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THINKING CATEGORICALLY ABOUT OTHERS: A CONJECTURAL EQUILIBRIUM APPROACH

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Abstract. Inspired by the social psychology literature, we study the implications of categorical thinking on decision making in the context of a large normal-form game. Every agent has a categorization (partition) of her opponents and can only observe the average behavior in each category. A strategy profile is a *Conjectural Categorical Equilibrium* (CCE) with respect to a given categorization profile if every player's strategy is a best response to some consistent conjecture about the strategies of her opponents.

We show that, for a wide family of games and for a particular categorization profile, every CCE becomes almost Nash as the number of players grows. An equivalence of CCE and Nash equilibrium is achieved in the settings of a non-atomic game. This highlights the advantage of categorization as a simplifying mechanism in complex environments. With much less information in their hands agents behave as if they see the full picture. Some properties of CCE when players categorize 'non-optimally' are also considered.

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1. Introduction

It is commonly accepted in the psychological literature[1] that people categorize the world around them. In particular, information about other people is often processed with the aid of social categories. As Macrae and Bodenhausen (2000) write

> " Given basic cognitive limitations and a challenging stimulus world, perceivers need some way to simplify and structure the person perception process. This they achieve through the activation and implementation of categorical thinking. Rather than considering individuals in terms of their unique constellations and proclivities, perceivers prefer instead to construe them on the basis of the social categories (e.g. race, gender, age) to which they belong..."

The purpose of the current paper is to study some issues related to categorical thinking in the context of decision making. Specifically, our concern here is with equilibrium behavior of agents in a non-cooperative normal form game. Equilibrium is viewed here as a steady state of a recurring interaction between agents with no strategic links among the repetitions. As such, it is highly sensitive to the information (and the way its being processed) that each agent has about the actions of her opponents. It is a key assumption of this paper that categorical thinking affects exactly this information.

To study the implications of categorization in such scenarios, we define a solution concept called *Conjectural Categorical Equilibrium* (CCE). This is a special case of Battigalli and Guaitoli's (1988) conjectural equilibrium. Each player i is equipped with an exogenously given partition of her opponents. This is the categorization that i uses in order to facilitate the process of information about the behavior of her opponents. As a consequence, i is unable to observe the actions taken by each individual player. Instead, she can only tell what is the average behavior within each category in her partition. Thus, when deciding what action to choose, player i is faces uncertainty as to the actual choices of her opponents. In this case it is natural to assume that i has some conjecture (which conforms to her information) about the profile of actions that her opponents actually play, and that she plays a best response to her conjecture. When all players behave in this way the resulting strategy profile is a CCE.

When an agent is categorizing her opponents there is a risk that it will lead her to make sub-optimal decisions and to lose utility. Therefore, in order for categorization to be efficient it should have the property that the loss of information incurred by the categorical representation of other agents will not result in choosing the wrong action. In other words, each agent wants to choose the action that she would choose had she known the entire strategy profile of her opponents. If the categorization of

[1]See Section 6 for references.

every player has this property then every CCE is also a Nash equilibrium. We call such a profile of categorizations *sufficient*. The main result of the paper concerns the existence of non-trivial sufficient categorization profiles.

Our solution concept is plausible only if the number of participating players is large. We therefore consider families of games with an increasing number of players. The result we obtain is asymptotic. It is shown that, with appropriate anonymity and continuity assumptions on the payoff functions, certain categorization profiles become close to being sufficient as the number of players grows to infinity. These categorization profiles are those in which each player lumps together players that have symmetric influence on her payoffs.

The aforementioned result can be interpreted in several ways. First, it highlights the advantage of categorization as a simplifying tool in complex environments. With much less information in their hands agents behave as if they see the full picture. The second interpretation is of a normative nature. The result can be seen as a recommendation for how one should categorize others when involved in a game-like situation. Finally, the result also increases the plausibility of Nash equilibrium in large games since it shows that an equilibrium must emerge even if players have limited information about the strategies of their opponents.

The model we use to obtain the asymptotic result is adopted from Kalai (2004). There is a finite universal set of actions $\mathbb{S}$. $\Gamma(\mathbb{S})$ is a family of normal-form games such that for every game G in $\Gamma(\mathbb{S})$ and for every player i in G the set of (pure) strategies available to i is some subset of $\mathbb{S}$. With a fixed family $\Gamma(\mathbb{S})$ in hand, one can very naturally define notions of *uniform* continuity and anonymity in $\Gamma(\mathbb{S})$. These are the key assumptions needed to obtain the asymptotic existence of a sufficient categorization profile. For a detailed discussion of the relation between our assumptions and results and those of Kalai (2004, 2005), see Section 6.

As noted before, CCE is appealing when the number of players is large. It is therefore natural to study it also in the setting of a game with a continuum of players. Working in the model of Schmeidler (1973), we define CCE for a non-atomic game similarly to its definition in the finite case. A simple sufficient condition for the existence of a sufficient categorization profile is provided. We then show that this condition holds for a dense set of non-atomic games. Thus, every non-atomic game can be approximated by a game in which a sufficient categorization profile exists.

The results described so far are of a 'positive' nature. They emphasize the advantages of categorization as an information processing mechanism. But these advantages may cease to exist if an agent makes use of the 'wrong' categorization. To illustrate this point we analyze two examples of non-atomic games in which agents categorize their opponents not as one may think they should. In the first example it is shown that this can lead to a CCE in which all the players get the worst possible payoff. In the second example there is a CCE which yield a higher *total* payoff for the society than the Nash equilibrium of the game (though the payoffs to some players are lower than their equilibrium payoffs).

The paper is organized as follows. In Section 2 we formally define CCE for both finite and non-atomic games. Section 3 contains the main results of the paper about sufficient categorization profiles. The influence that different categorizations may have on social efficiency is exemplified in Section 4. Some remarks about the model are in Section 5. These include a possible refinement of CCE and a result regarding CCE as a purifying device. Related literature is discussed in Section 6. All the proofs are in Section 7.

2. Definition of CCE

2.1. **Finite games.** A game G in normal form is defined by a triplet $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$. $N = \{1, \ldots, n\}$ is the set of players. For each $i \in N$, S_i is the finite set of pure strategies (actions) of player i. Denote by S the product $S = \times_{i\in N} S_i$ and for every player $i \in N$ let $S_{-i} = \times_{j\neq i} S_j$. A typical element of S $(S_i,\ S_{-i})$ will be denoted by[2] $\underline{s}$ $(s_i,\ s_{-i})$. $u_i : S \to \mathbb{R}$ is the utility function of player $i \in N$. Each player i may use a mixed strategy which is a probability distribution over S_i, usually denoted by[3] σ_i. If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a profile of strategies then σ_{-i} denotes the strategies of players other than i. As usual, u_i will also be used to denote expected utility whenever players use mixed strategies.

Assume that every player $i \in N$ categorizes the rest of the players according to some criteria. Formally, for every $i \in N$, let C_i be a partition of the set $N \setminus \{i\}$. That is, $C_i = \{B_1, \ldots, B_m\}$ where each B_j is a non-empty subset of $N \setminus \{i\}$, $j \neq k$ implies $B_j \cap B_k = \emptyset$, and $\cup_{j=1}^m B_j = N \setminus \{i\}$. A *categorization profile* is a vector $C = (C_1, \ldots, C_n)$, where each C_i is a partition of $N \setminus \{i\}$. For two categorization profiles $C = (C_1, \ldots, C_n)$ and $C' = (C'_1, \ldots, C'_n)$, we say that C is finer than C' if[4] C_i is finer than C'_i for every $i \in N$.

Assume that there is a finite universal set of actions $\mathbb{S}$ (not to be confused with the product set S) such that $S_i \subseteq \mathbb{S}$ for every $i \in N$. Every profile of (possibly mixed) strategies[5] $\sigma = (\sigma_1, \ldots, \sigma_n) \in \times_{i\in N} \Delta(S_i)$ and a non-empty set of players $B \subseteq N$ induce a probability distribution over $\mathbb{S}$, denoted σ^B, which is defined by[6] $\sigma^B(s) = \frac{1}{|B|} \sum_{i\in B} \sigma_i(s)$ for every $s \in \mathbb{S}$. Thus, $\sigma^B(s)$ is the expected proportion of players choosing s in the set B according to the profile of strategies $\{\sigma_i\}_{i\in B}$.

[2]The underline is used to emphasize that this is a vector of actions. The index s will be used for another purpose in the sequel.

[3]Thus, for every $s_i \in S_i$, $\sigma_i(s_i)$ is the probability of player i choosing the action s_i according to the mixed strategy σ_i.

[4]For two partitions P and P' of the same set, P is finer than P' (or equivalently, P' is coarser than P) if every cell of P' is a union of cells of P.

[5]If X is a finite set then $\Delta(X)$ denotes the family of all probability measures over X.

[6]$\sigma_i(s) = 0$ whenever $s \in \mathbb{S} \setminus S_i$.

Given a player $i \in N$, a categorization C_i of $N \setminus \{i\}$ and a profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$, let $F_{C_i}(\sigma_{-i}) = \{\tau_{-i} \ : \ \tau_{-i}^B = \sigma_{-i}^B$ for every $B \in C_i\}$ be the set of all strategy profiles of players other than i which induce the same distribution over $\mathbb{S}$ like σ in every set $B \in C_i$. Elements of $F_{C_i}(\sigma_{-i})$ are called *consistent conjectures* of player i at σ_{-i}.

Definition 1. *$\sigma = (\sigma_1, \ldots, \sigma_n)$ is a* Conjectural Categorical Equilibrium (CCE) w.r.t. the categorization profile *$C = (C_1, \ldots, C_n)$ if, for every $i \in N$, there exists a profile of strategies $\tau_{-i} \in F_{C_i}(\sigma_{-i})$ such that σ_i is a best response to τ_{-i}.*

Assuming that a categorization profile is exogenously given, a profile of strategies constitutes a CCE (w.r.t. the given categorization profile), if every player best responds to *some* conjecture about the strategies of the others. However, the conjecture of each player must be consistent with what she knows about the strategies of others, i.e., within the set $F_{C_i}(\sigma_{-i})$.

