VARs, common factors and the empirical validation of equilibrium business cycle models

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Abstract

This paper argues that factor models are better empirical tools than VARs for identifying and estimating impulse response functions and shows how to derive the latters from consistent estimates of the factor loadings. Our argument is based on two observations. First, equilibrium business cycle models imply fewer exogenous shocks than variables. Second, variables are measured with errors. We show that, with measurement error, impulse responses based on VARs are not consistent and quantify the empirical bias and mean squared error for both factor based and VAR based estimates using as data generating process a calibrated standard equilibrium business cycle model.

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1 Introduction

The basic econometric tool for empirical validation of macroeconomic models is the Vector Autoregressive Model (VAR). This model is easy to estimate and, once identification restrictions are imposed, can be used to evaluate the impact of economic shocks on key variables.

In structural VARs macroeconomics, variables are represented as driven by serially uncorrelated shocks, each having a different source or nature, like "demand", "supply", "technology", "monetary policy" and so on. Each variable reacts to a particular shock with a specific sign, intensity and lag structure, summarized by the so called "impulse-response function". Implications of economic theory not used for identification can then be compared with estimation results and tested.

A strong motivation for the use of VARs is that stochastic general equilibrium macroeconomic models have solution that can be represented in VAR form and therefore VAR econometrics provides the tool to bridge theory and data.

The typical theoretical macro model, however, has few shocks driving the key variables in the macroeconomy. In the first generation real business cycle models, for example, one shock – technology – is responsible for volatility of output, consumption and investment both in the short and long-run. In that stylized economy, there is only one source of variation. Other models take into account shocks in preferences or money, but sources of macro variations remain few.

A paradox of the macroeconometric literature is that this feature of macro theory has not been fully exploited in empirical modeling. Exceptions are few papers in the late eighties which have observed that the feature of having fewer shocks than variables, if combined with measurement error, implies a factor analytic structure for the solution of the theoretical models that can be analysed empirically with the econometric tools of the factor literature (Altug, 1989 and Sargent, 1989). The factor literature, which has wide applications in many fields other than economics, has been first introduced in macroeconomics by Sargent and Sims, 1977 and then further developed by Geweke, 1977, Geweke and Singleton, 1981 and Engle and Watson, 1983. Dynamic factor models imply a restriction on the spectral density of the observations whereby the latter can be expressed as the sum of two orthogonal components, the spectral density of the common component, of reduced rank, and the spectral density of the idiosyncratic component, of full rank. The former captures all the covariances of the observations at leads and lags while the latter is diagonal and can therefore represents non cross-correlated measurement error. Recently, factor models have been rediscovered in macroeconomics as a tool for analysing large panels of time series (Forni and Reichlin, 1998, Forni, Hallin, Lippi and Reichlin, 2000, Stock and Watson, 2002 and related literature). Both the traditional literature on factor estimation and the more recent one on factor models for large panels, develop different estimation techniques, but disregard issues of structural identification of shocks and propagation mechanisms. Exceptions are two recent papers by Giannone, Reichlin and Sala, 2002 and Forni, Lippi and Reichlin, 2002. This paper builds on ideas developed by these works, but provides a more general discussion on the matching between equilibrium business cycle models, factor estimates, identification and estimation of structural shocks and their impulses. Moreover, we provide an empirical comparison between VAR based estimates of impulse response functions and factor based ones.

The paper starts by recalling that the dynamic rank in equilibrium business cycle models depends on the characteristics of the exogenous processes with typical exogenous sources of variations being money, technology or preferences. Structural relations and assumptions on the dynamic characteristics of the economy, on the other hand, imply restrictions on the coefficients of the solution, which may generate another feature potentially important for empirical investigation, i.e. reduction of rank of the contemporaneous covariance matrix of the observables. The dimension of the latter is the static rank of the system and it is equal to the dimension of the state vector, determined by the number of lagged predetermined variables and exogenous variables. Dynamic characteristics of particular model economies can also be such that the lagged autoregressive matrices are of reduced rank. In this case the solution will have reduced rank representation as in Ahn and Reinsel, 1988, and Velu et al., 1986, or common features as defined in Engle and Kozicki, 1993, and, under further restrictions, common cycles as in Vahid and Engle, 1993.

A large class of equilibrium business cycle models have reduced static rank and with reduced static rank, VAR models are unfeasible. However, empirically VARs models are never collinear indicating that either the static reduced rank models are rejected by the data or that variables are measured with error. This empirical feature is to be contrasted with the finding that macroeconomic time series are often cointegrated, i.e. that they show reduction of rank of the spectral density of their first differences at zero frequency. In the long-run, measurement error is likely to be less sizeable and, as a consequence, underlying collinear relations more evident than at higher frequencies. Measurement error, we will argue, although "curing" the collinearity problem, makes VAR estimates inconsistent.

With measurement error, as mentioned, dynamic rank reduction implies a dynamic factor analytic structure and the factor loadings can be estimated consistently using available techniques. Since impulse response functions are continuous functions of the loadings, the latters can be estimated consistently. The intuition of this result is that the factor model helps to clean data from measurement error by exploiting the theoretical (and empirical) feature of stochastic rank reduction (for empirical evidence of stochastic rank reduction, see Altissimo et al, 2002 on European data and Giannone et al., 2002 on US data).

For illustration of these points, we then generate data from a simple business cycle model with and without measurement error and compare impulse response estimates from VAR and factor procedures. For the latter, we use a quasi-maximum likelihood estimator proposed by Doz and Lenglart, 1999.

The paper is organized as follows. In the first Section, we will describe the general linear solution of equilibrium business cycle models and then illustrate a special simple case. In the second Section, we discuss VAR and factor estimates with and without measurement errors. In the third, we perform the empirical experiment based on the simple model. The last Section, before the conclusions, is a general discussion which relates traditional factor models used in this paper with the more recent literature on factor models for large panels of time series.

2 A model economy and VAR analysis

2.1 Equilibrium business cycle models

A. General Structure

We will start by recalling the general structure of an equilibrium business cycle model. In this framework, as it is well known, the problem in the decentralized economy is the same as the social planner's. The latter maximizes the utility of the representative agent:

$$\max \ \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U(X_t, Y_t) \right]$$

subject to the feasibility constraints::

$$f(X_t, X_{t-1}, \cdots, Y_t, Y_{t-1}, \cdots, S_t, S_{t-1}, \cdots) \leq 0$$
$$S_t = g(\epsilon_t, \epsilon_{t-1}, \cdots)$$

where X_t is the $m \times 1$ vector of endogenous predetermined variables, Y_t is the $n \times 1$ vector of the endogenous non predetermined variables and S_t is the $q \times 1$ vector of exogenous variables (the number of variables considered is therefore $N = m + n + q$). The parameter β defines the discount factor and ϵ_t is a q dimensional i.i.d. normal process with mean 0 and variance Σ_ϵ.

