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Rational Preference and Rationalizable Choice*

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Abstract

We study a decision maker characterized by two binary relations. The first reflects his judgments about well-being, his *mental preferences*. The second describes the decision maker's choice behavior, his *behavioral preferences*, the ones that govern choice (see Rubinstein and Salant, 2008a,b). Specifically, in the context of decision making under uncertainty, we propose axioms that may describe the rationality of these two relations. These axioms allow a joint representation by a single set of probabilities and a single utility function. It is mentally rational to prefer f over g if and only if the expected utility of f is at least as high as that of g for *all* probabilities in the set. It is behaviorally rationalizable to choose f over g if and only if the expected utility of f is at least as high as that of g for *some* probability in the set. In other words, mental and behavioral preferences admit, respectively, a representation à la Bewley (2002) and à la Lehrer and Teper (2011).

Our results also provide a foundation for a decision analysis procedure called *robust ordinal regression* and proposed by Greco, Mousseau, and Słowiński (2008).

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1 Introduction

1.1 Mental and behavioral consistency

In the recent years there has been a renewed attention to the process of decision making, specifically to the transition from *mental preferences* to *behavioral preferences*.[1] Mental preferences are represented by a binary relation $\succsim^*$ on the set of alternatives that describes the decision maker's (DM) judgments about his own well-being. Behavioral preferences are represented by another binary relation $\succsim^\circ$ on the same set of alternatives that describes the DM's choice behavior.

Mental preferences "exist in the mind" of the DM, regardless of any choice to be made among some available alternatives. Thus, $f \succsim^* g$ means that the DM considers f *at least as good as* g. When $f, g \in \mathbb{R}^S$ represent state contingent payoffs, a natural example of a mental preference is weak Pareto dominance: $f \succsim^* g$ if and only if $f_s \geq g_s$ for all $s \in S$.[2]

Behavioral preferences, instead, rationalize the choice data available to an outside observer: $f \succ^\circ g$ means that f is *always chosen* from $\{f, g\}$, whereas $g \succsim^\circ f$ means the opposite, that is, g *can be chosen* from $\{f, g\}$.[3]

Arguably,[4] on these preferences it is reasonable to require:

Transitivity of $\succsim^*$: *If $f \succsim^* g$ and $g \succsim^* h$, then $f \succsim^* h$.*

Completeness of $\succsim^\circ$: *If $f \not\succsim^\circ g$, then $g \succsim^\circ f$.*

Consistency: *If $f \succsim^* g$, then $f \succsim^\circ g$.*

This latter assumption means that, whenever possible, mental preferences inform choice. Transitivity of $\succsim^*$ alludes to the fact that mental judgments are "rational". At the same time, comparing all alternatives may be impossible (for example because of some missing relevant information),[5] thus $\succsim^*$ is not assumed to be complete. In contrast, $\succsim^\circ$ is complete by definition (the burden of choice), but its transitivity is conceptually not very compelling, as the following example shows.

Example 1 (Beer & Wine) As Kreps (2012, p. 21) puts it "the consumer is allowed to say that 4 cans of beer and 11 bottles of wine is strictly better than 3 and 10, but both are incomparable to 20 cans of beer and 6 bottles of wine". In the dual preferences setup that we just introduced, we can think that $\succsim^*$ is the weak Pareto dominance. Since $(4, 11)$ strongly Pareto dominates $(3, 10)$, the consumer *always* chooses $(4, 11)$ from $\{(3, 10), (4, 11)\}$, that is, $(4, 11) \succ^\circ (3, 10)$. Now assume that he has to choose from $\{(3, 10), (20, 6)\}$ and from $\{(4, 11), (20, 6)\}$. Since the mental preference offers no guidance, he may *sometimes* choose $(3, 10)$ from $\{(3, 10), (20, 6)\}$ and

[1] See e.g. Mandler (2005), Rubinstein and Salant (2008a,b), and Gilboa et al. (2010).

[2] Pareto dominance is actually relevant for any choice among multidimensional alternatives, such as consumption bundles in consumer theory and attribute vectors in multi-criteria decision making.

[3] In terms of observed frequencies of choice, $f \succ^\circ g$ means that f is chosen from $\{f, g\}$ with frequency 1, while $g \succsim^\circ f$ means that such frequency is smaller than 1, that is, the frequency of choice of g from $\{f, g\}$ is not 0.

[4] See Mandler (2005), who calls $\succsim^*$ psychological preference and $\succsim^\circ$ revealed preference.

[5] See, e.g., Gilboa, Postlewaite, and Schmeidler (2009).

sometimes choose $(20, 6)$ from $\{(4, 11), (20, 6)\}$. In terms of behavioral preferences, this says that $(3, 10) \succsim^\circ (20, 6)$ and $(20, 6) \succsim^\circ (4, 11)$, which together with $(4, 11) \succ^\circ (3, 10)$ yields a violation of transitivity. ▲

Standard theory posits a single strict preference relation $\succ$ that implicitly reflects both strict mental and strict behavioral preferences (see Fishburn, 1970, and Kreps, 1988). In other words, $\succ^*$ and $\succ^\circ$ are assumed to coincide. Under the assumptions of Completeness of $\succsim^\circ$ and Consistency, it is easy to see that this coincidence is equivalent to the following two requirements:[6]

Possibility: *If $f \not\succsim^* g$, then $g \succsim^\circ f$.*

Strict Consistency: *If $f \succ^* g$, then $f \succ^\circ g$.*

Possibility says that g can possibly be chosen from $\{f, g\}$ whenever f is not mentally preferred to g, that is, whenever it is not a priori clear that f is better than g. As discussed in the previous example, this naturally explains the possible intransitivity of behavioral preference in that there are no compelling reasons for mental incomparability to be transitive.[7]

Strict Consistency corresponds to the assumption that the choice of a mentally dominated option can never be observed: "if $f \succ^* g$, then g is never chosen from $\{f, g\}$". This is more controversial. Indeed, Strict Consistency may result in choice behavior described by $\succ^\circ$ that is unstable under small "trembles" affecting the alternatives. Specifically, it may be the case that $f \succ^* g$, but arbitrarily small perturbations of f and g destroy such a strict mental preference. Under Strict Consistency, the strict behavioral preference $f \succ^\circ g$ may then be destroyed as well. For instance, when $\succsim^*$ is the weak Pareto dominance on $\mathbb{R}^2$, we have $(1, 0) \succ^* (0, 0)$, but in every neighborhood of $(0, 0)$ there exists an element $(-1/n, 1/n)$ such that $(1, 0) \not\succsim^* (-1/n, 1/n)$. Under Possibility and Strict Consistency, the unstable pattern

$$(-1/n, 1/n) \succsim^\circ (1, 0) \succ^\circ (0, 0) \qquad \text{for all } n \in \mathbb{N}$$

thus results for $\succ^\circ$. This instability of behavioral preference contrasts with its choice interpretation, according to which, $f \succ^\circ g$ means that f is always chosen from $\{f, g\}$ and g is never. Intuitively, the choice of f from $\{f, g\}$ in all circumstances presumes a stability with respect to small perturbations of the choice situation.

In order to avoid the highlighted instability, we first weaken Strict Consistency.[8] This is done by considering the robustification $\succ\!\!\succ^*$ of $\succ^*$, which is informally codified by

$$f \succ\!\!\succ^* g \iff f + \text{"specification error"} \succ^* g + \text{"specification error"}.$$

[6]See Lemma 2 for details.

[7]Observe that, under Possibility, mental incomparability of $(3, 10)$ and $(4, 11)$ with $(20, 6)$, not only implies $(3, 10) \succsim^\circ (20, 6)$ and $(20, 6) \succsim^\circ (4, 11)$, but also $(3, 10) \precsim^\circ (20, 6)$ and $(20, 6) \precsim^\circ (4, 11)$. Therefore, the resulting violation of transitivity actually applies to behavioral indifference.

[8]Of course, one could think of weakening Possibility too: this is done in Section 3.

Formally, with a convex set F of alternatives, we define $f \ggcurly^* g$ if and only if, for every $h, l \in F$, there exists $\varepsilon > 0$ such that

$$(1-\delta) f + \delta h \succ^* (1-\delta) g + \delta l \qquad \text{for all } \delta \in [0, \varepsilon]. \tag{1}$$

Accordingly, we replace Strict Consistency with the weaker:

Strong Consistency: *If $f \ggcurly^* g$, then $f \succ^\circ g$.*

Notice that when $\succsim^*$ is the weak Pareto dominance on $\mathbb{R}^S$, $\ggcurly^*$ coincides with strong Pareto dominance, that is, $f \ggcurly^* g$ if and only if $f_s > g_s$ for all $s \in S$. Furthermore, when alternatives are stochastic, the "specification error" view of (1) is the one used in robust statistics since Hampel (1974).

1.2 Representation

We consider Anscombe-Aumann acts $f : S \to X$ as alternatives, where S is a set of states and X a convex set of outcomes. In this way, we can interpret alternatives as describing state contingent (possibly random) payoffs, thus encompassing, inter alia, both the case in which alternatives belong to $\mathbb{R}^S$ and the one in which they are stochastic objects.

In a nutshell, our **first contribution** is to show that, under standard expected utility assumptions, Possibility and Strong Consistency are equivalent to the existence of an affine utility function u on outcomes and a set C of probabilities on states that jointly represent $\succsim^*$ and $\succsim^\circ$ by

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \text{for all } p \in C \tag{2}$$

$$f \ggcurly^{**} g \iff \int u(f)\, dp > \int u(g)\, dp \quad \text{for all } p \in C \tag{3}$$

and

$$f \succsim^\circ g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \text{for some } p \in C \tag{4}$$

$$f \succ^\circ g \iff \int u(f)\, dp > \int u(g)\, dp \quad \text{for all } p \in C \iff f \ggcurly^* g \tag{5}$$

respectively.

If we interpret u as describing the DM's material objectives and C as representing the DM's information about the nature of the uncertainty he is facing, the preference of f over g is mentally uncontroversial, i.e., $f \succsim^* g$, if and only if the expected utility of f is at least as high as that of g for *all* probabilistic scenarios consistent with the available information. On the other hand, f can be chosen over g, i.e., $f \succsim^\circ g$, if and only if the expected utility of f is at least as high as that of g for *some* of these scenarios (the choice is justified by scenarios, which are thus called *justifications*). Additionally, f is always chosen over g, i.e., $f \succ^\circ g$, if and only if the expected utility of f is strictly higher than that of g for *all* of these scenarios (the choice of g cannot be justified).

Notice that (5), by showing that $\succ^\circ$ coincides with $\succ\!\!\succ^*$, permits to derive behavioral preferences from mental ones: indeed, $f \succsim^\circ g$ if and only if $g \not\succ\!\!\succ^* f$. Our **second contribution** is about the converse, that is, the possibility of eliciting mental preferences from behavioral ones. Specifically, if $\succsim^\circ$ does not admit both a minimum and a maximum element in X, we show that $\succsim^*$ is the transitive core – or trace – of $\succsim^\circ$, in particular, it can be inferred from $\succsim^\circ$.[9]

Finally, we extend our main results by showing that, by weakening Possibility and retaining Strong Consistency, $\succsim^*$ maintains representation (2), while $\succsim^\circ$ takes the form

$$f \succsim^\circ g \iff \int u(f)\,dp \geq \int u(g)\,dp + c(p) \qquad \text{for some } p \in C \tag{6}$$

where $c : C \to [0,1]$ ranks justifications according to their plausibility: this is our **third contribution**. The intuition here is that the higher $c(p)$ is, the less plausible justification p is. Unless $c(p) = 0$, the condition $\int u(f)\,dp \geq \int u(g)\,dp$ is not sufficient to justify the choice of f over g, and the stronger condition $\int u(f)\,dp \geq \int u(g)\,dp + c(p)$ is required. Notably, in this case behavioral preferences are not completely determined by mental ones, but surprisingly it is still possible to infer mental preferences from behavioral preferences, since $\succsim^*$ turns out to be again the transitive core of $\succsim^\circ$.

1.3 Related literature

In the language of modern decision theory, $\succsim^*$ is a multiple prior (incomplete) preference à la Bewley (2002), and $\succsim^\circ$ is a (complete) justifiable preference in the sense of Lehrer and Teper (2011). Therefore our results provide, inter alia, a novel foundation of justifiable preferences under uncertainty.

We follow Gilboa et al. (GMMS, 2010) in considering a pair $(\succsim^*, \succsim^\circ)$ of binary relations.[10] In their paper, the first relation $\succsim^*$ is an incomplete preorder à la Bewley (2002), like in our case, whereas the second $\succsim^\circ$ is a maxmin multiple prior preference à la Gilboa and Schmeidler (1989), that is, (4) is replaced by

$$f \succsim^\circ g \iff \min_{p \in C} \int u(f)\,dp \geq \min_{p \in C} \int u(g)\,dp. \tag{7}$$

As discussed in detail in Section 4, the DM described by GMMS can choose f out of $\{f, g\}$ if and only if it is a maxminimizing strategy against a malevolent nature, while our DM can choose f out of $\{f, g\}$ if and only if it is a rationalizable strategy against a neutral Nature. In terms of departures from standard expected utility, in GMMS the "cost of completeness" is a loss of independence of $\succsim^\circ$, while in the present analysis this "cost" is a loss of transitivity of $\sim^\circ$.

[9] As recently shown by Cerreia-Vioglio and Ok (2015), the trace of $\succsim^\circ$ (see, for example, Bouyssou and Pirlot, 2005, and Nishimura, 2014) is the maximal "coherently transitive" subrelation of $\succsim^\circ$, see Section 2.5.

[10] Other works closely related to ours are Kopylov (2009), Cerreia-Vioglio (2016), Faro (2015), and Faro and Lefort (2015).

Our approach also provides a foundation of a popular procedure in decision analysis, called *robust ordinal regression*, in the case of choice under uncertainty. Here a decision analyst (DA) supports the preference formation of a DM. The ingredients available to the DA are:

- an observable ranking $\succsim^{\diamond}$ of *some* alternatives supplied by the DM himself, the data, in the form
$$f_i \succsim^{\diamond} g_i \qquad \text{for all } i \in I$$
(this ranking is typically *very* incomplete);

- some structural assumptions on the family $\mathcal{U} = \{U_p\}_{p\in P}$ of evaluation functionals the DM is willing to use to rank alternatives.

The problem is extending $\succsim^{\diamond}$ to the set of *all* alternatives. This lead Greco, Mousseau, Słowiński (2008) and Giarlotta and Greco (2013) to consider the family of parameters corresponding to evaluation functionals that are consistent with the data $\succsim^{\diamond}$, that is,

$$C = \{p \in P \mid U_p(f_i) \geq U_p(g_i) \text{ for all } i \in I\}.$$

After obtaining this set of parameters, the DA infers that f shall be *necessarily preferred* to g, denoted by $f \succsim^{*} g$, if and only if

$$U_p(f) \geq U_p(g) \qquad \text{for all } p \in C \tag{8}$$

that is, f outperforms g for all consistent evaluation functionals. On the other hand, the DA infers that f might be *possibly chosen* over g, denoted by $f \succsim^{\circ} g$, if and only if

$$U_p(f) \geq U_p(g) \qquad \text{for some } p \in C \tag{9}$$

that is, f outperforms g for some consistent evaluation functional. In this perspective, the present paper can be seen as an axiomatic foundation for the robust ordinal regression approach, in the special case in which the parameters $p \in P$ are probabilities on the state space S, and

$$U_p(f) = \int u(f)\, dp \tag{10}$$

for all acts f. Like in Lehrer and Teper (2014), a natural candidate for $\succsim^{\diamond}$ is an expected utility preference defined only on a restricted family of acts.

Finally, our results on the transitive core of $\succsim^{\circ}$ refine and non-trivially extend those of Nishimura (2014).

1.4 Organization of the results

The detailed exposition of the model and the main results are presented in the next Section 2. Section 3 is devoted to the more general model (6) and its properties. The final Section 4 clarifies the relation of our model with GMMS, and puts both of them in a "games against Nature" perspective. All the proofs are relegated to Appendix A.

