

Rest Unemployment and Unionization*

Fernando Alvarez

University of Chicago
f-alvarez1@uchicago.edu

Robert Shimer

University of Chicago
robert.shimer@gmail.com

June 5, 2008

PRELIMINARY

Abstract

This paper examines the impact of unions on unemployment and wages in a dynamic equilibrium search model. Unions impose a minimum wage on employers and ration jobs to ensure that their most senior members are employed. A minimum wage policy is optimal for a utilitarian union that cares equally about its employed and unemployed members. The combination of a minimum wage and rationing by seniority generates rest unemployment, where following a downturn in their labor market, unionized workers are willing to wait for jobs to reappear rather than search for a new labor market. Introducing unions into a dynamic equilibrium model has two implications, which others have argued are features of the data: the hazard of exiting unemployment at long unemployment durations is very low when the union-imposed minimum wage is high; and a high union-imposed minimum wage generates a compressed wage distribution and a high turnover rate of jobs.

*We are grateful for research assistance by Ezra Oberfield. Shimer's research is supported by a grant from the National Science Foundation.

1 Introduction

This paper examines the impact of unions on unemployment and wages. A union imposes a minimum wage on its members' employers. The minimum wage binds in at least some states of the world, in which event the union rations jobs to ensure that its most senior members are employed. Our focus is on the implications of such a union policy on workers' decision to enter and exit unionized labor markets. We prove that a laid-off union member will never immediately exit her labor market to search elsewhere for a job. Instead, she will endure a spell of rest unemployment, waiting for labor market conditions to improve. We find that the hazard rate of reentering employment generally declines during an unemployment spell, so unionized workers will experience both frequent short spells and infrequent long spells of unemployment.

Our modeling strategy closely follows [Alvarez and Shimer \(2008\)](#), which in turn builds on [Lucas and Prescott \(1974\)](#). The economy consists of a large number of labor markets that produce imperfect substitutes. There are many workers and firms in each labor market, so in the absence of unions, wages and output prices are determined competitively within each labor market. Productivity shocks induce workers to move between labor markets, but reallocation across markets is costly because of search frictions.

Both papers distinguish between rest and search unemployment. While in rest unemployment, individuals do not work, enjoying a value of leisure higher than working but lower than being outside the labor force. Moreover, the rest unemployed retain the possibility of returning instantly and at no cost to the labor market where they last worked. Search unemployment enables a worker to locate in any labor market. Our previous paper argued that the existence of rest unemployment may be important for understanding the dynamic behavior of wages. This paper focuses on the possibility that rest unemployment may arise because of unionization. We believe that the two explanations for rest unemployment are complementary. Still, it is interesting to note that if there is no leisure advantage to resting rather than working, there is rest unemployment if and only if the minimum wage is binding. In this sense, binding minimum wages create rest unemployment.

Technically, the main difference between the two papers is that in our earlier work, each labor market cleared at each point in time. Whenever a worker is rest unemployed, she weakly prefers rest unemployment to working in her labor market at that instant. In fact, that paper assumes that workers within a market are homogeneous and so all workers are indifferent about working whenever there is rest unemployment in their labor market. In this paper, union-mandated minimum wages and seniority rules make the rest unemployed worse off than the employed. This means we need to keep track of workers' seniority in order to

understand their decision to enter and leave labor markets.

We show that our model of unions generates rest unemployment. Whenever the minimum wage binds, workers with low seniority who are rationed out of a job decide to stay in the labor market, waiting for the conditions to improve so that they can return to work at the minimum wage. When labor market conditions are bad enough, workers with the lowest seniority among those who are rationed out of employment will leave. The prospects of a labor market are limited by the fact that as conditions improve, new workers will arrive via search. These newcomers will have the lowest seniority, and hence will be most vulnerable to bad shocks, but they will only arrive in a labor market when it is booming. The situation of newcomers depends on how high the minimum wage is. If it is not that high, so that it binds only for bad shocks, they will immediately start working. If the minimum wage is sufficiently high, it always binds. In this case, newcomers arrive when prospects are very good, but are forced to queue until enough good shocks have arrived before they can start work. In such a labor market, there is always a queue of workers waiting either to start or resume employment.

The aggregate behavior of the model is similar whether rest unemployment is a consequence of a high value of leisure and market clearing or of union-mandated minimum wages and seniority rules. But since the interpretation of the model is quite different, this paper connects with a number of different literatures on the impact of non-market-clearing wages. For example, [Blanchard and Summers \(1986\)](#) argue that for an “insider-outsider” theory of European unemployment, where unions run by insiders generate unemployment because wages are set to exclude disenfranchised outsiders. We find that a union that cares equally about insiders and outsiders opts for a minimum wage policy. More precisely, we consider a union which sets the wage, or equivalently the employment level, at each instant in order to maximize the utilitarian welfare of all of its members, insiders and outsiders. We show that the union’s policy is characterized by a constant minimum wage, where the minimum wage is a markup over the leisure value of rest unemployment. By setting this minimum wage, the union effectively restricts output so that it never exceeds the monopoly level. When the available number of workers is less the number needed to produce the monopoly output, all the union members are employed and the minimum wage does not bind. At other times the minimum wage binds and there is rest unemployment. Thus the difference between a monopoly producer and a monopoly union is simply an issue of who keeps the monopoly rents. In other words, we find that unions generate unemployment not because they exclude outsiders from the decision process, but because they can only raise the well-being of all their members by constraining output in some states of the world.

[Summers \(1986\)](#) argues that that union-induced wage rigidities can explain a large portion

of unemployment in the U.S. as well. Unemployed workers who lose their job because of sectoral shocks spend little time searching for jobs, but instead seem to be waiting either for wages to fall or for the shocks to be reversed. [Harris and Todaro \(1970\)](#) propose an extreme version of “wait unemployment” in less developed countries. When rural workers move to the city, they must queue for a job before they can start work. They are willing to do so even though the marginal product of labor is positive in the countryside. Both of these findings are consistent with our model. A spell of rest unemployment ends only if the shocks that caused it are reversed or if the worker becomes so discouraged that she leaves the labor market. In either case, workers can spend a considerable amount of time unemployed. If the minimum wage is sufficiently high, it will bind in all states of the world. Then even new entrants will not be able to go to work immediately. Instead, they must queue until productivity has risen sufficiently for their marginal product to exceed the minimum wage. While they are queueing, increases in productivity raise their seniority—their position in the queue—until they eventually reach the gates of the factory and get employed.

[Jacobson, LaLonde, and Sullivan \(1993\)](#) document that workers displaced from heavily unionized industries suffer unusually large and persistent income declines. This is consistent with the way we model seniority in unionized industries. In non-unionized industries, workers’ welfare is limited by possibility of new entrants coming to the industry. But in unionized industries, high seniority workers may be significantly better off than new entrants. When they are displaced, the consequences are then disproportionately severe.

Our model also addresses a large literature which argues that unions compress wages. [Blau and Kahn \(1996\)](#) observe that wages in the U.S. are more dispersed than in other OECD countries, particularly towards the bottom of the distribution and argue that this is due to the absence of centralized wage-setting mechanisms. [Mourre \(2005\)](#) confirms this using more recent and detailed data for the European Union. [Bertola and Rogerson \(1997\)](#) show that such wage compression may be important for understanding why other labor market institutions, especially restrictions on turnover, are not particularly correlated with measured job creation and destruction rates. In our model, unions can affect labor market institutions only by compressing wages and so we can confirm that high unemployment rates are associated with substantial wage compression.

Finally, our model is consistent with the finding in [Nickell and Layard \(1999\)](#) that unions raise the unemployment rate only in countries where they cannot effectively coordinate their bargaining. In our model, the equilibrium without unions is Pareto optimal. While any individual union can improve its workers’ well-being through a minimum wage, all workers are better off if unions do not exploit their monopoly power. Thus if unions can collude, they be able to avoid generating rest unemployment.

The next section of the paper presents our formal model. We characterize the equilibrium in [Section 3](#). We first prove that a minimum wage affects the wage distribution if and only if it generates rest unemployment. Then we show how minimum wages affect workers' decision to enter and exit labor markets. Finally, we characterize the search and rest unemployment rates and the hazard rate of exiting unemployment in a labor market with a binding minimum wage. [Section 4](#) explains why a union finds it optimal to impose a minimum wage. We finish in [Section 5](#) with a numerical example which illustrates the properties of the model.

2 Model

We consider a continuous time, infinite-horizon model. We focus for simplicity on an aggregate steady state and assume markets are complete.

2.1 Goods

There is a continuum of industries indexed by $n \in [0, 1]$. Within each industry, there is a continuum of goods indexed by $j \in [0, 1]$ and a large number of competitive producers of each good. Thus n_j is the name of a particular good produced in a particular industry. Each good is produced in a separate labor market with a constant returns to scale technology that uses only labor. In a typical labor market n_j at time t , there is a measure $l(n_j, t)$ workers. Of these, $e(n_j, t)$ are employed, each producing $Ax(n_j, t)$ units of good n_j , while the remaining $l(n_j, t) - e(n_j, t)$ are rest-unemployed. Competition forces firms to price each good at marginal cost, so the wage in labor market n_j , $w(n_j, t)$, is equal to the product of the price of good n_j , $p(n_j, t)$, and the productivity of each worker in labor market n_j , $Ax(n_j, t)$.

A is the aggregate component in productivity while $x(n_j, t)$ is an idiosyncratic shock that follows a geometric random walk,

$$d \log x(n_j, t) = \mu_n^x dt + \sigma_n^x dz(n_j, t), \quad (1)$$

where μ_n^x measures the drift of log productivity, $\sigma_n^x > 0$ measures the standard deviation, and $z(n_j, t)$ is a standard Wiener process, independent across goods.

To keep a well-behaved distribution of labor productivity, we assume that the market for good n_j shuts down according to a Poisson process with arrival rate δ_n , independent across goods and independent of good n_j 's productivity. When this shock hits, all the workers are forced out of the labor market. A new good, also named n_j , enters with positive initial productivity $x \sim F_n(x)$, keeping the total measure of goods in industry n constant. We

assume a law of large numbers, so the share of labor markets in each industry experiencing any particular sequence of shocks is deterministic.

2.2 Households

There is a representative household consisting of a measure 1 of members. The large household structure allows for full risk sharing within each household, a standard device for studying complete markets allocations.

At each moment in time t , each member of the representative household engages in one of the following mutually exclusive activities:

- $L(t)$ household members are located in one of the intermediate goods (or equivalently labor) markets.
 - $E(t)$ of these workers are employed at the prevailing wage and get leisure 0.
 - $U_r(t) = L(t) - E(t)$ of these workers are rest-unemployed and get leisure b_r .
- $U_s(t)$ household members are search-unemployed, looking for a new labor market and getting leisure b_s .
- The remaining $1 - E(t) - U_r(t) - U_s(t)$ household members are inactive, getting leisure b_i .

We assume $b_i > b_s$. Household members may costlessly switch between employment and rest unemployment and between inactivity and searching; however, they cannot switch intermediate goods markets without going through a spell of search unemployment. Workers exit their intermediate goods market for inactivity or search in three circumstances: first, they may do so endogenously at any time at not cost; second, they must do when their market shuts down, which happens at rate δ_n ; and third, they must do so when they are hit by an idiosyncratic shock, according to a Poisson process with arrival rate q_n , independent across individuals and independent of their labor market's productivity. We introduce the idiosyncratic "quit" shock q_n to account for separations that are unrelated to the state of the labor market. Finally, a worker in search unemployment finds a job according to a Poisson process with arrival rate α . When this happens, she may enter the intermediate goods market of her choice.

