though there is no obviously natural interpretation of D3, recall from Section 2 that it holds for a general class of probability of detection functions.[27]

A5 will be assumed so that a MPE exists. The following additional property is imposed which holds, for example, when the Bertrand price game is the stage game.

**D4** $V_i^{mpe}\left(\underline{P}, X\right)$ is non-increasing in $X$ and if $\underline{P} \neq \left(\widehat{P}, \ldots, \widehat{P}\right)$ and $X > 0$ then

$$\widehat{\pi}/\left(1-\delta\right) > V_i^{mpe}\left(\underline{P}, X\right) \geq \left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta X - F.$$

While D1-D3 do not imply the probability of detection is ever positive, such is implicit in D4. Define $\overline{\Lambda}\left(P\right)$ to be the maximal payoff from deviating when the cartel is in a steady-state of charging a price of $P$. This means that $P$ was charged both last period and this period and damages are at their steady-state level of $\gamma x\left(P\right)/\left(1-\beta\right)$.

$$\begin{aligned}\overline{\Lambda}\left(P\right) \quad \equiv \quad & \max_{P_i} \pi\left(P_i, P\right) + \delta\phi\left(\left(P, \ldots, P_i, \ldots, P\right), P\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta\left(\gamma x\left(P\right)/\left(1-\beta\right)\right) - F\right] \\ & + \delta\left[1 - \phi\left(\left(P, \ldots, P_i, \ldots, P\right), P\right)\right] V_i^{mpe}\left(\left(P, \ldots, P_i, \ldots, P\right), \beta\gamma x\left(P\right)/\left(1-\beta\right)\right).\end{aligned}$$

Note that A1-A5 imply that $\overline{\Lambda}\left(P\right)$ is defined. In D5, $P^*$ is defined to be the highest steady-state price path that is IC. By D2, the steady-state collusive payoff is $\pi\left(P\right)/\left(1-\delta\right)$.

**D5** $P^*$ exists and is unique where, if

$$\pi\left(P\right)/\left(1-\delta\right) \geq \overline{\Lambda}\left(P\right) \forall P \in \left[\widehat{P}, P^m\right]$$

then $P^* = P^m$ and, otherwise, $P^* \in \left[\widehat{P}, P^m\right)$ and is defined by

$$\pi\left(P\right)/\left(1-\delta\right) \gtreqless \overline{\Lambda}\left(P\right) \text{ as } P \lesseqgtr P^*, \forall P \in \left[\widehat{P}, P^m\right].$$

Furthermore, it is straightforward to show that $P^* \geq \widetilde{P}$ and if $P^m > \widetilde{P}$ then $P^* > \widetilde{P}$. It follows from D4 that

$$\overline{\pi}\left(\psi\left(P\right), P\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) > \overline{\Lambda}\left(P\right).$$

[27] Referring to the class of probability of detection functions described in Section 2, D3 does not require that $\widetilde{\phi}$ be weakly convex for price increases; it just requires that it be non-increasing for price decreases and non-decreasing for price increases.

It is then true that $\pi\left(\widetilde{P}\right)/(1-\delta) \geq \overline{\Lambda}\left(\widetilde{P}\right)$ which implies $P^* \geq \widetilde{P}$. If $\widetilde{P} < P^m$ then

$$\pi\left(\widetilde{P}\right)/(1-\delta) = \overline{\pi}\left(\psi\left(P\right), P\right) + \delta\left(\widehat{\pi}/\left(1-\delta\right)\right) > \overline{\Lambda}\left(\widetilde{P}\right)$$

and therefore $P^* > \widetilde{P}$.

Theorem 5 states that the price path is bounded below $P^*$ and converges to it. If ICCs are binding in the absence of antitrust laws, so that $\widetilde{P} < P^m$, then the introduction of antitrust laws causes the cartel to eventually price higher.

**Theorem 5** *Assume A1-A6 and D1-D5. If* $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ *is an OSSPE price path then* $\overline{P}^t \leq P^* \forall t$ *and* $\lim_{t\to\infty} \overline{P}^t = P^*$.

Given the prospects of detection, the cartel will tend to gradually raise price so as to reduce the likelihood of triggering suspicions that a cartel has formed. This could cause the cartel price path to initially lie below $\widetilde{P}$, which is the cartel price in the absence of antitrust laws. Theorem 5 establishes that eventually the cartel will price in excess of $\widetilde{P}$ *because* detection may occur and antitrust laws result in the levying of penalties. For example, suppose the MPE is infinite repetition of the static Nash equilibrium. The post-deviation period is then characterized by firms lowering their prices from some collusive level to $\widehat{P}$. This "price war" has associated with it some probability of triggering suspicions that firms may not be competing; leading to an investigation and the levying of costly antitrust penalties. These expected penalties represent an additional cost associated with deviation which serves to lower the payoff to deviating. Of course, detection can also occur with collusion which lowers the collusive payoff. However, since $\phi(P,P) = 0$ and the cartel price path eventually settles down, the probability of detection if firms continue colluding is approaching zero and, therefore, the collusive payoff is approaching that value which occurs without antitrust laws. In the long-run, antitrust laws then cause a loosening of ICCs which allows the cartel to support prices in excess of $\widetilde{P}$.[28]

As just argued, the assumption that $\phi(P,P) = 0$ means that antitrust penalties have no impact on the collusive payoff in the long-run because the probability of detection is converging to zero. However, they do have an impact on the payoff from deviating since deviation results in price discretely falling which means a positive probability of detection. If instead $\phi(P,P) > 0$ then the presence of an antitrust authority depresses

[28] Let me now comment on why I cannot *a priori* dismiss the possibility that an OSSPE path could entail prices in excess of $P^m$. By pricing above $P^m$, the cartel may make deviation less profitable as it could cause the MPE price path to involve bigger price decreases and thus be more likely to induce detection.

both the collusive payoff and the payoff from deviating so its effect on ICCs in the long-run is ambiguous. Still, by continuity, Theorem 5 would seem to hold as long as a deviation-induced price war is more likely to generate detection than the stable prices associated with continued collusion. The more general idea is that once parties engage in a conspiracy, detection is often more likely if they discontinue it - resulting in an abrupt change in behavior that might trigger suspicions - than if they continue with the charade. This perverse effect of antitrust policy on cartel pricing may then be quite general.[29]

A related result is derived in Cyrenne (1999) where the model of Green and Porter (1984) is modifying by assuming that the transition into a punishment phase entails an additional cost which is interpreted as an antitrust fine. He shows average price is increasing in the size of the fine. This result, however, is predicated on a very restrictive and nonsensical modelling of the detection process. As part of the standard Green-Porter mechanism, the cartel specifies a trigger price such that reversion to the static Nash equilibrium occurs when price falls below it. It is the process of price falling below the trigger price that brings forth cartel detection; no other element of the price series influences detection. If $P'$ is the trigger price then the probability of detection equals 1 if firms are colluding in $t$ and $P^t < P'$ and is zero otherwise. This has odd properties. For example, a small change in price can trigger detection - if price goes from being above $P'$ to below $P'$ - while a large change in price (up or down) can avoid detection as long as price remains above $P'$. Though Cyrenne (1999) motivates this specification by the notion that large price movements induce detection, his specification does not appear to capture that idea very well.

# 6 Concluding Remarks

This paper has enriched the classical repeated game model of collusion by taking account of how the manner in which a cartel prices may affect its detection and, in that event, the levying of penalties. Due to the complex way in which detection and penalties influence the conditions for the internal stability of the cartel, there is an array of implications. First, the introduction of antitrust laws can lower the prices set by the cartel but can also allow them to charge higher prices by loosening the incentive compatibility constraints associated with collusion. Second, while the optimal cartel price path is increasing when incentive compatibility constraints are not binding, when they bite the properties of the

[29] For very different reasons, McCutcheon (1997), Fershtman and Pakes (2000) and Athey and Bagwell (2001) identify some perverse effects of antitrust law with respect to price-fixing.

path depend on whether those constraints are loosening or tightening over time. When penalties are exogenously set, collusion becomes easier over time and this results in the price path being increasing. When penalties are endogenous but the probability of detection is fixed, collusion becomes more difficult over time as penalties accumulate. As a result, the cartel price path is decreasing over time, after initially being raised right after cartel formation. Combining these two dynamics, numerical analysis identifies two possible paths: i) the cartel price path is monotonically increasing and converges to a steady-state; and ii) the cartel gradually raises price but, after some point, lowers price and converges to a steady-state.

This is a rich area for further investigation. The focus of this paper has been on detection through the change in a common price; being motivated by the potentially suspicious nature of price increases. Another source of suspicions is parallel behavior by firms. One can also explore how cartel stability and detection are impacted by corporate leniency programs, which allow the first cartel member to report to avoid government fines and prison sentences (though not damages). Perhaps the most challenging direction is to model the role of buyers so as to endogenize the detection process.