2 Model and main results

2.1 Preliminaries

We use a stylized version of the Anscombe and Aumann (1963) framework introduced by Fishburn (1970). Here X is a convex set of *outcomes*, the set S of *states of the world* is endowed with an algebra Σ of *events*, and the set Δ of (finitely additive) *probabilities* on Σ is endowed with the event-wise convergence topology. The set of *acts* F consists of all simple measurable functions $f : S \to X$, that is

$$F = \left\{ \sum_{i=1}^{n} 1_{A_i} x_i \;\middle|\; n \in \mathbb{N},\ \{A_i\}_{i=1}^{n} \text{ is a partition of } S \text{ in } \Sigma,\ \{x_i\}_{i=1}^{n} \subseteq X \right\}.$$

The original version of Anscombe and Aumann is obtained by assuming X is the set of lotteries (that is, finitely supported probability distributions) over a set Z of deterministic prizes, and it is the most decision-theoretically relevant specification.

As anticipated, the DM is characterized by two binary relations $\succsim^*$ and $\succsim^\circ$ on F, the first representing mental preferences, and the second representing behavioral preferences. As usual, we denote by $\succ^*$ and $\succ^\circ$ the asymmetric parts of $\succsim^*$ and $\succsim^\circ$, and by $\sim^*$ and $\sim^\circ$ their symmetric parts. Finally, we extend $\succsim^*$ and $\succsim^\circ$ to X by identifying outcomes with constant acts.

2.2 The definition of strong mental preferences

Recall that we defined $f \ggcurly^* g$ if and only if for every $h, l \in F$ there is $\varepsilon > 0$ such that

$$(1 - \delta) f + \delta h \succ^* (1 - \delta) g + \delta l \qquad \text{for all } \delta \in [0, \varepsilon]. \tag{11}$$

This formalizes the intuition that

$$f \ggcurly^* g \iff f + \text{“specification error”} \succ^* g + \text{“specification error”} \tag{12}$$

for all specification errors that are “sufficiently small”, and it justifies the interpretation of $\ggcurly^*$ as representing strong mental preferences.

In order to better relate intuition and formalism, consider the case in which X is the set of lotteries on Z. Every $x \in X$ is a (finitely supported) probability measure on Z, and a *specification error* for x is a (finitely supported) signed measure m on Z such that $x + m$ is still a probability measure on Z, that is, $x + m$ still belongs to X.[11] This amounts to say that there exists $y \in X$ such that $x + m = y$ and $m = y - x$, in particular

$$x + \delta m = x + \delta (y - x) = (1 - \delta) x + \delta y \in X \qquad \text{for all } \delta \in [0, 1].$$

This makes clear the sense in which (11), written as

$$f + \delta (h - f) \succ^* g + \delta (l - g) \qquad \text{for all } \delta \in [0, \varepsilon]$$

[11] In particular $m(Z) = 0$, that is, m only redistributes the mass of x among the points of Z.

is the formal version of (12). In fact, for each s in S, $e(s) = h(s) - f(s)$ is the generic specification error for $f(s) \in X$, and $d(s) = l(s) - g(s)$ is the generic specification error for $g(s) \in X$.

2.3 The representation of mental preferences

Beyond technicalities, the assumptions we make on mental preferences amount to say that they admit a multiple prior representation (2) à la Bewley (2002). A first characterization of these preferences appears in GMMS (Theorem 1), and a second one on page 769 of the same paper. Here we propose a third minor variation – equivalent to the first two – which will be useful in comparing our analysis to theirs.

With the exception of the (existential) property of Non-triviality, all axioms below are intended as starting with the universal quantification "*Given any* f, g, h, l *in* F, ..."

Basic Conditions (BC)

Reflexivity: $f \succsim^* f$.

Monotonicity: $f(s) \succ^* g(s)$ *for all* $s \in S$ *implies* $f \succ^* g$.

Continuity: $\{\lambda \in [0,1] : \lambda f + (1-\lambda)g \succsim^* \lambda h + (1-\lambda)l\}$ *is closed.*

Non-triviality: *there exist constant* f *and* g *in* F *such that* $f \succ^* g$.

As discussed in the introduction, $\succsim^*$ is typically incomplete, but it enjoys some strong structural properties, listed below, which guarantee an "expected multi-utility" representation.

C-Completeness, Transitivity, and Independence

C-completeness: *if* f *and* g *are constant, then either* $f \succsim^* g$ *or* $g \succsim^* f$ *(or both).*

Transitivity: $f \succsim^* g$ *and* $g \succsim^* h$ *imply* $f \succsim^* h$.

Independence: $f \succsim^* g$ *implies* $\lambda f + (1-\lambda)h \succsim^* \lambda g + (1-\lambda)h$ *for all* λ *in* $(0,1)$.

Conceptually, C-completeness presumes that incompleteness of mental preferences is due to uncertainty. Indeed, the DM has complete preferences between outcomes, but not over uncertain acts.[12] On the other hand, Transitivity and Independence may be seen as assumptions on the rationality guiding the formation of mental preferences. Dubra and Ok (2002) refer to them as "procedures in going from simple decisions to complex ones", whereas GMMS call them "inference rules" with a similar intuition.

Lemma 1 *If a binary relation* $\succsim^*$ *on* F *satisfies the BC, C-Completeness, Transitivity, and Independence, then, given any* $f, g, h \in F$ *and any* λ *in* $(0,1)$,

[12] Indeed, also incompleteness about the ranking of outcomes could be relevant, depending on the decision problem at hand. See Aumann (1962), Kannai (1963), Richter (1966), Peleg (1970), and, more recently, Ok (2002), Dubra, Maccheroni, and Ok (2004), Nau (2006), Ok, Ortoleva, and Riella (2012), Galabaataar and Karni (2013).

(a) $f(s) \succsim^* g(s)$ *for all* $s \in S$ *implies* $f \succsim^* g$;

(b) $\lambda f + (1-\lambda)h \succsim^* \lambda g + (1-\lambda)h$ *implies* $f \succsim^* g$.

In particular, the BC, C-Completeness, Transitivity, and Independence are necessary and sufficient for the existence of a non-empty closed and convex set C *of probabilities on* Σ *and a non-constant affine function* $u : X \to \mathbb{R}$ *such that, for every* $f, g \in F$,

$$f \succsim^* g \iff \int u(f)\,dp \geq \int u(g)\,dp \qquad \text{for all } p \in C. \tag{13}$$

In this case, C *is unique,* u *is unique up to positive affine transformations, and, moreover*

$$f \succ\!\!\succ^* g \iff \int u(f)\,dp > \int u(g)\,dp \qquad \text{for all } p \in C. \tag{14}$$

As anticipated, this lemma shows that our conditions on $\succsim^*$ are equivalent to those of GMMS, and hence (13) follows from their Theorem 1. On the other hand (14) is novel and characterizes the algebraic interior of a multiple priors relation à la Bewley. Also notice that, since $\succ^*$ is the asymmetric part of $\succsim^*$, we have

$$f \succ^* g \iff \int u(f)\,dp \geq \int u(g)\,dp \quad \text{for all } p \in C \text{ with at least one strict inequality.} \tag{15}$$

Finally, observe that for S finite, $X = \mathbb{R}$, $u = \mathrm{id}_{\mathbb{R}}$, and $C = \Delta$, the mental preference $\succsim^*$ is simply the weak Pareto dominance $\geq$ on $\mathbb{R}^S$, the strict mental preference $\succ^*$ is the Pareto dominance $>$ on $\mathbb{R}^S$, and the strong mental preference $\succ\!\!\succ^*$ is the strong Pareto dominance $\gg$ on $\mathbb{R}^S$.

2.4 Main representation theorem

Theorem 1 *The following conditions are equivalent for a pair* $(\succsim^*, \succsim^\circ)$ *of binary relations on* F:

(i) $\succsim^*$ *satisfies the BC, C-Completeness, Transitivity, and Independence;*
$\succsim^\circ$ *satisfies Continuity;*
$(\succsim^*, \succsim^\circ)$ *satisfies Possibility and Strong Consistency.*

(ii) There exist a non-empty closed and convex set C *of probabilities on* Σ *and a non-constant affine function* $u : X \to \mathbb{R}$ *such that, for every* $f, g \in F$,

$$f \succsim^* g \iff \int u(f)\,dp \geq \int u(g)\,dp \qquad \text{for all } p \in C \tag{16}$$

and

$$f \succsim^\circ g \iff \int u(f)\,dp \geq \int u(g)\,dp \qquad \text{for some } p \in C. \tag{17}$$

(iii) $\succsim^*$ *satisfies the BC, C-Completeness, Transitivity, and Independence;*
$\succsim^\circ$ *is complete and* $\succ^\circ$ *coincides with* $\succ\!\!\succ^*$.

(iv) $\succsim^*$ *satisfies the BC, C-Completeness, Transitivity, and Independence;*
$f \succsim^\circ g$ *if and only if* $g \not\succ\!\!\succ^* f$.

In this case, C *is unique,* u *is unique up to positive affine transformations, and, moreover*

$$f \succ^\circ g \iff \int u(f)\,dp > \int u(g)\,dp \quad \text{for all } p \in C \iff f \succ\!\!\succ^* g. \tag{18}$$

The equivalence

$$f \succsim^\circ g \iff g \not\succ\!\!\succ^* f \tag{19}$$

in point (iv) describes how mental preferences fully determine choice behavior. In the next section we investigate the converse problem of how mental preferences can be inferred from choice behavior. Here we further clarify how the behavioral preferences characterized in Theorem 1 are a robust counterpart of the classical ones defined by

$$f \succsim^\circ g \iff g \not\succ^* f \tag{20}$$

and discussed in the Introduction (see also Lemma 2 in Appendix A). In order to distinguish between the two different behavioral preferences, we maintain the notation $f \succsim^\circ g$ for the ones defined by (19) that satisfy the assumptions of Theorem 1 and we simply write $g \not\succ^* f$ for those defined by (20). The next proposition essentially says that $f \succsim^\circ g$ if and only if there exist suitable (see the proof for details) sequences

$$\begin{array}{ccc} g_n & \not\succ^* & f_n \\ \downarrow & & \downarrow \\ g & & f. \end{array}$$

From a geometric viewpoint, under the assumptions of Theorem 1, $\succ^\circ$ is the algebraic interior $\succ\!\!\succ^*$ of $\succ^*$, and its complement $\precsim^\circ$ (that is, $\not\succ^\circ$) is the algebraic closure of the complement $\not\succ^*$ of $\succ^*$.

Proposition 1 *Under the assumptions of Theorem 1,* $f \succsim^\circ g$ *if and only if there exist* $h \not\prec^* l$ *such that*

$$(1-\gamma)\,g + \gamma l \not\succ^* (1-\gamma)\,f + \gamma h \qquad \text{for all } \gamma \in (0,1]\,.$$

2.5 From behavioral preferences to mental preferences

Under the assumptions of Theorem 1, $\succsim^*$ is a preorder and the pair $(\succsim^*, \succsim^\circ)$ satisfies another form of consistency, namely:

Transitive Consistency: *If either* $f \succsim^* g \succsim^\circ h$ *or* $f \succsim^\circ g \succsim^* h$*, then* $f \succsim^\circ h$.

In particular, under the assumptions of Theorem 1, the pair $(\succsim^*, \succsim^\circ)$ is a NaP-preference in the sense of Giarlotta and Greco (2013).

Cerreia-Vioglio and Ok (2015) have recently shown that, given any reflexive relation $\succsim^\circ$, the maximal subrelation $\succsim$ of $\succsim^\circ$ such that:

- $\succsim$ is a preorder, and
- $(\succsim, \succsim^\circ)$ satisfies Transitive Consistency,

exists, and it is given by

$$f \succsim^{\circ\circ} g \iff \begin{cases} h \succsim^\circ f \implies h \succsim^\circ g \\ g \succsim^\circ l \implies f \succsim^\circ l. \end{cases} \tag{21}$$

In the theory of semiorders and interval orders, the relation defined by (21) takes the name of *trace* of $\succsim^\circ$, and it is traditionally attributed to Duncan Luce and Peter Fishburn (see Bouyssou and Pirlot, 2005, and references therein). Its use in decision theory has been recently revived by Nishimura (2014), under the name of *transitive core* of $\succsim^\circ$.

The equivalence (21) is relevant to our analysis because it allows us to retrieve $\succsim^{\circ\circ}$ from $\succsim^\circ$. Next we show that a little strengthening of the assumptions of Theorem 1 guarantees that $\succsim^*$ coincides with $\succsim^{\circ\circ}$. This makes it possible to elicit $\succsim^*$ starting from $\succsim^\circ$.[13] Our result refines Proposition 5 of Nishimura (2014), and it applies to the original setting of Lehrer and Teper (2011).

Proposition 2 *Under the assumptions of Theorem 1, for every $f, g \in F$,*

$$f \succsim^* g \iff \begin{cases} h \succsim^\circ f \implies h \succsim^\circ g \\ g \succsim^\circ l \implies f \succsim^\circ l \end{cases}$$

provided $\succsim^\circ$ does not admit both a minimum and a maximum element in X.

The next example shows that this result is tight in that the assumption that $u(X)$ is not compact cannot be dropped.

Example 2 Let $S = \{-1, 1\}$, $X = [-1, 1]$, $f(s) = s$ and $g(s) = -s$ for all $s \in S$. Let $u(x) = x$ for all $x \in X$ and $C = \Delta$. Then, $h \succsim^\circ g$ for all $h \in F$, because $\int h d\delta_1 \geq -1 = \int g d\delta_1$, and $f \succsim^\circ l$ for all $l \in F$, because $\int f d\delta_1 = 1 \geq \int l d\delta_1$. By (21), this implies $f \succsim^{\circ\circ} g$, because automatically $h \succsim^\circ f$ implies $h \succsim^\circ g$ and $g \succsim^\circ l$ implies $f \succsim^\circ l$. But $\int f d\delta_{-1} = -1 < 1 = \int g d\delta_{-1}$ implies $f \not\succsim^* g$. ▲

The next proposition allows us to retrieve $\succsim^*$ from $\succsim^\circ$ without any requirement on $u(X)$.

Proposition 3 *Under the assumptions of Theorem 1, $f \succsim^* g$ if and only if there exist $h \succ^\circ l$ such that*

$$(1-\gamma) f + \gamma h \succ^\circ (1-\gamma) g + \gamma l \qquad \textit{for all } \gamma \in (0, 1].$$

Conceptually, this closes the loop of Theorem 1 by showing that, not only mental preferences determine choice behavior, but also choice behavior allows to infer mental preferences. Also the geometric loop is closed: Proposition 3 says that $\succsim^*$ is the algebraic closure of $\succ^\circ$ (which, in turn, is the algebraic interior of $\succsim^*$, as Proposition 4 in Appendix A shows).

[13] In the setting of GMMS, an analogous result holds by replacing the *transitive core* of $\succsim^\circ$ with the *unambiguous part* of $\succsim^\circ$, that is, the maximal subrelation of $\succsim^\circ$ satisfying independence (see Ghirardato, Maccheroni, and Marinacci, 2004).

3 Loosening the mental tie

So far mental preferences fully determine choice behavior: two DMs sharing the same mental preferences behave the same. This is due to the fact that, according to the representation (4) of $\succsim^\circ$ all justifications for choosing f from $\{f, g\}$ are equally good (or bad).

We now relax this dependence by permitting that different (consistent) behaviors may correspond to the same mental preferences. This is possible if the DM deems some justifications p in C more plausible or more convincing than others. Formally, it corresponds to the existence of a cost function $c : C \to [0, \infty)$ that penalizes less plausible justifications in a way that

$$f \succsim^\circ g \iff \int u(f)\, dp \geq \int u(g)\, dp + c(p) \qquad \text{for some } p \in C.$$

The less plausible is the justification — i.e., the higher $c(p)$ — the higher must be the difference between the expected utility of f and that of g to justify the choice of the former over the latter. In this more general setting two different DMs, who share the same mental preferences (same C), may have different plausibility rankings (different cost functions c). So, their choice behavior might well differ.