We represent the household's preferences via the utility function

$$\int_0^\infty e^{-\rho t} (\log \bar{C}(t) + b_i(1 - E(t) - U_r(t) - U_s(t)) + b_r U_r(t) + b_s U_s(t)) dt, \quad (2)$$

where $\rho > 0$ is the discount rate and $\bar{C}(t)$ is the household's consumption of an aggregate of all goods produced in all industries,

$$\log \bar{C}(t) = \int_0^1 \log C(n, t) dn, \quad (3)$$

$C(n, t)$ is the household's consumption of an aggregate of the goods in industry n ,

$$C(n, t) = \left(\int_0^1 c(n_j, t)^{\frac{\theta_n - 1}{\theta_n}} dj \right)^{\frac{\theta_n}{\theta_n - 1}}, \quad (4)$$

and $c(n_j, t)$ is the consumption of good n_j at time t . We assume that the elasticity of substitution between goods in industry n , θ_n , is greater than 1. The cost of this consumption is $\int_0^1 \int_0^1 p(n_j, t) c(n_j, t) dj dn$, which we assume the household finances using its labor income.

Standard arguments imply that the demand for good n_j satisfies

$$c(n_j, t) = \frac{C(n, t) P(n, t)^{\theta_n}}{p(n_j, t)^{\theta_n}}, \quad (5)$$

where

$$P(n, t) = \left(\int_0^1 p(n_j, t)^{1 - \theta_n} dj \right)^{\frac{1}{1 - \theta_n}} \quad (6)$$

is the price index in industry n . The demand for the consumption aggregator in industry n satisfies

$$C(n, t) = \frac{\bar{C}(t)}{P(n, t)}, \quad (7)$$

where we use the price of the aggregate consumption bundle \bar{C} as numeraire, or equivalently normalize

$$\int_0^1 \log P(n, t) dn = 0. \quad (8)$$

To ensure a well-behaved distribution of wages in each industry, we impose two restrictions on preferences and technology. First, we require

$$\delta > (\theta_n - 1) (\mu_n^x + (\theta_n - 1) \frac{1}{2} (\sigma_n^x)^2) \quad (9)$$

for all n , so industries exit sufficiently quickly to offset the drift in the stochastic process for productivity. If this condition failed, workers could attain infinite utility. Second, we require

$$X_n \equiv \left(\int_0^\infty x^{\theta_n - 1} dF_n(x) \right)^{\frac{1}{\theta_n - 1}} \in (0, \infty) \quad (10)$$

for all n , a restriction on the distribution of productivity in new labor markets. If this condition failed, the wage would be either zero or infinite.

2.3 Unions

Unions constrain the wage in labor market n_j , introducing a restriction $w(n_j, t) \geq \hat{w}(n_j)$. For most of our analysis, we treat the minimum wage $\hat{w}(n_j)$ as exogenous and show later how it reflects the power of the union in labor market n_j . To see whether the minimum wage constraint binds, first note that if all the workers in the industry were employed, they would produce $Ax(n_j, t)l(n_j, t)$ units of good n_j . Inverting the demand curve [equation \(5\)](#) and eliminating the price of industry n using [equation \(7\)](#), the relative price of good n_j would be

$$p(n_j, t) = \frac{\bar{C}(t)}{C(n, t)^{\frac{\theta_n-1}{\theta_n}} (Ax(n_j, t)l(n_j, t))^{1/\theta_n}}.$$

The wage in the industry would then be $p(n_j, t)Ax(n_j, t)$ or

$$w(n_j, t) = \frac{\bar{C}(t)(Ax(n_j, t))^{\frac{\theta_n-1}{\theta_n}}}{C(n, t)^{\frac{\theta_n-1}{\theta_n}} l(n_j, t)^{\frac{1}{\theta_n}}}. \quad (11)$$

This is increasing in the productivity of the labor market and decreasing in the number of workers. In particular, if there are too many workers in the market, the minimum wage constraint binds. In that case, $w(n_j, t) = \hat{w}(n_j)$ and employment is determined at the level that makes the price of good n_j equal to $\hat{w}(n_j)/Ax(n_j, t)$,

$$e(n_j, t) = \frac{\bar{C}(t)^{\theta_n} (Ax(n_j, t))^{\theta_n-1}}{C(n, t)^{\theta_n-1} \hat{w}(n_j)^{\theta_n}}, \quad (12)$$

increasing in productivity and decreasing in the minimum wage.

We assume that when the minimum wage constraint binds, more senior workers have the first option to work, where seniority is measured by the amount of time spent in the union. Consider a worker with relative seniority $s \in [0, 1]$, where $s = 1$ corresponds to the worker with the greatest seniority. Assuming she wants the job, she is guaranteed to be employed if $e(n_j, t)/l(n_j, t) \geq 1 - s$ or, from [equation \(12\)](#),

$$s \geq 1 - \frac{\bar{C}(t)^{\theta_n} (Ax(n_j, t))^{\theta_n-1}}{C(n, t)^{\theta_n-1} \hat{w}(n_j)^{\theta_n} l(n_j, t)}. \quad (13)$$

A worker with a given seniority is more likely to be employed when productivity is higher, the minimum wage is lower, or the number of workers in the industry is smaller.

Since workers are typically not indifferent about working, those with more seniority are weakly better off. Thus to analyze a worker’s decision to enter or stay in a labor market, we need to examine not only the behavior of wages in the market, but also how the entry and exit of other workers influences each worker’s seniority.

2.4 Equilibrium

We look for a competitive equilibrium of this economy, subject to the constraints imposed by minimum wages. At each instant, each household chooses how much of each good to consume and how to allocate its members between employment in each labor market, rest unemployment in each labor market, search unemployment, and inactivity, in order to maximize utility subject to technological constraints on reallocating members across labor markets and the minimum wage constraints, taking as given the stochastic process for wages and seniority in each labor market; and each goods producer n_j maximizes profits by choosing how many workers to hire taking as given the wage in its labor market and the price of its good. Moreover, the demand for labor from goods producers is equal to the supply from households in each market unless the minimum wage constraint binds, in which case labor demand may be less than labor supply; and households’ demand for goods is equal to the supply from from firms.

We look for a stationary equilibrium where all aggregate and industry-specific quantities and prices are constant, as is the joint distribution of wages, productivity, output, employment, and rest unemployment across labor markets within industries. We suppress the time argument as appropriate in what follows. With identical households and complete markets, consumption is equal to current labor income and hence we ignore financial markets in the remainder of this paper.

3 Characterization of Equilibrium

At any point in time, a typical labor market n_j is characterized by its productivity x and the number of workers l . We look for an equilibrium in which the ratio x^{θ_n-1}/l follows a Markov process. Workers enter labor markets when the ratio exceeds a threshold and exit labor markets when it falls below a strictly smaller threshold. Moreover, [equation \(13\)](#) shows that this ratio and a worker’s seniority determines whether she has the option to work.

3.1 The Marginal Value of Household Members

We start by computing the marginal value of an additional household member engaged in each of the three activities. These are related by the possibility of reallocating household members between activities.

Consider first a household member who is permanently inactive. It is immediate from [equation \(2\)](#) that he contributes

$$\underline{v} = \frac{b_i}{\rho} \quad (14)$$

to household utility. Since the household may freely shift workers between inactivity and search unemployment, this must also be the incremental value of a searcher, assuming some members are engaged in each activity. A searcher gets flow utility b_s and the possibility of finding a labor market at rate α , giving capital gain $\bar{v} - \underline{v}$, where \bar{v} is the value to the household of having a worker in the best labor market. This implies $\rho \underline{v} = b_s + \alpha (\bar{v} - \underline{v})$ or

$$\bar{v} = \underline{v} + b_i \kappa, \text{ where } \kappa \equiv \frac{b_i - b_s}{b_i \alpha} \quad (15)$$

is a measure of search costs, the percentage loss in current utility from searching rather than inactivity times the expected duration of search unemployment $1/\alpha$. Conversely, a worker may freely exit her labor market, and so the lower bound on the value of a household member in a labor market, either employed or search unemployed, is \underline{v} . If the household values a worker at some intermediate amount, it will be willing to keep her in her labor market rather than having her search for a new one.

Finally, consider the margin between employment and resting for a worker in a labor market paying a wage w . A resting worker generates b_r utils while an employed worker generates income valued at w/\bar{C} , where $1/\bar{C}$ is the marginal utility of the consumption aggregate. Since switching between employment and resting is costless, all workers prefer to work in any labor market with $w/\bar{C} > b_r$ and prefer to rest in any market with $w/\bar{C} < b_r$. This implies that if $\hat{w}/\bar{C} \leq b_r$, the minimum wage never binds because workers' willingness to enter rest unemployment endogenously keeps the wage above \hat{w} . Conversely, if $\hat{w}/\bar{C} > b_r$, the minimum wage may sometimes bind.

3.2 Wage and Labor Force Dynamics

Consider a labor market in industry n with l workers, productivity x , and a minimum wage \hat{w} . Let $P(l, x)$ denote the price of its good, $Q(l, x)$ denote the amount of the good produced, $W(l, x)$ denote the wage rate, and $E(l, x)$ denote the number of workers who are employed. Competition ensures that the wage is equal to the marginal product of labor, $W(l, x) =$

$P(l, x)Ax$, while the production function implies $Q(l, x) = E(l, x)Ax$. From [equation \(11\)](#), the wage solves

$$W(l, x) = \bar{C} \max\{e^{\hat{\omega}}, e^{\omega}\} \quad (16)$$

where

$$\omega \equiv \frac{(\theta_n - 1)(\log(Ax) - \log C(n)) - \log l}{\theta_n}, \quad (17)$$

is the logarithm of the “full-employment wage” measured in utils, the wage that would prevail if there were full employment in the labor market and

$$\hat{\omega} \equiv \max\{\log \hat{w} - \log \bar{C}, \log b_r\} \quad (18)$$

is the maximum of the log minimum wage expressed in utils and the utility from rest unemployment. From [equation \(12\)](#), the level of employment is $E(l, x) = le^{\theta_n(\omega - \hat{\omega})}$ if the minimum wage binds, $\omega < \hat{\omega}$, and l otherwise. Hence the amount of the good produced is

$$Q(l, x) = lAx \min\{1, e^{\theta_n(\omega - \hat{\omega})}\}. \quad (19)$$

When $\omega \geq \hat{\omega}$, the wage exceeds the minimum wage and so there is no rest unemployment. Otherwise, enough workers rest to raise the log wage in utils to $\hat{\omega}$.

Since the wage only depends on ω , we look for an equilibrium in which any labor market with $\omega > \bar{\omega}_n(\hat{\omega})$ immediately attracts new entrants to push the log full employment wage back to $\bar{\omega}_n(\hat{\omega})$ and workers with the least seniority immediately exit any labor market with $\omega < \underline{\omega}_n(\hat{\omega})$ until the log full employment wage increases to $\underline{\omega}_n(\hat{\omega})$. The thresholds $\underline{\omega}_n(\hat{\omega}) \leq \bar{\omega}_n(\hat{\omega})$ are endogenous and depend on both the industry n and the minimum wage $\hat{\omega}$. Workers neither enter nor endogenously exit from labor markets with $\omega \in (\underline{\omega}_n(\hat{\omega}), \bar{\omega}_n(\hat{\omega}))$, although a fraction of the workers $q_n dt$ quit during an interval of time dt . We allow for the possibility that $\underline{\omega}_n(\hat{\omega}) = -\infty$ so workers never exit labor markets. When a positive shock hits a labor market with $\omega = \bar{\omega}_n(\hat{\omega})$, ω stays constant and the labor force l increases. Conversely, negative shocks reduce ω , with l falling as workers exogenously quit the market. At $\underline{\omega}_n(\hat{\omega}) < \omega < \bar{\omega}_n(\hat{\omega})$, both positive and negative shocks affect ω , while l falls deterministically at rate q_n . When $\omega = \underline{\omega}_n(\hat{\omega})$, a negative shock reduces l without affecting ω , while a positive shock raises ω , with l falling due to quits.

