Simultaneous Ascending Bid Auctions with Unknown Budget Constraints*

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Abstract

The mere possibility, even if arbitrarily small, of binding budget constraints can reduce competition substantially because bidders can 'pretend' to be constrained, even if they are not. In these cases, measures restricting the participation of low-budget bidders, e.g. reserve prices, can increase social welfare.

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1 Introduction

Most auction theory ignores the possibility that bidders may be willing to pay for an object more than the amount of money they have available, i.e. that bidders may be budget constrained. Yet, budget constraints can play an important role in practice. For example, David Salant (see Salant [20], page 567), reporting on his experience in the bidding team at GTE during one of the FCC auctions for the sale of spectrum licenses, writes:

> We were very concerned about how budget constraints could affect bidding. Most of the theoretical literature ignores budget constraints. In the MTA [Major Trading Area] auction, budget constraints appeared to limit bids.

Salant also explains how, in order to formulate its strategy, the GTE bidding team used a simulation model in which possible budget levels of the different bidders entered as inputs.

In principle, if the bidders are interested in the objects for investment purposes (this was the case in the spectrum license auctions), and have access to well functioning capital markets, budget constraints should not matter. However, frictions in capital markets often make the amount of available internal funds relevant. Moreover, even when external funding is available at profitable rates, a bidder may be reluctant to borrow from a third party, because this might require disclosure of private information about its valuation for the goods, which in turn may put the bidder at a disadvantage in the auction. Also, a bidder may want to *choose* to be budget-constrained, in order to commit to a less aggressive bidding strategy and thus induce better outcomes in terms of final prices. A recent paper by Benoît and Krishna [5] highlights this effect by showing that in fact, at least in some cases, budget constraints may arise endogenously. Finally, financial constraints may emerge endogenously when bidders act as agents of financing principals (see for example Bolton and Scharfstein [3], or Holmström and Ricart i Costa [12]). These considerations provide good theoretical and empirical reasons to think that budget constraints play important roles in auctions.

The introduction of budget constraints in theoretical models of auctions is fairly recent. Pioneering work in this area is due to Che and Gale, [6] and [7]. They have analyzed single-object environments where each buyer has private information about both her willingness and her (possibly lower) ability to pay. One important insight that emerges from Che and Gale's work is that having a buyer with a budget w and a value v for an object is not the same, in general, as having a buyer with value $\min\{v, w\}$. Single-object second-price auctions with budget constrained bidders have been studied in Fang and Parreiras [10] and [11]. Zheng [23] studies a common value, (single object) first-price auction model in which the bidders can borrow at a given rate and default. Rhodes-Kropf and Viswanathan [18] analyze single object first-price auctions with privately known values and budgets, in which the bidders can finance their bids with cash or securities. Multiple objects auctions with budget constrained are analyzed in Benoît and Krishna [5], but only under the assumption of complete information about both willingness to pay and budgets.

We will study multi-unit simultaneous ascending-bid auctions under Che and Gale's information structure, i.e. under the assumption that each bidder has private information about both her willingness to pay and her budget. Since 1994, multi-unit simultaneous ascending-bid auctions have been used repeatedly by the US government to sell licenses for the use of parts of the electromagnetic spectrum. In a previous paper (Brusco and Lopomo [4]) we have shown that, for a large class of information and preference structures, these auctions provide the bidders with ample opportunities for collusion. The basic idea is that, for many distributions of the bidders' values, trying to win two objects often yields less expected surplus than buying a single object at a relatively low price.[1]

In the present paper we will focus on the effect that the *possibility* (even if small) of binding budget constraints has both on the highest level of competition sustainable in equilibrium. It is obvious that the presence of budget constrains reduces the maximum bids that bidder can post. But the possibility of budget constraints induces an additional 'demand reduction' effect, similar to the one seen in uniform price auctions. Once prices reach levels at which a budget-constrained bidder is unable to buy more than one object, a high-budget opponent can end the auction immediately by simply letting the low-budget bidder win one object. This is more profitable than trying to buy two objects for the high-budget bidder if the willingness to pay for a second object is relatively low.

Demand reduction effects in multiunit sealed-bid auctions with uniform pricing have been noted by Ausubel and Cramton [1] and Englebrecht-Wiggans and Kahn [9]; and the idea is also present in Wilson [22]. We study open ascending bid auctions, in which the prices of the objects need not be equal. Absent budget constraints, in such auctions there is an equilibrium in which the bidders simply raise the bid on each object up to their values, so that no demand reduction occurs. Therefore, in our model, the demand reduction effect is entirely attributable to the presence of

[1] The experimental results in Kwashnica and Sherstyuk [15] corroborate our theoretical results in the case with no complementarities. For a survey on recent experimantal work on collusion in mult-unit ascending bid auctions, see Sherstyuk [21]. There is now general consensus that collusion in mult-unit ascending bid auctions is empirically relevant. See, for example, Cramton and Schwartz [8], or Klemperer [13].

potentially binding budget constraints.

We find that for a large class of distributions, even if the probability of having potentially binding budget constraints is arbitrarily small, *all* high-budget types behave as if they were budget constrained, hence the bidders' behavior will be indistinguishable from the case in which it is common knowledge that all bidders are budget constrained. In these cases, imposing a reservation price for each object which is high enough to exclude any low-budget bidder from the auction increases not only the seller's revenue, but also the expected social surplus[2]. Without budget constraints, reservation prices unambiguously reduce social surplus because they prevent potential gains from trade from being realized. With potentially binding budget constraints however, there are distributions for which, even in non-collusive equilibria, the bidders split the objects, thus lowering the social surplus. Sufficiently high reservation prices in this case would prevent budget constrained bidders from participating in the bidding, thus making it common knowledge that all active bidders are unconstrained. Therefore, in a non-collusive equilibrium, each object ends up in the hands of a bidder with the highest value. For sufficiently small probabilities of having binding budget constraints, the expected gain in social surplus due to the better allocation of the objects is larger than the expected loss due to the exclusion of budget constrained types.

The insights of our model can also be applied to other situations in which two players compete for multiple 'prizes', and each player's type is characterized by two variables, one measuring the value attached to the prizes, and the other referring to a resource constraint which may or may not preclude the possibility of winning multiple prizes. Our analysis suggests that the mere possibility, no matter how unlikely, that each player may face a tight resource constraint can induce a significant reduction in competition, even in non-collusive equilibria. For example, the presence of capacity constraints in the multi-market contact model developed by Bernheim and Whinston [2] may induce firms to specialize in separate markets, i.e. to behave in a seemingly collusive fashion, even though they are using non-collusive equilibrium strategies.

Going outside the realm of economics, consider a military game in which two armies are trying to occupy two islands. Suppose that each army has private information about its military capacity, e.g. each army may be 'small' or 'large', small meaning able to occupy at most one island. This strategic situation is similar to the one we analyze in this paper, with the small army playing a

[2]Cramton and Schwartz [8] suggest that reservation prices may be used to upset collusion in multi-unit auctions. Their paper contains an example with complete information. We show that reservation prices can increase welfare in noncollusive equilibria when the possibility of binding budget constraints is admitted.

role similar to the budget constrained bidder. Our results suggest that, even if it is ex ante very unlikely that each army is small, the final outcome can entail a low degree of competition, with each army occupying one island.

The rest of the paper is organized as follows. Section **??** presents the model. For simplicity, we consider only two objects and two bidders, with constant marginal willingness to pay. Section 3 presents the main results, and section 5 concludes. All proofs are relegated to an appendix.

2 The Model

There are two objects, and two bidders. Each bidder $i = 1, 2$ is characterized by a type $\theta_i := (v_i, w_i)$, where v_i denotes the utility of each object and w_i is the maximum amount of money that she can spend in the auction. Therefore, the utility of a bidder who obtains n objects paying a total amount of m is $nv_i - m$, and m cannot exceed w_i.

The four variables (v_1, v_2, w_1, w_2) are independently distributed, with support $[0,1]^2 \times W_1 \times W_2$. The c.d.f. F of each variable v_i has a differentiable density f. The sets W_i are two point sets, $W_i = \{w_L, w_H\}$, with $w_L < w_H$, and we define $\Pr[w_i = w_L] := \lambda \in (0, 1)$.

The objects are sold using a "simultaneous ascending bid auction", which is a natural extension of the standard one-object English auction to environments with multiple objects. In each round $t = 1, 2, \ldots$, for each object $j = 1, 2$, each bidder i can either stay silent or raise the highest bid of the previous round by at least a minimum amount $\varepsilon > 0$. Formally, i's bid on object j in round t, denoted by $b_j^i(t)$, can either be $-\infty$, which is to be interpreted as "stay silent", or must be a number in the interval $[b_j(t-1) + \varepsilon,\ +\infty)$, where $b_j(t-1)$ denotes the "current outstanding bid", defined recursively by:

$$b_j(0) = 0 \quad \text{and} \quad b_j(t) := \max\left\{b_j(t-1),\ b_j^i(t);\ i \in N\right\}.$$

If at least one bidder increases the outstanding bid on at least one object, i.e. if $b_j(t) > b_j(t-1)$ for some j, then for each of these objects the new highest bid is identified, a potential winner is selected among the bidders who have made the new highest bid, and the auction moves to the next round, with the potential winner of all other objects unchanged. If instead all bidders stay silent on all objects, the auction ends, and each object is sold to the winner selected at the end of the previous round, for her last bid.

In our analysis we will consider the minimum bid increment ε negligibly small. This will simplify the statements and proofs of our propositions, essentially by eliminating the need to consider sub-cases in which a bidder's value is larger than the current outstanding bid but smaller than the current bid plus the minimum increment.

If $w_L \geq 2$, i.e. if each bidder's budget is above the highest total amount that she may be willing to spend in the auction, then the model is a special case of the model studied in Brusco and Lopomo [4]. In that paper we have established the existence of collusive equilibria which are sustained by the threat of reverting to non-collusive continuation strategies. Our focus here is on the effect that the possible presence of budget constraints has on the auction's equilibrium set. Thus we assume, without loss of generality, that $w_L < 2$. In order to simplify analysis we also assume $w_H > 2$ and $w_L > 1$. The first inequality implies that an high-budget bidder is financially unconstrained. The second implies that even a budget-constrained bidder has always enough money to bid up to her valuation on a single object. The assumption that budget constrained bidders can always bid up to their value for a single object (i.e. $1 < w_L$) is not essential, but it simplifies the analysis (by limiting the number of sub-cases to be considered).