Specifically, we first add two assumptions on $\succsim^*$ and $\succsim^\circ$:

Unboundedness: *If $f \succ^* g$ in F are constant, then there are constant h and l in F such that $\frac{1}{2}h + \frac{1}{2}g \succsim^* f \succ^* g \succsim^* \frac{1}{2}f + \frac{1}{2}l$.*

In words, there are arbitrarily good and bad outcomes. Mathematically, this is equivalent to assume that u in Lemma 1 be onto.

Strict Independence: *If $g \succeq^\circ l$ and $\lambda \in (0,1)$,*[14] *then*

$$f \succ^\circ h \implies \lambda f + (1-\lambda) g \succ^\circ \lambda h + (1-\lambda) l$$

and the converse implication is true when $g = l$.

Differently from the assumption of unboundedness, in view of (18) this condition is clearly satisfied if Theorem 1 holds. This is true also for the next three consistency conditions on the interplay between $\succsim^*$ and $\succsim^\circ$.

Strong Transitive Consistency: *$f \not\succ^* g$ and $g \succeq^\circ h$ imply $f \succ^\circ h$.*

Substitution Consistency: *if $f \sim^* h$ and $g \sim^* l$, then $f \succsim^\circ g$ implies $h \succsim^\circ l$.*

Weak Possibility: *For each g in F there exists $\tilde{g}$ in F such that $f \not\succsim^* g$ implies $\tilde{g} \succsim^\circ f$.*

We can now state the anticipated extension of Theorem 1. In reading it, recall that a function $c : C \to [0, \infty)$ is grounded if and only if $\inf_{p \in C} c(p) = 0$.

[14] By $g \succeq^\circ l$ we mean either $g \succ^\circ l$ or $g = l$.

Theorem 2 *The following conditions are equivalent for a pair* $(\succsim^*, \succsim^\circ)$ *of binary relations on* F*:*

(i) $\succsim^*$ *satisfies the BC, C-Completeness, Transitivity, Independence, and Unboundedness;* $\succsim^\circ$ *satisfies Completeness, Continuity, and Strict Independence;* $(\succsim^*, \succsim^\circ)$ *satisfies Strong Transitive Consistency, Substitution Consistency, and Weak Possibility.*

(ii) There exist a non-empty closed and convex set C *of probabilities on* Σ*, a grounded, convex, lower semicontinuous, bounded* $c : C \to [0, \infty)$*, and an onto affine function* $u : X \to \mathbb{R}$ *such that, for every* $f, g \in F$*,*

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \text{for all } p \in C$$

and

$$f \succsim^\circ g \iff \int u(f)\, dp \geq \int u(g)\, dp + c(p) \qquad \text{for some } p \in C.$$

In this case, $\succsim^*$ *coincides with the transitive core* $\succsim^{\circ\circ}$ *of* $\succsim^\circ$*. Moreover,* C *is unique,* u *is unique up to positive affine transformations, and* c *is unique given* u*.*

More specifically, Remark 4 shows that if u is replaced with the cardinally equivalent $\alpha u + \beta$ (with $\alpha > 0$ and $\beta \in \mathbb{R}$), then c must be replaced with αc. This implies that the preference model of Theorem 1 corresponds to the special case in which $c(p) = 0$ for all $p \in C$ and so $f \succ\!\succ^* g \iff f \succ^\circ g$. If, instead, $c(q) > 0$ for some $q \in C$, then the uniqueness properties of c imply that this equivalence is lost. Since Strong (Transitive) Consistency says that $f \succ\!\succ^* g \implies f \succ^\circ g$, it must be the case that there are pairs f and g in F such that

$$f \not\succ\!\succ^* g \text{ but } f \succ^\circ g.$$

That is, f is always chosen from $\{f, g\}$ although the mental preference does not provide robust arguments for such a strict behavioral preference. Some hesitation, not due to mental preferences, precludes the choice of g. Thus, as anticipated, mental preferences do not determine behavioral ones, but the latter still allow to infer the former by computation of the transitive core.

4 Games against Nature

In this final section, we provide a direct connection with GMMS by describing the rationality relation between mental and behavioral preferences. As anticipated our assumptions on $\succsim^*$ (the BC, C-Completeness, Transitivity, and Independence) coincide with those of GMMS. On the other hand, on $\succsim^\circ$ they assume Transitivity, while $\succsim^\circ$ in our Theorem 1 only satisfies:

C-Transitivity: *if* f*,* g*, and* h *are constant, then* $f \succsim^\circ g$ *and* $g \succsim^\circ h$ *imply* $f \succsim^\circ h$*.*

That is, we restrict Transitivity of $\succsim^{\circ}$ to constant acts, on the contrary GMMS restrict Possibility to constant acts and call it Caution, namely:

Caution (C-Possibility): *If f is constant and $g \not\succsim^{*} f$, then $f \succsim^{\circ} g$.*

The next variation on Theorem 1 shows how the replacement of our C-Transitivity and Possibility with theirs Transitivity and C-Possibility is the only formal difference between the two approaches. The conceptual differences are briefly discussed after the statement.

Theorem 3 *The following conditions are equivalent for a pair $(\succsim^{*}, \succsim^{\circ})$ of binary relations on F:*

(i) *$\succsim^{*}$ satisfies the BC, C-Completeness, Transitivity, and Independence;*
$\succsim^{\circ}$ satisfies the BC, Completeness, ***C-Transitivity****, and C-Independence;*[15]
$(\succsim^{*}, \succsim^{\circ})$ satisfies Transitive Consistency and ***Possibility****.*

(ii) *There exist a non-empty closed and convex set C of probabilities on Σ and a non-constant affine function $u : X \to \mathbb{R}$ such that, for every $f, g \in F$,*

$$f \succsim^{*} g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \text{for all } p \in C \tag{22}$$

and

$$f \succsim^{\circ} g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \text{for some } p \in C. \tag{23}$$

Moreover, replacing ***C-Transitivity*** *and* ***Possibility*** *with* ***Transitivity*** *and* ***C-Possibility*** *in (i) is equivalent to replace (23) in (ii) with*

$$f \succsim^{\circ} g \iff \min_{p \in C} \int u(f)\, dp \geq \min_{p \in C} \int u(g)\, dp. \tag{24}$$

Beyond the formal differences outlined above, conceptually GMMS focus on rationality in decision making under uncertainty and on a (more or less fictitious) dialogue between the DM and a DA. The choice of f from $\{f, g\}$ is *objectively rational*, $f \succsim^{*} g$, if the DM can convince the DA that she is right in making it. The choice of f from $\{f, g\}$ is *subjectively rational*, $f \succsim^{\circ} g$, if the DA cannot convince the DM that she is wrong in making it.

On the other hand, the focus here is on a possible relation between well-being judgements and choice behavior. A simple unifying perspective is the game theoretic one, according to which a decision problem under uncertainty can be seen as a game against Nature, in which Nature's available mixed strategies belong to C. In this perspective, the choice behavior predicted by GMMS corresponds to maxminimization if Nature is assumed to be malevolent, while the one predicted here corresponds to rationalizability if Nature is assumed to be neutral.

[15] C-Independence requires that: given any $\lambda \in (0, 1)$ and any constant h in F, then $f \succsim^{\circ} g$ if and only if $\lambda f + (1 - \lambda) h \succsim^{\circ} \lambda g + (1 - \lambda) h$.

This well known axiom is due to Gilboa and Schmeidler (1989), and it is shared by both this paper and GMMS.

A Proofs

A.1 Proof of Theorem 1 and related results

Lemma 2 *Let $\succsim^*$ and $\succsim^\circ$ be two binary relations on F such that $\succsim^\circ$ satisfies Completeness and $(\succsim^*, \succsim^\circ)$ satisfies Consistency. The following conditions are equivalent:*

(i) $f \succ^* g \iff f \succ^\circ g$;

(ii) $f \not\precsim^* g \implies g \succsim^\circ f$ *(Possibility) and* $f \succ^* g \implies f \succ^\circ g$ *(Strict Consistency);*

(iii) $g \succsim^\circ f \iff f \not\succ^* g$.

In this case, $f \sim^\circ g$ *if and only if either* f *and* g *are* $\succsim^*$ *incomparable or* $f \sim^* g$.

Completeness of $\succsim^\circ$ and its Consistency with $\succsim^*$ say that the former is a behavioral completion of the latter. The lemma shows that Possibility and Strict Consistency uniquely pin down $\succsim^\circ$: its asymmetric part coincides with that of $\succsim^*$, and its symmetric part is the union of $\succsim^*$ indifference and incomparability.

Proof of Lemma 2. Since $\succsim^\circ$ is complete, then $g \succsim^\circ f \iff f \not\succ^\circ g$. Thus (i) $f \succ^* g \iff f \succ^\circ g$ is equivalent to $f \not\succ^* g \iff f \not\succ^\circ g$ which is equivalent to $f \not\succ^* g \iff g \succsim^\circ f$ which is (iii). This shows (i) $\iff$ (iii).[16]

By (i) $f \succ^* g \implies f \succ^\circ g$, moreover $f \not\precsim^* g$ implies $f \not\succ^* g$ which by (iii) implies $g \succsim^\circ f$. So (i) $\implies$ (ii). Conversely, by (ii) $f \succ^* g \implies f \succ^\circ g$. Now if $f \succ^\circ g$, then $g \not\succsim^\circ f$, and (ii) implies $f \succsim^* g$; if we had $g \succsim^* f$, Consistency of $\succsim^\circ$ with $\succsim^*$ would imply $g \succsim^\circ f$ which is impossible, then $g \not\succsim^* f$ and $f \succ^* g$. This shows (ii) $\implies$ (i), concluding the proof of the first part of the statement.

Now if $f \not\sim^\circ g$, because of completeness of $\succsim^\circ$ either $f \succ^\circ g$ or $g \succ^\circ f$. Say $f \succ^\circ g$, by (i), $f \succ^* g$, so that f and g are neither $\succsim^*$ incomparable nor $\succsim^*$ indifferent. Conversely, if f and g are neither $\succsim^*$ incomparable nor $\succsim^*$ indifferent, comparability implies either $f \succsim^* g$ or $g \succsim^* f$, non-indifference means that they cannot both hold so that $f \succ^* g$ or $g \succ^* f$. Say $f \succ^* g$, by (i), $f \succ^\circ g$, so that f and g are not $\succsim^\circ$ indifferent. ■

Proof of Lemma 1. On constant acts, $\succsim^*$ is non-trivial and satisfies the axioms of Herstein and Milnor (1953). Therefore there exists a non-constant affine $u : X \to \mathbb{R}$ such that, given $x, y \in X$, $x \succsim^* y$ if and only if $u(x) \geq u(y)$.

(a) Take $x, y \in X$ such that $x \succ^* y$. If $f(s) \succsim^* g(s)$ for all $s \in S$, then

$$u(f(s)) \geq u(g(s)) \qquad \forall s \in S.$$

[16] Here we only used Completeness of $\succsim^\circ$ (not its Consistency with $\succsim^*$).

Therefore, for all $s \in S$ and all $\lambda \in (0,1)$,

$$\begin{gathered}\lambda u\left(f(s)\right) + (1-\lambda)\, u\left(x\right) > \lambda u\left(g(s)\right) + (1-\lambda)\, u\left(y\right) \Longrightarrow \\ u\left(\lambda f(s) + (1-\lambda)\, x\right) > u\left(\lambda g(s) + (1-\lambda)\, y\right) \Longrightarrow \\ \lambda f(s) + (1-\lambda)\, x \succ^* \lambda g(s) + (1-\lambda)\, y.\end{gathered}$$

By Monotonicity, this implies $\lambda f + (1-\lambda)\, x \succ^* \lambda g + (1-\lambda)\, y$ for all $\lambda \in (0,1)$, and Continuity delivers $f \succsim^* g$.

(b) This proof is due to Shapley and Baucells (1998) and we report it for the sake of completeness. Let $f, g, h \in F$ and $\lambda \in (0,1)$ be such that $\lambda f + (1-\lambda)h \succsim^* \lambda g + (1-\lambda)h$. Let

$$\bar{\alpha} = \sup\left\{\alpha \in [0,1] : \alpha f + (1-\alpha)h \succsim^* \alpha g + (1-\alpha)h\right\}.$$

Clearly $\bar{\alpha} \geq \lambda > 0$ and, by Continuity, $\bar{\alpha} f + (1-\bar{\alpha})h \succsim^* \bar{\alpha} g + (1-\bar{\alpha})h$. Now set $\beta = \dfrac{1}{1+\bar{\alpha}}$ and observe that:

- $\beta\bar{\alpha} = \dfrac{\bar{\alpha}}{1+\bar{\alpha}} = 1 - \dfrac{1}{1+\bar{\alpha}} = 1-\beta$ and $\beta\left(1-\bar{\alpha}\right) = \dfrac{1-\bar{\alpha}}{1+\bar{\alpha}}$,
- Independence yields

$$\begin{gathered}\beta\left(\bar{\alpha} f + (1-\bar{\alpha})h\right) + (1-\beta)\, f \succsim^* \beta\left(\bar{\alpha} g + (1-\bar{\alpha})h\right) + (1-\beta)\, f = \\ = \beta\bar{\alpha} g + \beta(1-\bar{\alpha})h + (1-\beta)\, f = (1-\beta)\, g + \beta(1-\bar{\alpha})h + \beta\bar{\alpha} f = \\ = \beta\bar{\alpha} f + \beta(1-\bar{\alpha})h + (1-\beta)\, g = \beta\left(\bar{\alpha} f + (1-\bar{\alpha})h\right) + (1-\beta)\, g \succsim^* \\ \succsim^* \beta\left(\bar{\alpha} g + (1-\bar{\alpha})h\right) + (1-\beta)\, g\end{gathered}$$

  so that, by Transitivity,

$$\begin{aligned}\frac{2\bar{\alpha}}{1+\bar{\alpha}} f + \frac{1-\bar{\alpha}}{1+\bar{\alpha}} h &= \beta\left(\bar{\alpha} f + (1-\bar{\alpha})h\right) + (1-\beta)\, f \succsim^* \beta\left(\bar{\alpha} g + (1-\bar{\alpha})h\right) + (1-\beta)\, g = \\ &= \frac{2\bar{\alpha}}{1+\bar{\alpha}} g + \frac{1-\bar{\alpha}}{1+\bar{\alpha}} h.\end{aligned}$$

But then, by definition of $\bar{\alpha}$, $\dfrac{2\bar{\alpha}}{1+\bar{\alpha}} \leq \bar{\alpha}$, that is, $\bar{\alpha}^2 - \bar{\alpha} \geq 0$. Since $\bar{\alpha} > 0$, we have $\bar{\alpha} = 1$, and hence $f \succsim^* g$.

Sufficiency of the axioms for representation (13) and its uniqueness properties follow from Theorem 1 of GMMS, necessity is easy to check. Finally, (14) is proved in Proposition 4 below. ■

The next proposition shows that if $\succsim^*$ is a multiple prior (incomplete) preference à la Bewley represented by u and C as in (13), then the algebraic interior $\succ\!\!\succ^*$ of $\succ^*$ admits the representation

$$f \succ\!\!\succ^* g \iff \int u\left(f\right) dp > \int u\left(g\right) dp \qquad \text{for all } p \in C$$

moreover it coincides with the algebraic interior of $\succsim^*$.

Proposition 4 *If C is a non-empty closed and convex set of probabilities on Σ, $u : X \to \mathbb{R}$ is a non-constant affine function, and, for every $h, l \in F$,*

$$h \succsim^* l \iff \int u(h)\, dp \geq \int u(l)\, dp \qquad \text{for all } p \in C$$

then the following conditions are equivalent for f and g in F:

(i) For every $x \succ^* y$ in X there exist ε in $(0,1)$ such that

$$(1-\varepsilon) f + \varepsilon y \succsim^* (1-\varepsilon) g + \varepsilon x.$$

(ii) There exist $x \succ^* y$ in X and ε in $(0,1)$ such that

$$(1-\varepsilon) f + \varepsilon y \succsim^* (1-\varepsilon) g + \varepsilon x.$$

(iii) $\int u(f)\, dp > \int u(g)\, dp$ for all $p \in C$.