If there is an equilibrium with this property, its definition in [equation \(17\)](#) implies ω is a regulated Brownian motion in each market n_j . When $\omega(n_j, t) \in (\underline{\omega}_n(\hat{\omega}), \bar{\omega}_n(\hat{\omega}))$, only productivity shocks change ω , so

$$d\omega(n_j, t) = \frac{\theta_n - 1}{\theta_n} d \log x(n_j, t) + \frac{q_n}{\theta_n} dt = \mu_n dt + \sigma_n dz(n_j, t), \quad (20)$$

where

$$\mu_n \equiv \frac{\theta_n - 1}{\theta_n} \mu_n^x + \frac{q_n}{\theta_n} \quad \text{and} \quad \sigma_n \equiv \frac{\theta_n - 1}{\theta_n} \sigma_n^x,$$

i.e., in this range $\omega(n_j, t)$ has drift μ_n and instantaneous standard deviation σ_n . When the thresholds $\underline{\omega}_n(\hat{\omega})$ and $\bar{\omega}_n(\hat{\omega})$ are finite, they act as reflecting barriers, since productivity shocks that would move ω outside the boundaries are offset by the entry and exit of workers.

3.3 The Value of a Worker

Now consider a typical worker in a labor market with log minimum wage $\hat{\omega}$ in industry n . The key to our analysis is to recognize that we can analyze the behavior of such a worker in isolation from the rest of the economy. For notational convenience, we suppress the dependence of the value function on industry-specific variables whenever there is no loss of clarity.

The worker's state is described by the log full employment wage in her labor market ω and her seniority s , as well as the characteristics of her labor market, including the log minimum wage, the stochastic process for productivity, and the substitutability of goods. But from the worker's perspective, it suffices to know that the log full employment wage is a regulated Brownian motion with endogenous, labor-market specific barriers $\underline{\omega} < \bar{\omega}$. Her seniority in her labor market is her percentile in the tenure distribution in the industry. When a worker arrives, she starts at $s = 0$. Subsequently when workers enter or exit the labor market, the seniority of all workers evolves so as to maintain a uniform distribution of s on $[0, 1]$. Thus s increases only when $\omega = \bar{\omega}$ and falls only when $\omega = \underline{\omega}$; [Figure 1](#) shows the dynamics of ω and s . Each worker exits at the first time $\tau(\underline{\omega}, 0)$ that her state hits $(\underline{\omega}, 0)$, i.e. the first time she is the least senior worker in a market with log full employment wage $\underline{\omega}$. She also exits exogenously at rate $\lambda \equiv q + \delta$, the sum of the quit rate and the rate at which the labor market shuts down.

To compute the value v of a worker in state (ω, s) , let

$$R(\omega, s) = \begin{cases} e^\omega & \text{if } \omega \geq \hat{\omega} \\ e^{\hat{\omega}} & \text{if } \omega < \hat{\omega} \text{ and } s \geq 1 - e^{\theta(\omega - \hat{\omega})} \\ b_r & \text{if } \omega < \hat{\omega} \text{ and } s < 1 - e^{\theta(\omega - \hat{\omega})} \end{cases} \quad (21)$$

denote the flow payoff of a worker in each state, where we suppress the dependence of the elasticity of substitution θ , and hence the return function R , on the industry n . [Figure 1](#) shows the flow payoff in (ω, s) space. If $\omega \geq \hat{\omega}$, all workers are employed at log wage ω . Otherwise, the most senior workers are employed at $\hat{\omega}$ and the less senior workers are unemployed and

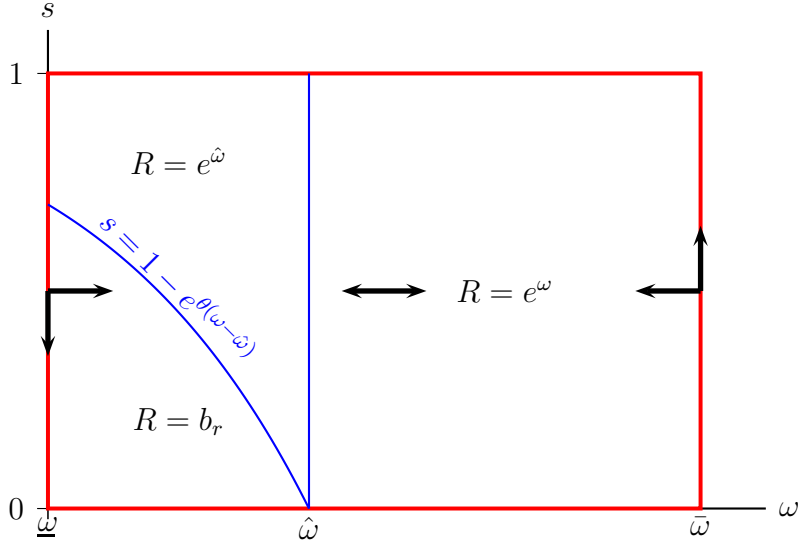


Figure 1: The dynamics of ω and s . All new markets enter at $(\bar{\omega}, 0)$. Markets with $\omega \geq \hat{\omega}$ have no unemployment, while markets with $\omega < \hat{\omega}$ have all workers with $s < 1 - e^{\theta(\omega - \hat{\omega})}$ unemployed.

get leisure b_r . By construction $b_r \leq e^{\hat{\omega}}$, so employed workers are always weakly better off than unemployed workers. Workers in a particular labor market are indifferent between employment and unemployment only if $b_r = e^{\hat{\omega}}$ and $\omega \leq \hat{\omega}$.

Using this expression, the value of a worker in state (ω_0, s_0) in a market characterized by log minimum wage $\hat{\omega}$ and thresholds $\underline{\omega} < \bar{\omega}$ is

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E} \left(\int_0^{\tau(\underline{\omega}, 0)} e^{-(\rho+\lambda)t} (R(\omega(t), s(t)) + \lambda \underline{v}) dt + e^{-(\rho+\lambda)\tau(\underline{\omega}, 0)} \underline{v} \mid (\omega(0), s(0)) = (\omega_0, s_0) \right), \quad (22)$$

where expectations are taken with respect to the random stopping time τ and the path of the state $(\omega(t), s(t))$ prior to the stopping time. Both the stopping time and the path of the state depends on the thresholds $\underline{\omega}$ and $\bar{\omega}$, while the period return function depends on $\hat{\omega}$. In equilibrium, workers must be willing to exit the labor market in state $(\underline{\omega}, 0)$ and to enter labor markets in state $(\bar{\omega}, 0)$. That is, $\underline{\omega}$ and $\bar{\omega}$ must satisfy

$$v(\underline{\omega}, 0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \underline{v} \quad (23)$$

$$v(\bar{\omega}, 0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \bar{v}, \quad (24)$$

where the values \underline{v} and \bar{v} are common to all labor markets and are determined by the leisure from search and inactivity and by the extent of search frictions; see [equations \(14\)–\(15\)](#). In

addition, workers must be willing to stay in labor markets in all other states, and to stay in labor markets otherwise,

$$v(\omega, s; \hat{\omega}, \underline{\omega}, \bar{\omega}) \geq \underline{v} \text{ for all } (\omega, s) \in [\underline{\omega}, \bar{\omega}] \times [0, 1]. \quad (25)$$

Note that in the presence of a binding minimum wage, workers in some states (ω, s) may attain a value strictly larger than \bar{v} . Workers from outside the labor market cannot move directly into such states because they do not have the requisite seniority.

In equilibrium, workers are just indifferent about exiting the labor market at the stopping time $\tau(\underline{\omega}, 0)$. This means that the value of a worker who stays in the labor market until she is hit by the exogenous quit shock is the same as the value of a worker who stays until either she is hit by the quit shock or the first time she reaches state $(\underline{\omega}, 0)$,

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E} \left(\int_0^\infty e^{-(\rho+\lambda)t} (R(\omega(t), s(t)) + \lambda \underline{v}) dt \mid (\omega(0), s(0)) = (\omega_0, s_0) \right), \quad (26)$$

when $(\underline{\omega}, \bar{\omega})$ solve [equations \(23\)](#) and [\(24\)](#) and all other workers follow the prescribed policy, exiting the first time they hit state $(\underline{\omega}, 0)$. The equivalence between the value functions in [equations \(22\)](#) and [\(26\)](#) simplifies our exposition.

3.4 Existence of Rest Unemployment

Our first result is that whenever the minimum wage binds, it generates some rest unemployment. As a starting point, consider the case where the minimum wage is zero, or equivalently $\hat{\omega} = -\infty$, the situation we analyzed in [Alvarez and Shimer \(2008\)](#). We proved in [Propositions 1 and 2](#) of that paper that, conditional on the other parameters in the model, there exists a threshold $\bar{b}_r > 0$ such that if $b_r < \bar{b}_r$, there is no rest unemployment. Moreover, in this case there exists a unique equilibrium characterized by thresholds $\bar{\omega}^* > \underline{\omega}^* > \log b_r$ where workers enter and exit labor markets so as to regulate wages in $[\underline{\omega}^*, \bar{\omega}^*]$. These thresholds and the associated value function satisfies [equations \(23\)–\(26\)](#).

Using these definitions, we prove that there will always be some rest unemployment if the minimum wage is higher than $\underline{\omega}^*$.

PROPOSITION 1. *A minimum wage $\hat{\omega} \leq \underline{\omega}^*$ does not bind, so $\bar{\omega} = \bar{\omega}^*$ and $\underline{\omega} = \underline{\omega}^*$. A minimum wage $\hat{\omega} > \underline{\omega}^*$ binds and causes some rest unemployment, $\hat{\omega} > \underline{\omega}$.*

Proof. First consider $\hat{\omega} \leq \underline{\omega}^*$. When $\hat{\omega} \leq \underline{\omega}$, $R(\omega, s) = e^\omega$ for all s and so the value function in [equation \(26\)](#) is independent of $\hat{\omega}$. Thus if $(\underline{\omega}^*, \bar{\omega}^*)$ solve [equations \(23\)–\(25\)](#) for $\hat{\omega} = -\infty$,

they solve the same equations for any $\hat{\omega} \leq \underline{\omega}^*$.

Now suppose $\hat{\omega} > \underline{\omega}^*$. To find a contradiction, suppose $\hat{\omega} \leq \underline{\omega}$. The argument in the previous paragraph implies that $R(\omega, s) = e^\omega$ for all s . But in this case we know from [Alvarez and Shimer \(2008\)](#) that the unique solution to [equations \(23\)–\(26\)](#) is $(\underline{\omega}^*, \bar{\omega}^*)$, and in particular $\underline{\omega} = \underline{\omega}^*$, a contradiction. \square

One might have imagined that a binding minimum wage simply raised the lower threshold for ω so $\underline{\omega} = \hat{\omega}$. This is not the case. Since the standard deviation of productivity per unit of time explodes when the time horizon is short, the option value of entering rest unemployment, at least briefly, always exceeds the option value of immediately exiting the labor market when productivity falls too far.

3.5 Characterization of the Value Function

We prove in [Appendix A.1](#) that the value function is twice differentiable on the interior of the state space, except at points where $R(\omega, s)$ is discontinuous, i.e. on the locus $s = 1 - e^{\theta(\omega - \hat{\omega})}$, where it is once differentiable. Moreover, taking the limit of a discrete time, discrete state space model, we show in [Appendix A.2](#) that the value function satisfies the following partial differential equations. First, for all (ω, s) ,

$$\rho v(\omega, s) = R(\omega, s) + \lambda(\underline{v} - v(\omega, s)) + \mu v_\omega(\omega, s) + \frac{1}{2}\sigma^2 v_{\omega\omega}(\omega, s). \quad (27)$$

At the highest and lowest wages and for all s ,

$$v_\omega(\underline{\omega}, s) = v_s(\underline{\omega}, s)(1 - s)\theta \quad (28)$$

$$v_\omega(\bar{\omega}, s) = v_s(\bar{\omega}, s)(1 - s)\theta. \quad (29)$$

For a worker who is at the exit threshold,

$$v_\omega(\underline{\omega}, 0) = 0. \quad (30)$$

Finally, the highest level of seniority is an absorbing state until the worker exits the labor market, which ensures that

$$v_\omega(\underline{\omega}, 1) = v_\omega(\bar{\omega}, 1) = 0. \quad (31)$$

These act as transversality conditions and are used in our proof that the thresholds uniquely determine the value function.