The set of all CCE in a game G w.r.t. a given categorization profile C is denoted by $CCE_G(C)$. NE_G is the set of Nash equilibria of the game G. The following observation is simple but important (the proof is omitted).

Lemma 1. *For every game G,*
(i) If C refines C' then $CCE_G(C) \subseteq CCE_G(C')$.
(ii) If C is the finest categorization profile in G (every cell of every categorization contains only one player) then $CCE_G(C) = NE_G$.

Corollary 1. *Every Nash equilibrium is a CCE w.r.t. any categorization profile.*

2.2. **A continuum of players.** By its nature, the concept of CCE is more plausible when the number of players is large. It is therefore natural to study this concept in the environment of a non-atomic game. As we shall see below, working in the limit with a continuum of players removes the need for many of the technical details involved in the finite model. As a consequence the results become sharper and clearer.

We follow the model and notation of Schmeidler (1973)[7]. The set of players is identified with the $T = [0, 1]$ interval equipped with the Lebesgue measure λ. There are n pure strategies, each of them represented by a vector e_i from the standard basis of $\mathbb{R}^n$. The set of possible mixed strategies of every player is[8] $P = conv(\{e_1, \ldots, e_n\})$. A T-strategy is (the equivalence class of) a measurable function $\hat{x}$ from T to P, specifying the strategy chosen by each of the players. $\hat{P}$ is the set of all T-strategies endowed with the L_1 weak topology.

The utility of player $t_0 \in T$ when she chooses e_i and almost every player in T plays according to the T-strategy $\hat{x}$ is $u^i(t_0, \hat{x})$. Denote $u(t_0, \hat{x}) = (u^1(t_0, \hat{x}), \ldots, u^n(t_0, \hat{x}))$.

[7] Since we are interested in games which are not necessarily anonymous, the models of Mas-Colell (1984) and of Rath (1992) are not suitable here.

[8] $conv(A)$ denotes the convex hull of the set A.

The payoff to player t_0 when almost every player in T plays according to $\hat{x}$ (and, of course, t_0 is playing $\hat{x}(t_0)$) is the scalar product $\hat{x}(t_0) \cdot u(t_0, \hat{x})$. Thus, a game with a continuum of players can be identified with the function $u : T \times \hat{P} \to \mathbb{R}^n$.

We will only consider games u with the following two properties:

(1) u is continuous on $\hat{P}$ for every $t \in T$; and
(2) u is measurable on T for every $\hat{x} \in \hat{P}$.

As in the finite case, assume that every player $t \in T$ has a finite and measurable partition C_t of the set T of players[9]. Define $F_{C_t}(\hat{x}) = \{\hat{y} \in \hat{P} \ : \ \int_B \hat{x} d\lambda = \int_B \hat{y} d\lambda$ for every $B \in C_t\}$. Again, if $\hat{y} \in F_{C_t}(\hat{x})$ we say that $\hat{y}$ is a consistent conjecture of player t at $\hat{x}$.

Definition 2. *A T-strategy $\hat{x} \in \hat{P}$ is a* Conjectural Categorical Equilibrium (CCE) w.r.t. the categorization profile $C = \{C_t\}_{t \in T}$ *if, for λ-almost every $t \in T$, there is a T-strategy $\hat{y}_t \in F_{C_t}(\hat{x})$ such that $\hat{x}(t) \cdot u(t, \hat{y}_t) \geq p \cdot u(t, \hat{y}_t)$ for every $p \in P$.*

Similarly to the finite case, we denote by $CCE_u(C)$ the set of all CCE in the game with a continuum of players u w.r.t. the categorization profile C. NE_u is the set of Nash equilibria of u. The following is the analogue of Lemma 1 for the case of a continuum of players.

Lemma 2. *For every game with a continuum of players u,*
(i) If C refines C' then $CCE_u(C) \subseteq CCE_u(C')$.
(ii) $NE_u \subseteq CCE_u(C)$ for every categorization profile C in u.

3. Sufficient categorization profiles

The current section contains the main results of the paper. It deals with a property of categorization profiles which we call *sufficiency*. A categorization profile is sufficient if a best response to every consistent conjecture of every player is also a best response to the actual profile of actions. When an agent categorizes her opponents according to a sufficient categorization she maintains her utility level with significantly less mental effort. Exact and approximated sufficiency are formally defined as follows.

Definition 3. *Fix a game (either finite or non-atomic) and let $\varepsilon \geq 0$. A categorization profile C is* ε-sufficient *if every CCE w.r.t. C is an ε-Nash equilibrium*[10] *of the game. A categorization profile is* sufficient *if it is 0-sufficient.*

[9] In this non-atomic setting it is not important for our purposes whether the partition is of T or of $T \setminus \{t\}$. We also assume that the measure of each set in the partition is strictly positive.

[10] A strategy profile constitute an ε-Nash equilibrium in a finite game if no player can gain more than ε by deviating. In the non-atomic case the same should hold almost everywhere.

The rest of this section discusses sufficient conditions for the existence of sufficient categorization profiles. Of course, the finest categorization (in the finite case) in which every category consists of only one agent is always sufficient. What we show, however, is that for a wide family of games there are also non-trivial sufficient categorization profiles. We start with finite games and then move on to the non-atomic case.

3.1. **Sufficiency in finite games.** We start with some notation. Fix a game G. For a profile of actions $\underline{s} = (s_1, \ldots, s_n) \in S$ and two players $j, k \in N$ with $S_j = S_k$, let $\underline{s}^{jk}$ be the profile of actions in which every player other than j and k plays the same as in $\underline{s}$ and players j and k exchange their choices. That is, player j plays s_k, player k plays s_j and every player $l \in N \setminus \{j, k\}$ plays s_l. For a player $i \in N$, we say that the players $j, k \in N \setminus \{i\}$ are *exchangeable for* i (denoted $j \sim_i k$) if $S_j = S_k$ and $u_i(\underline{s}) = u_i(\underline{s}^{jk})$ for every $\underline{s} \in S$.

If $j \sim_i k$ then player i only cares about the pair of actions taken by players j and k. She is not concerned with who plays what. Thus, assuming that i observes the distribution of actions in each cell of her categorization, it is natural for her to put j and k in the same cell.

It is easy to verify that $\sim_i$ is transitive and symmetric over $N \setminus \{i\}$. Let $\hat{C}_i$ be the partition of $N \setminus \{i\}$ to the equivalence classes of $\sim_i$ and let $\hat{C} = (\hat{C}_1, \ldots, \hat{C}_n)$. The element of $\hat{C}_i$ which contains player j will be denoted by $\hat{C}_i(j)$. Notice that our notation neglects the dependence of the categorization profile $\hat{C}$ on the game G. This is so since it will always be clear what is the relevant game. Notice also that $\hat{C}$ is endogenous: Nothing besides the description of the game is required in order to determine it.

If players were only allowed to play pure strategies and, in addition, players would always conjecture that their opponents play pure strategies then $\hat{C}$ would have been sufficient (see Lemma 6 in subsection 7.1). However, since players may randomize, some conditions on the game must be added in order to maintain the sufficiency of $\hat{C}$. Although restricting the generality of our discussion, these conditions are valid for a wide family of games.

Definition 4. *Fix a finite set of actions* $\mathbb{S}$. *Let* $\Gamma(\mathbb{S})$ *denote a family of normal form games such that, for every game* $G \in \Gamma(\mathbb{S})$ *and for every*[11] $i \in N$, $S_i \subseteq \mathbb{S}$.
(i) $\Gamma(\mathbb{S})$ *is* uniformly bounded *if there is* $M > 0$ *such that* $|u_i| \leq M$ *for every* $G \in \Gamma(\mathbb{S})$ *and for every utility function* $u_i \in G$.
(ii) $\Gamma(\mathbb{S})$ exhibits a diminishing effect of a single player *if there is* $M > 0$ *such that* $|u_i(\underline{s}) - u_i(s'_j \ ; \ s_{-j})| \leq \frac{M}{|N|}$ *for every* $G \in \Gamma(\mathbb{S})$, *every two players* $i, j \in N$, *every* $\underline{s} \in S$ *and every* $s'_j \in S_j$.

[11]Throughout the paper, a quantifier of the form "for every game $G \in \Gamma(\mathbb{S})$ and for every $i \in N$" should be understood as "for every game $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}) \in \Gamma(\mathbb{S})$ and for every $i \in N$".

The uniform boundness condition is standard. The diminishing effect of a single player is a continuity condition. It states that the effect of some player j changing his action on the payoff of another player i should be inversely proportional to the number of players in the game. Finally, we will also need to impose a restriction on the categorizations $\hat{C}_i$ in the family $\Gamma(\mathbb{S})$. Namely, the games in $\Gamma(\mathbb{S})$ should have a sufficient degree of anonymity. This intuition is captured by the following conditions.