To keep the formal statements of our results as simple as possible, we will often write that a given strategy profile σ "forms an equilibrium" to mean that there exists a consistent belief system μ such that the pair (σ, μ) constitutes a perfect Bayesian equilibrium. In most cases, given a strategy profile σ it will be easy to find a consistent belief system which supports σ as an equilibrium. We will be explicit about the belief system that goes together a given strategy profile only in some of our proofs.

3 The Effect of Budget Constraints

When there are no budget constraints (that is, $\lambda = 0$) there is a straightforward equilibrium in which each bidder bids up to her valuation on each object. This is a consequence of the fact that there are no complementarities and the prices of the two objects can be set independently. While this is not the unique equilibrium (see Brusco and Lopomo [4]), this is the equilibrium that guarantees the efficient allocation of the objects and maximizes the revenue of the seller subject to incentive compatibility. It is therefore of central interest to know whether an equilibrium with a similar outcome exists when the possibility of budget constraints is introduced.

To make the question more precise, observe that the probabilility that both agents are unconstrained is $(1-\lambda)^2$. In principle, when both agents are unconstrained it is a feasible outcome that both objects end up in the hands of the bidder who values them most. The issue is whether there is an equilibrium supporting this outcome, that is an equilibrium in which both objects go to the bidder with the highest value when both bidders are unconstrained. In particular, is it true that as λ goes to zero the probability that the bidder with the highest value gets both objects goes to one? We will show that the answer to the qustion is negative: as λ goes to zero, the probability that the objects are inefficiently split remains bounded away from zero.

3.1 Efficiency, Incentive Compatibility and Budget Constraints

Let us first define the efficient allocation in the presence of budget constraints. Let $h_L := \frac{w_L}{2}$ be half of the budget of the 'poor' bidder. Given our auction format, if the two bidders have values $v_i > h_L$ and they are both budget constrained then the only feasible allocation is that both bidders get exactly one object. The reason is that a bidder is unwilling to end the auction with zero objects unless the price for *both* objects is greater than v_i. But if $v_i > h_L$ and both prices are above v_i then a budget-constrained opponent does not have money enough to pay for both objects. Therefore, when both agents are budget constrained and $\min\{v_1, v_2\} > h_L$, each bidder must have exacty one object. Furthermore, the price paid must be h_L, as at that point (that is, when both prices reach h_L) the there is no excess demand for the objects.

Similarly, suppose that one bidder is unconstrained (say, $w_1 = w_H$) and the other is constrained (say, $w_2 = w_L$). Then, when $v_2 > v_1 > h_L$ it must again be the case that the the objects are split. The reason is the same as before: bidder 1 will not accept to let the auction end and win no object unless the prices of both objects are above v_1; but, since $v_1 > h_L$, this implies that bidder 2 will be able to pay for at most one object. In this case, the price for both objects will be v_1. These are natural restrictions on the feasible allocatons. Other than that, we would like both objects to end up in the ends of the highest type, paying for each object a price equal to the valuation of the lower type.

The question is whether this can be the outcome of a perfect Bayesian equilibrium of our auction. As mentioned earlier the answer is no. As a matter of fact, we are going to prove a more general result: No mechanism can implement this allocation, as it is not incentive compatible.

In order to prove that incentive compatibility fails, let us compute the expected utilities. Let

$U_1(v_1, w_1, v_2, w_2)$ be the utility of bidder 1 under the proposed allocation. We have:

$$U_1(v_1, w_H, v_2, w_2) = \begin{cases} 2(v_1 - v_2) & \text{if} \quad v_1 > v_2 \\ 0 & \text{if} \quad v_1 < v_2 \end{cases}$$

To see that the utility is zero whenever $v_1 < v_2$ notice that if $v_1 < h_L$ then the opponent will get both objects at v_1, while if $v_1 > h_L$ then each bidder will have one object at a price of v_1. In both cases the utility of the first bidder is zero. Consider now the low-budget bidder. We have

$$U_1(v_1, w_L, v_2, w_L) = \begin{cases} 2(v_1 - v_2) & \text{if} \quad v_1 > v_2 \text{ and } v_2 < h_L \\ v_1 - h_L & \text{if} \quad v_2 > h_L \text{ and } v_1 > h_L \\ 0 & \text{otherwise} \end{cases}$$

$$U_1(v_1, w_L, v_2, w_H) = \begin{cases} 2(v_1 - v_2) & \text{if} \quad v_1 > v_2 \text{ and } v_2 < h_L \\ v_1 - v_2 & \text{if} \quad v_1 > v_2 \text{ and } v_2 > h_L \\ 0 & \text{otherwise} \end{cases}$$

For this to be the outcome of a Bayesian equilibrium of some mechanism, it must be the case that in the direct mechanism in which the bidders report their types *and* their budgets, truthtelling is an equilibrium.

In a direct mechanism the bidders announce a pair (v_i, w_i), and the allocation is decided according to the above-mentioned rules. If the opponent is behaving truthfully then it is clearly optimal, *for any given budget announcement*, to tell the truth about the valuation v_i. The issue is therefore whether it can be profitable to announce a false budget. The assumption of 'money on the table' implies that the only deviation we have to worry about is that of an high-budget bidder pretending to be a low-budget bidder. We therefore have to check the following inequality:

$$E[U(v_1, w_H, v_2, w_2)] \geq E[(v_1, w_L, v_2, w_2)],$$

for each value v_1. The two sides of the inequality differ only if $v_1 > h_L$. In this case the inequality becomes:

$$\begin{aligned} 2\int_0^{v_1} (v_1 - v_2) f(v_2)\, dv_2 \quad \geq \quad & (1-\lambda)\left[\int_0^{h_L} 2(v_1 - v_2) f(v_2)\, dv_2 + \int_{h_L}^{v_1} (v_1 - v_2) f(v_2)\, dv_2\right] \\ & +\lambda\left[\int_0^{h_L} 2(v_1 - v_2) f(v_2)\, dv_2 + \int_{h_L}^{1} (v_1 - h_L) f(v_2)\, dv_2\right]. \end{aligned}$$

After rearrangement we obtain:

$$(1-\lambda)\int_{h_L}^{v_1} (v_1 - v_2) f(v_2)\, dv_2 \geq \lambda\left[(v_1 - h_L)(1 - F(h_L)) - 2\int_{h_L}^{v_1} (v_1 - v_2) f(v_2)\, dv_2\right] \quad (1)$$

This can be interpreted as follows. By pretending to be low-budget, a bidder in hgeneral loses when the opponent is high budget but has a chance to gain when the opponent is low-budget. On the RHS of inequality (1) we have the expected utility loss when a bidder pretends to be low-budget and meets a high-budget opponent. In this case she buys one object rather than 2, and pays the price v_2 whenever $v_2 < v_1$. One the LHS we have the net expected gain when a low-budget opponent is met. In this case, by pretending to be low-budget the bidder is able to get one object at a price h_L for each $v_2 \geq h_L$. Notice that in this case a positive utility of $v_1 - h_L$ is obtained even in the case $v_2 > v_1$. This has to be compared to the utility attained when the bidder declares a budget w_H and a low-budget opponent is met, which is $2\int_{h_L}^{v_1}(v_1 - v_2) f(v_2)\,dv_2$. Inequality (1) therefore requires that the expected cost from falsely announcing w_L be greater that the expected benefit.

At $v_1 = h_L$ the two sides are worth zero, so that the inequality is satisfied. Consider the function:

$$\psi(v_1, \lambda) = (1+\lambda)\int_{h_L}^{v_1}(v_1 - v_2) f(v_2)\,dv_2 - \lambda(v_1 - h_L)(1 - F(h_L))$$

Then the allocation is incentive compatible if $\psi(v_1, \lambda) \geq 0$ for each $v_1 \geq h_L$. For each λ we have $\psi(h_L, \lambda) = 0$ and:

$$\psi'(v_1, \lambda) = (1+\lambda)[F(v_1) - F(h_L)] - \lambda(1 - F(v_1))$$

At $v_1 = h_L$ we have $\psi'(h_L, \lambda) = -\lambda(1 - F(h_L))$. This implies that for each $\lambda > 0$, there will be a right neighborhood of h_L over which the incentive compatibility condition is not satisfied, and the high budget bidder finds it convenient to pretend to be a low budget bidder. The intuition can be better captured rewriting inequality (1) as:

$$(1-\lambda)\int_{h_L}^{v_1}\frac{v_1 - v_2}{v_1 - h_L} f(v_2)\,dv_2 \geq \lambda\left[(1 - F(h_L)) - 2\int_{h_L}^{v_1}\frac{v_1 - v_2}{v_1 - h_L} f(v_2)\,dv_2\right] \tag{2}$$

where the gains and losses are written as a proportion of $(v_1 - h_L)$. As v_1 converges to h_L the two integrals converge to $F(v_1) - F(h_L)$ (this is just the derivative of $\int_{h_L}^{v_1}(v_1 - s) f(s)\,ds$ with respect to v_1) and therefore to zero. Thus, for v_1 sufficiently close to h_L the only term that matters is $\lambda(1 - F(h_L))$. Intuitively, when v_1 is close to h_L the loss in utility due to the fact that one object less is bought at a price $v_2 \in (h_L, v_1)$ is of second order with respect to the gain in utility resulting from the fact that one object more is acquired at a price h_L for each $v_2 > v_1$ when the opponent has a low budget.

The conclusion is that the proposed allocation is not incentive compatible: there is an interval (h_L, v^*) such that types (v_i, w_H) with $v_i \in (h_L, v^*)$ who are better off announcing (v_i, w_L).

How important is this effect? The exact value of v^* depends on λ, and one may ask if the effect becomes negligible when the probability of meeting a low-budget opponent goes to zero. In other words, does v^* converge to h_L when λ goes to zero? We now show that this is not the case, that is $\lim_{\lambda\to 0} v^*(\lambda) > h_L$.

