Skills, Sunspots and Cycles*

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Abstract

This paper explores the ability of a class of one-sector, multi-input models to generate indeterminate equilibrium paths, and endogenous cycles. In particular, we consider a one sector economy in which there exist one type of capital stock, but a finite number (M) of heterogenous labor services. There are two main results. **First**, the model presents an original theoretical economic mechanism that explain sunspot-driven expansions; the mechanism does not require upward sloping labor demand schedules; the proposed mechanism differs from the customary one, and we consider it complementary to that one. **Second**, the model explains the labor market dynamics from the supply side, while endogenizing the capital productivity response to i.i.d. demand shocks through a change in the aggregate labor demand composition.

– PRELIMINARY - COMMENTS WELCOME –

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1 Introduction

In the last few years it has been recognized that indeterminacy of the equilibrium is a phenomenon that arises in a representative agent, infinite horizon economies if the assumption of constant returns to scale and/or perfect competition is relaxed.

The class of one-sector models with indeterminacy (e.g. Farmer [8], Farmer and Guo [7] or Benhabib and Farmer [2]) however, requires a degree of increasing returns which is too high with respect to what recent estimates seem to suggest (see, among the others, Basu and Fernald [1], Sbordone [24], Jimenez and Marchetti [15]). The high degree of increasing returns is also responsible for an undesirable properties of this class of models: specifically, in order to have local indeterminacy, the labor demand schedule must be upward sloping (Benhabib and Farmer [2]).

The economic literature proposes two classes of remedies to overcome these difficulty: the introduction of factor hoarding (e.g. Wen [26], Weder [25]);[1] or the explicit specification of a second sector[2] (e.g. Benhabib and Farmer [3], Perli [22]).

This paper explores the ability of a class of one-sector, multi-input models to generate indeterminate equilibrium paths, and endogenous cycles. In particular, we consider a one sector economy in which there exist one type of capital stock, but a finite number (M) of heterogenous labor services (we refer to this property as to "labor market segmentation"). Labor services are assumed to be heterogeneous along the skill/productivity dimension.[3]

In addition, this framework can be used to explain endogenous fluctuations of skilled and unskilled workers in bad and good times under indeterminacy; in particular, we investigate whether the model can we explain skill biased technical change from the labor supply side, and if a zero net labor reallocation has an aggregate impact over the economy.

Here is an overview of our results. **First**, the model presents an original theoretical

[1] The introduction of factor hoarding can sensibly reduce the amount of externality needed for having indeterminacy. For instance, in a model with variable capacity utilization, Weder [25] shows how indeterminacy can arise by assuming low externalities coupled with factor hoarding. Analogous results can be obtained by introducing the need for firms to devote a share of labor services to the maintenance of capital stock, as in Guo and Lansing [12]. See also Kim [17] for a survey on sources of externalities.

[2] The introduction of a second sector solves this problem. Perli [22] explicitly introduce an household production sector into a model with externalities and increasing returns. He shows that cycles driven by self-fulfilling prophecies can exist with external effects in labor and capital that are sensibly smaller than previously thought. He also shows that the equilibrium labor demand of his model is well behaved, in the sense that it slopes down. A similar result (indeterminacy with low externalities) has been obtained by Benhabib and Farmer [3] in a two sector model with sector specific instead of aggregate externalities. Their model, however, may have equilibria where consumption and hours are negatively correlated when the driving force is a sunspot rather than a technology shock.

[3] Notice, however, that what matters is the heterogeneity itself, and it is possible to obtain qualitatively analogous results for different kinds of heterogeneity (i.e. distinguishing between regular and underground labor services, or between labor services spatially separated).

economic mechanism that explain sunspot-driven expansions, which does not require upward sloping labor demand schedules; the proposed mechanism differs from the customary one, and we consider it complementary to that one. It turns out that the skill composition of aggregate labor demand drives expansionary i.i.d. demand shock, and that there exists a composition effect that it would not arise in a saddle path stable model. **Second**, the model explains the labor market dynamics from the supply side, while endogenizing the capital productivity response to changes in the aggregate labor demand composition, in the spirit of Acemoglu (2002). We do not present here a story of technology adoption but of endogenous increase in capital productivity, driven by a labor reallocation toward more skilled (and therefore) productive labor services.

The paper is organized as follows. Section 2 presents the theoretical model and its equilibrium; Section 3, then, discusses the topological properties of stationary state, derives conditions for indeterminacy and explains how the theoretical mechanism of the model works. This Section also derives conditions under which the model displays endogenous cycles, and discuss its economic intuition. Section 4, next, calibrates the model for the U.S. economy, and studies the model response to extrinsic uncertainty *via* generalized impulse response functions. Finally, Section 5 concludes. Proofs and derivation are included in the Appendices at the end of the paper.

2 The Model

There are two classes of agents in the models: firms and households. We assume that there exist a one aggregate capital stock, and M types of labor services, which are applied to the existing capital stock. In this sense the labor market is said to be *segmented*.

2.1 Firms

Assume that the production technology for the homogenous good uses $M+1$ inputs, $M > 1$: an aggregate capital stock k_t, and M different types of labor services, denoted as n^j, with $j = 1, 2, ..., M$; now, given these preliminaries, the $i-$th firm's production function reads:

$$y_{i,t} = A_t k_{i,t}^{\alpha_0} \left[\prod_{j=1}^{M} (n_{i,t}^j)^{\alpha_j} \right], \text{ with } \sum_{j=0}^{M} \alpha_j = 1.$$

The quantity A_t (defined below) represents an aggregate production externality

$$A_t = \underbrace{\{K_t^{\alpha_0}\}^{\omega}}_{\text{Marshallian Ext.}} \prod_{j=1}^{M} \underbrace{\left[(N_t^j)^{\alpha_j}\right]^{\eta_j}}_{\text{j-th labor Externality.}}, \quad \omega \neq \eta_\kappa \neq \eta_j, \quad \kappa \neq j,$$

where K_t and the $N_t^{j\prime}s$ are the economy-wide levels of the production inputs.