Definition 5. *Let $\Gamma(\mathbb{S})$ be as in Definition 4. Say that $\Gamma(\mathbb{S})$ satisfies condition*
(A1) *if for every $\varepsilon > 0$ there is $\rho > 0$ such that $\#\{j \in N \setminus \{i\} \ : \ |\hat{C}_i(j)| < \rho|N|\} < \varepsilon|N|$ for every game $G \in \Gamma(\mathbb{S})$ and for every $i \in N$.*
(A2) *if for every $r, \varepsilon > 0$ there is n_0 such that $|\hat{C}_i|e^{-rd(G)_i} < \varepsilon$ for every game $G \in \Gamma(\mathbb{S})$ with $|N| > n_0$ and for every $i \in N$, where $d(G)_i = \min_{B \in \hat{C}_i} |B|$.*

Theorem 1. *Consider a family $\Gamma(\mathbb{S})$ of normal form games which is uniformly bounded, exhibits a diminishing effect of a single player and satisfies at least one of the conditions (A1) or (A2) of Definition 5. For every $\varepsilon > 0$ there exists $n_0 = n_0(\Gamma(\mathbb{S}), \varepsilon)$ such that if $G \in \Gamma(\mathbb{S})$ satisfies $|N| > n_0$ then the categorization profile $\hat{C}$ in G is ε-sufficient.*

Remark 1. *If $\Gamma(\mathbb{S})$ satisfies the conditions of Theorem 1 then, by Lemma 1, every categorization profile which is finer than $\hat{C}$ is also ε-sufficient.*

Remark 2. *None of the conditions (A1) and (A2) implies the other. It is clear that (A1) doesn't imply (A2) since (A1) puts no restrictions on the size of the smallest set in $\hat{C}_i$. On the other hand, if the categorizations $\hat{C}_i$ contain $\sqrt{|N|}$ elements of size $\sqrt{|N|}$ each then (A2) will be satisfied while (A1) will not.*

Remark 3. *If the number of categories in each of the categorizations $\hat{C}_i$ in the family $\Gamma(\mathbb{S})$ is uniformly bounded*[12]*, then condition (A1) of Definition 5 is satisfied. Indeed, $\#\{j \in N \setminus \{i\} \ : \ |\hat{C}_i(j)| < \rho|N|\} \leq \rho|N||\hat{C}_i| \leq \rho|N|M$, so for a given $\varepsilon > 0$ one can take $\rho = \frac{\varepsilon}{M}$. In particular, if the game is anonymous (any two players are exchangeable for any third player) then (A1) is satisfied.*

We illustrate the result of Theorem 1 and the importance of the various conditions with the following examples.

Example 1. ***(village versus beach)*** *This example is taken from Kalai (2004, Example 1). The universal set of actions is $\mathbb{S} = \{v \text{ (village)}, b \text{ (beach)}\}$. The family $\Gamma(\mathbb{S})$ contains games with $|N| = 2n$ ($n \in \mathbb{N}$) players of which n are 'males' and n are 'females'. The payoff of a male is equal to the proportion of females his choice matches and the payoff of a female is equal to the proportion of males her choice mismatches.*

[12] Formally, there is $M > 0$ such that $|\hat{C}_i| < M$ for every $G \in \Gamma(\mathbb{S})$ and for every $i \in N$.

The categorization $\hat{C}_i$ of every player lumps together players of the same gender. Indeed, the payoff of every player is not changed if two males (or females) exchange their choices. Notice that the family $\Gamma(\mathbb{S})$ is uniformly bounded (by $M = 1$) and exhibits a diminishing effect of a single player (again, with $M = 1$). Moreover, $\Gamma(\mathbb{S})$ satisfies conditions (A1) and (A2) of Definition 5 (In fact, the condition in Remark 3 is also satisfied since always $|\hat{C}_i| = 2$). Thus, by Theorem 1, when the number of players is large every CCE w.r.t. the profile $\hat{C}$ is almost Nash.

As a matter of fact, in this particular example Theorem 1 is redundant and a stronger result can be achieved by a much simpler argument. The reason is that the signal that every player observes is the expected proportions of males and females in each of the locations v and b. But from this signal a player can deduce his/her payoff for every possible choice. Thus, if a player's choice is optimal w.r.t. some consistent conjecture then it is also optimal w.r.t. the true strategy profile of his/her opponents. It follows that in the village versus beach game, *no matter what is the number of players*, $\hat{C}$ is sufficient (and not just ε-sufficient).

Example 2. ***(A generalized village versus beach)*** *Let $\mathbb{S}$ be as in the previous example and fix two Lipschitz and non-decreasing functions $f, g : [0,1] \to \mathbb{R}$. We consider games of the following form. For each player $i \in N$ there is a set $F^i \subseteq N \setminus \{i\}$ of i's friends and a set $E^i \subseteq N \setminus \{i\}$ of i's enemies ($F^i \cap E^i = \emptyset$)[13]. The payoff to player i is $f(p)+g(q)$ where p is the proportion of i's friends that her choice matches and q is the proportion of i's enemies that her choice mismatches.*

It is clear that, without any further restrictions on the sets of friends and enemies, a family $\Gamma(\mathbb{S})$ of games in the above form will be uniformly bounded and will satisfy condition (A1) (since always $|\hat{C}_i| \leq 3$. See Remark 3). However, in order to make sure that the family of games exhibits a diminishing effect of a single player we need that, for every $i \in N$, the sets F^i and E^i contain a non-vanishing fraction of players.

If, on the other hand, F^i and E^i are not large enough then Theorem 1 may fail. Indeed, assume that $F^1 = \{2,3,4\}$, $E^1 = \emptyset$, and $F^i = E^i = \emptyset$ for every player $i > 1$. Moreover, assume that $f(p) = p^3$. The following strategy profile is a CCE (w.r.t. $\hat{C}$) which does not become close to being Nash equilibrium when the number of players increases. Player 1 plays v, players 2 and 3 play v with probability 3/4 and b with probability 1/4, and player 4 plays b (the strategies of the other players, if there are any, are arbitrary). The true payoff to player 1 in this case is $0 \cdot f(1) + 9/16 \cdot f(2/3) + 6/16 \cdot f(1/3) + 1/16 \cdot f(0) = 13/72$, whereas if she would switch to b she will get $0 \cdot f(0) + 9/16 \cdot f(1/3) + 6/16 \cdot f(2/3) + 1/16 \cdot f(1) = 14/72$. Thus, the action of player 1 is suboptimal.

[13] We do not assume that the relations 'to be a friend of' and 'to be an enemy of' are symmetric nor transitive.

To see that the above profile is a CCE notice that one of the consistent conjectures of player 1 is that players 2,3 play v with probability $1/4$ and b with probability $3/4$, and player 4 plays v. If this is the belief that player 1 has then it is optimal for her to play v since, by symmetry, this would give her a payoff of $14/72$. Switching to b, however, would reduce the payoff to $13/72$.

Example 3. *Let $\mathbb{S} = \{v, b\}$ as in the previous examples and consider a family $\Gamma(\mathbb{S})$ such that, for every positive integer n, $\Gamma(\mathbb{S})$ contains a game with $3n+1$ players defined as follows. For every $k = 1, 2, \ldots, n$ denote $A_k = \{3k-1, 3k, 3k+1\}$. The payoff to player 1 is $\frac{1}{n}\sum_{k=1}^{n}\left(\frac{a_k}{3}\right)^3$ where a_k is the number of players from the set A_k which player 1's choice matches ($k = 1, 2, \ldots, n$). The payoff functions of all other players are constant.*

The purpose of the this last example is to show that the uniform boundness and diminishing effect conditions alone are not sufficient for Theorem 1 to hold. Notice first that the payoffs in $\Gamma(\mathbb{S})$ are uniformly bounded by 1. Also, the maximal difference in player 1's payoff when some player $j \neq 1$ changes his action is $1 - \frac{8}{27n} = \frac{19}{27n} < \frac{10}{|N|}$ which means that the diminishing effect condition is satisfied. However, none of the conditions (A1) or (A2) is satisfied since, for every n, $\hat{C}_1 = \{A_1, A_2, \ldots, A_n\}$ and $|A_k| = 3$ for every $1 \leq k \leq n$.

Now, consider the following strategy profile. For every $k = 1, 2, \ldots, n$, players number $3k-1$ and $3k$ play v with probability $\frac{3}{4}$ and b with probability $\frac{1}{4}$ and player number $3k+1$ plays b with probability 1. The expected payoff to player 1 if he chooses v is $\frac{1}{n} \cdot n \cdot \left[0 \cdot 1^3 + \frac{9}{16} \cdot \left(\frac{2}{3}\right)^3 + \frac{6}{16} \cdot \left(\frac{1}{3}\right)^3 + \frac{1}{16} \cdot 0^3\right] = \frac{13}{72}$, while choosing b yields an expected payoff of $\frac{1}{n} \cdot n \cdot \left[0 \cdot 0^3 + \frac{9}{16} \cdot \left(\frac{1}{3}\right)^3 + \frac{6}{16} \cdot \left(\frac{2}{3}\right)^3 + \frac{1}{16} \cdot 1^3\right] = \frac{14}{72}$. However, a consistent conjecture for player 1 is that, for every $1 \leq k \leq n$, players $3k-1$ and $3k$ play v with probability $1/4$ and b with probability $3/4$, and player $3k+1$ plays v. By symmetry, the expected payoff to player 1 according to this conjecture is $\frac{14}{72}$ for playing v and $\frac{13}{72}$ for playing b. Thus, the lose of utility for player 1 doesn't vanish as the number of players in the game grows.

3.2. **Sufficiency in non-atomic games.** When there is a continuum of players it will be meaningless to define a relation $\sim_t$ analogous to the relation $\sim_i$ in the finite case. What we need in order to insure that a categorization profile will be sufficient is that the utility of every player only depends on the distribution of actions in each set of her partition. No other assumptions should be made and the sufficiency obtained is not approximated as in the finite case. Thus, we have the following result.

Theorem 2. *Let u be a game with a continuum of players. If $C = \{C_t\}_{t \in T}$ is a categorization profile such that, for every $t \in T$, $u(t, \hat{x})$ depends only on $\{\int_B \hat{x} d\lambda\}_{B \in C_t}$ then C is sufficient.*

Example 4. ***(A non-atomic generalized village versus beach)*** *The following example is taken (with cosmetic changes) from Schmeidler (1973)*[14]*. The number of possible actions for every player is $n = 2$. For $i = 1, 2$ the utility of player $t \in T$ when she chooses e_i and when the T-strategy is $\hat{x}$ is $u^i(t, \hat{x}) = -\int_0^t \hat{x}_i d\lambda$, where $\hat{x}_i$ is the i'th component of the vector function $\hat{x}$. Thus, every player t prefers the action which was less frequently used by her predecessors - the players $[0, t)$. This situation can be seen as a continuous analogue of the generalized village versus beach game, where for every player $t \in T$ the set of enemies is $[0, t)$ and the set of friends is empty.*

Notice that the utility of every player t depends only on the integral $\int_0^t \hat{x} d\lambda$. Thus, by Theorem 2 the profile of categorizations $C = \{C_t\}_{t \in T}$ defined by $C_t = \{[0, t], (t, 1]\}$ is sufficient.

