Dynamic Models of Social Interactions: Identification and Characterization

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Examples of Previous Work:

- the decision of a teen to commit a criminal act or to drop out of school; Case and Katz 1991, Glaeser, Sacerdote and Scheinkman 1996, and (Crane 1991);
- out-of-wedlock births; Crane 1991
- smoking habits; Jones 1994;
- neighborhood effects; Topa 2001, Krosnick and Judd 1982;
- local technological complementarities; Ellison and Fudenberg 1993, Durlauf 1993;
- urban agglomeration, segregation; Benabou 1993, Schelling 1972;
- spread of ideas and beliefs in the microstructure of financial markets; Brock 1993, Horst 2005.

- High variance across space and time and local spatial correlation is difficult to justify with changes in the fundamentals (eg. Crime, High School Achievement,...)

- The importance of social interactions for policy analysis relies on the fact that when social interactions are quantitatively important, policy interventions on single agents might have large effects - **social multiplier**.

The theoretical literature:

- economies with additive quadratic preferences, extreme value distributed shocks, and symmetric interaction effects, introduced by Blume 1993 and Brock 1993 (see also Brock and Durlauf 2001);

- economies with a finite number of agents, studied by Glaeser and Scheinkman 2000.

- Analysis of dynamic economies confined to the case of backward looking myopic dynamics, either as a simple explicit dynamic process with random sequential choice (Brock and Durlauf 2001, or as an equilibrium selection procedure (Glaeser and Scheinkman 2000, Blume and Durlauf 1998).

- In this paper, we contribute to this literature by extending the class of economies under study in three fundamental dimensions:

  - We study different interaction structures (only in static economies for the moment!)

  - We study the *rational expectations equilibria* of dynamic economies; agents anticipate the future effects of their present choices.

  - We study different information structures; e.g., agents having complete information only regarding agents in their own social group.

- The focus of Bisin-Horst-Özgür (2005) is on existence. The focus of the present work is on characterization and identifiability.

The Microeconomic Setup in a Static Model

- an *economy* is a vector $\mathcal{E} = (X, \Theta, u, \nu, N)$
- countably infinite set $\mathbb{A} := \mathbb{Z}$ of *agents*
- agent $a \in \mathbb{A}$ takes an *action* $x^a \in X \subset \mathbb{R}$
- $\theta^a \in \Theta$ denotes the random *type* of $a \in \mathbb{A}$; θ^a $(a \in \mathbb{A})$ are iid with law ν on $\mathbb{R}$
- agent $a \in \mathbb{A}$ interacts with agents $b \in N(a) \subset \mathbb{A}$
- agent $a \in \mathbb{A}$ enjoys the *utility* $u(x^a, \mathbf{x}^{N(a)}, \theta^a)$ where $u : X^{|N(a)|+1} \times \Theta \to \mathbb{R}$ is continuous and strictly concave in its first argument.
- Prior to his choice, each agent $a \in \mathbb{A}$ observes the realization of his own type θ^a as well as the realizations of the types θ^b of the agents $b \in M(a)$. Here $M(a) \subset \mathbb{N}^* \cup \{\infty\}$.
- For simplicity of exposition, consider the special case in which $N(a) = \{a+1\}$ and $M(a) = \{a+1, \ldots, a+N\}$ agent $a \in \mathbb{A}$ observes either

$$N = \infty \quad \textit{(complete information)}$$

or

$$|N| < \infty \quad \textit{(incomplete information)}$$

- Let $\mathcal{S} = (X, \Theta, u, \nu, N)$ be a static economy with local interactions.

  1. If $\mathcal{S}$ is an economy with complete information, $N = \infty$, then an *equilibrium* is a family $(g^{*a})_{a\in\mathbb{A}}$ of measurable mappings $g^{*a} : \boldsymbol{\Theta}^0 \to X$ such that

  $$g^{*a}\left(T^a\theta_N\right) = \arg\max_{x^a\in X} u\left(x^a, g^{*a+1}\left(T^{a+1}\theta_N\right), \theta^a\right) \tag{1}$$

     for all $a \in \mathbb{A}$.

  2. If $\mathcal{S}$ is an economy with incomplete information, $N < \infty$, then an *equilibrium* is a family $(g^{*a})_{a\in\mathbb{A}}$ of measurable mappings $g^{*a} : \Theta^{N+1} \to X$ such that

  $$g^{*a}\left(T^a\theta_N\right) = \arg\max_{x^a\in X} \int_\Theta u\left(x^a, g^{*a+1}\left(\theta^{a+1}, \ldots, \theta^{a+N}, \theta\right), \theta^a\right)\nu(d\theta) \tag{2}$$

     for all $a \in \mathbb{A}$.

  An equilibrium $(g^{*a})_{a\in\mathbb{A}}$ for an economy $\mathcal{S}$ is *symmetric* if

  $$g^{*a} = g^* \circ T^a \tag{3}$$

  for some mapping g^* and each $a \in \mathbb{A}$.

One-sided Interactions, Complete Information

- Conformity preferences:

  $u(x^a, x^{a+1}, \theta^a) := -\alpha_1(x^a - \theta^a)^2 - \alpha_2(x^a - x^{a+1})^2.$

  for $\alpha_1\ \alpha_2 \geq 0$.

- A generic agent a's optimal policy, in a symmetric equilibrium, solves

$$g(T^a\theta_N) = \arg\max_{x^a \in X} u(x^a,\ g(T^{a+1}\theta_N), \theta^a) \tag{4}$$

- A simple iterative induction argument shows that we have

$$g(T^a\theta_N) = \beta_1 \sum_{i=a}^{\infty} \beta_2^{i-a}\theta^i$$

One-sided Interactions, Incomplete Information - In Social Group

- Suppose agent $a \in \mathbb{A}$ observes

$$\theta^a,\ \theta^{a+1}, \cdots, \theta^{a+N}$$

but not

$$\theta^{a+N+1}, \ldots$$

- Agent a solves the problem in (4) with $T^a\theta_N = \{\theta^a, \theta^{a+1}, \cdots, \theta^{a+N}\}$

$$g(T^a\theta_N) = \arg\max_{x^a \in X} \int u(x^a,\ g(T^{a+1}\theta_N), \theta^a)\nu(d\theta^{a+N+1}) \tag{5}$$

- An induction argument similar to the one before gives

$$g(T^a\theta_N) = \beta_1 \sum_{i=a}^{a+N} \beta_2^{i-a}\theta^i + \beta_1 \frac{\beta_2^{N+1}}{1-\beta_2}\,\overline{\theta}$$

Two-sided Interactions - Complete/Incomplete Information

- Suppose agent $a \in \mathbb{A}$ has conformity preferences:

$$-\alpha_1(x^a - \theta^a)^2 - \frac{1}{2}\alpha_2(x^a - x^{a-1})^2 - \frac{1}{2}\alpha_2(x^a - x^{a+1})^2$$

- Suppose that agent $a \in \mathbb{A}$ observes only

$$\theta^{a-N}, \cdots, \ \theta^{a-1}, \ \theta^{a+1}, \cdots \theta^{a+N}$$

  Let $\theta_N = \{\theta^{-N}, \cdots, \theta^0, \cdots, \theta^N\}$. The policy function of this agent will be of the form

$$g(T^a \theta_N) = A \sum_{i=1}^{N} \delta^i (\theta^{a+i} + \theta^{a-i}) + B\theta^a + C\overline{\theta}$$

  where

$$\begin{aligned} \delta &= \frac{1 - \sqrt{1 - \beta_2^2}}{\beta_2} \\ A &= \frac{\beta_1 \beta_2}{\delta \left[2 - \beta_2(\delta + \beta_2)\right]} \\ B &= \beta_1 + \beta_2 A \delta \\ C &= \frac{\beta_2 A \delta^N}{1 - \beta_2} \\ \beta_i &= \frac{\alpha_i}{\alpha_1 + \alpha_2} \end{aligned}$$

- The complete information case is the limit for $N \to \infty$.