The aggregate external effect has $M+1$ different sources. The first one, without loss of generality, is related to the well known Marshallian effect, analogous to that of standard one-sector models (e.g. Farmer and Guo [7]). The other ones act through the various types of labor services. The model explicitly distinguishes among each labor-input-specific external effect: for example, the quantity $\left[(N_t^j)^{\alpha_j}\right]^{\eta_j}$ denotes the external effect associated to the $j-th$ type of labor. The parameters $(\omega, \eta_j, j = 1, 2, ..., M)$ are assumed to be different one the other. The idea is to exploit the peculiar characteristics that each production factor has.[4]

As firms are all identical, overall level of output for a given (and equal for all firms) level of inputs utilization is given by:

$$Y_t = A_t \int_i \left\{ k_{i,t}^{\alpha_0} \left[\prod_{j=1}^{M} (n_{it}^j)^{\alpha_j} \right] \right\} di = K_t^{\alpha_0(1+\omega)} \left[\prod_{j=1}^{M} (N_t^j)^{(1+\eta_j)\alpha_j} \right] \tag{1}$$

Increasing returns to scale are a pure aggregate phenomenon (as equation (1) suggests), and returns to scale faced by each firm in production are constant; formally, $\alpha_0 = 1 - \sum_{j=1}^{m} \alpha_j$. Assume, next, that each firm takes K, $N^1, \ldots, N^m$ as given.[5] As markets are competitive, firm's behavior is described by the $M+1$ first order conditions for the (expected) profit maximization, with respect to $k_{i,t}$, $n_{i,t}^1, ..., n_{i,t}^m$:

$$\begin{aligned} k_{i,t} &: \alpha_0 A_t \frac{\partial y_{i.t}}{\partial k_{i,t}} = r_t \\ n_{i,t}^1 &: \alpha_1 A_t \frac{\partial y_{i,t}}{\partial n_{i,t}^1} = w_t^1 \\ &\vdots \\ n_{i,t}^M &: \alpha_M A_t \frac{y_{i,t}}{n_{i,t}^M} = w_t^M. \end{aligned} \tag{2}$$

[4]This formulation adds to the analysis greater generality, as it encompasses a large class of one sector economies that do not explicitly distinguish among the input specific external effects. More details are offered in the following pages.

[5]In this context the externality A_t acts at pure aggregate-systemic level, as in Romer's [23] endogenous growth model.

The ordering of the α_j's parameters can differ form that of the externality parameters η_j: the latter are well suited to measure the productivity of labor - more skilled labor can be attached with a high level of external effect. All kinds of labor services must be employed in equilibrium, due to the Cobb-Douglas production structure, for having nonzero production; this can justified with the technical requirements of the division and of the specialization of labor. As additional rationale for this assumption, imagine that relatively more productive types of labor are also more costly for the firm and for the consumer/worker.[6]

2.2 Households

Suppose that there exist a continuum of identical households, indexed with super-script i, uniformly distributed over the unit interval. Suppose that each household supplies $j = 1, 2, ..., M$ different types of labor $n_{i,t}^j$. Assume that each household is complete, in the sense that all househeholds supply all types of labor services.

The households preferences are structured in the following way. The common consumption flows $c_{i,t}$ induces $\log(c_{i,t})$ level of utility; total labor $n_{i,t} = \sum_{j=1}^{M} n_{i,t}^j$ generates an overall disutility of work equal to: $Dn_{i,t}$; in addition each specific type of labor determines a specific amount of disutility, represented by $B_j n_{i,t}^j$, that captures the labor heterogeneity (or labor market segmentation). Without loss of generality, we can order the labor services along the disutility dimension assuming that $B_1 < B_2 < \cdots < B_M$. The quantities $B_j n^j$ represents the labor-specific effort exerted by each household. Labor types with higher B are assumed to be more costly for the consumers/workers. We can also assume that the higher the cost (i.e. the B_j), the higher the productivity of the worker. If we interpret labor heterogeneity as stemming from an un-modelled human capital stock and/or skills, the B_j disutility parameters would be associated to additional effort needed to acquired an higher education (or on-the-job training). Each different type of labor j may thus require a different cost for acquiring the related skills or characteristics. This formulation is not addressing a fully fledged "heterogeneity problem", but it is looking at a parsimonious model capable of capturing this issue.

Assuming separability, we specify the momentary household utility function as

$$\mathcal{V}_{i,t}\left(c_{i,t}, n_{i,t}^1, \cdots, n_{i,t}^M\right) = \log c_t^i - Dn_{i,t} - \sum_{j=1}^{M} B_j n_{i,t}^j$$

[6]A nested CES structure on production would allow for a more general analysis, and we should expect that the value of the elasticity of substitution (say σ) would play an interesting role. There exist, however, several difficulties to estimate this parameter because it captures substitution both within and across industries. Moreover, the majority of macro-estimates are between $\sigma = 1$ and $\sigma = 2$ (e.g. Freeman 1986), which correspond to the Cobb Douglas case.

The household' feasibility constraint ensures that consumption and investment $i_{i,t}$ do not exceed consumers' income,

$$c_{i,t} + i_{i,t} = r_t k_{i,t} + \sum_{j=1}^{M} w_t^j n_{i,t}^j.$$

Then, capital stock is accumulated according to a customary state equation, i.e.

$$k_{i,t+1} = (1-\delta)k_{i,t} + i_{i,t},$$

where δ denotes a quarterly capital stock depreciation rate and $i_{i,t}$ is the household's investment.

Imposing a constant subjective discount rate $0 < \beta < 1$, and defining $\mu_{i,t}$ as the costate variable, we form the Lagrangean of the household control problem:

$$\mathcal{L}_0^h = E_0 \sum_{t=0}^{\infty} \beta^t \mathcal{V}_{i,t} + E_0 \sum_{t=0}^{\infty} \mu_{i,t} \left(r_t k_{i,t} + \sum_{j=1}^{M} w_t^j n_{i,t}^j - c_{i,t} - i_{i,t} \right).$$

Household's optimal choice is determined by the following necessary and sufficient conditions:

$$\begin{aligned}
c_{i,t} &: \quad \beta^t c_{i,t}^{-1} = \mu_{i,t} \\
n_{i,t}^1 &: \quad \beta^t D + \beta^t B_1 = \mu_{i,t} w_t^1 \\
&\quad\ \vdots \\
n_{i,t}^M &: \quad \beta^t D + \beta^t B_M = \mu_{i,t} w_t^M \\
k_{i,t+1} &: \quad E_t \left\{ \mu_{i,t+1} \left[(1-\delta) + r_{t+1} \right] \right\} = \mu_{i,t} \\
&\quad\ \lim_{t \to \infty} E_0 \mu_{i,t} k_{i,t} = 0
\end{aligned} \tag{3}$$

The model collapses to the standard one sector model with aggregate increasing returns to scale (e.g. Farmer and Guo [7]); Just set $M = 1$; $\omega = \eta_1 = \eta$ into the previous equilibrium conditions.