The question naturally arises is how 'common' are games with the property that the utility of every player depends only on the average behavior of a finite number of groups of the participating players. Our next aim is to show that the set of games with this property is dense within the set of all non-atomic games. This implies that every non-atomic game can be approximated by a game in which a sufficient categorization profile exists.

We denote by Y the set of all possible (continuous) utility functions of a player. That is $Y = \{v : \hat{P} \to \mathbb{R}^n \mid v \text{ is continuous}\}$. Since $\hat{P}$ is compact we can define a norm in Y by $\|v\| = \sup_{\hat{x} \in \hat{P}} \|v(\hat{x})\|$, where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^n$. A non-atomic game u specifies the utility function of every player and is therefore an element of the product space Y^T. The set of all non-atomic games is denoted by $U \subseteq Y^T$ (since u should be a measurable function of t not every element of Y^T is a game). Let $\tilde{U} \subseteq U$ be the set of all games u with the property that, for each player t, there is a finite and measurable partition C_t of T such that $u(t, \hat{x}) = u(t, \hat{y})$ whenever $\int_B \hat{x} d\lambda = \int_B \hat{y} d\lambda$ for every $B \in C_t$.

Theorem 3. *$\tilde{U}$ is dense in U.*

4. CCE and social efficiency

The previous section considered the case in which every player categorizes her opponents "correctly" in the sense that players within each category are anonymous in the eyes of the categorizer. The aim of the current section is to study some of the effects that "wrong" categorizations may have. In particular, we are interested in the social efficiency of profiles of strategies which constitute a CCE in comparison to the efficiency of profiles which are Nash equilibria.

[14]The original purpose of this example was to show that not every non-atomic game has a pure strategy equilibrium.

There may be various reasons why agents categorize others according to payoff irrelevant criteria (that is, not according to the partitions $\hat{C}$). First, it may be that the partition $\hat{C}_i$ contains too many elements for player i to handle. If player i has a limited computational ability then the number of different categories that she can create in her mind is bounded. Thus, she cannot sort her opponents optimally if the number of categories she needs to do so is greater than her ability[15].

Another reason for sub-optimal categorization may be lack of information. Namely, player i may not know the effect that the actions taken by player j have on her payoff. This naturally brings up the question of how players (should) categorize in a game with incomplete information, which we will not discuss here.

As opposed to the previous section, here we do not pursue general results. Rather, we restrict attention to two examples which reflect the implications that categorization can have on social efficiency. The first example shows how CCE may cause *all* players to lose utility in comparison to their equilibrium payoffs (thus decreasing the social efficiency of the strategy profile). In the second example it is shown that a CCE may be more socially efficient than any Nash equilibrium. Both examples are of non-atomic *congestion games*[16] and can also be seen as special cases of the (generalized non-atomic) village versus beach game.

Definition 6. *Let u be a non-atomic game. The* social efficiency *of a strategy profile $\hat{x} \in \hat{P}$ is $eff(\hat{x}) = \int_T \hat{x}(t)u(t,\hat{x})d\lambda(t)$.*

Example 5. *Consider the following non-atomic game with 2 possible actions ($n = 2$). If $t \in [0, \frac{1}{2})$ then $u^1(t,\hat{x}) = \int_0^{\frac{1}{2}} \hat{x}_2 d\lambda$ and $u^2(t,\hat{x}) = \frac{1}{2} - u^1(t,\hat{x})$. For $t \in [\frac{1}{2}, 1]$ the utility function is $u^1(t,\hat{x}) = \int_{\frac{1}{2}}^1 \hat{x}_2 d\lambda$ and $u^2(t,\hat{x}) = \frac{1}{2} - u^1(t,\hat{x})$. We call the players in the interval $[0, \frac{1}{2})$ type 1 players and those in $[\frac{1}{2}, 1]$ are called type 2 players. Thus, each player tries to avoid players with his own type and is careless about the choices of players from the other type.*

Since payoffs depend only on the distribution of actions within each type of players, we may w.l.o.g. restrict attention to pure strategies. Denote by $p_1(\hat{x}) = \lambda(\{t \in [0, \frac{1}{2}) : \hat{x}(t) = e_1\})$ and $p_2(\hat{x}) = \lambda(\{t \in [\frac{1}{2}, 1] : \hat{x}(t) = e_1\})$ the proportions of players of types 1 and 2 respectively who choose the first action according to $\hat{x}$. Then the social efficiency of a T-strategy $\hat{x}$ is $eff(\hat{x}) = 2p_1(\hat{x})(\frac{1}{2} - p_1(\hat{x})) + 2p_2(\hat{x})(\frac{1}{2} - p_2(\hat{x}))$. Notice that, in any Nash equilibrium $\hat{x}$, it must be that $p_1(\hat{x}) = \frac{1}{4}$ and $p_2(\hat{x}) = \frac{1}{4}$. Therefore, the social efficiency in every equilibrium is $\frac{1}{4}$. Moreover, every equilibrium is *socially optimal* in the sense that there is no profile of strategies $\hat{x}$ with $eff(\hat{x}) > \frac{1}{4}$.

[15]Recall that one of the reasons for the need to categorize in the first place is to save mental resources. The issue of "optimal" categorization when there is a bound on the number of categories seems to be of self interest.

[16]For a general study of social optimality in non-atomic congestion games see Milchtaich (2004).

By Theorem 2 the categorization profile defined by $C_t = \{[0, \frac{1}{2}), [\frac{1}{2}, 1]\}$ for every $t \in T$, as well as any finer categorization profile, is sufficient. However, assume that players categorize their opponents differently and that the categorization of all the players is the same. For instance, this corresponds to the case where players are categorizing according to some publicly observed property (such as gender or skin color). For simplicity we restrict attention to the case in which the (common) categorization has only two elements (say, M=Males and F=Females) each of which have a measure of $\frac{1}{2}$.

Let $\alpha = \lambda(M \cap [0, \frac{1}{2}))$ be the measure of the set of type 1 males, and let $g(\alpha) = \min\{eff(\hat{x})\ ;\ \hat{x} \in CE_u(\{M, F\})\}$ be the *lowest* social efficiency of a CCE[17]. We have

Proposition 1.

$$g(\alpha) = \begin{cases} \frac{1}{4} - 16\alpha^2 & 0 \le \alpha \le \frac{1}{8} \\ 0 & \frac{1}{8} \le \alpha \le \frac{3}{8} \\ \frac{1}{4} - 16(\frac{1}{2} - \alpha)^2 & \frac{3}{8} \le \alpha \le \frac{1}{2} \end{cases}$$

Specifically, if the payoff relevant partition (type 1 versus type 2) and the actual categorization (males versus females) are statistically independent ($\alpha = \frac{1}{4}$) or not "too dependent" ($\frac{1}{8} \le \alpha \le \frac{3}{8}$) then there exists a CCE in which all the players get the worst possible payoff. As α approaches 0 or $\frac{1}{2}$ the social efficiency of any CCE increases to the optimal level.

Example 6. *Consider the following non-atomic game with 2 possible actions ($n = 2$). If $t \in [0, \frac{3}{4})$ then $u^1(t, \hat{x}) = \frac{1}{4} + \int_0^{\frac{3}{4}} \hat{x}_2 d\lambda + 2\int_{\frac{3}{4}}^1 \hat{x}_2 d\lambda$ and $u^2(t, \hat{x}) = \int_0^{\frac{3}{4}} \hat{x}_1 d\lambda + 2\int_{\frac{3}{4}}^1 \hat{x}_1 d\lambda$. For $t \in [\frac{3}{4}, 1]$ the utility function is $u^1(t, \hat{x}) = 2\int_0^{\frac{3}{4}} \hat{x}_2 d\lambda + \int_{\frac{3}{4}}^1 \hat{x}_2 d\lambda$ and $u^2(t, \hat{x}) = 1 + 2\int_0^{\frac{3}{4}} \hat{x}_1 d\lambda + \int_{\frac{3}{4}}^1 \hat{x}_1 d\lambda$. Players in the interval $[0, \frac{3}{4})$ are called type 1 players and those in $[\frac{3}{4}, 1)$ type 2 players. The payoff to players of each type equals the proportion of players of their own type that their choice mismatches, plus twice the proportion of players of the other type that their choice mismatches. In addition, players of type 1 get $\frac{1}{4}$ if they choose the first action, and players of type 2 get 1 if they choose the second action.*

As in the previous example, we restrict attention to pure strategy profiles and denote $p_1(\hat{x}) = \lambda(\{t \in [0, \frac{3}{4})\ :\ \hat{x}(t) = e_1\})$ and $p_2(\hat{x}) = \lambda(\{t \in [\frac{3}{4}, 1]\ :\ \hat{x}(t) = e_1\})$ the proportions of players of types 1 and 2 respectively who choose the first action according to $\hat{x}$. the efficiency of a T-strategy $\hat{x}$ is then given by $eff(\hat{x}) = -2p_1(\hat{x})^2 - 2p_2(\hat{x})^2 - 8p_1(\hat{x})p_2(\hat{x}) + \frac{11}{4}p_1(\hat{x}) + \frac{10}{4}p_2(\hat{x}) + \frac{1}{4}$. It is not hard to verify that there is a unique equilibrium in this game. Namely, all type 1 players choose the first action ($p_1(\hat{x}) = 3/4$) while all type 2 players choose the other option ($p_2(\hat{x}) = 0$). The social efficiency of the equilibrium strategy is equal to $\frac{19}{16} = 1.1875$.

[17]One can convince oneself that g indeed depends only on α and not on the choice of the sets M and F.

However, assume that the categorization of all the players is trivial. That is, $C_t = \{T\}$ for all $t \in T$. In this case there is a CCE which is more efficient than the Nash equilibrium. Indeed, it is a simple exercise to check that the set of CCE in this case is $\{\hat{x} \ : \ \frac{1}{2} \leq p_1(\hat{x}) \leq \frac{3}{4},\ p_2(\hat{x}) = 0\}$. Taking $\hat{x}$ to be a profile with $p_1(\hat{x}) = \frac{11}{16}$ and $p_2(\hat{x}) = 0$ gives a CCE with social efficiency of $\frac{153}{128} \cong 1.1953$. Moreover, such a profile is socially optimal.

