INSTANTANEOUS GRATIFICATION

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Abstract. We propose a tractable continuous-time model that captures the key psychological properties of the discrete-time quasi-hyperbolic discount function. Like the discrete-time model, our 'instantaneous-gratification model' reflects consumers' preference to act impatiently in the short run and patiently in the long run. Unlike the discrete-time model, the instantaneous-gratification model generates continuous, monotonic, policy functions and admits only one equilibrium. We illustrate these useful properties using a standard consumption model with liquidity constraints. The instantaneous-gratification model eliminates the problematic and counterfactual properties of the discrete-time hyperbolic model, but preserves the model's desirable psychological features.

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1. Introduction

Consumers purportedly have a special taste for instant gratification. In popular culture we talk about "living only for the moment." We say that consumers "want it now" and have a hard time postponing immediate gratification even if they will receive significant (delayed) rewards for doing so.

Psychological and economic researchers have long noted the heightened payoffs that are associated with immediate rewards. For example, Freud (1915) identified a "pleasure principle" that drives people to seek out immediate pleasures even if those pleasures are associated with large delayed costs. Even before Freud, Smith (1759, VI.i.11) wrote about the troubling appeal of immediate gratification. Smith "approve[s] and even applaud[s] that proper exertion of self-command, which enables [men] to act as if their present and their future situation affected them nearly in the same manner in which they affect an [impartial spectator]." Smith also wrote that "we are capable, it may be said, of resolving, and even of taking measures to execute, many things which, when it comes to the point, we feel ourselves altogether incapable of executing."

Nearly two centuries later Robert Strotz (1956) developed the first mathematical model that explains the appeal of instant gratification. He suggested that discount rates are higher in the short run than in the long run. His formulation implies that delaying current consumption by a period produces proportionately more devaluation than a one-period delay of future consumption. Most experimental studies on time preference have supported Strotz's conjecture (Ainslie 1992, Loewenstein and Read 2001), though debate continues about the shape of the discount function (Frederick, Loewenstein, and O'Donoghue 2002).

To capture the taste for instant gratification, Laibson (1997a) adopted a discrete-time discount function, $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots\}$, which Phelps and Pollak (1968) had previously used to model intergenerational time preferences. With $\beta < 1$, this so called 'quasi-hyperbolic' discount function captures the gap between a high short-run discount rate and a low long-run rate. O'Donoghue and Rabin (199?) call these "present-biased" preferences, emphasizing the heightened weight they place on current consumption. In the last several years, the quasi-hyperbolic discount function has been used to study a wide range of behaviors, including consumption, procrastination, addiction, and job search.[1]

However, the quasi-hyperbolic discounting model has two significant drawbacks. First, the discount function does not have a natural generalization to continuous-time. The model does not scale with the length of the time period. Making matters even worse, generic continuous-time variants of the model are not analytically tractable since they generate preferences that are not recursive.

The second drawback of the quasi-hyperbolic model arises because of strategic considerations. Since the pathbreaking work of Strotz (1956), most researchers have analyzed dynamically inconsistent preferences by treating the individual as a sequence of independent selves whose choices are modelled as an intrapersonal game. Although this game-theoretic framework has proved generally fruitful, a recurrent problem has plagued many of these quasi-hyperbolic applications: strategic interaction among intrapersonal selves often generates counterfactual policy functions. Hyperbolic consumption functions need not be globally monotonic in wealth, and may even drop discontinuously at a countable number of points. Numerous authors, including Laibson (1997b), Morris and Postlewaite (1997), O'Donoghue and Rabin (1999a), Harris and Laibson (2001b), and Krusell and Smith (2000) have identified hyperbolic examples in which the consumption function has negatively sloped intervals or downward discontinuities. Figure 1 plots examples of such 'pathological' consumption functions.

The current paper resolves all of these problems.[2] We present a continuous-time discount function that captures the taste for instant gratification, admits analytically tractable analysis, and eliminates all of the pathologies listed above.

Our continuous-time model captures the qualitative properties of the original discrete-time quasi-hyperbolic model. Our continuous-time model distinguishes between the 'present' and the 'future'. The present is valued discretely more than the future, mirroring the one-time drop in valuation implied by the discrete-time

[1] For some examples, see O'Donoghue and Rabin (1999b), Angeletos, Laibson, Repetto, Tobacman and Weinberg (2001), and Della Vigna and Paserman (2000).

[2] Two other solutions to these problem have been proposed. First, Harris and Laibson (2001b) point out that pathologies occur only when the model is calibrated in a limited region of the parameter space (notably when the coefficient of relative risk aversion lies well below unity). Second, O'Donoghue and Rabin (1999a) point out that pathologies do not arise if consumers naively believe that their preferences are dynamically consistent. Any partial knowledge of future dynamic inconsistency reinstates the pathologies.

quasi-hyperbolic discount function (Phelps and Pollak 1998, Laibson 1997) and its continuous-time generalizations (Barro 1999, Luttmer and Mariotti 2000). In addition, we assume that the transition from the present to the future is determined by a constant hazard rate. This simplifying assumption enables us to reduce our problem to a system of two differential equations that characterize present and future value functions.

We show that our continuous-time model has a limit case that is analytically tractable and psychologically relevant. This is the case in which the present is vanishingly short. By focusing on this psychologically important limit case, we take the phrase 'instant gratification' literally. We analyze a model in which individuals prefer gratification in the present instant discretely more than consumption in the momentarily delayed future. This model is a useful benchmark that captures the essence of neighboring models in which the present is short, but not precisely instantaneous.

Third, we show that the instantaneous-gratification model, which is dynamically *inconsistent*, shares the same value function as a related dynamically *consistent* optimization problem with a wealth-contingent utility function. Using this partial equivalence, we can show both existence and *uniqueness* of the hyperbolic equilibrium. However, our economy is not observationally equivalent to the related dynamically consistent optimization problem. The partial equivalence applies to the value functions but *not* to the policy functions.

We also show that the equilibrium consumption function of the hyperbolic problem is continuous and monotonic in wealth. The monotonicity property relies on the condition that the long-run discount rate is weakly greater than the interest rate. When this inequality is satisfied, all of the pathological properties of discrete-time hyperbolic models are eliminated by our continuous-time model.

Two other sets of authors have analyzed hyperbolic preferences in continuous time. Barro (1999) analyzes the choices of hyperbolic agents with constant relative risk aversion. He focuses on the general equilibrium implications of hyperbolic discounting and the ways in which hyperbolic economies may be observationally equivalent to exponential economies. Luttmer and Mariotti (2000) analyze the choices of agents with arbitrary discount functions, constant relative risk aversion, and stochastic asset returns. Luttmer-Mariotti generalize Barro's observational-equivalence result, but also identify particular endowment processes for which the hyperbolic model has interesting new asset-pricing implications (e.g., an elevated equity premium). Luttmer and Mariotti work with general discount functions and consider numerous special cases. They have independently identified some properties of the particular case in which the present is vanishingly short. However, their findings do not overlap with ours.

Barro and Luttmer-Mariotti both restrict their analysis to *linear* policy rules. The existence of a linear equilibrium depends on special preference assumptions (constant relative risk aversion) and market assumptions (complete markets enabling sales of future labor income). We do not make restrictive assumptions of this kind: we work with a broad class of preferences; and we introduce the constraint that consumers may not borrow against future labor income. We pursue these generalizations for greater realism. Our problem does not admit a linear equilibrium. We have to contend with the pathologies that arise in our general setting, but do not arise under the Barro/Luttmer-Mariotti simplifying assumptions in either discrete or continuous time.

Our results also differ from Barro and Luttmer-Mariotti in that we are able to prove uniqueness of Markov equilibrium in the class of *all* policy rules. This is a desirable and unexpected result, since the hyperbolic model is a dynamic game, and can therefore generate non-uniqueness. For example, Krusell and Smith (2000) have shown that hyperbolic Markov equilibria are *not* unique in a deterministic discrete-time setting. In the current paper, we provide two uniqueness results. First, we prove uniqueness in a class of continuous-time models with stochastic asset returns. Second, we propose a refinement that uses the unique equilibrium in the stochastic setting to select a sensible unique equilibrium in the deterministic setting. This refinement takes the natural approach of selecting the limiting equilibrium obtained as the noise in the asset returns vanishes.

The rest of the paper formalizes these claims. In Section 2 we present our general continuous-time model and formulate some of the properties of this model. In Section 3 we present the consumption model that will provide the principal application of the paper. In Section 4 we describe an important limit case of our model. We call this limit case the instantaneous-gratification model. In Section 5 we show that the instantaneous-gratification model has the same *value function* as a particular dynamically consistent optimization problem. We call this latter problem the 'equivalent problem', but note that it is *not* observationally equivalent to the hyperbolic problem. The instantaneous-gratification model shares the same long-run discount rate as the equivalent problem, but the two problems have different instantaneous utility functions and different equilibrium policy functions.[3] In Section 6, we use our partial equivalence result to derive several important properties of the instantaneous-gratification problem, including equilibrium existence, equilibrium uniqueness, consumption-function continuity, and consumption-function monotonicity. In Section 6 we also derive the deterministic version of the instantaneous-gratification model, and provide a complete analysis of the case of constant relative risk aversion. In Section 7 we further generalize our results in Section 8 we conclude.

2. Intertemporal Preferences

2.1. The Basic Model of Preferences. In the standard discrete-time formulation of quasi-hyperbolic time preferences, it is natural to divide time into two subperiods: the present period and all future periods. All future periods are valued less than the present period by a uniform factor $\beta < 1$. In addition, the agent discounts all periods exponentially. In total, a period $n \geq 1$ steps into the future is discounted with the overall discount factor $\beta\delta^n$ (Phelps and Pollak 1968, Laibson 1997).

This model can be extended to continuous-time and generalized in two ways. First, instead of having the present last for exactly one period, the present can last for an arbitrary duration τ. Second, the duration τ can be random. Consider an economic self born at date t_0. The preferences of this agent is divided into two subperiods. A 'present', which lasts from t_0 to $t_0 + \tau_0$, and a future that lasts from date $t_0 + \tau_0$ to ∞. Think of the present as the period of control and the future as the period after which the self has passed control to the next self. The length of the present, τ_0, is stochastic with an exponential distribution parameterized by $\lambda \in [0, +\infty)$. Hence, λ represents the arrival rate of transitions from the present to the future.

When self t_0 enters its future at $t_0 + \tau_0$, a new self is born and takes control of decision-making. Call this self $t_1 = t_0 + \tau_0$. The preferences of this new self can

[3] By contrast, see Barro (1999), Laibson (1996), and Luttmer and Mariotti (2000) for the special case — log utility and no liquidity constraints — in which observational equivalence of the policy functions *does* hold.

also be divided into subperiods. Self t_1 has a present that lasts from date t_1 to date $t_1+\tau_1$, and a future that lasts from $t_1+\tau_1$ to ∞. Extending this idea, we assume that at each juncture of present and future a new self is born, yielding a sequence of selves born at dates $\{t_0, t_1, t_2, ...\}$, with respective present subperiods of duration $\{\tau_0, \tau_1, \tau_2, ...\}$. This structure generates countable generations of selves.

We assume that all selves discount exponentially at rate $\gamma > 0$. In addition, they value the future discretely less than the present. For example, consider a self with birth at date t and present of duration τ. This self's preferences are given by

$$\mathrm{E}_t \left[\int_0^{\tau} e^{-\gamma s} u(c(t+s))ds + \alpha \int_{\tau}^{\infty} e^{-\gamma s} u(c(t+s))ds \right], \tag{1}$$

where $\alpha \in (0,1]$, and u represents a utility function. Because the transition date $t+\tau$ is stochastic, self t has a stochastic discount function,

$$D(s) = \left\{ \begin{array}{ll} e^{-\gamma s} & \text{if} \quad s \in [0,\tau) \\ \alpha e^{-\gamma s} & \text{if} \quad s \in [\tau, \infty) \end{array} \right\}.$$

$D(s)$ decays exponentially at rate γ up to time $t+\tau$, drops discontinuously at $t+\tau$ to a fraction α of its level just prior to $t+\tau$, and decays exponentially at rate γ thereafter. Hence, self t discounts all flows in the 'future' — i.e., flows that come after time $t+\tau$ — with an extra factor of α. This continuous-time formalization is close to some of the deterministic discount functions used in Barro (1999) and Luttmer and Mariotti (2000). However, we assume that the duration of the present, τ, is stochastic. Figure 2 plots a single realization of this discount function, with $t=0$ and $\tau = 3.4$.