Information and Interaction

- What is the effect of information on the relative weights of the policy function?

- For an economy where agent a has information about

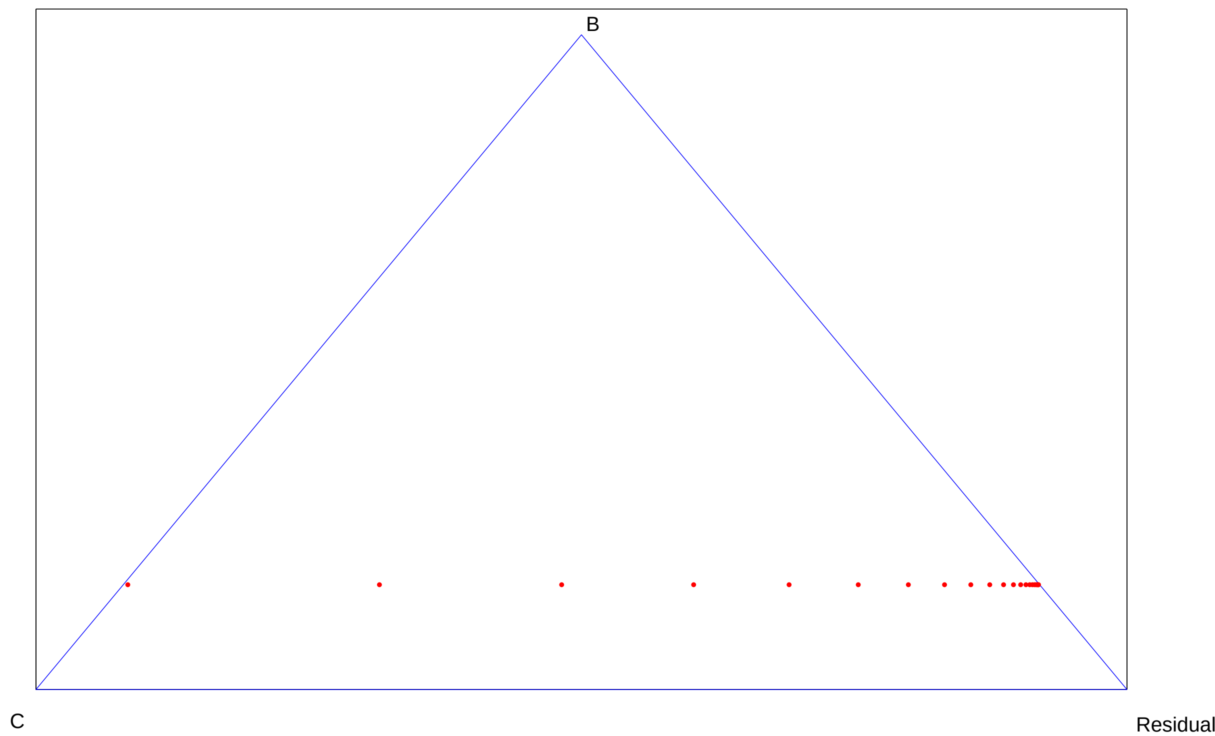
  $$\theta_N = \{\theta^{a-N}, \cdots, \theta^{a-1}, \theta^{a}, \theta^{a+1}, \cdots, \theta^{a+N}\}$$

  his policy function is of the form

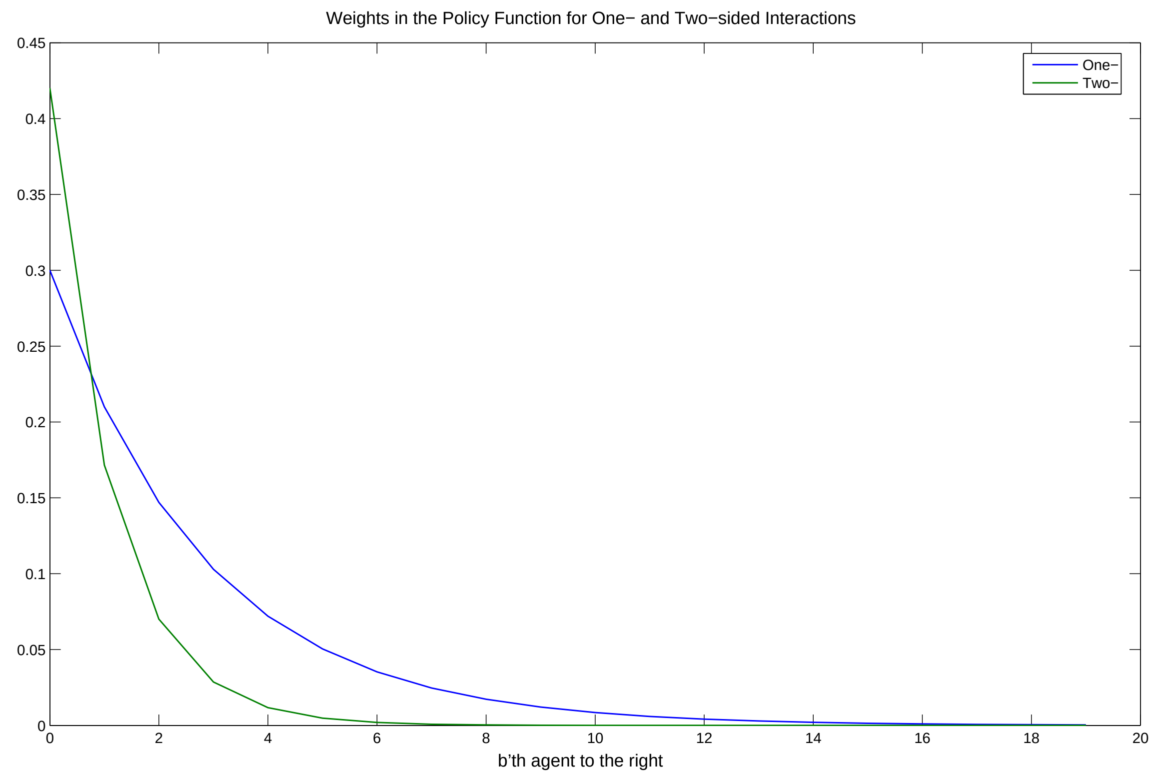
  $$g(T^a \theta_N) = A \sum_{i=1}^{N} \delta^i (\theta^{a+i} + \theta^{a-i}) + B\theta^a + C\overline{\theta}$$

  with $N = \infty$ in the case of complete information.