PROPOSITION 2. For any $\bar{\omega} > \underline{\omega}$, $v(\omega, s)$ is uniquely determined by [equations \(27\)–\(31\)](#) and

the condition that it is almost everywhere twice differentiable.

The proof in [Appendix A.3](#) provides a closed form solution for the value function in a typical industry with an arbitrary minimum wage. Using that solution, we can prove algebraically that the value function is monotone in the log full employment wage and the worker’s seniority:

PROPOSITION 3. The value function $v(\omega, s)$ is strictly increasing in ω and is strictly increasing in s if $\hat{\omega} > \max\{\log b_r, \underline{\omega}\}$ and independent of s otherwise.

The proof will be in the next version of the paper. For the case with $\lambda = 0$, see pages 10–12 of the file `minwagesolutionApril17.pdf`. Note that monotonicity of the value function ensures that [equation \(25\)](#) is satisfied. The fact that the value function is independent of seniority when $\hat{\omega} \leq \max\{\log b_r, \underline{\omega}\}$ is straightforward to verify algebraically. Economically, seniority matters only if the minimum wage sometimes binds, in the sense that unemployed workers are worse off than employed workers within the same market.

PROPOSITION 4. In any industry n and for any minimum wage $\hat{\omega}$, there exists thresholds $\underline{\omega} < \bar{\omega}$ such that [equations \(23\)](#) and [\(24\)](#) are satisfied.

The proof will be in the next version of the paper. For the case with $\lambda = 0$ and a moderate minimum wage ($\underline{\omega} < \hat{\omega} < \bar{\omega}$), see pages 21–22 of the file `minwagesolutionApril17.pdf`. Note that [Proposition 2](#) explains how to compute \underline{v} and \bar{v} given $\underline{\omega}$ and $\bar{\omega}$. [Proposition 4](#) tells us that the mapping is invertible, i.e. that given \underline{v} and \bar{v} , we can find $\underline{\omega}$ and $\bar{\omega}$. We conjecture that thresholds are unique, but only have a proof in the case when the minimum wage does not bind, $\hat{\omega} = \log b_r$ (see [Alvarez and Shimer, 2008](#), Proposition 1), and when the minimum wage always binds, $\hat{\omega} \geq \bar{\omega}$.

3.6 Aggregation

Consider a typical industry n and minimum wage rate $\hat{\omega}$. Denote the thresholds for that industry by $\underline{\omega}_n(\hat{\omega})$ and $\bar{\omega}_n(\hat{\omega})$. Given these, we can compute the fraction of workers at each value of $\omega \in [\underline{\omega}_n(\hat{\omega}), \bar{\omega}_n(\hat{\omega})]$. Note that this is different than the fraction of labor markets at each value of ω , since there are typically more workers in labor markets with a higher log full employment wage.

PROPOSITION 5. The steady state density of workers’ log full employment wage in industry

n , minimum wage $\hat{\omega}$ is

$$f_n(\omega; \hat{\omega}) = \frac{\sum_{i=1}^2 |\eta_{i,n} + \theta_n| e^{\eta_{i,n}(\omega - \underline{\omega}_n(\hat{\omega}))}}{\sum_{i=1}^2 |\eta_{i,n} + \theta_n| \frac{e^{\eta_{i,n}(\bar{\omega}_n(\hat{\omega}) - \underline{\omega}_n(\hat{\omega}))} - 1}{\eta_i}}, \quad (32)$$

where $\eta_{1,n} < \eta_{2,n}$ solve the characteristic equation $\delta_n + q_n = -\mu_n \eta_n + \frac{\sigma_n^2}{2} \eta_n^2$ and $\underline{\omega}_n(\hat{\omega}) < \bar{\omega}_n(\hat{\omega})$ are the thresholds for that industry and minimum wage.

The proof of this result is identical to Proposition 3 in [Alvarez and Shimer \(2008\)](#) and hence omitted. That proposition also shows how to close the model to compute the number of workers labor markets and the consumption of each good, results that we do not repeat here. Note that under condition (9), $\eta_{1,n} \leq -\theta$ and $\eta_{2,n} > 0$.

Using this result, we can compute the rest and search unemployment rates for each industry n and minimum wage \hat{w} . To reduce the notation, we suppress the dependence of the thresholds on n and \hat{w} . If $\hat{w} \leq \underline{\omega}$, there is no rest unemployment in any such labor market. Otherwise when $\omega < \hat{w}$, all workers with seniority $s < 1 - e^{\theta_n(\omega - \hat{w})}$ are rest unemployed. This gives the rest unemployment rate in such a labor market. Integrating across markets using [equation \(32\)](#) gives the industry- and minimum wage-specific rest unemployment rate

$$\frac{U_{r,n}(\hat{w})}{L_n(\hat{w})} = \int_{\underline{\omega}}^{\min\{\hat{w}, \bar{\omega}\}} (1 - e^{\theta_n(\omega - \hat{w})}) f_n(\omega; \hat{w}) d\omega.$$

where $U_{r,n}(\hat{w})$ is the number of rest unemployed and $L_n(\hat{w})$ is the number of (employed or unemployed) workers in such labor markets. This gives

$$\frac{U_{r,n}(\hat{w})}{L_n(\hat{w})} = \frac{\frac{\theta}{\eta_{2,n}}(e^{\eta_{2,n}(\hat{w} - \underline{\omega})} - 1) - \frac{\theta}{\eta_{1,n}}(e^{\eta_{1,n}(\hat{w} - \underline{\omega})} - 1)}{\sum_{i=1}^2 |\theta_n + \eta_{i,n}| \frac{e^{\eta_{i,n}(\bar{\omega} - \underline{\omega})} - 1}{\eta_{i,n}}}, \quad (33)$$

when $\hat{w} \in (\underline{\omega}, \bar{\omega})$ and

$$\frac{U_{r,n}(\hat{w})}{L_n(\hat{w})} = \frac{(1 - e^{\theta_n(\bar{\omega} - \hat{w})} + \frac{\theta_n}{\eta_{2,n}})(e^{\eta_{2,n}(\hat{w} - \underline{\omega})} - 1) - (1 - e^{\theta_n(\bar{\omega} - \hat{w})} + \frac{\theta_n}{\eta_{1,n}})(e^{\eta_{1,n}(\hat{w} - \underline{\omega})} - 1)}{\sum_{i=1}^2 |\theta_n + \eta_{i,n}| \frac{e^{\eta_{i,n}(\bar{\omega} - \underline{\omega})} - 1}{\eta_{i,n}}}, \quad (34)$$

when $\hat{w} \geq \bar{\omega}$. Using these equations, we can easily compute how the level of the minimum wage affects the unemployment rate within an industry and how a given minimum wage affects the unemployment rate in different industries.

Now we turn to the search unemployed connected to a particular industry n and minimum wage \hat{w} . Let $N_{s,n}(\hat{w})$ be the number of workers that leave their labor market per unit of time, either because conditions are sufficiently bad or because they exogenously quit or because

their labor market has exogenously shut down. We prove in [Alvarez and Shimer \(2008\)](#) that this satisfies

$$N_{s,n}(\hat{\omega}) = \left(\frac{\theta_n \sigma_n^2}{2} f_n(\underline{\omega}; \hat{\omega}) + \delta_n + q_n \right) L_n(\hat{\omega}). \quad (35)$$

The first term gives the fraction of workers who leave their labor market to keep ω above $\underline{\omega}$. The second term is the fraction of workers who exogenously leave their market. In steady state, the fraction of workers who leave labor markets must balance the fraction of workers who arrive in labor markets. The latter is given by the fraction of workers engaged in searching for this industry and minimum wage, $U_{s,n}(\hat{\omega})$, times the rate at which they arrive to the labor market α , so $\alpha U_{s,n}(\hat{\omega}) = N_{s,n}(\hat{\omega})$. Solve [equation \(35\)](#) using [equation \(32\)](#) to obtain an expression for the ratio of search unemployment to workers in labor markets:

$$\frac{U_{s,n}(\hat{\omega})}{L_n(\hat{\omega})} = \frac{1}{\alpha} \left(\frac{\theta_n \sigma_n^2}{2} \frac{\eta_{2,n} - \eta_{1,n}}{\sum_{i=1}^2 |\theta_n + \eta_{i,n}| \frac{e^{\eta_{i,n}(\bar{\omega} - \underline{\omega})} - 1}{\eta_{i,n}}} + \delta_n + q_n \right) \quad (36)$$

To compute the aggregate rest and search unemployment rates, simply aggregate across minimum wages and industries.

In some special cases, the formulae for search and rest unemployment rates simplify further. Consider an industry with $\mu_n^x = -(\theta_n - 1)(\sigma_n^x)^2/2$, or equivalently $\mu_n = q_n/\theta_n - \theta_n \sigma_n^2/2$. Then one can show that the search and rest unemployment rates are well behaved even if goods last forever, $\delta_n \rightarrow 0$. Although the variance of the productivity distribution explodes, the roots of the characteristic equation in [Proposition 5](#) converge to $\eta_{1,n} = -\theta_n$ and $\eta_{2,n} = 2q_n/\theta_n \sigma_n^2$. Substituting into [equation \(32\)](#), we find

$$f(\omega) = \frac{\eta_{2,n} e^{\eta_{2,n}(\omega - \underline{\omega})}}{e^{\eta_{2,n}(\bar{\omega} - \underline{\omega})} - 1}.$$

If $q = 0$ as well, this simplifies further to $f(\omega) = 1/(\bar{\omega} - \underline{\omega})$, i.e. f is uniform on its support, while for positive q the density is increasing in ω .¹ Using this, we can compute the search and rest unemployment rates. When $\delta \rightarrow 0$, these converge to

$$\frac{U_{r,n}(\hat{\omega})}{L_n(\hat{\omega})} = \frac{e^{\eta_{2,n}(\min\{\bar{\omega}, \hat{\omega}\} - \underline{\omega})} \left(1 - \frac{\eta_{2,n}}{\theta_n + \eta_{2,n}} e^{\theta_n \min\{\bar{\omega} - \hat{\omega}, 0\}} \right) - 1 + \frac{\eta_{2,n}}{\theta_n + \eta_{2,n}} e^{-\theta_n(\hat{\omega} - \underline{\omega})}}{e^{\eta_{2,n}(\bar{\omega} - \underline{\omega})} - 1}, \quad (37)$$

$$\frac{U_{s,n}(\hat{\omega})}{L_n(\hat{\omega})} = \frac{q_n}{\alpha(1 - e^{-\eta_{2,n}(\bar{\omega} - \underline{\omega})})}. \quad (38)$$

These expressions simplify further when there are no quits, $q_n = 0$ and so $\eta_{2,n} \rightarrow 0$.

¹This result does not depend on the order in which δ and q converge to 0.

3.7 Hazard Rate of Exiting Unemployment

When there is no rest unemployment, the hazard of exiting unemployment is simply α . This section characterizes the hazard of exiting unemployment when there is rest unemployment, $\hat{\omega} > \underline{\omega}$. A worker who just switched between employment and rest unemployment is at the margin between the two states. A small shock will move her back. But the longer a worker remains unemployed, the more likely her labor market has suffered a series of adverse shocks, reducing the hazard of finding a job. The low hazard rate of exiting long-term unemployment may be important for understanding the coexistence of many workers who move easily between jobs and a relatively small number of workers who suffer extended unemployment spells (Juhn, Murphy, and Topel, 1991).