(iv) For every h, l in F, there exist ε in $(0,1)$ such that

$$(1-\delta) f + \delta h \succ^* (1-\delta) g + \delta l \qquad \text{for all } \delta \in [0, \varepsilon]$$

that is, $(f, g) \in \operatorname{int}(\succ^*)$, here denoted $f \ggcurly^* g$.

(v) For every h, l in F, there exist ε in $(0,1)$ such that

$$(1-\delta) f + \delta h \succsim^* (1-\delta) g + \delta l \qquad \text{for all } \delta \in [0, \varepsilon]$$

that is, $(f, g) \in \operatorname{int}(\succsim^*)$.

Proof of Proposition 4. (i) obviously implies (ii).

(ii) implies (iii). By (ii) there are $x \succ^* y$ in X and ε in $(0,1)$ such that

$$\int u((1-\varepsilon) f + \varepsilon y)\, dp \geq \int u((1-\varepsilon) g + \varepsilon x)\, dp \qquad \forall p \in C$$

but then

$$(1-\varepsilon) \int u(f)\, dp + \varepsilon u(y) \geq (1-\varepsilon) \int u(g)\, dp + \varepsilon u(x) \qquad \forall p \in C$$

$$(1-\varepsilon) \int u(f)\, dp \geq (1-\varepsilon) \int u(g)\, dp + \varepsilon (u(x) - u(y)) \qquad \forall p \in C$$

$$\int u(f)\, dp \geq \int u(g)\, dp + \frac{\varepsilon}{1-\varepsilon} (u(x) - u(y)) \qquad \forall p \in C$$

and so $\int u(f)\, dp > \int u(g)\, dp$ for all $p \in C$.

(iii) implies (iv). If $\int u(f)\, dp > \int u(g)\, dp$ for all $p \in C$, then

$$\int [u(f) - u(g)]\, dp > 0 \qquad \forall p \in C.$$

But, C is weak*-compact and $p \mapsto \int [u(f) - u(g)]\, dp$ is weak*-continuous. Hence, we have

$$\int [u(f) - u(g)]\, dp \geq \eta \qquad \forall p \in C$$

where $\eta = \min_{p \in C} \int [u(f) - u(g)]\, dp > 0$. For every h, l in F let $x, y \in X$ be such that $u(x) \geq u(l(s))$ and $u(y) \leq u(h(s))$ for all $s \in S$. Without loss of generality, assume that $u(x) \geq u(y)$.[17] Choose $\varepsilon \in (0,1)$ such that

$$\frac{\varepsilon}{1-\varepsilon}(u(x) - u(y)) < \eta$$

and consider any $\delta \in [0, \varepsilon]$, then

$$\int u(f)\, dp - \int u(g)\, dp \geq \eta > \frac{\varepsilon}{1-\varepsilon}(u(x) - u(y)) \geq \frac{\delta}{1-\delta}(u(x) - u(y)) \qquad \forall p \in C$$

$$\int u(f)\, dp > \int u(g)\, dp + \frac{\delta}{1-\delta}(u(x) - u(y)) \qquad \forall p \in C$$

$$(1-\delta)\int u(f)\, dp + \delta u(y) > (1-\delta)\int u(g)\, dp + \delta u(x) \qquad \forall p \in C$$

$$\begin{aligned}
(1-\delta)\int u(f)\, dp + \delta \int u(h)\, dp &\geq (1-\delta)\int u(f)\, dp + \delta u(y) > \\
&> (1-\delta)\int u(g)\, dp + \delta u(x) \geq \\
&\geq (1-\delta)\int u(g)\, dp + \delta \int u(l)\, dp \qquad \forall p \in C
\end{aligned}$$

$$\int u((1-\delta) f + \delta h)\, dp > \int u((1-\delta) g + \delta l)\, dp \qquad \forall p \in C.$$

(iv) implies (v) and (v) implies (i) are trivial observations. ■

Proof of Theorem 1. (i) implies (ii) and (iii) and (18). By Lemma 1, there exist a non-empty closed and convex set C of probabilities on Σ and a non-constant affine $u : X \to \mathbb{R}$ such that, for every $f, g \in F$,

$$\begin{aligned}
f \succsim^* g \quad &\iff \quad \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C \\
f \succ\!\!\succ^* g \quad &\iff \quad \int u(f)\, dp > \int u(g)\, dp \qquad \forall p \in C.
\end{aligned} \tag{25}$$

Next we show that $(\succsim^*, \succsim^\circ)$ satisfy Consistency. Assume $f \succsim^* g$, and choose $x \succ^* y$ in X, then for every $\varepsilon \in (0,1)$

$$(1-\varepsilon)\int u(f)\, dp + \varepsilon u(x) > (1-\varepsilon)\int u(g)\, dp + \varepsilon u(y) \qquad \forall p \in C$$

$$\int u((1-\varepsilon) f + \varepsilon x)\, dp > \int u((1-\varepsilon) g + \varepsilon y)\, dp \qquad \forall p \in C$$

$$(1-\varepsilon) f + \varepsilon x \succ\!\!\succ^* (1-\varepsilon) g + \varepsilon y$$

[17] If $u(x) < u(y)$, leave y unchanged and replace x with $x' = y$ so that $u(x') = u(y) \geq u(x) \geq u(l(s))$ for all $s \in S$.

and Strong Consistency implies

$$(1-\varepsilon)\, f+\varepsilon x \succ^{\circ} (1-\varepsilon)\, g+\varepsilon y \qquad \forall \varepsilon \in (0,1)$$

whence $(1-\varepsilon)\, f+\varepsilon x \succsim^{\circ} (1-\varepsilon)\, g+\varepsilon y$ for all $\varepsilon \in (0,1)$ and $f \succsim^{\circ} g$ follows by Continuity of $\succsim^{\circ}$.

Now, Possibility and Consistency imply that $\succsim^{\circ}$ satisfies Completeness. In turn, Continuity and Completeness of $\succsim^{\circ}$ imply that given any f, g, h, l in F,

$$\{\lambda \in [0,1] : (1-\lambda)f+\lambda h \succ^{\circ} (1-\lambda)g+\lambda l\}$$

is open in $[0,1]$. If $f \succ^{\circ} g$, then 0 belongs to the set for every h, l in F, and so there is $\varepsilon > 0$ such that

$$\begin{aligned} (1-\lambda)f+\lambda h &\succ^{\circ} (1-\lambda)g+\lambda l \qquad \forall \lambda \in [0,\varepsilon] \\ (1-\lambda)g+\lambda l &\not\succsim^{\circ} (1-\lambda)f+\lambda h \qquad \forall \lambda \in [0,\varepsilon] \end{aligned}$$

by Possibility

$$(1-\lambda)f+\lambda h \succsim^{*} (1-\lambda)g+\lambda l \qquad \forall \lambda \in [0,\varepsilon]$$

by Proposition 4,

$$\int u(f)\, dp > \int u(g)\, dp \qquad \forall p \in C$$

that is, $f \ggcurly^{*} g$. Summing up, $f \succ^{\circ} g$ implies $f \ggcurly^{*} g$ and the converse is true by Strong Consistency. This shows that (iii) holds because, as already observed, $\succsim^{\circ}$ is complete. By (25)

$$f \succ^{\circ} g \iff f \ggcurly^{*} g \iff \int u(f)\, dp > \int u(g)\, dp \quad \forall p \in C$$

which is (18) and Completeness of $\succsim^{\circ}$ again yields (17). That is (ii) holds.

(ii) implies (iii). The properties of $\succsim^{*}$ follow from Lemma 1, Completeness of $\succsim^{\circ}$ from (17), in turn (17) and Completeness of $\succsim^{\circ}$ yield

$$f \succ^{\circ} g \iff \int u(f)\, dp > \int u(g)\, dp \quad \forall p \in C.$$

Then (16) and Proposition 4 deliver $f \succ^{\circ} g \iff f \ggcurly^{*} g$.

(iii) implies (i) and (iv). We only have to prove that $\succsim^{\circ}$ satisfies Continuity and $(\succsim^{*}, \succsim^{\circ})$ satisfies Possibility. By Lemma 1, there exist a non-empty closed and convex set C of probabilities on Σ and a non-constant affine $u : X \to \mathbb{R}$ such that, for every $f, g \in F$,

$$f \succsim^{*} g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C \tag{26}$$

$$f \ggcurly^{*} g \iff \int u(f)\, dp > \int u(g)\, dp \qquad \forall p \in C. \tag{27}$$

Coincidence of $\succ^{\circ}$ with $\ggcurly^{*}$ implies

$$f \succ^{\circ} g \iff \int u(f)\, dp > \int u(g)\, dp \qquad \forall p \in C$$

and Completeness of $\succsim^\circ$ delivers

$$f \succsim^\circ g \iff \int u(f)\,dp \geq \int u(g)\,dp \qquad \text{for some } p \in C \tag{28}$$

and Possibility follows from (26) and (28). Moreover, (28) and (27) show that (iii) implies (iv). Given any f, g, h, l in F, set

$$\Lambda = \{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim^\circ \lambda h + (1-\lambda) l\}$$

and assume λ_n is a sequence in Λ that converges to λ. Set $\varphi_n = u(\lambda_n f + (1-\lambda_n) g)$, $\psi_n = u(\lambda_n h + (1-\lambda_n) l)$, $\varphi = u(\lambda f + (1-\lambda) g)$, $\psi = u(\lambda h + (1-\lambda) l)$ and observe that, in supnorm, $\varphi_n \to \varphi$ and $\psi_n \to \psi$.[18] Since λ_n is a sequence in Λ, then

$$\lambda_n f + (1-\lambda_n) g \succsim^\circ \lambda_n h + (1-\lambda_n) l \qquad \forall n \in \mathbb{N} \tag{29}$$

and by (28), for every $n \in \mathbb{N}$, there exists p_n in C such that

$$\int \varphi_n dp_n \geq \int \psi_n dp_n.$$

But C is a weak* compact subset of the space $ba(\Sigma)$ of all bounded and finitely additive set functions on Σ, then there exists a subnet p_{n_β} of p_n that weak* converges to some p in C. Since clearly $\varphi_{n_\beta} \to \varphi$ and $\psi_{n_\beta} \to \psi$ in supnorm, and p_{n_β} is norm bounded in $ba(\Sigma)$, then

$$\begin{array}{ccc} \int \varphi_{n_\beta} dp_{n_\beta} & \geq & \int \psi_{n_\beta} dp_{n_\beta} \\ \downarrow & & \downarrow \\ \int \varphi dp & \geq & \int \psi dp \end{array}$$

that is, $\lambda f + (1-\lambda) g \succsim^\circ \lambda h + (1-\lambda) l$ and so $\lambda \in \Lambda$. This proves that Λ is closed and $\succsim^\circ$ is continuous.

(iv) implies (ii). By (13) of Lemma 1, $\succsim^*$ admits representation (16), while (14) of Lemma 1 and coincidence of $\succsim^\circ$ with $\not\succ^*$ delivers (17).

Uniqueness of C and cardinal uniqueness of u follow from Lemma 1. ■

Proof of Proposition 1. Assume there exist $l \not\succ^* h$ in F, such that for every γ in $(0,1]$

$$(1-\gamma)\, g + \gamma l \not\succ^* (1-\gamma)\, f + \gamma h$$

then there is a sequence $\varepsilon_n \to 0$ such that

$$(1-\varepsilon_n)\, g + \varepsilon_n l \not\succ^* (1-\varepsilon_n)\, f + \varepsilon_n h \qquad \forall n \in \mathbb{N}.$$

[18] In fact, since $\varphi_n = u(g) + \lambda_n (u(f) - u(g))$ and $\varphi = u(g) + \lambda (u(f) - u(g))$, then

$$\|\varphi_n - \varphi\| = \|(\lambda_n - \lambda)(u(f) - u(g))\| = |\lambda_n - \lambda|\, \|(u(f) - u(g))\|$$

and the same argument applies to ψ_n and ψ.

If

$$(1-\varepsilon_n)\,g+\varepsilon_n l \sim^* (1-\varepsilon_n)\,f+\varepsilon_n h$$

for infinitely many n's, then there exists a subsequence ε_{n_k} of ε_n such that

$$(1-\varepsilon_{n_k})\,g+\varepsilon_{n_k} l \sim^* (1-\varepsilon_{n_k})\,f+\varepsilon_{n_k} h \qquad \forall k\in\mathbb{N}$$

by Consistency

$$(1-\varepsilon_{n_k})\,f+\varepsilon_{n_k} h \sim^\circ (1-\varepsilon_{n_k})\,g+\varepsilon_{n_k} l \qquad \forall k\in\mathbb{N}$$

and Continuity of $\succsim^\circ$ delivers $f\succsim^\circ g$. Otherwise, eventually

$$(1-\varepsilon_n)\,g+\varepsilon_n l \nsim^* (1-\varepsilon_n)\,f+\varepsilon_n h \text{ and } (1-\varepsilon_n)\,g+\varepsilon_n l \not\succ^* (1-\varepsilon_n)\,f+\varepsilon_n h$$

then eventually

$$(1-\varepsilon_n)\,g+\varepsilon_n l \not\succsim^* (1-\varepsilon_n)\,f+\varepsilon_n h$$

by Possibility

$$(1-\varepsilon_n)\,f+\varepsilon_n h \succsim^\circ (1-\varepsilon_n)\,g+\varepsilon_n l$$

and Continuity of $\succsim^\circ$ delivers $f\succsim^\circ g$.

Conversely, assume $f\succsim^\circ g$, then there exists $q\in C$ such that

$$\int u(f)\,dq \geq \int u(g)\,dq$$

take $x\succ^* y$ (so that $y\not\succ^* x$), then for every ε in $(0,1]$

$$(1-\varepsilon)\int u(f)\,dq+\varepsilon u(x) > (1-\varepsilon)\int u(g)\,dq+\varepsilon u(y)$$
$$\int u((1-\varepsilon)\,f+\varepsilon x)\,dq > \int u((1-\varepsilon)\,g+\varepsilon y)\,dq$$
$$(1-\varepsilon)\,g+\varepsilon y \not\succ^* (1-\varepsilon)\,f+\varepsilon x$$

so that the proof is concluded by setting $l=y$ and $h=x$. ■

Proof of Proposition 2. Let $f,g\in F$. First observe that if $f\succsim^* g$, then

$$\int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p\in C. \tag{30}$$

But then $h\succsim^\circ f$ implies that there exists $q\in C$ such that

$$\int u(h)\,dq \geq \int u(f)\,dq \geq \int u(g)\,dq$$

where the last inequality follows from (30), that is, $h\succsim^\circ g$. Analogously, $g\succsim^\circ l$ implies that there exists $q\in C$ such that

$$\int u(f)\,dq \geq \int u(g)\,dq \geq \int u(l)\,dq$$

where the first inequality follows from (30), that is, $f \succsim^{\circ} l$. This shows that $f \succsim^{*} g$ implies $f \succsim^{\circ\circ} g$.

As to the converse, notice that, under the assumptions of Theorem 1, $\succsim^{\circ}$ is represented by u on X. Consider first the case in which the interval $u(X)$ does not admit a maximum point, and assume –per contra– that there exist $f \succsim^{\circ\circ} g$ such that $f \not\succsim^{*} g$. Therefore there exists $q \in C$ such that

$$\int u(g)\,dq > \int u(f)\,dq.$$

For all $\varepsilon \in (0,1)$, set

$$f^{\varepsilon} = (1-\varepsilon) f + \varepsilon x$$

where $x \succ^{\circ} f(s)$ for every $s \in S$ (such an x exists because $u(X)$ does not admit maximum). Notice that $u(f^{\varepsilon}(s)) = (1-\varepsilon) u(f(s)) + \varepsilon u(x)$ for all $s \in S$, therefore

(a) $\int u(f^{\varepsilon})\,dq \longrightarrow \int u(f)\,dq$ as $\varepsilon \to 0$;

(b) $u(f^{\varepsilon}(s)) = u(f(s)) + \varepsilon(u(x) - u(f(s))) > u(f(s))$ because $u(x) - u(f(s)) > 0$ for all $s \in S$.