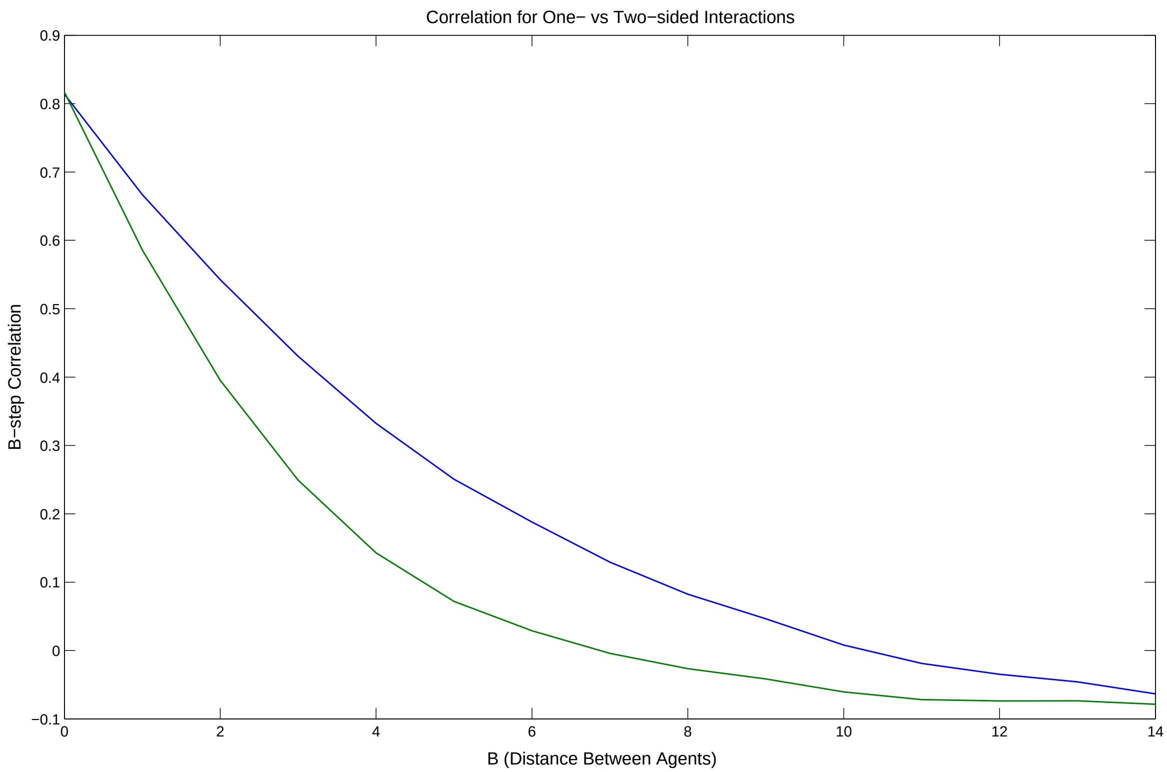
- We represent the policy function on the simplex with vertices: $C, B, 2A\sum_{i=1}^{N} \delta^i$



$N \to \infty$, $\alpha_1 = .05$ and $\alpha_2 = .95$



Complete Info, $\alpha_1 = .3$ and $\alpha_2 = .7$



$N = 20$, $\alpha_1 = .2$ and $\alpha_2 = .8$

Identifiability

- Are social interaction models identifiable? that is, Can the structural parameters of the economy be recovered from observable behavior? and in particular, Are predictions of the model distinct from those of models (without social interaction) whose allocations are instead Pareto efficient.

- It is well known since the work of Manski (1993) that in fact identifiability is problematic for social interaction models. Two main problems arise.

  The *reflection problem*:' regressing x^a on x^{a+1} is problematic if x^{a+1} depends directly on x^a or if they both depend on some aggregate measure of actions (e.g., on the average action $E(x^a)$ in the population).

  Identifiability might fail in the presence of unobservable heterogeneity when such heterogeneity is determined by a random factor which is spatially correlated.

- The reflection problem can be successfully dealt with by studying the stochastic properties of the equilibrium configuration of action, $\{x^0, \ldots, x^a, \ldots\}$, e.g., the variance of the average action across different sub-groups of the population, as in Glaeser-Scheinkman 1999 and Graham-Hahn 2003.

- We then abstract from two-sided interactions and from the possible dependence of actions on means, so that the reflection problem per se does not arise.

  We concentrate instead on the identifiability problems which arise if each agent action depends on some unobservable factor which is spatially correlated.

- Related literature:

  Glaeser-Sacerdote-Scheinkman 1996 explicitly assume no spatially correlated unobserved heterogeneity in their analysis of crime, as do Glaeser-Scheinkman 1999 when discussing their approach to identification of social interactions;

  Graham-Hahn 2003 also assume no spatially correlated unobserved heterogeneity (implicitly, by requiring that an instrument exists which is orthogonal to correlated group effects; see the discussion at the end of Section 3.2, page 7.)

  Explicit modeling of endogenous selection provide additional restrictions on equilibrium configurations; see Evans-Oates-Schwab 1992, Ioannides-Zabel 2002, Zanella 2004.

- Representation of the problems that spatially correlated unobserved heterogeneity generates for identifiability:

  Consider our simple economy with one-sided interaction and complete information. Agent a's policy function is

  $$x^a = \gamma x^{a+1} + (1-\gamma)\theta^a.$$

  Consider now an equivalent model in which each agent a's choice, perhaps because of endogenous selection, only depends on the realization of an unobservable factor u^a

  $$x^a = \eta u^a,$$

  which is however spatially correlated,

  $$u^a = \rho u^{a+1} + \epsilon^a,$$

  for some i.i.d. zero-mean process ϵ^a.

  With

  $$\rho = \gamma, \ \eta = 1-\gamma$$

  the economy with unobserved spatial correlation and no social interaction is observationally equivalent to the economy with social interaction, and identifiability fails.

Two Positive Identifiability Results

- Identifiability due to incomplete information.
  - Suppose it is known to the econometrician that, if social interactions obtain they are characterized by incomplete information. In particular (but this is inessential for the result), suppose that each agent a can only observe the realization of the preference shocks of the agents whom he interacts with (that is, of agent $a+1$ in our economy).
  - In this case, agent a's policy function is

$$x^a = \delta_1 \theta^{a+1} + \delta_0 \theta^a;$$

while, with correlated unobserved spatial heterogeneity, his equilibrium action is determined by

$$x^a = \eta u^a, \;\; u^a = \rho u^{a+1} + \epsilon^a.$$

  - Identifiability then obtains from the properties of the equilibrium configuration: In particular,

$$E(x^a x^{a+2}) = 0$$

in the economy with social interactions, while

$$E(x^a x^{a+2}) > 0$$

if equilibrium actions are determined by correlated unobserved spatial heterogeneity.