# 7 Appendix

**Proof of Theorem 1:** Suppose $\Gamma$ is empty. As the choice set is the singleton $\{No\ Cartel\}$, the OSSPE price path is $\widehat{P}$ forever. For the remainder of the proof, suppose $\Gamma$ is non-empty. Consider the payoff function in (2). Since $\pi(\cdot)$ and $x(\cdot)$ are bounded functions and $\delta, \beta \in (0,1)$, the payoff function is defined for all price paths. The payoff function is continuous in $\{P^t\}_{t=1}^{\infty}$ by the continuity of $\pi(\cdot)$, $x(\cdot)$, and $\phi(\cdot)$. To show that $\Gamma$ is a compact set, first note that it is a subset of $\Omega^{\infty}$ which, by the compactness of $\Omega$ and Tychonoff's Product Theorem, is itself compact. The lhs expression of the ICC is continuous in $\{P^t\}_{t=1}^{\infty}$. Under (i) of A1, the rhs expression is continuous (using A5). Under (ii), the rhs takes the form:

$$\max_{P_i \le P^t} n\pi(P_i) + \delta\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \sum_{j=1}^{t-1}\beta^{t-j}\gamma x\left(P^j\right) - F\right]$$

$$+\delta\left[1 - \phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\right] V_i^{mpe}((P^t, \ldots, P_i, \ldots, P^t), \sum_{j=1}^{t-1}\beta^{t-j}\gamma x\left(P^j\right))$$

which is also continuous in $P^t$. It follows that $\Gamma$ is a closed set. Since $\Gamma$ is a closed subset of a compact set, $\Gamma$ is compact. There is then a solution to (2) as it involves maximizing a continuous function over a non-empty compact set. If the associated payoff exceeds $\widehat{\pi}/(1-\delta)$ then such a solution is an OSSPE price path. If it does not exceed $\widehat{\pi}/(1-\delta)$ then an OSSPE price path is $\widehat{P}$ forever. ⧫

**Proof of Theorem 2:** The proof is comprised of two steps. Suppose $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is an OSSPE path. First, it is shown that if $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$ then it is IC to keep price constant and thereby price at $\overline{P}^{t'}$ in $t'+1$. Note that the ICC when price is raised to $\overline{P}^{t'}$ and when it is kept constant at $\overline{P}^{t'}$ are identical in terms of current profit and the future payoff but differ only in terms of the current probability of detection. With B1, cheating on the cartel more favorably affects the probability of detection when price is raised to $\overline{P}^{t'}$ than when it is kept fixed at $\overline{P}^{t'}$. Thus, if it is IC to raise price to some level then it is IC to keep it at that level. Second, if, contrary to the theorem, this price path has a decreasing subsequence then, by the first step, one can substitute that decreasing subsequence with a constant price path which is IC and yields a strictly higher payoff. This produces the desired contradiction.

Given this OSSPE, let $V\left(\overline{P}^t\right)$ denote the associated payoff starting with period $t+1$.[30] In performing the first step, let us initially show that if $\overline{P}^{t'-1} \leq \overline{P}^{t'}$ and $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/(1-\delta)$ then it is IC to keep price at $\overline{P}^{t'}$. There are two cases to consider: i) $V\left(\overline{P}^{t'-1}\right) > V\left(\overline{P}^{t'}\right)$; and ii) $V\left(\overline{P}^{t'-1}\right) \leq V\left(\overline{P}^{t'}\right)$. Starting with case (i) and recognizing that the lhs of (6) is $V\left(\overline{P}^{t'-1}\right)$, we have

$$\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - F\right] + \delta\left[1-\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right]V\left(\overline{P}^{t'}\right) > V\left(\overline{P}^{t'}\right). \tag{6}$$

This implies

$$\frac{\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - F\right]}{1-\delta\left[1-\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right]} > V\left(\overline{P}^{t'}\right). \tag{7}$$

Substituting the lhs of (7) for $V\left(\overline{P}^{t'}\right)$ in the expression for $V\left(\overline{P}^{t'-1}\right)$ on the lhs of (6), the following upper bound for $V\left(\overline{P}^{t'-1}\right)$ is derived:

$$\begin{aligned} V\left(\overline{P}^{t'-1}\right) &< \pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - F\right] \\ &\quad + \delta\left[1-\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right]\left[\frac{\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - F\right]}{1-\delta\left(1-\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right)}\right]. \end{aligned}$$

[30] Throughout this paper, $V(\cdot)$ denotes the payoff in period $t$ from an OSSPE. This is not a value function and it is only required to be defined for values of the state variables on the OSSPE path.

Re-arranging yields

$$V\left(\overline{P}^{t'-1}\right) < \frac{\pi\left(\overline{P}^{t'}\right) + \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]}{1 - \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right]} \tag{8}$$

which gives us an upper bound on $V\left(\overline{P}^{t'-1}\right)$.

Now consider a constant price path of $\overline{P}^{t'}$ starting in period $t'+1$. The payoff, denoted $W\left(\overline{P}^{t'}\right)$, is defined by:

$$W\left(\overline{P}^{t'}\right) = \pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\right] W\left(\overline{P}^{t'}\right),$$

and solving for $W\left(\overline{P}^{t'}\right)$:

$$W\left(\overline{P}^{t'}\right) = \frac{\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]}{1 - \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\right]}. \tag{9}$$

$W\left(\overline{P}^{t'}\right) > V\left(\overline{P}^{t'-1}\right)$ follows from (8) and (9) since $\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right) \leq \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)$ by A4. Given that, by supposition, $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/\left(1-\delta\right)$ then $W\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/\left(1-\delta\right)$. The next step is to show that this constant price path is IC. The ICC for period $t'$ for the original OSSPE path is:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] \\
&\quad +\delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] V\left(\overline{P}^{t'}\right) \quad \geq \\
&\max_{P_i \in \Omega} \overline{\pi}\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] \\
&+\delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right).
\end{aligned} \tag{10}$$

As $W\left(\overline{P}^{t'}\right) > V\left(\overline{P}^{t'-1}\right) > V\left(\overline{P}^{t'}\right)$ then (10) continues to hold if $W\left(\overline{P}^{t'}\right)$ replaces $V\left(\overline{P}^{t'}\right)$:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] \\
&\quad +\delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] W\left(\overline{P}^{t'}\right) \quad \geq \\
&\max_{P_i \in \Omega} \overline{\pi}\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] \\
&+\delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right).
\end{aligned} \tag{11}$$

Now consider the ICC for a constant price path of $\overline{P}^{t'}$:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&\quad+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]W\left(\overline{P}^{t'}\right) \quad \geq \\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right).
\end{aligned} \tag{12}$$

I want to show that (11) implies (12). Note that we only need to be concerned with $P_i < \overline{P}^{t'}$ as deviating with a price in excess of $\overline{P}^{t'}$ cannot yield a higher payoff than colluding as current profit is weakly lower by A1, the probability of detection is weakly higher, and, by B2, $W\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/\left(1-\delta\right)$ implies $W\left(\overline{P}^{t'}\right) \geq V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)$ so that the MPE payoff is weakly lower than the future collusive payoff. Re-arranging (11) and (12), I want to show that:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)-\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\left[W\left(\overline{P}^{t'}\right)-V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)\right] \\
\geq\;&\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} \\
&-\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\times \\
&\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}
\end{aligned} \tag{13}$$

implies

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)-\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\left[W\left(\overline{P}^{t'}\right)-V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)\right] \\
\geq\;&\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} \\
&-\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\times \\
&\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}, \\
\forall P_i\;<\;&\overline{P}^{t'}.
\end{aligned} \tag{14}$$

As the lhs of (13) and (14) are identical, (13) implies (14) if the rhs of (13) is at least as great as the rhs of (14):

$$\begin{aligned}
&\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} \\
&-\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} \\
\geq\;&\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} \\
&-\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}.
\end{aligned}$$

Re-arranging this inequality,

$$\begin{aligned}\left[\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]\left\{W\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} &\geq \\ \left[\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]\times \\ \left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}.\end{aligned} \tag{15}$$

Since

$$W\left(\overline{P}^{t'}\right)\geq\widehat{\pi}/\left(1-\delta\right)\geq V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)\geq\left(\widehat{\pi}/\left(1-\delta\right)\right)-F$$

and $\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\geq\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)$ then (15) holds if:

$$\begin{aligned}\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right) &\geq \\ \phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right),\end{aligned}$$

or, equivalently,

$$\begin{aligned}\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right) &\geq \\ \phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right).\end{aligned} \tag{16}$$

Given that $\overline{P}^{t'}\geq P_i,\overline{P}^{t'-1}$, (16) holds by B1. Having shown that a constant price path of $\overline{P}^{t'}$ starting from $t'+1$ is IC and yields a payoff strictly greater than $V\left(\overline{P}^{t'}\right)$, we have a contradiction that the original price path is an OSSPE path. Therefore, if $\overline{P}^{t'}\geq\overline{P}^{t'-1}$ and $V\left(\overline{P}^{t'-1}\right)\geq\widehat{\pi}/\left(1-\delta\right)$, it cannot be true that $V\left(\overline{P}^{t'-1}\right)>V\left(\overline{P}^{t'}\right)$.