Stated at this level of generality, the model encompasses several examples in the literature, from the simple real business cycle model á la King, Plosser and Rebelo, 1991, to the time-to-build economy á la Kydland and Prescott, 1983 to the model with heterogenous capital (Campbell, 1997). Indicating with small letters the difference between the log of the variables and their non-stochastic steady state, the solution of such models has the following recursive structure:

$$\Psi(L)\mathbf{s_t} = \epsilon_t$$

$$C(L)\mathbf{x}_t = D(L)\mathbf{s}_t$$

$$\mathbf{y}_t = \Lambda_1(L)\mathbf{x}_t + \Lambda_2(L)\mathbf{s}_t$$

where:

$$C(L) = C_0 + C_1 L + \ldots + C_{p_c} L^{p_c}$$
$$D(L) = D_0 + D_1 L + \ldots + D_{p_d} L^{p_d}$$
$$\Lambda_1(L) = \Lambda_{1,1} L + \ldots + \Lambda_{1,p_{\Lambda_1}} L^{p_{\Lambda_1}}$$
$$\Lambda_2(L) = \Lambda_{2,0} + \Lambda_{2,1} L + \ldots + \Lambda_{2,p_{\Lambda_2}} L^{p_{\Lambda_2}}.$$

It should be noticed that this solution form applies even to a larger class of models than those based on the maximization problem described above. As Christiano, 2001 has pointed out, more complex models with heterogenous agents and different informations sets, also have the same solution structure. This can be understood by noticing that the length of the filters $\Lambda_1(L)$ and $C(L)$ is determined by the lags of predetermined variables necessary for the determination of the endogenous and the predetermined variables while the filters $\Lambda_2(L)$ and $D(L)$ accomodate for the possibility that endogenous variables are determined on the basis of different information sets.

The solution, written in its dynamic state space representation, is:

$$\begin{pmatrix} \mathbf{y}_t \\ \mathbf{x}_t \\ \mathbf{s}_t \end{pmatrix} = \Phi(L)\mathbf{s}_t \tag{2.1}$$

$$\Psi(L)\mathbf{s_t} = \epsilon_t$$

where: $\Phi(L) = [\Phi_1(L)' \ \Phi_2(L)' \ I_q]'$ and:

$$\Phi_1(L) = \Lambda_1(L)C(L)^{-1}D(L) + \Lambda_2(L)$$
$$\Phi_2(L) = C(L)^{-1}D(L).$$

In this representation the endogenous variables at t are expressed as linear filters of the q exogenous state variables and therefore as linear filters of the q dimensional white noise exogenous shock.

Alternatively, we can express the solution in its constrained VAR form as:

$$\begin{pmatrix} I_n & -\Lambda_1(L) & -\Lambda_2(L) \\ 0 & C(L) & -D(L) \\ 0 & 0 & \Psi(L) \end{pmatrix} \begin{pmatrix} \mathbf{y}_t \\ \mathbf{x}_t \\ \mathbf{s}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I_q \end{pmatrix} \epsilon_t. \tag{2.2}$$

Let us define the vector of all observables as $\mathbf{w}_t = [\mathbf{y}_t' \ \mathbf{x}_t' \ \mathbf{s}_t']'$. The dynamic rank of this system of equations, defined as the rank of the spectral density matrix of $\mathbf{w}_t$ is q, with $q < N$. The model, therefore, has reduced dynamic rank.

It is also customary to write the solution in its static state space representation where the vector of state variables includes the lagged predetermined variables, and current and lagged exogenous variables. The latter is defined as $F_t = [\mathbf{x}_{t-1}' \dots \mathbf{x}_{t-p_x}' \ \mathbf{s}_t' \dots \mathbf{s}_{t-p_s}']'$, where $p_x = \max\{p_{\Lambda_1}, p_c\}$ and $p_s = \max\{p_{\Lambda_2}, p_d\}$, while the variables in the vector $\mathbf{w}_t$ are expressed as contemporaneous linear combinations of F_t:

$$\mathbf{w}_t = \Lambda F_t \tag{2.3}$$

with:

$$H(L)F_t = K\epsilon_t. \tag{2.4}$$

The dimension of the vector of state variables in this static representation is $r = mp_x + q(p_s + 1)$ and it therefore depends on the p_x and p_s lags included in the model as well as q and m. This is also equal to the rank of the contemporaneous variance-covariance matrix of $\mathbf{w}_t$, $\Gamma_{\mathbf{w}}(0) = \mathrm{E}\mathbf{w}'\mathbf{w}$ and defines the static rank of the system. Notice that we have $r \geq q$ and that an economy with reduced stochastic rank, does not have, in general, reduced static rank.

Static and dynamic rank reveal different features of the model economies. Reduced dynamic rank q tells us that only q shocks matter for dynamics and therefore is a consequence of the characteristics of the exogenous forces driving the economy, while the static rank depends in general on the structure of the economy (the zero restrictions on the coefficients of the VAR form) and on the number of lags included[1]. Typically, models with rich dynamics, such as, for example, the time-to build model á la Kydland and Prescott, 1983, have reduced stochastic rank but may have full static rank while simpler models have both reduced static and dynamic rank. Static and dynamic ranks must be thought as restrictions, in principle testable, derived from theory. Moreover, rank reduction has implications for estimation that we will develop below.

To clarify the structure of the model and the role of the filters, as well as the role of rank reduction it will be useful to discuss a specific example of the general model. The same example will be used in the empirical Section.

B. *The basic business cycle model*

What we illustrate here is a simplified version of King, Plosser and Rebelo, 1991, which is also the textbook example analyzed by Uhlig, 1998, to which we refer for all details.

The model can be seen as a special case of what discussed in A. where there is only one source of variability – technology –, labor is exogenous, there are no time to build features, agents are homogeneous and have the same information set. We have: $n = 3$, $m = 1$, $q = 1$ and $\Psi(L) = 1 - \psi L$. The only exogenous state variable is productivity, z_t, which, with lagged capital stock k_{t-1}, form the vector of the state variables. By using a standard functional form for the utility function, we can write the maximization problem as:

$$\max U = \mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1 - \eta} \right]$$

subject to:

$$C_t + K_t = Z_t + K_{t-1}^{\rho} + (1 - \delta) K_{t-1}$$

$$\log(Z_t) = \log \bar{Z} + \psi \log(Z_{t-1}) + \epsilon_t$$

where C_t, K_t define consumption and the capital stock and Z_t is the productivity exogenous process. The parameters δ, ρ, η and ψ define, respectively, the discount

[1] A different restriction, as mentioned in the introduction, is *common feature* and it implies rank reduction of the lagged VAR matrices

factor, the depreciation rate, the capital share, the coefficient of relative risk aversion and the autoregressive parameter governing persistence of the technology shock in the productivity equation.

Notice that in this case $p_{\Lambda_1} = p_c = 1$, $p_{\Lambda_2} = p_d = 0$. We have $\mathbf{y}_t = [r_t\ c_t\ y_t]'$ where r_t is the real interest rate and y_t is output; moreover, $\mathbf{x}_t = k_t$ and $\mathbf{s}_t = z_t$ (lower cases define, as before, variables in log and deviation from their non stochastic steady state).

The VAR solution can be written as:

$$\Pi(L)\mathbf{w}_t = B\epsilon_t$$

where:

$$\Pi(L) = \begin{pmatrix} I_3 & -\Lambda_1 L & -\Lambda_2 \\ 0 & (1-CL) & -D \\ 0 & 0 & (1-\psi L) \end{pmatrix}$$

and:

$$B = \begin{pmatrix} 0_{(4\times 1)} \\ 1 \end{pmatrix}$$

where $\Pi(L)$ can be written as $[I - A(L)]$. Then $\mathbf{w}_t$ has a VAR(1) structure: $[I - A(L)]\mathbf{w}_t = B\epsilon_t$. Obviously, the coefficients of the A and B matrices depend on the deep parameters ρ, β, δ, η, the parameter ψ governing technology and the steady state value of the level of productivity.

The vector of the state variables is $F_t = [k_{t-1}\ z_t]'$ and:

$$F_t = HF_{t-1} + K\epsilon_t.$$

Notice that the number of state variables is less than the dimension of the model and it is equal to two. This also implies that the rank r of $\Gamma_{\mathbf{w}}(0)$, the static rank, is equal to 2. This model therefore has both dynamic and static reduced rank. Notice also that, for this example, the rank of A is equal to 2 so that the static rank is the same as the rank of the autoregressive lagged matrix therefore implying that the model has common features.