If the value of v^* belongs to the interval $(h_L, 1)$ then it must be the value at which a high-budget bidder is indifferent between announcing w_H and w_L *when all the high-budget bidders with valuation in* (h_L, v^*) *announce* w_L. Otherwise, we set $v^* = 1$, so that all high-budget bidders are better off announcing a low budget. If it is true that $\lim_{\lambda\to 0} v^*(\lambda) = h_L$ then, for λ sufficiently small, the value of $v^*(\lambda)$ is determined by the following equation:

$$\begin{aligned} 2\int_{h_L}^{v^*} (v^* - v_2)\, f(v_2)\, dv_2 &= (1-\lambda)\left[\int_{h_L}^{v^*} (v^* - h_L)\, f(v_2)\, dv_2\right] \\ &\quad +\lambda\left[\int_{h_L}^{1} (v^* - h_L)\, f(v_2)\, dv_2\right]. \end{aligned}$$

or:

$$2\int_{h_L}^{v^*} (v^* - v_2)\, f(v_2)\, dv_2 = (v^* - h_L)\left[(F(v^*) - F(h_L)) + \lambda(1 - F(v^*))\right].$$

One solution to this equation is h_L and we look at the solution larger than h_L, if it exists. Define:

$$\Gamma(v^*) = 2\int_{h_L}^{v^*} (v^* - v_2)\, f(v_2)\, dv_2 - (v^* - h_L)\left[F(v^*) - F(h_L) + \lambda(1 - F(v^*))\right].$$

Then:

$$\Gamma'(v^*) = [F(v^*) - F(h_L)] - \lambda(1 - F(v^*)) - (v^* - h_L)(1-\lambda)\, f(v^*).$$

so that $\Gamma'(h_L) = -\lambda(1 - F(h_L)) < 0$. The second derivative is:

$$\Gamma''(v^*) = 2\lambda f(v^*) - (v^* - h_L)(1-\lambda)\, f'(v^*).$$

Define $v^0(\lambda)$ be the first point at which $\Gamma'(v^0) = 0$. Since we are assuming that $v^*(\lambda) \in (h_L, 1)$ such a point must exists. In fact, since $\Gamma(h_L) = 0$ and $\Gamma'(h_L) < 0$, it must be the case that $v^*(\lambda) > v^0(\lambda)$, that is the first point at which the function reaches a value of zero must be to the right of the first local minimum. Therefore, if we can prove $\lim_{\lambda\to 0} v^0(\lambda) > h_L$ we are done. The function $v^0(\lambda)$ satisfies the equation:

$$\left[F\left(v^0\right) - F(h_L)\right] = \lambda\left(1 - F\left(v^0\right)\right) + \left(v^0 - h_L\right)(1-\lambda)\, f\left(v^0\right) \tag{3}$$

for each λ. Since we have assumed that the density f is differentiable, $v^0(\lambda)$ is differentiable. Suppose that than $\lim_{\lambda \to 0} v^0(\lambda) = h_L$. Dividing both sides by λ we have:

$$\frac{F(v^0) - F(h_L)}{\lambda} = \left(1 - F\left(v^0\right)\right) + \frac{(v^0 - h_L)}{\lambda}(1-\lambda) f\left(v^0\right)$$

Taking the limit as $\lambda \to 0$ on both sides and assuming $v^0 \to h_L$ we obtain:

$$f(h_L) \left.\frac{dv^0}{d\lambda}\right|_{\lambda=0} = (1 - F(h_L)) + f(h_L) \left.\frac{dv^0}{d\lambda}\right|_{\lambda=0}$$

which is impossible since $h_L < 1$.

The conclusion is that the measure of the set of types who inefficiently split the objects remains bounded away from zero as λ goes to zero, that is $\lim_{\lambda \to 0} v^*(\lambda) \geq \lim_{\lambda \to 0} v^0(\lambda) > h_L$.

Since at $\lambda = 0$ there is a Bayesian equilibrium in which the objects are allocated efficiently, it is important to understand why there is such a discontinuity. The reason is that at $\lambda > 0$, no matter how small, the new incentive compatibility constraint (1) has to be satisfied. We have seen that for every λ there is an interval $\left[h_L, v^{(1)}\right]$ of high-budget types who do not satisfy the constraint. Thus, the probabilty that a bidder faces an opponent who behaves as a low-budget type goes to $\lambda + (1-\lambda)\left[F\left(v^{(1)}\right) - F(h_L)\right]$. We can now compute another interval $\left[h_L, v^{(2)}\right]$ of high-budget types who prefer to behave as low-budget. Continuing this process, we converge to a value $v^*(\lambda)$ such that all high-budget types in the interval $[h_L, v^*(\lambda)]$ prefer to behave as low-budget types. In other words, given any $\lambda > 0$ there is a 'multiplier effect': some high-budget types will always pretend to be low-budget, and this in turn induces more high-budget types to pretend to be low-budget and so on. When $\lambda = 0$ the incentive compatibility condition 1 disappears, and there is no 'starting point' on which to build the multiplier effect.

4 Equilibria in the Auction

The previous analysis implies that, when budget constraints are possible, some inefficient splitting of the objects occurs even in the case in which both agents happen to be unconstrained.

In this section we want to analyze the existence of perfect Bayesian equilibria implementing the allocation described above (that is, the objects are split if the types are low-budget or high-budget with a type in an interval $[h_L, v^*]$, for some $v^* > h_L$).

The bidders' behavior on the equilibrium path is as follows. The auction starts with both outstanding bids increasing at the same pace, up to $\min\{v_1, v_2, h_L\}$. More precisely, for each

$i = 1, 2$, bidder i increases the outstanding bid by the minimum increment ε on object i in any odd round, and on object $3 - i$ in any even round, up to $\min\{v_i, h_L\}$. Thus the auction progresses with each bidder being the potential winner of one object in each round, until the outstanding bids reach either the lowest of the bidders' values, or h_L. In the first case, the bidder with the lowest value stays silent, and the auction ends with her opponent winning both objects. Otherwise, i.e. if $h_L < \min\{v_1, v_2\}$, the behavior of each bidder depends on whether her type is "*tough*", i.e. high-budget and with value above a threshold $v^* > h_L$, or "*soft*", i.e. either low-budget, or high-budget and with value between h_L and v_*. Once the bids reach h_L, all soft types remain silent. All tough types instead continue to raise the outstanding bid on any object which is assigned to the opponent, up to a threshold which depends on the opponent's behavior.

Thus, if both bidders are soft, they stay silent and the auction ends with each bidder buying one object and paying h_L. If both bidders are tough, the bidding continues as in the initial phase and the bidder with the highest value wins both objects. Finally, if one bidder is tough and the other is soft, the soft bidder starts to bid "defensively," i.e. she bids on one of the objects with the lowest current outstanding bid if she is losing both objects, and stays silent otherwise. The tough bidder instead tries to win both objects until the bids reach an optimally chosen threshold. The auction then ends with the tough bidder buying both objects if her threshold is above her opponent's value, and with the bidders splitting the objects otherwise.

Define the "stopping" function $s(v_i, w_i; b_1, b_2)$ which determines the highest price that a bidder of type (v_i, w_i) who has observed a pair of bids (b_1, b_2) is willing to pay in order to get both objects. Notice that with incomplete information on the budget levels, each bidder's optimal stopping time depends on her opponent's behavior. While it is clear that a low-budget opponent will bid defensively, the behavior of a high-budget opponent will depend on her beliefs. Some, but not necessarily all, high-budget types will play "soft," thus mimicking the behavior of low-budget bidders. Therefore each bidder needs to formulate a conjecture about the stopping function used by her opponent in order to compute her own stopping function, and the symmetric equilibrium stopping function has to be computed as the solution of a fixed point problem.

For all low budget types things are simple. The function $s(v_i, w_L; b_1, b_2)$ prescribes to stop immediately if any of the bids is above $\min\{v_i, h_L\}$, and keep trying to buy both objects otherwise. Since the function s is defined for $v_i \geq h_L$, we have:

$$s(v_i, w_L; b_1, b_2) \equiv v_i.$$

For the high budget types things are more complicated. We first characterize the optimal stopping function on the equilibrium path. Suppose that the current bids are (b, b), with $b > h_L$, and that, when the bids reached the level (h_L, h_L), bidder 1 played 'tough', meaning she raised the bid on the object she was not winning, and bidder 2 played 'soft', i.e. she remained silent. The beliefs of the two bidders are as follows. Bidder 2 believes that $w_1 = w_H$ with probability 1, and that v_1 is distributed according to the posterior c.d.f. determined by the optimal stopping function and by F. Bidder 1's beliefs about her opponent's are as follows:

$$\Pr\left(w_2 = w_L \middle| \, \text{soft}\right) = \frac{\lambda\left[1 - F\left(b\right)\right]}{\lambda\left[1 - F\left(b\right)\right] + \left(1 - \lambda\right)\max\left\{F\left(v_*\right) - F\left(b\right), 0\right\}},$$

$$\Pr\left(w_2 = w_H \middle| \, \text{soft}\right) = \frac{\left(1 - \lambda\right)\max\left\{F\left(v_*\right) - F\left(b\right), 0\right\}}{\lambda\left[1 - F\left(b\right)\right] + \left(1 - \lambda\right)\max\left\{F\left(v_*\right) - F\left(b\right), 0\right\}};$$

and the conditional densities on v_2 are:

$$g\left(v_2 \middle| \, b, w_L\right) \equiv \begin{cases} \frac{f(v)}{1-F(b)} & v_2 \in \left[b, 1\right], \\ 0 & \text{otherwise;} \end{cases}$$

$$g\left(v_2 \middle| \, b, w_H\right) \equiv \begin{cases} \frac{f(v)}{F(v_*)-F(b)} & v_2 \in \left[\min\left\{b, v_*\right\}, v_*\right], \\ 0 & \text{otherwise.} \end{cases}$$

Letting $G\left(\cdot \middle| \, \cdot, \cdot\right)$ denote the c.d.f. corresponding to the densities g, we can write bidder 1's expected surplus from stopping at s as:

$$\begin{aligned} U\left(v_1, s; v_*, b\right) \;=\;& \Pr\left(w_L \middle| \, \text{soft}\right)\left[2\int_b^s \left(v_1 - y\right)\, dG\left(y \middle| \, b, w_L\right) + \left(v_1 - s\right)\left[1 - G\left(s \middle| \, b, w_L\right)\right]\right] \\ &+ \Pr\left(w_H \middle| \, \text{soft}\right)\left[2\int_b^s \left(v_1 - y\right)\, dG\left(y \middle| \, b, w_H\right) + \left(v_1 - s\right)\left[1 - G\left(s \middle| \, b, w_H\right)\right]\right] \end{aligned}$$

After substitutions and ignoring multiplicative constants we can write the objective function as:

- for $b < v_*$:

$$\begin{aligned} U\left(v_1, s; v_*, b\right) = \lambda\left[2\int_b^s \left(v_1 - y\right)\, dF\left(y\right) + \left(v_1 - s\right)\left[1 - F\left(s\right)\right]\right] \quad (4) \\ + \left(1 - \lambda\right)\left[2\int_b^{\min\{s, v_*\}} \left(v_1 - y\right)\, dF\left(y\right) + \left(v_1 - s\right)\left[F\left(v_*\right) - F\left(\min\left\{s, v_*\right\}\right)\right]\right]; \end{aligned}$$

- for $b \geq v_*$:

$$U\left(v_1, s; v_*, b\right) = 2\int_b^s \left(v_1 - y\right) dF\left(y\right) + \left(v_1 - s\right)\left[1 - F\left(s\right)\right]. \quad (5)$$

We now define the set

$$R(v_1, w_H; v_*, b) \equiv \arg\max_s \; U(v_1, s; v_*, b), \tag{6}$$

and, since we are interested in the 'most competitive' equilibrium, we consider the 'stopping rule' given by:

$$r(v_1; v_*, b) \equiv \max \; R(v_1, w_H; v_*, b). \tag{7}$$

When v_* is an equilibrium value, and the bids $b_1 = b_2 = b$ are reached along the equilibrium path, we set:

$$s(v_1, w_H; b, b) \equiv r(v_1; v_*, b).$$

For any given threshold v_* we can compute the expected surplus of each player from playing 'tough' and 'soft' (once the bids reach h_L) when her opponent conjectures that all high-budget types above v_* play tough and all high-budget types below v_* play soft. In equilibrium, v_* must be such that the conjecture is confirmed. We now discuss more in depth the existence of an equilibrium with the characteristics just described.