Let us now examine case (ii): $V\left(\overline{P}^{t'-1}\right)\leq V\left(\overline{P}^{t'}\right)$. Consider keeping price at $\overline{P}^{t'}$ in period $t'+1$ but then continuing with the original OSSPE path. The ICC at $t'+1$ is:

$$\begin{aligned}&\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\ &+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]V\left(\overline{P}^{t'}\right) \\ \geq\;&\max_{P_i\in\Omega}\pi\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right] \\ &+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right).\end{aligned} \tag{17}$$

Using the same series of steps as with case (i), (10) implies (17) if

$$\begin{aligned}\left[\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]\left\{V\left(\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} &\geq \\ \left[\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]\times \\ \left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\}.\end{aligned}$$

The same argument is used to show that this inequality holds. The important point to note is that $V\left(\overline{P}^{t'}\right) \geq V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)$ because $V\left(\overline{P}^{t'}\right) \geq V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/\left(1-\delta\right)$. Hence, if $\overline{P}^{t'} \geq \overline{P}^{t'-1}$ then it is IC to keep price at $\overline{P}^{t'}$ and, in addition, $V\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/\left(1-\delta\right)$.

To summarize, it has been shown that, on an OSSPE path, if $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/\left(1-\delta\right)$ and $\overline{P}^{t'-1} \leq \overline{P}^{t'}$ then: i) it is IC to keep price at $\overline{P}^{t'}$; and ii) $V\left(\overline{P}^{t'}\right) \geq \widehat{\pi}/\left(1-\delta\right)$. Arguing by strong induction, I will show that if the price path is non-decreasing then it is IC to keep price constant in the future. First note that, by the conditions of an OSSPE, $V\left(P^0\right) \geq \widehat{\pi}/\left(1-\delta\right)$. Since, by supposition, $P^0 \leq \overline{P}^1$, it then follows that it is IC to keep price at $\overline{P}^1$. Also note that $V\left(\overline{P}^1\right) \geq \widehat{\pi}/\left(1-\delta\right)$. Now suppose $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$. By strong induction, it follows from $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'-1}$ that $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/\left(1-\delta\right)$. Since $V\left(\overline{P}^{t'-1}\right) \geq \widehat{\pi}/\left(1-\delta\right)$ and, by supposition, $\overline{P}^{t'} \geq \overline{P}^{t'-1}$, it is IC to keep price at $\overline{P}^{t'}$. This shows that, on a non-decreasing price path, it is IC to keep price constant. Also note that as long as an OSSPE price path is non-decreasing then so is the value to colluding: if $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$ then $V\left(P^0\right) \leq \cdots \leq V\left(\overline{P}^{t'}\right)$.

Armed with this property, the second step is to suppose that $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is not non-decreasing and show that there exists another IC path which yields a strictly higher payoff. Suppose the price path declines at some time and let $t'+1$ be the first period in which it does so, $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'} > \overline{P}^{t'+1}$. Define $t''+1$ as the first period after $t'$ for which price is at least as great as in $t'$: $\overline{P}^t < \overline{P}^{t'}$ $\forall t \in \{t'+1,\ldots,t''\}$ and $\overline{P}^{t''+1} \geq \overline{P}^{t'}$. $t''$ might be $\infty$. Now consider an alternative price path in which price equals $\overline{P}^{t'}$ for periods $t'+1,\ldots,t''$ and is identical to the original path starting at $t''+1$. First note that this alternative path yields a strictly higher payoff than the original path since it generates strictly higher profit in periods $t'+1,\ldots,t''$ (here I use the property that price does not exceed $P^m$ so that a higher price means higher profit) and the same profit thereafter. Furthermore, by A4, it results in a weakly lower probability of detection in periods $t'+1,\ldots,t''+1$ because, with this alternative path, price doesn't change over $t'+1,\ldots,t''$ and, with respect to $t''+1$, the price rise is $\overline{P}^{t''+1} - \overline{P}^{t'}$ with the alternative path as opposed to a higher price rise of $\overline{P}^{t''+1} - \overline{P}^{t''}$ with the original path which means a weakly lower probability of detection.

Having established that this alternative price path yields a strictly higher payoff, let me argue that it is IC. Consider incentive compatibility over $t'+1,\ldots,t''$. If $t'=0$ then, since $P^0 = \widehat{P}$, a constant price path of $P^{t'}$ over $t'+1,\ldots,t''$ is certainly IC. If $t' \geq 1$ then $\overline{P}^{t'-1} \leq \overline{P}^{t'}$ and, by our previous analysis, a constant price of $\overline{P}^{t'}$ starting with period

$t'+1$ is IC. It is also IC for periods after $t''+1$ since the previous period's price and the current period's price are the same as with the original path which, by supposition, is IC. The only remaining ICC is for period $t''+1$. The period $t''+1$ price is the same for both paths but with the original path the lagged price is $\overline{P}^{t''}$ and with the alternative path it is $\overline{P}^{t'}$ where $\overline{P}^{t'} > \overline{P}^{t''}$. The ICC for $t''+1$ for the original path is:

$$\begin{aligned} & \pi\left(\overline{P}^{t''+1}\right) + \delta\phi\left(\overline{P}^{t''+1}, \overline{P}^{t''}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] + \delta\left[1 - \phi\left(\overline{P}^{t''+1}, \overline{P}^{t''}\right)\right] V\left(\overline{P}^{t''+1}\right) \\ \geq\ & \overline{\pi}\left(P_i, \overline{P}^{t''+1}\right) + \delta\phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t''}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] \\ & + \delta\left[1 - \phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t''}\right)\right] V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \\ \forall P_i \leq\ & P^{t''+1} \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \pi\left(\overline{P}^{t''+1}\right) - \overline{\pi}\left(P_i, \overline{P}^{t''+1}\right) \\ & +\delta\left[V\left(\overline{P}^{t''+1}\right) - V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right)\right] \quad \geq \\ & \delta\phi\left(\overline{P}^{t''+1}, \overline{P}^{t''}\right)\left\{V\left(\overline{P}^{t''+1}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) + F\right]\right\} \\ & -\delta\phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t''}\right) \times \\ & \left\{V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\}, \forall P_i \leq \overline{P}^{t''+1}. \end{aligned} \tag{18}$$

The ICC for the alternative path at $t''+1$ is:

$$\begin{aligned} & \pi\left(\overline{P}^{t''+1}\right) + \delta\phi\left(\overline{P}^{t''+1}, \overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] + \delta\left[1 - \phi\left(\overline{P}^{t''+1}, \overline{P}^{t'}\right)\right] V\left(\overline{P}^{t''+1}\right) \\ \geq\ & \overline{\pi}\left(P_i, \overline{P}^{t''+1}\right) + \delta\phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right] \\ & + \delta\left[1 - \phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t'}\right)\right] V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \\ \forall P_i \leq\ & P^{t''+1} \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \pi\left(\overline{P}^{t''+1}\right) - \overline{\pi}\left(P_i, \overline{P}^{t''+1}\right) \\ & +\delta\left[V\left(\overline{P}^{t''+1}\right) - V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right)\right] \quad \geq \\ & \delta\phi\left(\overline{P}^{t''+1}, \overline{P}^{t'}\right)\left\{V\left(\overline{P}^{t''+1}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\} \\ & -\delta\phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t'}\right) \times \\ & \left\{V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right) - \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\right\}, \forall P_i \leq \overline{P}^{t''+1}. \end{aligned} \tag{19}$$

I then want to show that the rhs of (18) is at least as great as the rhs of (19):

$$\begin{aligned}
&\left[\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)-\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\right]\left\{V\left(\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\} \qquad (20)\\
\geq\ &\left[\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right)\right]\times\\
&\left\{V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-F\right]\right\},\ \forall P_i\leq\overline{P}^{t''+1}.
\end{aligned}$$

Let me first argue that $V\left(\overline{P}^{t''+1}\right)\geq V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)$. As the OSSPE price path is non-decreasing over $1,\ldots,t'$ then, by our earlier argument, $V\left(\overline{P}^{t'}\right)\geq\widehat{\pi}/\left(1-\delta\right)$. Next note that since a constant price path of $\overline{P}^{t'}$ is IC - and recalling that $W\left(\overline{P}^{t'}\right)$ denotes the associated payoff - then the conditions of an OSSPE imply $V\left(\overline{P}^{t'}\right)\geq W\left(\overline{P}^{t'}\right)$. Since the expected income stream from the OSSPE path is less than that from the constant price path over $t'+1,\ldots,t''$ (recall that the former generates strictly lower profit and a weakly higher probability of detection in those periods), it must deliver a higher payoff stream after $t''$. Since $V\left(\overline{P}^{t''}\right)$ is the payoff associated with the stream after $t''$, it follows that $V\left(\overline{P}^{t''}\right)>V\left(\overline{P}^{t'}\right)$. We then have $V\left(\overline{P}^{t''}\right)\geq\widehat{\pi}/\left(1-\delta\right)$ and since $\overline{P}^{t''+1}\geq\overline{P}^{t''}$ implies $V\left(\overline{P}^{t''+1}\right)\geq\widehat{\pi}/\left(1-\delta\right)$, it follows that $V\left(\overline{P}^{t''+1}\right)\geq V_i^{mpe}\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right)$. Since $\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)\geq\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)$, a sufficient condition for (20) to hold is:

$$\begin{aligned}
&\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)-\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)\\
\geq\ &\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right),
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\phi\left(\overline{P}^{t''+1},\overline{P}^{t''}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t''}\right)\\
\geq\ &\phi\left(\overline{P}^{t''+1},\overline{P}^{t'}\right)-\phi\left(\left(\overline{P}^{t''+1},\ldots,P_i,\ldots,\overline{P}^{t''+1}\right),\overline{P}^{t'}\right).
\end{aligned}$$

Since $\overline{P}^{t''+1}>\overline{P}^{t'}>\overline{P}^{t''}$ and $\overline{P}^{t''+1}>P_i$, this condition follows from B1. ♦

**Proof of Lemma 3:** The method of proof is to presume that $\exists t'$ such that $\overline{X}^{t'-1}<\overline{X}^{t'}>\overline{X}^{t'+1}$ and derive a contradiction. Associated with such a path of damages is a relatively high level of current damages (and, therefore, a high price) in $t'$, $\overline{X}^{t'}-\beta\overline{X}^{t'-1}$, and a relatively low level, $\overline{X}^{t'+1}-\beta\overline{X}^{t'}$, in $t'+1$. However, as $\pi\left(\xi\left(\cdot\right)\right)$ is concave in damages then it is more profitable to have more incremental changes in damages. More specifically, it is shown that if current damages of $\overline{X}^{t'}-\beta\overline{X}^{t'-1}$ is preferred to $\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}$ in $t'$ then it must be true that $\overline{X}^{t'}-\beta\overline{X}^{t'}$ is preferred to $\overline{X}^{t'+1}-\beta\overline{X}^{t'}$ in $t'+1$ which gives us a contradiction.