3 Business Cycle Empirics

What is the best estimation procedure to recover the dynamic structure of the model economy? We will here compare two alternative strategies. The first is VAR analysis and consists in estimating a reduced form autoregressive model on $\mathbf{w}_t$, identifying the exogenous shocks using a minimal set of (just-identifying) restrictions and then matching the resulting impulse response functions with the theoretical ones (for a survey of this line of research applied to the study of the effects of monetary policy shocks, see

Christiano, Eichenbaum and Evans, 1999). The second exploits explicitely stochastic rank reduction and consists in the estimation of a dynamic factor model. This strategy was first advocated in the macroeconomic literature by Sargent and Sims, 1977 and used for structural analysis by Altug, 1989 and Sargent, 1989. That literature, however, while showing how to test for restrictions on the covariances of the data, did not go as far as showing how to estimate impulse response functions and identify shocks as in VARs. This is why factor models have not been popular tools for empirical structural and policy analysis. In what follows, we show how to do so and compare the estimates with those based on VARs.

3.1 VAR Analysis

For VAR estimation to be feasible, we must have full static rank since the estimation of A requires the inversion of $\Gamma_{\mathbf{w}}(0)$.

As we have seen, for the simple model, but this is true for a wide class of models, the model has static reduced rank and so has the VAR. In the case of the basic model $\Gamma_{\mathbf{w}}(0)$ has rank 2, so that a 5 dimensional VAR cannot be estimated, as $\Gamma_{\mathbf{w}}(0)$ cannot be inverted. Reduced static rank might be a characteristic of the theory, but it never occurs empirically, most likely because presence of measurement error in the data hide "fuzzifies" collinear relations.

With static rank reduction and measurement error, we can estimate a VAR for a block of variables of dimension r provided that the VAR representation for that block exists. Alternatively, we can assume measurement error, and estimate a VAR on the whole system. Let us now analyse the two cases.

A. *No measurement Error*

When variables are cleaned from measurement error, estimation can be performed on a block of $\mathbf{w_t}$, call it $\mathbf{w}_t^B$, so as to obtain a full rank covariance matrix of the variables in the block $\Gamma_{\mathbf{w}}(0)^B$.

Let us analyse this case for the general model and call the dimension of the block N_B. It is easily seen that any block has a VMA representation:

$$\mathbf{w}_t^B = \Theta^B(L)\epsilon_t$$

For example, if only the non predetermined variables are included in the block, then:

$$\Theta^B(L) = [\Lambda_1^B(L)C(L)^{-1}D(L) + \Lambda_2^B(L)]\Psi(L)^{-1}$$

For a VAR representation to exist, the following condition must hold.

Fundamentalness condition. There exists a $q \times N_B$ matrix of filters $\alpha(L)$ in non-negative powers of L such that:

$$\alpha(L)\Theta^B(L) = I_q.$$

This point has been made by Hansen and Sargent, 1990 and Lippi and Reichlin, 1993. For further insight into this issue, see Forni, Lippi and Reichlin, 2002.

If $p_x = 1$, $p_s = 0$ and the exogenous process is fundamental, this condition is satisfied. Hence, for our simple model the condition holds.

If the fundamentalness condition is satisfied, we can approximate the VMA representation with a finite order VAR:

$$\left(I_{N_B} - A^B(L)\right) \mathbf{w}_t^B = \mathbf{v}_t^B$$

where $A^B(L)$ is a finite order $N_B \times N_B$ matrix of filters and $\mathbf{v}_t^B = B^B \epsilon_t$, with B^B being a $N_B \times q$ matrix[2].

Notice that B^B is an orthonormal rotation of the first q principal component of $\Gamma_{v^B}(0)$. Defining as V the $q \times q$ matrix containing its first q eigenvalues and as J the $N_B \times q$ matrix of the corresponding eigenvectors, we have: $\epsilon_t = R' J^{-1/2} V' \mathbf{v}_t^B$, $\Gamma_{v_B}(0) = VJV' = B^B B^{B'}$, $B^B = VJ^{1/2}R$ where $RR' = I_q$.

The impulse response function are hence given by:

$$\mathbf{w}_t^B = \left(I_{N_B} - A^B(L)\right)^{-1} VJ^{1/2}R,$$

Notice that once we have consistent estimates of $A^B(L)$, the impulse response functions can be consistently estimated since the eigenvalues and the eigenvectors are continuous functions of of the matrix entries.

An important remark is that the dimension of the rotation matrix, and hence the degree of indeterminacy due to observational equivalence of alternative structures, depends only on the dimension q of the vector of exogenous shocks and not on the dimension of the subsystem N_B.

B. *Measurement Error.*

If the variables have independent measurement error, collinearity disappears and the estimation of the full system is always possible. VAR estimates, however, are no longer consistent.

Let us assume that measurement error comes in its simplest form, i.e. as a white noise process $\xi_t \sim WN(0, \Gamma_\xi(0))$ orthogonal to the vector of the variables of interest $\mathbf{w}_t$. Let us refer to the simple model. The vector of measured variables is:

$$\tilde{\mathbf{w}}_t = \mathbf{w}_t + \xi_t. \tag{3.5}$$

To prove that estimated parameters from a VAR on $\tilde{\mathbf{w}}_t$ are not consistent, it suffices to analyse OLS estimates for the VAR(1) case.

A VAR(1) for $\mathbf{w}_t$ implies the following model for the measured equation:

$$\tilde{\mathbf{w}}_t = A\mathbf{w}_{t-1} + \mathbf{u}_t + \xi_t \tag{3.6}$$

[2]Due to the approximation, $\mathrm{E}\mathbf{v}_t^B \mathbf{v}_t^{B'} = \Gamma_v(0)$ is not exactly of reduced rank.

where $\mathbf{u}_t = B\epsilon_t$.

The OLS estimate of A is:

$$\hat{A} = (\hat{\Gamma}_{\mathbf{w}}(0) + \hat{\Gamma}_{\xi}(0))^{-1}\hat{\Gamma}_{\mathbf{w}}(1)$$

where $\hat{\Gamma}_{\mathbf{w}}(0)$, $\hat{\Gamma}_{\mathbf{w}}(1)$ and $\hat{\Gamma}_{\xi}(0)$ are consistent estimates of the related population covariances. We have:

$$\hat{A} = A\hat{\Gamma}_{\mathbf{w}}(0)(\hat{\Gamma}_{\mathbf{w}}(0) + \hat{\Gamma}_{\xi}(0))^{-1}$$

and therefore:

$$\text{plim}(\hat{A}) = A\Gamma_{\mathbf{w}}(0)(\Gamma_{\tilde{\mathbf{w}}})^{-1} \neq A.$$

Given that we cannot recover the matrix A, it is evident that we cannot recover the structural impulse response functions.

More generally, consider a sub-system for $\tilde{\mathbf{w}}_t^B$. Given that $\tilde{\mathbf{w}}_t^B$ is stationary it has an $VMA(\infty)$, Wold, representation:

$$\tilde{\mathbf{w}}_t^B = \tilde{\Theta}^B(L)\tilde{\mathbf{v}}_t^B$$

where, because of the presence of measurement errors, $\tilde{\Theta}^B(L) \neq \Theta^B(L)$ even if the fundamentalness condition is satisfied. The Wold representation can be approximated, and hence estimated, through a VAR of finite order:

$$(I_{N_B} - \tilde{A}^B(L))\tilde{\mathbf{w}}_t^B = \tilde{\mathbf{v}}_t^B$$

Defining $\mathrm{E}\tilde{\mathbf{v}}_t^B\tilde{\mathbf{v}}_t^{B\prime} = \Gamma_{\tilde{v}}(0)$, the covariance matrix of the residuals, the impulse response functions are given by:

$$\mathbf{w}_t^B = \left(I_{N_B} - \tilde{A}^B(L)\right)(\Gamma_{\tilde{v}}(0))^{1/2}\,\tilde{R},$$

with $\tilde{R}\tilde{R}' = I_{N_B}$. Notice that the rotation matrix $\tilde{R}$ is of dimension N_B. The reason is that, because of the presence of measurement error, $\Gamma_{\tilde{v}}(0)$ and hence $(\Gamma_{\tilde{v}}(0))^{1/2}$ is of full rank.