As $\lambda \to 0$ our discount function reduces to the standard exponential discount function, namely

$$\lim_{\lambda \to 0} D(s) = e^{-\gamma s} \text{ for all } s \in [0, \infty).$$

As $\lambda \to \infty$ the discount function converges to a deterministic jump function with a jump at $s=0$, namely

$$\lim_{\lambda \to \infty} D(s) = \left\{ \begin{array}{ll} 1 & \text{if} \quad s = 0 \\ \alpha e^{-\gamma s} & \text{if} \quad s \in (0, \infty) \end{array} \right\}.$$

We shall return to this case below.

2.2. An Alternative Interpretation of Preferences. The arguments in this paper are consistent with a second interpretation of the time preferences described above. In particular, one can assume that a new self is born every instant, and that each self has a *deterministic* discount function equal to the expected value of the *stochastic* discount function described above. Specifically, this expectation is given by

$$\overline{D}(s) \equiv \mathrm{E}D(s) = e^{-\lambda s}e^{-\gamma s} + (1 - e^{-\lambda s})\alpha e^{-\gamma s}.$$

The instantaneous discount rate is the rate of decline of the deterministic discount function $\overline{D}(s)$:

$$-\frac{\overline{D}'(s)}{\overline{D}(s)} = \gamma + \frac{\Pr(s < \tau)\lambda\,(1-\alpha)\,D(s|s<\tau)}{\overline{D}(s)} \tag{2}$$

Note that $\Pr(s < \tau)$ represents the probability that s is less than an exponentially distributed random variable τ with density $f(\tau) = \lambda\exp(-\lambda\tau)$. The right-hand-side of equation (2) contains two terms. The first term is just the long-run (exponential) discount rate. The second term reflects the additional discounting that arises from the fact that $D(s)$ will eventually drop at date τ. Note that $\Pr(s < \tau)\lambda$ is the unconditional instantaneous hazard of a drop in $D(s)$ at horizon s. Finally, $(1-\alpha)\,D(s|s<\tau)$ is the size of the drop in $D(s)$, if it occurs at horizon s.

Two special cases should be emphasized. First, in the immediate present ($s \to 0$) the instantaneous discount rate converges to

$$\lim_{s\to 0} -\frac{\overline{D}'(s)}{\overline{D}(s)} = \gamma + \lambda\,(1-\alpha) > 0.$$

By contrast as the horizon goes to infinity the discount rate converges to

$$\lim_{s\to\infty} -\frac{\overline{D}'(s)}{\overline{D}(s)} = \gamma.$$

Figure 3 plots $\overline{D}(s)$ for a set of λ values: $\lambda \in \{0, 0.1, 1, 10, \infty\}$.

2.3. Comparison of the two interpretations. The "basic" and "alternative" models described respectively in subsections 2.1 and 2.2 at first appear to be quite distinct. After all, the basic model uses a stochastic discount function with a present of expected duration $\frac{1}{\lambda}$, while the alternative model uses a deterministic discount function with a present of zero measure. Likewise, the basic model implies a countable number of selves while the alternative model implies a continuum of selves. However, we will show that the two approaches turn out to be essentially equivalent.

We principally emphasize the basic model because it requires no ancillary assumptions. Under the basic model the selves make choices that have a non-negligible impact on utility, enabling us to pin down those choices with standard optimality conditions. Under the alternative model the selves make choices that have a zero measure impact on utility, requiring that we impose a standard optimality condition — e.g., the envelope property (cf. subsection 3.4) — to pin down behavior.

3. A Continuous-Time Consumption Model

The consumer is modeled as a sequence of autonomous selves. Each self controls consumption in the 'present' and cares about but does not directly control consumption in the 'future.'

3.1. Dynamics. Our consumption dynamics incorporate liquidity constraints, an important qualitative feature of consumers' planning problems (cf. Deaton 1991, Carroll 1992, 1997).

At a point in time $t \in [0, \infty)$, the consumer has stock of wealth $x \in [0, \infty)$ and receives a flow of labor income $y \in (0, \infty)$. If $x > 0$, then the consumer is not liquidity constrained and may choose a consumption flow $c \in (0, \infty)$. In other words, since wealth is a stock and consumption is a flow, she is free to choose any level of consumption she wishes. If $x = 0$, then she may only choose a consumption flow $c \in (0, y]$. In other words, since she has no wealth, she cannot consume more than her labor income. In particular, she may never borrow.

Whatever the consumer does not consume is invested in an asset, the returns on which are distributed normally with mean $\mu \, dt$ and variance $\sigma^2 \, dt$, where $\mu \in$

$(-\infty, +\infty)$ and $\sigma \in (0, +\infty)$. The change in her wealth at time t is therefore

$$dx = (\mu\, x + y - c)\, dt + \sigma\, x\, dz,$$

where z is a standard Wiener process.

We could easily generalize this framework by adding a stochastic source of labor income. For example, stochastic increments of labor income could follow a Poisson arrival process. We do not pursue this generalization, since it would not qualitatively change the analysis that follows.

3.2. Equilibrium. Suppose that $c : [0, +\infty) \to (0, +\infty)$ is a consumption function. Then we may then define the continuation-value function $v : [0, +\infty) \to \mathbb{R}$ by the formula

$$v(x_0) = \mathrm{E}\left[\int_0^{+\infty} e^{-\gamma\, t}\, u(c(x(t)))\, dt\right],$$

where x is the timepath of wealth starting at x_0 when the consumption function is c; and we may define the current-value function $w : [0, +\infty) \to \mathbb{R}$ by the formula

$$w(x_0) = \mathrm{E}\left[\int_0^{\tau} e^{-\gamma\, t}\, u(c(x(t)))\, dt + \alpha\, e^{-\gamma\, \tau}\, v(x(\tau))\right].$$

The continuation-value function v discounts utility flows exponentially, with discount rate γ. The current-value function w discounts utility flows up to the stochastic transition time τ exponentially, again with discount rate γ. It discounts the continuation-value function v obtained after τ by the composite discount factor $\alpha\, e^{-\gamma\, \tau}$. The component α reflects the one-time discounting that arises from the transition from the "present" to the "future". The component $e^{-\gamma\, \tau}$ is the standard exponential discount factor.

Using this notation, we can define equilibrium as follows.

Definition 1. *A consumption function $c : [0, +\infty) \to (0, +\infty)$ is an* ***equilibrium*** *iff:*

1. *For all consumption functions $\widetilde{c} : [0, +\infty) \to (0, +\infty)$ and all $x_0 \in [0, +\infty)$, we*

have

$$w(x_0) \geq \mathrm{E}\left[\int_0^{\tau} e^{-\gamma t}\, u(\widetilde{c}(\widetilde{x}(t)))\, dt + \alpha\, e^{-\gamma \tau}\, v(\widetilde{x}(\tau))\right],$$

where $\widetilde{x}$ is the timepath of wealth starting at x_0 when the consumption function is $\widetilde{c}$.

2. *For all $x_0 \in [0, +\infty)$, we have $v(x_0) \geq \frac{1}{\gamma}\, u(y)$.*

The first condition in this definition of equilibrium reflects our assumption that the current self maintains control of the consumption decision for the duration of the present – i.e., until the next stochastic transition date τ periods in the future. The second condition requires that equilibrium continuation-payoff functions be bounded below by the payoff function associated with the myopic policy "always consume deterministic labor income y". This requirement rules out equilibria supported by policy functions that generate expected utility of $-\infty$. Such infinitely bad policy functions can in general be equilibria since no single self has an incentive to deviate.[4]

3.3. Characterization of Equilibrium. Since the current self controls consumption over the non-trivial interval $[0, \tau)$, equilibrium can be characterized in the usual way using dynamic programming. We begin this subsection with a heuristic derivation of the Bellman system for our economy. There are three parts to this Bellman system: an equation for the continuation-value function of the current self, an equation for the current-value function of the current self, and a first-order condition determining the consumption chosen by the current self.

We begin with the equation for the continuation-value function v. Suppose that future selves use the consumption function c, and let the current state be x. Then $v(x)$ has two components: the current payoff $u(c(x))\, dt$, and the expected discounted continuation payoff $\mathrm{E}\left[\exp(-\gamma\, dt)\, v(x + dx)\right]$. We therefore have

$$v(x) = u(c(x))\, dt + \mathrm{E}\left[\exp(-\gamma\, dt)\, v(x + dx)\right]. \tag{3}$$

[4] Our equilibrium concept is essentially perfect equilibrium in stationary Markov strategies. However, we depart from the usual definition in only allowing deviations to stationary Markov strategies. (The standard definition allows deviations to arbitrary non-stationary and history-dependent strategies.) We do this for expositional convenience. (It should be intuitively clear that the set of equilibria is unaffected by this departure.)

Multiplying through by $\exp(\gamma\,dt)$ and subtracting $v(x)$ from both sides, we obtain

$$(\exp(\gamma\,dt) - 1)\,v(x) = \exp(\gamma\,dt)\,u(c(x))\,dt + \mathrm{E}\left[v(x + dx) - v(x)\right].$$

Now

$$\exp(\gamma\,dt) = 1 + \gamma\,dt + O(dt^2)$$

and

$$\mathrm{E}\left[v(x + dx) - v(x)\right] = ((\mu\,x + y - c(x))\,v'(x) + \tfrac{1}{2}\,\sigma^2\,x^2\,v''(x))\,dt + O(dt^2)$$

(cf. Itô's Lemma). Hence, dropping terms of order dt^2 and higher, dividing through by dt and suppressing the arguments of v and c,

$$\gamma\,v = u(c) + (\mu\,x + y - c)\,v' + \tfrac{1}{2}\,\sigma^2\,x^2\,v''. \tag{4}$$

The term $\gamma\,v$ represents the expected value of instantaneous changes in v arising from exponential discounting at rate γ; the term $u(c)$ represents the instantaneous value of the consumption flow; the term $(\mu\,x + y - c)\,v'$ represents the expected value of instantaneous changes in v arising from the deterministic component of the returns process; and the term $\frac{1}{2}\,\sigma^2\,x^2\,v''$ represents the expected value of instantaneous changes in v arising from the stochastic component of the returns process.

An analogous derivation can be used to obtain the equation for the current-value function w. We decompose $w(x)$ into the current payoff $u(c(x))\,dt$, the expected discounted continuation payoff $\mathrm{E}\left[\exp(-\gamma\,dt)\,w(x + dx)\right]$ and the expected change in welfare associated with the possibility of a transition from present to future $\lambda\,dt\,(\alpha\,v(x) - w(x))$. Only this last term is new relative to the decomposition in equation 3 above. The new term represents the expected value of instantaneous changes in w arising from the stochastic arrival (with hazard rate λ) of a transition between the "present", with current value w, and the "future", with continuation value $\alpha\,v$. Proceeding as above, we obtain

$$\gamma\,w = u(c) + (\mu\,x + y - c)\,w' + \tfrac{1}{2}\,\sigma^2\,x^2\,w'' + \lambda(\alpha\,v - w). \tag{5}$$

Equation (5) is analogous to equation (4).

Finally, we turn to derivation of the equilibrium policy function c. Consumption is chosen by the current self, so:

$$c(x) = \arg\max_c u(c) + (\mu\, x + y - c)\, w' + \tfrac{1}{2}\,\sigma^2\, x^2\, w'' + \lambda(\alpha v - w).$$

Hence, when $x > 0$, c is unconstrained and

$$u'(c) = w'.$$

This is just a statement of the envelope theorem. When $x = 0$, c may be constrained and so we write

$$u'(c) = \max\{w', u'(y)\}\,.$$

When $x > 0$, consumption is chosen so as to equate the marginal utility of consumption to the marginal value of saving. When $x = 0$, consumption cannot exceed y. Hence the marginal utility of consumption must be at least $u'(y)$.

Drawing all of the conditions together, we have the system of equations that define the equilibrium in our economy.

Definition 2. *The Bellman system of the finite-λ model is the system*

$$0 = \tfrac{1}{2}\,\sigma^2\, x^2\, v'' + (\mu\, x + y - c)\, v' - \gamma\, v + u(c), \tag{6}$$

$$0 = \tfrac{1}{2}\,\sigma^2\, x^2\, w'' + (\mu\, x + y - c)\, w' - \gamma\, w + u(c) - \lambda\,(w - \alpha\, v), \tag{7}$$

where

$$u'(c) = \left\{ \begin{array}{ll} w' & \text{if } x > 0 \\ \max\{w', u'(y)\} & \text{if } x = 0 \end{array} \right\}. \tag{8}$$

3.4. The Alternative Interpretation of Equilibrium. We can also derive an equivalent Bellman system for the alternative model with deterministic discount function $\overline{D}(s)$ (cf. subsection 2.2). Since the analysis of this subsection simply recovers the results above, many readers will wish to jump immediately to the next section.