A critical property that will be used is that if, on an OSSPE path, the cartel prices at $P'$ and the damage state variable at the end of the period is $X'$ then pricing at $P$ with end-of-period damages of $X$ is also IC if $P \leq P'$ and $X \leq X'$. To see this, consider the ICC for $(P^t, X^t) = (P', X')$:

$$\pi(P') + \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} (1-\phi^o)^{\tau-t} \left[\pi\left(P^t\right) - \theta^c \gamma x\left(P^\tau\right)\right] + \kappa^c \left[\left(\widehat{\pi}/(1-\delta)\right) - F\right] - \theta^c \beta X' \geq$$
$$\overline{\pi}\left(\psi(P'), P'\right) + \delta\left(\widehat{\pi}/(1-\delta)\right) - \theta^d X' + \theta^d \gamma x(P') - \kappa^d F.$$

Since, by deviating rather than colluding, a firm avoids current damages of $\gamma x(P')$, if the end-of-period damages are $X'$ when a firm colludes then they are $[X' - \gamma x(P')]$ when it deviates. By C1-C2, the lhs decreases at a weakly faster rater with respect to $X'$ than the rhs. Hence, if $X'$ is replaced with a lower value for the damage variable, this condition still holds. By C3-C4, $\overline{\pi}(\psi(P), P) + \theta^d \gamma x(P) - \pi(P)$ is increasing in $P$. Hence, this ICC holds if $P'$ is replaced with a lower price. I conclude that, on an OSSPE path, if $(P^t, X^t)$ is replaced with a lower price and/or lower damage variable then the ICC at $t$ still holds.

Since $X^0 = 0$, if $\overline{X}^1 = 0$ then, by the stationarity of the policy function, $\overline{X}^t = 0 \forall t$ and thus, trivially, damages are non-decreasing.[31] Next suppose that $X^0 < \overline{X}^1$. If Lemma 3 is not true then $\exists t' \geq 1$ such that $X^0 < \overline{X}^1 < \cdots < \overline{X}^{t'} > \overline{X}^{t'+1}$. (Note that if damages are constant from one period to the next then they are constant in all future periods by stationarity.) Given the path of damages on an OSSPE path, the associated prices in $t'$ and $t'+1$ are defined by $\overline{P}^{t'} = \xi\left(\overline{X}^{t'} - \beta \overline{X}^{t'-1}\right)$ and $\overline{P}^{t'+1} = \xi\left(\overline{X}^{t'+1} - \beta \overline{X}^{t'}\right)$. That is, $\xi\left(\overline{X}^{t'} - \beta \overline{X}^{t'-1}\right)$ is the price that results in damages of $\overline{X}^{t'}$ given inherited damages of $\beta \overline{X}^{t'-1}$.

Since, by supposition, $\overline{X}^{t'} > \overline{X}^{t'+1}$ and furthermore $\overline{X}^{t'+1} \geq \beta \overline{X}^{t'} > \beta \overline{X}^{t'-1}$ then $\overline{X}^{t'+1} \in \left[\beta \overline{X}^{t'-1}, \overline{X}^{t'}\right]$. Hence, it was feasible to set price in $t'$ so that damages equalled $\overline{X}^{t'+1}$ at $t'$ and the price that would have done this is $\xi\left(\overline{X}^{t'+1} - \beta \overline{X}^{t'-1}\right)$. Since $\overline{X}^{t'} - \beta \overline{X}^{t'-1} > \overline{X}^{t'+1} - \beta \overline{X}^{t'-1}$ and $\xi$ is increasing then $\xi\left(\overline{X}^{t'} - \beta \overline{X}^{t'-1}\right) > \xi\left(\overline{X}^{t'+1} - \beta \overline{X}^{t'-1}\right)$. Given that, by supposition, charging a price of $\xi\left(\overline{X}^{t'} - \beta \overline{X}^{t'-1}\right)$ with resulting total damages of $\overline{X}^{t'}$ is IC (as it is part of an OSSPE) then the price-damage pair $\left(\xi\left(\overline{X}^{t'+1} - \beta \overline{X}^{t'-1}\right), \overline{X}^{t'+1}\right)$ is also IC as it involves a lower collusive price and lower damages. Since $\left(\xi\left(\overline{X}^{t'} - \beta \overline{X}^{t'-1}\right), \overline{X}^{t'}\right)$ was selected in $t'$ and, as just argued, the cartel could have chosen $\left(\xi\left(\overline{X}^{t'+1} - \beta \overline{X}^{t'-1}\right), \overline{X}^{t'+1}\right)$, I conclude that the former yields at least as high a payoff. Letting $V(X)$ denote the payoff associated with the OSSPE when damages are $X$, the previous statement is then repre-

[31] The assumption $X^0 = 0$ could be replaced with the condition that, on the optimal path, $X^0 < \overline{X}^1$.

sented as:

$$\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right)\right)+\delta\phi^{o}\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\overline{X}^{t'}-F\right]+\delta\left(1-\phi^{o}\right)V\left(\overline{X}^{t'}\right)\geq$$
$$\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}\right)\right)+\delta\phi^{o}\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\overline{X}^{t'+1}-F\right]+\delta\left(1-\phi^{o}\right)V\left(\overline{X}^{t'+1}\right)\quad\Leftrightarrow$$

$$\begin{aligned}&\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right)\right)-\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}\right)\right)\geq\\&\delta\left(1-\phi^{o}\right)\left[V\left(\overline{X}^{t'+1}\right)-V\left(\overline{X}^{t'}\right)\right]+\delta\phi^{o}\left(\overline{X}^{t'}-\overline{X}^{t'+1}\right).\end{aligned}\tag{21}$$

Next note that $\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right),\overline{X}^{t'}\right)$ being IC implies $\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right),\overline{X}^{t'}\right)$ is as well since $\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right)>\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right)$. Given that $\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right),\overline{X}^{t'+1}\right)$ was chosen in $t'+1$, it follows that $\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right),\overline{X}^{t'+1}\right)$ yields at least as high a payoff as $\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right),\overline{X}^{t'}\right)$:

$$\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right)\right)+\delta\phi^{o}\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\overline{X}^{t'+1}-F\right]+\delta\left(1-\phi^{o}\right)V\left(\overline{X}^{t'+1}\right)\geq$$
$$\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right)\right)+\delta\phi^{o}\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\overline{X}^{t'}-F\right]+\delta\left(1-\phi^{o}\right)V\left(\overline{X}^{t'}\right)\quad\Leftrightarrow$$

$$\begin{aligned}&\delta\left(1-\phi^{o}\right)\left[V\left(\overline{X}^{t'+1}\right)-V\left(\overline{X}^{t'}\right)\right]+\delta\phi^{o}\left(\overline{X}^{t'}-\overline{X}^{t'+1}\right)\geq\\&\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right)\right)-\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right)\right).\end{aligned}\tag{22}$$

(21)-(22) imply:

$$\begin{aligned}\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right)\right)-\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}\right)\right)&\geq\\\pi\left(\xi\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right)\right)-\pi\left(\xi\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right)\right).&\end{aligned}\tag{23}$$

Note that the difference in the arguments on the lhs of (23) is

$$\left(\overline{X}^{t'}-\beta\overline{X}^{t'-1}\right)-\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}\right)=\overline{X}^{t'}-\overline{X}^{t'+1},$$

and on the rhs is:

$$\left(\overline{X}^{t'}-\beta\overline{X}^{t'}\right)-\left(\overline{X}^{t'+1}-\beta\overline{X}^{t'}\right)=\overline{X}^{t'}-\overline{X}^{t'+1}.$$

By the concavity of $\pi\left(\xi\left(\cdot\right)\right)$, it then follows from (23) that:

$$\overline{X}^{t'+1}-\beta\overline{X}^{t'-1}\leq\overline{X}^{t'+1}-\beta\overline{X}^{t'}\Leftrightarrow\overline{X}^{t'}\leq\overline{X}^{t'-1},$$

which is a contradiction. This proves that $\overline{X}^{t}$ is non-decreasing on an OSSPE path. ♦

**Proof of Theorem 4:** Let $\left\{\overline{X}^{t}\right\}_{t=1}^{\infty}$ denote the path of damages associated with $\left\{\overline{P}^{t}\right\}_{t=1}^{\infty}$. Recall that $\left(P^{0},X^{0}\right)=\left(\widehat{P},0\right)$.[32] If $\overline{P}^{1}\leq\widehat{P}$ then, since $X^{0}=0$, $\overline{X}^{1}=0$ (by

[32] The assumption $P^{0}=\widehat{P}$ can be replaced with $\overline{P}^{1}>P^{0}$ on the optimal path.