The presence of measurement error implies that $\tilde{A}^B(L) \neq A^B(L)$, and that ϵ_t cannot be recovered for $\tilde{\mathbf{v}}_t^B$. Hence, there exists no rotation matrix $\tilde{R}$ for which one of the structural shocks has the same impulse response function of the "true" one.

It is interesting to stress that this problem is deeper than the typical identification indeterminacy that pervades the VAR literature (see Christiano, Eichnbaum and Evans, 1999). Even if the researcher knew perfectly the economic model and knew where to impose the "right" restrictions, the presence of measurement error will make inference impossible.

3.2 Factor model estimation

As Altug, 1989 and Sargent, 1989 have observed, if we add measurement error, the model economy has a factor analytic structure.

The dynamic state space representation for the general case becomes:

$$\tilde{\mathbf{w}}_t = \Phi(L)\Psi(L)^{-1}\epsilon_t + \xi_t = G(L)\epsilon_t + \xi_t$$

where ϵ_t is a vector of common shocks of dimension q and ξ_t is an idiosyncratic process of dimension N (see, for example, Sargent and Sims, 1977).

When the dynamic lag structure is finite, we can write the model in static form, by stacking lagged variables and we obtain[3]:

$$\begin{aligned} \tilde{\mathbf{w}}_t &= \Lambda F_t + \xi_t \\ H(L)F_t &= K\epsilon_t \end{aligned} \tag{3.7}$$

where Λ is a $N \times r$ matrix, F_t is $r \times 1$ and ΛF_t represents the "common component" of $\tilde{\mathbf{w}}_t$, of dimension r, while ξ_t is the "idiosyncratic component" of dimension N. The impulse response functions are defined, up to a rotation as: $G(L) = \Lambda H(L)^{-1}K$. The model written in this way, is the static state representation discussed earlier. It can be shown that in the case which the order of the AR process for $\mathbf{s}_t$ is less or equal than p_s, the filter $H(L)$ in (3.7) is of order 1 so that the states have an AR(1) representation (on this point, see Giannone, Reichlin and Sala, 2002).

Let us analyse the estimation of this model for the basic model example where $F_t = HF_{t-1} + K\epsilon_t$ (the result can be easily generalized).

Under the orthogonal measurement error assumption, the model is identified once we impose the normalization condition $\mathrm{E}FF' = I$. The covariances of the variables of interest and of measured data are:

$$\Gamma_{\tilde{\mathbf{w}}}(0) = \Gamma_{\mathbf{w}}(0) + \Gamma_{\xi}(0)$$

$$\Gamma_{\mathbf{w}}(0) = \Lambda\Lambda'$$

$$\Gamma_{\mathbf{w}}(1) = \Gamma_{\tilde{\mathbf{w}}}(1)$$

We will now show how we can recover the impulse response functions $(I - AL)^{-1}B$ from the covariances.

¿From equation (3.6) and the Yule-Walker equations we have:

$$\Gamma_{\mathbf{w}}(1) = A\Gamma_{\mathbf{w}}(0) = A\Lambda\Lambda'.$$

Moreover:

$$\Gamma_u(0) = \Gamma_{\mathbf{w}}(0) - A\Gamma_{\mathbf{w}}(1).$$

[3]For a discussion of the difference between a static and a dynamic factor representation, see Forni, Lippi and Reichlin, 2002

Our problem is to estimate the parameters of A and B which can be expressed in terms of the factor loadings Λ's. We have:

$$A = \Lambda H(\Lambda'\Lambda)^{-1}\Lambda', \qquad B = \Lambda K$$

To estimate A we will have to compute the generalized inverse of $\Gamma_{\mathbf{w}}(0)$, i.e.:

$$\Gamma_{\mathbf{w}}(0)^{-1} = [\Lambda\Lambda']^{+} = \Lambda(\Lambda'\Lambda)^{-2}\Lambda'.$$

The estimator for A is:

$$\hat{A} = \hat{\Gamma}_{\mathbf{w}}(1)[\hat{\Gamma}_{\mathbf{w}}(0)]^{+}.$$

To obtain a consistent estimate of A we need a consistent estimate of Λ.

Once the Λ's are estimated consistently, consistency of A follows from consistency of $\hat{\Gamma}_{\mathbf{w}}(1)$. We have:

$$\text{plim}\hat{A} = \Lambda H(\Lambda'\Lambda)^{-1}\Lambda'$$

We obtain B as an orthonormal rotation of the first q principal component of $\Gamma_u(0)$. Defining as M the $q \times q$ diagonal matrix containing the q largest eigenvalues of $\Gamma_u(0)$ and as P the $N_B \times q$ matrix of the corresponding eigenvectors, we have: $\Gamma_u(0) = PMP' = BB'$ and $B = PM^{1/2}R$ where $RR' = I_q$. The consistency of the empirical counterpart, $\hat{B}$, is a consequence of consistency of $\hat{A}$, $\hat{\Gamma}_{\mathbf{w}}(0)$ and $\hat{\Gamma}_{\mathbf{w}}(1)$ since $\Gamma_u(0) = \Gamma_{\mathbf{w}}(0) - A\Gamma_{\mathbf{w}}(1)$.

4 Empirical comparison

The exercise here is as follows. We generate the model economy and then estimate it using the VAR procedure and the factor model procedure with and without measurement error. We generate 500 vector time-series $\mathbf{w}_t = (c_t,\, r_t,\, y_t,\, k_t,\, z_t)'$ for our model economy with a sample size $T = 200$.

We compute impulse response functions for alternative estimation procedures and report bias, mean square errors and confidence bands.

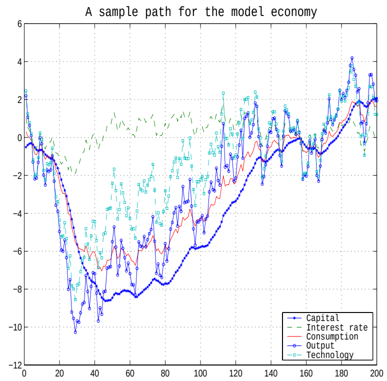
The particular model economy is the simple business cycle model where we use the same calibrated parameters as in Uhlig, 1998. They are reported on the Table below.

Table 1. Calibrated Parameters

β	.99
ρ	.36
η	1
δ	.025
ψ	.95

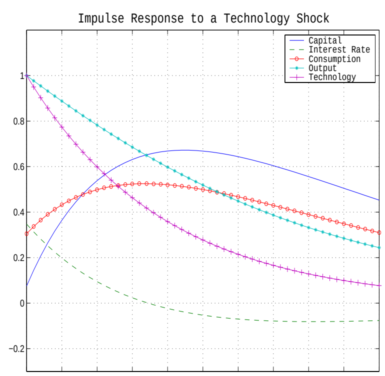
In Figure 1 we show the sample paths of the five variables for one simulation of the model.

Figure 1. Simulated Path



In Figure 2 we show the theoretical impulse response functions in response to a unitary technology shock generated by our model.