2.3 Symmetric perfect foresight equilibrium

We focus on a symmetric perfect foresight equilibrium in which firms make zero profits. In equilibrium the aggregate consistency requires that $y_{i,t} = Y_t$, $k_{i,t} = K_t$, $n_{i,t}^j = N_t^j$, $c_t = C_t$, where capital letters denote aggregate equilibrium quantities[7]. An equilibrium is a sequence of prices $\left\{w_t^1, \cdots, w_t^m, r_t\right\}_{t=0}^{\infty}$ and a sequence of quantities $\left\{N_t^1, \cdots, N_t^m, K_{t+1}, C_t, \right\}_{t=0}^{\infty}$ such that firms and households solve their optimization problems and the resource constraints are satisfied. As a result, the first order conditions characterizing the equilibrium are given by:

$$
\begin{aligned}
D + B_1 &= (C_t)^{-1} \alpha_1 \frac{Y_t}{N_t^1} \\
&\vdots \\
D + B_M &= (C_t)^{-1} \alpha_M \frac{Y_t}{N_t^M} \\
(C_{t+1})^{-1} \left((1-\delta) + \alpha_0 \frac{Y_{t+1}}{K_{t+1}} \right) \beta &= (C_t)^{-1} \\
K_{t+1} &= K_t^{\alpha_0(1+\omega)} \prod_{j=1}^{M} N_t^{\alpha_j(1+\eta_j)} + (1-\delta) K_t - C_t \\
\lim_{T\to\infty} (C_T)^{-1} K_T &= 0
\end{aligned}
$$

note that the first M equations imply that all wages have to be equated, net of the idiosyncratic cost B_i.

The model has a unique deterministic steady state; let x be any variable of the model; the deterministic steady state is defined as the locus in which $\bar{x} = x_t = x_{t+1} = ... = x_{t+T}$ for all t, T.We can state, next, the following two results.

Proposition 1 *Consider the deterministic version of the model; then, there exist a unique stationary state.*

Proof. Application of fixed point theorem; Appendix A. ■

Corollary 1 *Approximating the quantity* $\frac{N_i^{\alpha_i(1+\eta_i)}}{N_M^{\alpha_i(1+\eta_M)}}$ *as* $\left(\frac{N_i}{N_M}\right)^{\alpha_i(1+\eta_i)}$*, it is possible to derive a closed form deterministic steady state.*

Proof. Algebraic derivation; Appendix A. ■

[7]The aggregate resource constraint holds: $C_t + I_t = Y_t$.

3 Topological Properties and Endogenous Cycles

3.1 Topological Properties

To solve the model, we log-linearize the economy-wide version of first order conditions (2) and (3) around the steady state (as in King et al.[16]). To study how investors "animal spirits" operate into an economy with indeterminacy, production externalities and two types of labor input, we arrange the system of linearized equations in a way such that consumption rather than the Lagrangian multiplier appears in the state vector.

Denoting with S_t as the vector $(K_t; C_t)$, the model can be reduced to the following system of linear difference equations (where hat-variables denote percentage deviations from their steady state values):[8]

$$\hat{S}_{t+1} = \mathbf{F}\hat{S}_t + \Omega\mathcal{E}_{t+1}, \tag{4}$$

where $\mathcal{E}_{t+1}$ is a 2×1 vector of one step ahead forecasting errors satisfying $E_t\mathcal{E}_{t+1} = 0$ and Ω is a coefficient matrix. Its first element $\widehat{K}_{t+1} - E_t\widehat{K}_{t+1}$ equals zero, since $\widehat{K}_{t+1}$ is known at period t; denote the second element with $\tilde{\varepsilon}_c = \widehat{C}_{t+1} - E_t\widehat{C}_{t+1}$. Now, when the model has a unique equilibrium (i.e., one of the eigenvalues of $\mathbf{F}$ lies outside the unit circle), the optimal decision rule for investment does not depend on the forecasting error, $\tilde{\varepsilon}_c$.[9]

If both eigenvalues of $\mathbf{F}$ lie inside the unit circle, however, the model is indeterminate in the sense that any value of $\widehat{C}_t$ is consistent with equilibrium given $\widehat{K}_t$. Hence, the forecasting error $\tilde{\varepsilon}_c$ can play a role in determining the equilibrium level of consumption. Under indeterminacy the decision rule for consumption at time t take the special form

$$\widehat{C}_t = f_{21}\widehat{K}_{t-1} + f_{22}\widehat{C}_{t-1} + \omega_2\tilde{\varepsilon}_{c,t}$$

where f_{21} and f_{22} are the second row elements of the matrices $\mathbf{F}$ and Ω. The condition $E_t\tilde{\varepsilon}_{c,t+1} = 0$ ensures that rational agents do not make systematic mistakes in forecasting future based on current information. Since $\tilde{\varepsilon}_{c,t}$ can reflect a purely extraneous shock, it can be interpreted as shock to autonomous consumption (that is the "animal spirits hypothesis").

[8] The procedure used to obtain (4) is shown in Appendix B.

[9] Specifically, in that case $\hat{c}_t$ can be solved forward under the expectation operator E_t to eliminate any forecasting errors associated with future investment. Then the optimal decision rules at time t depend only on the current capital stock k_t

3.2 Conditions for Local Indeterminacy of the Equilibrium Path

Necessary and Sufficient Conditions (for local indeterminacy of the equilibrium path) are derived in Theorem 2 below. To present a neat economic interpretation it is convenient to write them in terms of elasticities and cross-elasticities of the demand schedules for capital, and for the various types of labor with respect to the $M+1$ production inputs.

Theorem 2 *The equilibrium of system (4) is locally indeterminate when the following necessary and sufficient (NSC) condition holds:*

$$\boldsymbol{NSC}: \max\left(\frac{1}{\beta(1-\delta)}, \frac{\underline{\mathcal{R}}}{\underline{\mathcal{R}}-1}\right) < \Phi < \frac{\overline{\mathcal{R}}}{\overline{\mathcal{R}}-1},$$

where $\Phi = \sum_{j=1}^{M}(1+\eta_j)\alpha_j$ *is the sum of the cross elasticities of the linearized labor demand functions,* $\underline{\mathcal{R}} = \frac{\delta(1-s_I)[1-\beta(1-\delta)](1-\alpha_0(1+\omega))+2[\delta\alpha_0(1+\omega)+s_I(2-\delta)]}{\delta(1-s_I)[1-\beta(1-\delta)](1-\alpha_0(1+\omega))+2[\delta\alpha_0(1+\omega)+s_I(1-\beta(1-\delta))]} > 1$, $\overline{\mathcal{R}} = \frac{\delta s_I - \delta\alpha_0(1+\omega)}{s_I[1-\beta(1-\delta)]-\delta\alpha_0(1+\omega)} > 1$, *and* $s_I = I^*/Y^*$ *denotes the (steady state) share of investment.*

Proof. see Appendix B. ■

Condition **NSC** is enlightening about the nature of the economic process at basis of indeterminacy in our model. Rewriting Φ in terms of cross elasticities of labor demand yields $\sum_{j=1}^{M}(1+\eta_j)\alpha_j = \sum_{j=1, j\neq\kappa}^{M}\tilde{\varepsilon}_{j,\kappa}$.