To derive the Bellman System for the alternative model, consider the (discounted) value function, $z(x, s)$, that represents the discounted continuation value to the self at date t of having cash on hand x at date $t+s$. Using standard methods (cf. Ito's Lemma), one can show that the associated Bellman Equation is

$$0 = \frac{1}{2}\sigma^2 x^2 z_{xx} + (\mu x + y - c) z_x + z_s + \overline{D}u(c)$$

This Bellman Equation is satisfied when

$$z(x, s) = e^{-\lambda s} e^{-\gamma s} w(x) + (1 - e^{-\lambda s})\alpha e^{-\gamma s} v(x)$$

where $v(x)$ and $w(x)$ respectively satisfy equations 6 and 7. Hence, the value function $z(x, s)$ is simply a weighting of $e^{-\gamma s} w(x)$ and $\alpha e^{-\gamma s} v(x)$, where the associated weights, $e^{-\lambda s}$ and $1 - e^{-\lambda s}$, represent the respective probabilities of being in the present or in the future in the basic model.

In addition, note that $z_x(x, 0) = w'(x)$. This alternative model then yields identical results to the basic countable-self model as long as we close the alternative model with the heuristic equilibrium condition $u'(c) = z_x(x, 0)$: marginal utility of consumption equals the marginal value of wealth.

4. The Instantaneous-Gratification Model

The continuous-time consumption model presented in the last subsection has an immediate advantage over its discrete-time analogue: equilibrium consumption functions are everywhere continuous. However, the principal pathology of the discrete-time hyperbolic consumption model remains: there may be intervals on which the consumption function is downward sloping.[5]

Fortunately, we need not be interested in the general case of the continuous-time consumption model. The urge for "instantaneous gratification" suggests that the present – i.e. the interval $[t, t+\tau)$ during which consumption is highly valued – is very short. Since the arrival rate of τ is λ, this is the same as saying that λ is very

[5] The jumps that can occur in equilibrium consumption functions of the discrete-time model are always downward. As such, they are simply mathematically extreme versions of downward slopes. The Brownian noise in the continuous-time model eliminates the mathematical pathology of jumps, but fails to eliminate the economic pathology of downward slopes.

large. We are therefore led to consider the limiting case $\lambda \to +\infty$. We refer to this case as the instantaneous-gratification case, or IG case for short.

Suppose that the triple $(v_\lambda, w_\lambda, c_\lambda)$ solves the Bellman system of the finite-λ model. Suppose further that $(v_\lambda, w_\lambda, c_\lambda) \to (w, v, c)$ as $\lambda \to +\infty$. Then, letting $\lambda \to +\infty$ in equation (6), we obtain

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - c)\,v' - \gamma\,v + u(c). \tag{9}$$

In other words, v is the expected present discounted value obtained when the discount rate is γ, and when consumption is chosen according to the exogenously given consumption function c. Next, dividing equation 7 through by λ and rearranging, we obtain

$$w_\lambda - \alpha\,v_\lambda = \tfrac{1}{\lambda}\left(\tfrac{1}{2}\,\sigma^2\,x^2\,w_\lambda'' + (\mu\,x + y - c_\lambda)\,w_\lambda' - \gamma\,w_\lambda + u(c_\lambda)\right).$$

Hence, letting $\lambda \to +\infty$,

$$w - \alpha\,v = 0. \tag{10}$$

This reflects the fact that, as $\lambda \to +\infty$, the discount function drops essentially immediately to a fraction α of its initial value, and that the current-value function w is therefore α times the continuation-value function v. Finally, letting $\lambda \to +\infty$ in equation (8), we obtain

$$u'(c) = \left\{ \begin{array}{ll} w' & \text{if } x > 0 \\ \max\{w', u'(y)\} & \text{if } x = 0 \end{array} \right\}. \tag{11}$$

In other words, consumption is chosen optimally by the current self.

This derivation motivates the following definition:

Definition 3. *The Bellman equation of the IG model is the equation*

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - c)\,v' - \gamma\,v + u(c), \tag{12}$$

where

$$u'(c) = \left\{ \begin{array}{ll} \alpha\,v' & \text{if } x > 0 \\ \max\{\alpha\,v', u'(y)\} & \text{if } x = 0 \end{array} \right\}. \tag{13}$$

Equation (12) is identical to equation (9). Equation (13) is obtained by expressing w in terms of v using equation (10), (i.e., replacing w with αv).

Further understanding of the Bellman equation of the IG model can be obtained by comparing it with the Bellman equation of the exponential model, i.e. the Bellman equation that would be obtained if the consumer were dynamically consistent.

Definition 4. *The Bellman equation of the exponential model is the equation*

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - c)\,v' - \gamma\,v + u(c), \tag{14}$$

where

$$u'(c) = \left\{ \begin{array}{ll} v' & \textit{if } x > 0 \\ \max\{v', u'(y)\} & \textit{if } x = 0 \end{array} \right\}. \tag{15}$$

As this definition makes clear, the Bellman equation of the exponential model is simply the special case of the Bellman equation of the IG model obtained by putting $\alpha = 1$.

In general, the consumer applies a discount factor $\alpha < 1$ to the shadow value of saving. Since u is concave, this means that, for any given shadow value of saving, the IG consumer will consume more than the exponential consumer. This increase in consumption drives a wedge between the value functions of the two problems.

5. Value-Function Equivalence

In the present section we show that the value function v of the IG consumer with the original utility function u is also the value function of an exponential consumer with a new utility function $\widehat{u}$. More explicitly, we show that the Bellman equation of the IG model with utility function u is identical to the Bellman equation of the exponential model with utility function $\widehat{u}$.

5.1. Assumptions.

We shall need the following assumptions:

A1 $u : (0, +\infty) \to \mathbb{R}$ is three times continuously differentiable;

A2 $u'(c) > 0$ for all $c \in (0, +\infty)$;

A3 there exist $0 < \underline{\rho} \leq \overline{\rho} < +\infty$ such that $\underline{\rho} \leq \frac{-c\,u''(c)}{u'(c)} \leq \overline{\rho}$ for all $c \in (0, +\infty)$;

A4 there exist $-\infty < \underline{\pi} \leq \overline{\pi} < +\infty$ such that $\underline{\pi} \leq \frac{-c\,u'''(c)}{u''(c)} \leq \overline{\pi}$ for all $c \in (0, +\infty)$;

A5 $\alpha + \underline{\rho} - 1 > 0$;

A6 $(2-\alpha)\,\underline{\rho} - (1-\alpha)\,\overline{\pi} > 0$;

A7 $\gamma > \max_{\rho \in [\underline{\rho}_{\widehat{u}}, \overline{\rho}_{\widehat{u}}]} (1-\rho)(\mu - \frac{1}{2}\,\rho\,\sigma^2)$, where

$$\underline{\rho}_{\widehat{u}} = \frac{(\alpha + \overline{\rho} - 1)\,\overline{\rho}}{(2-\alpha)\,\underline{\rho} - (1-\alpha)\,\overline{\pi}} \text{ and } \overline{\rho}_{\widehat{u}} = \frac{(\alpha + \underline{\rho} - 1)\,\underline{\rho}}{(2-\alpha)\,\overline{\rho} - (1-\alpha)\,\underline{\pi}}.$$

Assumption A1 is technical. Assumption A2 is self-explanatory. Assumption A3 means that the consumer has bounded relative risk aversion, or BRRA for short. Assumption A4 means that the consumer has bounded relative prudence. Assumptions A5 and A6 ensure that a utility function $\widehat{u}$ with the necessary properties exists. Assumption A7 ensures that, in the exponential model with utility function $\widehat{u}$, the discount rate γ exceeds the rate at growth of the utility of wealth when wealth grows at the risk-adjusted rate of return $\mu - \frac{1}{2}\,\rho\,\sigma^2$.

Assumptions A1-A7 can be dramatically simplified if the consumer has constant relative risk aversion ρ. In that case we have: $\underline{\rho} = \overline{\rho} = \rho$; $\underline{\pi} = \overline{\pi} = \rho + 1$; and $\underline{\rho}_{\widehat{u}} = \overline{\rho}_{\widehat{u}} = \rho$. Hence Assumptions A1-A7 reduce to:

B1 $\rho > 0$;

B2 $\alpha + \rho - 1 > 0$;

B3 $\gamma > (1-\rho)(\mu - \frac{1}{2}\,\rho\,\sigma^2)$.

Assumption B1 is self-explanatory. Assumption B2 will usually be satisfied in practice: empirical estimates of the coefficient of relative risk aversion ρ typically lie between 1 and 5; and the short-run discount factor α is typically thought to be at least 0.5.[6] However, for completeness, we discuss the case $\alpha + \rho - 1 < 0$ in Section 7. Assumption B3 simply means that the discount rate γ exceeds the rate at growth of the utility of wealth when wealth grows at the risk-adjusted rate of return $\mu - \frac{1}{2}\,\rho\,\sigma^2$.

[6] See Laibson et al (1998) and Ainslie (1992).

5.2. Reduced Form of the Bellman Equations. The first step in the proof of the value-function equivalence theorem is to eliminate consumption from the Bellman equations of the exponential model and the IG model. In this way we obtain a reduced form of the two equations that is ideally suited for our analysis.

Let the functions $f_+ : (0, +\infty) \to (0, +\infty)$ and $f_0 : (-\infty, +\infty) \to (0, y]$ be defined implicitly by the equations

$$\begin{aligned} u'(f_+(\phi)) &= \phi, \\ u'(f_0(\phi)) &= \max\{\phi, u'(y)\}. \end{aligned}$$

In other words, let them be the functions obtained by inverting the first-order conditions for $x > 0$ and $x = 0$ respectively. Also, let the functions $h_+ : (0, +\infty) \to (0, +\infty)$ and $h_0 : (-\infty, +\infty) \to (0, +\infty)$ be defined by the formulae

$$\begin{aligned} h_+(\phi) &= u(f_+(\phi)) - f_+(\phi)\,\phi, \\ h_0(\phi) &= u(f_0(\phi)) - f_0(\phi)\,\phi. \end{aligned}$$

Then the Bellman equation of the exponential model becomes

$$\begin{aligned} 0 &= \tfrac{1}{2}\sigma^2 x^2 v'' + (\mu x + y - c)\,v' - \gamma v + u(c) \\ &= \tfrac{1}{2}\sigma^2 x^2 v'' + (\mu x + y)\,v' - \gamma v + u(c) - c\,v' \\ &= \tfrac{1}{2}\sigma^2 x^2 v'' + (\mu x + y)\,v' - \gamma v + u(f_+(v')) - f_+(v')\,v' \\ &= \tfrac{1}{2}\sigma^2 x^2 v'' + (\mu x + y)\,v' - \gamma v + h_+(v') \qquad (16) \end{aligned}$$

for $x \in (0, +\infty)$, with boundary condition

$$\begin{aligned} 0 &= (y - c)\,v' - \gamma v + u(c) \\ &= y\,v' - \gamma v + u(c) - c\,v' \\ &= y\,v' - \gamma v + u(f_0(v')) - f_0(v')\,v' \\ &= y\,v' - \gamma v + h_0(v') \qquad (17) \end{aligned}$$

at $x = 0$.

Next, let the functions $\widehat{h}_+ : (0, +\infty) \to (0, +\infty)$ and $\widehat{h}_0 : (-\infty, +\infty) \to (0, +\infty)$ be defined by the formulae

$$\begin{aligned} \widehat{h}_+(\phi) &= u(f_+(\alpha\,\phi)) - f_+(\alpha\,\phi)\,\phi, \\ \widehat{h}_0(\phi) &= u(f_0(\alpha\,\phi)) - f_0(\alpha\,\phi)\,\phi. \end{aligned}$$

Then a similar derivation shows that the Bellman equation of the IG model becomes

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y)\,v' - \gamma\,v + \widehat{h}_+(v') \tag{18}$$

for $x \in (0, +\infty)$, with boundary condition

$$0 = y\,v' - \gamma\,v + \widehat{h}_0(v') \tag{19}$$

at $x = 0$.

Comparing equations (16-17) with equations (18-19), we see that the only difference between the Bellman equation of the exponential model and the Bellman equation of the IG model is that the former involves the functions h_+ and h_0, whereas the latter involves the functions $\widehat{h}_+$ and $\widehat{h}_0$.

5.3. Equivalence of the Two Equations in the Interior. The second step in the proof of the value-function equivalence theorem is to show that, for a suitable choice of utility function $\widehat{u}_+$, equation (18) with utility function u is identical to equation (16) with utility function $\widehat{u}_+$.

Note first that, by construction, $f_+(\phi)$ is the unique maximizer over $c \in (0, +\infty)$ of the expression $u(c) - \phi\,c$. Hence

$$h_+(\phi) = \max_{c \in (0,+\infty)} u(c) - \phi\,c.$$

In other words, h_+ is the dual – in the sense of convex analysis – of u. In order to establish that equation (18) with utility function u is identical to equation (16) with utility function $\widehat{u}_+$, it therefore suffices to choose $\widehat{u}_+$ in such a way that

$$\widehat{h}_+(\phi) = \max_{\widehat{c}\in(0,+\infty)} \widehat{u}_+(\widehat{c}) - \phi\,\widehat{c}.$$

In other words, it suffices to choose $\widehat{u}_+$ in such a way that $\widehat{h}_+$ is the dual of $\widehat{u}_+$.