To see how this hazard is determined, consider a worker with seniority s who is rest unemployed whenever $s < 1 - e^{\theta(\omega - \hat{\omega})}$. Using the definition of ω in equation (17) and suppressing the dependence of these variables on the industry and minimum wage, we can write this as a condition relating the number of more senior workers in the market, $l(1 - s)$, to the current productivity of the market x_0 ,

$$l(1 - s) > \left(\frac{Ax_0}{C} \right)^{\theta-1} e^{-\theta\hat{\omega}}.$$

The worker exits rest unemployment and returns to this market the next time this inequality is violated, i.e. when productivity reaches \hat{x} solving

$$l(1 - s) = \left(\frac{A\hat{x}}{C} \right)^{\theta-1} e^{-\theta\hat{\omega}}.$$

Conversely, she exits rest unemployment and leaves the market when she first reaches state $(\underline{\omega}, 0)$, which occurs at the productivity level \underline{x} satisfying

$$l(1 - s) = \left(\frac{A\underline{x}}{C} \right)^{\theta-1} e^{-\theta\underline{\omega}},$$

so the log full employment wage is $\underline{\omega}$ if there are $l(1 - s)$ workers left in the market. She also exits the market exogenously if she quits or the market breaks down, at rate $\lambda = q + \delta$. Thus the hazard of ending a spell of rest unemployment depends on competing hazards of productivity rising to \hat{x} or falling to \underline{x} . The key observation is that the ratio of these two thresholds is monotone in the distance between $\hat{\omega}$ and $\underline{\omega}$,

$$\hat{\omega} - \underline{\omega} = \frac{\theta - 1}{\theta} (\log \hat{x} - \log \underline{x}),$$

and so is the same for all workers in an industry, regardless of their seniority.

Now let $h(t)$ denote the hazard of ending a (rest or search) unemployment spell of duration t . This solves

$$h(t) = \hat{h}_r(t) \frac{u_r(t)}{u_r(t) + u_s(t)} + \alpha \frac{u_s(t)}{u_r(t) + u_s(t)},$$

where $\frac{u_r(t)}{u_r(t) + u_s(t)}$ is the probability that a worker with unemployment duration t is rest-unemployed. For a search-unemployed worker, spells end at rate α , independent of the duration of the spell. For a rest-unemployed worker, her spell ends when local labor market conditions improve enough for her to reenter employment. We let $\hat{h}_r(t)$ denote that hazard rate of this event. It is also useful to let $\underline{h}_r(t)$ denote the hazard of endogenously exiting rest unemployment for search unemployment. The previous logic suggests that these hazards depend only on $\hat{\omega} - \underline{\omega}$; indeed we prove in [Alvarez and Shimer \(2008\)](#) that these solve

$$\begin{aligned} \hat{h}_r(t) &= \frac{\sum_{m=1}^{\infty} m^2 e^{-t\psi_m}}{\sum_{m=1}^{\infty} \frac{m^2}{\psi_m} e^{-t\psi_m} \left(1 - (-1)^m e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}\right)} \\ \underline{h}_r(t) &= \frac{-\sum_{m=1}^{\infty} m^2 e^{-t\psi_m} (-1)^m e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}}{\sum_{m=1}^{\infty} \frac{m^2}{\psi_m} e^{-t\psi_m} \left(1 - (-1)^m e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}\right)}, \end{aligned} \quad (39)$$

where

$$\psi_m \equiv \frac{1}{2} \left(\frac{\mu^2}{\sigma^2} + \frac{m^2 \pi^2 \sigma^2}{(\hat{\omega} - \underline{\omega})^2} \right).$$

These sums are easily calculated numerically.

We then compute the duration-contingent unemployment rates by solving a system of two ordinary differential equations with time-varying coefficients:

$$\dot{u}_r(t) = -u_r(t)(\delta + q + \underline{h}_r(t) + \hat{h}_r(t)) \quad \text{and} \quad \dot{u}_s(t) = -u_s(t)\alpha + u_r(t)(\delta + q + \underline{h}_r(t)) \quad (40)$$

for all $t > 0$. The number of workers in rest unemployment falls as markets shut down and workers exogenously quit, as they exit the market for search unemployment, and as they reenter employment. In the first three events, they become search unemployed, while search unemployment falls at rate α as these workers find jobs. To solve these differential equations, we require two boundary conditions; however, to compute the share of rest unemployed in the unemployed population with duration t , $\frac{u_r(t)}{u_r(t) + u_s(t)}$, we need only a single boundary condition,

$$\frac{\int_0^{\infty} u_r(t) dt}{\int_0^{\infty} u_s(t) dt} = \frac{U_r}{U_s}, \quad (41)$$

where U_r and U_s are given in [equations \(33\)](#) and [\(36\)](#).

The hazard rate is particularly easy to characterize both at short and long durations. When t is small, we find that $\hat{h}_r(t) \approx \frac{1}{2t}$. Intuitively, consider a worker on the threshold of rest unemployment, $s = 1 - e^{\theta(\omega - \hat{\omega})}$. After a short time interval—short enough that the variance of the Brownian motion dominates the drift—there is a $\frac{1}{2}$ probability that ω has increased, so the worker is reemployed, and a $\frac{1}{2}$ chance it has fallen. But a one-half probability over any horizon t implies a hazard rate $1/2t$. Thus our model predicts that unionized workers will experience many short spells of unemployment, which perhaps can be interpreted as temporary layoffs.

When t is large, the first term of the partial sum in [equation \(39\)](#) dominates,

$$\lim_{t \rightarrow \infty} \hat{h}_r(t) = \frac{\psi_1}{1 + e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}} \text{ and } \lim_{t \rightarrow \infty} h_r(t) = \frac{\psi_1 e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}}{1 + e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}}.$$

In addition, if $\alpha > \delta + q + \psi_1$,

$$\lim_{t \rightarrow \infty} \frac{u_r(t)}{u_s(t)} = \frac{(\alpha - \psi_1 - \delta - q) \left(1 + e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}\right)}{\delta + q + (\delta + q + \psi_1) e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}},$$

while otherwise the limiting ratio is zero. Together this implies $\lim_{t \rightarrow \infty} h(t) = \min\{\alpha, \psi_1 + \delta + q\}$, a function only of the slower exit rate. Since ψ_1 is decreasing in $\hat{\omega} - \underline{\omega}$, the asymptotic exit rate from rest unemployment may be extremely low at long unemployment durations, so unionized workers will sometimes remain unemployed for years with little chance of reemployment.

4 Unions and Minimum Wages

Consider a monopoly union representing the $l(n_j, t)$ workers in labor market n_j at time t . The union's objective is to maximize the total flow utility of those workers,

$$e(n_j, t)w(n_j, t)\frac{1}{\bar{C}} + (l(n_j, t) - e(n_j, t))b_r,$$

where $e(n_j, t)$ is the measure of workers who are employed, $w(n_j, t)$ is the wage, and $1/\bar{C}$ is the marginal utility of consumption. For example, we can think of the union setting the wage and then letting competitive firms determine how many workers to hire. From the analysis

in [Section 2.3](#), we know that employment is

$$e(n_j, t) = \min \left\{ l(n_j, t), \frac{\bar{C}(t)^{\theta_n} (Ax(n_j, t))^{\theta_n - 1}}{C(n, t)^{\theta_n - 1} w^{\theta_n}} \right\}.$$

The solution to the union's problem is to set $w(n_j, t) = \bar{C}e^{\hat{\omega}}$ where

$$\hat{\omega} = \log b_r + \log(\theta_n / (\theta_n - 1)) \tag{42}$$

if this leaves some workers unemployed and otherwise to set a higher level of wages consistent with full employment, $w(n_j, t) = \bar{C}e^{\omega}$, where ω is the log full employment wage defined in [equation \(17\)](#). In other words, the union sets a constant minimum wage which leaves a gap between the utility of the members who work and those who are rest unemployed. The minimum wage is time-invariant, although it will vary across industries depending on the elasticity of substitution θ_n . This is exactly the type of policy that we have analyzed in this paper; the analysis here simply provides a link between the minimum wage and the preference parameters b_r and θ_n .

According to this model, the economy would be perfectly competitive in the absence of unions. By monopolizing a labor market, a union can extract the monopoly rent. It does this by raising wages in order to restrict employment and output and hence raise the price of the good produced by the industry. It achieves exactly the same outcome as would be attained by a monopoly producer facing a competitive industry.² The model predicts that unions will be more successful at raising wages in industries producing goods that have poor substitution possibilities, θ_n close to 1.

Some observers have noted that, while unionization raises unemployment rates, the effects are mitigated if unions coordinate their activities ([Nickell and Layard, 1999](#)). Our model suggests that this may be because coordinated unions are able to internalize the impact of exploiting their monopoly power on other workers. The Pareto optimal allocation is achieved by dropping the minimum wage constraints, so a worker can work whenever $\omega = \log b_r$ (see [Alvarez and Shimer, 2008](#), Appendix B.2). Perhaps coordinated unions are able to avoid the incentive to restrict output in individual labor markets.

²We do not analyze the interaction between a monopoly producer a monopoly union. In this case, setting a wage and allowing the firm to determine employment is generally inefficient. The two monopolists should agree on both a wage and a level of employment. Still, it seems likely that the equilibrium outcome will be a wage floor.

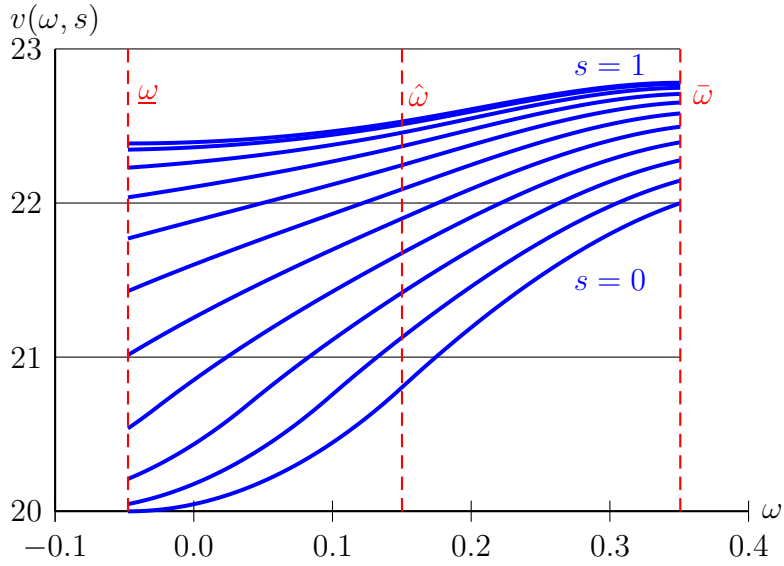


Figure 2: Value functions for $s = 0, 0.1, \dots, 1$. Parameters are in the text, with $\hat{\omega} = 0.15$.

5 Example

We set parameters broadly in line with those in our previous paper. Consider an industry with an elasticity of substitution $\theta = 2$. Let the discount rate be $\rho = 0.05$, the leisure value of inactivity be $b_i = 1$, so $\underline{v} = 20$, and the search cost be $\kappa = 2$ so $\bar{v} = 22$. Set the leisure from rest unemployment to $b_r = 0.7$. Fix the standard deviation of wages at $\sigma = 0.12$ and the quit rate at 0.04. Then let $\mu = q/\theta - \theta\sigma^2/2 \approx 0.0056$ so that we can focus on the limit as $\delta \rightarrow 0$. Finally, set the job finding rate for searchers to $\alpha = 3.2$. Since our exploration of parameters is cursory, the results that follow should be considered preliminary.