- Identifiability due to observable heterogeneity.
  - Suppose it is known to the econometrician that the policy function of each agent a depends on an observable factor y. Suppose information is complete.
  - In this case agent a's policy function is

$$x^a = \gamma x^{a+1} + (1-\gamma)\theta^a + \sigma y^a;$$

    while, with correlated unobserved spatial heterogeneity, his equilibrium action is determined by

$$x^a = \eta u^a + \sigma y^a, \;\; u^a = \rho u^{a+1} + \epsilon^a.$$

  - Once again, identifiability obtains from the properties of the equilibrium configuration:

$$E\left(x^a y^{a+1} \;\mid\; y^a, x^{a+1}\right) = 0$$

    in the economy with social interactions, while

$$E\left(x^a y^{a+1} \;\mid\; y^a, x^{a+1}\right) < 0$$

    if equilibrium actions are determined by correlated unobserved spatial heterogeneity.

Dynamic Economies with Local Interactions

- The theoretical literature on dynamic economies with local interactions has so far concentrated on models with *ad hoc* myopic dynamics.

- In this case the resulting equilibrium process for $\{x_t^a\}_{t \in \mathbb{N}}$ has been intensively investigated in the mathematical literature on interacting particle systems. Conditions for asymptotic stability of these processes have been established under suitable weak interaction and average contraction conditions; see e.g., Liggett 1985, Kindermann and Snell 1980 or Föllmer and Horst 2001.

- In this paper, we instead study economies with forward looking agents and consider rational expectations equilibrium dynamics.

  At time t agent $a \in \mathbb{A}$ takes *action* $x_t^a \in X$

  Random *type* of $a \in \mathbb{A}$ at time t is $\theta_t^a \in \Theta$

  Agent $a \in \mathbb{A}$ interacts with agent $a + 1 \in \mathbb{A}$

  Period t instantaneous *utility* of agent $a \in \mathbb{A}$:

$$u(x_{t-1}^a, x_t^a, x_t^{a+1}, \theta_t^a)$$

  In particular we restrict to quadratic preferences:

$$-\alpha_1 \left(x_t^a - x_{t-1}^a\right)^2 - \alpha_2 \left(x_t^a - \theta_t^a\right)^2 - \alpha_3 \left(x_t^a - x_t^{a+1}\right)^2$$

  Information:

  - Complete Information of Present Shocks; Complete Information of Past Actions: At time t, agent $a \in \mathbb{A}$ observes

$$(x_{t-1}^b)_{b \geq a} \quad \text{and } (\theta_t^b)_{b \geq a}$$

  - Incomplete Information of Present Shocks; Complete Information of Past Actions: At time t, agent $a \in \mathbb{A}$ observes

$$(x_{t-1}^b)_{b \geq a} \text{ and } \quad (\theta_t^b)_{b=a+1}^{a+N}$$

  Discount factor is $\beta < 1$.

- An *economy* is a vector $\mathcal{E} = (X, \Theta, u, \nu, \beta)$ Agent 0 *believes* that agent $a \geq 1$ chooses

  $$x^a = g^a(x_t, \theta_t^a) = g(\{x_t^b\}_{b \geq a}, \theta^a) \in X$$

  according to a decision rule $g : X^{\mathbb{N}} \times \Theta \to X$.

- The conditional distribution of x_{t+1}^a is then

  $$\pi_g^a(x_t; \cdot) := \pi_g(T^a x_t; \cdot) = \int \delta_{g(T^a x_t, \theta)}(\cdot) \nu(d\theta).$$

- The conditional distribution of the configuration $\{x_{t+1}^a\}_{a \geq 1}$ takes the product form

  $$K_g(x_t; \cdot) = \prod_{a \geq 1} \pi_g(T^a x_t; \cdot).$$

- The *maximization problem* of agent $0 \in \mathbb{A}$ is

  $$\max_{\{x_t^0\}} \sum_{t \in \mathbb{N}} \beta^t \int \int u(x_{t-1}^0, x_t^0, y^1, \theta^0) K_g^t(x, dy) \nu(d\theta^0).$$

- The conditional optimal action $g^*(x, \theta^0)$ is given by the solution to a dynamic program:

  $$g^*(x, \theta^0) = \arg\max_{y^0 \in X} \Big\{ \int u(x^0, y^0, y^1, \theta^0) \pi_g(x; dy^1) + \beta \int \int V_g(y, \theta^0) K_g(x; dy) \nu(d\theta) \Big\}$$

- Any fixed point of the operator

  $$\widehat{V} g(x, \theta^0) = \arg\max_{y^0 \in X} \Big\{ \int u(x^0, y^0, x^1, \theta^0) \pi_g(x; dy^1) + \beta \int \int V_g(y, \theta^0) K_g(x; dy) \nu(d\theta) \Big\}$$

  defines a symmetric equilibrium.

- **Lemma** Assume that the choice map g is continuous. Under our assumptions on the utility function u, the functional fixed point equation has a unique bounded and continuous solution V_g on $\mathbf{X^0} \times \Theta$. Moreover, the map $V_g(\cdot, Tx_{t-1}, \theta_t^0)$ is strictly concave on X and there exists a unique continuous policy function $\widehat{g}_g : \mathbf{X^0} \times \Theta \to X$ that satisfies

$$\begin{aligned} \widehat{g}_g \left(x_{t-1}, \theta_t^0 \right) \quad = \quad & \operatorname{arg\,max}_{x_t^0 \in X} \left\{ \int u \left(x_{t-1}^0, x_t^0, y_t^1, \theta_t^0 \right) \pi_g(Tx_{t-1}; dy_t^1) \right. \\ & \left. + \beta \int V_g(x_t^0, \widehat{x}_t, \theta^1) \Pi_g(Tx_{t-1}; d\widehat{x}_t) \nu(d\theta^1) \right\}. \end{aligned}$$

- **Definition** For $C > 0$, let

$$L_+^C := \{ \mathbf{c} = (c_a)_{a \in \mathbb{N}} \ : \ c_a \geq 0, \ \sum_{a \in \mathbb{A}} c_a \leq C \}$$

denote the class of all non-negative sequences whose sum is bounded from above by C. A sequence $\mathbf{c} \in L_+^C$ will be called a *correlation pattern* with total impact C.

Each correlation pattern $\mathbf{c} \in L_+^C$ gives rise to a metric

$$d_{\mathbf{c}}(x, y) := \sum_{a \in \mathbb{N}} c_a |x^a - y^a|$$

that induces the product topology on $\mathbf{X^0}$. Thus, $(d_{\mathbf{c}}, \mathbf{X^0})$ is a compact metric space. In particular, the class

$$\mathrm{Lip}_{\mathbf{c}}^C := \{ f : \mathbf{X^0} \to \mathbb{R} : |f(x) - f(y)| \leq d_{\mathbf{c}}(x, y) \}$$

of all functions $f : \mathbf{X^0} \to \mathbb{R}$ which are Lipschitz continuous with constant 1 with respect to the metric $d_{\mathbf{c}}$ is compact in the topology of uniform convergence.