At this point it is obvious what to do: we need to let $\widehat{u}_+$ be the dual of $\widehat{h}_+$. In other words, we need to define $\widehat{u}_+ : (0,+\infty) \to \mathbb{R}$ by the formula

$$\widehat{u}_+(\widehat{c}) = \min_{\phi\in(0,+\infty)} \widehat{h}_+(\phi) + \widehat{c}\,\phi.$$

Duality will then ensure that $\widehat{h}_+$ is also the dual of $\widehat{u}_+$.

In order to verify that this approach works, we need two lemmas.

Lemma 5. *We have:*

1. $\widehat{h}_+ : (0,+\infty) \to \mathbb{R}$ *is twice continuously differentiable;*
2. $\widehat{h}'_+(\phi) < 0$ *for all* $\phi \in (0,+\infty)$*;*
3. $\overline{\rho}_{\widehat{u}}^{-1} \le \frac{-\phi\,\widehat{h}''_+(\phi)}{\widehat{h}'_+(\phi)} \le \underline{\rho}_{\widehat{u}}^{-1}$ *for all* $\phi \in (0,+\infty)$*.*

In particular: $\widehat{h}'_+(0+) = -\infty$; $\widehat{h}'_+(+\infty) = 0$; and $\widehat{h}_+$ is BRRA.

Proof. See Appendix A.1. ■

Remark 6. *It is Assumption A5 which ensures that* $\widehat{h}'_+ < 0$*, and Assumption A6 which ensures that* $\widehat{h}''_+ > 0$*.*

Lemma 7. *We have:*

1. $\widehat{u}_+ : (0,+\infty) \to \mathbb{R}$ *is twice continuously differentiable;*
2. $\widehat{u}'_+(\widehat{c}) > 0$ *for all* $\widehat{c} \in (0,+\infty)$*;*
3. $\underline{\rho}_{\widehat{u}} \le \frac{-\widehat{c}\,\widehat{u}''_+(\widehat{c})}{\widehat{u}'_+(\widehat{c})} \le \overline{\rho}_{\widehat{u}}$ *for all* $\widehat{c} \in (0,+\infty)$*.*

In particular: $\widehat{u}'_+(0+) = +\infty$; $\widehat{u}'_+(+\infty) = 0$; and $\widehat{u}_+$ is BRRA.

Proof. See Appendix A.2. ■

We can now prove:

Theorem 8. *For all $\phi \in (0, +\infty)$, we have $\widehat{h}_+(\phi) = \max_{\widehat{c}\in(0,+\infty)} \widehat{u}_+(\widehat{c}) - \phi\,\widehat{c}$.*

Proof. Application of Fenchel's convex duality Theorem (cf Rockafellar 1970, section 31). ■

5.4. Equivalence of the Two Equations on the Boundary. The third step in the proof of the value-function equivalence theorem is to show that, for a suitable choice of utility function $\widehat{u}_0$, equation (19) with utility function u is identical to equation (17) with utility function $\widehat{u}_0$.

Proceeding as before, we note that

$$h_0(\phi) = \max_{c\in(0,y]} u(c) - \phi\, c.$$

We therefore need to choose $\widehat{u}_0$ in such a way that

$$\widehat{h}_0(\phi) = \max_{\widehat{c}\in(0,y]} \widehat{u}_0(\widehat{c}) - \phi\,\widehat{c}.$$

To achieve this, we define $\widehat{u}_0 : (0, y] \to \mathbb{R}$ by the formula

$$\widehat{u}_0(\widehat{c}) = \min_{\phi\in(0,+\infty)} \widehat{h}_0(\phi) + \widehat{c}\,\phi.$$

Duality will then ensure that $\widehat{h}_0$ is also the dual of $\widehat{u}_0$.

In order to verify that this approach works, we again need two lemmas.

Lemma 9. *We have:*

$$\widehat{h}_0(\phi) = \left\{ \begin{array}{ll} u(y) - \phi\, y & \text{if } \phi \in \left(0, \frac{1}{\alpha}\, u'(y)\right] \\ \widehat{h}_+(\phi) & \text{if } \phi \in \left[\frac{1}{\alpha}\, u'(y), +\infty\right) \end{array} \right\}.$$

Moreover $\widehat{h}'_0(\frac{1}{\alpha}\, u'(y)-) \le \widehat{h}'_0(\frac{1}{\alpha}\, u'(y)+)$.

In other words: $\widehat{h}_0$ is affine on $\left(0, \frac{1}{\alpha}\, u'(y)\right]$ with slope $-y$; $\widehat{h}_0$ coincides with $\widehat{h}_+$ on $\left[\frac{1}{\alpha}\, u'(y), +\infty\right)$; and the slope of $\widehat{h}_0$ either jumps up or remains constant at $\frac{1}{\alpha}\, u'(y)$. In particular, $\widehat{h}_0$ is strictly decreasing and convex.

Proof. See Appendix A.3. ■

Lemma 10. *We have*

$$\widehat{u}_0(\widehat{c}) = \left\{ \begin{array}{ll} \widehat{u}_+(\widehat{c}) & \text{if } \widehat{c} \in (0, \psi\, y] \\ \widehat{u}_+(\psi\, y) + (\widehat{c} - \psi\, y)\, \widehat{u}'_+(\psi\, y) & \text{if } \widehat{c} \in [\psi\, y, y] \end{array} \right\},$$

where

$$\psi = \frac{\alpha + \rho(y) - 1}{\rho(y)}.$$

Moreover $\widehat{u}_0(y) = u(y)$.

In other words: $\widehat{u}_0$ coincides with $\widehat{u}_+$ on $(0, \psi\, y]$; $\widehat{u}_0$ is affine on $[\psi\, y, y]$ with slope $\widehat{u}'_+(\psi\, y)$; and $\widehat{u}_0$ coincides with u at y. In particular, $\widehat{u}_0$ is strictly increasing and concave.

Proof. See Appendix A.4. ■

We can now prove:

Theorem 11. *For all* $\phi \in (0, +\infty)$, *we have* $\widehat{h}_0(\phi) = \max_{\widehat{c} \in (0,y]} \widehat{u}_0(\widehat{c}) - \phi\, \widehat{c}$.

Proof. Application of Fenchel's convex duality Theorem (cf Rockafellar 1970, section 31). ■

5.5. The Equivalent Exponential Problem. Combining Theorems 8 and 11, we see that the Bellman equation of the IG model with utility function u is identical to the Bellman equation of the exponential model with utility function $\widehat{u}$ given by the formula

$$\widehat{u}(\widehat{c}, \widehat{x}) = \left\{ \begin{array}{ll} \widehat{u}_+(\widehat{c}) & \text{if } x > 0 \text{ and } \widehat{c} \in (0, +\infty) \\ \widehat{u}_0(\widehat{c}) & \text{if } x = 0 \text{ and } \widehat{c} \in (0, y] \end{array} \right\}.$$

In particular, the value function v of the hyperbolic consumer of the IG model is also the value function of an equivalent exponential consumer whose wealth evolves according to the same dynamics as in the original problem, but whose preferences are given by

$$\mathrm{E}_t\left[\int_t^{+\infty} e^{-\gamma\,(s-t)}\,\widehat{u}(\widehat{c}(s),\widehat{x}(s))\,ds\right].$$

In other words, the equivalent exponential consumer uses a standard discount function that decays exponentially at rate γ, but uses a non-standard utility function $\widehat{u}$ that depends on her wealth.

Remark 12. *We denote consumption and wealth in the equivalent problem by $\widehat{c}$ and $\widehat{x}$ in order to emphasize the fact that the consumption choices of the equivalent exponential consumer are different from those of the original hyperbolic consumer. In other words, the equivalent exponential problem is not observationally equivalent to the original hyperbolic problem.*

Figure 4 shows $\widehat{u}$ and $\widehat{u}_0$ in the special case in which $u(c)$ takes the form $\frac{1}{1-\rho}\,c^{1-\rho}$ with $\rho \neq 1$ (i.e. u has constant relative risk aversion ρ). For this special case, we have the closed-form solutions

$$\widehat{u}_+(\widehat{c}) = \tfrac{\psi}{\alpha}\,u(\tfrac{1}{\psi}\,\widehat{c})$$

and

$$\widehat{u}_0(\widehat{c}) = \left\{\begin{array}{ll} \frac{\psi}{\alpha}\,u(\frac{1}{\psi}\,\widehat{c}) & \text{if } \widehat{c}\in(0,\psi\,y] \\ \frac{\psi}{\alpha}\,u(y) + \frac{1}{\alpha}\,(\widehat{c}-\psi\,y)\,u'(y) & \text{if } \widehat{c}\in[\psi\,y,y] \end{array}\right\},$$

where

$$\psi = \frac{\alpha+\rho-1}{\rho} \in (0,1).$$

We also have

$$\begin{aligned} h_+(\phi) &= -\tfrac{1}{1-\rho^{-1}}\,\phi^{1-\rho^{-1}}, \\ \widehat{h}_+(\phi) &= \tfrac{\psi}{\alpha}\,h_+(\alpha\,\phi). \end{aligned}$$

Hence

$$\widehat{f}_+(\phi) = -\widehat{h}'_+(\phi) = -\psi\,h'_+(\alpha\,\phi) = \psi\,f_+(\alpha\,\phi).$$

In particular, when $x > 0$,

$$c(x) = \tfrac{1}{\psi}\,\widehat{c}(x).$$

When $x = 0$, there are two cases. If $\alpha\, v'(0) \geq u'(y)$, then

$$\begin{aligned} \widehat{c}(0) &= \widehat{c}(0+), \\ c(0) &= c(0+) = \tfrac{1}{\psi}\,\widehat{c}(0+). \end{aligned}$$

If $\alpha\, v'(0) < u'(y)$, then

$$c(0) = \widehat{c}(0) = y.$$

Moreover it can be shown that

$$\begin{aligned} \widehat{c}(0+) &= \overline{\psi}\, y > y, \\ c(0+) &= \tfrac{1}{\psi}\,\widehat{c}(0+) = \tfrac{1}{\psi}\,\overline{\psi}\, y, \end{aligned}$$

where $\overline{\psi}$ is the larger of the two solutions of the equation $\widehat{u}_+(\widehat{c}) + (1-\widehat{c})\,\widehat{u}'_+(\widehat{c}) = u(1)$.

Remark 13. *This example can easily be extended to the cover the case* $\rho = 1$.

6. Some Features of the IG Model

In the present section, we exploit the value-function equivalence result of Section 5 to investigate the IG model in more detail. We establish the existence and uniqueness of equilibrium, the continuity of the consumption function in the interior of the wealth space, a sufficient condition for the monotonicity of the consumption function, and a generalized Euler equation governing the evolution of the marginal utility of consumption. Assumptions A1-A7 will be in force throughout the section.

6.1. Existence and Uniqueness of Equilibrium.

Theorem 14. *The Bellman equation of the IG model has a unique solution* (v, c).