C4). Hence, by stationarity, an OSSPE price path then involves pricing at $\overline{P}^1$ in period 2 and every period thereafter. As this contradicts the optimality of colluding, it is inferred that $\overline{P}^1 > \widehat{P}$ and, therefore, $\overline{P}^1 > P^0$.[33]

If Theorem 4 is not true then $\exists t' \geq 1$ such that $P^0 < \overline{P}^1 \geq \cdots \geq \overline{P}^{t'} < \overline{P}^{t'+1}$. For the OSSPE price path under consideration, let $\overline{P}^{t'} = P'$ and $\overline{P}^{t'+1} = P''$ where $P' < P''$. The analysis will involve comparing the original price path - $\{\overline{P}^1, \ldots, \overline{P}^{t'-1}, P', P'', \overline{P}^{t'+2}, \ldots\}$ - with an alternative price path - $\{\overline{P}^1, \ldots, \overline{P}^{t'-1}, P'', P', \overline{P}^{t'+2}, \ldots\}$ - which has the prices in $t'$ and $t'+1$ switched. It'll be shown that if a price path has price rise from one period to the next then an alternative price path in which those two prices are switched yields a strictly higher collusive payoff and if the original price path was IC then so is this one. This contradicts the original price path being induced by an OSSPE and thus contradicts the supposition that an OSSPE price path has an increasing sub-sequence after period 1.

The first step is to show that an OSSPE price path is bounded from above by $P^+$ (which is defined in C6). Suppose not so that in some period price exceeds $P^+$. Consider an alternative price path which is identical except that it has a price of $P^+$ in those periods for which price exceeded $P^+$. By C6, the collusive payoff, which is expressed in (5), is strictly higher since $\pi(P^+) - \Delta\gamma x(P^+)$ exceeds the comparable expression when price exceeded $P^+$. By C4, accumulated damages are lower. As ICCs are loosened when damages are reduced, if the original price path is IC then so is this one. In that a price path has been constructed which generates a higher payoff and is IC, it contradicts the supposition that the original path was generated by an OSSPE. I conclude that an OSSPE price path is bounded from above by $P^+$.

Given $P' < P'' \leq P^+$, it follows from C6 that $\pi(P'') - \theta^c\gamma x(P'') > \pi(P') - \theta^c\gamma x(P')$. Inspection of (5) then reveals that, due to discounting, the alternative price path yields a strictly higher payoff as it has the cartel receive $\pi(P'') - \theta^c\gamma x(P'')$ in period $t'$ and $\pi(P') - \theta^c\gamma x(P')$ in $t'+1$; which is the reverse of the original path. The remainder of the proof involves showing that if the original price path is IC then so is the alternative price path.

I begin with the supposition that the original path is IC in all periods. With the alternative path, the ICCs over periods $1, \ldots, t'-1$ are still satisfied since the collusive payoff is higher and the deviation payoff is unchanged. Next consider the period $t$ constraint where $t \geq t'+2$. As the current and future price path is the same as with the original

[33] If $F > 0$ then colluding and pricing at or below $\widehat{P}$ is clearly inferior to not colluding. If $F = 0$ then it could be optimal to collude and price at $\widehat{P}$ though that is a non-generic result.

path, the only difference in the constraint is lagged damages. Note that accumulated damages at $t$, where $t \geq t' + 2$, under the alternative path and under the original path are identical in all terms except for the damages incurred in periods $t'$ and $t' + 1$. The difference between the accumulated damages at $t$, where $t \geq t' + 2$, under the alternative path and under the original path then equals:

$$\begin{aligned} & \left[\beta^{t-t'}\gamma x\left(P''\right) + \beta^{t-t'-1}\gamma x\left(P'\right)\right] - \left[\beta^{t-t'}\gamma x\left(P'\right) + \beta^{t-t'-1}\gamma x\left(P''\right)\right] \\ = \quad & -\beta^{t-t'-1}\left(1-\beta\right)\gamma\left[x\left(P''\right) - x\left(P'\right)\right] < 0. \end{aligned}$$

Since, compared to the original path, the alternative path substitutes higher current damages in $t'$ for lower ones in $t' + 1$, accumulated damages are lower after $t' + 1$. Given that damages are lower under the alternative price path, the path is IC for $t \geq t' + 2$.

Next consider the ICC at $t' + 1$. With the original price path, price is $P''$ and damages are $\beta^2\overline{X}^{t'-1} + \beta\gamma x\left(P'\right) + \gamma x\left(P''\right)$ at $t' + 1$. With the alternative price path, price is $P'$ and damages are $\beta^2\overline{X}^{t'-1} + \beta\gamma x\left(P''\right) + \gamma x\left(P'\right)$. As price is lower then, by C3, this loosens the ICC. As damages are lower, this also serves to loosen the ICC. I conclude that the ICC is satisfied at $t' + 1$ for the alternative price path.

Finally, consider the ICC at $t'$. Using Lemma 3, it'll be shown that if the original price path is IC at $t' + 1$ then the alternative path is IC at $t'$. As an initial step, compare the damages at $t' + 1$ for the original path with those at $t'$ for the alternative path. The latter is weakly smaller iff $\beta\overline{X}^{t'} + \gamma x\left(P''\right) \geq \beta\overline{X}^{t'-1} + \gamma x\left(P''\right)$. As $\overline{X}^{t'-1} \leq \overline{X}^{t'}$ by Lemma 3, this is then indeed true. Since then damages at $t'$ for the alternative path are weakly lower than damages at $t' + 1$ for the original path, *ceteris paribus*, if the original path is IC at $t' + 1$ then the alternative path is IC at $t'$. For the next step, recall that the collusive payoff at $t'$ for the alternative path exceeds the collusive payoff at $t'$ for the original path. Since $\overline{X}^{t'} \leq \overline{X}^{t'+1}$, it must then be true, for the original path, that $V\left(\overline{X}^{t'}\right) \geq V\left(\overline{X}^{t'+1}\right)$.[34] Holding fixed the level of accumulated damages, it follows that the collusive payoff at $t'$ for the alternative path exceeds the collusive payoff at $t' + 1$ for the original path. Still holding fixed the level of accumulated damages, since the price at $t'$ for the alternative path is the same as the price at $t' + 1$ for the original path, the deviation payoffs are the same. Finally, since the accumulated damages at $t'$ for the alternative path are weakly lower than the accumulated damages at $t' + 1$ for the original path, the ICC being satisfied at $t' + 1$ for the original path then implies it holds at $t'$ for the alternative path.

[34] The reason is that, at $t'$, the cartel can use the price path starting at $t' + 1$ and, since damages are weakly lower in $t'$, the collusive payoff must be weakly higher.

It has then been shown that the incentive compatibility of the original price path implies the incentive compatibility of the alternative price path. As the latter yields a strictly higher payoff, this contradicts the original path being generated by an OSSPE and thereby establishes that an OSSPE price path cannot have an increasing sub-sequence after period 1. ♦

**Proof of Theorem 5:** Most of the proof works to show that if $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is an OSSPE price path then it converges. Define $\mathcal{P}^t \equiv \max\left\{\overline{P}^0, \overline{P}^1, \ldots, \overline{P}^t\right\}$ to be the maximum price set over the first $t$ periods. As an initial step, it is shown that, on an OSSPE path, if the current period's price is at least as great as all past prices, $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$, then $\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ is a lower bound on the value in that period: $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right) \geq \pi\left(\overline{P}^{t'}\right)/(1-\delta)$; where $\overline{X}^t$ is the value of the state variable on the OSSPE path. Intuitively, if it was IC to change price to $\overline{P}^{t'}$ then it is IC to keep price at $\overline{P}^{t'}$ since the probability of detection is zero from doing so (by D2). The next step argues that, generally, $\pi\left(\mathcal{P}^t\right)/(1-\delta)$ is a lower bound on the equilibrium payoff. Since $\left\{\mathcal{P}^t\right\}_{t=1}^{\infty}$ and $\left\{\pi\left(\mathcal{P}^t\right)/(1-\delta)\right\}_{t=1}^{\infty}$ are both non-decreasing bounded sequences (with the latter following from the former because the price space has an upper bound of $P^m$), they have a limit. From this we can argue that $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ has a limit. It is then straightforward to show that $\lim_{t\to\infty}\overline{P}^t = P^*$.