4.1 Existence of the Equilibrium

In order to complete the analysis we have to accomplish two tasks. First, we have to show that a threshold value v_* exists, i.e. we have to show that a fixed point exists. That is, it must be true that when bidder 1 conjectures a threshold value v_* for the opponent, then all types $v_1 < v_*$ are willing to play soft and all types $v_1 > v_*$ are willing to play tough. As we will see, this requires an additional assumption. Second, we have to describe the out of equilibrium behavior.

For the moment, let it be taken for granted that a threshold value v_* exists, so that an optimal stopping function $s(v_i, w_H; b, b)$ can be computed along the equilibrium path. We now proceed to generalize the bidding behavior for any arbitrary pair (b_1, b_2).

As in the previous two sections, we specify that a bidder stays silent when winning both objects, and raises the bid on (one of) the lowest priced object(s) when losing both objects (provided the lowest bid is below her value). Thus, it remains to specify the behavior at pairs (b_1, b_2) at which the two bidders are winning one object each.

Consider the following three cases:

1. $\max\{b_1, b_2\} < h_L$. In this case the strategies are as in the standard 'competitive' equilibrium, i.e. $s(v_i, w_i; b_1, b_2) = v_i$.

2. $\max\{b_1, b_2\} \geq h_L$ and (b_1, b_2) can be reached on the equilibrium path. In this case the beliefs are updated using Bayes' rule. The stopping rule for all types with $v_i \geq v_*$ remains the same. Those with $v_i < v_*$ play defensively and have no interest in triggering the 'competitive' equilibrium, since the opponent has a higher value.

3. $\max\{b_1, b_2\} \geq h_L$ and (b_1, b_2) is out of the equilibrium path. We specify the beliefs so that, whenever a bidder observes the other deviating, she puts probability 1 on $w_i = w_H$, and this belief is maintained in case further deviations are observed. Furthermore, low-budget bidders cannot hope to win both objects, so that they stay silent whenever they win at least one. Note that this implies that any attempt on the part of a bidder to buy both objects signals that the bidder has a high budget. There are 3 sub-cases, depending on how many bidders have deviated.

   (a) Both bidders deviated from the prescribed strategy. In this case both bidders assign probability 1 to the fact that the other bidder has a high budget, and this fact is common knowledge. In this case the 'competitive' equilibrium is triggered. Therefore we set $s(v_i, w_H; b_1, b_2) = v_i$.

   (b) If the other bidder deviated then bidder i assigns probability 1 to $v_{3-i} = \widehat{b}$, where $\widehat{b}$ is the highest bid ever made by bidder $3-i$, and assumes that bidder $3-i$ will never make a bid on any object if she becomes convinced that the type of the other bidder is higher. Since by making a bid on the other object bidder i signals that her type is greater than $\widehat{b}$ (remember that $\max\{b_1, b_2\} \geq \widehat{b}$ and i bids on both objects) then it is rational for i to bid myopically on both objects, i.e. assuming that the other bidder will not make any further bid. This in turn justifies a myopic behavior on the part of bidder $3-i$. Notice that this cannot make a deviation profitable, since by deviating bidder $3-i$ only obtains a more aggressive behavior on the part of bidder i. We can therefore set $s(v_i, w_H; b_1, b_2) = v_i$ in this case as well.

   (c) The last case we have to deal with is the one in which a deviation occurred only on part of agent i. Since $\max\{b_1, b_2\} > h_L$ any counterbid by $3-i$ signals that she is of type w_H and $v_{3-i} \geq \widehat{b}(i)$, where $\widehat{b}(i)$ is the highest bid ever made by i up to that round. Also, in that case bidder $3-i$ starts bidding myopically. Then bidding myopically is a best reply on part of agent i. The conclusion is that in this case as well we can set $s(v_i, ; v_*, b_1, b_2) = v_i$.

Essentially, out of the equilibrium path the bidders raise the bids whenever the value of the object is superior to their current bids. Along the equilibrium path, the bidders adopt optimal stopping times.

We now come to the issue of the existence of a threshold value v_*. Let $\mu(v_i) \equiv \frac{1-F(v_i)}{f(v_i)}$ be the inverse hazard rate. We make the following assumption.

Assumption 1 *For each $v_i > h_L$, we have:*

$$2f(v_i) \geq f(v_i + \mu(v_i)) \left[1 + \mu'(v_i)\right]$$

whenever $v_i + \mu(v_i) < 1$ and $1 + \mu'(v_i) > 0$.

Assumption 1 is immediately satisfied when either $v_i + \mu(v_i) \geq 1$ or $1 + \mu'(v_i) \leq 0$ for each $v_i \geq h_L$. The latter holds, for example, in the uniform case. More generally, if $\mu'(v_i) \leq 0$ (non-decreasing hazard rate), then a sufficient condition for Assumption 1 is that, for all $v_i > h_L$, we have

$$2f(v_i) \geq f(x), \text{ for each } x > v_i.$$

This is satisfied by any distribution without large peaks. In particular, if $v_{\min}$ and $v_{\max}$ are respectively the points at which the density achieves the maximum and the minimum over the interval $[h_L, 1]$ then a sufficient condition is:

$$2f(v_{\min}) \geq f(v_{\max}).$$

The next proposition characterizes the non-collusive equilibrium.

Proposition 1 *If Assumption 1 is satisfied, then there exists a value v_* and a corresponding stopping function $s(v_i, w_i; b_1, b_2)$ such that the following strategy profile forms an equilibrium. At any stage in which the current outstanding bids are b_1 and b_2, each type (v_i, w_i) of bidder i increases the bid by the minimum increment:*

- *on the object with the lowest outstanding bid, breaking ties in favor of object i, if she is not the winner on any object, the current outstanding bids are different, and*

$$\min\{b_1, b_2\} < \min\{v_i, w_i\};$$

- *on object j, if she is winning object $3-j$ only, and*

$$b_j < \min\left\{s\left(v_i, w_i;\, b_1,\, b_2\right),\, \frac{w_i}{2},\, w_i - b_{3-j}\right\};$$

- *on no object, otherwise.*

As in the case with known and asymmetric budgets, a "demand reduction" effect is present in the non-collusive equilibrium of Proposition 1. On the equilibrium path, the stopping function is such that $s\left(v_i, w_H;\, h_L,\, h_L\right) = h_L$, for all $v_i \in [h_L, v_*)$. Thus a set of high-budget types with sufficiently low values mimic the behavior of low budget types. Furthermore, $s\left(v_i, w_H;\, h_L,\, h_L\right) < v_i$ for $v_i \geq v_*$, hence even high budget types with high values reduce their demand.

We now show that, if the density function f is non-decreasing on $[h_L, 1]$, the demand reduction is actually quite dramatic.

4.2 Non-decreasing Densities

Suppose that the density f is non-decreasing on the interval $[h_L, 1]$. We want to show that in this case the strategy described in Proposition 1 is an equilibrium if, and only if, $v_* = 1$. First, we show that there is no equilibrium with $v_* < 1$.

For any $v_* \in (h_L, 1]$, the problem of bidder 1's type (v_1, w_H), conditional on the bids having reached (h_L, h_L), and the opponent having played soft, can be written as:

$$\max_s \; U\left(v_1, s; v_*, h_L\right) = \lambda\left[2\int_{h_L}^{s}\left(v_1 - v_2\right)\, dF\left(v_2\right) + \left(v_1 - s\right)\,\left[1 - F\left(s\right)\right]\right]$$

$$+\left(1-\lambda\right)\left[2\int_{h_L}^{\min\{s, v_*\}}\left(v_1 - v_2\right)\, dF\left(v_2\right) + \left(v_1 - s\right)\,\left[F\left(v_*\right) - F\left(\min\left\{s, v_*\right\}\right)\right]\right].$$

Suppose that $v_* < 1$, and consider type $v_1 = v_*$. Since the derivative $\frac{\partial U}{\partial s}$, evaluated at $v_1 = v_*$, and for $s < v_*$, is proportional to:

$$\left(v_* - s\right) f\left(s\right) - \left[\lambda + \left(1-\lambda\right) F\left(v_*\right) - F\left(s\right)\right],$$

we have that the (left) derivative at $s = v_*$ is strictly negative. Furthermore, the second derivative $\frac{\partial^2 U}{\partial^2 s}$ on the interval (h_L, v_*) is proportional to:

$$\left(v_* - s\right)\, f'\left(s\right).$$

Given the assumption that $f'(s) \geq 0$ for each $s \geq h_L$, we have that $\frac{\partial U}{\partial s}$ is non-decreasing on (h_L, v_*), and strictly negative at v_*; hence strictly negative over (h_L, v_*). Therefore, the optimal stopping time for v_* must be h_L. This remains true for types $v_* + \delta'$, with δ' small enough. Therefore, a set of types $(v_*, v_* + \delta)$, with $\delta > 0$, will choose a stopping time of h_L. This is a contradiction, since Lemma 5 in the Appendix establishes that the stopping time must be strictly greater than h_L. Thus, it can never be the case that $v_* < 1$.

