AN AXIOMATIC CHARACTERIZATION OF THE COLEMAN INDEX OF THE POWER OF A COLLECTIVITY TO ACT

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Abstract

An index of the power of a voting body to act determines the degree of ease with which the interests of the body members in a division can be transformed into actual decisions. Coleman(1971) suggested an index of 'the power of a collectivity to act', which can be regarded as measuring the extent of deference of the concerned voting body to the passage of a resolution. This paper develops an axiomatic characterization of the Coleman index and investigates its properties analytically. It is proved that the axioms used for the characterization exercise are independent.

Key words: voting game, power of a voting body, Coleman index, properties, characterization.

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1. Introduction

An indicator of the power of a voting body to act, under a given decision rule, is a quantification of the extent to which the body is able to control the outcome of a division of it. This index will measure the propensity of the voting body to a proposed resolution in an unambiguous way. Thus, it is a characteristic of the voting body itself, rather than of any particular member.

Several attempts have been made in the voting power literature to suggest indicators of a voter's ability to control a decision (See, for example, Shapley and Shubik, 1954; Banzhaf, 1965; Coleman, 1971; Deegan and Packel, 1978; Johnston, 1978; Burgin and Shapley 2001 and Barua, Chakravarty and Roy, 2002). Coleman (1971) also suggested an indicator of the 'power of a collectivity to act' for assessing the decision rule of a voting body. It equals the a-priori probability that a proposed resolution will be accepted by the decision-making committee under consideration. It has a decreasing monotonic association with the resistance coefficient introduced by Felsenthal and Machover (1998, 2001); for a given size of the voting body an increase in the resistance coefficient is equivalent to a reduction in the Coleman index. Thus, the resistance coefficient represents the voting body's willingness to retain the status quo position by blocking a proposed bill.

The objective of this paper is to characterize the Coleman index axiomatically and study its properties analytically. An axiomatic characterization will give a set of necessary and sufficient conditions for identifying the index uniquely. It provides a greater insight of the index in a more elaborate way through the axioms employed in the characterization exercise. The problem of independence of these axioms is also considered in the paper. Independence means that the given set of axioms is minimal in the sense that none of its proper subset can characterize the Coleman index.

The paper is organized as follows. The next section introduces the preliminaries, considers the Coleman index and discusses its properties. The characterization exercise is presented in section 3. This section also demonstrates independence of the axioms employed in the characterization theorem. Finally, section 4 concludes.

2. Notation, Definitions and Preliminaries

It is possible to model a voting situation as a coalitional form game, the hallmark of which is that any subgroup of players can make contractual agreements among its members independently of the remaining players. Let $N = \{A_1, A_2, ...A_n\}$ be a set of players. The power set of *N*, that is, the collection of all subsets of N is denoted by 2^N. Any member of 2^N is called a coalition. A coalitional form game with player set N is a pair $(N;V)$, where $V : 2^N \to R$ such that $V(\phi) = 0$, where R is the real line. For any coalition S, the real number $V(S)$ is the worth of the coalition, that is, this is the amount that S can guarantee to its members. For any set *S*, $|S|$ will denote the number of elements in *S*.

We frame a voting system as a coalitional form game by assigning the value 1 to any coalition which can pass a bill and 0 to any coalition which cannot. In this context, a player is a voter and the set $N = \{A_1, A_2, ...A_n\}$ is called the set of voters. Throughout the paper we assume that voters are not allowed to abstain from voting. A coalition S will be called winning or losing according as it can or cannot pass a resolution.

Definition1: Given a set of voters N, a voting game associated with N is a pair $(N;V)$, where $V : 2^N \to \{0,1\}$ satisfies the following conditions:

(i) $V(\phi) = 0$.

(ii) $V(N) = 1$.

(iii) If $S \subseteq T$, $S,T \in 2^N$, then $V(S) \le V(T)$.

The above definition formalizes the idea of a decision-making committee in which decisions are made by vote. It follows that the empty coalition ϕ is losing (condition (i)) and the grand coalition N is winning (condition (ii)). All other coalitions are either winning or losing. Condition (iii) ensures that if a coalition S can pass a bill, then any superset T of S can pass it as well. A voting game $(N;V)$ is called proper if for $S,T \in 2^N$, $V(S) = V(T) = 1$ implies that $S \cap T \ne \phi$. According to this condition two winning coalitions cannot be disjoint. The collection of all voting games is denoted by **F**.