Assume $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$ in which case $\overline{P}^{t'} \geq \overline{P}^{t'-1}$. The ICC for period $t'$ is

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \beta\overline{X}^{t'-1} - \gamma x\left(\overline{P}^{t'}\right) - F\right] \\
&\quad + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] V\left(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x\left(\overline{P}^{t'}\right)\right) \quad \geq \\
&\max_{P_i\in\Omega} \overline{\pi}\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/(1-\delta)\right) - \beta\overline{X}^{t'-1} - F\right] \\
&\quad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'-1}\right).
\end{aligned} \tag{24}$$

We want to make two substitutions in (24). First, replace $\left[\left(\widehat{\pi}/(1-\delta)\right) - \beta\overline{X}^{t'-1} - \gamma x\left(\overline{P}^{t'}\right) - F\right]$ on the lhs with $\left[\left(\widehat{\pi}/(1-\delta)\right) - \beta\overline{X}^{t'-1} - F\right]$. Second, suppose, contrary to the claim that $\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ is a lower bound on the collusive payoff, we have $V\left(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x\left(\overline{P}^{t'}\right)\right) < \pi\left(\overline{P}^{t'}\right)/(1-\delta)$ and replace $V\left(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x\left(\overline{P}^{t'}\right)\right)$ with $\pi\left(\overline{P}^{t'}\right)/(1-\delta)$ on the lhs of (24). If (24) holds then it is still true after these two substitutions:

$$
\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right] \\
&\quad+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right) \quad \geq \\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right] \\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\right]V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right).
\end{aligned} \tag{25}
$$

The objective is to show that pricing at $\overline{P}^{t'}$ from $t'+1$ onward is IC and thus $\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)$ is a lower bound on $V\left(\overline{P}^{t'},\overline{X}^{t'}\right)$ which gives us the desired contradiction.

As an alternative price path, consider the firm maintaining price at the $t'$ level; that is, pricing at $\overline{P}^{t'}$ in period $t, \forall t\geq t'+1$. The ICC for period $t'+1$ is

$$
\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\left(\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\right) \quad \geq \\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'}-F\right] \\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\right]V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'}\right),
\end{aligned} \tag{26}
$$

where $\overline{X}^{t'}=\beta\overline{X}^{t'-1}+\gamma x\left(\overline{P}^{t'}\right)$. Note that $\overline{X}^{t'}\geq\overline{X}^{t'-1}$ since $\overline{P}^{t'}$ is the highest price charged thus far. In addition, damages are no longer present in the collusive payoff as, by D2, $\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)=0$. For both (25) and (26), the ICC holds when $P_i>\overline{P}^{t'}$ as pricing above $\overline{P}^{t'}$ weakly lowers current profit (by A1), weakly raises the probability of detection (by A4), and it'll be shown that the MPE payoff does not exceed the collusive payoff.

I want to show that (25) implies (26) which will establish that if the original price path was IC at $t'$ then so is a price of $\overline{P}^{t'}$ at $t'+1$. Since the rhs of (26) is non-increasing in damages (using D4), a sufficient condition for (26) to hold is

$$
\begin{aligned}
&\pi\left(\overline{P}^{t'}\right)+\delta\left(\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\right) \quad \geq \\
&\max_{P_i\in\Omega}\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right] \\
&+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\right]V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right),
\end{aligned} \tag{27}
$$

where $\beta\overline{X}^{t'}$ has been replaced with $\beta\overline{X}^{t'-1}$. Let us then show that (25) implies (27). This is true if the rhs minus the lhs of (27) is at least as great as the rhs minus the lhs of (25):

$$\begin{aligned}
& \pi\left(\overline{P}^{t'}\right)+\delta\left(\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\right)-\overline{\pi}\left(P_i,\overline{P}^{t'}\right) \\
& -\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right] \\
& -\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right) \\
\geq\ & \pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right] \\
& +\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right]\left(\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\right) \\
& -\overline{\pi}\left(P_i,\overline{P}^{t'}\right)-\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right] \\
& -\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\right]V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right), \\
\forall P_i\ <\ & \overline{P}^{t'}.
\end{aligned}$$

Eliminating common terms on both sides and re-arranging:

$$\begin{aligned}
& \phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left\{\left(\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right]\right\} \\
\geq\ & \left[\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]\times \\
& \left\{V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right)-\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F\right]\right\}.
\end{aligned} \tag{28}$$

As D4 implies

$$\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)\geq V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right)\geq\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta\overline{X}^{t'-1}-F$$

then (28) holds if

$$\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\geq\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right). \tag{29}$$

Since $\overline{P}^{t'} > P_i$ and $\overline{P}^{t'} \geq \overline{P}^{t'-1}$ then (29) is true by D3. We conclude that a constant price path of $\overline{P}^{t'}$ is IC in period $t'+1$. As far as $t > t'+1$, the ICC is as specified in (26) except that $\overline{X}^{t'}$ is replaced with a weakly higher level of damages. Since the rhs of (26) is decreasing in damages and the lhs is independent of them, the ICC holds. In summary, if pricing at $\overline{P}^{t'}$ is IC in $t'$, where $\overline{P}^{t'}$ exceeds all past prices, then a constant price path of $\overline{P}^{t'}$ starting in period $t'+1$ is IC. This implies $V\left(\overline{P}^{t'},\overline{X}^{t'}\right)\geq\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)$ which gives us our desired contradiction. We have then show that if $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$ then $V\left(\overline{P}^{t'},\overline{X}^{t'}\right)\geq\pi\left(\overline{P}^{t'}\right)/\left(1-\delta\right)$.

The next step is to show that, for all periods, a lower bound on the value function at the end of period $t$ is $\pi\left(\mathcal{P}^t\right)/\left(1-\delta\right)$. The proof is by induction. Start with period $t'$ and suppose that a lower bound on the value function is $\pi\left(\mathcal{P}^{t'}\right)/(1-\delta)$. Note that $t'$ exists since

$$V\left(P^0, X^0\right) \geq \pi\left(P^0\right)/(1-\delta) = \pi\left(\mathcal{P}^0\right)/(1-\delta).$$

If $\overline{P}^{t'+1} \geq \mathcal{P}^{t'}$ then the result is immediate by the previous analysis. Next suppose $\overline{P}^{t'+1} < \mathcal{P}^{t'}$. By definition,

$$\begin{aligned} V\left(\overline{P}^{t'}, \overline{X}^{t'}\right) &= \pi\left(\overline{P}^{t'+1}\right) + \delta\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\left[(\widehat{\pi}/(1-\delta)) - \overline{X}^{t'+1} - F\right] \\ &\quad + \delta\left[1 - \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\right] V\left(\overline{P}^{t'+1}, \overline{X}^{t'+1}\right). \end{aligned}$$

Since, by the inductive step, $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right) \geq \pi\left(\mathcal{P}^{t'}\right)/(1-\delta)$ then

$$\begin{aligned} &\pi\left(\overline{P}^{t'+1}\right) + \delta\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\left[(\widehat{\pi}/(1-\delta)) - \overline{X}^{t'+1} - F\right] \\ &+ \delta\left[1 - \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\right] V\left(\overline{P}^{t'+1}, \overline{X}^{t'+1}\right) \geq \pi\left(\mathcal{P}^{t'}\right)/(1-\delta). \end{aligned} \tag{30}$$

Given $\overline{P}^{t'+1} < \mathcal{P}^{t'}$ then $\pi\left(\overline{P}^{t'+1}\right) < \pi\left(\mathcal{P}^{t'}\right)$ (here we use the fact that the upper bound on the price space is $P^m$) which, using (30), implies

$$\begin{aligned} &\delta\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\left[(\widehat{\pi}/(1-\delta)) - \overline{X}^{t'+1} - F\right] \\ &+ \delta\left[1 - \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\right] V\left(\overline{P}^{t'+1}, \overline{X}^{t'+1}\right) > \delta\pi\left(\mathcal{P}^{t'}\right)/(1-\delta). \end{aligned} \tag{31}$$

Given that

$$V\left(\overline{P}^{t'+1}, \overline{X}^{t'+1}\right) \geq (\widehat{\pi}/(1-\delta)) - \overline{X}^{t'+1} - F$$

then (31) implies

$$V\left(\overline{P}^{t'+1}, \overline{X}^{t'+1}\right) > \pi\left(\mathcal{P}^{t'}\right)/(1-\delta).$$

Since $\mathcal{P}^{t'+1} = \mathcal{P}^{t'}$ when $\overline{P}^{t'+1} < \mathcal{P}^{t'}$, we then have

$$V\left(\overline{P}^{t'+1}, \overline{X}^{t'+1}\right) > \pi\left(\mathcal{P}^{t'+1}\right)/(1-\delta)$$

which is the desired result.

For an OSSPE path, $\pi\left(\mathcal{P}^t\right)/(1-\delta)$ is then a lower bound for $V\left(\overline{P}^t, \overline{X}^t\right)$. Since $\pi$ is increasing in price (here we use the fact that the price path does not exceed $P^m$) and $\mathcal{P}^t$ is non-decreasing over time (being the maximum of all prices over the first $t$ periods), this lower bound for the value function is a non-decreasing sequence. As it has an upper bound of $\pi\left(P^m\right)/(1-\delta)$, the sequence of lower bounds converges. Call $\overline{V}$ the value to which it converges.