Thus the only possible candidate for an equilibrium is $v_* = 1$. In fact, we can readily check that we have an equilibrium for $v_* = 1$. In this case, the expected utility of playing soft is $v_1 - h_L$. Playing tough is now an out of equilibrium action, and we specify that, faced with a tough opponent, each bidder plays defensively. (This is optimal for any belief which assigns a high probability to high values of the opponent). Then the highest utility which can be obtained by opening tough and then choosing s is obtained solving:

$$\max_s \left[2 \int_{h_L}^{s} (v_1 - v_2) f(v_2) \, dv_2 + (v_1 - s)(1 - F(s)) \right].$$

Again, it can checked that at $s = v_1$ the derivative is negative, and that, since $f(v_2)$ is non-decreasing, the derivative must be negative over (h_L, v_1). Thus, the optimal stopping time turns out to be $s = h_L$. The deviation is therefore not profitable.

The equilibrium has the remarkable property that it does not depend on λ, the fraction of budget-constrained players. That is, for *any* $\lambda > 0$ the most competitive equilibrium has all the high-budget bidders mimicking the low-budget bidders when the bids reach (h_L, h_L). This implies a discontinuity in the equilibrium set. When $\lambda = 0$ then a "competitive" equilibrium exists in which each bidder pushes up the bid on each object up to their value. However, for any $\lambda > 0$ this equilibrium disappears, and it becomes impossible to induce competition among bidders at prices higher than h_L.

4.3 Increasing Welfare by Excluding Low-budget Bidders

The outcome of the non-collusive equilibrium described above for non-decreasing densities is inefficient. As the probability $(1 - \lambda)^2$ that both bidders are not constrained increases toward 1, efficiency requires that the probability with which both objects be assigned to the bidder with the highest value also approach 1. In the limit, the welfare loss is equal to the expected value of $|v_2 - v_1|$, conditional on $\min\{v_1, v_2\} \geq h_L$.

For small values of λ, measures restricting the participation of low-budget bidders can increase the expected social surplus, as well as the seller's expected revenue. For example, the seller may impose a reserve price for each object above w_L, or require each bidder to deposit an amount of w_H at the beginning of the auction. Once the possibility that any participating bidder is budget constrained is ruled out, the high-budget bidders cannot 'hide' behind budget constrained types; hence the non-collusive equilibrium produces the socially efficient outcome. For sufficiently small values of λ, the cost of excluding low-budget bidders is lower than the gain in social surplus obtained by inducing the efficient allocation of the objects.

As an example, suppose that the distribution is $F(v) = v^4$, and take $h_L = 0.35$. First, we check that there is no equilibrium with $v_* < 1$, and that it is an equilibrium for the bidders to split the objects when the two values are above 0.35. This does not follow immediately from the previous analysis because we now have $w_L < 1$. The additional complication in this case is that the low-budget types with value greater than w_L can offer at most w_L for a single object. We now show that, as in the case where $w_L \geq 1$, the equilibrium threshold v_* cannot be strictly less than 1.

First note that, if $v_* < 0.7$, or $s < 0.7$, the analysis of the previous subsection applies immediately. Thus consider $v_* \in (0.7, 1)$ and $s > 0.7$. Then the optimal stopping time for the type v_* is obtained solving:

$$\max_s \lambda \left[\int_{0.35}^{0.7} 2(v_* - y)\left(4y^3\right) dy + 2(v_* - 0.7)\left(1 - (0.7)^4\right)\right] +$$
$$(1-\lambda)\left[\int_{0.35}^{\min\{s,v_*\}} 2(v_* - y)\left(4y^3\right) dy + (v_* - s)\left(v_*^4 - s^4\right)\right]$$

The derivative is strictly negative for each $s < v_*$, so that the optimal stopping time is 0.7. The expected utility of playing tough is therefore:

$$U(tough) = \lambda \left[\int_{0.35}^{0.7} 2(v_* - y)\frac{4y^3}{1-(0.35)^4} dy + 2(v_* - 0.7)\frac{1-(0.7)^4}{1-(0.35)^4}\right] +$$
$$(1-\lambda)\left[\int_{0.35}^{0.7} 2(v_* - y)\frac{4y^3}{1-(0.35)^4} dy + (v_* - 0.7)\frac{v_*^4 - (0.7)^4}{1-(0.35)^4}\right]$$
$$= \left[\int_{0.35}^{0.7} 2(v_* - y)\frac{4y^3}{1-(0.35)^4} dy\right] +$$
$$+\left[2\lambda\left(1-(0.7)^4\right) + (1-\lambda)\left(v_*^4 - (0.7)^4\right)\right]\frac{v_* - 0.7}{1-(0.35)^4}$$

The expected utility of opening soft is

$$U(soft) = \left(\lambda + (1-\lambda)\frac{v_*^4 - (0.35)^4}{1-(0.35)^4}\right)(v_* - 0.35)$$

As λ goes to zero we have:

$$U(tough) = \left[\int_{0.35}^{0.7} 2(v_* - y)\frac{4y_2^3}{1-(0.35)^4}dy\right] + (v_* - 0.7)\frac{v_*^4 - (0.7)^4}{1-(0.35)^4}$$

$$U(soft) = \frac{v_*^4 - (0.35)^4}{1-(0.35)^4}(v_* - 0.35)$$

It can now be checked that:

$$U(soft) > U(tough)$$

for each $v_* \in [0.7, 1]$. Thus, any equilibrium must have $v_* = 1$. It is now straightforward to check that $v_* = 1$ can in fact be supported in equilibrium.

When λ is close to 1, the expected social welfare is approximately:

$$W^a = \int_0^{0.35}\int_0^{v_1} 2v_1\left(4v_2^3\right)dy + \int_{v_1}^1 2v_2\left(4v_2^3\right)dv_2\left(4v_1^3\right)dv_1$$

$$+\int_{0.35}^1\left(\int_0^{0.35} 2v_1\left(4v_2^3\right)dv_2 + \int_{0.35}^1 (v_1+v_2)\left(4v_2^3\right)dv_2\right)\left(4v_1^3\right)dv_1 = 1.6156.$$

If a reservation price of 0.7 is imposed, then all low budget types, as well as the high budget types with value lower than 0.7, do not participate. In this case, since it is common knowledge that the participants are not budget constrained, there is a competitive equilibrium in which the bidder with the highest value wins both objects, and the expected social welfare is:

$$W^b = (0.7)^4 \times \left(\int_{0.7}^1 2v_2\left(4v_2^3\right)dv_2\right) +$$

$$+\int_{0.7}^1\left(\int_0^{v_1} 2v_1\left(4v_2^3\right)dv_2 + \int_{v_1}^1 2v_2\left(4v_2^3\right)dv_2\right)\left(4v_1^3\right)dv_1 \cong 1.706$$

To see how the expression is computed, notice that when $v_1 < 0.7$, which happens with probability $(0.7)^4$, then the two objects go to bidder 2 iff $v_2 > 0.7$; this is the first term. When $v_1 > 0.7$ then the two objects go to the highest bidder (second term). Since $W^b > W^a$, in this case the imposition of a reservation price increases efficiency.

4.4 Other Changes in the Auction Format

In this paper we have examined a specific auction format, and have not tried to find the optimal mechanism (either under the point of view of social welfare or under the point of view of revenue) for the sale of multiple objects when bidders can be budget constrained. The analysis of section

3, however, was general and it applies to any auction format. Therefore, we can conclude that the allocation in which the two objects are assigned to the highest-valuation bidder, at the price of the second valuation, whenever the agents are not budget constrained cannot be the outcome of any auction format. Quite simply, the problem is that such an allocation is not incentive compatible.

While a discussion of the optimal mechanism is beyond the scope of this paper, it is instructive to briefly analyze what would happen under some simple modifications of the auction rules.

One possible modification is irrevocable exit. This would correspond to a 'button auction' in which each bidder controls two buttons, each one corresponding to one of the objects. According to this auction format, the price on a given object goes up as long as both bidders keep pressing the corresponding button; as soon as one bidder i releases the button, the object is adjudicated to bidder $3-i$ at the price at which the other bidder drops out.

Let us first observe that the equilibrium analyzed in section 4 would remain virtually unchanged if the buttons referred to the *number* of objects that a bidder is willing to buy, as opposed to a specific object. In this case each bidder has one button saying 'two objects' and one button saying 'one object'. Each agent starts pressing the 'two' button, and a switch to the 'one' button is irreversible. However, the price on both objects goes up as long as there is excess demand (in particular, one agent pressing 'two' and the other pressing 'one'). Under this auction format, the same incentives for price reduction present in the equilibrium discussed in section 4 would remain. As a matter of fact, as this would be basically a uniform-price auction, the results of Ausubel and Cramton [1] apply: demand reduction would be present even in the absence of budget constraints.

Suppose now that each button is object-specific, so that whenever a button is released the corresponding object is assigned to the remaining bidder at the drop-out price of the other bidder. Consider the following strategy:

- keep the button on object i pressed as lond as $p_i \leq \min\left\{v_i, \frac{w_i}{2}\right\}$.
- as soon as p reaches $\frac{w_i}{2}$ then exit the auction on object i. Remain in the auction for the other object as long as the price is $p \leq v_i$.

The allocation resulting from this strategy is the following:

- A high-budget bidder gets $2\max\left\{v_i - v_{3-i}, 0\right\}$ when the opponent is also high-budget, and $\max\left\{v_i - v_{3-i}, 0\right\} + \max\left\{v_i - \min\left\{v_{3-i}, h_L\right\}, 0\right\}$ when the opponent is low-budget.

- A low-budget bidder gets $2\max\{v_i - v_{3-i}, 0\}$ if $v_{3-i} < h_L$. If the opponent is high-budget with $v_{3-i} \geq h_L$ then she gets $\max\{v_i - v_{3-i}, 0\}$, and the opponent is low-budget with $v_{3-i} \geq h_L$ then she gets $\max\{v_i - h_L, 0\}$.

It can be checked that this allocation is incentive compatible, and in particular that the inequality:

$$E\left[U\left(v_i, w_H, v_{3-i}, w_{3-i}\right)\right] \geq E\left[\left(v_i, w_L, v_{3-i}, w_{3-i}\right)\right],$$

is satisfied for a high budget type. Also, it is not difficult to find beliefs and out-of-equilibrium strategies making this allocation the outcome of a perfect Bayesian equilibrium in the object-specific button auction.

This however may not be revenue maximizing for the seller. The trade off is due to the fact that a tough high-budget bidder in the auction in which re-entering is allowed always pays the opponent player value v_i on both objects, while the irrevocable exit auction implies a payment of v_i on one object and h_L on the other object.