- **Theorem** Assume that there exists $C < \infty$ such that the following holds:

  1. For any $\mathbf{c} \in L_{+}^{C}$, for all $\theta^0 \in \Theta$ and for each choice function $g(\cdot, \theta^0) \in \mathrm{Lip}_{\mathbf{c}}^{C}$, there exists $F(\mathbf{c}) \in L_{+}^{C}$ such that the unique policy function $\widehat{g}_g(\cdot, \theta^0)$ which solves the dynamic program above, is Lipschitz continuous with respect to the metric $d_{F(\mathbf{c})}$ uniformly in $\theta^0 \in \Theta$.

  2. The map $F : L_{+}^{C} \to L_{+}^{C}$ is continuous.

  3. We have $\lim_{n\to\infty} \|\widehat{g}_{g_n}(\cdot, \theta^0) - \widehat{g}_g(\cdot, \theta^0)\|_\infty = 0$ if $\lim_{n\to\infty} \|g_n - g\|_\infty = 0$.

  Then the dynamic economy with local interactions has a symmetric Markov perfect equilibrium g^* and the function $g^*(\cdot, \theta^0)$ is Lipschitz continuous uniformly in θ^0.

- Our dynamic economy with conformity preferences satisfies the assumptions of the theorem. Hence a symmetric Markov Perfect Equilibrium exists and the optimal policy functions can be fully characterized.

Characterization

- An equilibrium is characterized by the following policy functions.
  - Complete Information of Present Shocks; Complete Information of Past Actions:

    The policy function of a generic agent a is given by

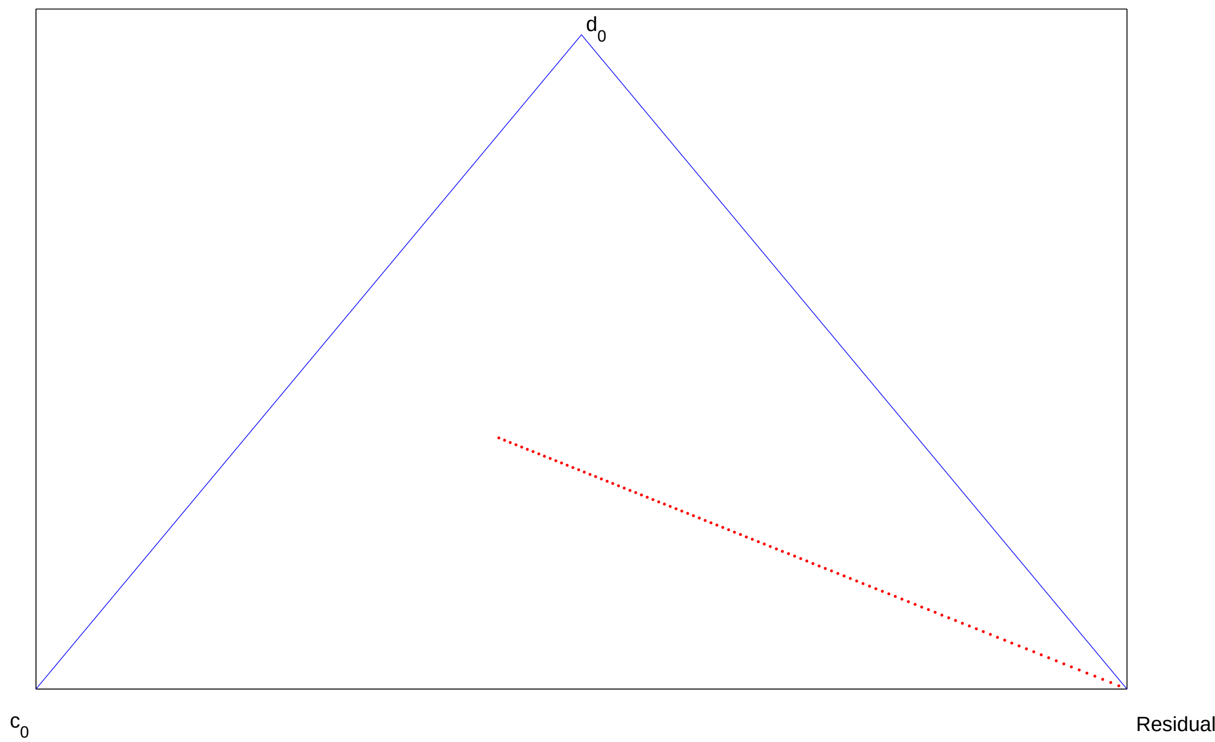
    $$g^*(T^a\, x_{t-1},\ T^a\, {}_t\theta_N) = \sum_{i=0}^{\infty} c_i\, x_{t-1}^{a+i} + \sum_{j=0}^{\infty} d_j\, \theta_t^{a+j}$$

    where ${}_t\theta_N = \{\ldots, \theta_t^0, \ldots,\}$
  - Incomplete Information of Present Shocks; Complete Information of Past Actions:

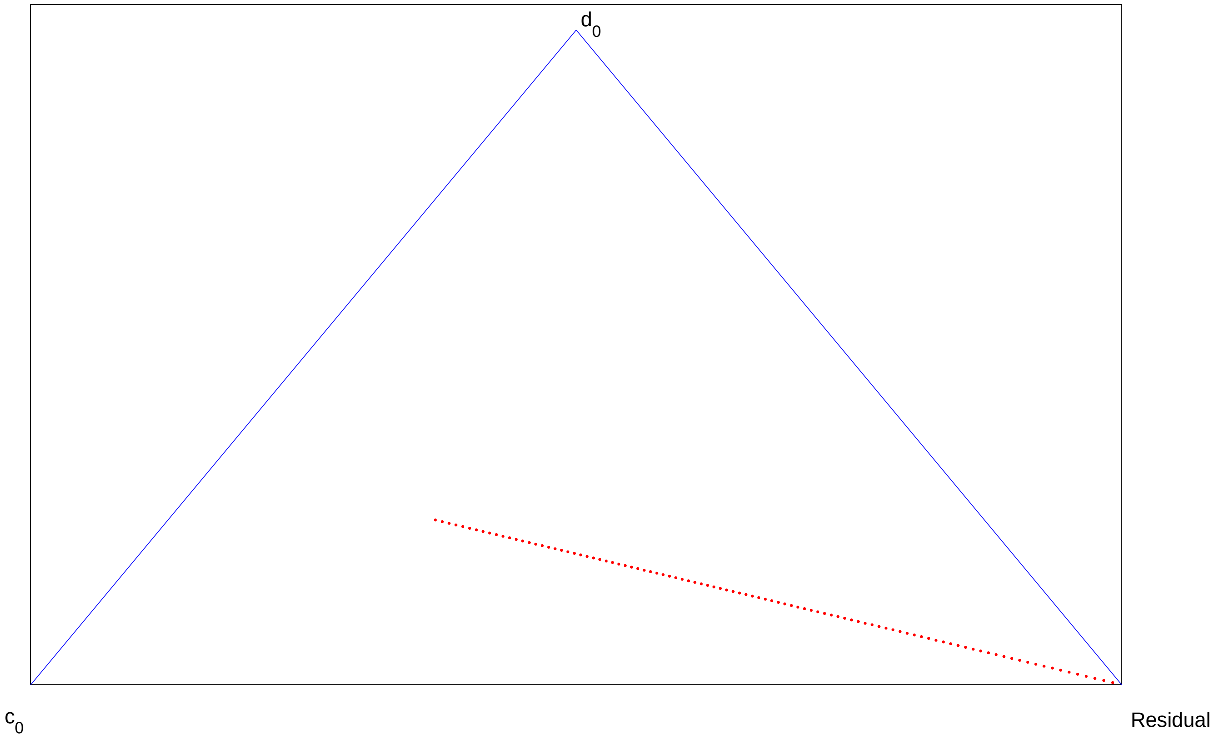
    Optimal policy of agent a in this case is

    $$g^*(T^a x_{t-1},\ T^a\, {}_t\theta_N) = \sum_{i=0}^{\infty} c_i\, x_{t-1}^{a+i} + \sum_{j=0}^{N} d_j\, \theta_t^{a+j} + e\,\overline{\theta}$$

    where ${}_t\theta_N = \{\theta_t^{-N}, \ldots, \theta_t^0, \ldots, \theta_t^N\}$
  - We represent the policy function on the simplex with vertices: $c_0, d_0, \sum_{i=1}^{\infty} c_i + \sum_{j=1}^{N} d_j + e$