Since $\mathcal{P}^t$ is non-decreasing and bounded, it converges and let $\mathcal{P}^\infty \equiv \lim_{t\to\infty} \mathcal{P}^t$. Thus, $\overline{V} = \pi\left(\mathcal{P}^\infty\right)/(1-\delta)$. An OSSPE price path is bounded from above by $\mathcal{P}^\infty$. If it does not converge to $\mathcal{P}^{\infty}$ then $V^t$ is bounded below $\pi(\mathcal{P}^t)/(1-\delta)$ as $t \to \infty$ but this contradicts $\pi(\mathcal{P}^t)/(1-\delta)$ being a lower bound on the value function. Therefore, an OSSPE price path must converge to $\mathcal{P}^{\infty}$. For incentive compatibility to hold, it must then be true that

$$\lim_{t\to\infty}\left[\left(\pi\left(\mathcal{P}^t\right)/(1-\delta)\right)-\Lambda\left(\mathcal{P}^t\right)\right]\geq 0. \tag{32}$$

By the definition of $P^*$ being the highest constant price path that is IC in the steady-state (that is, with damages equal to their steady-state value of $\gamma x(P^*)/(1-\beta)$), it follows from (32) that $\mathcal{P}^{\infty} \leq P^*$. The final step is to show $\mathcal{P}^{\infty} = P^*$.

If $\mathcal{P}^{\infty} < P^*$ then

$$\lim_{t\to\infty}\left[\left(\pi\left(\overline{P}^t\right)/(1-\delta)\right)-\Lambda\left(\overline{P}^t\right)\right]>0.$$

Recall that the cartel payoff is

$$\sum_{t=1}^{\infty}\delta^{t-1}\Pi_{j=1}^{t-1}\left[1-\phi\left(P^j,P^{j-1}\right)\right]\pi\left(P^t\right)$$
$$+\sum_{t=1}^{\infty}\delta^{t}\phi\left(P^t,P^{t-1}\right)\Pi_{j=1}^{t-1}\left[1-\phi\left(P^j,P^{j-1}\right)\right]\left[\left(\widehat{\pi}/(1-\delta)\right)-\beta^tX^0-\sum_{j=1}^{t}\beta^{t-j}\gamma x\left(P^j\right)-F\right].$$

Taking the derivative of it with respect to $P^{t'}$and evaluating it at $P^{t'}=\overline{P}^{t'}$, if the ICC is not binding at $t'$ then optimality requires that:

$$\pi'\left(\overline{P}^{t'}\right)+\delta\left(\frac{\partial\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)}{\partial P^t}\right)\left(\left(\frac{\widehat{\pi}}{1-\delta}\right)-\beta^{t'}X^0-\sum_{j=1}^{t'}\beta^{t'-j}\gamma x\left(\overline{P}^j\right)-F\right) \tag{33}$$
$$+\delta^2\left[\left(\frac{\partial\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)}{\partial P^{t-1}}\right)\left(1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right)-\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)\left(\frac{\partial\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)}{\partial P^t}\right)\right]\times$$
$$\left[\left(\frac{\widehat{\pi}}{1-\delta}\right)-\beta^{t'+1}X^0-\sum_{j=1}^{t'+1}\beta^{t'+1-j}\gamma x\left(\overline{P}^j\right)-F\right]$$

$$-\left[\left(\frac{\partial\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)}{\partial P^t}\right)\left(1-\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)\right)+\left(1-\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\right)\left(\frac{\partial\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)}{\partial P^{t-1}}\right)\right]\times$$
$$\sum_{t=t'+2}^{\infty}\delta^{t-t'+1}\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\Pi_{j=t'+2}^{t-1}\left[1-\phi\left(\overline{P}^j,\overline{P}^{j-1}\right)\right]\left[\left(\widehat{\pi}/(1-\delta)\right)-\beta^tX^0-\sum_{j=1}^{t}\beta^{t-j}\gamma x\left(\overline{P}^j\right)-F\right]$$
$$-\sum_{t=t'}^{\infty}\delta^{t-t'+1}\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\Pi_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^j,\overline{P}^{j-1}\right)\right]\beta^{t-t'}\gamma x'\left(\overline{P}^{t'}\right)=0.$$

As $t' \to \infty$, $\left(\overline{P}^{t'} - \overline{P}^{t'-1}\right) \to 0$ which implies, by D1-D2, that

$$\begin{aligned} \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) &\to 0, \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right) \to 0 \\ \frac{\partial \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t} &\to 0, \frac{\partial \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)}{\partial P^{t-1}} \to 0 \end{aligned}$$

Thus, (33) implies $\lim_{t' \to \infty} \pi'\left(\overline{P}^{t'}\right) = 0$. However, $P^* \leq P^m$ and, by supposition, $\lim_{t\to\infty} \overline{P}^t < P^*$ so that $\lim_{t'\to\infty} \pi'\left(\overline{P}^{t'}\right) > 0$. This contradiction proves that our original claim that $\mathcal{P}^\infty < P^*$ is false. I conclude that $\lim_{t\to\infty} \overline{P}^t = P^*$. ♦

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Figure 1 – Value and Policy Functions (Benchmark Case)

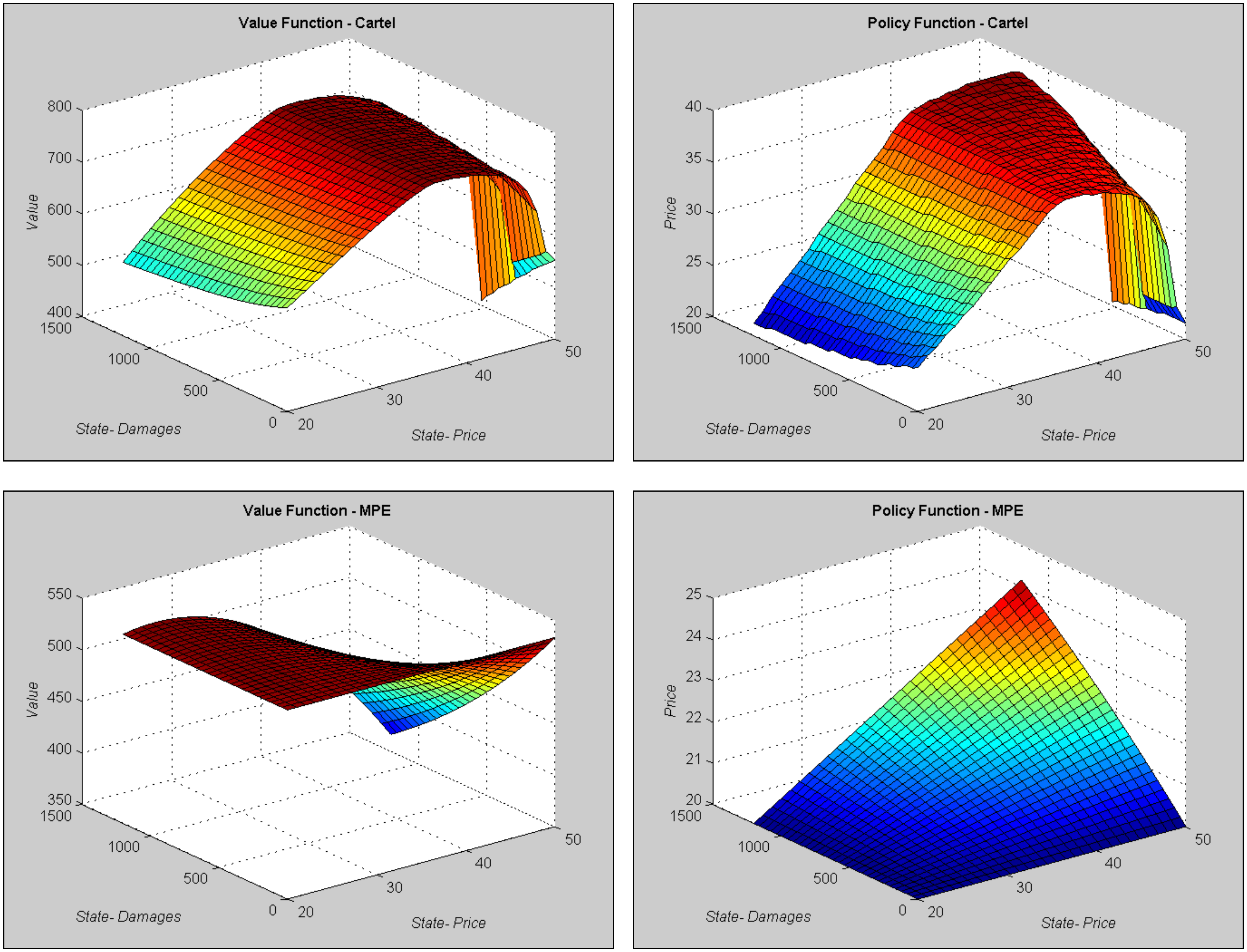

Figure 2 – Time Paths (Benchmark Case)