Consider now a sequential auction. Suppose that in both auctions each bidder bids up to $\min\left\{v_i, \frac{w_i}{2}\right\}$. Then, whenever two bidders are not budget constrained the two auctions have the same outcome, the bidder with the highest value gets both objects. Is the allocation of the irrevocable exit auction an equilibrium?

Suppose the low-budget bidder goes for the object in the first period. This cannot be an equilibrium because you have an incentive to increase the price in order to deplete your opponent's budget, in case it is budget constrained. Now suppose that a low-budget agent bids up to h_L in the first auction and up to the valuation (in case no object was won) in the second auction. This appear to be an equilibrium. If I allow my opponent to win at h_L nothing changes if she is high-budget (I can get at most one object at a price v_{3-i}, so my utility is $\max\{v_i - v_{3-i}, 0\}$). If she is low-budget then the price will be h_L in the second period as well. Therefore, a low budget bidder is indifferent between winning one object in the first period or in the second (this argument does not work if there is discounting, as the low budget bidder would prefer to win in the first period; in this case, the equilibrium would probably be that the price in the first auction when two low-budget meet is higher than in the second auction).

5 Conclusions

We have explored the effects that the possibility of binding budget constraints may have on auctions with multiple objects. It is clear that budget constraints reduce the level of competition because the bidders have a lower ability to pay: we have shown that competition is further reduced due to strategic reasons. In fact even the slightest possibility of having binding budget constraints implies that a set with strictly positive measure of high-budget bidders will pretend to be low-budget, so that the objects are inefficiently split far more often than what would be implied by the pure effect of budget constraints. In fact, the outcome of the auction appears to be collusive, with the bidders splitting the objects at low prices. This effect appears to be robust to the auction format, as it stems from the fact that the budget-constrained efficient allocation is not incentive-compatible. Thus, in any equilibrium of any mechanism some high-budget types will pretend to be low-budget types. The problem is created by the fact that low-budget bidders have strictly positive probability, thus introducing an additional incentive compatibility constraint. It follows that measures which exclude budget-constrained bidders from participating can be welfare enhancing, since they stimulate competition and favor a more efficient allocation of the objects.

Appendix

Proof of Proposition 1. In order to prove that an equilibrium exists, we have to show that a type $v_* \geq h_L$ exists, such that all types $v_i \in [h_L, v_*)$ prefer to play 'soft' when prices reach h_L, while all types $[v_*, 1]$ prefer to play 'tough'. We begin by establishing some preliminary results. Let $U(v_1, s; v_*, b)$ be the function defined by (4) and (5), $R(v_1; b, v_*)$ the correspondence defined by (6) and $s(v_1; b, v_*)$ the function defined by (7). In our first lemma we characterize some properties of the optimal stopping rule.

Lemma 1 *The optimal stopping rule satisfies the following properties:*

- *The correspondence $R(v_1; b, v_*)$ is upper-hemicontinuous in v_1 and v_* and compact valued.*
- *The function $s(v_1; b, v_*)$ is non-decreasing and $s(v_1; b, v_*) < v_1$ whenever $v_1 \in (b, 1)$.*
- *If $s(v_1; b, v_*)$ is constant over an interval $[v_1, v_1 + \delta)$ then either $s(v_1; b, v_*) = b$ or $s(v_1; b, v_*) = v_*$.*
- *Let $K \subset [b, 1]$ be the set of points of discontinuity of $s(v_1; b, v_*)$ and let ψ be a measure defined on $[b, 1]$ which is absolutely continuous with respect to the Lebesgue measure. Then $\psi(K) = 0$.*

Proof. The properties of the correspondence R follow from the Maximum Theorem and the continuity of $U(v_1, s; v_*, b)$.

Since the function $U(v_1, s; v_*, b)$ satisfies increasing differences in $(s; v_1)$, we have that $s(v_1; b, v_*)$ is non-decreasing in v_1 (see e.g. Milgrom and Shannon [16]). It is obvious that $s(v_1; b, v_*) \leq v_1$. To see that $s(v_1; b, v_*) < v_1$ whenever $v_1 \in (b, 1)$ observe that $U(v_1, s; v_*, b)$ is always left-differentiable with respect to s at $s = v_1$, and the left derivative $\left.\frac{\partial^- U(v_1, s; v_*, b)}{\partial s}\right|_{s=v_1}$ is strictly negative whenever $v_1 < 1$. Thus $s = v_1$ cannot be optimal.

Suppose now that $s(v_1; b, v_*)$ is constant over an interval $[v_1, v_1 + \delta)$. When $b \geq v_*$ then the function $U(v_1, s; v_*, b)$ is everywhere differentiable in s. Therefore, for s to be optimal it must be the case that:

$$\frac{\partial U}{\partial s} = (v_1 - s) f(s) - [1 - F(s)] \leq 0.$$

Let $\overline{s} = s(v_1; b, v_*)$. If $\frac{\partial U}{\partial s}(v_1, \overline{s}; b, v_*) = 0$ then, for each $v_1' > 0$ we have $\frac{\partial U}{\partial s}(v_1', \overline{s}; b, v_*) > 0$, so that $\overline{s}$ cannot be the optimal stopping time on an interval $(v_1, v_1 + \delta)$. If $\frac{\partial U}{\partial s}(v_1, \overline{s}; b, v_*) < 0$ then it must be the case that $\overline{s} = b$.

Consider now the case $b < v_*$. The function is differentiable in s except at $s = v_*$. The derivative at $s \neq v_*$ is:

$$\frac{\partial U}{\partial s} = \begin{cases} (v_1 - s) f(s) - [\lambda + (1-\lambda) F(v_*) - F(s)] & \text{if } s < v_* \\ \\ \lambda [(v_1 - s) f(s) - (1 - F(s))] & \text{if } s > v_* \end{cases}$$

Thus, suppose that at v_1 we have $\overline{s} = s(v_1; b, v_*)$. If $\overline{s} < v_*$ then we can apply the same reasoning as above to conclude that the function can only be constant if $\overline{s} = b$. Suppose now $\overline{s} = v_*$. Notice that at v_* the function U is both left and right differentiable, and we have:

$$\begin{aligned} \left.\frac{\partial^- U}{\partial s}\right|_{s=v_*} &= (v_1 - v_*) f(v_*) - \lambda (1 - F(v_*)) \\ \left.\frac{\partial^+ U}{\partial s}\right|_{s=v_*} &= \lambda [(v_1 - v_*) f(v_*) - (1 - F(v_*))] \end{aligned}$$

Since $\lambda \in (0,1)$, it is possible to have $\left.\frac{\partial^- U}{\partial s}\right|_{s=v_*} > 0 > \left.\frac{\partial^+ U}{\partial s}\right|_{s=v_*}$ over a set $[v_1, v_1 + \delta)$. In this case v_* can be the optimal stopping time for each v_1 in the set, and the optimal stopping time can therefore be constant. Next, suppose $\overline{s} > v_*$. Then $\overline{s}$ can be optimal only if the derivative is zero, and this in turn implies that $\overline{s}$ cannot be the optimal stopping time if $v_1' > v_1$.

To prove the last point observe that a non-decreasing function defined on a compact set has at most countably many points of discontinuity (Kolmogorov-Fomin, page 316, Theorem 3), and a countable set has Lebesgue measure zero. ■

Suppose now that we are on the equilibrium path, and the bids have just reached the level h_L. Each bidder is winning one object. At this point, each bidder has to signal whether she is "soft," by remaining silent, or "tough," by bidding on the object she is not winning. Fix an arbitrary threshold $v_* \in (h_L, 1]$, and assume that bidder 1 conjectures that her opponent plays soft if and only if $w_2 = w_L$, or $w_2 = w_H$ and $v_2 < v_*$. Let $G(v_2) \equiv \frac{F(v_2) - F(h_L)}{1 - F(h_L)}$.

Suppose that bidder 1 plays tough. If bidder 2 is not budget constrained, then with probability $1 - G(v_*)$ she also plays tough: each bidder then bids up to her value for both objects, and bidder 1 earns $2 \max\{v_1 - v_2, 0\}$. With probability $G(v_*)$, the high-budget opponent plays soft. In this case, by trying to win both objects until the bids arrive at level s bidder 1 earns $2(v_1 - v_2)$ if $v_2 < s$, and $v_1 - s$ otherwise. Thus the expected payoff for bidder 1 when facing a high-budget opponent is:

$$T_H(v_1; s, v_*) = 2\int_{h_L}^{\min\{v_*,s\}} (v_1 - v_2)\, dG(v_2) + \int_{\min\{v_*,s\}}^{v_*} (v_1 - s)\, dG(v_2)$$
$$+2\int_{v_*}^{1} \max\{v_1 - v_2, 0\}\, dG(v_2).$$

If instead bidder 2 is budget constrained, she will also play soft, and bidder 1 can push both bids up to s, thus earning on average:

$$T_L(v_1; s, v_*) \equiv 2\int_{h_L}^{s} (v_1 - v_2)\, dG(v_2) + (v_1 - s)[1 - G(s)].$$

The overall expected payoff of playing tough, and selecting a stopping time s against a soft opponent is:

$$\lambda T_L(v_1; s, v_*) + (1-\lambda) T_H(v_1; s, v_*).$$

Now let

$$T(v_1; v_*) \equiv \max_{s\in[h_L,1]} \lambda T_L(v_1; s, v_*) + (1-\lambda) T_H(v_1; s, v_*).$$

This function is bidder 1's expected surplus of opening tough when she conjectures that her opponent plays tough if and only if $w_2 = w_H$ and $v_2 > v_*$.

Lemma 2 *The function $T(v_1; v_*)$ is continuous in (v_1, v_*).*

Proof. This follows from the Maximum Theorem and the fact that the function $\lambda T_L(v_1; s, v_*) + (1-\lambda) T_H(v_1; s, v_*)$ is continuous in v_1, s and v_*. ■

Suppose now that bidder 1 plays soft. If $w_2 = w_L$, or if $w_2 = w_H$ and $v_2 < v_*$, bidder 2 also plays soft, and the auction ends immediately with one object going to each bidder. Bidder 1's surplus in this case is $v_1 - h_L$. The probability of this happening is $\lambda + (1-\lambda)\, G(v_*)$.

If instead $w_2 = w_H$ and $v_* \leq v_2$, bidder 2 plays tough, i.e. bids on her second object, and continues to do so until the bids reach $s(v_2; h_L, v_*)$, if bidder 1 responds by bidding "defensively", i.e. if by bidding on one object only when she is losing both. At any given round however, bidder 1 may decide to bid on a second object. This constitutes an out of equilibrium action, and our equilibrium specifies that in this case the two bidders will simply bid up to their values.