5. Discussion

5.1. **Simple conjectures: A refinement.** In a CCE the conjecture of an agent is limited only by the signal she observed (and by the restriction that agents play independently of each other). One may want to restrict the belief that an agent can have even more by requiring that it will be simple in some sense[18]. By so doing, a refinement of CCE can be obtained.

Among all the possible conjectures of a player there is one which can quite naturally be considered as the simplest, namely, the conjecture in which all players in each cell of her partition are playing the same strategy. A player holding this belief can be seen as having a prototypical agent for each set in her partition. All the players in each set are playing the same as their representing prototype. The common strategy in each cell is then uniquely determined by the signal the player observed.

Such a refinement leads (in finite games) to a special case of Jehiel's (2005) *Analogy Based Expectation Equilibrium* (ABEE). The scope of ABEE is much wider since it is defined for extensive form games and for any partition of opponents' nodes in the game tree. Notice that it may well be that all Nash equilibria will not survive the suggested refinement. However, by a standard fixed point argument (or by Jehiel's theory in the finite case) existence is guaranteed.

5.2. **Correlated conjectures.** The sufficiency result of subsection 3.1 relies heavily on the assumption that a player takes into account only *independent* profiles of strategies of her opponents. That is, we rule out the possibility that some player thinks that other players correlate their strategies, even though this correlated strategy might be consistent with the signal that this player observes. The fact that correlated conjectures are not allowed enables us to use the power of the laws of large numbers, which otherwise fail.

To emphasize this point we return to the generalized village versus beach game (Example 2). Assume that, for some player $i \in N$, $F^i = N \setminus \{i\}$ and $E^i = \emptyset$, and that $F^j = E^j = \emptyset$ for all other players j. As opposed to Example 2, we do not assume

[18]This idea is certainly not new. Eliaz (2003) and Spiegler (2002 and 2004) are examples of papers in which the solution concept takes into account the complexity of the belief of an agent about what others will do.

that the function f is non-decreasing[19]. Specifically, consider the function f defined by $f(x) = 3x$ for $0 \le x \le \frac{1}{3}$ and $f(x) = \frac{4}{3} - x$ for $\frac{1}{3} \le x \le 1$.

If correlated conjectures were allowed then the following profile of strategies would constitute a CCE w.r.t. the categorization profile $\hat{C}$. Player i plays v with probability 1 and every other player plays v with probability $\frac{2}{3}$ and b with probability $\frac{1}{3}$. Indeed, a consistent (correlated) conjecture of player i is that either all the players play v (with probability $\frac{2}{3}$) or all the players play b (with probability $\frac{1}{3}$). For this conjecture the best response of i is v since $\frac{2}{3}f(1) + \frac{1}{3}f(0) > \frac{2}{3}f(0) + \frac{1}{3}f(1)$. However, this profile of strategies does not become approximately Nash as the number of players increases. This is because the payoff to player i will be close to $f(\frac{2}{3}) = \frac{2}{3}$ while deviating to b would result in a payoff close to $f(\frac{1}{3}) = 1$.

5.3. **Pure equilibrium.** The fact that Nash equilibrium in pure strategies may fail to exist is seen by many as a drawback of this solution concept. Even in games with a continuum of players, a certain degree of anonymity is required in order to insure the existence of a pure equilibrium (see Remarks 2 and 3 in Schmeidler, 1973). The reason that players need to randomize in equilibrium is to hide their action from their opponents. The same goal can be achieved by using the CCE solution concept. The fact that an agent cannot predict accurately her opponents behavior in some cases eliminates the need for randomization in equilibrium. This phenomenon is demonstrated in the following theorem.

Theorem 4. *Let u be a game with a continuum of players, and let C^* be a (finite) partition of T. Assume that $C = \{C_t\}_{t \in T}$ is a profile of categorizations such that C^* is finer than C_t for every $t \in T$. Then there is a pure CCE w.r.t. C.*

To illustrate the idea consider Example 4. Assume that $C_t = \{T\}$ for every $t \in T$. Thus, the signal to every player is just the average behavior of the entire set of players. The pure T-strategy $\hat{x}$ defined by $\hat{x}(t) = e_1$ for $0 \le t \le \frac{1}{2}$ and $\hat{x}(t) = e_2$ for $\frac{1}{2} < t \le 1$ is a CCE w.r.t. $C = \{C_t\}_{t \in T}$. Indeed, for every $t \in T$, $F_{C_t}(\hat{x})$ contains a T-strategy for which $\hat{x}(t)$ is a best response.

Finally, we note that the condition that C_t is coarser than some C^* for all $t \in T$ is necessary for the theorem to hold. This can be seen by considering the categorization profile $C_t = \{[0, t], (t, 1]\}$, $t \in T$ in the above example.

5.4. **Self-categorization.** Throughout the discussion of finite games the categorization of player i is of the set $N \setminus \{i\}$. Thus, i doesn't include herself in any of the groups of her categorization. The reason for this modeling choice is the common assumption that every agent knows the action he plays. Inserting i into one of the cells of her partition C_i (say B) can create a situation in which i's conjecture (about

[19] A possible interpretation is that a player wants to be with her friends, but not with too many of them.

what the players in B are playing) is consistent with her signal (the average behavior within B) but not with the action which she actually plays. Leaving i out of her own partition prevents such an awkward situation.

It should be noted, however, that individuals do not exclude themselves from their categorical perception of the society. In fact, self-categorization and identity are among the most studied subjects in social psychology (for references see Ellemers et al., 2002). The social categories to which one belongs and the way these categories are seen by the society can have significant implications on one's choices. This important issue is not addressed by the current paper.

6. Related literature

Our main result (Theorem 1) is inspired by the works of Kalai (2004, 2005) on the robustness of equilibria in large games. There, it is shown that, when the number of players is large, Nash equilibria of a wide family of games are immune to many modifications of the game. These modifications include various extensive form versions of the game such as sequential play (instead of simultaneous play) and versions in which players can revise their initial choices. The main difference between the current paper and Kalai's is that here we keep the game unchanged while allowing players' beliefs about their opponents strategies to be incorrect. Since in general it is impossible to construct an extensive version of a game in which Nash equilibria corresponds to CCE of the original game, the results of Kalai do not imply ours.

It is interesting to compare our model and assumptions to those of Kalai. The first difference is that we study only complete information games while Kalai allows players to be of several *types* (though a key assumption in his paper is that types are drawn independently from some universal finite set). Another difference is in the anonymity and continuity assumptions used. In Kalai's paper the payoff to a player depends on his own type and action and on the empirical distribution of type-action characters of the other players. This implies that, with probability 1, *the categorization $\hat{C}_i$ will be the same for all the players*[20]. In the current paper 'types' of players are *subjective* in the sense that the categorization $\hat{C}_i$ is likely to depend on i. Moreover, the number of different 'types' (elements of the categorization $\hat{C}_i$) is not bounded and can grow to infinity as $|N|$ grows to infinity[21].

The condition of diminishing effect of a single player (Definition 4 (ii)) is slightly different from the uniform equicontinuity condition of Kalai (2004, Definition 3). None of them implies the other. Uniform equicontinuity implies semi-anonymity in the sense of Kalai (2004, Definition 2) which will make our analysis trivial. But

[20]More accurately, for every two players i, j the partitions $\hat{C}_i$ excluding player j and $\hat{C}_j$ excluding player i will be identical. Notice, however, that when different profiles of types are realized this common categorization may very well change.

[21]If $|\hat{C}_i|$ is uniformly bounded in $\Gamma(\mathbb{S})$ then, by Remark 3, Theorem 1 holds.

the diminishing effect condition requires that the change in payoff when one player changes his action will be *inversely proportional* to the number of players in the game. This property is not implied by uniform equicontinuity.

The concept of Nash equilibrium in normal form games with a continuum of players originated in the work of Schmeidler (1973). The main result there is the existence of a pure equilibrium when the game is anonymous. Similar results in different models were obtained by Mas-Colell (1984) and by Rath (1992). Theorem 4 shows that CCE can in some cases eliminate the need for randomization even if the game is not anonymous. A comprehensive survey of the literature about games with a continuum of players can be found in Ali Khan and Sun (2002).

CCE is a special case of Battigalli and Guaitoli's (1988) *Conjectural Equilibrium* (CE). Rubinstein and Wolinsky (1994) introduced the notion of *Rationalizable Conjectural Equilibrium* (RCE) by adding common knowledge of rationality, thus refining CE. One may ask why we use CE and not RCE as our solution concept. Two reasons justify this choice. The first is that the games we analyze have many players. It is natural to assume that in this case players do not "get into the head" of their opponents and draw conclusions which change their beliefs, simply because it is too complicated to do so. Second, every RCE is also a CE. Thus, our main results wouldn't change had we defined CCE using RCE and not CE.

Each one of these solution concepts is weaker than Nash equilibrium. In the final section of their paper Rubinstein and Wolinsky (1994) suggest that the plausibility of Nash equilibrium increases when every RCE is also Nash. They write

> "... In games with this property the Nash equilibrium concept is more compelling, because in a sense the equilibrium requires less information on the part of the players. It may therefore be of interest to identify conditions under which, for some natural signal function such as one's own payoff, RCE and Nash equilibria are equivalent."

Theorems 1 and 2 provide precisely the kind of conditions that Rubinstein and Wolinsky are talking about. The signal function of a player, however, is not her own payoff but the expected behavior of groups of her opponents.

Attempts to weaken the assumption that agents predict accurately the actions of their opponents have been made in settings other than normal form games. For extensive form games Fudenberg and Levine's (1993) *self-confirming equilibrium* is based on the fact that agents' beliefs are correct only along the equilibrium path of play[22]. For repeated games, Kalai and Lehrer (1993a, 1993b) introduced the notion of *subjective equilibrium* where player's beliefs are not contradicted by the observed choices of their opponents.