Figure 2. Theoretical Impulse Response Functions



The measurement error is generated as:

$$\xi_t \sim i.i.d.N\left(\mathbf{0}_5, \mathrm{diag}[\gamma_r, \gamma_c, \gamma_y, \gamma_k, \gamma_z]\right)$$

with the γ_i's calibrated so that the degree of commonality, given by the ratio $\frac{Var(\tilde{\mathbf{w}}_t^i)}{Var(\mathbf{w}_t^i)} = 1 - \frac{\gamma_i}{Var(\mathbf{w}_t^i)}$, is the same for $i = r, \ldots, z$ and is equal to: $VR = [1, .9, .8, .6]$.

A. *VAR analysis*

As a full-size VAR on $\mathbf{w}_t$ cannot be estimated without introducing measurement error, we concentrate on the sub-block $\mathbf{w}_t^{yc} = (y_t, c_t)'$.

We estimated the VAR on the sub-block by assuming to know the lag length (in this case, 1). Given that we have just one shocks there are no identification problems in absence of measurement errors. For the case with measurement error we identify the structural impulse response function by choosing one column of the orthonormal rotation matrix $\tilde{R}$:

$$\tilde{R} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad \theta \in [-\pi, \pi].$$

Our choice is to set θ so as to minimize the sum of the Euclidean distances between the true and the estimated impulse responses for 12 period after the shock for both y_t and c_t.

B. *Factor estimates*

To obtain consistent estimates of the loadings Λ, we could use either a Kalman filter maximum likelihhod procedure (e.g. Engle and Watson, 1993) or a procedure recently proposed by Doz and Lenglart, 1999. We opt for the latter since it is well suited for the benchmark business cycle model estimated here because it exploits static rank reduction and it presents computational advantages over the Kalman filter. The method is a quasi maximum likelihood procedure where the likelihood is first defined as if both the factors and the idiosyncratic components were not autocorrelated and then it is shown that in a stationary framework such estimates are still consistent (see appendix for details). For the dynamic case, this method does not work since we cannot exploit static rank reduction. In this case, we can either use any Kalman filter based method if n is sufficiently small or, if n is large, principal components methods as suggested by Forni, Hallin, Lippi and Reichlin, 2000 or Stock and Watson, 2002.

Let us remark that, as we have done for the number of lags in VAR estimates, we assume here that r and q are known. Notice, however, that since we are using a likelihood procedure, we could have used the test for the number of common factors proposed by Doz and Lenglart, 1999.

C. *Comparison*

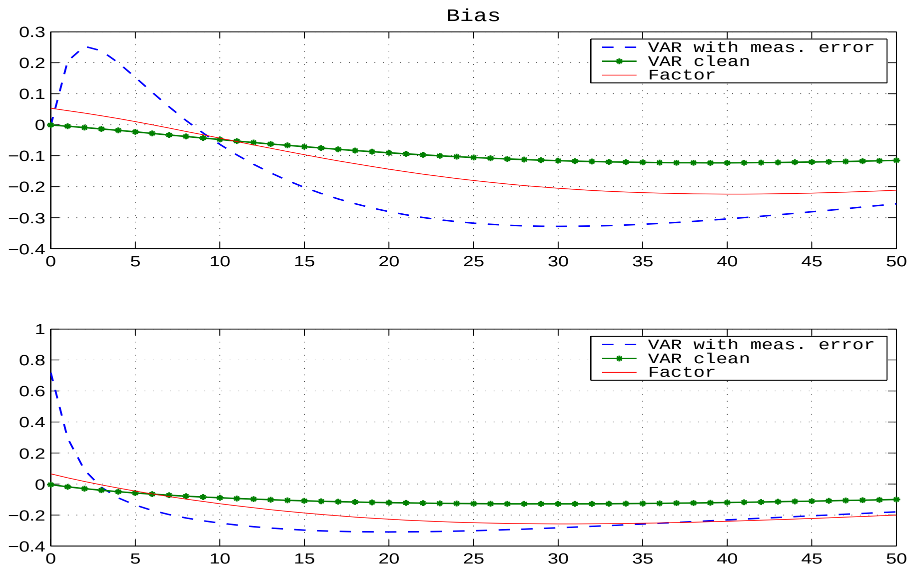
Let us now provide a comparison between the two alternative methods is provided in what follows. For each value of VR we report 4 figures. These figures display, respectively, the bias, the mean square error and the confidence bands[4] for the estimates of the impulse response functions for the two variables under scrutiny, y_t and c_t and for both the VAR and the factor model, taking as a benchmark the bivariate VAR estimation without measurement error.

For example, Figures 4 and 5 show the Bias and the MSE of the estimated impulse response functions in the case $VR = .9$ for c_t (top) and y_t (bottom). The statistics for the "clean" VAR, the dotted line, are reported for comparison.

Figures 6 and 7 report the true impulse response functions and the confidence bands computed from, respectively, a VAR on $\mathbf{w}_t^{yc}$ (without measurement error) a VAR on $\tilde{\mathbf{w}}_t^{yc}$ (with measurement error) and the factor model (from left to right). The true impulse response functions are reported for comparison (bold line).

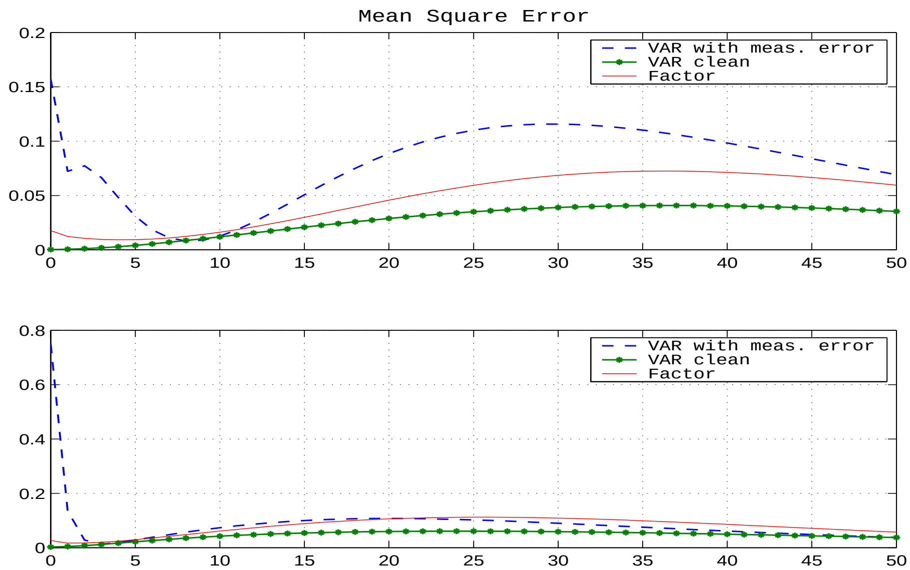
[4]95% confidence bands were computed from the empirical distribution function by taking the 2.5-th and the 97.5-th percentile.