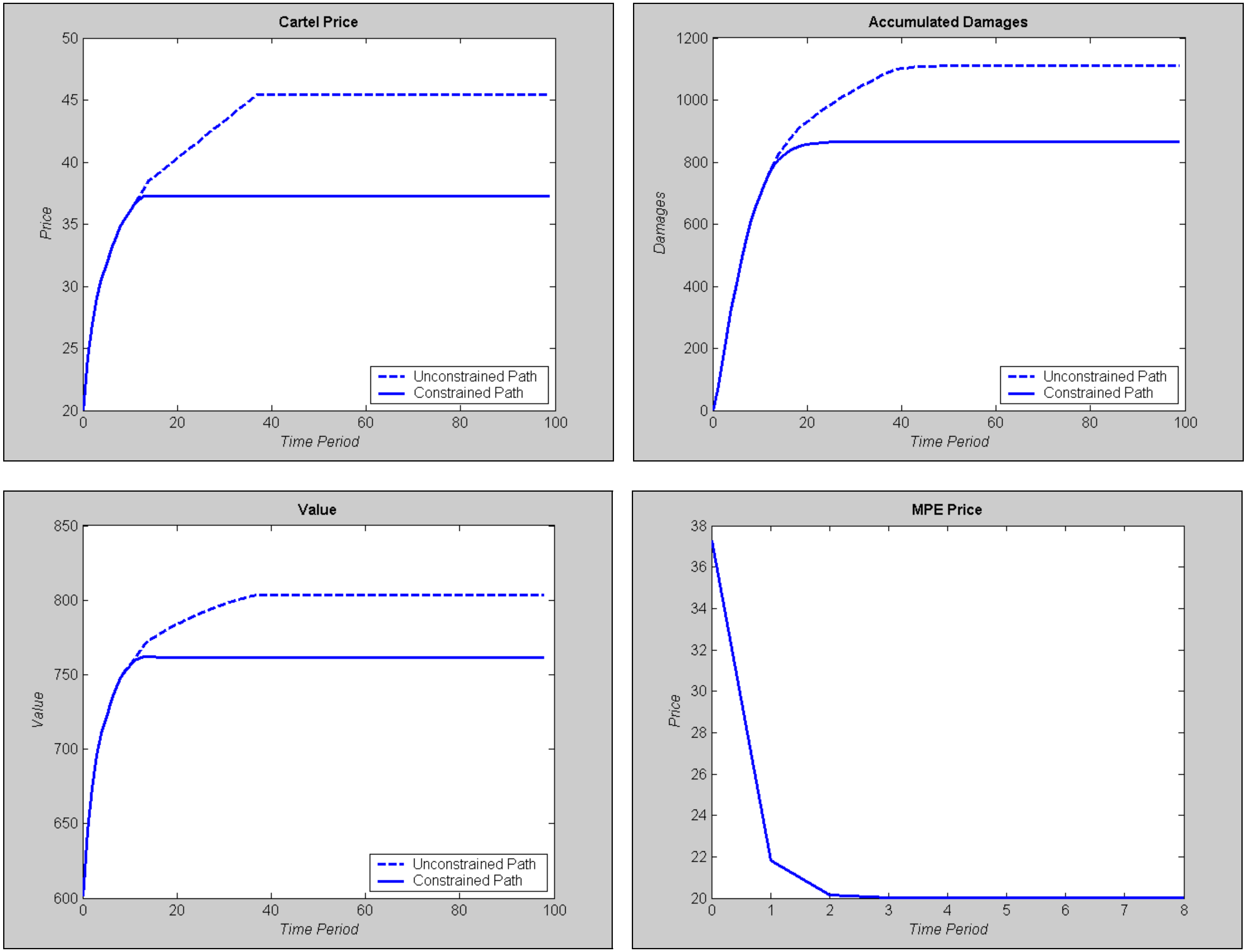

Figure 3 – Non-monotonic Cartel Price Path

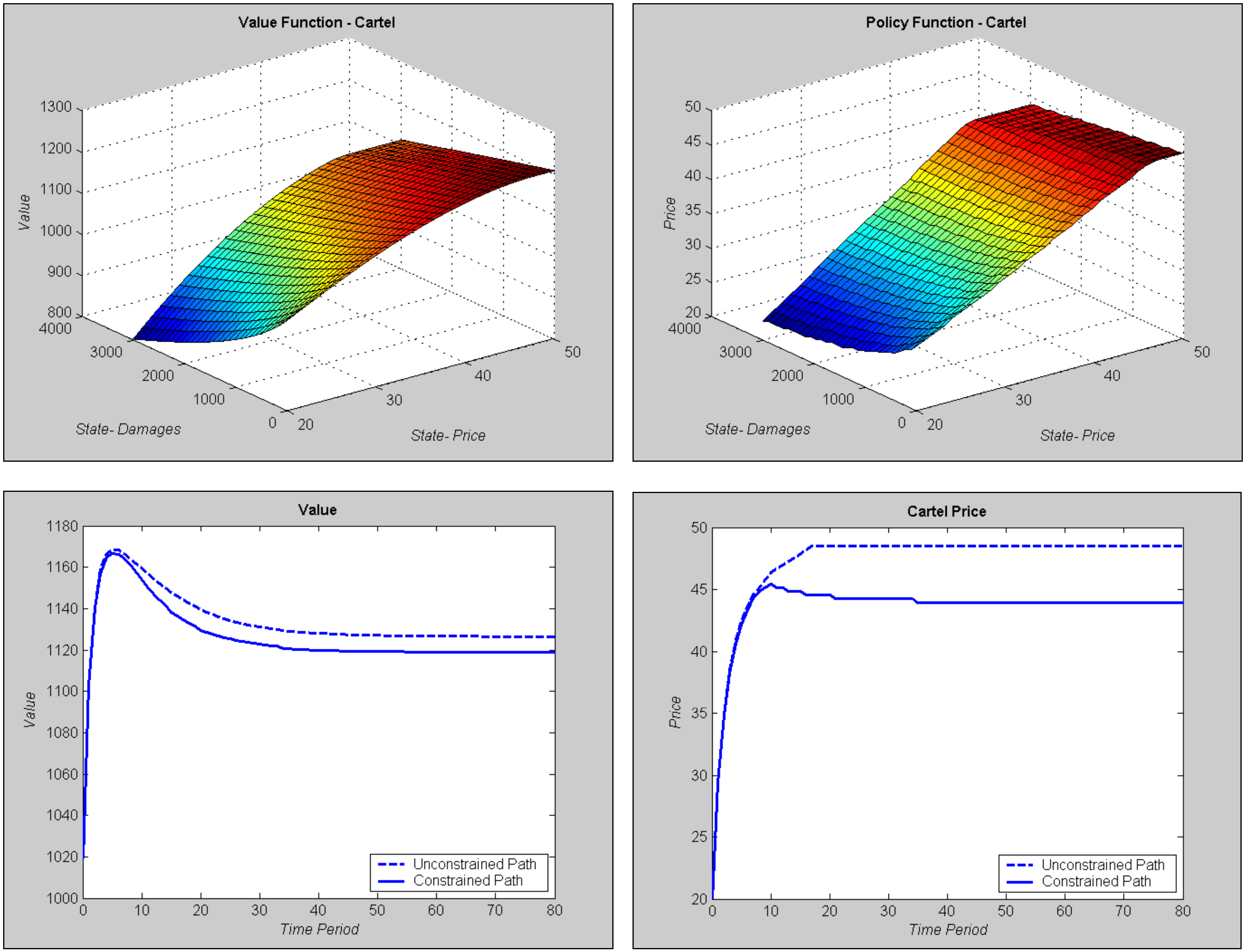

Figure 4 – Effect of Time Preferences on the Cartel Price Path

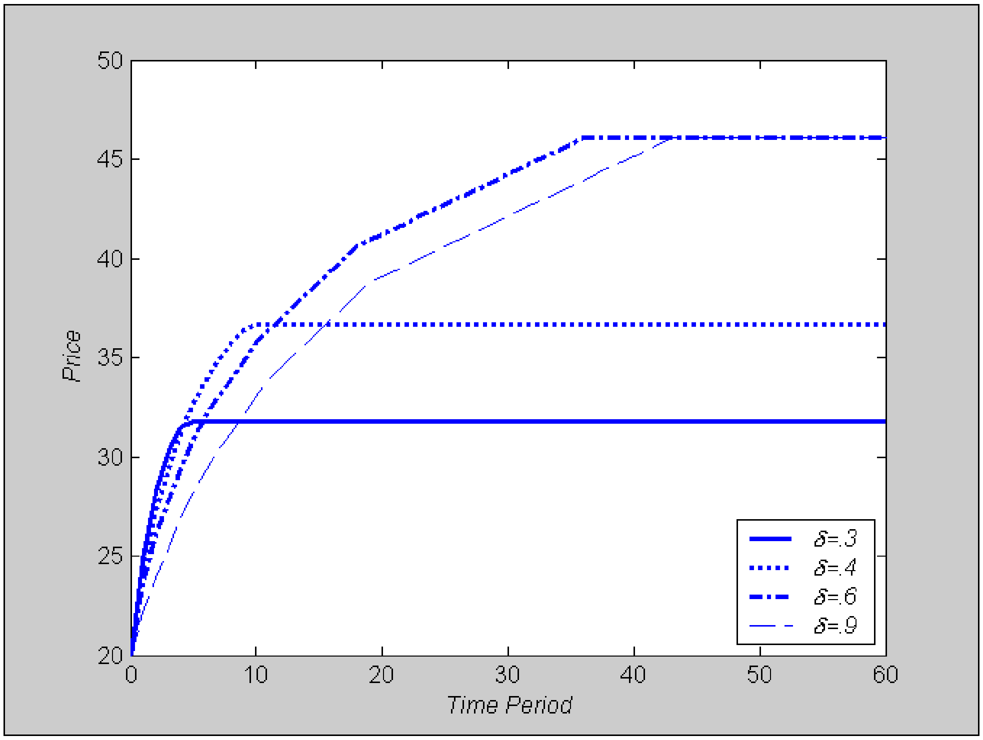


Figure 5 – Effect of Market Structure on the Cartel Price Path