Therefore, we can choose $\varepsilon \in (0,1)$ small enough so that

$$\int u(g)\,dq > \int u(f^{\varepsilon})\,dq > \int u(f)\,dq$$

and $g \succsim^{\circ} f^{\varepsilon}$. But $f \succsim^{\circ\circ} g$ and $g \succsim^{\circ} f^{\varepsilon}$ imply $f \succsim^{\circ} f^{\varepsilon}$ which is absurd because

$$\int u(f^{\varepsilon})\,dp > \int u(f)\,dp \qquad \forall p \in C.$$

Next consider the case in which the interval $u(X)$ does not admit a minimum point, and assume –per contra– that there exist $f \succsim^{\circ\circ} g$ such that $f \not\succsim^{*} g$. Therefore there exists $q \in C$ such that

$$\int u(g)\,dq > \int u(f)\,dq.$$

For all $\varepsilon \in (0,1)$, set

$$g^{\varepsilon} = (1-\varepsilon) g + \varepsilon x$$

where $g(s) \succ x$ for every $s \in S$ (such an x exists because $u(X)$ does not admit minimum). Notice that $u(g^{\varepsilon}(s)) = (1-\varepsilon) u(g(s)) + \varepsilon u(x)$ for all $s \in S$, therefore

(a) $\int u(g^{\varepsilon})\,dq \longrightarrow \int u(g)\,dq$ as $\varepsilon \to 0$;

(b) $u(g^{\varepsilon}(s)) = u(g(s)) - \varepsilon(u(g(s)) - u(x)) < u(g(s))$ because $u(g(s)) - u(x) > 0$ for all $s \in S$.

Therefore, we can choose ε small enough so that

$$\int u(g)\,dq > \int u(g^{\varepsilon})\,dq > \int u(f)\,dq$$

and $g^{\varepsilon} \succsim^{\circ} f$. But $f \succsim^{\circ\circ} g$ and $g^{\varepsilon} \succsim^{\circ} f$ imply $g^{\varepsilon} \succsim^{\circ} g$ which is absurd because

$$\int u(g^{\varepsilon})\,dp < \int u(g)\,dp \qquad \forall p \in C.$$

Summing up, $f \succsim^{\circ\circ} g$ implies $f \succsim^{*} g$. ■

The next proposition promptly delivers **Proposition 3** as a corollary.

Proposition 5 *If C is a non-empty closed and convex set of probabilities on Σ, $u : X \to \mathbb{R}$ is a non-constant affine function, and, for every $h, l \in F$,*

$$h \succsim^{*} l \iff \int u(h)\,dp \geq \int u(l)\,dp \qquad \text{for all } p \in C$$

then the following conditions are equivalent for f and g in F:

(i) For every $h \ggcurly^{*} l$ in F and every γ in $(0,1]$

$$(1-\gamma) f + \gamma h \ggcurly^{*} (1-\gamma) g + \gamma l.$$

(ii) There exist $h \ggcurly^{*} l$ in F such that for every γ in $(0,1]$

$$(1-\gamma) f + \gamma h \ggcurly^{*} (1-\gamma) g + \gamma l,$$

that is $(f,g) \in \mathrm{cl}(\ggcurly^{*})$.

(iii) $f \succsim^{*} g$.

In particular, under the assumptions of Theorem 1, $\ggcurly^{*}$ coincides with $\succ^{\circ}$, and the equivalence between (iii) and (ii) above means that $f \succsim^{*} g$ if and only if there exist $h \succ^{\circ} l$ such that

$$(1-\gamma) f + \gamma h \succ^{\circ} (1-\gamma) g + \gamma l \qquad \text{for all } \gamma \in (0,1].$$

Proof of Proposition 5. (i) obviously implies (ii) because $\ggcurly^{*}$ is non-trivial.

(ii) implies (iii). Since $h, l \in F$ are such that $(1-\gamma) f + \gamma h \ggcurly^{*} (1-\gamma) g + \gamma l$ for every $\gamma \in (0,1)$, then

$$\int u((1-\gamma) f + \gamma h)\,dp > \int u((1-\gamma) g + \gamma l)\,dp \qquad \forall p \in C$$

$$(1-\gamma)\int u(f)\,dp + \gamma \int u(h)\,dp > (1-\gamma)\int u(g)\,dp + \gamma \int u(l)\,dp \qquad \forall p \in C$$

$$(1-\gamma)\int u(f)\,dp > (1-\gamma)\int u(g)\,dp + \gamma \int [u(l) - u(h)]\,dp \qquad \forall p \in C$$

$$\int u(f)\,dp > \int u(g)\,dp + \frac{\gamma}{1-\gamma}\int [u(l) - u(h)]\,dp \qquad \forall p \in C$$

and so, by passing to the limits as $\gamma \to 0$, $\int u(f)\,dp \geq \int u(g)\,dp$ for all $p \in C$.

(iii) implies (i). If $\int u(f)\,dp \geq \int u(g)\,dp$ for all $p \in C$, then, for every $h \succ\!\!\succ^* l$ in F and every γ in $(0,1]$

$$(1-\gamma)\int u(f)\,dp + \gamma \int u(h)\,dp > (1-\gamma)\int u(g)\,dp + \gamma\int u(l)\,dp \qquad \forall p \in C$$
$$\int u((1-\gamma) f + \gamma h)\,dp > \int u((1-\gamma) g + \gamma l)\,dp \qquad \forall p \in C$$

that is, $(1-\gamma) f + \gamma h \succ\!\!\succ^* (1-\gamma) g + \gamma l$. ■

A.2 Proof of Theorem 2

A.2.1 Utility profiles

Here we denote by $B_0(S,\Sigma)$ the vector space of all simple and measurable functions $\varphi : S \to \mathbb{R}$, and given an element $k \in \mathbb{R}$, we denote by k both the real number and the constant function in $B_0(S,\Sigma)$ that takes value k. Given two functions $\varphi, \psi \in B_0(S,\Sigma)$, we define

$$\varphi \gg \psi \iff \varphi(s) > \psi(s) \qquad \forall s \in S.$$

We also define $B_0^{++}(S,\Sigma) = \{\varphi \in B_0(S,\Sigma) : \varphi \gg 0\}$. Consider two binary relations $\succcurlyeq^\circ$ and $\succcurlyeq^*$ on $B_0(S,\Sigma)$. Assume that $\succcurlyeq^*$ is such that

$$\varphi \succcurlyeq^* \psi \iff \int \varphi dp \geq \int \psi dp \qquad \forall p \in C \tag{31}$$

where $C \neq \varnothing$ is a convex and closed subset of Δ. Define also

$$\varphi \succ\!\!\succ^* \psi \iff \int \varphi dp > \int \psi dp \qquad \forall p \in C. \tag{32}$$

Assume that $\succcurlyeq^*$ and $\succcurlyeq^\circ$ satisfy the following properties:

0. $\succcurlyeq^\circ$ is complete;

1. If $\varphi_2 \succeq^\circ \psi_2$ and $\lambda \in (0,1)$,[19] then

$$\varphi_1 \succ^\circ \psi_1 \implies \lambda\varphi_1 + (1-\lambda)\varphi_2 \succ^\circ \lambda\psi_1 + (1-\lambda)\psi_2$$

   and the converse is true when $\varphi_2 = \psi_2$;

2. If $\varphi \succ\!\!\succ^* \psi$ and $\psi \succ^\circ \zeta$, then $\varphi \succ^\circ \zeta$;

3. If $\varphi \succ\!\!\succ^* \psi$, then $\varphi \succ^\circ \psi$;

[19] By $\varphi_2 \succeq^\circ \psi_2$ we mean either $\varphi_2 \succ^\circ \psi_2$ or $\varphi_2 = \psi_2$.

4. If $\varphi \succ^\circ \psi$, then for each $\varepsilon > 0$ there exists $\hat{\lambda} \in (0,1)$ such that $\varphi \succ^\circ \lambda\psi + (1-\lambda)\varepsilon$ for all $\lambda \in \left(\hat{\lambda}, 1\right)$;

5. There exists $\tilde{\varphi}$ such that
$$\varphi \not\gg^* 0 \implies \tilde{\varphi} \not\succeq^\circ \varphi.$$

Define
$$A = \{\phi \in B_0(S,\Sigma) : \phi = \varphi - \psi \text{ with } \varphi \succ^\circ \psi\}$$
and
$$K^{++} = \left\{\phi \in B_0(S,\Sigma) : \int \phi dp > 0 \quad \forall p \in C\right\} = \{\phi \in B_0(S,\Sigma) : \phi \ggcurly^* 0\}.$$
It is immediate to see that $K^{++} \supseteq B_0^{++}(S,\Sigma)$.

Lemma 3 *The set A has the following properties:*

1. *$B_0^{++}(S,\Sigma) \subseteq K^{++} \subseteq A$, in particular, $A \neq \varnothing$;*

2. *A is convex;*

3. *$A + K^{++} \subseteq A$;*

4. *$A \cap -B_0^{++}(S,\Sigma) = \varnothing$.*

Proof. We already observed that $B_0^{++}(S,\Sigma) \subseteq K^{++}$. Moreover, if $\phi \in K^{++}$, then $\phi \ggcurly^* 0$, and by Property 3, $\phi \succ^\circ 0$, thus $\phi = \phi - 0 \in A$. This proves Point 1.

Consider $\phi_1, \phi_2 \in A$ and $\lambda \in (0,1)$. It follows that there exist $\varphi_i, \psi_i \in B_0(S,\Sigma)$ such that $\varphi_i \succ^\circ \psi_i$ and $\phi_i = \varphi_i - \psi_i$ for $i = 1,2$. By Property 1, we have that $\lambda\varphi_1 + (1-\lambda)\varphi_2 \succ^\circ \lambda\psi_1 + (1-\lambda)\psi_2$, then
$$\begin{aligned}\lambda\phi_1 + (1-\lambda)\phi_2 &= \lambda(\varphi_1 - \psi_1) + (1-\lambda)(\varphi_2 - \psi_2) \\ &= \lambda\varphi_1 + (1-\lambda)\varphi_2 - (\lambda\psi_1 + (1-\lambda)\psi_2) \in A.\end{aligned}$$
This proves Point 2.

Next, consider $\eta \in K^{++}$ and $\phi \in A$. By definition of A, $\phi = \varphi - \psi$ with $\varphi \succ^\circ \psi$. But then $\varphi + \eta \ggcurly^* \varphi$ and $\varphi \succ^\circ \psi$. By Property 2, it follows that $\varphi + \eta \succ^\circ \psi$, then $\varphi + \eta - \psi \in A$ and $\phi + \eta \in A$. This proves Point 3.

By contradiction, notice that if $\phi \in A \cap -B_0^{++}(S,\Sigma)$, then there would exist $\varphi, \psi \in B_0(S,\Sigma)$ such that $\varphi \succ^\circ \psi$, and $\phi = \varphi - \psi \ll 0$. But then, $\psi \ggcurly^* \varphi$ (because $\psi \gg \varphi$), and, by Property 3, $\psi \succ^\circ \varphi$, a contradiction with $\varphi \succ^\circ \psi$. This proves Point 4. ■

Remark 1 *Notice that if $\phi_1 \gg \phi_2$ and $\phi_2 \in A$, then $\phi_1 \in A$. For, if we define $\eta = \phi_1 - \phi_2$, then $\eta \gg 0$ and $\eta \in K^{++}$, therefore $\phi_1 = \phi_2 + \eta \in A$.*

Remark 2 *$\varphi \succ^\circ \psi \iff \varphi - \psi \in A$. In fact, by definition of A, if $\varphi \succ^\circ \psi$, then $\varphi - \psi \in A$. Conversely, if $\varphi - \psi \in A$, there exists $\bar{\varphi}, \bar{\psi} \in B_0(S, \Sigma)$ such that $\varphi - \psi = \bar{\varphi} - \bar{\psi}$ and $\bar{\varphi} \succ^\circ \bar{\psi}$. Then*

$$\frac{1}{2}\varphi + \frac{1}{2}\bar{\psi} = \frac{1}{2}\bar{\varphi} + \frac{1}{2}\psi \tag{33}$$

by Property 1

$$\frac{1}{2}\bar{\varphi} + \frac{1}{2}\psi \succ^\circ \frac{1}{2}\bar{\psi} + \frac{1}{2}\psi \tag{34}$$

but (33) and (34) yield

$$\frac{1}{2}\varphi + \frac{1}{2}\bar{\psi} \succ^\circ \frac{1}{2}\psi + \frac{1}{2}\bar{\psi}$$

and Property 1 again implies $\varphi \succ^\circ \psi$.

Set

$$I(\phi) = \sup\{k \in \mathbb{R} : \phi - k \in A\} \qquad \forall \phi \in B_0(S, \Sigma). \tag{35}$$

Lemma 4 *If I is defined as in (35), then I is a normalized concave niveloid. Moreover,*

1. *$\varphi \succ^\circ \psi$ if and only if $I(\varphi - \psi) > 0$.*
2. *$\phi \in A$ if and only if $I(\phi) > 0$*
3. *$\varphi \succcurlyeq^* \psi$ implies $I(\varphi) \geq I(\psi)$.*

Proof. Consider $\phi \in B_0(S, \Sigma)$.

Since $A \supseteq B_0^{++}(S, \Sigma)$, $A \cap -B_0^{++}(S, \Sigma) = \varnothing$, and $A + B_0^{++}(S, \Sigma) \subseteq A$, it follows that $A_\phi = \{k \in \mathbb{R} : \phi - k \in A\}$ is a non-empty and bounded above half line such that

$$\left(-\infty, \min_{s \in S} \phi(s)\right) \subseteq A_\phi \subseteq \left(-\infty, \max_{s \in S} \phi(s)\right] \tag{36}$$

thus $I(\phi) = \sup A_\phi \in \mathbb{R}$, and I is well defined. Moreover, (36) implies that $I(\bar{k}) = \bar{k}$ for all $\bar{k} \in \mathbb{R}$, that is, I is normalized.

For every $\bar{k} \in \mathbb{R}$, $A_{\phi + \bar{k}} = A_\phi + \bar{k}$, then

$$I(\phi + \bar{k}) = \sup A_{\phi + \bar{k}} = \sup(A_\phi + \bar{k}) = \sup A_\phi + \bar{k} = I(\phi) + \bar{k}.$$

Since $\bar{k}$ and ϕ were arbitrarily chosen, we can conclude that $I(\phi + \bar{k}) = I(\phi) + \bar{k}$ for all $\phi \in B_0(S, \Sigma)$ and for all $\bar{k} \in \mathbb{R}$. That is, I is translation invariant.

If $\phi_1 \gg \phi_2$, then $\phi_2 - k \in A$ implies $\phi_1 - k \gg \phi_2 - k$ also belongs to A. This means

$$\{k \in \mathbb{R} : \phi_2 - k \in A\} \subseteq \{k \in \mathbb{R} : \phi_1 - k \in A\},$$

whence $I(\phi_1) \geq I(\phi_2)$. If $\phi_1 \geq \phi_2$, then $\phi_1 \gg \phi_2 - \frac{1}{n}$ for all $n \in \mathbb{N}$ and so

$$I(\phi_1) \geq I\left(\phi_2 - \frac{1}{n}\right) = I(\phi_2) - \frac{1}{n}$$

for all $n \in \mathbb{N}$, thus $I(\phi_1) \geq I(\phi_2)$. That is, I is monotone.