It is clear that, if $v_1 \leq v_*$, no such deviation is profitable for bidder 1, since her opponent has a higher value: $v_1 \leq v_* < v_2$. If instead $v_* < v_1$, suppose that bidder 1 bid defensively until the bids reach level b_a and then try to win both objects. In this case her expected payoff is:

$$S(v_1, b_a; v_*) = [\lambda + (1-\lambda) G(v_*)] \ (v_1 - h_L) + (1-\lambda) \ S_H(v_1, b_a; v_*),$$

where:

$$S_H(v_1, b_a; v_*) \equiv \int_{v_*}^{v(b_a)} [v_1 - s(v_2; h_L, v_*)] \ dG(v_2) + 2 \int_{\min\{v(b_a), v_1\}}^{v_1} (v_1 - v_2) \ dG(v_2).$$

and:

$$v(b_a) := \sup \{ v_2 | \, s(v_2; h_L, v_*) \leq b_a \}$$

is the highest type of bidder 2 with a stopping time inferior to b_a.

To be part of a sequentially rational strategy the point b_a has to be chosen optimally. In order to analyze this problem, it is useful to reformulate it in terms of the choice of an optimal "stopping type" $v_a = v(b_a)$. In this case we write

$$S_H(v_1, v_a; v_*) \equiv \int_{v_*}^{v_a} [v_1 - s(v_2; h_L, v_*)] \ dG(v_2) + 2 \int_{\min\{v_a, v_1\}}^{v_1} (v_1 - v_2) \ dG(v_2).$$

For a given v_* and corresponding function $s(v_2; h_L, v_*)$, define:

$$v^+ = \inf \{ v_2 \in [v_*, 1] | \, s(v_2; h_L, v_*) > h_L \}$$

$$\underline{v} = \sup \{ v_2 \in [v_*, 1] | \, s(v_2; h_L, v_*) < v^* \}$$

$$\overline{v} = \inf \{ v_2 \in [v_*, 1] | \, s(v_2; h_L, v_*) > v^* \}$$

By Lemma 1 the function $s(v_2; h_L, v_*)$ can be flat only over an initial interval $[v_*, v^+]$ and another interval $[\underline{v}, \overline{v}]$ at which $s = v_*$, and it is strictly increasing elsewhere. Therefore choosing a "trigger point" b_a is equivalent to choosing a "trigger type" v_a in the set:

$$A(v_*) = \{v_*\} \cup [v^+, \underline{v}] \cup [\overline{v}, 1].$$

Recall that we are analyzing what happens in the round after the bids have reached (h_L, h_L), bidder 2 has played tough, and bidder 1 has remained silent.

At this point, a choice of v_* can be interpreted as "trigger the fight immediately", by bidding on both objects (this is equivalent to choosing $b_a = h_L + \varepsilon$ as triggering bid). A choice of v^+ can be interpreted as bidding defensively after the opponent has made a bid to $h_L+\varepsilon$, so that the bids reach $(h_L+\varepsilon, h_L+\varepsilon)$, then wait to see if the opponent counterbids and in that case trigger the war (equivalent to choosing $b_a = h_L + 2\varepsilon$ as triggering bid). A choice of $v_a \in (v^+, \underline{v}]$ simply means "trigger the fight as soon as the bids reach $s(v_a; h_L, v_*)$", where $s(v_a; h_L, v_*) \in (h_L, v_*]$. A choice of $\overline{v}$ means "trigger the fight as soon as the bids reach $v_*+\varepsilon$, and so on. Observe that the function $S_H(v_1, v_a; v_*)$ is continuous in v_a and that $A(v_*)$ is compact. Now define:

$$S_H(v_1; v_*) = \max_{v_a \in A(v_*)} \quad S_H(v_1, v_a; v_*) .$$

We have the following result.

Lemma 3 *The function $S(v_1; v_*)$ is continuous in (v_1, v_*).*

Proof. It suffices to show that the function $S_H(v_1, v_a; v_*)$ is continuous with respect to $(v_1, v_a; v_*)$, and that the correspondence $A(v_*)$ is continuous. Then the result follows from the Maximum Theorem.

Continuity in v_1 and v_a is immediate. In order to show that $S_H(v_1, v_a; v_*)$ is continuous in v_* it is enough to show that:

$$H(v_1, v_a; v_*) \equiv \int_{v_*}^{v_a} s(v_2; h_L, v_*)\ dG(v_2)$$

is continuous in v_*, which in turn is implied by the fact that s is continuous almost everywhere, since G is atomless. To establish continuity of s almost everywhere, suppose that at a point v_2 we have

$$\lim_{n\to\infty} \quad s(v_2; h_L, v_n) = s^* \neq s(v_2; h_L, v_*) ,$$

where $\{v_n\}$ is a sequence converging to v_*. It must be $s^* \in R(v_2; h_L, v_*)$, since the correspondence $R(v_2; h_L, v_n)$ is upper-hemicontinuous in v_n (Lemma 1). This in turn implies $s^* < s(v_2; h_L, v_*)$, since $s(v_2; h_L, v_*)$ is the maximum of $R(v_2; h_L, v_*)$. Thus v_2 must be a point of discontinuity of $s(v_2; h_L, v_*)$. But we have already established in Lemma 1 that, since s is non-decreasing in v_2, the set of discontinuity points has measure zero.

We come now to the continuity of $A(v_*)$. The only complications here are created by the 'flat' parts of the stopping function s, which generate 'gaps' in the interval. We will analyze the case in which the only flat part is at v_*, as it is always the case in equilibrium. Extending the analysis to incorporate the possibility of a flat part at h_L is immediate.

With no flat part at h_L, we have $A(v_*) = [h_L, \underline{v}] \cup [\overline{v}, 1]$. Consider a sequence $\{v^n\}$ converging to v_*, and let $A(v^n) = [h_L, \underline{v}^n] \cup [\overline{v}^n, 1]$. To prove the continuity of the correspondence $A(\cdot)$ it suffices to show that $\underline{v}^n \to \underline{v}$ and $\overline{v}^n \to \overline{v}$.

As a preliminary result, we first prove that if $\underline{v} < \overline{v}$, so that there is an open set $(\underline{v}, \overline{v})$ of types having v_* as optimal stopping time, then v_* is the unique optimal stopping time for all types in the set. Suppose not, so that $s^* \neq v_*$ is also an optimal stopping time for a type $v' \in (\underline{v}, \overline{v})$. This means $U(v', s^*; h_L, v_*) = U(v', v_*; v_*, h_L)$, since both s^* and v_* are optimal stopping times. Furthermore, since v_* is the highest stopping time, it must be $s^* < v_*$, so that:

$$U(v', s^*; v_*, h_L) = 2\int_{h_L}^{s^*} (v' - y)\, dF(y) + (v' - s)(\lambda + (1-\lambda) F(v_*) - F(s^*))$$

Now observe that:

$$\frac{\partial U(v', s^*; v_*, h_L)}{\partial v'} = 2F(s^*) - 2F(h_L) + (\lambda + (1-\lambda) F(v_*) - F(s^*))$$

On the other hand, when the optimal stopping time is v_* we have:

$$U(v', v_*; v_*, h_L) = 2\int_{h_L}^{v_*} (v' - y)\, dF(y) + \lambda (v' - s)(1 - F(v_*))$$

so that:

$$\frac{\partial U(v', v_*; v_*, h_L)}{\partial v'} = 2F(v_*) - 2F(h_L) + \lambda(1 - F(v_*))$$

Thus

$$\frac{\partial U(v', s^*; v_*, h_L)}{\partial v'} < \frac{\partial U(v', v_*; v_*, h_L)}{\partial v'}$$

Since $U(v', s^*; h_L, v_*) = U(v', v_*; v_*, h_L)$, this implies that there is a type v'' sufficiently close to v' such that $v'' \in (\underline{v}, \overline{v})$ and $U(v'', s^*; h_L, v_*) > U(v'', v_*; v_*, h_L)$, a contradiction.

We now come back to proving the continuity of the correspondence $A(\cdot)$. Suppose first that $\underline{v} = \overline{v}$. This happens when there is a single value $\widehat{v}$ such that $s(\widehat{v}; h_L, v_*) = v_*$ (that is, no flat part), so that we have $A(v_*) = [h_L, 1]$. Suppose now that $\lim_{n\to\infty} \underline{v}^n = \widehat{\underline{v}} < \lim_{n\to\infty} \overline{v}^n = \widehat{\overline{v}}$. For each type $v' \in \left(\widehat{\underline{v}}, \widehat{\overline{v}}\right)$ it must be the case that $\lim_{n\to\infty} s(v'; v^n, h_L) = k$, a constant, and $k \in R(v'; v_*, h_L)$. Since no open interval of types can have a common optimal stopping time other than v_*, we conclude that v_* is an optimal stopping time for types in $\left(\widehat{\underline{v}}, \widehat{\overline{v}}\right)$. Then $\widehat{v} < \widehat{\underline{v}}$, since $s(\widehat{v}; h_L, v_*) = v_*$ and s is non-decreasing. Moreover, all types in $v' \in \left(\widehat{\underline{v}}, \widehat{\overline{v}}\right)$ must have both v_* and another (higher) point as optimal stopping times. But this cannot be true since, as proved above, if v_* is an optimal stopping time for an open interval of types then it has to be unique.

We now come to the case in which there is an open set $(\underline{v}, \overline{v})$ of types having v_* as optimal stopping time when the threshold is v_*. We will prove that for each $v' \in (\underline{v}, \overline{v})$, there is N large enough such that v' has v^n as optimal stopping time for each $n > N$. Furthermore, if v' and v'' have v^n as optimal stopping time then all types in the set (v', v'') have v^n as optimal stopping time. This is turn implies that $\underline{v}^n \to \underline{v}$ and $\overline{v}^n \to \overline{v}$.