For any $G = (N;V)$, we write $\mathbf{W}_G(\mathbf{L}_G)$ for the set of all winning (losing) coalitions associated with *G*. Thus, for any $S \subseteq N$, $V(S) = 1(0)$ is equivalent to the condition that $S \in \mathbf{W}_G(\mathbf{L}_G)$.

Definition 2: The unanimity game $(N;U_N)$ associated with a given set of voters *N* is the game whose only winning coalition is the grand coalition *N*.

Definition 3: Given a set of voters N, let $(N;V)$ be a voting game.

(i) For any coalition $S \in 2^N$, we say that $i \in N$ is swing in S if $V(S) = 1$ but $V(S - \{i\}) = 0$.

(ii) For any coalition $S \in 2^N, i \in N$ is said to be swing outside S if $V(S) = 0$ but $V(S \cup \{i\}) = 1$.

(iii) A coalition $S \in 2^N$ is said to be minimal winning if $V(S) = 1$ but there does not exist $T \subset S$ such that $V(T) = 1$.

Thus, voter i is swing, also called pivotal or key, in the winning coalition S if his deletion from S makes the resulting coalition $S - \{i\}$ losing. Similarly, voter i is swing outside the losing coalition S if his addition to S makes the resulting coalition $S \cup \{i\}$ winning. For any voter i, the number of winning coalitions in which he is swing is same as the number of losing coalitions outside which he is swing (Burgin and Shapley, 2001, Corollary 4.1).

Definition 4: For a set of voters N, let $(N;V)$ be a voting game. A voter $i \in N$ is called a dummy in $(N;V)$ if he is never swing in the game. A voter $i \in N$ is called a nondummy in $(N;V)$ if he is not dummy in $(N;V)$.

Following Felsenthal and Machover (1998) we have

Definition 5: For a voting game $(N;V)$ with the set of voters N, a voter $i \in N$ is called a dictator if $\{i\}$ is the sole minimal winning coalition in the game.

A dictator in a game is unique. If a game has a dictator, then he is the only swing voter in the game.

A very important voting game is a weighted majority game.

Definition 6: For a set of voters $N = \{A_1, A_2, ... A_n\}$, a weighted majority game is a quadruplet $G = (N; V; \mathbf{w}; q)$, where $\mathbf{w} = (w_1, w_2, ... w_n)$ is the vector of nonnegative weights of the $|N|$ voters in *N*, *q* is a positive real number quota such that $q \leq \sum_{i=1}^{|N|} w_i$ and for any $S \in 2^N$,

$$V(S) = 1 \quad \text{if} \sum_{i \in S} w_i \geq q$$

$$= 0 \quad \text{otherwise.}$$

That is, the i^{th} voter casts w_i votes and *q*, satisfying the condition $q \leq \sum_{i=1}^{|N|} w_i$, is the quota of votes needed to pass a bill. A weighted majority game will be proper if $\sum_{i=1}^{|N|} \frac{w_i}{2} < q$. Note that a weighted majority game satisfies conditions (i)-(iii) of definition 1. (See Felsenthal and Machover, 1998, for further discussions on definitions 1 and 6.)

An index of the power of a voting body to act (**PTA**) is non-negative real valued function P defined on **F**, that is, $P : \mathbf{F} \to R_+$, the non-negative part of the real line. For any $G \in \mathbf{F}$, $P(G)$ is a summary statistic of the positive inclination of the underlying voting body towards the passage of a proposed act.

The Coleman index of **PTA** is given by

$$C(G) = \frac{|\mathbf{W}_G|}{2^{|N|}} = \frac{\sum_{S \subseteq N} V(S)}{2^{|N|}}, \tag{1}$$

where $G = (N; V) \in \mathbf{F}$ is arbitrary. Since $|\mathbf{W}_G|$ is the total number of winning coalitions and $2^{|N|}$ is the total number of coalitions (including the empty one) in the game *G*, $C(G)$ is the prior probability of a positive outcome, that is, the probability that a resolution will be adopted by the voting body. *C* is closely related to the Felsenthal-Machover resistance coefficient *M* defined by

$$M(G) = \frac{2^{|N|-1} - |\mathbf{W}_G|}{2^{|N|-1} - 1}, \tag{2}$$

where $G=(N;V)\in\mathbf{F}$ is arbitrary. We can rewrite *M* as

$$M(G)=\frac{2^{|N|-1}-2^{|N|}C(G)}{2^{|N|-1}-1} \quad . \qquad (3)$$

Thus, with a given set of voters, an increase in *M* implies and is implied by a reduction in *C*. Clearly, while *C* represents an extent of complaisance of the concerned decision-making body with respect to a resolution, *M* gives a quantification of the opposite of complaisance.