In the absence of a minimum wage, we find that $\underline{\omega} = -0.258$, higher than $\log b_r = -0.357$. Therefore any minimum wage below this level has no effect. **Figure 2** shows the value function $v(\omega, s)$ for different seniorities when $\hat{\omega} = 0.15$. More senior workers are always better off than less senior workers and all workers are better off when the log full employment wage ω is higher, although more senior workers' value function is less sensitive to ω .

Figure 3 shows how the thresholds change as functions of the minimum wage. The lower bound $\underline{\omega}$ increases in $\hat{\omega}$, with a slope less than 1. Put differently, when the minimum wage is higher, the maximum number of workers willing to stay in the industry is smaller for any value of productivity. On the other hand, the upper bound initially falls with $\hat{\omega}$, indicating that a modest degree of monopolization attracts workers to the industry for a given level of productivity. This is true even though the last entrant to the union is the first worker laid off. A monopoly union sets $\hat{\omega} = 0.34$, consistent with a $\bar{\omega} > \hat{\omega} > \underline{\omega}$.

Note that unions in our model generate not only unemployment, but also wage compression (**Blau and Kahn, 1996**; **Bertola and Rogerson, 1997**). The range of log wages is given

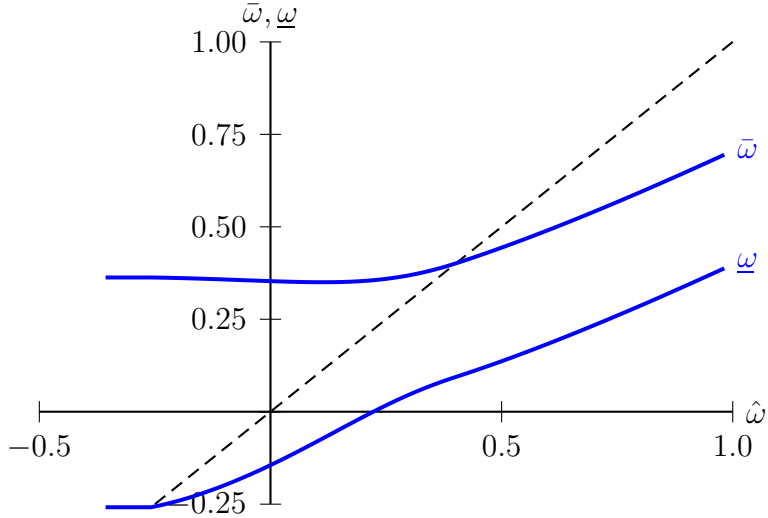


Figure 3: Thresholds as functions of the minimum wage $\hat{\omega}$. Parameters are in the text.

by the distance between the dashed 45 degree line and $\bar{\omega}$. This is declining in the minimum wage, eventually disappearing once all workers are paid $\hat{\omega}$.

Using the computed thresholds, it is straightforward to find out how the rest and search unemployment rates in this industry vary with $\hat{\omega}$ (Figure 4). Initially there is no rest unemployment, although search unemployment is necessary to sustain the industry. As the minimum wage rises, the rest unemployment rate starts to increase while the search unemployment rate is approximately unchanged. We conclude from this exercise that union-mandated minimum wages provide a powerful mechanism for generating rest unemployment.

Figure 5 shows the annual hazard rate for two markets with different minimum wages but the other parameters fixed at the benchmark level. In one labor market, the minimum wage is set at $\hat{\omega} = 0$ while in the other it is at the monopoly level, $\hat{\omega} \approx 0.34$. The overall hazard rates are similar at short durations, roughly $1/2t$, because in both cases most unemployed workers are in rest unemployment. With a low minimum wage, few workers get trapped in long-term unemployment because the gap between the minimum wage $\hat{\omega}$ and the exit threshold $\underline{\omega}$ is not that large. That is, most workers either quickly find a job or exit the industry, so mean unemployment duration is 0.33 years and the median duration is 0.21 years. Asymptotically, the exit rate from unemployment converges to $\alpha = 3.2$. But with the monopoly union wage, more workers get stuck in long-term unemployment. In this case, the mean unemployment duration is 0.83 years, the median duration is 0.48 years, and the exit hazard converges to 1.01. In a labor market with such a high minimum wage, the efficiency of search affects the hazard of exiting long-term unemployment only indirectly, through its influence on the distance between the rest unemployment boundaries $\hat{\omega} - \bar{\omega}$.

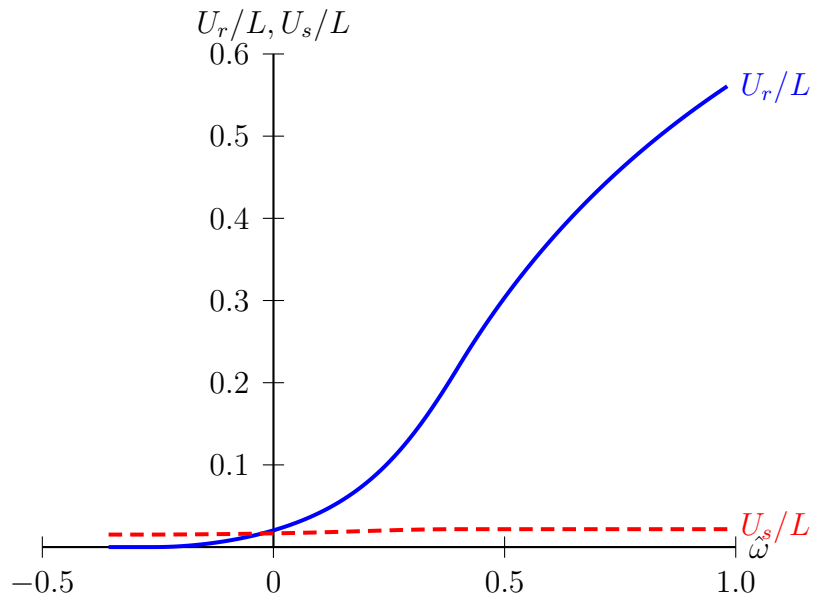


Figure 4: Unemployment as functions of the minimum wage \hat{w} . Parameters are in the text.

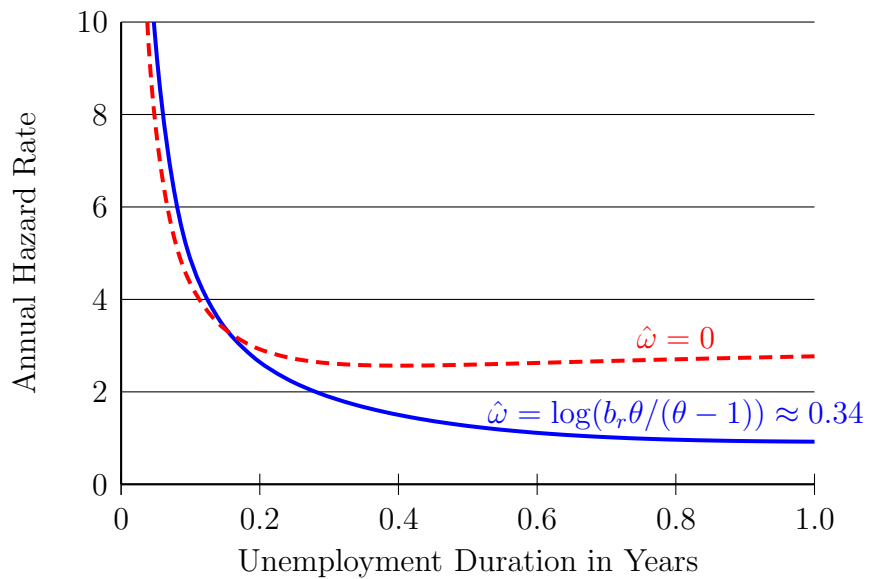


Figure 5: Hazard rate of finding a job as a function of unemployment duration. The parameter values are in the text. The blue solid line uses the monopoly union minimum wage, while in the red dashed line, the minimum wage is $\hat{w} = 0$.

Finally, it is straightforward to perform simple comparative statics. Take, for example, an industry producing a good that is easy to substitute, $\theta_n = 3$, but with all other parameters unchanged. We find this has little effect on the curves in [Figure 3](#) and [Figure 4](#). Leaving $\hat{\omega}$ fixed at its monopoly value in the industry with $\theta_n = 2$, we find that U_r/L falls from 0.164 to 0.159. But if the minimum wage falls to its new monopoly value, $\hat{\omega} = 0.05$, the rest unemployment rate falls substantially to $U_r/L = 0.031$, while the search unemployment rate is virtually unchanged at $U_s/L = 0.022$.

A Appendix

A.1 Differentiability of the Value Function

Note that for $\omega \in (\underline{\omega}, \bar{\omega})$, a worker's seniority is constant; although some workers exit the market exogenously, they are drawn uniformly from the population of workers. Now let $\tau(\underline{\omega})$ and $\tau(\bar{\omega})$ denote the (stochastic) time when ω first hits $\underline{\omega}$ and $\bar{\omega}$, infinite if it hits the other boundary first. Then we can rewrite [equation \(26\)](#) as

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \mathbb{E} \left(\int_0^{\min\{\tau(\underline{\omega}), \tau(\bar{\omega})\}} e^{-(\rho+\lambda)t} (R(\omega(t), s_0) + \lambda \underline{v}) dt + v(\underline{\omega}, s_0) e^{-(\rho+\lambda)\tau(\underline{\omega})} + v(\bar{\omega}, s_0) e^{-(\rho+\lambda)\tau(\bar{\omega})} \mid \omega(0) = \omega_0 \right).$$

Now let $\pi(\omega|\omega_0)$ be the discounted local time of a Brownian motion prior to the first time it hits the boundary. Rewrite the value function as

$$v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega}) = \int_{\underline{\omega}}^{\bar{\omega}} (R(\omega(t), s_0) + \lambda \underline{v}) \pi(\omega|\omega_0) dt + \mathbb{E}(v(\underline{\omega}, s_0) e^{-(\rho+\lambda)\tau(\underline{\omega})} + v(\bar{\omega}, s_0) e^{-(\rho+\lambda)\tau(\bar{\omega})} \mid \omega(0) = \omega_0).$$

Since π is everywhere continuous and is continuously differentiable except at ω_0 , differentiating the previous expression gives

$$\begin{aligned} \frac{\partial v(\omega_0, s_0; \hat{\omega}, \underline{\omega}, \bar{\omega})}{\partial \omega_0} &= \int_{\underline{\omega}}^{\bar{\omega}} (R(\omega(t), s_0) + \lambda \underline{v}) \frac{\partial \pi(\omega|\omega_0)}{\partial \omega_0} dt \\ &\quad + \left(\lim_{\omega \uparrow \omega_0} R(\omega, s_0) - \lim_{\omega \downarrow \omega_0} R(\omega, s_0) \right) \pi(\omega_0|\omega_0) \\ &\quad + v(\underline{\omega}, s_0) \frac{\partial \mathbb{E} e^{-(\rho+\lambda)\tau(\underline{\omega})}}{\partial \omega_0} + v(\bar{\omega}, s_0) \frac{\partial \mathbb{E} e^{-(\rho+\lambda)\tau(\bar{\omega})}}{\partial \omega_0} \mid \omega(0) = \omega_0. \end{aligned}$$

This in turn is continuously differentiable if $R(\omega, s_0)$ is continuous at $\omega = \omega_0$, i.e. if $s_0 \neq 1 - e^{\theta(\omega - \hat{\omega})}$.