Since v_* is the unique optimum for v', it must be the case that:

$$\left.\frac{\partial^- U(v', s; h_L, v_*)}{\partial s}\right|_{s=v_*} = (v' - v_*) f(v_*) - \lambda (1 - F(v_*)) > 0$$
$$\left.\frac{\partial^+ U(v', s; h_L, v_*)}{\partial s}\right|_{s=v_*} = \lambda \left[(v' - v_*) f(v_*) - (1 - F(v_*))\right] < 0$$

This in turn implies that

$$\left.\frac{\partial^- U(v', s; h_L, v^n)}{\partial s}\right|_{s=v^n} = (v' - v^n) f(v^n) - \lambda (1 - F(v^n)) > 0$$
$$\left.\frac{\partial^+ U(v', s; h_L, v^n)}{\partial s}\right|_{s=v^n} = \lambda \left[(v' - v^n) f(v^n) - (1 - F(v^n))\right] < 0$$

for n large enough, and it also implies that if the inequalities hold for two types v' and v'' then they must hold for all types in the interval (v', v''). This in turn implies that v^n is a local maximum for an open interval of types (v', v''). Now suppose that there is a different global maximum $s(v'; h_L, v^n)$. Since v^n is the only point of non-differentiability with respect to s, it must be the case that the derivative with respect to s computed at the optimal point $s(v'; h_L, v^n) \neq v^n$ must be zero.

Fix now a neighborhood of $I(v_*)$ such that $\frac{\partial U(v', s; h_L, v_*)}{\partial s} \neq 0$ for each $s \in I(v_*)$ at which U is differentiable. For n large enough, we also have $\frac{\partial U(v', s; h_L, v^n)}{\partial s} \neq 0$ for each $s \in I(v_*)$ such that U is differentiable. Now observe that, given the upper-hemicontinuity of the best response correspondence, it must be the case that $s(v'; h_L, v^n) \to v_*$, since v_* is the unique optimal stopping time $s(v'; h_L, v_*)$. This in turn implies that for n large enough, we have $s(v'; h_L, v^n) \in I(v_*)$. This is a contradiction, since at $s(v'; h_L, v^n)$ the derivative is supposed to be zero. ∎

Lemma 4 *For any $v_* > h_L$ there exists a $\delta > 0$ such that all types $v_1 \in (h_L, h_L + \delta)$ prefer playing soft to playing tough.*

Proof. For any v_* we have $T(h_L; v_*) = S(h_L; v_*) = 0$. Assume $v_* > h_L$ and consider δ such that $h_L + \delta < v_*$. Then the utility of playing soft is:

$$S(v_1; v_*) = (\lambda + (1 - \lambda) G(v_*)) (v_1 - h_L) + (1 - \lambda) S_H(v_1; v_*)$$

The utility of playing tough for a type v_1 is:

$$T(v_1; s, v_*) = \lambda \left[2 \int_{h_L}^{s} (v_1 - v_2)\, dG(v_2) + (v_1 - s)(1 - G(s)) \right]$$

$$+ (1-\lambda) \left[2 \int_{h_L}^{s} (v_1 - v_2)\, dG(v_2) + (v_1 - s)(G(v_*) - G(s)) \right]$$

Now observe that, for a small enough δ and for all types $v_1 < h_L + \delta$,

$$\frac{\partial T}{\partial s} = \lambda \left[(v_1 - s)\, g(s) - (1 - G(s)) \right]$$

$$+ (1-\lambda) \left[(v_1 - s)\, g(s) - (G(v_*) - G(s)) \right] < 0,$$

for each $s \in [h_L, h_L + \delta]$. Then the utility of playing tough is exactly:

$$H_L(v_1; s, v_*) = (\lambda + (1-\lambda) G(v_*)) \left[(v_1 - h_L) \right].$$

This is less than or equal to $S(v_1; v_*)$. ■

Lemma 5 *Suppose that v_* is an equilibrium threshold, and let $s(v_1; h_L, v_*)$ be the corresponding stopping function defined for $v_1 \in [v_*, 1]$. Then $\lim_{v_1 \downarrow v_*} s(v_1; h_L v_*) > h_L$.*

Proof. Since $s(v_1; h_L v_*)$ is monotone a limit exists. Suppose that the claim of the lemma is not true, so that $\lim_{v_1 \downarrow v_*} s(v_1; h_L, v_*) = h_L$. It must be the case that all types $v_1 > v_*$ prefer playing tough to playing soft. When we consider the utility of playing soft we have:

$$\begin{aligned} \lim_{v_1 \downarrow v_*} S(v_1; v_*) &= (\lambda + (1-\lambda) G(v_*))(v_* - h_L) \\ &\quad + (1-\lambda) \int_{v_*}^{1} \max\{v_* - s(v_2; h_L, v_*), 0\}\, dG(v_2) \end{aligned}$$

while when we look at the utility of playing tough we have:

$$\lim_{v_1 \downarrow v_*} T(v_1; v_*) = (\lambda + (1-\lambda) G(v_*))(v_* - h_L)$$

Since there is a set with positive measure such that $s(v_2; h_L, v_*) < v_*$, we conclude that:

$$\lim_{v_1 \downarrow v_*} S(v_1; v_*) > \lim_{v_1 \downarrow v_*} T(v_1; v_*)$$

a contradiction. ■

The last lemma implies that, when v_* is actually an equilibrium value, then the corresponding stopping function $s(v_1; h_L; v_*)$ cannot take the value h_L over an interval. Combined with Lemma 1, it implies that the only value at which the stopping function can be constant is v^*.

Lemma 6 *For any equilibrium threshold value v_*, there is a set $[h_L, h_L + \delta]$ such that for each $v_1 \in [h_L, h_L + \delta]$ we have*

$$S(v_1; v_*) = T(v_1; v_*)$$

Proof. By Lemma 5, for every possible equilibrium threshold v_* there is $\delta' > 0$ such that $\lim_{v_1 \downarrow v_*} s(v_1; v_*) = h_L + \delta'$. This implies that all types $v_1 \in [h_L, h_L + \delta']$, when playing soft or tough can possibly win something only if they meet a soft type. Therefore

$$S(v_1; v_*) = (\lambda + (1 - \lambda) G(v_*)) (v_1 - h_L).$$

When we look at the utility of playing tough, we know by Lemma 4 that for a set of types $[h_L, h_L + \delta]$ the optimal stopping time is h_L. Therefore, for this set $T(v_1; v_*) = S(v_1; v_*)$, thus yielding the result. ■

Define now:

$$D = \{\delta \in [h_L, 1] \mid S(v_1; \delta) \geq T(v_1; \delta) \text{ for all } v_1 \in [h_L, \delta]\}$$

We know by Lemma 6 that the set D is non-empty. Since by Lemma 3 the functions $S(v_1; \delta)$ and $T(v_1; \delta)$ are continuous in δ, the set D is a closed interval. Then we define:

$$v_* = \max \quad D \tag{8}$$

If $v_* = 1$ then we are done.

Suppose now $v_* < 1$. By definition of v_*, all types $v_1 < v_*$ prefer to play soft. The final step is to show that, for all types $v_1 \geq v_*$ we have $T(v_1; v_*) \geq S(v_1; v_*)$. This is done in the next lemma, which makes use of Assumption 1.

Lemma 7 *Let v_* be defined by (8), and suppose $v_* < 1$. Then, if assumption 1 is satisfied, we have $T(v_1; v_*) \geq S(v_1; v_*)$ for each $v_1 > v_*$.*

Proof. Consider the function

$$\Psi(v_1) = T(v_1; v_*) - S(v_1; v_*).$$

Since both T and S are continuous, so is Ψ. Also, we know that there exists $\varepsilon > 0$ such that $T(v_1; v_*) > S(v_1; v_*)$ for each $v_1 \in (v_*, v_* + \varepsilon)$. Thus it is enough to show that

$$\frac{\partial \Psi(v_1)}{\partial v_1} = \frac{\partial T(v_1; v_*)}{\partial v_1} - \frac{\partial S(v_1; v_*)}{\partial v_1} \geq 0$$

at each point of differentiability of Ψ. We start observing that, since $T(v_*; v_*) = S(v_*; v_*)$, then for any $\varepsilon > 0$ there must be some $v_1 \in (v_*, v_* + \varepsilon)$ such that $\frac{\partial \Psi(v_1)}{\partial v_1} > 0$.

By the previous analysis we have

$$\frac{\partial T}{\partial v_1} = \lambda (1 + G(s)) + (1 - \lambda)[2G(v_1) - G(v_*) + G(\min\{s, v_*\})],$$

and

$$\frac{\partial S}{\partial v_1} = \lambda + (1 - \lambda)[G(v_a) + 2\max\{G(v_1) - G(v_a), 0\}].$$

Suppose first $v_a \leq v_1$. Then

$$\frac{\partial \Psi(v_1)}{\partial v_1} = \lambda G(s) + (1 - \lambda)[G(v_a) - G(v_*) + G(\min\{s, v_*\})]$$

which is positive since $v_a \geq v^*$.

Consider next $v_a > v_1$ (in which case $v_a = \sup\{v_2 \,|s(v_2; h_L, v_*) \leq v_1\}$). Now

$$\frac{\partial \Psi(v_1)}{\partial v_1} = \lambda G(s) + (1 - \lambda)[2G(v_1) - G(v_*) - G(v_a) + G(\min\{s, v_*\})]$$

Since $v_a > v_1 > v_*$ then v_1 must be on a strictly increasing part of the stopping function, hence the following first order condition must hold :

$$(v_a - v_1)\, g(v_1) = 1 - G(v_1).$$

This is the condition ensuring that v_1 is the optimal stopping time for type v_a. (Since $v_1 > v_*$, the FOC that we apply is the one relative to the case $s > v_*$). The FOC can be rewritten as

$$v_a = v_1 + \mu(v_1).$$

Substituting into the expression above for $\frac{\partial \Psi(v_1)}{\partial v_1}$, we have

$$\begin{aligned} \frac{\partial \Psi(v_1)}{\partial v_1} &= \lambda G(s) + (1 - \lambda)[G(\min\{s, v_*\}) - G(v_*)] \\ &\quad + (1 - \lambda)[2G(v_1) - G(v_1 + \mu(v_1))]. \end{aligned}$$

Now observe that, since s is increasing, the function

$$\lambda G(s) + (1 - \lambda)[G(\min\{s, v_*\}) - G(v_*)]$$

is increasing. Furthermore, assumption 1 implies that

$$2G(v_1) - G(v_1 + \mu(v_1))$$

is increasing. We can then conclude that the expression of the derivative in this case is increasing. Thus, in order to prove that $\frac{\partial \Psi(v_1)}{\partial v_1}$ is positive, it is enough to show that at any $v_1' < v_1$ the expression is positive. Now remember that in a right neighborhood of v_* the function Ψ is strictly increasing. Furthermore, for v_1' sufficiently close to v_* it must be the case that $v_a > v_1'$ (it would not make sense to trigger a war). Then, there must be some point v_1' at which $\frac{\partial \Psi(v_1')}{\partial v_1} > 0$. This completes the proof. ∎

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