We will now state some properties for a general **PTA** index *P* and examine the Coleman index in the light of these postulates. By assumption the grand coalition is always winning. Suppose that other than this, the voting game does not show any obligingness to the passage of a resolution. That is, the total number of winning (losing) coalitions in the game is one (maximum). Clearly, in such a case *P* should take on the minimum value. Conversely, we can argue that if the value of P is minimum, then the total number of winning (losing) coalitions in the game should be one (maximum), which means that the game under consideration is an unanimity game.

Thus, we have

Minimality(MIN): For all $G=(N;V)\in\mathbf{F}$, $P(G)$ attains its minimum value if and only if one of the following equivalent conditions holds:

(a) $|\mathbf{W}_G|=1$.

(b) $|\mathbf{L}_G|=2^{|N|}$-1.

(c) *G* is the unanimity game.

If a voting body has complete consent to the acceptability of a bill, that is, the number of winning (losing) coalitions in the game is maximum(minimum), then the **PTA** index should achieve its maximum value.

We therefore have

Maximality(MAX): For all $G=(N;V)\in\mathbf{F}$, $P(G)$ should attain its maximum value if all nonempty coalitions in the game are winning, that is, whenever one of the following equivalent conditions holds:

(a) $|\mathbf{W}_G|=2^{|N|}-1$.

(b) $|\mathbf{L}_G| = 1$.

However, if the voting game G is proper, then $P(G)$ is maximized if $|\mathbf{W}_G| = |\mathbf{L}_G| = 2^{|N|-1}$, because for a proper voting game, $2^{|N|-1}$ is an attainable upper bound on the number of winning coalitions (see proposition 1 below).

Proposition 1: Let $G = (N;V)$ be a proper voting game. Then $|\mathbf{W}_G| \leq 2^{|N|-1}$.

Proof: The mapping $S \to N - S$ is a one-one mapping of $\mathbf{W}_G$ into $\mathbf{L}_G$. Therefore, $|\mathbf{W}_G| \leq |\mathbf{L}_G|$, which gives $2|\mathbf{W}_G| \leq |\mathbf{L}_G| + |\mathbf{W}_G| = 2^{|N|}$.

The result now follows from the latter inequality. □

The third property is anonymity.

Anonymity (ANY): Let $G = (N;V)$ and $\overline{G} = (\overline{N};\overline{V}) \in \mathbf{F}$ be two isomorphic games. That is, there exists a bijection *f* of *N* onto $\overline{N}$ such that for all $S \subseteq N$, $V(S) = 1$ if and only if $\overline{V}(f(S)) = 1$, where $f(S) = \{f(x) : x \in S\}$. Then $P(G) = P(\overline{G})$.

Anonymity means that *P* remains invariant under any reordering of the voters. Thus, all irrelevant characteristics, e.g., the names of the voters, should not be taken into account while measuring the power of a voting body to act.

The next property is concerning a change in the number of winning coalitions. Of two voting games with the same set of voters, assume that one has a lower number of winning coalitions than the other. Then it is reasonable to expect that the latter demonstrates a higher extent of **PTA** than the former. For instance, suppose that the weighted majority game $\hat{G}_0 = (N;V;1,2,4;6)$ is obtained from the game $G_0 = (N;V;1,2,4;5)$ by increasing the quota from 5 to 6, where $N = \{A_1, A_2, A_3\}$. The number of winning coalitions in $\hat{G}_0(G_0)$ is 2(3). We then appeal that $P(G_0) > P(\hat{G}_0)$. This is formally stated as

Monotonicity (MON): Let $G, G' \in \mathbf{F}$ be two voting games with the same voter set *N* and $|\mathbf{W}_G| > |\mathbf{W}_{G'}|$. Then $P(G) > P(G')$.