Proof. The equivalence result of Section 5 shows that (v, c) solves the Bellman equation of the IG model iff: (i) v solves the Bellman equation of the equivalent exponential problem; (ii) $c = f_+(\alpha\, v')$ when $x > 0$; and (iii) $c = f_0(\alpha\, v')$ when $x = 0$. Moreover standard considerations show that the Bellman equation of the equivalent exponential problem possesses a unique solution. ■

6.2. Continuity of the Consumption Function.

Theorem 15. *We have:*

1. *c is continuous when $x > 0$;*
2. *there exists $\mu_{\text{crit}} \in (-\infty, +\infty)$ such that*
   (a) *if $\mu < \mu_{\text{crit}}$ then $c(0) < c(0+)$, and*
   (b) *if $\mu \geq \mu_{\text{crit}}$ then $c(0) = c(0+)$.*

Proof. Note first that c is continuous in the interior because $c = f_+(\alpha\, v')$ there. Secondly, $c(0) = f_0(\alpha\, v'(0)) = \min\{f_+(\alpha\, v'(0)), y\} \leq f_+(\alpha\, v'(0)) = c(0+)$. Hence $c(0) \leq c(0+)$, with equality iff $v'(0) \geq \frac{1}{\alpha}\, u'(y)$. Thirdly, let $\widetilde{v}$ be the value function of the restricted version of the equivalent consumption problem in which the consumer has utility function $\widehat{u}_+$ instead of $\widehat{u}_0$ when her wealth is 0. It can be shown that $v'(0) \geq \frac{1}{\alpha}\, u'(y)$ iff $\widetilde{v}(0) \geq \frac{1}{\gamma}\, u(y)$. Moreover: $\widetilde{v}(0)$ is strictly increasing in μ; $\widetilde{v}(0) = \frac{1}{\gamma}\, \widehat{u}(y) < \frac{1}{\gamma}\, u(y)$ for all μ sufficiently small; and $\widetilde{v}(0) \to +\infty$ as $\mu \to +\infty$. ■

Remark 16. *In the case $\mu < \mu_{\text{crit}}$, the consumer dissaves when her wealth is low, spends all her wealth in finite time, and experiences a discontinuous drop in consumption when her wealth runs out. In the case $\mu \geq \mu_{\text{crit}}$, asset returns are high enough to induce the consumer to save when her wealth is low.*

6.3. Monotonicity of the Consumption Function.

Theorem 17. *Suppose that $\gamma \geq \mu$. Then $c' > 0$ when $x > 0$.*

Proof. Note first that $c = f_+(\alpha\, v')$ in the interior. Hence c is continuously differentiable there, and $c' = f_+'(\alpha\, v')\, \alpha\, v''$. Hence $c' > 0$ iff $v'' < 0$. Secondly, differentiating equation (18) with respect to x, we obtain

$$\tfrac{1}{2}\, \sigma^2\, x^2\, v''' + (\mu\, x + y)\, v'' - \gamma\, v' + \sigma^2\, x\, v'' + \mu\, v' + \widehat{h}_+'(v')\, v'' = 0$$

or

$$v''' = \frac{2}{\sigma^2\, x^2}\, ((\gamma - \mu)\, v' - ((\mu + \sigma^2)\, x + y + \widehat{h}_+'(v'))\, v'').$$

In particular, if $v'' = 0$, then

$$v''' = \frac{2}{\sigma^2 x^2} (\gamma - \mu) v' \geq 0.$$

Hence, if there exists $x_1 \in (0, +\infty)$ such that $v''(x_1) \geq 0$, then $v'' \geq 0$ on $(x_1, +\infty)$. Thirdly, if there exists $x_1 \in (0, +\infty)$ such that $v''(x_1) \geq 0$ on $(x_1, +\infty)$, then v grows at least linearly; and this contradicts the fact that $\rho_{\widehat{u}} \geq \underline{\rho}_{\widehat{u}} > 0$. Overall, then, we must have $v'' < 0$ on $(0, +\infty)$. ■

Remark 18. *Theorem 23 below shows that, if $\gamma < \mu$, then it may happen that $c' < 0$ at some wealth levels.*

6.4. The Generalized Euler Equation. Since $u'(c)$ may have a discontinuity at 0, we cannot use Itô's Lemma to study its dynamics. We can, however, use Itô's Lemma to study the dynamics of $m = \alpha v'$. These dynamics are very closely related to those of $u'(c)$. Indeed, we have $u'(c) = m$ for $x > 0$. Moreover:

1. if $c(0+) = c(0)$, then the dynamics of m are identical to those of $u'(c)$;
2. if $c(0+) > c(0)$ and $x(0) \in (0, +\infty)$, and if T is the first time that x hits 0, then the dynamics of m are identical to those of $u'(c)$ on the interval $(0, T)$; and
3. if $c(0+) > c(0)$ and $x(0) = 0$, then the dynamics of m are identical to those of $u'(c)$.

The two dynamics only differ if $c(0+) > c(0)$ and $x(0) \in (0, +\infty)$, in which case $u'(c)$ jumps up at T (whereas m does not jump down).

Theorem 19. *We have:*

$$\frac{dm}{m} = \left(\gamma - \mu + \sigma^2 \rho(c) \frac{x\, c'}{c} + (1 - \alpha)\, c' \right) dt - \sigma\, \rho(c) \frac{x\, c'}{c}\, dz$$

if either $x > 0$ or $x = 0$ and $c(0+) = c(0)$; and

$$\frac{dm}{m} = 0$$

if $x = 0$ and $c(0+) > c(0)$.

This theorem gives an exact expression for the rate of growth of m. The equation includes deterministic terms (i.e. the terms which include dt) and a stochastic term (i.e. the final term, which includes dz). The stochastic term captures the negative effect that positive wealth shocks have on marginal utility.

The term $\gamma\, dt$ implies that marginal utility rises more quickly the higher the long-run discount rate γ. The term $-\mu\, dt$ implies that marginal utility rises more slowly the higher the rate of return μ. The term $\sigma^2\, \rho(c)\, \frac{x\, c'}{c}\, dt$ captures two separate effects. First, asset income uncertainty σ^2 affects the savings decision. Second, since marginal utility is non-linear in consumption, asset income uncertainty affects the average value of future marginal utility. The net impact of these two effects is always positive. The term $(1-\alpha)\, c'\, dt$ captures the effect of hyperbolic discounting. Naturally, when $\alpha = 1$, this effect vanishes and the model coincides with the standard exponential discounting case.

Proof. See Appendix A.5. ■

6.5. The Deterministic IG Model. Up to now we have assumed that the standard deviation of asset returns $\sigma > 0$. In other words, we have been focussing on the *stochastic* IG model. In the present section, we investigate the case $\sigma = 0$. In other words, we focus on the *deterministic* IG model. We show that, by viewing the deterministic IG model as a limiting case of the stochastic IG model, we are able to pinpoint a unique value function for the deterministic IG model. More precisely, we have the following theorem. The proof, which follows standard lines, is omitted.

Theorem 20. *Let v_σ be the value function of the stochastic IG model. Then:*

1. *there is a continuous function $v : [0, +\infty) \to \mathbb{R}$ such that $v_\sigma \to v$ uniformly on compact subsets of $[0, +\infty)$ as $\sigma \to 0+$;*

2. *v is the unique viscosity solution[7] of the Bellman equation*

$$0 = (\mu\, x + y)\, v' - \gamma\, v + \widehat{h}_+(v') \tag{20}$$

[7] See Crandall et al (1992) for a "user's guide" to viscosity solutions.

for $x \in (0, +\infty)$*, with boundary condition*

$$0 = y\, v' - \gamma\, v + \widehat{h}_0(v') \tag{21}$$

at $x = 0$. ■

We refer to equation (20) as the Bellman equation of the deterministic IG model, and to equation (21) as the boundary condition of the deterministic IG model.

The equilibrium consumption function c of the deterministic IG model can be determined from the value function v using the first-order condition. More precisely, we have

$$u'(c) = \left\{ \begin{array}{ll} \alpha\, v' & \text{if } x > 0 \\ \max\left\{\alpha\, v', u'(y)\right\} & \text{if } x = 0 \end{array} \right\}.$$

In other words, by letting $\sigma \to 0+$ in the stochastic IG model, we select a unique sensible equilibrium of the deterministic IG model.

Remark 21. *Krusell and Smith (2000) consider a deterministic discrete-time hyperbolic consumption model. They show that equilibrium is indeterminate in their model. Our results suggest that this indeterminacy could be resolved by a refinement analogous to the one that we have used here.*

Remark 22. *The Bellman equation of the deterministic IG model is simpler than than that of the stochastic IG model: it is a first-order ordinary differential equation, whereas the latter is a second-order ordinary differential equation.*

6.6. The Deterministic IG Model when u is CRRA. In the present section, we shall investigate the deterministic IG model in the case that u is CRRA. More precisely, we shall make the following assumptions:[8]

C1 $\sigma = 0$;

C2 $u(c)$ takes the form $\frac{1}{1-\rho}\, c^{1-\rho}$ with $\rho \neq 1$;

[8] Assumptions C1-C3 are made in addition to the standing assumptions A1-A7. The latter accordingly reduce to Assumptions B1-B3.

C3 $\mu > 0$.

Under these assumptions, the Bellman equation possesses a symmetry that allows us to transform it from a *non-autonomous* ordinary differential equation into an *autonomous* ordinary differential equation. We are therefore able to provide a complete analysis of equilibrium in this case.

In the deterministic IG model, total income is $\mu\, x + y$. If c is the equilibrium consumption function, we therefore put

$$APC(x) = \frac{c(x)}{\mu\, x + y}.$$

We then have:

Theorem 23. *Suppose that Assumptions C1-C3 hold. Then there exists* $\overline{\gamma} \in (\mu, +\infty)$ *such that:*

1. *If* $\gamma \in ((1-\rho)\,\mu, \alpha\,\mu)$*, then* APC *is constant and strictly less than* 1*. In particular,* c *is strictly increasing and affine.*

2. *If* $\gamma \in (\alpha\,\mu, \mu)$*, then there exists* $x_1 \in (0, +\infty)$ *such that* APC *jumps up at* 0*, is strictly decreasing and strictly greater than* 1 *on* $(0, x_1)$*, jumps down at* x_1*, and is constant and strictly greater than* 1 *on* $(x_1, +\infty)$*. Moreover* c *jumps up at* 0*, is strictly decreasing on* $(0, x_1)$*, jumps down at* x_1*, and is strictly increasing and affine on* $(x_1, +\infty)$*.*

3. *If* $\gamma \in (\mu, \overline{\gamma})$*, then* APC *jumps up at* 0*, and is strictly decreasing and strictly greater than* 1 *on* $(0, +\infty)$*. Moreover* c *jumps up at* 0*, and is strictly increasing on* $(0, +\infty)$*.*

4. *If* $\gamma \in (\overline{\gamma}, +\infty)$*, then* APC *jumps up at* 0*, and is strictly increasing and strictly greater than* 1 *on* $(0, +\infty)$*. Moreover* c *jumps up at* 0*, and is strictly increasing on* $(0, +\infty)$*.*

In particular, the condition $\gamma \geq \mu$ used in the proof of monotonicity of the consumption function is necessary.

Proof. See Appendix A.6 ■

Remark 24. *The analysis of this section can easily be extended to the cover the case* $\rho = 1$.

Remark 25. *There is a unique solution to the dynamics even when* $\gamma \in (\alpha\,\mu, \mu)$. *Indeed, in this case we have* $APC > 1$, *and* x *therefore decreases strictly with time.*

7. Derivation of the IG Model Revisited

7.1. A Limit Theorem. Our derivation of the IG model from the finite-λ model in Section 4 was deliberately heuristic. The relationship between the two models can, however, be made rigorous. The following theorem, which we state without proof, gives the flavor of the link between the two models.

Theorem 26. *Suppose that Assumptions A1-A7 hold. Then there exists* $\lambda_0 \in (0, +\infty)$ *such that, for all* $\lambda \in [\lambda_0, +\infty)$, *the finite-*$\lambda$ *model possesses a unique equilibrium* c_λ. *If* w_λ *is the current-value function associated with* c_λ, *then* w_λ *is continuous on* $[0, +\infty)$. *Moreover* $\frac{1}{\alpha}\,w_\lambda$ *converges uniformly on compact subsets of* $[0, +\infty)$ *as* $\lambda \to +\infty$ *to a limit function* v, *which is the unique viscosity solution*[9] *of the Bellman equation (16) for* $x \in (0, +\infty)$, *with boundary condition (17) at* $x = 0$. ■

7.2. A Complementary Theorem. Theorem 26 covers the case in which u has constant relative risk aversion $\rho > 1 - \alpha$. It is also possible to prove a limit theorem that covers the case in which u has constant relative risk aversion $\rho < 1 - \alpha$. In order to formulate such a theorem, we introduce the following assumptions, which complement Assumptions A5 and A6:

A5′ $\alpha + \overline{\rho} - 1 < 0$;

A6′ $(2 - \alpha)\overline{\rho} - (1 - \alpha)\underline{\pi} < 0$.

The theorem, which we state without proof, is then as follows.

Theorem 27. *Suppose that Assumptions A1-A4, A5′, A6′ and A7 hold. Then there exists* $\lambda_0 \in (0, +\infty)$ *such that, for all* $\lambda \in [\lambda_0, +\infty)$, *the finite-*$\lambda$ *model possesses a unique equilibrium* c_λ. *If* w_λ *is the current-value function associated with* c_λ, *then* w_λ *is continuous on* $[0, +\infty)$. *Moreover* $\frac{1}{\alpha}\,w_\lambda \to \frac{1}{\gamma}\,u(y)$ *uniformly on compact subsets of* $[0, +\infty)$ *as* $\lambda \to +\infty$. ■

[9] See Crandall et al (1992) for a “user’s guide” to viscosity solutions.

This theorem reflects the following behavior. For large λ, the consumer quickly consumes all her wealth. Thereafter, she consumes her labor income. Because her utility function has the property that $\frac{1}{c}\,u(c) \to 0$ as $c \to +\infty$, the initial consumption binge does not contribute much to the expected present discounted value of her consumption flow. It is the subsequent consumption of her labor income that matters. The expected present discounted value of this consumption is of course simply $\frac{1}{\gamma}\,u(y)$.