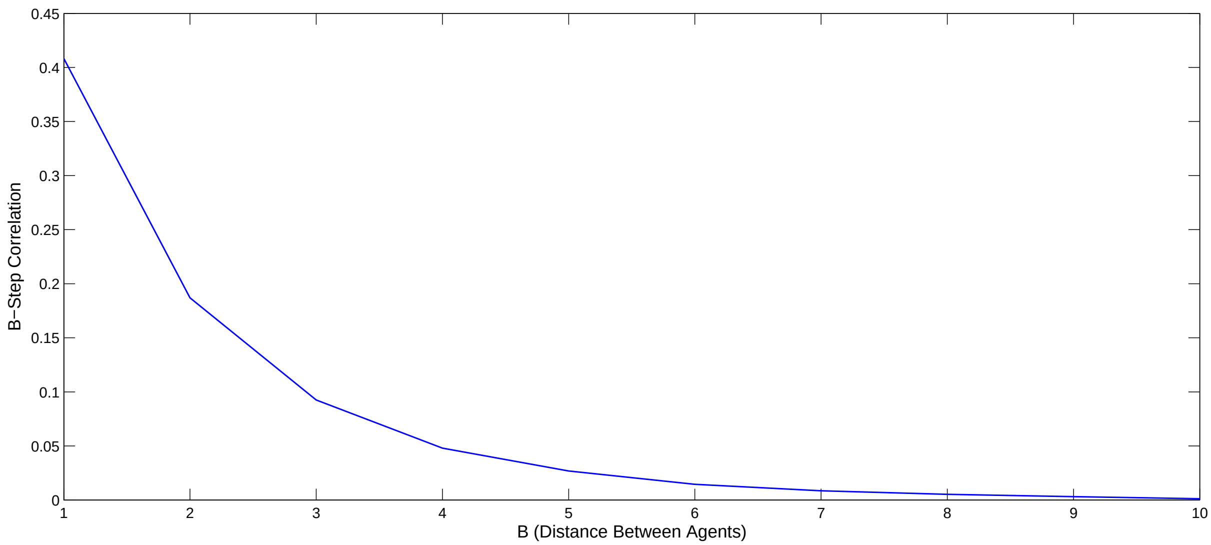
$\alpha_3 \to 1$ where $\alpha_1 = \alpha_2 = \frac{1}{2}(1 - \alpha_3)$



$\alpha_3 \to 1$ where $\alpha_1 = 2\alpha_2 = \frac{1}{3}(1 - \alpha_3)$

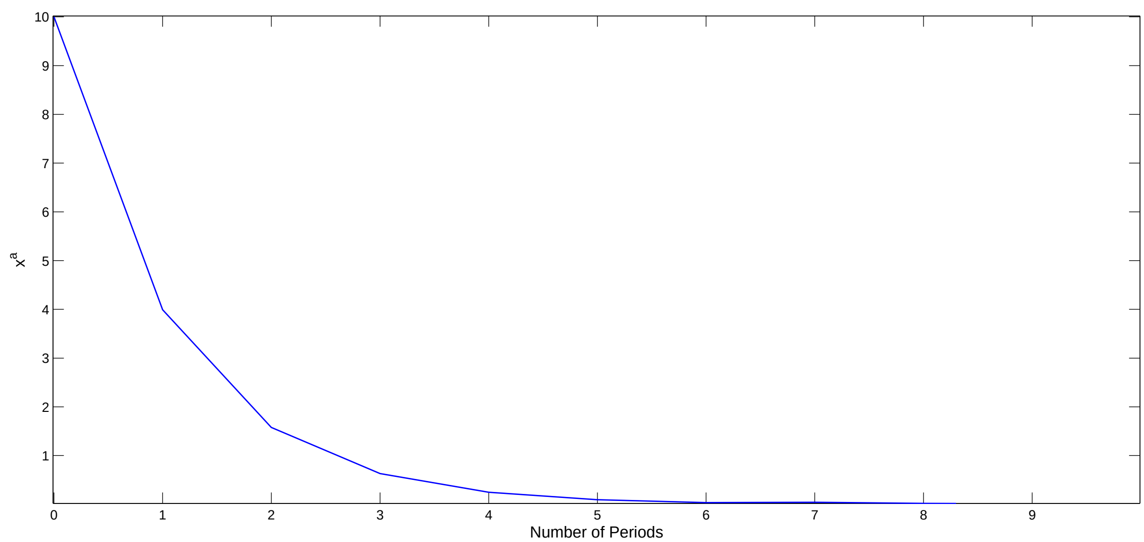
- We represent equilibrium configuration via their spatial correlation structure:

$$\frac{E\left(x_t^a, x_t^{a+b}\right)}{var\left(x_t^a\right)}, \ for \ \ any \ \ b \geq 1$$