Since a dummy can never affect the outcome of voting, it is natural to expect that the collective power of voters remains unaltered if a dummy is excluded from the game. In view of this we can state the following:

Dummy Exclusion Principle (DEP): For any $G = (N;V) \in \mathbf{F}$ and for any dummy $d \in N$, $P(G) = P(G_{-d})$, where G_{-d} is the game obtained from G by excluding the dummy $d \in N$.

Likewise we can have a **Dummy Inclusion Principle (DIP)**, which requires that if *d* is not a voter in the game $G \in \mathbf{F}$, then $P(G) = P(G_{+d})$, where G_{+d} is the expanded game obtained from *G* by including *d* as a dummy.

The following theorem summarizes the behaviour of the Coleman index *C* with respect to the above properties.

<u>Theorem1:</u> The Coleman PTA index C given by (1) **satisfies MIN, MAX, ANY, MON, DEP and DIP.**

<u>Proof:</u> C obviously satisfies **MIN**, **MAX** and **MON**. Since a reordering of the voters does not change the number of winning coalitions of the game, C fulfils **ANY** also.

To check satisfaction of **DEP** by *C*, let $d \in N$ be a dummy voter in $(N;V) \in \mathbf{F}$ and let $G_{-d} = (N - \{d\}; V')$, where $V'(S) = V(S), S \subseteq N - \{d\}$. Note that since *d* is a dummy, $V(S) = V(S \cup \{d\})$.

We can write $\mathbf{W}_G$ as $\mathbf{W}_G = \mathbf{W}_G^1 \cup \mathbf{W}_G^2$, where $\mathbf{W}_G^1 = \{S \in \mathbf{W}_G : d \in S\}$ and $\mathbf{W}_G^2 = \{S \in \mathbf{W}_G : d \notin S\}$. Clearly, $\mathbf{W}_G^2$ coincides with $\mathbf{W}_{G-d}$, the collection of all winning coalitions corresponding to the game G_{-d}.

Since $S \subseteq N$ is winning in *G* if and only if $S - \{d\}$ is winning, it follows that the mapping $S \to S - \{d\}$ is a bijection of $\mathbf{W}_G^1$ onto $\mathbf{W}_G^2$.

Hence,

$$|\mathbf{W}_G| = |\mathbf{W}_G^1| + |\mathbf{W}_G^2| = 2|\mathbf{W}_G^2| = 2|\mathbf{W}_{G-d}|.$$

Thus,

$C(G)=\frac{|\mathbf{W}_G|}{2^{|N|}}=\frac{2|\mathbf{W}_{G-d}|}{2^{|N|}}=\frac{|\mathbf{W}_{G-d}|}{2^{|N|-1}}=C(G_{-d})$, which shows that C meets **DEP**. An analogous argument will establish that C verifies **DIP** also. □

Examples of **PTA** indices other than the Coleman index C that fulfil the above properties are

$$P_1(G)=(C(G))^r,\ r>0, r\neq 1, \qquad (4)$$

$$P_2(G)=e^{C(G)}, \qquad (5)$$

$$P_3(G)=\frac{C(G)}{1+C(G)}\ . \qquad (6)$$

However, because of its expositional ease and probabilistic interpretation, the Coleman index appears to be more attractive than such indices. We therefore characterize this index axiomatically in the next section.

3. The Characterization Theorem

In order to present the axioms that characterize the Coleman index, we need the following definitions.

Definition 7: Given $G_1=(N_1;V_1), G_2=(N_2;V_2)\in\mathbf{F}$, we define $G_1\vee G_2$ as the game with the set of voters $N_1\cup N_2$, where a coalition $S\subseteq N_1\cup N_2$ is winning if and only if $V_1(S\cap N_1)=1$ or $V_2(S\cap N_2)=1$.

Definition 8: Given $G_1=(N_1;V_1), G_2=(N_2;V_2)\in\mathbf{F}$, we define $G_1\wedge G_2$ as the game with the set of voters $N_1\cup N_2$, where a coalition $S\subseteq N_1\cup N_2$ is winning if and only if $V_1(S\cap N_1)=1$ and $V_2(S\cap N_2)=1$.

Thus, in order to win in $G_1\vee G_2$ a coalition must win in either G_1 or G_2, whereas to win in $G_1\wedge G_2$ it has to win in both G_1 and G_2. Clearly, given $G_1, G_2\in\mathbf{F}$; $G_1\vee G_2, G_1\wedge G_2\in\mathbf{F}$.