This behavior arises because, when $\rho < 1-\alpha$, the utility function is not sufficiently bowed to dampen the feedback effects that arise in hyperbolic models. Instead, these feedback effects drive consumption to infinity. In effect, the current self knows that subsequent selves are going to consume at a very high rate, and therefore chooses to consume at a very high rate herself in order to preempt the consumption by the later selves.

7.3. A Stronger Limit Theorem. Finally, note that Theorem 26 continues to hold when Assumptions A5 and A6 are replaced by the following, significantly weaker, assumptions:

A5″ $\alpha + \liminf_{c\to+\infty} \rho(c) - 1 > 0$;

A6″ $(2-\alpha)\liminf_{c\to+\infty} \rho(c) - (1-\alpha)\limsup_{c\to+\infty} \pi(c) > 0$.

These assumptions ensure that $\widehat{h}_+$ is decreasing and convex near 0. This is enough to ensure that consumption remains bounded as $\lambda \to +\infty$. These assumptions are, however, consistent with $\widehat{h}_+$ being increasing or concave away from 0. In other words, for some BRRA utility functions, the IG problem is not value-function equivalent to any exponential consumption problem.

8. Conclusions

We have described a continuous-time model of hyperbolic discounting. Our model allows for a general class of preferences, includes liquidity constraints, and places no restrictions on equilibrium policy functions. The model is also psychologically relevant. We take the phrase "instantaneous gratification" literally. We analyze a model in which individuals prefer gratification in the present instant discretely more than consumption in the momentarily delayed future. In this simple setting, equilibrium is unique and the consumption function is continuous. When the long-run discount rate weakly exceeds the interest rate, the consumption function is also monotonic. All of the pathologies that characterize discrete-time hyperbolic models vanish.

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A. Appendix

A.1. Proof of Lemma 5. Note first that

$$\begin{aligned}\widehat{h}_+(\phi) &= u(f_+(\alpha\,\phi)) - f_+(\alpha\,\phi)\,\phi \\ &= u(f_+(\alpha\,\phi)) - \alpha\,f_+(\alpha\,\phi)\,\phi - (1-\alpha)\,f_+(\alpha\,\phi)\,\phi \\ &= h(\alpha\,\phi) - (1-\alpha)\,f_+(\alpha\,\phi)\,\phi.\end{aligned}$$

Hence

$$\begin{aligned}\widehat{h}'_+(\phi) &= \alpha\,h' - (1-\alpha)\,f_+ - (1-\alpha)\,\alpha\,f'_+\,\phi \\ &= -\alpha\,f_+ - (1-\alpha)\,f_+ - (1-\alpha)\,\alpha\,f'_+\,\phi \\ &= -f_+ - (1-\alpha)\,f'_+\,\alpha\,\phi \\ &= -f_+\left(1 + (1-\alpha)\,\frac{f'_+\,\alpha\,\phi}{f_+}\right) \\ &= -f_+\left(1 + (1-\alpha)\,\frac{u'(f_+)}{f_+\,u''(f_+)}\right) \\ &= -f_+\left(1 - \frac{1-\alpha}{\rho(f_+)}\right) \\ &= \frac{-(\alpha + \rho(f_+) - 1)\,f_+}{\rho(f_+)},\end{aligned}$$

where we have suppressed the dependence of h and f_+ on $\alpha\,\phi$. Assumption A5 therefore implies that $\widehat{h}'_+ < 0$.

Second, as shown above, we have

$$\widehat{h}'_+(\phi) = -f_+ - (1-\alpha)\,f'_+\,\alpha\,\phi.$$

Hence

$$\begin{aligned}\widehat{h}''_+(\phi) &= -\alpha\, f'_+ - (1-\alpha)\, f'_+\, \alpha - (1-\alpha)\, \alpha\, f''_+\, \alpha\, \phi \\ &= -\alpha\, f'_+ \left(1 + (1-\alpha)\left(1 + \frac{f''_+\, \alpha\, \phi}{f'_+}\right)\right) \\ &= \frac{-\alpha}{u''(f_+)} \left(1 + (1-\alpha)\left(1 - \frac{u'''(f_+)\, u'(f_+)}{u''(f_+)^2}\right)\right) \\ &= \frac{-\alpha}{u''(f_+)} \left(1 + (1-\alpha)\left(1 - \frac{\pi(f_+)}{\rho(f_+)}\right)\right) \\ &= \frac{-\alpha}{u''(f_+)\, \rho(f_+)} \left((2-\alpha)\, \rho(f_+) - (1-\alpha)\, \pi(f_+)\right).\end{aligned}$$

Assumption A6 therefore implies that $\widehat{h}''_+(\phi) > 0$.

Third, using the final expressions obtained above for $\widehat{h}'_+(\phi)$ and $\widehat{h}''_+(\phi)$, we have

$$\frac{-\phi\, \widehat{h}''_+(\phi)}{\widehat{h}'_+(\phi)} = \frac{(2-\alpha)\, \rho(f_+) - (1-\alpha)\, \pi(f_+)}{(\alpha + \rho(f_+) - 1)\, \rho(f_+)}.$$

Hence

$$\overline{\rho}_{\widehat{u}}^{-1} \leq \frac{-\phi\, \widehat{h}''_+(\phi)}{\widehat{h}'_+(\phi)} \leq \underline{\rho}_{\widehat{u}}^{-1},$$

as required. ■

A.2. Proof of Lemma 7. Put

$$\widehat{g}_+(\widehat{c}) = \operatorname{argmin}_{\phi \in (0,+\infty)} \widehat{h}_+(\phi) + \widehat{c}\, \phi.$$

Then

$$\widehat{u}'_+(\widehat{c}) = \widehat{g}_+(\widehat{c}), \qquad \widehat{u}''_+(\widehat{c}) = \frac{-1}{\widehat{h}''_+(\widehat{g}_+(\widehat{c}))}$$

and

$$\frac{-\widehat{c}\, \widehat{u}''_+(\widehat{c})}{\widehat{u}'_+(\widehat{c})} = \frac{\widehat{h}'_+(\widehat{g}_+(\widehat{c}))}{-\widehat{g}_+(\widehat{c})\, \widehat{h}''_+(\widehat{g}_+(\widehat{c}))}.$$

We may therefore apply part 3 of Lemma 5 to conclude that

$$\underline{\rho}_{\widehat{u}} \leq \frac{-\widehat{c}\,\widehat{u}''_+(\widehat{c})}{\widehat{u}'_+(\widehat{c})} \leq \overline{\rho}_{\widehat{u}},$$

as required. ■

A.3. Proof of Lemma 9. The first statement is immediate from the definition of $\widehat{h}_0$. It implies that

$$\begin{aligned}\widehat{h}'_0(\tfrac{1}{\alpha}\,u'(y)+) = \widehat{h}'_+(\tfrac{1}{\alpha}\,u'(y)) &= \frac{-(\alpha+\rho(f_+(u'(y)))-1)\,f_+(u'(y))}{\rho(f_+(u'(y)))} \\ &= -\left(\frac{\alpha+\rho(y)-1}{\rho(y)}\right) y \geq -y = \widehat{h}'_0(\tfrac{1}{\alpha}\,u'(y)-).\end{aligned}$$

This completes the proof of the lemma. ■

A.4. Proof of Lemma 10. We have

$$\widehat{h}_0(\phi) = \left\{\begin{array}{ll} u(y)-\phi\,y & \text{if } \phi\in\left(0,\tfrac{1}{\alpha}\,u'(y)\right] \\ \widehat{h}_+(\phi) & \text{if } \phi\in\left[\tfrac{1}{\alpha}\,u'(y),+\infty\right) \end{array}\right\}$$

and

$$\widehat{h}'_0(\phi)\left\{\begin{array}{ll} =-y & \text{if } \phi\in\left(0,\tfrac{1}{\alpha}\,u'(y)\right) \\ \in[-y,-\psi\,y] & \text{if } \phi=\tfrac{1}{\alpha}\,u'(y) \\ =\widehat{h}'_+(\phi) & \text{if } \phi\in\left(\tfrac{1}{\alpha}\,u'(y),+\infty\right) \end{array}\right\}.$$

Hence

$$\min_{\phi\in(0,+\infty)} \widehat{h}_0(\phi)+\widehat{c}\,\phi = \left\{\begin{array}{ll} \min_{\phi\in(0,+\infty)}\widehat{h}_+(\phi)+\widehat{c}\,\phi & \text{if } \widehat{c}\in(0,\psi\,y) \\ \widehat{h}_+(\tfrac{1}{\alpha}\,u'(y))+\tfrac{1}{\alpha}\,\widehat{c}\,u'(y) & \text{if } \widehat{c}\in[\psi\,y,y] \\ -\infty & \text{if } \widehat{c}\in(y,+\infty) \end{array}\right\}.$$

Moreover

$$\min_{\phi\in(0,+\infty)} \widehat{h}_+(\phi)+\widehat{c}\,\phi = \widehat{u}(\widehat{c})$$

and

$$\begin{aligned}\widehat{h}_+(\tfrac{1}{\alpha}\,u'(y)) + \tfrac{1}{\alpha}\,\widehat{c}\,u'(y) &= \widehat{h}_+(\tfrac{1}{\alpha}\,u'(y)) + \tfrac{1}{\alpha}\,\psi\,y\,u'(y) + \tfrac{1}{\alpha}\,(\widehat{c} - \psi\,y)\,u'(y) \\ &= \widehat{u}_+(\psi\,y) + (\widehat{c} - \psi\,y)\,\widehat{u}'_+(\psi\,y).\end{aligned}$$

Finally,

$$\widehat{h}_+(\tfrac{1}{\alpha}\,u'(y)) + \tfrac{1}{\alpha}\,y\,u'(y) = u(y) - \tfrac{1}{\alpha}\,y\,u'(y) + \tfrac{1}{\alpha}\,y\,u'(y) = u(y).$$

This completes the proof of the lemma. ∎

A.5. Proof of Theorem 19. We begin by applying Itô's Lemma to m to obtain

$$dm = (\tfrac{1}{2}\,\sigma^2\,x^2\,m'' + (\mu\,x + y - c)\,m')dt + \sigma\,x\,m'\,dz.$$

Next, we put $\widetilde{c} = f(m)$. Then, differentiating equation (9) with respect to x, we have

$$\tfrac{1}{2}\,\sigma^2\,x^2\,m'' + (\mu\,x + y - \widetilde{c})\,m' - \gamma\,m + \sigma^2\,x\,m' + \mu\,m - \widetilde{c}'\,m + \alpha\,u'(\widetilde{c})\,\widetilde{c}' = 0$$

when $x > 0$. Moreover this equality extends by continuity to the case $x = 0$. Hence

$$\begin{aligned}\tfrac{1}{2}\,\sigma^2\,x^2\,m'' + (\mu\,x + y - c)\,m' &= \tfrac{1}{2}\,\sigma^2\,x^2\,m'' + (\mu\,x + y - \widetilde{c})\,m' + (\widetilde{c} - c)\,m' \\ &= \gamma\,m - \sigma^2\,x\,m' - \mu\,m + \widetilde{c}'\,m - \alpha\,u'(\widetilde{c})\,\widetilde{c}' + (\widetilde{c} - c)\,m' \\ &= \gamma\,m - \sigma^2\,x\,m' - \mu\,m + \widetilde{c}'\,m - \alpha\,m\,\widetilde{c}' + (\widetilde{c} - c)\,m' \\ &= (\gamma - \mu + (1-\alpha)\,\widetilde{c}')\,m - (\sigma^2\,x - (\widetilde{c} - c))\,m'\end{aligned}$$

and

$$\begin{aligned}\frac{dm}{m} &= \left(\gamma - \mu + (1-\alpha)\,\widetilde{c}' - \sigma^2\,x\,\frac{m'}{m} + (\widetilde{c} - c)\,\frac{m'}{m}\right) dt + \sigma\,x\,\frac{m'}{m}\,dz \\ &= \left(\gamma - \mu + (1-\alpha)\,\widetilde{c}' + \sigma^2\,\rho(\widetilde{c})\,\frac{x\,\widetilde{c}'}{\widetilde{c}} - (\widetilde{c} - c)\,\rho(\widetilde{c})\,\frac{\widetilde{c}'}{\widetilde{c}}\right) dt - \sigma\,\rho(\widetilde{c})\,\frac{x\,\widetilde{c}'}{\widetilde{c}}\,dz\end{aligned}$$

since

$$\frac{m'}{m} = \frac{u''(\widetilde{c})\,\widetilde{c}'}{u'(\widetilde{c})} = \frac{\widetilde{c}\,u''(\widetilde{c})}{u'(\widetilde{c})}\,\frac{\widetilde{c}'}{\widetilde{c}} = -\rho(\widetilde{c})\,\frac{\widetilde{c}'}{\widetilde{c}}.$$

In particular, we have the first statement of the Theorem.