A.2 Discrete Time, Discrete State Space Model

We consider a discrete time, discrete state space model. The length of the time period is Δt , the discount factor is $1 - \rho \Delta t$, and the exogenous exit probability is $\lambda \Delta t$. We imagine that ω lies on the grid $\{\underline{\omega}, \underline{\omega} + \Delta\omega, \underline{\omega} + 2\Delta\omega, \dots, \bar{\omega}\}$ while $s \in [0, 1]$. If at the start of the

period ω lies on the interior of the grid, it increases by $\Delta\omega$ with probability $\frac{1}{2}(1 + \Delta p)$ and otherwise decreases by $\Delta\omega$, while s stays constant. If $\omega = \underline{\omega}$, it increases to $\underline{\omega} + \Delta\omega$ with probability $\frac{1}{2}(1 + \Delta p)$. Otherwise, there is a negative shock. ω is unchanged and all workers with seniority $s < 1 - e^{-\theta\Delta\omega}$ exit the labor market. This ensures that log employment falls by $\theta\Delta\omega$, which according to [equation \(17\)](#) is enough to leave the log full employment wage constant. The seniority of all other workers changes as well, falling from s to

$$s' = \frac{s - 1 + e^{-\theta\Delta\omega}}{e^{-\theta\Delta\omega}}. \quad (43)$$

Conversely, if $\omega = \bar{\omega}$, a negative shock reduces it to $\bar{\omega} - \Delta\omega$ with probability $\frac{1}{2}(1 - \Delta p)$. Otherwise there is a positive shock. ω is unchanged, but log employment rises by $\theta\Delta\omega$ in order to leave the log full employment wage constant. The seniority of all workers rises from s to

$$s' = \frac{s - 1 + e^{\theta\Delta\omega}}{e^{\theta\Delta\omega}}. \quad (44)$$

Finally, assume $\Delta t = (\Delta\omega/\sigma)^2$ and $\Delta p = \mu\Delta\omega/\sigma^2$. We focus on the limiting behavior as $\Delta\omega$ converges to 0 holding fixed μ and σ , which corresponds to the stochastic process that we study in the body of the paper.

First take ω on the interior of the grid and an arbitrary s . The Bellman equation implies

$$v(\omega, s) = R(\omega, s)\Delta t + (1 - \rho\Delta t) \left(\lambda\Delta t \underline{v} + (1 - \lambda\Delta t) \left(\frac{1}{2}(1 + \Delta p)v(\omega + \Delta\omega, s) + \frac{1}{2}(1 - \Delta p)v(\omega - \Delta\omega, s) \right) \right).$$

Take a second order Taylor expansion to $v(\omega \pm \Delta\omega, s)$ around $v(\omega, s)$ and simplify:

$$v(\omega, s) = R(\omega, s)\Delta t + (1 - \rho\Delta t) \left(\lambda\Delta t \underline{v} + (1 - \lambda\Delta t) \left(v(\omega, s) + v_\omega(\omega, s)\Delta p\Delta\omega + \frac{1}{2}v_{\omega\omega}(\omega, s)\Delta\omega^2 \right) \right).$$

Subtract $(1 - \rho\Delta t)(1 - \lambda\Delta t)v(\omega, s)$ from both sides of this equation. Since $\Delta p\Delta\omega = \mu\Delta t$ and $\Delta\omega^2 = \sigma^2\Delta t$, we can divide through by any $\Delta t > 0$ to obtain

$$(\rho + \lambda - \rho\lambda\Delta t)v(\omega, s) = R(\omega, s) + (1 - \rho\Delta t) \left(\lambda\underline{v} + (1 - \lambda\Delta t) \left(\mu v_\omega(\omega, s) + \frac{1}{2}\sigma^2 v_{\omega\omega}(\omega, s) \right) \right).$$

Taking the limit as $\Delta t \rightarrow 0$ gives [equation \(27\)](#).

Now consider $\omega = \underline{\omega}$. For $s > 1 - e^{-\theta\Delta\omega}$, the Bellman equation solves

$$v(\underline{\omega}, s) = R(\underline{\omega}, s)\Delta t + (1 - \rho\Delta t) \left(\lambda\Delta t \underline{v} + (1 - \lambda\Delta t) \left(\frac{1}{2}(1 + \Delta p)v(\underline{\omega} + \Delta\omega, s) + \frac{1}{2}(1 - \Delta p)v(\underline{\omega}, s') \right) \right).$$

where s' solves [equation \(43\)](#). Now take a first order Taylor expansion of v around $(\underline{\omega}, s)$; higher order terms would disappear from the expression. Also approximate $e^{\theta n\Delta\omega} - 1 = \theta\Delta\omega$.

$$v(\underline{\omega}, s) = R(\underline{\omega}, s)\Delta t + (1 - \rho\Delta t) \left(\lambda\Delta t \underline{v} + (1 - \lambda\Delta t) \left(v(\underline{\omega}, s) + \frac{1}{2}(1 + \Delta p)v_\omega(\underline{\omega}, s)\Delta\omega - \frac{1}{2}(1 - \Delta p)v_s(\underline{\omega}, s)(1 - s)\theta\Delta\omega \right) \right).$$

Subtract $(1 - \rho\Delta t)(1 - \lambda\Delta t)v(\underline{\omega}, s)$ from both sides of this equation. Recall that $\Delta t = \Delta\omega^2/\sigma^2$ and $\Delta p = \mu\Delta\omega/\sigma^2$ and divide through by $\Delta\omega > 0$:

$$\left(\rho + \lambda - \rho\lambda\frac{\Delta\omega}{\sigma^2} \right) \frac{\Delta\omega}{\sigma^2} v(\underline{\omega}, s) = R(\underline{\omega}, s)\frac{\Delta\omega^2}{\sigma^2} + \left(1 - \rho\frac{\Delta\omega^2}{\sigma^2} \right) \left(\lambda\frac{\Delta\omega}{\sigma^2} \underline{v} + \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2} \right) \frac{1}{2} \left(\left(1 + \frac{\mu\Delta\omega}{\sigma^2} \right) v_\omega(\underline{\omega}, s) - \left(1 - \frac{\mu\Delta\omega}{\sigma^2} \right) v_s(\underline{\omega}, s)(1 - s)\theta \right) \right).$$

Taking the limit as $\Delta\omega \rightarrow 0$ gives [equation \(28\)](#). The derivation of [equation \(29\)](#) is almost identical and hence omitted.

Now take $\omega = \underline{\omega}$ and $s \leq 1 - e^{-\theta\Delta\omega}$. In this case, the Bellman equation solves

$$v(\underline{\omega}, s) = R(\underline{\omega}, s)\Delta t + (1 - \rho\Delta t) \left(\lambda\Delta t \underline{v} + (1 - \lambda\Delta t) \left(\frac{1}{2}(1 + \Delta p)v(\underline{\omega} + \Delta\omega, s) + \frac{1}{2}(1 - \Delta p)\underline{v} \right) \right).$$

Taking a first order Taylor expansion of v around $(\underline{\omega}, 0)$ and using $v(\underline{\omega}, 0) = \underline{v}$ gives

$$\underline{v} + v_s(\underline{\omega}, 0)s = R(\underline{\omega}, s)\Delta t + (1 - \rho\Delta t) \left(\lambda\Delta t \underline{v} + (1 - \lambda\Delta t) \left(\underline{v} + \frac{1}{2}(1 + \Delta p)(v_\omega(\underline{\omega}, 0)\Delta\omega + v_s(\underline{\omega}, 0)s) \right) \right).$$

Subtract $(1 - \rho\Delta t)(1 - \lambda\Delta t)\underline{v}$ from both sides of this equation. Recall that $\Delta t = \Delta\omega^2/\sigma^2$

and $\Delta p = \mu\Delta\omega/\sigma^2$:

$$\begin{aligned} \left(\rho + \lambda - \rho\lambda\frac{\Delta\omega^2}{\sigma^2}\right) \frac{\Delta\omega^2}{\sigma^2} \underline{v} + v_s(\underline{\omega}, 0)s &= R(\underline{\omega}, s) \frac{\Delta\omega^2}{\sigma^2} + \left(1 - \rho\frac{\Delta\omega^2}{\sigma^2}\right) \left(\lambda\frac{\Delta\omega^2}{\sigma^2} \underline{v} \right. \\ &\quad \left. + \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right) \frac{1}{2} \left(1 + \frac{\mu\Delta\omega}{\sigma^2}\right) (v_\omega(\underline{\omega}, 0)\Delta\omega + v_s(\underline{\omega}, 0)s) \right). \end{aligned}$$

Taking the limit as $\Delta\omega \rightarrow 0$ gives $v_s(\underline{\omega}, 0) = 0$ for all $s \leq 0$. Combining with [equation \(28\)](#) gives [equation \(30\)](#).

Finally we handle the case of $s = 1$. Since the seniority of such a worker never changes (see [equations 43](#) and [44](#)), we have

$$\begin{aligned} v(\underline{\omega}, 1) &= R(\underline{\omega}, 1)\Delta t + (1 - \rho\Delta t) \left(\lambda\Delta t \underline{v} \right. \\ &\quad \left. + (1 - \lambda\Delta t) \left(\frac{1}{2}(1 + \Delta p)v(\underline{\omega} + \Delta\omega, 1) + \frac{1}{2}(1 - \Delta p)v(\underline{\omega}, 1) \right) \right). \end{aligned}$$

Take a first order Taylor expansion of v around $(\underline{\omega}, 1)$:

$$v(\underline{\omega}, 1) = R(\underline{\omega}, 1)\Delta t + (1 - \rho\Delta t) \left(\lambda\Delta t \underline{v} + (1 - \lambda\Delta t) \left(v(\underline{\omega}, 1) + \frac{1}{2}(1 + \Delta p)v_\omega(\underline{\omega}, 1)\Delta\omega \right) \right).$$

Subtract $(1 - \rho\Delta t)(1 - \lambda\Delta t)v(\underline{\omega}, 1)$ from both sides of this equation. Recall that $\Delta t = \Delta\omega^2/\sigma^2$ and $\Delta p = \mu\Delta\omega/\sigma^2$ and divide through by $\Delta\omega$:

$$\begin{aligned} \left(\rho + \lambda - \rho\lambda\frac{\Delta\omega^2}{\sigma^2}\right) \frac{\Delta\omega}{\sigma^2} v(\underline{\omega}, 1) &= R(\underline{\omega}, 1) \frac{\Delta\omega}{\sigma^2} \\ &\quad + \left(1 - \rho\frac{\Delta\omega^2}{\sigma^2}\right) \left(\lambda\frac{\Delta\omega}{\sigma^2} \underline{v} + \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right) \frac{1}{2} \left(1 + \frac{\mu\Delta\omega}{\sigma^2}\right) v_\omega(\underline{\omega}, 1) \right). \end{aligned}$$

Taking the limit as $\Delta\omega \rightarrow 0$ gives $v_\omega(\underline{\omega}, 1) = 0$. A similar logic at $(\bar{\omega}, 1)$ gives $v_\omega(\bar{\omega}, 1) = 0$, establishing [equation \(31\)](#).

A.3 Value Function

A.3.1 High Minimum Wage

We first tackle the case where $\hat{\omega} \geq \bar{\omega}$. We claim that the value function satisfies

$$v(\omega, s) = \begin{cases} \frac{e^{\hat{\omega}} + \lambda \underline{v}}{\rho + \lambda} + \hat{c}_1(s)e^{\zeta_1(\omega - \hat{\omega})} + \hat{c}_2(s)e^{\zeta_2(\omega - \hat{\omega})} & \text{if } s \geq 1 - e^{\theta(\omega - \hat{\omega})} \\ \frac{b_r + \lambda \underline{v}}{\rho + \lambda} + \underline{c}_1(s)e^{\zeta_1(\omega - \hat{\omega})} + \underline{c}_2(s)e^{\zeta_2(\omega - \hat{\omega})} & \text{if } s > 1 - e^{\theta(\omega - \hat{\omega})}, \end{cases} \quad (45)$$

where $\zeta_1 < 0 < \zeta_2$ are the roots of the characteristic equation

$$\rho + \lambda - \zeta\mu - \zeta^2 \frac{\sigma^2}{2} = 0 \quad (46)$$

and the univariate functions of integration satisfy

$$\hat{c}_1(s) = - \left(\frac{\zeta_2}{\zeta_2 - \zeta_1} \right) \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda} \right) (1 - e^{-(\zeta_2 - \zeta_1)(\hat{\omega} - \underline{\omega})}) (1 - s)^{-\zeta_1/\theta} \quad (47)$$

$$\hat{c}_2(s) = 0 \quad (48)$$

$$\underline{c}_1(s) = \hat{c}_1(s) + \left(\frac{\zeta_2}{\zeta_2 - \zeta_1} \right) \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda} \right) (1 - s)^{-\zeta_1/\theta} \quad (49)$$

$$\underline{c}_2(s) = \hat{c}_2(s) + \left(\frac{-\zeta_1}{\zeta_2 - \zeta_1} \right) \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda} \right) (1 - s)^{-\zeta_2/\theta}, \quad (50)$$

where we leave the expressions in a convenient form.