Figure 4. Bias - VR = 0.9



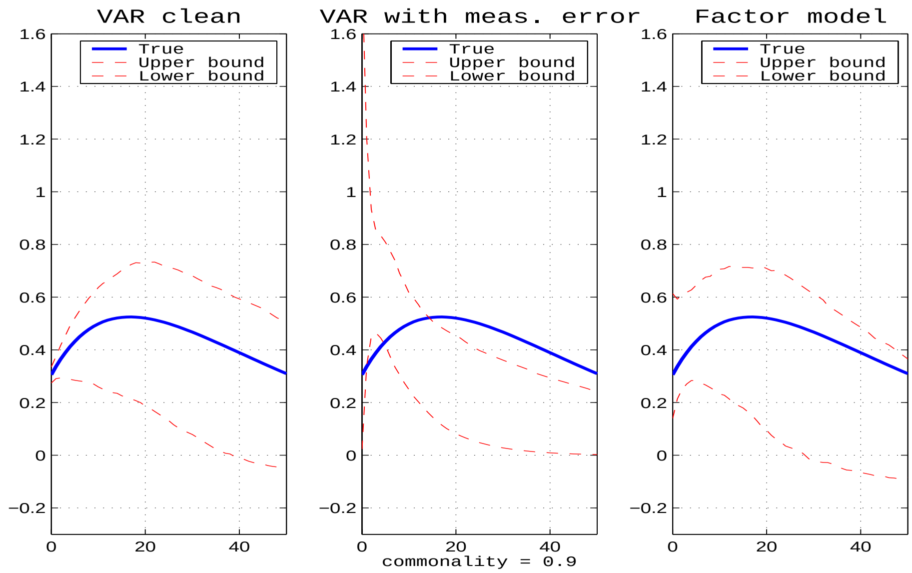
top: Consumption - bottom: Output

Figure 5. Mean Square Error - VR = 0.9



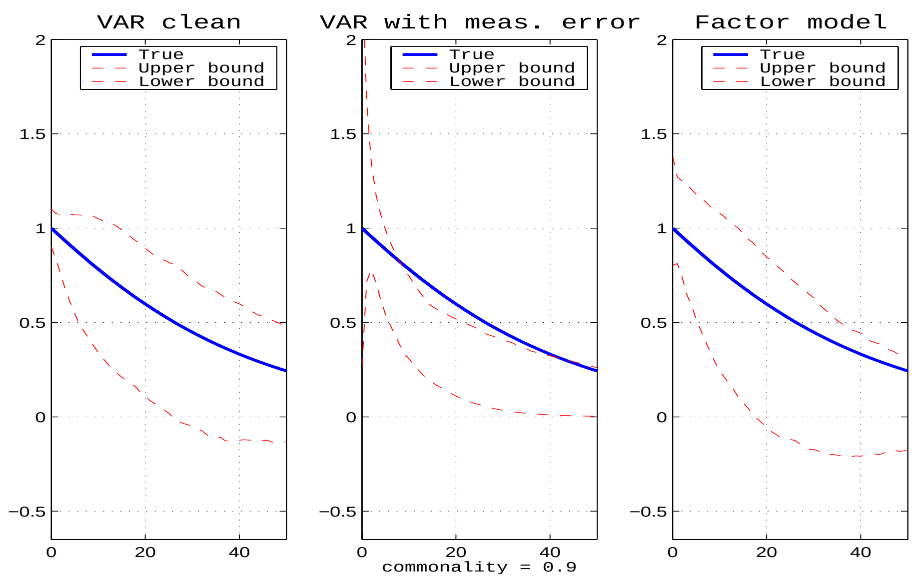
top: Consumption - bottom: Output

Figure 5. Comparison of Impulse Response Functions - VR = 0.9



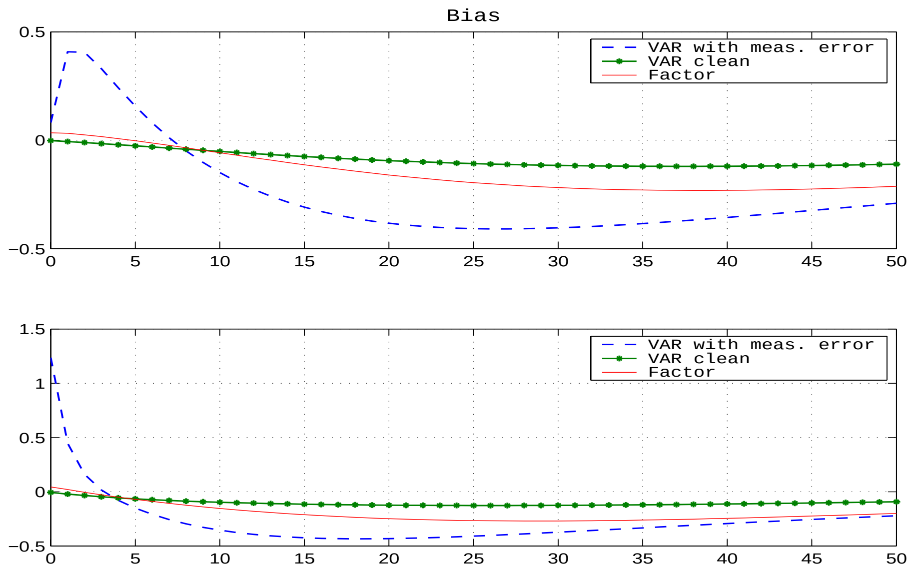
Consumption

Figure 7. Comparison of Impulse Response Functions - VR = 0.9



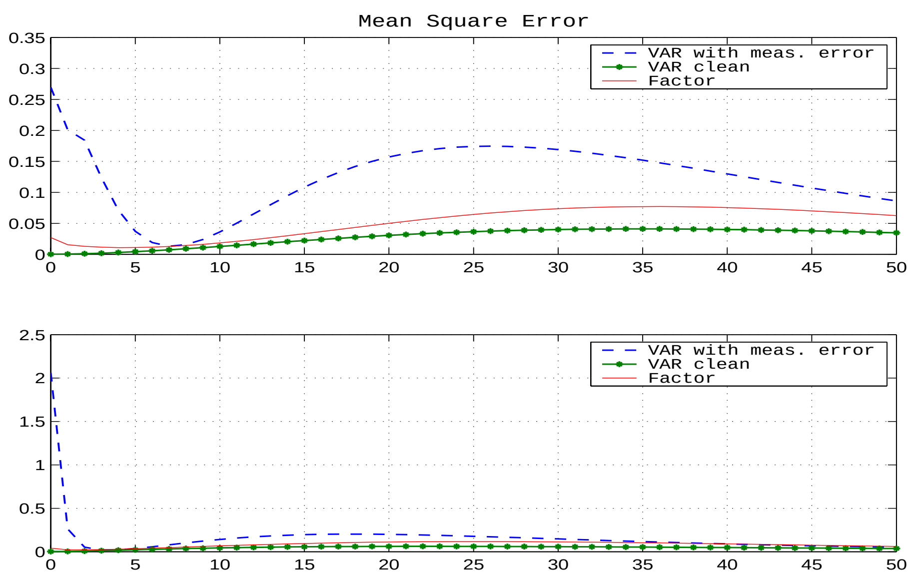
Output

Figure 8. Bias - VR = 0.8



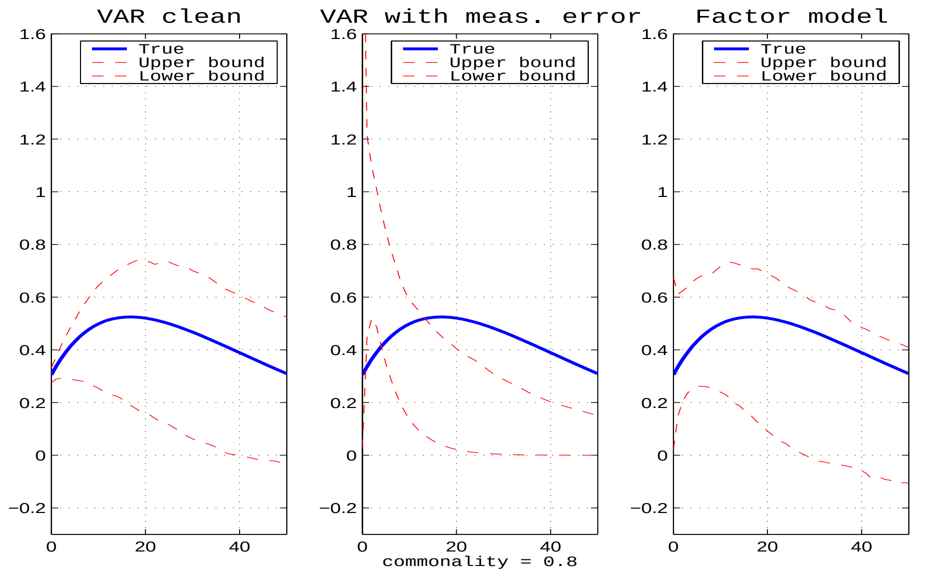
top: Consumption - bottom: Output

Figure 9. Mean Square Error - VR = 0.8



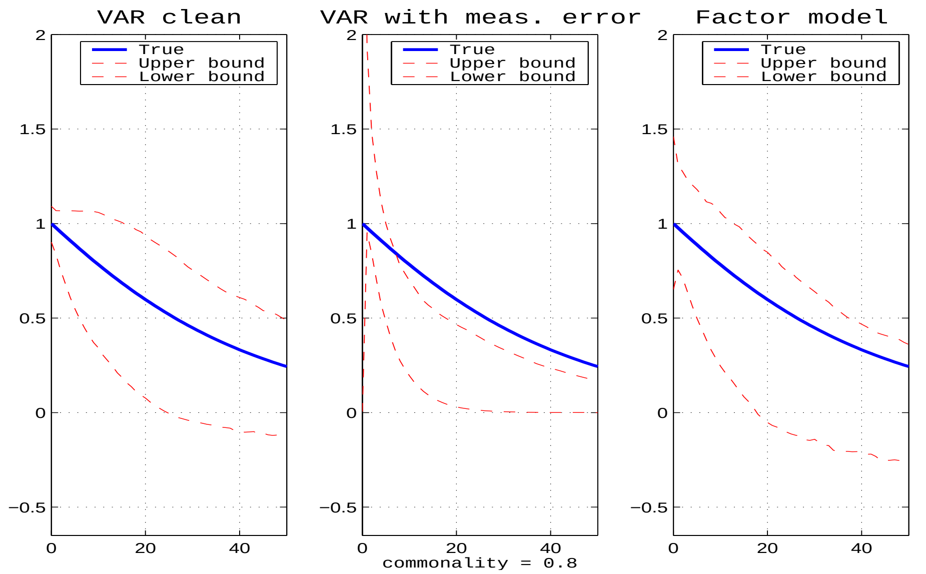
top: Consumption - bottom: Output

Figure 10. Comparison of Impulse Response Functions - VR = 0.8



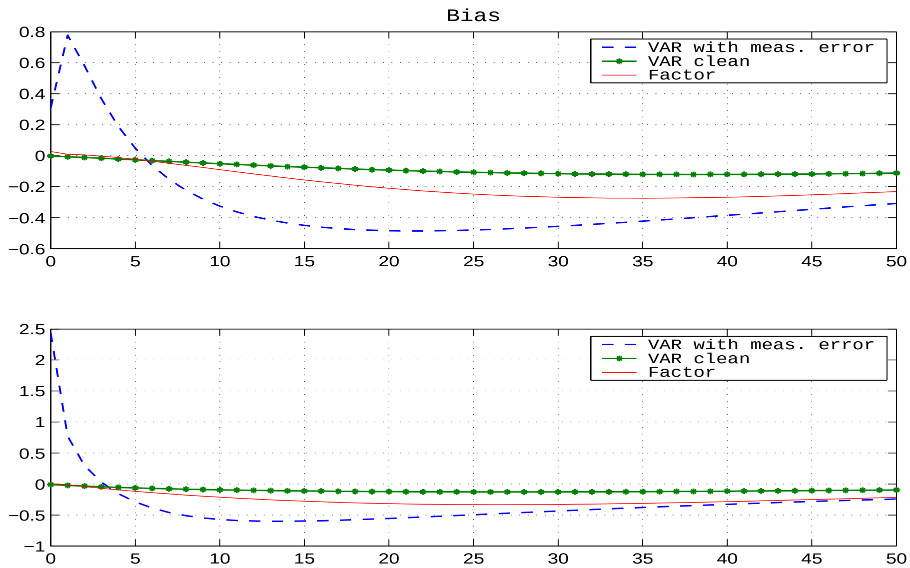
Consumption

Figure 11. Comparison of Impulse Response Functions - VR = 0.8



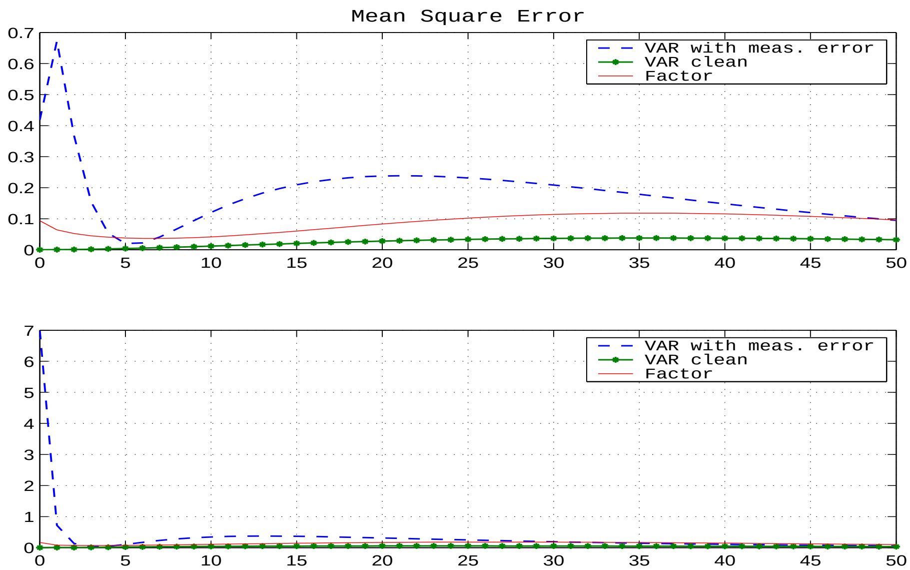
Output

Figure 12. Bias - VR = 0.6



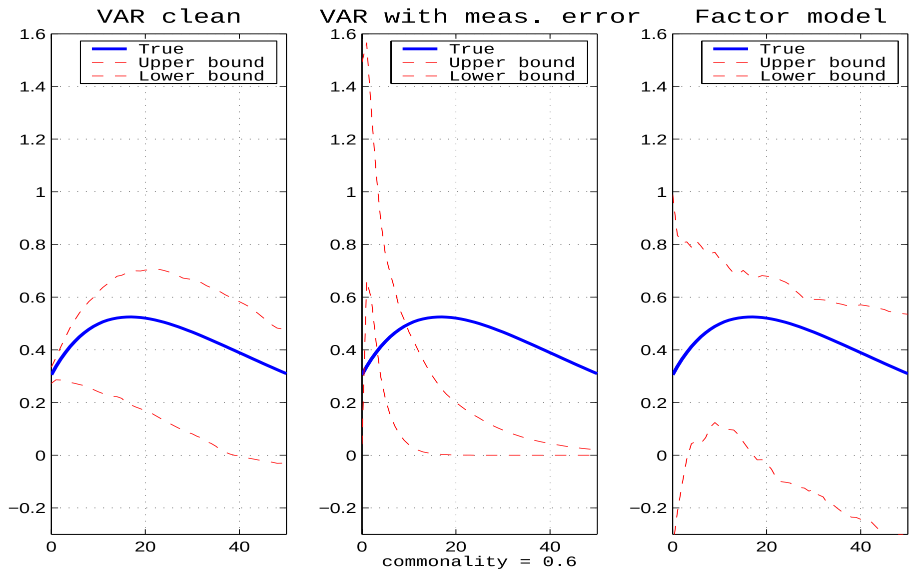
top: Consumption - bottom: Output

Figure 13. Mean Square Error - VR = 0.6



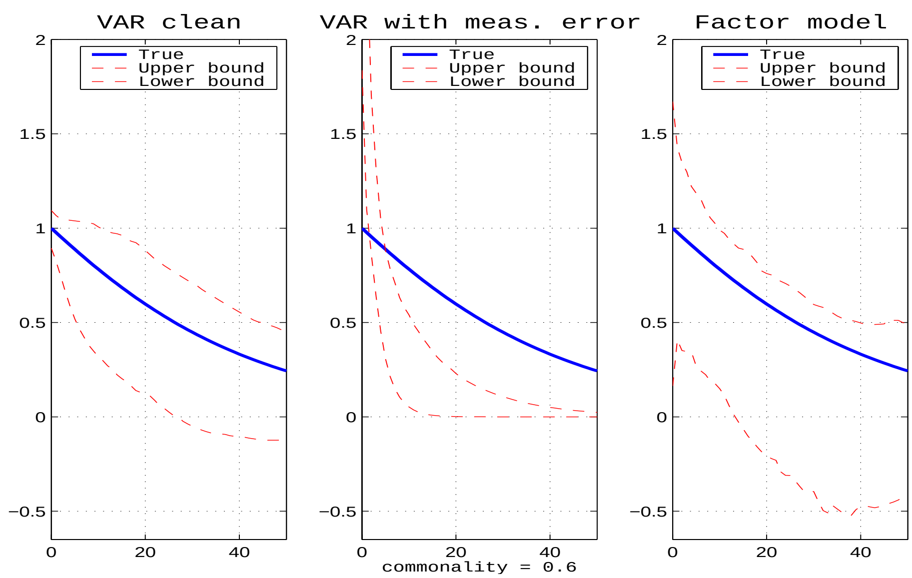
top: Consumption - bottom: Output

Figure 14. Comparison of Impulse Response Functions - VR = 0.6



Consumption

Figure 15. Comparison of Impulse Response Functions - VR = 0.6



Output

The results show the following features.

First, in all the experiments considered and at all horizons, both the bias and the mean square error are larger in the case of the "contaminated" VAR.

Second, and this demonstrates empirically the results of the previous Section, even a small measurement error ($VR = .9$) is sufficient to spoil the inference drawn from the

VAR. Consider for example Figures 5 and 6. It is evident from the confidence intervals that the VAR does not consistently estimate the true dynamics of the system: the true response almost always lies out of the confidence bands. As VR becomes smaller, the factor model has harder and harder times in estimating efficiently the impulse responses, as one can see from the fact that confidence bands widen. However, factor estimates are always within the bands.

5 n large: Discussion