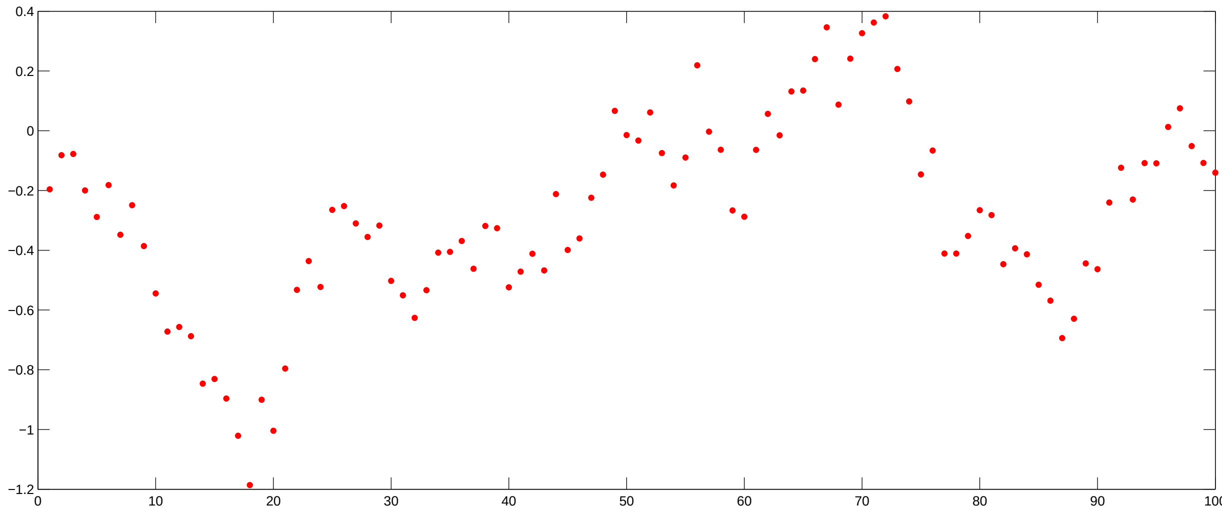


$$\alpha_1 = \alpha_2 = \alpha_3 = .33$$

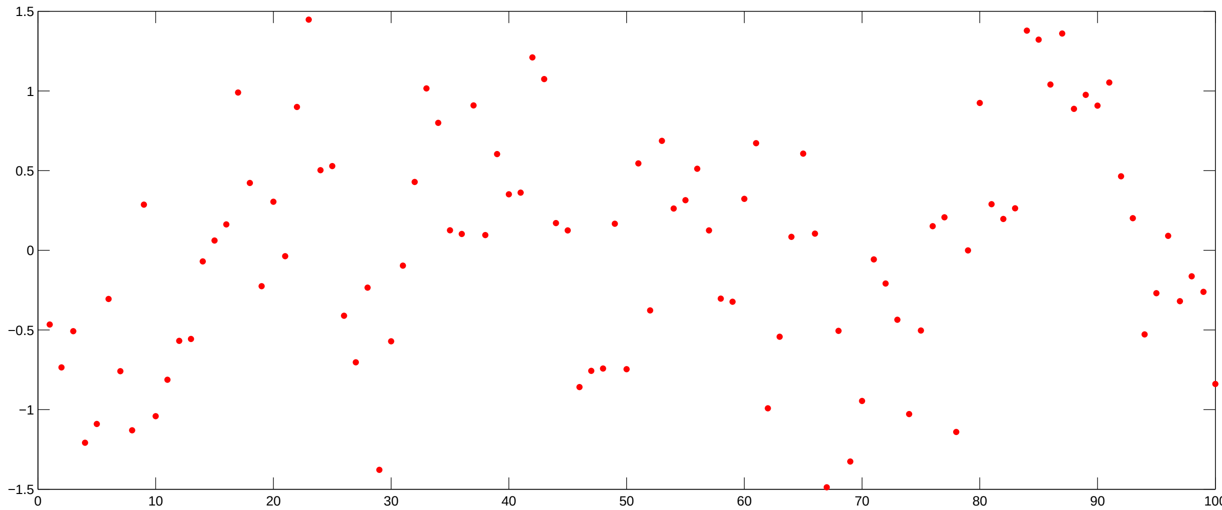
- or simply by a realization of the spatial configuration:
  - Details of the simulation algorithm
  - We represent the convergence to the ergodic distribution via convergence to the mean action in the configuration:



$$\alpha_1 = \alpha_2 = \alpha_3 = .33$$



Economy with strong history dependence;
$\alpha_1 = .9, \alpha_2 = .02, \alpha_3 = .08$



Economy with strong history dependence;

$\alpha_1 = .9, \alpha_2 = .08, \alpha_3 = .02$

Rational vs. Myopic Interactions

We compare equilibrium configurations of economies with rational agents with those of economies with two different kind of myopic, backward looking agents:

- Backward Looking: A generic agent a solves

$$\max_{x_t^a \in X} -\alpha_1 (x_t^a - x_{t-1}^a)^2 - \alpha_2 (x_t^a - \theta_t^a)^2 - \alpha_3 (x_t^a - x_{t-1}^{a+1})^2$$

- Myopic: Agent a solves

$$\max_{x_t^a \in X} -\alpha_1 (x_t^a - x_{t-1}^a)^2 - \alpha_2 (x_t^a - \theta_t^a)^2$$
$$-\alpha_3 \int (x_t^a - x_t^{a+1})^2 \pi_{g^M}(T^{a+1} x_{t-1}; dx_t^{a+1})$$

Rational Policy Function:

$$x_t^a = \sum_{i=0}^{\infty} c_i^R \, x_{t-1}^{a+i} + \sum_{j=0}^{N} d_j^R \, \theta_t^{a+j} + e^R \, \overline{\theta}$$

Backward Looking Policy Function:

$$x_t^a = \beta_1 x_{t-1}^a + \beta_2 \theta_t^0 + \beta_3 x_{t-1}^1$$

Myopic Policy Function:

$$x_t^a = \sum_{i=0}^{\infty} c_i^M \, x_{t-1}^{a+i} + \sum_{j=0}^{N} d_j^M \, \theta_t^{a+j} + e^M \, \overline{\theta}$$

	Own-Past	*Own-Shock*	*Neighbor's Past*	*Other Neighbors' Past*	*Other Neighbors' Shocks*	Mean θ
Rational	.32	.16	.11	.13	0	.28
Backward Looking	.4	.2	.4	0	0	0
Myopic	.4	.2	.16	.11	0	.13

Table 1: $\alpha_1 = .4$, $\alpha_2 = .2$ and $\alpha_3 = .4$, $N = 0$.

	Own-Past	*Own-Shock*	*Neighbor's Past*	*Other Neighbors' Past*	*Other Neighbors' Shocks*	Mean θ
Rational	.05	.86	.002	$\simeq 0$	0	$\simeq .08$
Backward Looking	.05	.95	.05	0	0	0
Myopic	.05	.9	.0025	$\simeq 0$	0	$\simeq .05$