Consider $\phi_1, \phi_2 \in B_0(S, \Sigma)$ and arbitrarily choose $\lambda \in (0,1)$. If $k_1, k_2 \in \mathbb{R}$ are such that $\phi_i - k_i \in A$ for $i = 1, 2$ (that is, $k_i \in A_{\phi_i}$ for $i = 1, 2$). Since A is convex, it follows that

$$A \ni \lambda(\phi_1 - k_1) + (1-\lambda)(\phi_2 - k_2) = (\lambda\phi_1 + (1-\lambda)\phi_2) - (\lambda k_1 + (1-\lambda) k_2).$$

It follows that

$$I(\lambda\phi_1 + (1-\lambda)\phi_2) \geq \lambda k_1 + (1-\lambda) k_2$$

for all $k_1 \in A_{\phi_1}$, and $k_2 \in A_{\phi_2}$, yielding that

$$I(\lambda\phi_1 + (1-\lambda)\phi_2) \geq \lambda I(\phi_1) + (1-\lambda) I(\phi_2),$$

proving that I is concave.

1. If $\varphi \succ^\circ \psi$, then $\varphi - \psi \in A$. Let ε be such that $\varepsilon > \max\{\max_{s\in S} \psi(s), 0\}$. By Property 4, there exists $\hat{\lambda} \in (0,1)$ such that $\varphi \succ^\circ \lambda\psi + (1-\lambda)\varepsilon$ for all $\lambda \in \left(\hat{\lambda}, 1\right)$. In particular, we have that $\varphi - (\lambda\psi + (1-\lambda)\varepsilon) \in A$ for all $\lambda \in \left(\hat{\lambda}, 1\right)$, that is,

$$(\varphi - \psi) + (1-\lambda)(\psi - \varepsilon) \in A.$$

Fix such a λ and notice that $\eta = (1-\lambda)(\psi - \varepsilon) \ll 0$ is such that

$$(\varphi - \psi) + \eta \in A.$$

Now setting $d = \frac{\max_{s\in S} \eta(s)}{2}$, we have

$$0 > d = \frac{\max_{s\in S} \eta(s)}{2} > \max_{s\in S} \eta(s)$$

therefore $0 > d \gg \eta$ and

$$(\varphi - \psi) + d \gg (\varphi - \psi) + \eta \in A$$

delivers $(\varphi - \psi) + d \in A$ or $(\varphi - \psi) - (-d) \in A$. By definition of I, we have that $I(\varphi - \psi) \geq -d > 0$.

Viceversa, by definition of I, if $I(\varphi - \psi) > 0$, then $(\varphi - \psi) - k \in A$ for some $k > 0$. It follows that $\varphi - \psi \in A$, because $\varphi - \psi \gg (\varphi - \psi) - k \in A$. By Remark 2, $\varphi \succ^\circ \psi$.

2. By Remark 2, $\phi \in A$ if and only if $\phi \succ^\circ 0$, which, by Point 1, is equivalent to $I(\phi) > 0$.

3. Recall that $A + K^{++} \subseteq A$, assume $\varphi \ggcurly^* \psi$, then $\eta = \varphi - \psi \ggcurly^* 0$ and $\varphi = \psi + \eta$. Now

$$\psi - k \in A \implies \psi - k + \eta \in A \implies \varphi - k \in A$$

then $A_\psi \subseteq A_\varphi$ and $I(\psi) \leq I(\varphi)$. If $\varphi \succcurlyeq^* \psi$, then $\varphi \ggcurly^* \psi - \frac{1}{n}$ for all $n \in \mathbb{N}$ and so

$$I(\varphi) \geq I\left(\psi - \frac{1}{n}\right) = I(\psi) - \frac{1}{n}$$

for all $n \in \mathbb{N}$, thus $I(\varphi) \geq I(\psi)$. ■

Define $\bar{I} : B_0(S, \Sigma) \to \mathbb{R}$ as

$$\bar{I}(\phi) = -I(-\phi) \qquad \forall \phi \in B_0(S, \Sigma). \tag{37}$$

Observe that $-\bar{I}(\phi) = I(-\phi)$ for all $\phi \in B_0(S, \Sigma)$.

Proposition 6 *If $\bar{I}$ is defined as in (37), then*

$$\varphi \succcurlyeq^\circ \psi \iff I(\psi - \varphi) \leq 0 \iff \bar{I}(\varphi - \psi) \geq 0.$$

Proof. Since $\succcurlyeq^\circ$ is complete (Property 0), $\varphi \succcurlyeq^\circ \psi \iff \psi \not\succ^\circ \varphi$, thus

$$\begin{aligned}\varphi \succcurlyeq^\circ \psi &\iff I(\psi - \varphi) \not> 0 \iff I(\psi - \varphi) \leq 0 \iff I(-(\varphi - \psi)) \leq 0 \\ &\iff -\bar{I}(\varphi - \psi) \leq 0\end{aligned}$$

as wanted. ■

Remark 3 *Maccheroni, Marinacci, and Rustichini (2006, henceforth MMR) show that if $I : B_0(S, \Sigma) \to \mathbb{R}$ is a normalized concave niveloid, there exists a unique, grounded, convex, and lower semicontinuous function $c : \Delta \to [0, \infty]$ such that*

$$I(\phi) = \min_{p \in \Delta} \left\{ \int \phi dp + c(p) \right\} \qquad \forall \phi \in B_0(S, \Sigma).$$

Specifically, for each $p \in \Delta$,

$$c(p) = \sup \{ I(\psi) - \langle \psi, p \rangle : \psi \in B_0(S, \Sigma) \}. \tag{38}$$

Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011b) show that, for each $p \in \Delta$,

$$c(p) = \sup \{ I(\varphi) : \langle \varphi, p \rangle = 0 \} = \sup \{ I(\phi) : \langle \phi, p \rangle \leq 0 \} = \sup \{ I(\eta) : \langle \eta, p \rangle < 0 \}.$$

Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011a) show that if D is a convex and closed subset of Δ such that

$$\int \phi_1 dp \geq \int \phi_2 dp \quad \forall p \in D \qquad \Longrightarrow \qquad I(\phi_1) \geq I(\phi_2), \tag{39}$$

then $\mathrm{cl}(\mathrm{dom}\, c) \subseteq D$.

Proposition 7 *Let I and $\bar{I}$ be defined as in (35) and (37). The following statements are true:*

1. *There exists a unique $c : \Delta \to [0, \infty]$ grounded, convex, and lower semicontinuous such that*

$$\bar{I}(\phi) = \max_{p \in \Delta} \left\{ \int \phi dp - c(p) \right\} \qquad \forall \phi \in B_0(S, \Sigma). \tag{40}$$

2. $\mathrm{cl}\,(\mathrm{dom}\, c) \subseteq C$.

3. *If* $\varphi \succ^{\circ} \psi$ *and* $\psi \succcurlyeq^{*} \zeta$, *then* $\varphi \succ^{\circ} \zeta$.

4. $\mathrm{cl}\,(\mathrm{dom}\, c) = C$.

5. c *is bounded on* $\mathrm{cl}\,(\mathrm{dom}\, c)$. *In particular,* $\mathrm{cl}\,(\mathrm{dom}\, c) = \mathrm{dom}\, c = C$.

Proof. 1. By MMR and since I is a normalized concave niveloid, there exists a unique grounded, convex, and lower semicontinuous function $c : \Delta \to [0, \infty]$ such that

$$I(\phi) = \min_{p \in \Delta} \left\{ \int \phi dp + c(p) \right\} \qquad \forall \phi \in B_0(S, \Sigma). \tag{41}$$

By definition of $\bar{I}$, (40) follows.

2. We already observed that $\phi_1 \succcurlyeq^{*} \phi_2$ implies $I(\phi_1) \geq I(\phi_2)$, by Remark 3, we can conclude that $C \supseteq \mathrm{cl}\,(\mathrm{dom}\, c)$.

3. Consider $\varphi \succ^{\circ} \psi$ and $\psi \succcurlyeq^{*} \zeta$. We have that $I(\varphi - \psi) > 0$ and

$$\varphi - \zeta = (\varphi - \psi) + (\psi - \zeta) \succcurlyeq^{*} \varphi - \psi$$

thus $I(\varphi - \zeta) \geq I(\varphi - \psi) > 0$. By Lemma 4, $\varphi \succ^{\circ} \zeta$.

4. By contradiction, assume that $C \supset \mathrm{cl}\,(\mathrm{dom}\, c)$. Thus, there exists $\bar{p} \in C \backslash \mathrm{cl}\,(\mathrm{dom}\, c)$. Since $\mathrm{cl}\,(\mathrm{dom}\, c)$ is convex and closed, there exists $\psi \in B_0(S, \Sigma)$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$\int \psi d\bar{p} \leq \alpha - \varepsilon < \alpha + \varepsilon \leq \min_{p \in \mathrm{cl}(\mathrm{dom}\, c)} \int \psi dp.$$

Setting $\varphi = \psi - \alpha$, we have

$$\int \varphi d\bar{p} \leq -\varepsilon < \varepsilon \leq \min_{p \in \mathrm{cl}(\mathrm{dom}\, c)} \int \varphi dp. \tag{42}$$

If we define $\varphi_n = n\varphi$ for all $n \in \mathbb{N}$, then φ_n satisfies (42) with ε replaced by $n\varepsilon$. By (42), it follows that, for all $n \in \mathbb{N}$, $\varphi_n \not\succcurlyeq^{*} 0$ and

$$I(\varphi_n) = \inf_{p \in \mathrm{dom}\, c} \left\{ \int \varphi_n dp + c(p) \right\} \geq \inf_{p \in \mathrm{dom}\, c} \int \varphi_n dp \geq n\varepsilon > \frac{n\varepsilon}{2} > 0.$$

This implies that $I\left(\varphi_n - \frac{n\varepsilon}{2}\right) > 0$, that is, $\varphi_n \succ^{\circ} \frac{n\varepsilon}{2}$ for all $n \in \mathbb{N}$. So far we have found a sequence φ_n in $B_0(S, \Sigma)$ such that, for all $n \in \mathbb{N}$, $\varphi_n \not\succcurlyeq^{*} 0$ and $\varphi_n \succ^{\circ} \frac{n\varepsilon}{2}$. At the same time, if we choose $\bar{n}$ large enough, we have that $\frac{\bar{n}\varepsilon}{2} \geq \tilde{\varphi}$ and, in particular, $\frac{\bar{n}\varepsilon}{2} \succcurlyeq^{*} \tilde{\varphi}$. By point 3, we have that $\varphi_{\bar{n}} \succ^{\circ} \tilde{\varphi}$, a contradiction with Property 5 which implies $\tilde{\varphi} \succcurlyeq^{\circ} \varphi_{\bar{n}}$ because $\varphi_{\bar{n}} \not\succcurlyeq^{*} 0$.

5. We next show that there exists $k \geq 0$ such that $c(p) \leq k$ for all $p \in \mathrm{cl}\,(\mathrm{dom}\, c)$. By contradiction, assume that for each $n \in \mathbb{N}$ there exists $p_n \in \mathrm{cl}\,(\mathrm{dom}\, c)$ such that $c(p_n) > n$. By Remark 3, $c(p_n) = \sup \{I(\phi) : \langle \phi, p_n \rangle < 0\}$. It follows that for each $n \in \mathbb{N}$ there exists φ_n

such that $\langle \varphi_n, p_n \rangle < 0$ and $I(\varphi_n) > n$. This implies that $I(\varphi_n - n) > 0$ for all $n \in \mathbb{N}$. Since $\langle \varphi_n, p_n \rangle < 0$, $\varphi_n \not\succcurlyeq^* 0$, but $I(\varphi_n - n) > 0$ implies that $\varphi_n \succ^\circ n$. By Property 5, we can conclude that $\tilde{\varphi} \succcurlyeq^\circ \varphi_n$ for all $n \in \mathbb{N}$. At the same time, if we choose $\bar{n}$ large enough, $\bar{n} \geq \tilde{\varphi}$, that is, $\bar{n} \succcurlyeq^* \tilde{\varphi}$. By point 3 and since $\varphi_{\bar{n}} \succ^\circ \bar{n}$, we have that $\varphi_{\bar{n}} \succ^\circ \tilde{\varphi}$, a contradiction. ■

Theorem 4 *Let $C \neq \varnothing$ be a convex and closed subset of Δ, $\succcurlyeq^*$ be the binary relation on $B_0(S, \Sigma)$ defined by (31), and $\succcurlyeq^\circ$ be another binary relation on $B_0(S, \Sigma)$ that satisfies Properties 0–5, then there exists a unique function $\gamma : C \to [0, \infty]$ which is grounded, lower semicontinuous, convex, and bounded, such that*

$$\varphi \succcurlyeq^\circ \psi \iff \max_{p \in C} \left\{ \int \varphi dp - \int \psi dp - \gamma(p) \right\} \geq 0.$$

Proof. Consider the normalized concave niveloid I of (35) and its conjugate functional $\bar{I}$ defined in (37). By Proposition 6

$$\varphi \succcurlyeq^\circ \psi \iff \bar{I}(\varphi - \psi) \geq 0$$

by Proposition 7, there exists $c : \Delta \to [0, \infty]$ grounded, convex, and lower semicontinuous such that

$$\bar{I}(\phi) = \max_{p \in \Delta} \left\{ \int \phi dp - c(p) \right\} \qquad \forall \phi \in B_0(S, \Sigma).$$

Moreover, $C = \mathrm{cl}(\mathrm{dom}\, c) = \mathrm{dom}\, c$ and c is bounded on C, so that,

$$\bar{I}(\phi) = \max_{p \in C} \left\{ \int \phi dp - c(p) \right\} \qquad \forall \phi \in B_0(S, \Sigma).$$

The function γ in the statement is simply the restriction of c to its domain C.

Assume $\delta : C \to [0, \infty]$ is another grounded, lower semicontinuous, convex, and bounded function such that

$$\varphi \succcurlyeq^\circ \psi \iff \max_{p \in C} \left\{ \int \varphi dp - \int \psi dp - \delta(p) \right\} \geq 0.$$

Since $\succcurlyeq^\circ$ is complete

$$\phi \succ^\circ 0 \iff \min_{p \in C} \left\{ \int \phi dp + \delta(p) \right\} > 0. \tag{43}$$

Setting

$$d(p) = \begin{cases} \delta(p) & p \in C \\ \infty & p \in \Delta \setminus C \end{cases}$$

it is easy to check that $d : \Delta \to [0, \infty]$ is a grounded, convex, and lower semicontinuous function and so

$$J(\phi) = \min_{p \in \Delta} \left\{ \int \phi dp + d(p) \right\} \qquad \forall \phi \in B_0(S, \Sigma)$$

defines a normalized concave niveloid $J : B_0(S, \Sigma) \to \mathbb{R}$ such that

$$J(\phi) > 0 \iff \phi \succ^\circ 0 \iff I(\phi) > 0.$$

But then, for all $\varphi \in B_0(S,\Sigma)$

$$\begin{aligned} I(\varphi) &= \sup\{t \in \mathbb{R} : I(\varphi) > t\} = \sup\{t \in \mathbb{R} : I(\varphi) - t > 0\} \\ &= \sup\{t \in \mathbb{R} : I(\varphi - t) > 0\} = \sup\{t \in \mathbb{R} : J(\varphi - t) > 0\} = J(\varphi). \end{aligned}$$

Because of the uniqueness of the representation of concave niveloids obtained by MMR, it follows $c = d$. ■

A.2.2 Main body of the proof

(i) implies (ii). By Lemma 1, there exist a (unique) non-empty closed and convex set C of probabilities on Σ and a (cardinally unique) non-constant affine $u : X \to \mathbb{R}$ such that, for every $f, g \in F$,

$$f \succsim^* g \iff \int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C \tag{44}$$

$$f \succ^* g \iff \int u(f)\,dp > \int u(g)\,dp \qquad \forall p \in C. \tag{45}$$

So that u represents $\succsim^*$ on X. Unboundedness and Lemma 59 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) guarantee that $u(X) = \mathbb{R}$.