As mentioned in the introduction, factor models have recently gained popularity as a parsimonios method to estimate dynamic relations in large panels of time series. This recent literature advocates principal component methods for the estimation of the factor space and provide consistency results and rates as the dimension n of the cross section and the time dimension T go to infinity. The model analysed in this literature are more flexible than standard factor models since the idiosyncratic component is typically allowed to contain cross-correlated elements. Under these conditions, common and idiosyncratic components are not identified for n fixed and this makes it impossible to use maximum likelihood estimation; higher flexibility in the parametrization of the model comes at a cost of possible loss of efficiency. Comparing finite sample estimation performance for this class of econometric models and the likelihood based ones is an interesting project, but not the problem addressed here. Let us here instead make few remarks on what we would gain in adopting the large cross section approach, instead of the n fixed approach used in this paper, for the purpose of the structural analysis we have discussed.

First of all, let us oberve that the model with n large corresponds to a model with a large number of states such as, for example, the model with heterogenous capital (Campbell, 1997) or models with heterogenous agents and different information sets. These models are therefore more complex than the benchmark business cycle example analysed here, but they belong to the same family. From the econometric point of view, information contained in a large cross-section helps in cleaning from measurement error. The intuition is that principal components are linear combinations of the measured variables of the panel and they become increasingly collinear with the underlying variables of interest as n increases (the idiosyncratic component capturing measurement error dies on average by a law of large number mechanism). This desirable effect, moreover, is obtained for more general models than the traditional factor model with orthogonal idiosyncratic components and therefore allows to take into account more complex measurement error than the one assumed in this paper. In fact, in the more flexible specification, the error term could also be thought as an error in model specification as, for example, in Watson, 1993. For what concerns shocks identification, the strategy is the same as the one followed here. To obtain consistent estimates of the impulses, we just need consistent estimates of the factor loadings and the shocks are identified as rotations of principal components of the residuals of the VAR on the factors (see Giannone, Reichlin and Sala, 2002 and Forni, Lippi and Reichlin, 2002). As n increases, the number of shocks to identify remains fixed at q. No matters how large is the number of states, the complexity of the identification problem depends on the number of exogenous shocks q.