[22]See also Dekel et al. (1999).

Some recent papers discuss issues related to categorical thinking in the context of decision making. Fryer and Jackson (2004) develop a model of how past experiences are sorted into categories and show that certain biases in decision making emerge from this process. Pęski (2006) shows that in symmetric environments categorization is an optimal way for predicting properties of future instances based on past instances. Azrieli and Lehrer (2007) axiomatize categorizations that are generated by proximity to a set of prototypical cases. Finally, for surveys about categorization in social psychology see Fiske (1998) and Macrae and Bodenhausen (2000).

7. Proofs

7.1. **Proof of Theorem 1.** We start with several lemmas. Fix a family of games $\Gamma(\mathbb{S})$ which is uniformly bounded (by a constant $M > 0$) and exhibits a diminishing effect of a single player (w.l.o.g. with the same constant M).

For the following Lemmas 3-8 and 10-11 fix a game $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}) \in \Gamma(\mathbb{S})$, a player $i \in N$, a profile of strategies σ and two positive numbers $\delta, \rho > 0$. A typical element of the categorization $\hat{C}_i$ will be denoted by B. Let

$$E_\rho = \left\{B \in \hat{C}_i \ : \ |B| \geq \rho|N|\right\} \text{ and } H_\rho = \hat{C}_i \setminus E_\rho = \left\{B \in \hat{C}_i \ : \ |B| < \rho|N|\right\}.$$

For every $B \in \hat{C}_i$ and for every $s \in \mathbb{S}$ let

$$D_\delta(B, s) = \left\{s_{-i} \in S_{-i} \ : \ \left|\frac{\#\{j \in B \ : \ s_j = s\}}{|B|} - \sigma^B(s)\right| \geq \delta\right\},$$

and denote

$$D_{\delta,\rho} = \bigcup_{B\in E_\rho} \bigcup_{s\in\mathbb{S}} D_\delta(B, s), \ D_\delta = \bigcup_{B\in\hat{C}_i} \bigcup_{s\in\mathbb{S}} D_\delta(B, s).$$

Let $\mathbb{P}_{\sigma_{-i}}$ denote the probability measure on S_{-i} induced by the profile of strategies σ_{-i}.

Lemma 3. $\mathbb{P}_{\sigma_{-i}}(D_\delta(B, s)) \leq 2e^{-2|B|\delta^2}$ *For every $B \in \hat{C}_i$ and for every $s \in \mathbb{S}$.*

Proof. For a given $B \in \hat{C}_i$ and $s \in \mathbb{S}$ consider the sequence of independent random variables $(X_j)_{j\in B}$ defined by $X_j = 1$ if player j's realized strategy is s and $X_j = 0$ otherwise. Let $X = \sum_{j\in B} X_j$. We have $D_\delta(B, s) = \left\{\left|\frac{X}{|B|} - \sigma^B(s)\right| \geq \delta\right\}$ where $\sigma^B(s)$ is the expected value of the random variable $\frac{X}{|B|}$. By a classical bound of Hoeffding (see for instance Petrov, 1975 chapter III) the probability of this event is not greater than $2e^{-2|B|\delta^2}$. ■

Lemma 4. $\mathbb{P}_{\sigma_{-i}}(D_{\delta,\rho}) \leq \frac{2|\mathbb{S}|e^{-2|N|\rho\delta^2}}{\rho}$ *and* $\mathbb{P}_{\sigma_{-i}}(D_\delta) \leq 2|\mathbb{S}||\hat{C}_i|e^{-2d(G)_i\delta^2}$.

Proof. Using the previous lemma, and by the definition of E_ρ

$$\begin{aligned}\mathbb{P}_{\sigma_{-i}}(D_{\delta,\rho}) &\leq \sum_{B\in E_\rho}\sum_{s\in\mathbb{S}} 2e^{-2|B|\delta^2} = 2|\mathbb{S}|\sum_{B\in E_\rho} e^{-2|B|\delta^2} \\ &\leq 2|\mathbb{S}|\sum_{B\in E_\rho} e^{-2\rho\delta^2|N|} \leq \frac{2|\mathbb{S}|e^{-2|N|\rho\delta^2}}{\rho}.\end{aligned}$$

Also,

$$\mathbb{P}_{\sigma_{-i}}(D_\delta) \leq \sum_{B\in\hat{C}_i}\sum_{s\in\mathbb{S}} 2e^{-2|B|\delta^2} = 2|\mathbb{S}|\sum_{B\in\hat{C}_i} e^{-2|B|\delta^2} \leq 2|\mathbb{S}||\hat{C}_i|e^{-2d(G)_i\delta^2}.$$

■

Lemma 5. *If $\tau_{-i} \in F_{\hat{C}_i}(\sigma_{-i})$ then the bounds of Lemma 4 hold when $\mathbb{P}_{\sigma_{-i}}$ is replaced with $\mathbb{P}_{\tau_{-i}}$. In particular, $|\mathbb{P}_{\sigma_{-i}}(D_{\delta,\rho}) - \mathbb{P}_{\tau_{-i}}(D_{\delta,\rho})| \leq \frac{2|\mathbb{S}|e^{-2|N|\rho\delta^2}}{\rho}$ and $|\mathbb{P}_{\sigma_{-i}}(D_\delta) - \mathbb{P}_{\tau_{-i}}(D_\delta)| \leq 2|\mathbb{S}||\hat{C}_i|e^{-2d(G)_i\delta^2}$.*

Proof. $\tau_{-i} \in F_{\hat{C}_i}(\sigma_{-i})$ means that $\sigma^B(s) = \tau^B(s)$ for every $B \in \hat{C}_i$ and for every $s \in \mathbb{S}$. Thus, Lemma 4 may be applied without any change to $\mathbb{P}_{\tau_{-i}}$. It follows that both $\mathbb{P}_{\tau_{-i}}(D_{\delta,\rho})$ and $\mathbb{P}_{\sigma_{-i}}(D_{\delta,\rho})$ are in the interval $\left[0, \frac{2|\mathbb{S}|e^{-2|N|\rho\delta^2}}{\rho}\right]$. Similarly, $\mathbb{P}_{\tau_{-i}}(D_\delta)$ and $\mathbb{P}_{\sigma_{-i}}(D_\delta)$ are both in the interval $\left[0, 2|\mathbb{S}||\hat{C}_i|e^{-2d(G)_i\delta^2}\right]$. ■

Lemma 6. *Fix two profiles of actions $\underline{s}, \underline{s}' \in S$. If $s_{-i} \in F_{\hat{C}_i}(s'_{-i})$ then $u_i(\underline{s}) = u_i(\underline{s}')$.*

Proof. $s_{-i} \in F_{\hat{C}_i}(s'_{-i})$ means that $\#\{j \in B : s_j = s\} = \#\{j \in B : s'_j = s\}$ for every $s \in \mathbb{S}$ and for every $B \in \hat{C}_i$. Thus, for every $B \in \hat{C}_i$, there is a permutation of players' names in the set B which transforms the restriction of $\underline{s}$ to B to the restriction of $\underline{s}'$ to B. However, every such permutation of players in the set B can be achieved by a sequence of exchanges of pairs of players. By the definition of the partition $\hat{C}_i$, every such exchange doesn't affect the payoff of player i. The assertion follows. ■

Lemma 7. *If $s_{-i}, s'_{-i} \in S_{-i} \setminus D_\delta$ then $|u_i(s_i\ ;\ s_{-i}) - u_i(s_i\ ;\ s'_{-i})| \leq \delta|\mathbb{S}|M$ for every $s_i \in S_i$.*

Proof. Since both s_{-i} and s'_{-i} are not in D_δ it follows that $|\#\{j \in B : s_j = s\} - \#\{j \in B : s'_j = s\}| \leq 2\delta|B|$ for every $B \in \hat{C}_i$ and for every $s \in \mathbb{S}$. Thus, there is a profile of actions $s''_{-i} \in F_{\hat{C}_i}(s'_{-i})$ such that s''_{-i} is obtained from s_{-i} by no more than $\sum_{B\in\hat{C}_i}\sum_{s\in\mathbb{S}} \delta|B| = \delta|\mathbb{S}|(|N|-1)$ changes in players' actions. By the previous lemma, we have that $|u_i(s_i\ ;\ s_{-i}) - u_i(s_i\ ;\ s'_{-i})| = |u_i(s_i\ ;\ s_{-i}) - u_i(s_i\ ;\ s''_{-i})|$. By the diminishing effect assumption, the influence of some player $j \neq i$ changing her action on the utility function u_i is not greater than $\frac{M}{|N|}$. It follows that $|u_i(s_i\ ;\ s_{-i}) - u_i(s_i\ ;\ s''_{-i})| \leq \delta|\mathbb{S}|\,(|N|-1)\,\frac{M}{|N|} \leq \delta|\mathbb{S}|M$. ■

Lemma 8. *If $s_{-i}, s'_{-i} \in S_{-i} \backslash D_{\delta,\rho}$ then $|u_i(s_i\ ;\ s_{-i}) - u_i(s_i\ ;\ s'_{-i})| \leq M \left(\delta|\mathbb{S}| + \frac{\sum_{B\in H_\rho}|B|}{|N|}\right)$ for every $s_i \in S_i$.*

Proof. Repeat the same argument as in the previous lemma for every set $B \in E_\rho$. The total number of players which belong to one of the sets in H_ρ (the complement of E_ρ) is $\sum_{B\in H_\rho}|B|$. By the diminishing effect assumption the total influence of these players on the payoff of i is bounded by $\frac{M\sum_{B\in H_\rho}|B|}{|N|}$. The assertion follows. ■

Lemma 9. *Let Ω be a finite set, P, Q two probability measures on Ω, $X : \Omega \to \mathbb{R}$ a random variable and $\varepsilon > 0$. Let $A \subseteq \Omega$ be an event such that $|P(A) - Q(A)| \leq \varepsilon$, and assume that $r \leq X(\omega) \leq R$ for every $\omega \in A$ ($r \leq R$ are two constants). Then $\left|\sum_{\omega\in A} X(\omega)(P(\omega) - Q(\omega))\right| \leq R - r + \varepsilon \max(|r|, |R|)$.*

Proof. Denoting $\mathbb{E}_P$ ($\mathbb{E}_Q$) the expectation operator w.r.t. to the measure P (Q), one has

$$\begin{aligned}
&\left|\sum_{\omega\in A} X(\omega)(P(\omega) - Q(\omega))\right| = |P(A)\mathbb{E}_P(X|A) - Q(A)\mathbb{E}_Q(X|A)| \\
\leq\ & |P(A)\mathbb{E}_P(X|A) - Q(A)\mathbb{E}_P(X|A)| + |Q(A)\mathbb{E}_P(X|A) - Q(A)\mathbb{E}_Q(X|A)| \\
=\ & |P(A) - Q(A)||E_P(X|A)| + Q(A)|E_P(X|A) - E_Q(X|A)| \\
\leq\ & \varepsilon \max(|R|, |r|) + R - r.
\end{aligned}$$

■

Lemma 10. *If $\tau_{-i} \in F_{\hat{C}_i}(\sigma_{-i})$ then*

$$|u_i(s_i\ ;\ \sigma_{-i}) - u_i(s_i\ ;\ \tau_{-i})| \leq M|\mathbb{S}|\left(\delta + 6|\hat{C}_i|e^{-2d(G)_i\delta^2}\right)$$

for every $s_i \in S_i$.