Finally, we have

Definition 9: Given $(N;V) \in \mathbf{F}$, suppose that the voters $i, j \in N$ are amalgamated into one voter ij. Then the post-merger voting game is the pair $(N';V') \in \mathbf{F}$, where

$N' = N - \{i, j\} \cup \{ij\}$ and

$$V'(S) = V(S) \quad \text{if } S \subseteq N' - \{ij\},$$
$$= V((S - \{ij\}) \cup \{i, j\}) \quad \text{if } ij \in S.$$

We are now in a position to present three axioms on a **PTA** index P that will uniquely isolate the Coleman index. The first axiom is taken from Dubey (1975) (see also Dubey and Shapley, 1979). It shows how the sum of the **PTA**s in the games $G_1 \vee G_2$ and $G_1 \wedge G_2$ are related to the **PTA**s in G_1 and G_2.

Axiom A1 (Sum Criterion): For any $G_1, G_2 \in \mathbf{F}$,

$$P(G_1 \vee G_2) + P(G_1 \wedge G_2) = P(G_1) + P(G_2).$$

The next axiom is concerning the change of **PTA** in a unanimity game under merger of two voters. In an unanimity game there is only one winning coalition. But the number of voters in different unanimity games may be different. Consider two such games with number of voters being 1 and k (>>1) respectively. Thus, while in the former, one individual enjoys the capability of making the coalition winning, in the latter it is shared by many individuals. It is therefore reasonable to expect that the former demonstrates a higher extent of power to act than the latter. In view of this we can argue that **PTA** increases under a merger of two voters in an unanimity game and the following axiom gives a formulation along this line.

Axiom A2 (Merger Criterion): Let $G' \in \mathbf{F}$ be the game obtained from the unanimity game $G = (N;U_N) \in \mathbf{F}$ by merging any two voters $i, j \in N$. Then

$$P(G') = 2P(G). \qquad (7)$$

The third axiom, which specifies the value of P in a game with a dictator, is a normalization condition.

Axiom A3 (Normalization): If $G \in \mathbf{F}$ has a dictator, then $P(G)$ takes on the value $1/2$.

Note that if $G = (N;V) \in \mathbf{F}$ is proper, then by proposition 1, maximum number of winning coalitions in G is $2^{|N|-1}$. Moreover, if G has a dictator, G is proper and

$\left|\mathbf{W}_G\right| = 2^{|N|-1}$ (see also Burgin and Shapley, 2001). Hence, $\frac{1}{2}$ is the maximum value of $P(G)$ if G has a dictator.

We then have

<u>Theorem 2:</u> A PTA index P satisfies axioms A1-A3 if and only if P is the Coleman index C given by (1).

<u>Proof:</u> We first shows that C satisfies axioms **A1-A3**.

Let $G_1 = (N_1;V_1), G_2 = (N_2;V_2) \in \mathbf{F}$. By definition $S \subseteq N_1 \cup N_2$ is winning in $G_1 \vee G_2$ if and only if $S \cap N_1 \in \mathbf{W}_{G_1}$ or $S \cap N_2 \in \mathbf{W}_{G_2}$, where $\mathbf{W}_{G_i}$ is the set of all winning coalitions in G_i, $i = 1,2$. Hence we can write $\mathbf{W}_{G_1 \vee G_2}$, the family of all winning coalitions in $G_1 \vee G_2$, as

$$\mathbf{W}_{G_1 \vee G_2} = \mathbf{W}_1 \cup \mathbf{W}_2, \qquad (8)$$

where, $\mathbf{W}_1 = \{S_1 \cup S_2 : S_1 \subseteq N_1, S_2 \subseteq N_2 - N_1 \text{ and } S_1 \in \mathbf{W}_{G_1}\}$,

$\mathbf{W}_2 = \{S_1 \cup S_2 : S_1 \subseteq N_1 - N_2, S_2 \subseteq N_2 \text{ and } S_2 \in \mathbf{W}_{G_2}\}$.