6 Conclusion

This paper argues that factor models are better empirical tools than VARs for identifying and estimating impulse response functions. The reasons are twofold. First, equilibrium business cycle models imply fewer exogenous shocks than variables. Second, variables are measured with errors.

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7 Appendix: Doz-Lenglart procedure

Doz and Lenglart, 1999, consider the quasi-likelihood of the model (3.7), computed under a Gaussian assumption as if neither the factors nor the idiosyncratic component were autocorrelated. The quasi-likelihood can be written, up to a constant term, as:

$$L_T\left(\mathbf{w}_1^M, \ldots, \mathbf{w}_T^M; \Lambda, \Gamma_\xi(0)\right) = -\frac{1}{2}\ln\left(\det\left(\Lambda\Lambda' + \Gamma_\xi(0)\right)\right) - \frac{1}{2}\mathrm{trace}\left(\left(\Lambda\Lambda' + \Gamma_\xi(0)\right)^{-1}\hat{\Gamma}_{\mathbf{w}^M}(0)\right)$$

The maximum likelihood estimates, $\hat{\Lambda}$ and $\hat{\Gamma}_\xi(0)$, are the solution of the following system (cfr. Magnus and Neudecker, 1988):

$$\Lambda = \hat{\Gamma}_{\mathbf{w}^M}(0)\left(\Gamma_\xi(0) + \Gamma_\xi(0)\right)^{-1}\Lambda \tag{7.8}$$

$$\Gamma_\xi(0) = \mathrm{diag}(\hat{\Gamma}_{\mathbf{w}^{\mathrm{M}}}(0) - \Lambda\Lambda') \tag{7.9}$$

If $\Gamma_\xi(0)$ were known, (7.8) would be satisfied if:

$$\hat{\Lambda} = \left(\Gamma_\xi(0)\right)^{1/2} V_r J_r \tag{7.10}$$

where J_r is the $r \times r$ diagonal matrix conaining the r largest eigenvalues of

$$\left(\Gamma_\xi(0)\right)^{-1/2}\left(\hat{\Gamma}_{\mathbf{w}^M}(0) - \Gamma_\xi(0)\right)\left(\Gamma_\xi(0)\right)^{-1/2}$$

and V_r is the $N \times r$ matrix of corresponding orthonormal eigenvectors.

Thus, (7.10) provides an explicit solution for Λ as a function of $\Gamma_\xi(0)$ and (7.9) gives $\Gamma_\xi(0)$ as an explicit solution of Λ. The solution of the system (7.8) and (7.9) can hence be found iteratively choosing an appropiate starting value for $\Gamma_\xi(0)$.

Doz and Lenglart, 1999, show that if F_t and ξ_t are weakly stationary, then $\hat{\Lambda}$ and $\hat{\Gamma}_\xi(0)$ are consistent:

$$\mathrm{plim}\hat{\Lambda} = \Lambda, \quad \mathrm{plim}\hat{\Gamma}_\xi(0) = \Gamma_\xi(0), \text{ as T} \to \infty$$