Consider the lower bound of **NSC** first, $\left(\max\left(\frac{1}{\beta(1-\delta)}, \frac{\underline{\mathcal{R}}}{\underline{\mathcal{R}}-1}\right) < \sum_{j=1, j\neq\kappa}^{M}\tilde{\varepsilon}_{j,\kappa}\right)$. It suggests that the labor demand schedules should have a *sufficiently large* response to changes in equilibrium employment. But, at the same time, that this response should *not be too large*; the upper bound suggests that $\sum_{j=1, j\neq\kappa}^{M}\tilde{\varepsilon}_{j,\kappa} < \frac{\overline{\mathcal{R}}}{\overline{\mathcal{R}}-1}$. This condition represents a building block of the theoretical mechanism supporting self-fulfilling properties (see. Section 3.4).

The second inequality turns out to be particularly relevant. Corollary 2 below recast it as follows:

Corollary 2

$$\frac{\partial\widehat{w}^i}{\partial\widehat{K}} > \frac{s_I}{\delta}\left[1+(1-\beta)(1-\delta)\sum_{j=1}^{M}\frac{\partial\widehat{w}^i}{\partial\widehat{N}^j}\right] \tag{5}$$

Proof. Algebraic derivation; Appendix A. ■

Condition (5) suggests that labor demand functions should react more to changes in capital stock rather than changes in labor services, *ceteris paribus*.[10] In other words, each

[10] Technically speaking, for the generic inverse demand function of labor of type i, $\frac{\partial\widehat{w}^i}{\partial\widehat{K}}$ should be larger than $\sum_{j=1}^{M}\frac{\partial\widehat{w}^i}{\partial\widehat{N}^j}$, which is also reduced by quantities $\frac{s_I}{\delta}$ and $(1-\delta)(1-\beta)$.

labor demand schedule should display a large enough response to variation in capital stock for expectation to be self-fulfilled.

3.3 Dynamics of a segmented labor market

The explicit disentanglement of the labor input into different categories, endowed with specific technical features, as is the case for qualified labor, has a particularly welcome results.

The first one is that both labor demand schedules are well behaved (in the sense that they slope down), compared to standard one-sector economy models where labor demand is upward sloping. Linearized labor demand functions can be written as functions $\widehat{w}_t^j = \widehat{w}^j\left(\widehat{N}_t^1, \cdots, \widehat{N}_t^M\right)$. A labor demand function is said to be well behaved when it slopes down over its wage domain, that is when the partial derivative with respect the corresponding labor input is negative; pick, WLOG, the $h-th$ labor input:

$$\frac{\partial \widehat{w}^h}{\partial \widehat{N}_t^h} < 0,\ \widehat{N}_t^h = \left(\widehat{N}_t^1, \cdots, \widehat{N}_t^M\right)$$

It is then possible to derive a set of restrictions on selected parameters to ensure that these inequalities hold. A natural choice is to use labor elasticities. For our production technology (equation (1)) the previous condition reads:

$$\frac{\partial \widehat{w}^h}{\partial \widehat{N}_t^h} < 0 \Leftrightarrow \eta_h < \frac{1-\alpha_h}{\alpha_h},$$

for each type of labor: $h = 1, \cdots, M$.

The introduction of labor input heterogeneity eases the conditions for having well behaved labor demand schedules. Denote with $\eta_h^{**} = \frac{1-\alpha_h}{\alpha_h}$ the largest degrees of input-specific increasing returns to scale ensuring that local indeterminacy arises, *and* that labor demand schedules are well behaved. Recall that the production function, apart form the externality effect, has constant returns to scale: $\alpha_0 + \sum_{j=1}^M \alpha_j = 1$. Rewriting each labor shares as $\alpha_h = 1 - \alpha_0 - \sum_{j\neq h}^M \alpha_j$, the previous inequality reads

$$\frac{\partial \widehat{w}^h}{\partial \widehat{N}_t^h} < 0 \Leftrightarrow \eta_h < \frac{\alpha_0 + \sum_{j\neq h}^M \alpha_j}{1 - \alpha_0 - \sum_{j\neq h}^M \alpha_j},$$

and the threshold level equals to $\eta_h^{**} = \left(\alpha_0 + \sum_{j\neq h}^M \alpha_j\right) \diagup \left(1 - \alpha_0 - \sum_{j\neq h}^M \alpha_j\right)$. Now, if the number M of labor types shrinks that upper bound decreases for the remaining labor inputs, while reducing, by this end, the parameter´s region in which the equilibrium is locally indeterminate and the labor demand schedules are well behaved at the same time.

A numerical example from Busato, Charini and Marchetti [5], may further clarify this claim. When $\alpha_0 = 0.23$ and $\alpha_1 = 0.088$, which are two reasonable figures when N^1 and N^2 are interpreted as regular and underground labor shares, the upper bound of the regular labor externality equals $\eta_1^{**} = 0.4662$; *without* underground sector it goes down to $\eta_1^{**}|_{\alpha_2=0} = 0.23$.

This in an important implication since Farmer and Guo [7] show that for having indeterminacy they need to have a very large externality parameter. To display indeterminacy their model needs a high degree of increasing returns to scale, which equals $\eta = 0.785$, which is way above their threshold ($\eta^* = 0.23$) for having a well behaved demand schedule. Basically, the reason why the $m-$input model is more easily characterized by demand functions that slope down rests in the underlying necessary condition for indeterminacy. As shown in Appendix (theorem **??**), for indeterminacy to arise it must be $\left[\sum_{j=1}^{M} \alpha_j(1+\eta_j)\right] - 1 > 0$; in a single labor input case, as in Farmer and Guo [7], this condition would read: $\alpha_1(1+\eta_1) - 1 > 0$, meaning that the demand function for the (unique) labor input should be negatively sloped; when there is more than one type of labor input this is no more needed for indeterminacy to arise.

3.4 The Model Theoretical Mechanism

The result shown in Theorem 2 has important implications for the economic mechanism explaining the model reaction to a stochastic shock, particularly to an i.i.d. sunspot. The very idea of the "animal spirits hypothesis" is that expectations are self-fulfilled under local indeterminacy of the equilibrium path. This means, that following a positive sunspot shock *today*, a rational consumers should expect a higher income *tomorrow*;[11] the self-fulfilling mechanism, generated under indeterminacy, should indeed push the economy into an expansionary pattern. In Farmer and Guo [7] a positive sunspot shock $\hat{\varepsilon}_t$ on the labor supply $\hat{w}_t = \widehat{C}_t + \hat{\varepsilon}_t$ shifts upward the wage $\hat{w}_t$; as the labor demand is upward sloping, this induces an increase in equilibrium labor, thus creating a self-fulfilling expansionary push on the economy.