Clearly, $\mathbf{W}_{G_1 \wedge G_2} = \mathbf{W}_1 \cap \mathbf{W}_2$. $\qquad$ (9)

Hence, by Inclusion-Exclusion Principle,

$$\begin{aligned}\left|\mathbf{W}_{G_1 \vee G_2}\right| &= \left|\mathbf{W}_1\right| + \left|\mathbf{W}_2\right| - \left|\mathbf{W}_1 \cap \mathbf{W}_2\right| \\ &= \left|\mathbf{W}_{G_1}\right| 2^{|N_2 - N_1|} + \left|\mathbf{W}_{G_2}\right| 2^{|N_1 - N_2|} - \left|\mathbf{W}_{G_1 \wedge G_2}\right|. \qquad (10)\end{aligned}$$

Therefore,

$$\frac{\left|\mathbf{W}_{G_1 \vee G_2}\right|}{2^{|N_1 \cup N_2|}} = \frac{\left|\mathbf{W}_{G_1}\right|}{2^{|N_1|}} + \frac{\left|\mathbf{W}_{G_2}\right|}{2^{|N_2|}} - \frac{\left|\mathbf{W}_{G_1 \wedge G_2}\right|}{2^{|N_1 \cup N_2|}},$$

or, $C(G_1 \vee G_2) = C(G_1) + C(G_2) - C(G_1 \wedge G_2)$,

which on rearrangement gives

$$C(G_1 \vee G_2) + C(G_1 \wedge G_2) = C(G_1) + C(G_2). \qquad (11)$$

Thus, C verifies **A1**.

Next, let $G = (N;U_N)$ and let $G' = (N';U_{N'})$ be the merged game obtained from G by merging any two voters $i, j \in N$, where G' is given by definition 9.

Hence $C(G)=\frac{1}{2^{|N|}}$ and $C(G')=\frac{1}{2^{|N|-1}}$, so that $C(G')=2C(G)$, which demonstrates fulfillment of **A2** by *C*.

If $\{i\}$ is a dictator in $G=(N;V)\in \mathbf{F}$, then any winning coalition must contain i. Hence $|\mathbf{W}_G|=2^{|N|-1}$, which gives $P(G)=\frac{2^{|N|-1}}{2^{|N|}}=\frac{1}{2}$. Thus, *C* meets **A3** also.

We will now show that if a **PTA** index *P* satisfies axioms **A1**-**A3**, then it must be the Coleman index. First note that any *P* satisfying axiom **A1** is uniquely determined by its values on unanimity games. This is because for any $G\in \mathbf{F}$, $G=G_{S_1}\vee G_{S_2}\vee ....\vee G_{S_m}$, where $S_1,S_2,....,S_m$ are minimal winning coalitions of G and G_{S_i} is the unanimity game corresponding to $S_i, i=1,2,...,m$. Thus, by **A1**, $P(G)$ is determined if $P(G_{S_1})$, $P(G_{S_2}\vee G_{S_3}\vee ....G_{S_m})$ and $P(G_{S_1}\wedge(G_{S_2}\vee ...\vee G_{S_m}))$ are known. But, $G_{S_1}\wedge(G_{S_2}\vee \ldots \vee G_{S_m})=G_{S_1\cup S_2}\vee \ ...\vee G_{S_1\cup S_m}$ and hence, by induction hypothesis, both $P(G_{S_2}\vee \ldots \vee G_{S_m})$ and $P(G_{S_1}\wedge(G_{S_2}\vee ...\vee G_{S_m}))$ are determined. So $P(G)$ is determined.

In view of the above discussion we can now say that it is enough to determine $P(N;U_N)$ for any unanimity game $(N;U_N)$. We will show by induction on $|N|$ that if $G=(N;U_N)$, then $P(G)=1/2^{|N|}$. If $|N|=1$, then G has a dictator and hence by **A3**, $P(G)=1/2$. So assume $|N|>1$ and the result for all games $(\hat{N};U_{\hat{N}})$ where $|\hat{N}|<|N|$.