For any $(\varphi, \psi) \in B_0(S,\Sigma)$ it is convenient to set

$$F(\varphi,\psi) = \left\{(f,g) \in F^2 : u(f) = \varphi \text{ and } u(g) = \psi\right\}$$

and to observe that $u(X) = \mathbb{R}$ implies that $F(\varphi,\psi)$ is non-empty. We also write R^* (resp. R°) to denote $\succsim^*$ (resp. $\succsim^\circ$) when regarded as a subset of F^2.

Lemma 5 *The following conditions are equivalent for $\varphi, \psi \in B_0(S,\Sigma)$:*

(a) There are $f, g \in F$ such that $u(f) = \varphi$, $u(g) = \psi$ and $f \succsim^* g$ (i.e., $F(\varphi,\psi) \cap R^* \neq \varnothing$).

(b) $f' \succsim^* g'$ for all $f', g' \in F$ such that $u(f') = \varphi$, $u(g') = \psi$ (i.e., $F(\varphi,\psi) \subseteq R^*$).

(c) $\int \varphi dp \geq \int \psi dp$ for all $p \in C$.

In this case write $\varphi \succcurlyeq^* \psi$, this is consistent with (31), and notice that, by points (a) and (b) above

$$f \succsim^* g \iff u(f) \succcurlyeq^* u(g).$$

Proof. Notice that if $f, f', g, g' \in F$, $u(f) = u(f')$, $u(g) = u(g')$, then

$$\begin{aligned} f \succsim^* g &\iff \int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C \\ &\iff \int u(f')\,dp \geq \int u(g')\,dp \qquad \forall p \in C \iff f' \succsim^* g'. \end{aligned}$$

(a) implies (b). If there are $f, g \in F$ such that $u(f) = \varphi$, $u(g) = \psi$ and $f \succsim^* g$, then for any $f', g' \in F$ such that $u(f') = \varphi$, $u(g') = \psi$ we have $u(f) = \varphi = u(f')$, $u(g) = \psi = u(g')$, and as we observed $f' \succsim^* g'$.

(b) implies (c). Take $f', g' \in F$ such that $u(f') = \varphi$, $u(g') = \psi$, they exist because $F(\varphi, \psi) \neq \varnothing$, by (b) we have $f' \succsim^* g'$, by (44) we have that (c) holds.

(c) implies (a). Take $f, g \in F$ such that $u(f) = \varphi$, $u(g) = \psi$, they exist because $F(\varphi, \psi) \neq \varnothing$, by (c) we have $\int u(f)\, dp \geq \int u(g)\, dp$ for all $p \in C$, by (44) we have $f \succsim^* g$ and (a) holds. ■

An almost identical argument yields:

Lemma 6 *The following conditions are equivalent for $\varphi, \psi \in B_0(S, \Sigma)$:*

(a) There are $f, g \in F$ such that $u(f) = \varphi$, $u(g) = \psi$ and $f \succ\!\succ^* g$.

(b) $f' \succ\!\succ^* g'$ for all $f', g' \in F$ such that $u(f') = \varphi$, $u(g') = \psi$.

(c) $\int \varphi dp > \int \psi dp$ for all $p \in C$.

In this case we write $\varphi \succ\!\succ^* \psi$, this is consistent with (32), and notice that, by points (a) and (b) above

$$f \succ\!\succ^* g \iff u(f) \succ\!\succ^* u(g).$$

Lemma 7 *The following conditions are equivalent for $\varphi, \psi \in B_0(S, \Sigma)$:*

(a) There are $f, g \in F$ such that $u(f) = \varphi$, $u(g) = \psi$ and $f \succsim^\circ g$ (i.e., $F(\varphi, \psi) \cap R^\circ \neq \varnothing$).

(b) $f' \succsim^\circ g'$ for all $f', g' \in F$ such that $u(f') = \varphi$, $u(g') = \psi$ (i.e., $F(\varphi, \psi) \subseteq R^\circ$).

In this case write $\varphi \succcurlyeq^\circ \psi$, and notice that, by points (a) and (b) above

$$f \succsim^\circ g \iff u(f) \succcurlyeq^\circ u(g).$$

Proof. (a) implies (b). If there are $f, g \in F$ such that $u(f) = \varphi$, $u(g) = \psi$ and $f \succsim^\circ g$, then for any $f', g' \in F$ such that $u(f') = \varphi$, $u(g') = \psi$ we have $u(f) = \varphi = u(f')$, $u(g) = \psi = u(g')$, and by (44) $f \sim^* f'$ and $g \sim^* g'$. Substitution Consistency yields $f' \succsim^\circ g'$.

(b) implies (a). Take $f', g' \in F$ such that $u(f') = \varphi$, $u(g') = \psi$, they exist because $F(\varphi, \psi) \neq \varnothing$, by (b) we have $f' \succsim^\circ g'$. ■

In particular, for any, $\varphi, \psi \in B_0(S, \Sigma)$, taking $f, g \in F$ such that $u(f) = \varphi$ and $u(g) = \psi$, either $f \succsim^\circ g$ and so $\varphi \succcurlyeq^\circ \psi$, or $g \succsim^\circ f$ and so $\psi \succcurlyeq^\circ \varphi$. Thus $\succcurlyeq^\circ$ is complete. This fact and the previous lemma readily imply the following result.

Lemma 8 *The following conditions are equivalent for $\varphi, \psi \in B_0(S, \Sigma)$:*

(a) *There are* $f, g \in F$ *such that* $u(f) = \varphi$, $u(g) = \psi$ *and* $f \succ^\circ g$ *(i.e.,* $F(\psi, \varphi) \cap (R^\circ)^c \neq \varnothing$*).*

(b) $f' \succ^\circ g'$ *for all* $f', g' \in F$ *such that* $u(f') = \varphi$, $u(g') = \psi$ *(i.e.,* $F(\psi, \varphi) \subseteq (R^\circ)^c$*).*

(c) $\varphi \succ^\circ \psi$ *(i.e.,* $\psi \not\succsim^\circ \varphi$*).*

Notice that, by points (a), (b), and (c) above

$$f \succ^\circ g \iff u(f) \succ^\circ u(g).$$

Lemma 9 *The pair* $(\succsim^*, \succsim^\circ)$ *satisfies properties 0–5 (of page 24).*

Proof. By Lemma 5, $\succsim^*$ can be represented as in (31).

Property 0. We already observed that $\succsim^\circ$ is complete.

Property 1. Let $\varphi_1 = u(f_1)$, $\psi_1 = u(g_1)$, $\varphi_2 = u(f_2)$, $\psi_2 = u(g_2)$, and $\lambda \in (0,1)$. Observe that $\lambda\varphi_1 + (1-\lambda)\varphi_2 = u(\lambda f_1 + (1-\lambda) f_2)$ and $\lambda\psi_1 + (1-\lambda)\psi_2 = u(\lambda g_1 + (1-\lambda) g_2)$. If $\varphi_2 \succ^\circ \psi_2$ and $\varphi_1 \succ^\circ \psi_1$, then $f_1 \succ^\circ g_1$ and $f_2 \succ^\circ g_2$. By Strict Independence,

$$\lambda f_1 + (1-\lambda) f_2 \succ^\circ \lambda g_1 + (1-\lambda) g_2$$

but as observed

$$\begin{aligned} \lambda f_1 + (1-\lambda) f_2 \succ^\circ \lambda g_1 + (1-\lambda) g_2 &\iff u(\lambda f_1 + (1-\lambda) f_2) \succ^\circ u(\lambda g_1 + (1-\lambda) g_2) \\ &\iff \lambda\varphi_1 + (1-\lambda)\varphi_2 \succ^\circ \lambda\psi_1 + (1-\lambda)\psi_2. \end{aligned}$$

This shows that: If $\varphi_2 \succ^\circ \psi_2$ and $\lambda \in (0,1)$, then

$$\varphi_1 \succ^\circ \psi_1 \implies \lambda\varphi_1 + (1-\lambda)\varphi_2 \succ^\circ \lambda\psi_1 + (1-\lambda)\psi_2.$$

On the other hand, if $\varphi_2 = \psi_2$ and $\lambda \in (0,1)$, we can choose $f_2 = g_2$; then, by Strict Independence again,

$$\begin{aligned} \varphi_1 \succ^\circ \psi_1 \iff f_1 \succ^\circ g_1 &\iff \lambda f_1 + (1-\lambda) f_2 \succ^\circ \lambda g_1 + (1-\lambda) g_2 \\ &\iff u(\lambda f_1 + (1-\lambda) f_2) \succ^\circ u(\lambda g_1 + (1-\lambda) g_2) \\ &\iff \lambda\varphi_1 + (1-\lambda)\varphi_2 \succ^\circ \lambda\psi_1 + (1-\lambda)\psi_2 \end{aligned}$$

as wanted.

Property 2. Let $\varphi = u(f)$, $\psi = u(g)$, $\zeta = u(h)$. If $\varphi \ggcurly^* \psi$ and $\psi \succ^\circ \zeta$, then $u(f) \ggcurly^* u(g)$ and $u(g) \succ^\circ u(h)$, that is, $f \ggcurly^* g$ and $g \succ^\circ h$, by Strong Transitive Consistency, $f \succ^\circ h$ and $u(f) \succ^\circ u(h)$, that is, $\varphi \succ^\circ \zeta$.

Property 3. Let $\varphi = u(f)$ and $\psi = u(g)$. If $\varphi \ggcurly^* \psi$, then $f \ggcurly^* g$ and, by Strong Transitive Consistency, $f \succ^\circ g$, thus $u(f) \succ^\circ u(g)$, that is, $\varphi \succ^\circ \psi$.

Property 4. Let $\varphi = u(f)$, $\psi = u(g)$, $\varepsilon = u(x)$. If $\varphi \succ^\circ \psi$, then $f \succ^\circ g$, thus $g \not\succsim^\circ f$ and

$$\begin{aligned}
1 &\notin \{\lambda \in [0,1] : \lambda g + (1-\lambda)x \succsim^\circ \lambda f + (1-\lambda)f\} \\
1 &\in \{\lambda \in [0,1] : \lambda g + (1-\lambda)x \succsim^\circ \lambda f + (1-\lambda)f\}^c \\
1 &\in \{\lambda \in [0,1] : \lambda f + (1-\lambda)f \succ^\circ \lambda g + (1-\lambda)x\}
\end{aligned}$$

and, by Continuity of $\succsim^\circ$, the latter set is open in $[0,1]$ (because it is the complement of a closed set). Therefore there exists $\hat{\lambda} \in (0,1)$ such that

$$\left(\hat{\lambda}, 1\right] \subseteq \{\lambda \in [0,1] : \lambda f + (1-\lambda)f \succ^\circ \lambda g + (1-\lambda)x\}$$

that is

$$\begin{aligned}
f \succ^\circ \lambda g + (1-\lambda)x \qquad & \forall \lambda \in \left(\hat{\lambda}, 1\right] \\
u(f) \succ^\circ u(\lambda g + (1-\lambda)x) \qquad & \forall \lambda \in \left(\hat{\lambda}, 1\right] \\
u(f) \succ^\circ \lambda u(g) + (1-\lambda)u(x) \qquad & \forall \lambda \in \left(\hat{\lambda}, 1\right] \\
\varphi \succ^\circ \lambda \psi + (1-\lambda)\varepsilon \qquad & \forall \lambda \in \left(\hat{\lambda}, 1\right]
\end{aligned}$$

as wanted.

Property 5. Let $0 = u(x)$, by Weak Caution, there exists $\tilde{g}$ in F such that $f \not\succsim^* x$ implies $\tilde{g} \succsim^\circ f$. Set $\tilde{\varphi} = u(g)$, and take any $\varphi = u(f)$, then

$$\varphi \not\succcurlyeq^* 0 \iff \neg\left[u(f) \succcurlyeq^* u(x)\right] \iff \neg\left[f \succsim^* x\right] \implies \tilde{g} \succsim^\circ f \iff u(g) \succcurlyeq^\circ u(f) \iff \tilde{\varphi} \succcurlyeq^\circ \varphi$$

as wanted. ■

By Theorem 4, since $C \neq \varnothing$ is a convex and closed subset of Δ, $\succcurlyeq^*$ is the binary relation on $B_0(S, \Sigma)$ defined by (31), and $\succcurlyeq^\circ$ is another binary relation on $B_0(S, \Sigma)$ that satisfies Properties 0–5, there exists a unique function $\gamma : C \to [0, \infty]$ which is grounded, lower semicontinuous, convex, and bounded such that

$$\varphi \succcurlyeq^\circ \psi \iff \max_{p \in C} \left\{ \int \varphi dp - \int \psi dp - \gamma(p) \right\} \geq 0$$

then

$$\begin{aligned}
f \succsim^\circ g \iff u(f) \succcurlyeq^\circ u(g) &\iff \max_{p \in C} \left\{ \int u(f)\, dp - \int u(g)\, dp - \gamma(p) \right\} \geq 0 \\
&\iff \int u(f)\, dp \geq \int u(g)\, dp + \gamma(p) \qquad \text{for some } p \in C.
\end{aligned}$$

This concludes the proof of (i) implies (ii).

Moreover, if $\delta : C \to [0, \infty]$ is a grounded, convex, lower semicontinuous, and bounded function such that, for every $f, g \in F$,

$$f \succsim^\circ g \iff \int u(f)\, dp \geq \int u(g)\, dp + \delta(p) \qquad \text{for some } p \in C$$

then, for every $\varphi = u(f)$ and $\psi = u(g)$ in $B_0(S, \Sigma)$,

$$\begin{aligned} \varphi \succcurlyeq^\circ \psi \quad &\iff \quad u(f) \succcurlyeq^\circ u(g) \iff f \succsim^\circ g \iff \int u(f)\, dp \geq \int u(g)\, dp + \delta(p) \qquad \text{for some } p \in C \\ &\iff \quad \max_{p \in C} \left\{ \int u(f)\, dp - \int u(g)\, dp - \delta(p) \right\} \geq 0 \\ &\iff \quad \max_{p \in C} \left\{ \int \varphi dp - \int \psi dp - \delta(p) \right\} \geq 0 \end{aligned}$$

and $\delta = \gamma$, by Theorem 4. This shows that γ is unique given u.

Remark 4 *If $\alpha > 0$ and $\beta \in \mathbb{R}$, then (ii) and simple algebra yield*

$$\begin{aligned} f \succsim^* g &\iff \int [\alpha u(f) + \beta]\, dp \geq \int [\alpha u(g) + \beta]\, dp \qquad \forall p \in C \\ f \succsim^\circ g &\iff \int [\alpha u(f) + \beta]\, dp \geq \int [\alpha u(g) + \beta]\, dp + \alpha c(p) \qquad \textit{for some } p \in C. \end{aligned}$$

This means that when u is replaced by $v = \alpha u + \beta$, c must be replaced with αc, because αc delivers the desired representation, and given v, there can be only one grounded, convex, lower semicontinuous, and bounded cost function with this property.

In particular, if c is not identically 0 for some u, it can never be identically zero.

(ii) implies (i). Assume that there exist a non-empty closed and convex set C of probabilities on Σ, a grounded, convex, lower semicontinuous, and bounded $c : C \to [0, \infty)$, and an onto affine function $u : X \to \mathbb{R}$, such that, for every $f, g \in F$,

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C$$

and

$$f \succsim^\circ g \iff \int u(f)\, dp \geq \int u(g)\, dp + c(p) \qquad \text{for some } p \in C.$$

By Lemma 1, $\succsim^*$ satisfies the BC, C-Completeness, Transitivity, Independence. Since u represents $\succsim^*$ on X and $u(X) = \mathbb{R}$, Lemma 59 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) guarantee that Unboundedness is satisfied too.

Since c is grounded, there exists $\bar{p} \in C$ such that $c(\bar{p}) = 0$. It follows that for any $f, g \in F$ either $\int u(f)\, d\bar{p} \geq \int u(g)\, d\bar{p} + c(\bar{p})$ or $\int u(g)\, d\bar{p} \geq \int u(f)\, d\bar{p} + c(\bar{p})$, that is, $\succsim^\circ$ is complete.