Proof.

$$\begin{aligned}
|u_i(s_i\ ;\ \sigma_{-i}) - u_i(s_i\ ;\ \tau_{-i})| \ \leq\ & \left|\sum_{s_{-i}\in D_\delta} u_i(s_i\ ;\ s_{-i})\left(\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})\right)\right| \\
& + \left|\sum_{s_{-i}\in S_{-i}\backslash D_\delta} u_i(s_i\ ;\ s_{-i})\left(\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})\right)\right|.
\end{aligned}$$

The first sum can be estimated by

$$\begin{aligned}
&\left|\sum_{s_{-i}\in D_\delta} u_i(s_i\ ;\ s_{-i})\left(\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})\right)\right| \\
&\qquad \leq M \cdot \sum_{s_{-i}\in D_\delta} |\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})| \ \leq\ M \cdot \left(\mathbb{P}_{\sigma_{-i}}(D_\delta) + \mathbb{P}_{\tau_{-i}}(D_\delta)\right) \\
&\qquad \leq\ M \cdot 4|\mathbb{S}||\hat{C}_i|e^{-2d(G)_i\delta^2},
\end{aligned}$$

where the first inequality is due to the fact that $\Gamma(\mathbb{S})$ is uniformly bounded by M, and the third inequality is by Lemmas 4 and 5. In order to estimate the second sum we use Lemma 9 with $\Omega = S_{-i},\ P = \mathbb{P}_{\sigma_{-i}},\ Q = \mathbb{P}_{\tau_{-i}},\ X(\omega) = u_i(s_i\ ;\ \omega),\ A = S_{-i} \setminus D_\delta$ and $\varepsilon = 2|\mathbb{S}||\hat{C}_i|e^{-2d(G)_i\delta^2}$. Notice that, by Lemma 7, the utility $u_i(s_i\ ;\ \cdot)$ is bounded in an interval of length not larger than $\delta|\mathbb{S}|M$. Thus,

$$\left|\sum_{s_{-i}\in S_{-i}\setminus D_\delta} u_i(s_i\ ;\ s_{-i})\left(\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})\right)\right| \le \delta|\mathbb{S}|M + M\cdot 2|\mathbb{S}||\hat{C}_i|e^{-2d(G)_i\delta^2}.$$

Summing up the two inequalities gives the desired bound. ■

Lemma 11. *If $\tau_{-i} \in F_{\hat{C}_i}(\sigma_{-i})$ then*

$$|u_i(s_i\ ;\ \sigma_{-i}) - u_i(s_i\ ;\ \tau_{-i})| \le M\left(\delta|\mathbb{S}| + \frac{\sum_{B\in H_\rho}|B|}{|N|} + \frac{6|\mathbb{S}|e^{-2|N|\rho\delta^2}}{\rho}\right)$$

for every $s_i \in S_i$.

Proof. Similarly to the previous proof,

$$\begin{aligned}|u_i(s_i\ ;\ \sigma_{-i}) - u_i(s_i\ ;\ \tau_{-i})| &\le \left|\sum_{s_{-i}\in D_{\delta,\rho}} u_i(s_i\ ;\ s_{-i})\left(\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})\right)\right| \\ &+ \left|\sum_{s_{-i}\in S_{-i}\setminus D_{\delta,\rho}} u_i(s_i\ ;\ s_{-i})\left(\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})\right)\right|.\end{aligned}$$

By Lemmas 4 and 5,

$$\left|\sum_{s_{-i}\in D_{\delta,\rho}} u_i(s_i\ ;\ s_{-i})\left(\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})\right)\right| \le M\cdot\frac{4|\mathbb{S}|e^{-2|N|\rho\delta^2}}{\rho}.$$

The second sum is estimated by Lemma 9 with $\Omega = S_{-i},\ P = \mathbb{P}_{\sigma_{-i}},\ Q = \mathbb{P}_{\tau_{-i}},\ X(\omega) = u_i(s_i\ ;\ \omega),\ A = S_{-i}\setminus D_{\delta,\rho}$ and $\varepsilon = \frac{2|\mathbb{S}|e^{-2|N|\rho\delta^2}}{\rho}$. Notice that, by Lemma 8, the utility $u_i(s_i\ ;\ \cdot)$ is bounded in an interval of length not larger than $M\left(\delta|\mathbb{S}| + \frac{\sum_{B\in H_\rho}|B|}{|N|}\right)$. Thus,

$$\left|\sum_{s_{-i}\in S_{-i}\setminus D_{\delta,\rho}} u_i(s_i\ ;\ s_{-i})\left(\mathbb{P}_{\sigma_{-i}}(s_{-i}) - \mathbb{P}_{\tau_{-i}}(s_{-i})\right)\right| \le M\left(\delta|\mathbb{S}| + \frac{\sum_{B\in H_\rho}|B|}{|N|} + \frac{2|\mathbb{S}|e^{-2|N|\rho\delta^2}}{\rho}\right).$$

Summing up the two inequalities gives the desired bound. ■

The proof of Theorem 1 with condition (A1):
Assume that $\Gamma(\mathbb{S})$ satisfies (A1) and fix $\varepsilon > 0$. Choose $0 < \delta < \frac{\varepsilon}{6M|\mathbb{S}|}$. By (A1), there is $\rho > 0$ such that $\frac{\sum_{B\in H_\rho}|B|}{|N|} < \frac{\varepsilon}{6M}$ for every $G \in \Gamma(\mathbb{S})$ and for every $i \in N$. After

fixing such δ and ρ, let n_0 be large enough so that $\frac{6|\mathbb{S}|e^{-2n_0\rho\delta^2}}{\rho} < \frac{\varepsilon}{6M}$. Fix $G \in \Gamma(\mathbb{S})$ with $|N| > n_0$ and let σ be a CCE w.r.t. $\hat{C}$ in G.

Then, for every $i \in N$, there is a strategy profile $\tau_{-i} \in F_{\hat{C}_i}(\sigma_{-i})$ such that σ_i is a best response to τ_{-i}. It follows from Lemma 11 that, for every $i \in N$ and for every $s_i \in S_i$, $|u_i(s_i\ ;\ \sigma_{-i}) - u_i(s_i\ ;\ \tau_{-i})| \leq \frac{\varepsilon}{2}$. Thus,

$$u_i(s_i\ ;\ \sigma_{-i}) \leq u_i(s_i\ ;\ \tau_{-i}) + \frac{\varepsilon}{2} \leq u_i(\sigma_i\ ;\ \tau_{-i}) + \frac{\varepsilon}{2} \leq u_i(\sigma_i\ ;\ \sigma_{-i}) + \varepsilon.$$

The proof of Theorem 1 with condition (A2):
Assume that $\Gamma(\mathbb{S})$ satisfies (A2) and fix $\varepsilon > 0$. Fix $0 < \delta < \frac{\varepsilon}{4M|\mathbb{S}|}$. By (A2) there is n_0 large enough so that $6|\hat{C}_i|e^{-2d(G)_i\delta^2} < \frac{\varepsilon}{4M|\mathbb{S}|}$ for every $G \in \Gamma(\mathbb{S})$ with $|N| > n_0$ and for every $i \in N$. Fix $G \in \Gamma(\mathbb{S})$ with $|N| > n_0$ and let σ be a CCE w.r.t. $\hat{C}$ in G.

Then, for every $i \in N$, there is a strategy profile $\tau_{-i} \in F_{\hat{C}_i}(\sigma_{-i})$ such that σ_i is a best response to τ_{-i}. It follows from Lemma 10 that, for every $i \in N$ and for every $s_i \in S_i$, $|u_i(s_i\ ;\ \sigma_{-i}) - u_i(s_i\ ;\ \tau_{-i})| \leq \frac{\varepsilon}{2}$. Thus,

$$u_i(s_i\ ;\ \sigma_{-i}) \leq u_i(s_i\ ;\ \tau_{-i}) + \frac{\varepsilon}{2} \leq u_i(\sigma_i\ ;\ \tau_{-i}) + \frac{\varepsilon}{2} \leq u_i(\sigma_i\ ;\ \sigma_{-i}) + \varepsilon.$$

7.2. **Proof of Theorem 2.** Assume that $\hat{x} \in \hat{P}$ is a CCE w.r.t. a categorization profile $C = \{C_t\}_{t\in T}$ which satisfy the conditions of the theorem. For every player $t \in T$, the payoff function $u(t, \cdot)$ is constant on the set $F_{C_t}(\hat{x})$. Also, it is clear that $\hat{x} \in F_{C_t}(\hat{x})$. Thus, since, for almost every $t \in T$, $\hat{x}(t)$ is a best response to some $\hat{y} \in F_{C_t}(\hat{x})$, it follows that, for almost every $t \in T$, $\hat{x}(t)$ is also a best response to $\hat{x}$. This means that $\hat{x}$ is also a Nash equilibrium.