For the merged game $G'=(N';U_{N'})$ obtained from $G=(N;U_N)$ by merging any two voters $i,j\in N$, induction hypothesis gives $P(G')=1/2^{|N|-1}$. By axiom **A2**, $P(G)=1/2\,P(G')=1/2\times\frac{1}{2^{|N|-1}}=\frac{1}{2^{|N|}}$. This proves that *P* coincides with *C* on all unanimity games and hence on all voting games. □

We may now give an example to illustrate how the Coleman index in a game can be calculated from its minimal winning coalitions. Consider the weighted majority game $G=(N;V;\mathbf{w};q)$ with the voter set $N=\{A_1,A_2,A_3\}$; $\mathbf{w}=(1,2,3)$ and *q*=4. The minimal winning coalitions in this game are $S_1 = \{A_1, A_3\}$ and $S_2 = \{A_2, A_3\}$. Hence $C(G)$ = $C(G_{S_1}) + C(G_{S_2}) - C(G_N)$, with G_{S_i} being the unanimity game corresponding to S_i, where i = 1,2, and $G_N = (N;U_N)$. Then $C(G)$ = $\frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{2^3} = \frac{3}{8}$.

Finally we show that axioms **A1**-**A3** are independent. Independence says that none of these axioms implies or is implied by a second one. Demonstration of independence will require that if one of these axioms is dropped, then there will exist a **PTA** index that will satisfy the remaining two axioms but not the dropped one. More formally we have

Theorem 3: Axioms A1-A3 are independent.

Proof: Let $G = (N;V) \in \mathbf{F}$ be arbitrary. Then consider the **PTA** indices given by

$$I_1(G) = \frac{q|\mathbf{W}_G|}{2^{|N|}}, \qquad (12)$$

where $q > 0, q \neq 1$ is a constant,

$$I_2(G) = \frac{|\mathbf{W}_G|}{2^{|N|-1}} - \frac{1}{2}, \qquad (13)$$

and

$$I_3(G) = \frac{\log_2|\mathbf{W}_G|}{2(|N|-1)}, \; |N| > 2. \qquad (14)$$

It is easy to see that I_1 meets **A1** and **A2** but not **A3,** whereas I_2 meets **A1** and **A3** but not **A2**. One can also check that I_3 fulfils **A2** and **A3** but not **A1**. □

4.Conclusion

The Coleman index of the power of a collectivity to act is a summary measure of the degree of concurrence of the underlying decision making body with a proposed bill. This paper has axiomatized the index to get a deeper understanding of it. The index has decreasing monotonic association with the Felsenthal-Machover resistance coefficient that shows the level of disagreement of the voting body to a given act. Since we can express the resistance coefficient in terms of the set of losing coalitions as

$$M(G) = \frac{|\mathbf{L}_G| - 2^{|N|-1}}{2^{|N|-1} - 1} \ , \qquad (15)$$

a worthwhile exercise will be to find a family of intuitively reasonable desiderata using losing coalitions for characterizing it. We leave this a future research programme.

References

1. Banzhaf, J. F. (1965): Weighted Voting Doesn't Work: A Mathematical Analysis, Rutgers Law Review, 19, 317-343.
2. Barua, R., S.R. Chakravarty and S. Roy (2002): Measuring Power in Weighted Majority Games, Mimeographed, Indian Statistical Institute, Calcutta.
3. Burgin, M. and L.S. Shapley (2001): Enhanced Banzhaf Power Index and Its Mathematical Properties, WP-797, Department of Mathematics, UCLA.
4. Coleman, J.S. (1971): Control of Collectives and the Power of a Collectivity to Act, in B. Lieberman (ed.) Social Choice, Gordon and Breach, New York, pp. 269-298.
5. Deegan, J. and E.W. Packel (1978): A New Index of Power for Simple n- Person Games, International Journal of Game Theory, 7, 113-123.
6. Dubey, P. (1975): On the Uniqueness of the Shapley Value, International Journal of Game Theory, 4, 131-140.
7. Dubey, P. and L.S. Shapley (1979): Mathematical Properties of the Banzhaf Power Index, Mathematics of Operations Research, 4,99-131.
8. Felsenthal, D.S. and M. Machover (1995): Postulates and Paradoxes of Relative Voting Power – A Critical Reappraisal, Theory and Decision, 38, 195 – 229.
9. Felsenthal, D. and M. Machover (1998): The Measurement of Voting Power, Edward Elgar, Cheltenham.
10. Felsenthal, D. and M. Machover (2001): The Treaty of Nice and Qualified Majority Voting.
11. Johnston, R.J.(1978): On the Measurement of Power: Some Reactions to Laver, Environment and Planning A, 10, 907-914.
12. Shapley, L.S. and M.J. Shubik (1954): A Method for Evaluating the Distribution of Power in a Committee System, American Political Science Review, 48, 787-792.