Define $I : B_0(S, \Sigma) \to \mathbb{R}$ by

$$I(\phi) = \min_{p \in C} \left\{ \int \phi dp + c(p) \right\} \qquad \forall \phi \in B_0(S, \Sigma).$$

It is immediate to see that I is a normalized concave niveloid and

$$\bar{I}(\phi) = -I(-\phi) = \max_{p \in C} \left\{ \int \phi dp - c(p) \right\} \qquad \forall \phi \in B_0(S, \Sigma)$$

is a normalized convex niveloid.

Moreover,

$$\begin{aligned} f \succsim^\circ g &\iff \int u(f)\, dp - \int u(g)\, dp - c(p) \geq 0 \qquad \text{for some } p \in C \\ &\iff \max_{p \in C} \left\{ \int u(f)\, dp - \int u(g)\, dp - c(p) \right\} \geq 0 \\ &\iff \bar{I}(u(f) - u(g)) \geq 0 \\ &\iff -I(u(g) - u(f)) \geq 0 \\ &\iff I(u(g) - u(f)) \leq 0 \end{aligned}$$

while

$$g \succ^\circ f \iff \neg (f \succsim^\circ g) \iff I(u(g) - u(f)) > 0.$$

Continuity. Consider $f, g, h, l \in F$ and a sequence λ_n in $[0,1]$ such that $\lambda_n \to \lambda$. If $\lambda_n f + (1 - \lambda_n) g \succsim^\circ \lambda_n h + (1 - \lambda_n) l$ for all $n \in \mathbb{N}$, then

$$\begin{aligned} 0 &\leq \bar{I}(u(\lambda_n f + (1 - \lambda_n) g) - u(\lambda_n h + (1 - \lambda_n) l)) \\ &= \bar{I}(\lambda_n u(f) + (1 - \lambda_n) u(g) - (\lambda_n u(h) + (1 - \lambda_n) u(l))). \end{aligned}$$

Since I is continuous, the inequality also holds for λ, proving Continuity.

Strict Independence. Assume that $g \succeq^\circ l$ and $\lambda \in (0,1)$. For all $f, h \in F$, we have that

$$\begin{aligned} I(u(\lambda f + (1 - \lambda) g) - u(\lambda h + (1 - \lambda) l)) &= I(\lambda u(f) + (1 - \lambda) u(g) - (\lambda u(h) + (1 - \lambda) u(l))) \\ &= I(\lambda (u(f) - u(h)) + (1 - \lambda)(u(g) - u(l))) \\ &\geq \lambda I(u(f) - u(h)) + (1 - \lambda) I(u(g) - u(l)). \end{aligned}$$

If $f \succ^\circ h$, then $I(u(f) - u(h)) > 0$ and $g \succeq^\circ l$ implies $I(u(g) - u(l)) \geq 0$ whence

$$I(u(\lambda f + (1 - \lambda) g) - u(\lambda h + (1 - \lambda) l)) > 0$$

proving that $\lambda f + (1 - \lambda) g \succ^\circ \lambda h + (1 - \lambda) l$. Conversely, if $g = l$ and $\lambda \in (0,1)$, then

$$\lambda f + (1 - \lambda) g \succ^\circ \lambda h + (1 - \lambda) l$$

$$\iff$$

$$I(u(\lambda f + (1 - \lambda) g) - u(\lambda h + (1 - \lambda) g)) > 0$$

$$\iff$$

$$I(\lambda [u(f) - u(h)]) > 0.$$

But I is convex and $I(0)=0$, hence, for all $\phi \in B_0(S,\Sigma)$ and $\lambda \in (0,1)$,

$$\lambda I(\phi) = \lambda I(\phi) + (1-\lambda) I(0) \geq I(\lambda\phi + (1-\lambda)0) = I(\lambda\phi)$$

in particular, $I(\lambda\phi) > 0$ implies $I(\phi) > 0$. Therefore

$$I(\lambda[u(f) - u(h)]) > 0 \implies I(u(f) - u(h)) > 0 \implies f \succ^\circ h.$$

Strong Transitive Consistency. If $f \ggcurly^* g$ and $g \succeq^\circ h$, then

$$\int u(f)\,dp > \int u(g)\,dp \text{ and } \int u(h)\,dp \leq \int u(g)\,dp + c(p) \qquad \forall p \in C$$

(where the equality part in the second relation accounts for the case $g = h$ because $c \geq 0$) this implies

$$\int u(h)\,dp \leq \int u(g)\,dp + c(p) < \int u(f)\,dp + c(p) \qquad \forall p \in C$$

that is, $f \succ^\circ h$.

Substitution Consistency. If $f \sim^* h$, $g \sim^* l$, and $f \succsim^\circ g$ imply

$$\begin{aligned} \int u(f)\,dp &= \int u(h)\,dp \qquad \forall p \in C \\ \int u(g)\,dp &= \int u(l)\,dp \qquad \forall p \in C \\ \int u(f)\,dq &\geq \int u(g)\,dq + c(q) \qquad \text{for some } q \in C \end{aligned}$$

whence $\int u(h)\,dq \geq \int u(l)\,dq + c(q)$ for some $q \in C$ and $h \succsim^\circ l$.

Weak Caution. Assume that $f \not\succsim^* g$, it follows that there exists $\bar{p} \in C$ such that

$$\int u(f)\,d\bar{p} < \int u(g)\,d\bar{p}$$

but then, setting $k = \sup_{p \in C} c(p)$,

$$\int u(g)\,d\bar{p} + k \geq \int u(f)\,d\bar{p} + c(\bar{p})$$

and so it is sufficient to consider $\tilde{g}$ such $u(\tilde{g}) = u(g) + k$ to have $\tilde{g} \succsim^\circ f$ for all $f \not\succsim^* g$.

It only remains to prove that $\succsim^*$ is the transitive core of $\succsim^\circ$.

First assume that $f \succsim^* g$, then

$$\int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C. \tag{46}$$

But then $h \succsim^\circ f$ implies that there exists $q \in C$ such that

$$\int u(h)\,dq \geq \int u(f)\,dq + c(q) \geq \int u(g)\,dq + c(q)$$

where the last inequality follows from (46), that is, $h \succsim^{\circ} g$. Analogously, $g \succsim^{\circ} l$ implies that there exists $q \in C$ such that

$$\int u(f)\, dq \geq \int u(g)\, dq \geq \int u(l)\, dq + c(q)$$

where the first inequality follows from (46), that is, $f \succsim^{\circ} l$. This shows that $f \succsim^{*} g$ implies $f \succsim^{\circ\circ} g$.

As to the converse, assume that $f \succsim^{\circ\circ} g$, then given $h \in F$,

$$g \succsim^{\circ} h \implies f \succsim^{\circ} h \tag{47}$$

that is

$$I(u(h) - u(g)) \leq 0 \implies I(u(h) - u(f)) \leq 0.$$

Now given $\eta \in B_0(S, \Sigma)$ and $k \in \mathbb{R}$, the above relation delivers

$$\begin{aligned} I(\eta - u(g)) \leq k \implies & I(\eta - k - u(g)) \leq 0 \implies I(u(h_{\eta,k}) - u(g)) \leq 0 \\ \implies & I(u(h_{\eta,k}) - u(f)) \leq 0 \implies I(\eta - k - u(f)) \leq 0 \\ \implies & I(\eta - u(f)) \leq k \end{aligned}$$

where $h_{\eta,k}$ is an element of F such that $u(h_{\eta,k}) = \eta - k$. Therefore, given $\eta \in B_0(S, \Sigma)$ and $k \in \mathbb{R}$, if $I(\eta - u(g)) \leq k$, then also $I(\eta - u(f)) \leq k$. In particular, taking any $\eta \in B_0(S, \Sigma)$, since $I(\eta - u(g)) \leq I(\eta - u(g))$, then also $I(\eta - u(f)) \leq I(\eta - u(g))$. We have shown that

$$f \succsim^{\circ\circ} g \implies I(\eta - u(g)) - I(\eta - u(f)) \geq 0 \qquad \forall \eta \in B_0(S, \Sigma). \tag{48}$$

Recall that, for every $\psi \in B_0(S, \Sigma)$,

$$\partial I(\psi) = \left\{ p \in \Delta : I(\varphi) - I(\psi) \leq \int (\varphi - \psi)\, dp \quad \forall \varphi \in B_0(S, \Sigma) \right\}$$

then for every $\varphi \in B_0(S, \Sigma)$ we have

$$\int (\varphi - \psi)\, dp \geq I(\varphi) - I(\psi) \qquad \forall p \in \partial I(\psi).$$

Then for every $\eta \in B_0(S, \Sigma)$, setting $\psi_\eta = \eta - u(f)$, and $\varphi_\eta = \eta - u(g)$, equation (48) implies

$$\int (u(f) - u(g))\, dp = \int \left(\varphi_\eta - \psi_\eta\right) dp \geq I\left(\varphi_\eta\right) - I\left(\psi_\eta\right) = I(\eta - u(g)) - I(\eta - u(f)) \geq 0$$

for all $p \in \partial I(\eta - u(f))$. We have shown that

$$\begin{aligned} f \succsim^{\circ\circ} g \quad \implies & \int (u(f) - u(g))\, dp \geq 0 \qquad \forall p \in \bigcup_{\eta \in B_0(S,\Sigma)} \partial I(\eta - u(f)) \\ \implies & \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in \bigcup_{\zeta \in B_0(S,\Sigma)} \partial I(\zeta) \\ \implies & \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in \mathrm{cl}\left(\mathrm{co} \bigcup_{\zeta \in B_0(S,\Sigma)} \partial I(\zeta)\right) \end{aligned}$$

but the results of MMR and Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini (2015, henceforth CMMR), guarantee that $\text{cl}\left(\text{co}\bigcup_{\zeta\in B_0(S,\Sigma)}\partial I(\zeta)\right) = C$,[20] so that $f \succsim^* g$. $\blacksquare$

A.3 Proof of Theorem 3

(i) implies (ii). By Lemma 1, there exist a non-empty closed and convex set C of probabilities on Σ and a non-constant affine $u : X \to \mathbb{R}$ such that, for every $f, g \in F$,

$$f \succsim^* g \iff \int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C.$$

Moreover, given $x, y \in X$, by Reflexivity of $\succsim^\circ$ and Transitive Consistency,

$$x \succsim^* y \implies x \succsim^* y \succsim^\circ y \implies x \succsim^\circ y.$$

That is, on constant acts, $\succsim^*$ is a subrelation of $\succsim^\circ$, and both relations are non-trivial and satisfy the axioms of Herstein and Milnor (1953). By Corollary B.3 of Ghirardato, Maccheroni, and Marinacci (2004), these relations coincide and are both represented by u. For this reason, we often omit the superscripts * and $^\circ$ when comparing constant acts.

Next we show that, for every $f, g \in F$,

$$g \succsim^\circ f \iff \int u(g)\,dp \geq \int u(f)\,dp \qquad \text{for some } p \in C.$$

Assume first that there exists $q \in C$ such that

$$\int u(g)\,dq \geq \int u(f)\,dq.$$

By Proposition 4, it is not true that, for every $x \succ y$ in X, there exist ε in $(0,1)$ such that

$$(1-\varepsilon) f + \varepsilon y \succsim^* (1-\varepsilon) g + \varepsilon x.$$

Then, there are $x \succ y$ in X such that for every ε in $(0,1)$

$$(1-\varepsilon) f + \varepsilon y \not\succsim^* (1-\varepsilon) g + \varepsilon x$$

[20] Specifically, adopting the notation of CMMR we observe that the canonical extension of c to Δ

$$\gamma(p) = \begin{cases} c(p) & p \in C \\ \infty & p \notin C \end{cases}$$

is grounded, lower semicontinuous and convex, hence by Lemma 26 of MMR it is the only function with these properties such that $I(\phi) = \min_{p\in C}\left\{\int \phi\,dp + \gamma(p)\right\}$ for all $\phi \in B_0(S,\Sigma)$. Then, the set called $\mathcal{C}$ on page 16 of CMMR is a singleton, thus $c^\star = d^\star = \gamma$ in their Theorem 3. By point 5 of the same theorem and Corollary 5 of CMMR, it follows

$$C = \text{dom}\,\gamma = \text{cl}\,(\text{dom}\,d^\star) = \text{cl}\left(\text{co}\bigcup_{\zeta\in B_0(S,\Sigma)}\partial I(\zeta)\right).$$

and by Possibility

$$(1-\varepsilon)\, g+\varepsilon x \succsim^{\circ} (1-\varepsilon)\, f+\varepsilon y \qquad \forall \varepsilon \in (0,1)$$

and Continuity of $\succsim^{\circ}$ delivers $g \succsim^{\circ} f$. This shows that, if $\int u(g)\, dp \geq \int u(f)\, dp$ for some $p \in C$, then $g \succsim^{\circ} f$. Conversely, assume – per contra – that there exist $f, g \in F$ such that $g \succsim^{\circ} f$ and

$$\int u(g)\, dp < \int u(f)\, dp \qquad \forall p \in C.$$

Then, by Proposition 4 again, there exist $x \succ y$ in X and ε in $(0,1)$ such that

$$(1-\varepsilon)\, f+\varepsilon y \succsim^{*} (1-\varepsilon)\, g+\varepsilon x.$$

By C-Independence of $\succsim^{\circ}$, and since $g \succsim^{\circ} f$, it follows

$$(1-\varepsilon)\, f+\varepsilon y \succsim^{*} (1-\varepsilon)\, g+\varepsilon x \succsim^{\circ} (1-\varepsilon)\, f+\varepsilon x$$

so that by Transitive Consistency

$$(1-\varepsilon)\, f+\varepsilon y \succsim^{\circ} (1-\varepsilon)\, f+\varepsilon x \tag{49}$$

which by Monotonicity of $\succsim^{\circ}$ leads to a contradiction. In fact, $x \succ y$ implies

$$u\left((1-\varepsilon)\, f(s)+\varepsilon x\right) > u\left((1-\varepsilon)\, f(s)+\varepsilon y\right)$$

for all $s \in S$, that is, $[(1-\varepsilon)\, f+\varepsilon x](s) \succ [(1-\varepsilon)\, f+\varepsilon y](s)$ for all $s \in S$, and $(1-\varepsilon)\, f+\varepsilon x \succ^{\circ} (1-\varepsilon)\, f+\varepsilon y$, contradicting (49).

(ii) implies (i). By Theorem 1, $\succsim^{*}$ satisfies the BC, C-Completeness, Transitivity, and Independence, $\succsim^{\circ}$ satisfies Continuity, and $(\succsim^{*}, \succsim^{\circ})$ satisfies Possibility. It remains to show that $(\succsim^{*}, \succsim^{\circ})$ satisfies Transitive Consistency, and $\succsim^{\circ}$ satisfies Completeness, C-Transitivity, and C-Independence, Reflexivity, Monotonicity, and Non-Triviality. The verification is routinely obtained by using the representations (22) and (23) and observing that:

- given any $x, y \in X$,

$$x \succsim^{*} y \iff x \succsim^{\circ} y \iff u(x) \geq u(y)\, ;$$

- given any two simple measurable functions $\varphi, \psi : S \to \mathbb{R}$,

$$\varphi(s) > \psi(s) \qquad \forall s \in S \implies \int \varphi dp > \int \psi dp \qquad \forall p \in C.$$

Finally, replace **C-Transitivity** and **Possibility** with **Transitivity** and **C-Possibility** in (i) of our statement, it is then easy to check that the conditions in point (i) of Theorem 3 of GMMS are satisfied.[21] Representations (22) and (24) follow.

While, the converse implication can be routinely obtained from representations (22) and (24). ■

[21] We only observe that the first part of the proof of Lemma 1 can be used to show that if $f(s) \succsim^{\circ} g(s)$ for all $s \in S$, then $f \succsim^{\circ} g$.

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