As for the second statement, note that if $x = 0$ and $c(0+) > c(0)$ then $c(0) = y$. Wealth therefore remains constant at 0 forever, $m = u'(y)$ forever and $dm = 0$. ■

A.6. Proof of Theorem 23. We proceed in steps. First, put $X = \log(\mu\, x + y)$, $v(x) = (\mu\, x + y)^{1-\rho}\, V(X)$, $\ell = \log(y)$ and, in a slight abuse of notation,

$$\widehat{h}_0(\phi, y) = \left\{ \begin{array}{ll} u(y) - \phi\, y & \text{if } \phi \in \left(0, \frac{1}{\alpha}\, u'(y)\right] \\ \widehat{h}_+(\phi) & \text{if } \phi \in \left[\frac{1}{\alpha}\, u'(y), +\infty\right) \end{array} \right\}.$$

Then

$$0 = (\mu\, x + y)\, v' - \gamma\, v + \widehat{h}_+(v')$$

(using the Bellman equation of the deterministic IG model, namely (20))

$$= (\mu\, x + y)^{1-\rho}\, (\mu\, V' + (1-\rho)\, \mu\, V - \gamma\, V) + \widehat{h}_+((\mu\, x + y)^{-\rho}\, (\mu\, V' + (1-\rho)\, \mu\, V))$$

(since $v' = (\mu\, x + y)^{-\rho}\, (\mu\, V' + (1-\rho)\, \mu\, V)$)

$$= (\mu\, x + y)^{1-\rho} \left(\mu\, V' + (1-\rho)\, \mu\, V - \gamma\, V + \widehat{h}_+(\mu\, V' + (1-\rho)\, \mu\, V)\right)$$

(because $\widehat{h}_+$ is homogeneous of degree $1 - \frac{1}{\rho}$). Also

$$0 = y\, v'(0) - \gamma\, v(0) + \widehat{h}_0(v'(0))$$

(using the boundary condition of the deterministic IG model, namely (21))

$$= y^{1-\rho}\, (\mu\, V'(\ell) + (1-\rho)\, \mu\, V(\ell) - \gamma\, V(\ell)) + \widehat{h}_0(y^{-\rho}\, (\mu\, V'(\ell) + (1-\rho)\, \mu\, V(\ell)), y)$$

(since $v'(0) = y^{-\rho}\, (\mu\, V'(\ell) + (1-\rho)\, \mu\, V(\ell))$)

$$= y^{1-\rho} \left(\mu\, V'(\ell) + (1-\rho)\, \mu\, V(\ell) - \gamma\, V(\ell) + \widehat{h}_0(\mu\, V'(\ell) + (1-\rho)\, \mu\, V(\ell), 1)\right).$$

Hence v is the value function of the deterministic IG model iff V satisfies the Bellman equation

$$0 = \mu\, V' + (1-\rho)\, \mu\, V - \gamma\, V + \widehat{h}_+(\mu\, V' + (1-\rho)\, \mu\, V) \tag{22}$$

for $X \in (\ell, +\infty)$ with boundary condition

$$0 = \mu V' + (1-\rho)\,\mu V - \gamma V + \widehat{h}_0(\mu V' + (1-\rho)\,\mu V, 1) \tag{23}$$

at $X = \ell$.

Second, put $\zeta = \mu V' + (1-\rho)\,\mu V$. Then equation (22) determines a curve C_+ in (V', V)-space parametrized by $\zeta \in (0, +\infty)$. If $\rho < 1$, then: for $\zeta \in (0, \widehat{u}'_+(1))$, V' is increasing in ζ and V is decreasing in ζ; and, for $\zeta \in (\widehat{u}'_+(1), +\infty)$, both V' and V are increasing in ζ. If $\rho > 1$, and if we put

$$a = \frac{\gamma - (1-\rho)\,\mu}{(\rho - 1)\,\mu} > 1,$$

then: for $\zeta \in (0, \widehat{u}'_+(a))$, both V' and V are decreasing in ζ;[10] for $\zeta \in (\widehat{u}'_+(a), \widehat{u}'_+(1))$, V' is increasing in ζ and V is decreasing in ζ; and, for $\zeta \in (\widehat{u}'_+(1), +\infty)$, both V' and V are increasing in ζ.

Third, V is minimized when $\zeta = \widehat{u}'_+(1)$, at which point $V = \widehat{u}_+(1)$. Hence there are two points on C_+ at which $V = u(1) > \widehat{u}_+(1)$. We denote the corresponding values of ζ by $\widehat{u}'_+(\overline{\psi})$ and $\widehat{u}'_+(\psi)$.[11] It is easy to verify that

$$\psi = \frac{\alpha + \rho - 1}{\rho} < 1 < \overline{\psi},$$

but there is no closed-form expression for $\overline{\psi}$.

Fourth, equation (23) determines a curve C_0 in (V', V)-space parametrized by $\zeta \in (-\infty, +\infty)$. For $\zeta \in (-\infty, \widehat{u}'_+(\psi))$, V' is increasing in ζ and V is constant and equal to $u(1)$; and, for $\zeta \in (\widehat{u}'_+(\psi), +\infty)$, C_0 coincides with C_+ (in particular, both V' and V are increasing in ζ).

Fifth, there is a unique point on C_+ at which $V' = 0$. We denote the corresponding value of ζ by $\widehat{u}'_+(b)$. It is easy to verify that

$$b = \frac{\gamma - (1-\rho)\,\mu}{\rho\,\mu}.$$

[10] Since $V, V' \to 0-$ as $\zeta \to 0+$, we have $V' < 0$ for $\zeta \in (0, \widehat{u}'_+(A))$.

[11] The points ψ and $\overline{\psi}$ are the two solutions of the equation $\widehat{u}_+(\widehat{c}) + (1-\widehat{c})\,\widehat{u}'_+(\widehat{c}) = u(1)$.

It can also be shown that $V' = \frac{\gamma - \alpha\,\mu}{\gamma\,\mu}\,\widehat{u}'_+(\psi)$ when $\zeta = \widehat{u}'_+(\psi)$, $V' = \frac{\gamma-\mu}{\gamma\,\mu}\,\widehat{u}'_+(1)$ when $\zeta = \widehat{u}'_+(1)$, $V' = \frac{\gamma-(1-\rho)\,\mu-\rho\,\mu\,\overline{\psi}}{\gamma\,\mu}\,\widehat{u}'_+(\overline{\psi})$ when $\zeta = \widehat{u}'_+(\overline{\psi})$, and, if $\rho > 1$, then $V' = \frac{\gamma-(1-\rho)\,\mu}{\gamma\,(1-\rho)\,\mu}\,\widehat{u}'_+(a) < 0$ when $\zeta = \widehat{u}'_+(a)$.[12]

Sixth, it is easy to verify that b is increasing in γ. Moreover there exists $\overline{\gamma} \in (\mu, +\infty)$ such that: $b \in (0, \psi)$ iff $\gamma \in ((1-\rho)\,\mu, \alpha\,\mu)$; $b \in (\psi, 1)$ iff $\gamma \in (\alpha\,\mu, \mu)$; $b \in (1, \overline{\psi})$ iff $\gamma \in (\mu, \overline{\gamma})$; and $b \in (\overline{\psi}, +\infty)$ iff $\gamma \in (\overline{\gamma}, +\infty)$.[13]

Seventh, we complete the analysis of APC using a phase diagram. Equation (22) is a first-order autonomous ordinary differential equation. A one-dimensional phase diagram (in V-space) would therefore appear to be appropriate. However, there may be upward jumps in V'.[14] It is therefore preferable to work with a two-dimensional phase diagram (in (V', V)-space). The phase curve corresponding to the equilibrium starts on the curve C_0, is confined to the curve C_+, and converges to the steady state $V' = 0$ as $X \to +\infty$.

Eighth, we complete the analysis of c by noting that, when $x > 0$: the first-order condition of the equivalent exponential problem gives $\widehat{u}''_+(\widehat{c})\,\widehat{c}' = v''$; and the Bellman equation of the deterministic IG model gives $0 = (\mu\,x + y - \widehat{c})\,v'' - (\gamma - \mu)\,v'$. We therefore have

$$\widehat{c}' = \frac{(\gamma - \mu)\,v'}{(\mu\,x + y)\,(1 - \widehat{APC})\,\widehat{u}''_+(\widehat{c})},$$

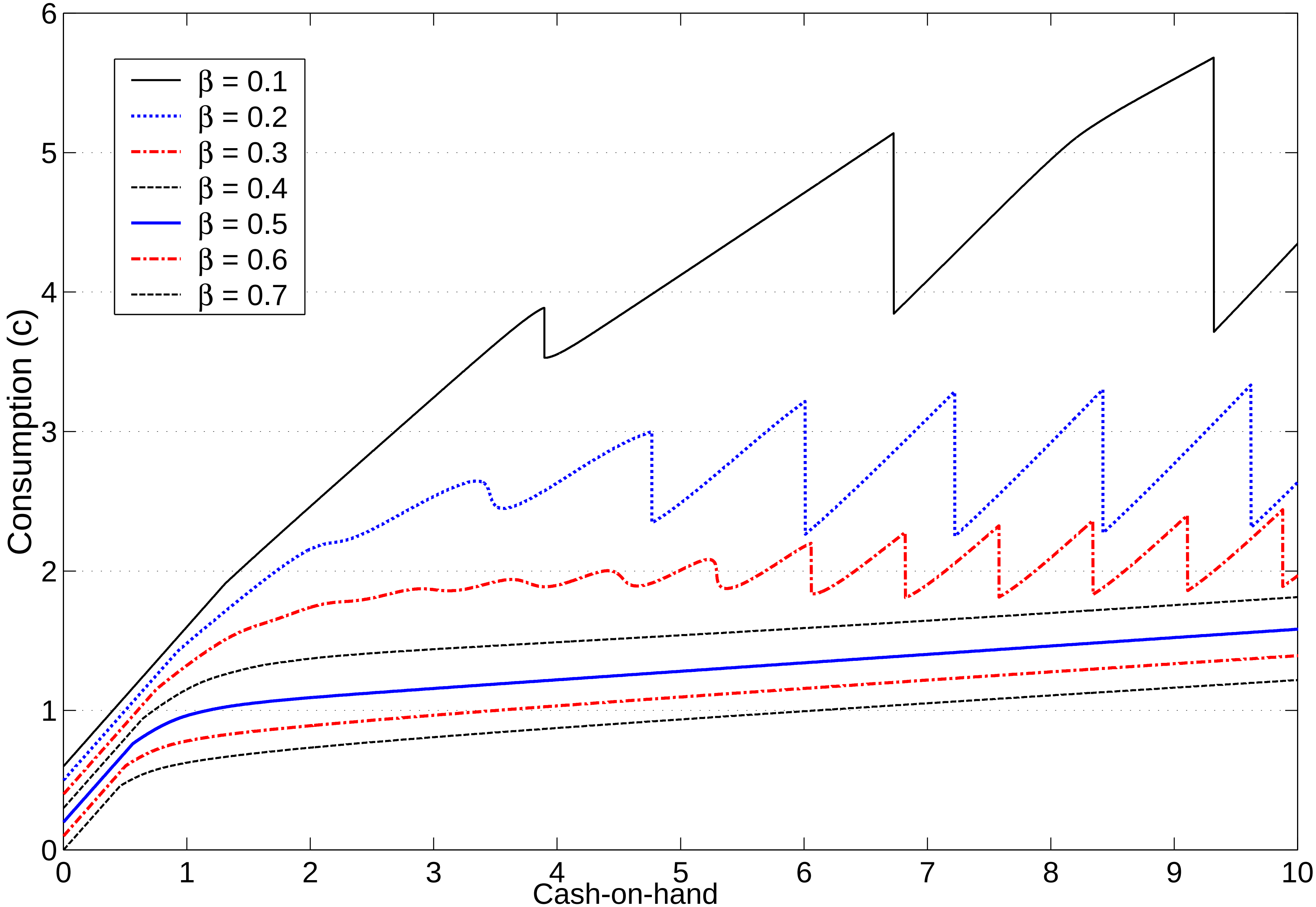
where $\widehat{APC} = \frac{\widehat{c}}{(\mu\,x+y)}$. ■

[12] These values of V' are found from the formula $V' = \frac{\gamma-(1-\rho)\,\mu-\rho\,\mu\,\widehat{c}}{\gamma\,\mu}\,\widehat{u}'_+(\widehat{c})$. The corresponding values of V can be found from the formula $V = \frac{1-\rho+\rho\,\widehat{c}}{\widehat{c}}\,\widehat{u}_+(\widehat{c})$.