In our model the final consequences of a shock $\hat{\varepsilon}_t$ are the same, but the interaction between input markets is different. Suppose, for simplicity, that we have only two types of labor, skilled and unskilled labor services The sunspot shock affects the two labor supplies in the same way as in Farmer and Guo [7], but, as the labor demands are well behaved, this would induce a *reduction* in the equilibrium levels of type 1 and type 2 labor, which

[11]This is represented by the forecasting error previously defined, $\tilde{\varepsilon}_{c,t}$. It can reflect a purely extraneous shock, and it can be interpreted as shock to autonomous consumption.

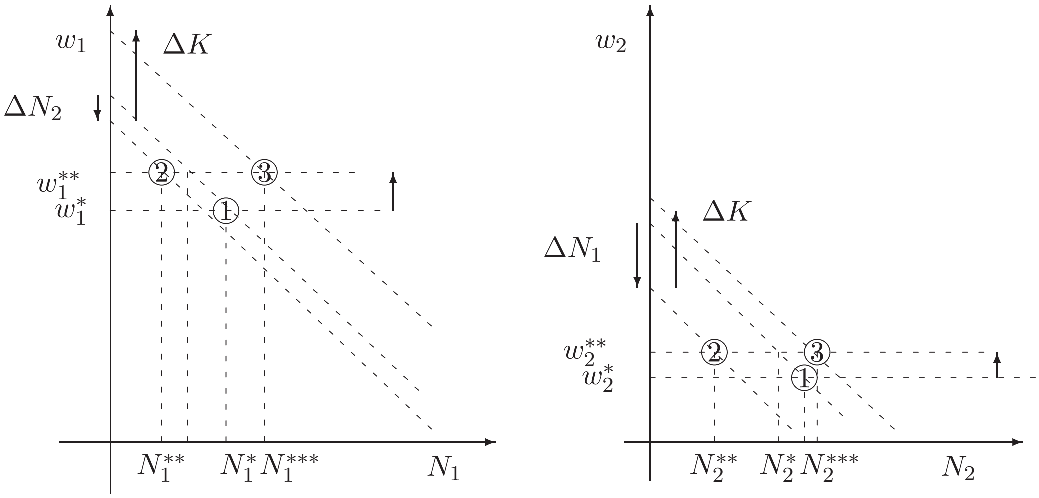

Figure 1: **Theoretical Mechanism**. Skilled and unskilled labor supply schedules shift upward after an i.i.d. sunspot shock; the economy would enter into a recession as labor demands are negatively sloped. The cross-interaction between labor markets would further strengthen the inward shifts of labor demands. But, in a perfect foresight equilibrium, the labor input reallocation toward the relative more skilled labor input would increase capital productivity. This triggers the capital accumulation ($\Delta K > 0$) that shifts out both labor demand schedules, driving the economy into the conjectured expansion.

is reinforced *via* labor demands' cross elasticities[12]. The labor market response following a positive sunspot shock is presented in **Figure 1** below.

Suppose the economy is at the steady state in which $\frac{N_1^*}{N_2^*} = \phi^* < 1$, and consider a sunspot shock. Now, the households are willing to have a higher consumption flow $\uparrow C$, and, at the same time, to work less. It means that all labor supply schedules shift upward. Demand schedules have different slope, and therefore the resulting change in equilibrium

[12]To see this more clearly, consider the inverse (linearized) demand for type 1 labor can be written:

$$\widehat{w}_t^1 = [(1+\omega)\alpha_0]\,\widehat{K}_t + [(1+\eta_1)\alpha_1 - 1]\,\widehat{N}_t^1 + [(1+\eta_2)\,(1-\alpha_0-\alpha_1)]\,\widehat{N}_t^2$$

i.e. as a function $w^1 = L_1^D\{N^1; N^2, K, \}$, whose partial derivatives have the following signs: $\frac{\partial L_1^D}{\partial N^1} < 0$, $\frac{\partial L_1^D}{\partial N^2} > 0$. Symmetrically, the other wage - the demand for type 2 labor - equals:

$$\widehat{w}_t^2 = [(1+\omega)\alpha_0]\,\widehat{K}_t + [(1+\eta)\alpha_1]\,\widehat{N}_t^1 + [(1+\eta_2)\,(1-\alpha_0-\alpha_1) - 1]\,\widehat{N}_t^2$$

and it is written as $w^2 = L_2^D\{N^1, N^2, K\}$ where $\frac{\partial L_2^D}{\partial N^2} < 0$, $\frac{\partial L_2^D}{\partial N^1} > 0$. Now, the initial fall in each sector equilibrium labor services (that is a movement along each sector demand schedule) induces a further reduction in each sector employment through an inward shift of demand schedules (that is a schedule shift, induced by a change in the other-sector equilibrium employment).

labor services differ across labor market segment. In our example, in which N_1 denotes skilled labor and N_2 the unskilled counterpart, $\downarrow N_1$ and $\downarrow\downarrow N_2$. This implies that the ratio$\frac{N_1^{**}}{N_2^{**}} = \phi^{**} > \phi^{*}$ and $\phi^{**} \gtreqless 1$;

The economy reaches a phase in which the composition of the labor demand is changed towards a more qualified combination used labor services. This makes the the capital more productive (capital skill complementarity), and the interest rate increases, and households increase capital accumulation. Now, recall the economic intuition behind Corollary 1: the outward shift of labor demands (driven by an increase of aggregate capital stock) offset the initial inward shit (triggered by the desired higher consumption), and the economy enters an expansion. In summary the increase in capital stock, is capable to offset the initial decrease in the labor demands. Labor demands are pulled right-upward (via increase in the use of capital): $L^D(1) \rightarrow L^D(2)$. Eventually wages (and r) increase, as well as equilibrium levels of capital, labor 1 and labor 2. The overall increase in inputs usage drives the economy into a self-fulfilling expansion.

This mechanism is distinctive of a class of models with heterogenous labor. Indeed, an increase in capital stock would work against the self fulfillment of the expansionary prophecies in the standard model with increasing returns to scale.[13]. An idea behind the increase in capital stock is that to consume more tomorrow a rational consumer needs to produce more, and since labors fall after a sunspot shock, capital should substitute labor services. In other words, agents formulate a conjecture on future income and consumption, according to which they believe to be more wealthy. They want to consume more and - initially - work less. But they realize that to sustain increased consumption and income, factor prices must be higher, so that at the end an increase in the demand for the three inputs must take place: this leads to a general expansion, which fulfills the initial prophecy.