7.3. **Proof of Theorem 3.** Let $\Pi_t : Y^T \to Y$ be the projection function on the coordinate t. Since we work with the product topology in Y^T it will be sufficient to show that $\Pi_t(\tilde{U})$ is dense in Y for every $t \in T$. Since $\Pi_t(\tilde{U})$ is independent of t we denote $\mathcal{A} = \Pi_t(\tilde{U})$.

Recall that $\mathcal{A}$ is the set of all continuous functions $v : \hat{P} \to I\!R^n$ with the property that there exists a finite measurable partition R of T such that $v(\hat{x}) = v(\hat{y})$ whenever $\int_B \hat{x} d\lambda = \int_B \hat{y} d\lambda$ for every $B \in R$. In order to prove that $\mathcal{A}$ is dense in Y it is sufficient to do so for each one of the n coordinates of v separately. Thus, with abuse of notation, we let $Y = \{v : \hat{P} \to I\!R \mid v \text{ is continuous}\}$ and $\mathcal{A} \subseteq Y$ is the class of (real valued) functions which depend only on the integral of the T-strategy over a finite number of sets in T.

Claim 1. *$\mathcal{A}$ is a vector subspace of Y.*

Proof. Let $v_1, v_2 \in \mathcal{A}$. Then there are finite partitions R_1, R_2 of T such that v_1 depends only on $\left\{\int_B \hat{x} d\lambda\right\}_{B\in R_1}$ and v_2 depends only on $\left\{\int_B \hat{x} d\lambda\right\}_{B\in R_2}$. let R be a finite partition which is finer than both R_1 and R_2. If $\int_B \hat{x} d\lambda = \int_B \hat{y} d\lambda$ for every $B \in R$ then $\int_B \hat{x} d\lambda = \int_B \hat{y} d\lambda$ for every $B \in R_1 \cup R_2$. Thus, for such $\hat{x}, \hat{y} \in \hat{P}$,

$v_1(\hat{x}) + v_2(\hat{x}) = v_1(\hat{y}) + v_2(\hat{y})$. This implies that $v_1 + v_2 \in \mathcal{A}$. Finally, if $\alpha \in I\!R$ then obviously $\alpha v_1 \in \mathcal{A}$. ■

Claim 2. *$\mathcal{A}$ is a subalgebra of (the algebra) Y. Moreover, $\mathcal{A}$ contains the constant functions.*

Proof. To prove that if $v_1, v_2 \in \mathcal{A}$ then $v_1 \cdot v_2 \in \mathcal{A}$ repeat the argument of Claim 1. Also, it is clear that every constant function is in $\mathcal{A}$. ■

Claim 3. *$\mathcal{A}$ separates points of $\hat{P}$. That is, for every $\hat{x} \neq \hat{y} \in \hat{P}$ there is $v \in \mathcal{A}$ such that $v(\hat{x}) \neq v(\hat{y})$.*

Proof. Fix $\hat{x} \neq \hat{y} \in \hat{P}$. For $i = 1, \ldots, n$ define the sets $B_i^1 = \{t \in T \ : \ \hat{x}_i(t) > \hat{y}_i(t)\}$ and $B_i^2 = \{t \in T \ : \ \hat{x}_i(t) < \hat{y}_i(t)\}$. $\hat{x} \neq \hat{y}$ implies that there is $1 \leq i \leq n$ such that $\lambda(B_i^1) > 0$ or $\lambda(B_i^2) > 0$. Assume w.l.o.g. that $\lambda(B_1^1) > 0$. Consider the function $v \in Y$ defined by $v(\hat{z}) = \int_{B_1^1} \hat{z}_1 d\lambda$ for every $\hat{z} \in \hat{P}$.

First, notice that $v \in \mathcal{A}$ since it only depends on the integral of the T-strategy over the set B_1^1. Also, by construction, $\hat{x}_1(t) > \hat{y}_1(t)$ for every $t \in B_1^1$. Since B_1^1 has a positive measure it must be that $\int_{B_1^1} \hat{x}_1 d\lambda > \int_{B_1^1} \hat{y}_1 d\lambda$. Thus, $v(\hat{x}) > v(\hat{y})$ and the claim is proved. ■

Finally, recall that $\hat{P}$ is a compact set. By the previous claims the conditions of the Stone-Weierstrass theorem (see e.g. Ha, 2006, Theorem 5.4.1 in page 398) hold. Thus, we can conclude that $\mathcal{A}$ is dense in Y. This proves the theorem.

7.4. **Proof of Proposition 1.** First, it is clear that g is symmetric around $\frac{1}{4}$, so $g(\alpha) = g(\frac{1}{2} - \alpha)$ for every $\alpha \in [0, \frac{1}{4}]$. It is therefore sufficient to compute g on the interval $[0, \frac{1}{4}]$.

Fix $\frac{1}{8} \leq \alpha \leq \frac{1}{4}$ and consider the T-strategy where all type 1 players choose e_1 and all type 2 players e_2. Obviously, the efficiency of such profile is 0. We claim that this is a CCE. To see this, let F' be a set of type 1 females with measure α, and let M' be a set of type 2 males with measure α. Then the T-strategy $\hat{Y}$ defined by $\hat{Y}(t) = e_1$ for $t \in \big([0, \frac{1}{2}) \setminus (M \cup F')\big) \cup M' \cup \big(F \cap [\frac{1}{2}, 1]\big)$ and $\hat{Y}(t) = e_2$ otherwise is a consistent conjecture for every player. The best response for such a belief is e_1 for type 1 players and e_2 for type 2 players.

Next, consider the case where $0 \leq \alpha \leq \frac{1}{8}$. Let F' be a set of type 1 females with measure $\frac{1}{4} + \alpha$, and let M' be a set of type 2 males with measure $\frac{1}{4} + \alpha$. Define $\hat{x}$ by $\hat{x}(t) = e_1$ for $t \in F' \cup \big(M \cap [0, \frac{1}{2})\big) \cup \big([\frac{1}{2}, 1] \setminus (F \cup M')\big)$ and $\hat{x}(t) = e_2$ otherwise. We have $p_1(\hat{x}) = \frac{1}{4} + 2\alpha$ and $p_2(\hat{x}) = \frac{1}{4} - 2\alpha$ so $eff(\hat{x}) = 4(\frac{1}{4} + 2\alpha)(\frac{1}{4} - 2\alpha) = \frac{1}{4} - 16\alpha^2$.

To see that $\hat{x}$ is a CCE, consider the following T-strategy $\hat{y}$. The set $M \cap [0, \frac{1}{2})$ of type 1 males play e_2 and not e_1 as in $\hat{x}$. Instead, a set of measure α of type 2 males who played e_2 according to $\hat{x}$ is switching to e_1. Similarly, the set of type 2 females is switching from e_2 in $\hat{x}$ to e_1 in $\hat{y}$ and a set of measure α of type 1 females is switching from e_1 to e_2. It is clear that $\hat{y} \in F_{C_t}(\hat{x})$ for every player t. Moreover,

$p_1(\hat{y}) = \frac{1}{4}$ and $p_2(\hat{y}) = \frac{1}{4}$ so for every player both actions give the same payoff (thus both are best responses to $\hat{y}$).

Finally, notice that the efficiency of a T-strategy $\hat{x}$ is decreasing in the distance of $p_1(\hat{x})$ and $p_2(\hat{x})$ from $\frac{1}{4}$. It can therefore be verified that there is no CCE which yields a lower social efficiency.

7.5. **Proof of Theorem 4.** For a partition R of T and a player $t \in T$, let $R(t)$ denotes the element of R which contains t. If $\hat{x} \in \hat{P}$ is a T-strategy, let $f_R(\hat{x})$ denote the T-strategy defined by $f_R(\hat{x})(t) = \frac{1}{\lambda(R(t))} \int_{R(t)} \hat{x} d\lambda$. Notice that always $f_R(\hat{x}) \in F_R(\hat{x})$. Before proving the theorem we need the following lemma.

Lemma 12. *For every finite partition R of T the map $f_R : \hat{P} \to \hat{P}$ is continuous in the L_1 weak topology on $\hat{P}$.*

Proof. Since f_R is linear, continuity in the weak topology is equivalent to continuity in the metric topology (see Dunford and Schwartz, 1988, Theorem 15, page 422). The latter is guaranteed by the inequality $\int_T |f_R(\hat{x}) - f_R(\hat{y})| d\lambda \leq \int_T |\hat{x} - \hat{y}| d\lambda$, which follows from the fact that f_R is the projection to the subspace of T-strategies which are measurable w.r.t. the partition R. ■

Fix a game u and a categorization profile $C = \{C_t\}_{t \in T}$, with C_t coarser than some C^* for every $t \in T$. By Lemma 2, it is sufficient to show that there is a pure CCE w.r.t. the categorization profile $\{C_t = C^*\}_{t \in T}$.

Consider the map $f_{C^*} : \hat{P} \to \hat{P}$. By Lemma 12 it is continuous. It follows that the game $\bar{u}$ defined by $\bar{u}(t, \hat{x}) = u(t, f_{C^*}(\hat{x}))$ satisfies assumptions (1) and (2) of subsection 2.2. Moreover, $\bar{u}$ is an anonymous game, so by Theorem 2 in Schmeidler (1973, see also Remark 2 there) there is a pure Nash equilibrium in $\bar{u}$.

Denote by $\hat{x}$ one such pure equilibrium of $\bar{u}$. We claim that $\hat{x}$ is CCE in u. Indeed, for almost every $t \in T$, $\hat{x}(t) \cdot \bar{u}(t, \hat{x}) \geq p \cdot \bar{u}(t, \hat{x})$ for all $p \in P$. For every such t, by definition of $\bar{u}$, $\hat{x}(t) \cdot u(t, f_{C^*}(\hat{x})) \geq p \cdot u(t, f_{C^*}(\hat{x}))$ for all $p \in P$.

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