[13] The critical value of γ can be found by solving the equation $\frac{\gamma}{\alpha\,\mu} = \left(\frac{\gamma-(1-\rho)\,\mu}{\psi\,\rho\,\mu}\right)^{\rho}$. This equation has two solutions, $\alpha\,\mu$ and $\overline{\gamma}$.

[14] There are no downward jumps. Intuitively speaking, this is because V is the upper envelope of smooth functions, and can therefore have convex kinks but not concave kinks.

Figure 1: Consumption functions for $\beta \in \{0.1,0.2,...,0.7\}$*



*These consumption functions are taken from discrete time simulations in Harris and Laibson (2001b). These simulations assume iid income, a risk-free asset, and CRRA. The short-run discount factor is $\beta\delta$. The long-run discount factor is δ=.95. The plotted consumption functions are shifted upward (in increments of .1) so they do not overlap.

Figure 2: Realization of discount function (α=0.7, γ=0.1)

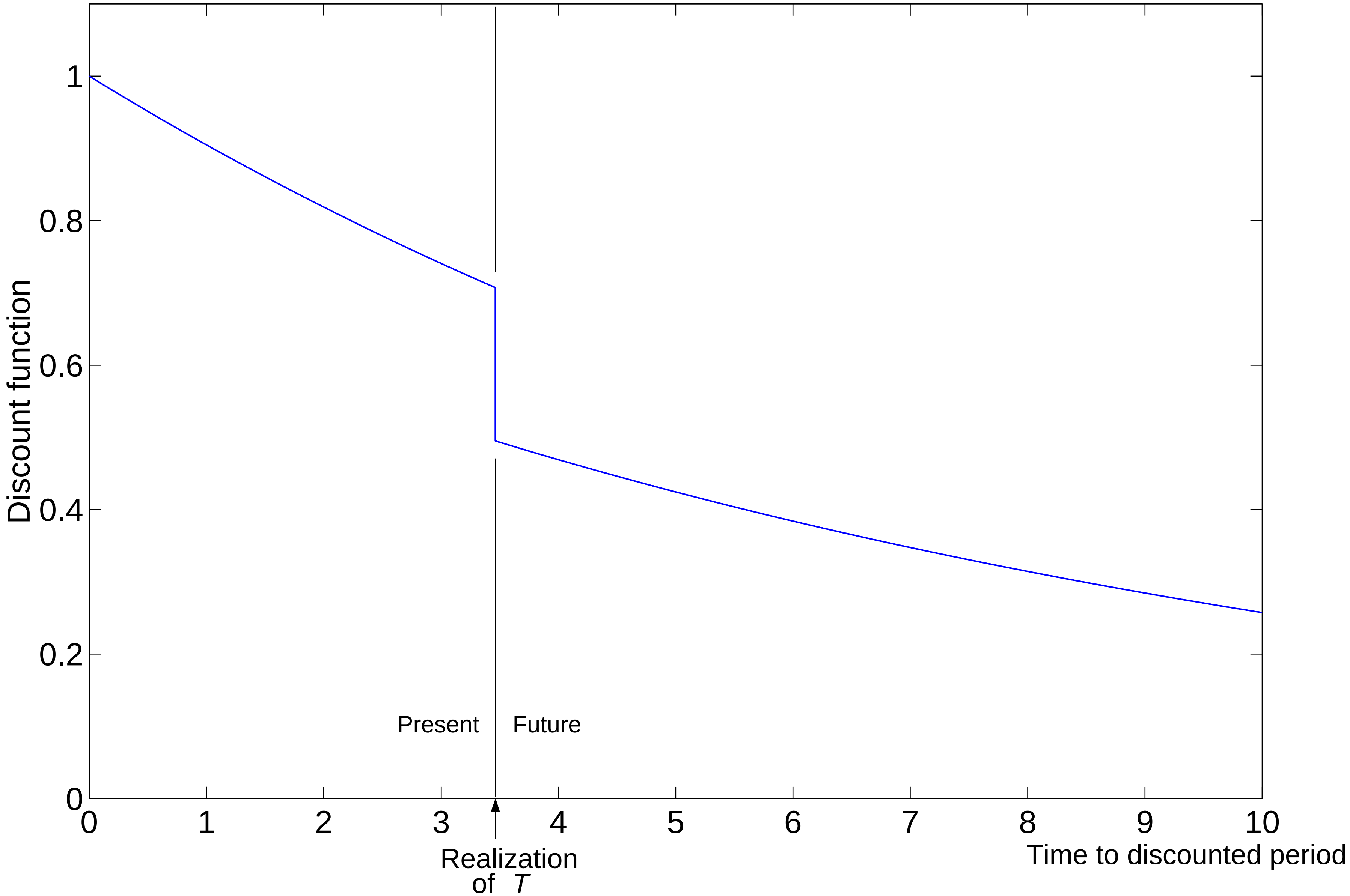
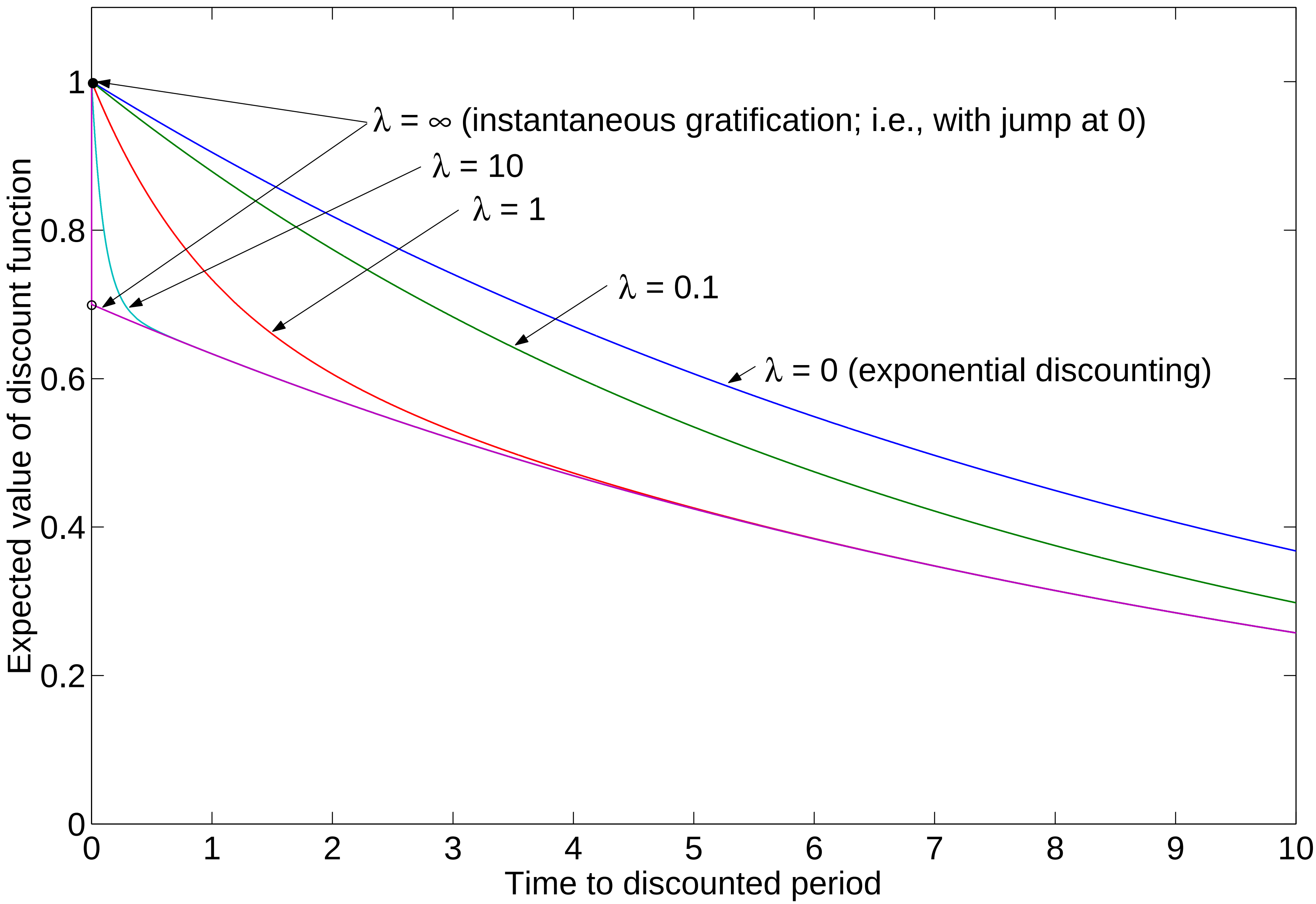


Figure 3: Expected value of discount function for $\lambda \in \{0, 0.1, 1, 10, \infty\}$



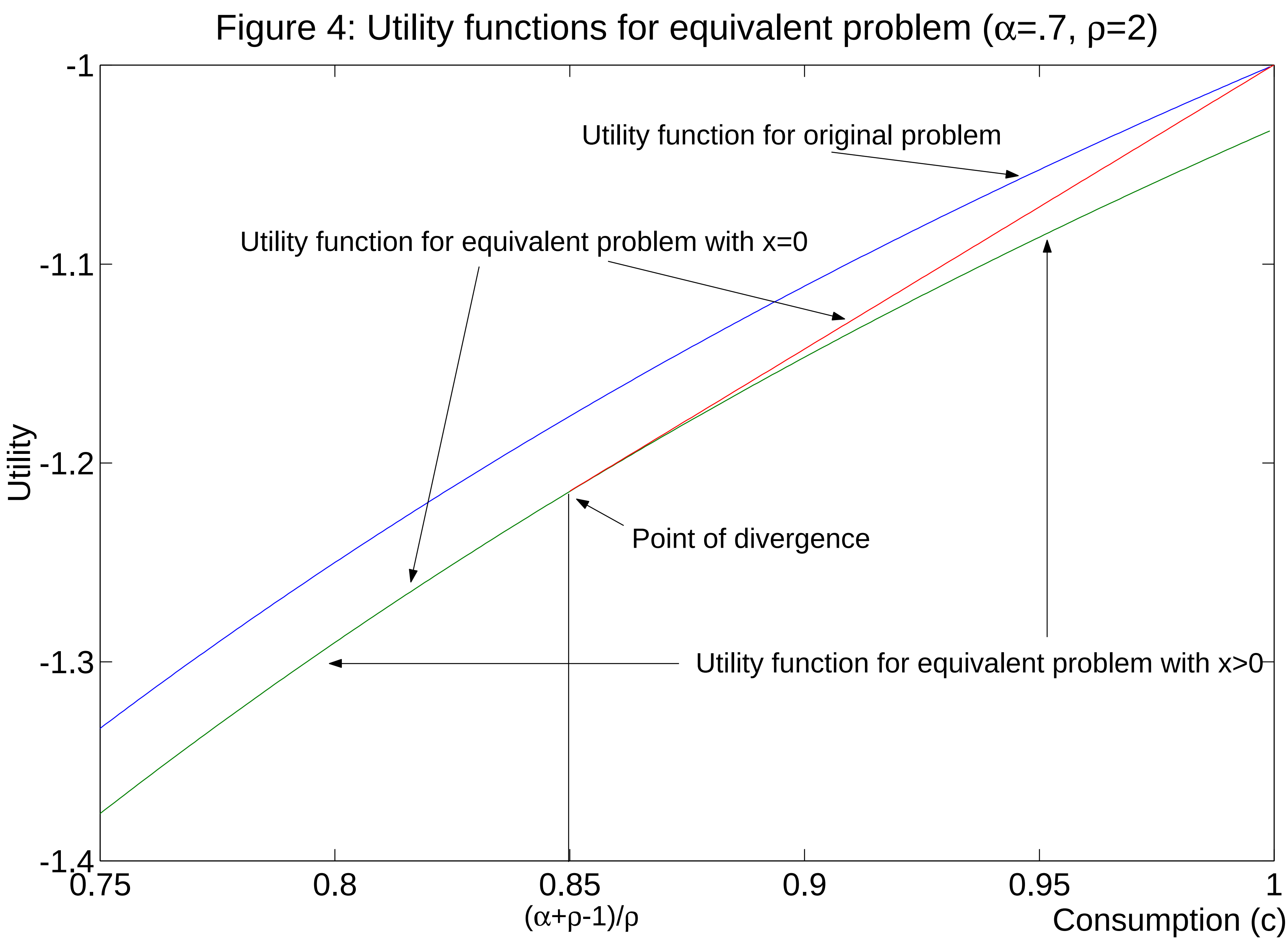


Figure 4: Utility functions for equivalent problem (α=.7, ρ=2)
Utility function for original problem
Utility function for equivalent problem with x=0
Point of divergence
Utility function for equivalent problem with x>0
Utility
-1
-1.1
-1.2
-1.3
-1.4
0.75
0.8
0.85
0.9
0.95
1
(α+ρ-1)/ρ
Consumption (c)