A theoretical implication of this mechanism it that a sunspot shock under indeterminacy should make investment more appealing (in order to self fulfill the expansionary expectations). A natural way to verify this issue is to rewrite the Euler equation, isolating the covariance term between marginal utility of consumption and investment returns $Cov\left(C_{t+1}^{-1}, R_{t+1}\right)$, i.e.

$$E_t\left(C_t^{-1}\right) = \beta E_t\left(C_{t+1}^{-1} R_{t+1}\right) \Rightarrow E_t\left(C_t^{-1}\right) = \beta E_t\left(C_{t+1}^{-1}\right) E_t\left(R_{t+1}\right) + \beta Cov\left(C_{t+1}^{-1}, R_{t+1}\right)$$

Now, investment becomes more appealing when the returns to saving is high in times

[13]This is a consequence of upward sloping labor demand schedule. Specifically, an increase in equilibrium capital stock would induce and inward shift in the labor demand schedule, pushing the economy into a recession.

when marginal utility of consumption is low. Hence, the lower the covariance $Cov\left(C_{t+1}^{-1}, R_{t+1}\right)$, the more appealing investment is. We should expect, therefore, that the stronger the self-fulfilling prophecies, the higher the correlation between consumption and returns. A numerical exercise confirms this claim. When self fulfilling prophecies are low (....) then $Cov\left(C_{t+1}^{-1}, R_{t+1}\right) = 0.63$; consistently with our claim, when self fulfilling prophecies get stronger (....) then $Cov\left(C_{t+1}^{-1}, R_{t+1}\right)$ monotonically decrease to 0.23.

This indirect and subtle effect is made possible by the increasing returns and the presence of a further type of labor, endowed with its own externality. The more complex structure of the model allows for the possibility of a self-fulfilling mechanism acting trough the interdependencies of the three inputs, but not necessarily inducing "ill-behaved" demand functions.

3.5 Endogenous cycles: Deterministic closed orbits

Although the model can easily display fluctuations due to non-intrinsic uncertainty, it is also interesting to ask wether there can be cycles due to endogenous deterministic dynamics. In other words, we wish to see if (4) has a stable set different from a point, i.e. if it has a closed invariant orbit. This amount to know if (4) undergoes to a Hopf bifurcation as some among the deep parameters $(\alpha_j, \beta, \delta, \eta_j, \omega)$ varies.

We can make use of the following theorem[14]:

Theorem 3 (Hopf bifurcation - existence part) *Let the mapping* $\mathbf{x}_{t+1} = \mathbf{F}(\mathbf{x}_t, \mu)$, $\mathbf{F} \in \Re^2 \rightarrow \Re^2$, $\mu \in \Re$ *(μ is a parameter) have a smooth family of fixed points* $\mathbf{x}^*(\mu)$ *at which the linear approximation* $\mathbf{x}_{t+1} = F(\mu)\mathbf{x}_t$, $F = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}$ *has complex conjugate eigenvalues λ. If there exist a μ_0 such that:*

$$\begin{aligned} mod\lambda(\mu_0) &= 1 \quad but \quad \lambda^n(\mu_0) \neq \pm 1, \quad n = 1,2,3,4 \\ and: & \quad \frac{d(mod\lambda(\mu_0))}{d\mu} > 0 \end{aligned}$$

then there is an invariant closed curve bifurcating from $\mu = \mu_0$.

By applying this theorem 3 to system (4), we can state the following:

Theorem 4 *If there exist a string of parameters values $(\bar{\eta}_{j=1,2\cdots n})$ such that:* ***i)*** *eigenvalues of F are complex conjugate;* ***ii)*** $\alpha_0(1+\omega) = \frac{s_I}{\delta}\left[1 + \bar{\Phi}(1-\beta)(1-\delta)\right]$, $\bar{\Phi} = \sum_{j=1}^{M}(1+\bar{\eta}_j)\alpha_j$;

[14] Which is taken from Iooss [14] and Guckenheimer and Holmes [10]. See also Lorenz [18], p. 115.

iii) *the trace of F computed in $\Phi = \bar{\Phi}$, i.e.:$Tr(F)|_{\bar{\Phi}} = Tr_{\bar{\Phi}}$,*[15] *satisfies the following conditions: $Tr(F)|_{\bar{\Phi}} \neq \pm 1, \pm\sqrt{2}, \pm 2$; then there is an invariant closed curve bifurcating from $\bar{\Phi}$.*

Proof. Appendix B. ■

Thus there can be some parameter configuration that enable the model economy to oscillate indefinitely, also without being hit by external shocks: if the system sits in this closed orbit, its long run dynamic will have a cyclical shape.

The economic interpretation such deterministic cycles can be clarified by considering condition **ii)** of the above proposition 4. For having a closed orbit as stable set, it is necessary that:

$$\begin{aligned} \frac{\partial \hat{w}_t^i}{\partial \hat{k}_t} &= \frac{s_I}{\delta}\left[1 + \Phi\left(1-\beta\right)\left(1-\delta\right)\right] \text{ or also:} \\ \Phi &= \delta \frac{\left(\frac{\partial \hat{w}_t^i}{\partial \hat{k}_t} - \frac{s_I}{\delta}\right)}{s_I\left(1-\beta\right)\left(1-\delta\right)} \end{aligned}$$

Recall that $\Phi = \sum_{j=1}^{M}\left(1+\eta_j\right)\alpha_j = \sum_{j=1}^{M}\frac{\partial \hat{w}^i}{\partial \hat{N}^j}$ is the sum of the cross elasticities of the inverse demand for labor of type i (such a sum is equal for all types i). This condition suggests that each labor demand function should react to changes in capital stocks in a specific way, i.e. in the "right" proportion with respect to changes in labor services. To better evaluate this condition, just recall the "upper" inequality of condition NSC: $\frac{\partial \hat{w}_t^i}{\partial \hat{k}_t} > \frac{s_I}{\delta}\left[1 + \Phi\left(1-\beta\right)\left(1-\delta\right)\right]$; it says that the impact $\frac{\partial \hat{w}_t^i}{\partial \hat{k}_t}$ must be "high enough" with respect to the aggregate impact Φ when the system is (locally) stabe and the attractor is a sink. When the system's stable set is instead a closed orbit, the impact of capital $\frac{\partial \hat{w}_t^i}{\partial \hat{k}_t}$ must be smaller than required in the former case.

In condition NSC, a relatively strong (and positive) impact of the capital stock on labor demand was required to offset the initial negative effect due to cross elasticities of labor when the system is off its point-like attractor (e.g. when a positive sunspot shock pushes away the economy from the steady state). If the stable set has instead to be a closed orbit, the same effect $\partial \hat{w}_t^i / \partial \hat{k}_t$ must be "not so big": it has to counteract the effect of labor types' cross elasticities, but to a lower extent. In this case the system steadily "bounces" from increasing values of the inputs' usage to reductions and slow-downs.