To prove this, note first that [equation \(45\)](#) is the general solution to [equation \(27\)](#). All that remains is to characterize the four functions of integration. The condition that the value function is continuously differentiable even at the boundary between work and rest unemployment, i.e. points of the form $(\omega, 1 - e^{\theta(\omega - \hat{\omega})})$, yields [equations \(49\)](#) and [\(50\)](#). [Equations \(28\)](#) and [\(29\)](#) then reduce to $\hat{c}_i(s)\zeta_i = \hat{c}'_i(s)(1 - \theta)s$ for $i = 1, 2$, or equivalently $\hat{c}_i(s) = c_i(1 - s)^{-\zeta_i/\theta}$.

To pin down the two constants c_1 and c_2 , we use two more boundary conditions. Differentiate [equation \(45\)](#) to get

$$v_\omega(\omega, s) = c_1(1 - s)^{-\zeta_1/\theta} \zeta_1 e^{\zeta_1(\omega - \hat{\omega})} + c_2(1 - s)^{-\zeta_2/\theta} \zeta_2 e^{\zeta_2(\omega - \hat{\omega})}$$

when $s \geq 1 - e^{\theta(\omega - \hat{\omega})}$. In particular, [equation \(31\)](#) implies that this should converge to 0 as $s \rightarrow 1$ at $\omega = \underline{\omega}$ or $\omega = \bar{\omega}$. Since $\zeta_2 > 0 > \zeta_1$, this implies $c_2 = 0$, which delivers [equation \(48\)](#). Note that this implies $v_\omega(\omega, 1) = 0$ for all ω , since a worker at $s = 1$ will earn $\hat{\omega}$ regardless of the subsequent sequence of shocks. Finally, use [equation \(30\)](#) to pin down the last constant c_1 , yielding [equation \(47\)](#).

A.3.2 Moderate Minimum Wage

Next we turn to the case where $\bar{\omega} > \hat{\omega} \geq \underline{\omega}$. The basic approach is similar. We claim that the value function satisfies

$$v(\omega, s) = \begin{cases} \frac{e^\omega}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \frac{\lambda v}{\rho + \lambda} + \bar{c}_1(s)e^{\zeta_1(\omega - \hat{\omega})} + \bar{c}_2(s)e^{\zeta_2(\omega - \hat{\omega})} & \text{if } \omega \geq \hat{\omega} \\ \frac{e^{\hat{\omega}} + \lambda v}{\rho + \lambda} + \hat{c}_1(s)e^{\zeta_1(\omega - \hat{\omega})} + \hat{c}_2(s)e^{\zeta_2(\omega - \hat{\omega})} & \text{if } \omega < \hat{\omega} \text{ and } s \geq 1 - e^{\theta(\omega - \hat{\omega})} \\ \frac{b_r + \lambda v}{\rho + \lambda} + \underline{c}_1(s)e^{\zeta_1(\omega - \hat{\omega})} + \underline{c}_2(s)e^{\zeta_2(\omega - \hat{\omega})} & \text{if } \omega < \hat{\omega} \text{ and } s > 1 - e^{\theta(\omega - \hat{\omega})}, \end{cases} \quad (51)$$

where $\zeta_1 < \zeta_2$ solve [equation \(46\)](#) and

$$\bar{c}_1(s) = c_1(1-s)^{-\zeta_1/\theta} + \frac{e^{\hat{\omega}} \left(\frac{e^{-(\hat{\omega} - \underline{\omega})(\zeta_2 - \zeta_1)}(1 - e^{-(\bar{\omega} - \hat{\omega})(\zeta_2 - 1)})}{-\zeta_1} + \frac{(1 - e^{-(\hat{\omega} - \underline{\omega})(\zeta_2 - \zeta_1)})(\rho + \lambda - (\mu + \frac{1}{2}\sigma^2)\zeta_2)}{(\rho + \lambda)(\zeta_2 - \zeta_1)} \right)}{(1 - e^{-(\bar{\omega} - \underline{\omega})(\zeta_2 - \zeta_1)})(\rho + \lambda - \mu - \frac{1}{2}\sigma^2)} \quad (52)$$

$$\bar{c}_2(s) = c_2(1-s)^{-\zeta_2/\theta} - \frac{e^{\hat{\omega}} \left(\frac{e^{(\hat{\omega} - \underline{\omega})(\zeta_2 - \zeta_1)}(1 - e^{-(\bar{\omega} - \hat{\omega})(1 - \zeta_1)})}{\zeta_2} + \frac{(1 - e^{(\hat{\omega} - \underline{\omega})(\zeta_2 - \zeta_1)})(\rho + \lambda - (\mu + \frac{1}{2}\sigma^2)\zeta_1)}{(\rho + \lambda)(\zeta_2 - \zeta_1)} \right)}{(1 - e^{-(\bar{\omega} - \underline{\omega})(\zeta_2 - \zeta_1)})(\rho + \lambda - \mu - \frac{1}{2}\sigma^2)} \quad (53)$$

$$\hat{c}_1(s) = \bar{c}_1(s) - \left(\frac{e^{\hat{\omega}}}{\rho + \lambda} \right) \left(\frac{\rho + \lambda - (\mu + \frac{1}{2}\sigma^2)\zeta_2}{(\rho + \lambda - \mu - \frac{1}{2}\sigma^2)(\zeta_2 - \zeta_1)} \right) \quad (54)$$

$$\hat{c}_2(s) = \bar{c}_2(s) + \left(\frac{e^{\hat{\omega}}}{\rho + \lambda} \right) \left(\frac{\rho + \lambda - (\mu + \frac{1}{2}\sigma^2)\zeta_1}{(\rho + \lambda - \mu - \frac{1}{2}\sigma^2)(\zeta_2 - \zeta_1)} \right) \quad (55)$$

$$\underline{c}_1(s) = \hat{c}_1(s) + \left(\frac{\zeta_2}{\zeta_2 - \zeta_1} \right) \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda} \right) (1-s)^{-\zeta_1/\theta} \quad (56)$$

$$\underline{c}_2(s) = \hat{c}_2(s) - \left(\frac{\zeta_1}{\zeta_2 - \zeta_1} \right) \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda} \right) (1-s)^{-\zeta_2/\theta}. \quad (57)$$

with

$$c_1 = - \left(\frac{\zeta_2}{\zeta_2 - \zeta_1} \right) \left(\frac{e^{\hat{\omega}} - b_r}{\rho + \lambda} \right) (1 - e^{-(\hat{\omega} - \underline{\omega})(\zeta_2 - \zeta_1)})$$

and $c_2 = 0$.

Once again, [equation \(51\)](#) is the general solution to [equation \(27\)](#). Continuity and differentiability of the value function again allow us to relate $\hat{c}_i(s)$ to $\bar{c}_i(s)$ and $\underline{c}_i(s)$ to $\hat{c}_i(s)$. Then [equations \(28\)](#) and [\(29\)](#) yield a pair of differential equations which can be solved for $\bar{c}_1(s)$ and $\bar{c}_2(s)$ as a function of parameters and the two constants c_1 and c_2 . This gives [equations \(52\)–\(56\)](#). [Equation \(31\)](#), evaluated either at $\underline{\omega}$ or $\bar{\omega}$, again pins down $c_2 = 0$ so that $\bar{c}_2(s)$ remains finite as $s \rightarrow 1$. Finally, use [equation \(30\)](#) to pin down c_1 .

A.3.3 Low Minimum Wage

Finally we study $\hat{\omega} < \underline{\omega}$. This is equivalent to the case when $\hat{\omega} = \underline{\omega}$, since in both situations the minimum wage never binds. Applying the analysis with a moderate minimum wage gives

$$v(\omega, s) = \frac{e^{\omega}}{\rho + \lambda - \mu - \frac{1}{2}\sigma^2} + \frac{\lambda v}{\rho + \lambda} + c_1 e^{\zeta_1(\omega - \underline{\omega})} + c_2 e^{\zeta_2(\omega - \underline{\omega})} \quad (58)$$

where $\zeta_1 < \zeta_2$ solve [equation \(46\)](#) and

$$c_1 = \frac{e^{\underline{\omega}} (1 - e^{-(\bar{\omega} - \underline{\omega})(\zeta_2 - 1)})}{-\zeta_1 (1 - e^{-(\bar{\omega} - \underline{\omega})(\zeta_2 - \zeta_1)}) (\rho + \lambda - \mu - \frac{1}{2}\sigma^2)} \quad (59)$$

$$c_2 = -\frac{e^{\underline{\omega}} (1 - e^{(\bar{\omega} - \underline{\omega})(1 - \zeta_1)})}{\zeta_2 (1 - e^{(\bar{\omega} - \underline{\omega})(\zeta_2 - \zeta_1)}) (\rho + \lambda - \mu - \frac{1}{2}\sigma^2)}. \quad (60)$$

Note that the value function does not depend on the worker's seniority.

References

- Alvarez, Fernando, and Robert Shimer, 2008. “Search and Rest Unemployment.” NBER Working Paper 13772.
- Bertola, Giuseppe, and Richard Rogerson, 1997. “Institutions and Labor Reallocation.” *European Economic Review*. 41 (6): 1147–1171.
- Blanchard, Olivier J., and Lawrence H. Summers, 1986. “Hysteresis and the European Unemployment Problem.” *NBER Macroeconomics Annual*. 1 (1): 15–78.
- Blau, Francine D., and Lawrence M. Kahn, 1996. “International Differences in Male Wage Inequality: Institutions versus Market Forces.” *Journal of Political Economy*. 104 (4): 791–837.
- Harris, John R., and Michael P. Todaro, 1970. “Migration, Unemployment and Development: A Two-Sector Analysis.” *American Economic Review*. 60 (1): 126–142.
- Jacobson, Louis S., Robert J. LaLonde, and Daniel G. Sullivan, 1993. “Earnings Losses of Displaced Workers.” *American Economic Review*. 83 (4): 685–709.
- Juhn, Chinhui, Kevin M. Murphy, and Robert H. Topel, 1991. “Why has the Natural Rate of Unemployment Increased over Time?.” *Brookings Papers on Economic Activity*. 1991 (2): 75–142.
- Lucas, Robert E. Jr., and Edward C. Prescott, 1974. “Equilibrium Search and Unemployment.” *Journal of Economic Theory*. 7: 188–209.
- Mourre, Gilles, 2005. “Wage Compression and Employment in Europe: First Evidence from the Structure of Earnings Survey 2002.” European Commission Economic Papers 232.
- Nickell, Stephen, and Richard Layard, 1999. “Labor Market Institutions and Economic Performance.” in O. Ashenfelter, and D. Card (ed.), *Handbook of Labor Economics*, vol. 3, chap. 46, pp. 3029–3084. Elsevier.
- Summers, Lawrence H., 1986. “Why is the Unemployment Rate So Very High Near Full Employment?.” *Brookings Papers on Economic Activity*. 2: 339–383.