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[15] The expression for $Tr_{\bar{\Phi}}$ is: $\frac{\left[1+\left[1-\beta(1-\delta)\right]\frac{\Phi}{1-\Phi}\right]\left[\delta\alpha_0(1+\omega)\left(1+\frac{\Phi}{1-\Phi}\right)+(1-\delta)s_I\right]+s_I-\delta\left[1-\beta(1-\delta)\right]\left\{\alpha_0(1+\omega)\left[1+\frac{\Phi}{1-\Phi}\right]-1\right\}\left[\left(\frac{\Phi}{1-\Phi}\right)+(1-s_I)\right]}{s_I\left[1+\left[1-\beta(1-\delta)\right]\frac{\Phi}{1-\Phi}\right]}$.

Note that this particular expression for the trace depends on the externality parameters η_j; included in the expression $\frac{\Phi}{1-\Phi}$.

4 Calibration and dynamic response

In this section we deliver a numeric example to show the model's dynamic response under indeterminacy and its sensitivity to changes in crucial parameters (the externalities η_j, ω). In doing so, we adopt a two-types -of-labor version of the model (skilled and unskilled: N^2 and N^1) and specify a simplified calibration by imposing an "empiric" value for the ratio of the two labor type N^1/N^2.[16]

For the **preference parameters** B_1, B_2, β , we set $\beta = 0.984$, $B_1^* = 0.1$. We obtain a value of $B_2^* = 0.1422 > B_1^*$ so to have equilibrium in the labor market for the given values of $\left(\frac{N^1}{N}\right)^*$ and N^*.

Technology parameters α, ρ, δ are calibrated as follows. The capital share α^* is set to 0.23, a standard calibration for a one sector economy with aggregate increasing returns (i.e. Farmer and Guo [7]). Papageorgiou [21] estimates a production function with skilled and unskilled labor components for the US economy; his results suggest that the share of skilled labor α_2^* can be calibrated to 0.36, and the unskilled labor share α_1 equals 0.41. Quarterly capital depreciation rate is set to $\delta = 0.025$.

There are three, input-specific, **externality parameters** $\eta_1 = \eta, \eta_2 = \zeta$,and ω. These parameters are set to $\omega^* = 0.076$, $\eta^* = 0.27$, $\zeta^* = 0.6$. The overal degree of increasing returns equals 1.3442, which is lower than the customary value for the standard one-sector model with increasing returns.

Skilled-unskilled labor have been chosen by using the OECD data for the U.S. economy[17]; according to these data, the avegare value (for the 1997-2000 period) of the share of total labor force with higher education (ISCED 5A6 - 5B) equals 34.03%, giving rise to a steady state ratio for $\left(\frac{N^1}{N^2}\right)^*$ of 1.94.

For such parametrization, the model's attractor is a sink so that the linearized system (in reduced form and excluding the shocks) in capital and consumption is:

$$\begin{bmatrix} \widehat{K}_{t+1} \\ \widehat{C}_{t+1} \end{bmatrix} = \begin{bmatrix} 0.0123 & 3.4847 \\ -0.2678 & 1.8534 \end{bmatrix} \begin{bmatrix} \widehat{K}_t \\ \widehat{C}_t \end{bmatrix}.$$

The dynamical model has two complex conjugated eigenvalues: the two roots equals $0.9329 - 0.2929i$ and $0.9329 + 0.2929i$, thus the system's attractor is a sink.

The explanation of the basic mechanism detailed in section 3.4 is confirmed by an

[16] Qualitative results would not change much if we determine the steady state values endogenously.

[17] Data source: OECD [20], table 4 Labor Force Statistics by educational attainement (for the United States). List of time series: ISCED 0/1 Series Name U17 E0 2032; ISCED 2 Series Name U17 E0 2232; ISCED 3A Series Name U17 E0 2432; ISCED 5A/6 Series Name U17 E0 2B32; ISCED 5B Series Name U17 E0 2C32;

inspection of the impulse response functions for the aggregate variables: consumption, capital, output and total employment (Figure 2) when the system is disturbed by an i.i.d. sunspot shock:

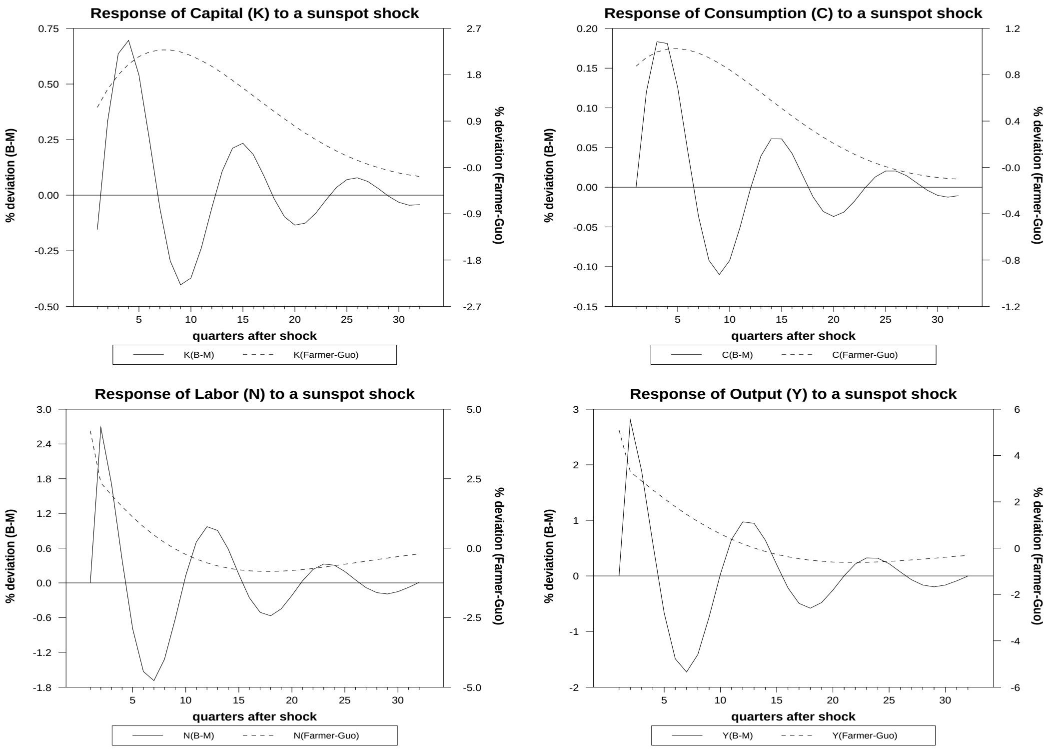

Figure 2: Impulse response function for the multi-input and the standard Farmer and Guo [7] Models. For the multi-input model, the IRFs have been calculated by imposing values for the standard deviation and the scaling factor on the sunspot variable so to obtain a 1% response of consumption to a 1% shock in the sunspot.

A sunspot shock leads to an increase in capital, consumption, equilibrium employment and production output. Notice that all responses show a wave-like pattern reverting to the stationary state; this is due to the imaginary part of the dynamic system roots. Also the magnitude of the response is consistent with the actual data for the US economy.

Figure 2 also offers a comparison with the dynamic behavior of Farmer and Guo [7] model, highlighting both similarities and some important difference. The presence of imaginary eigenvalues makes the dynamic patterns of the two models somehow similar; the two models reaction are pretty similar as far as amplitude is concerned; but the responses differ with respect to the period of the endogenous oscillations. The multi-input model shows a frequency of the oscillations higher than that of Farmer-Guo; this is probably due to the specific calibration values for the externality parameters and to the presence of the of a second input, magnifying the effect of increasing returns.

Sensitivity analysis shows that the model's dynamic response appear to be rather sensitive to small changes in externality parameters. Figure 3 shows the multi-input model IRF when the values of the three externality parameters is increased by a 10%.

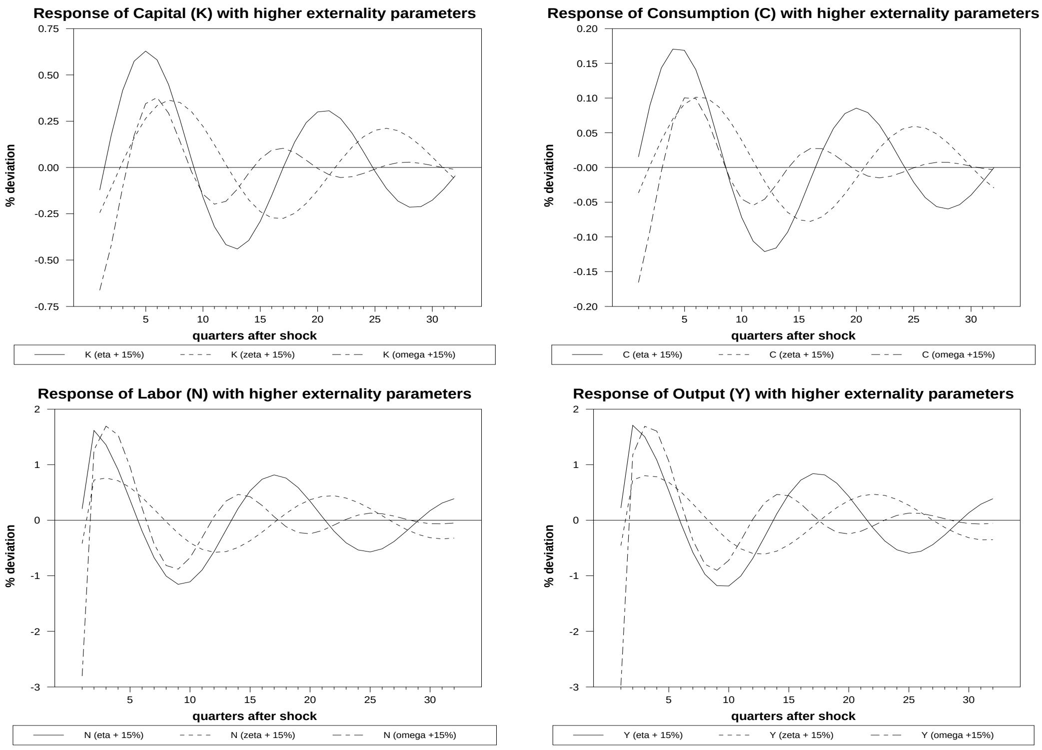

Figure 3: Sensitivity analysis. The impulse response function for the multi-input model have been calculated by imposing values for the standard deviation and the scaling factor on the sunspot variable so to obtain a 1% response of consumption to a 1% shock in the sunspot.

The impact of a greater externality on dynamic responses is both of quantitative and qualitative type. In the first case, a greater value for η, ζ or ω changes period and amplitude of the fluctuations (the same happens when an analogous sensitivity analysis is performed in the Farmer-Guo mode). An increase in the value of one the externality parameter reduces the initial impact on endogenous variables, for a small interval of variation in η, ζ and ω. In such an interval there is an asymmetry between effects of η with respect to the other two externality parameters. An increase in η determines a dynamic response of the "standard type", in the sense that a positive sunspot shock induces a self-fulfilling upturn.

Equivalent increases in the other parameters, ω and particularly ζ, seems instead to disrupt the mechanism: the initial effect is a recession[18]. This feature is common to Farmer and Guo model: also in that case relatively small changes in the externality parameter can alter the dynamic nature of the model. Anyway, the model's reactions to variations in the externality parameter are rather non-linear: for greater increases in the parameters' values - particularly in ζ - the model tends to loose local stability, giving rise to complex eigenvalues with modulus greater than one.

5 Conclusions

This paper proposes a new class of one-sector dynamic general equilibrium models with increasing returns and self-fulfilling prophecies.Two different types of labor input, each endowed with its own external effect, are explicitly included in a Farmer and Guo [7]- type model, so that increasing returns to scale can induce sunspots and indeterminacy. One of the possible interpretations of the labor heterogeneity (the one which we adopt for the model's numerical simulations) is skilled and unskilled labor. We obtain two main results.

First, we can describe fluctuations driven by self-fulfilling mechanism different form that of Farmer and Guo [7], as it relies upon the interdependency of the various inputs demand functions and it doesn't requires the latter to be ill-behaved.

Furthermore, we investigate the model's topological properties, showing how indeterminacy is possible in a wide region of the space of the labor demand elasticities values, and how relatively small increases in the externality parameters allow for significant enlargements of the same region. Also sensitivity analysis of the impulse response functions with respect to increases in the externality parameters is performed. Finally, we show the conditions on the externality parameters needed for the system to posses a closed orbit as stable set. This latter case can depict economic fluctuations driven by an intrinsic deterministic dynamics.

[18]This is probably due to the relative weight o the demand elasticity parameters α, α_2 and α_1: the latter (to which is attached the externality η) is bigger than the other two.

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Technical Appendix

All appendices are available upon request.

Appendix A

Proposition 1. Existence of a deterministic steady stateand Log-Linearized Equations,

Corollary 1. Derivation of a closed form for the deterministic steady state;

Log-linearization of each equation, and derivation of the dynamical system.

Appendix B:

Theorem 1. Necessary and sufficient condition for indeterminacy;

Corollary 2. conditions on labor elasticity wrt capital stock

Theorem 3. Hopf bifurcation; existence part.