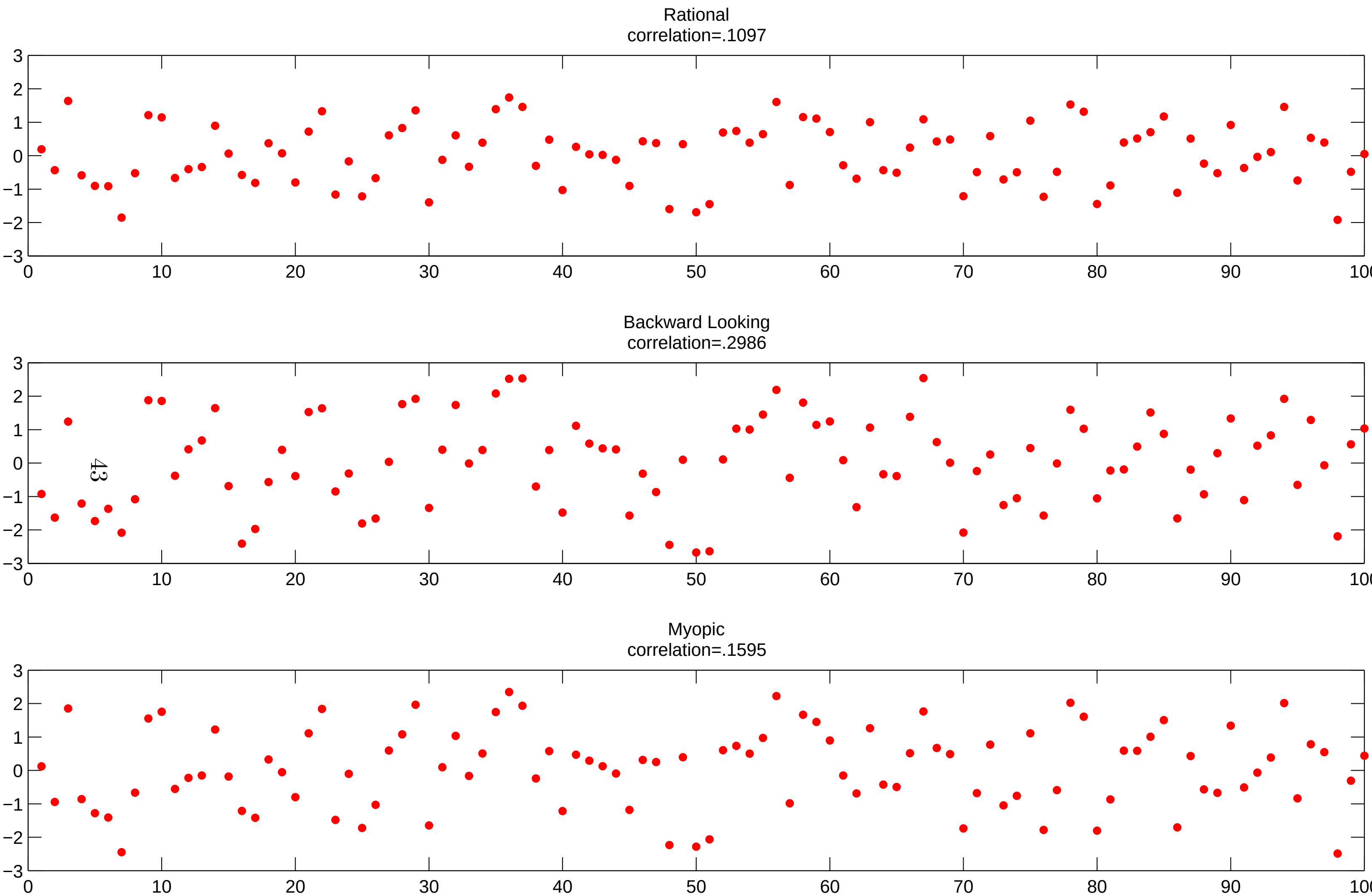
Table 2: $\alpha_1 = .05$, $\alpha_2 = .95$ and $\alpha_3 = .05$, $N = 0$

	Own-Past	*Own-Shock*	*Neighbor's Past*	*Other Neighbors' Past*	*Other Neighbors' Shocks*	Mean θ
Rational	.17	.09	.11	.5643	0	$\simeq .07$
Backward Looking	.2	.1	.7	0	0	0
Myopic	.2	.1	.14	.3267	0	$\simeq .23$

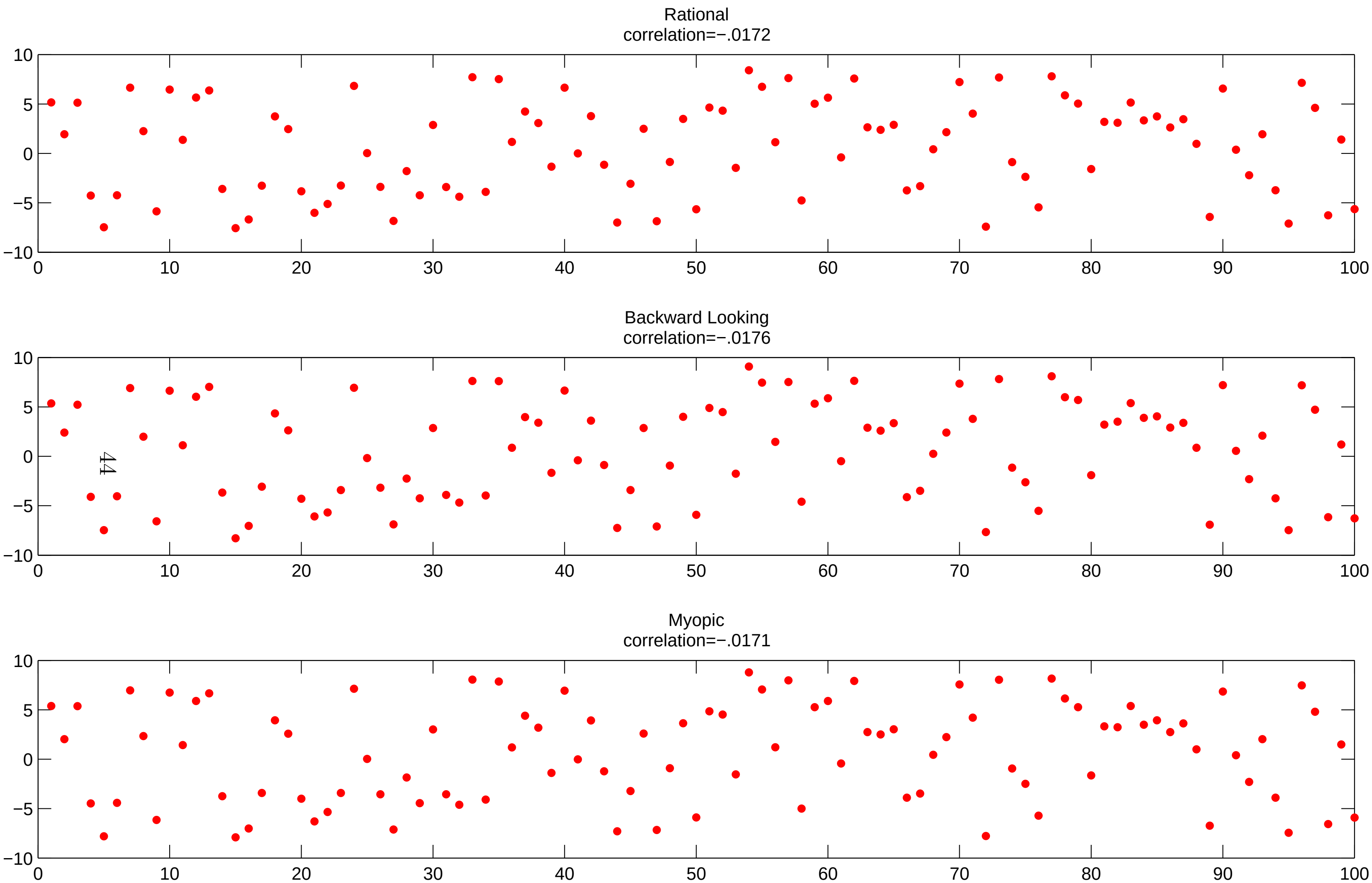
Table 3: $\alpha_1 = .2$, $\alpha_2 = .1$ and $\alpha_3 = .7$, $N = 0$.

	Own-Past	*Own-Shock*	*Neighbor's Past*	*Other Neighbors' Past*	*Other Neighbors' Shocks*	Mean θ
Rational	.32	.16	.11	.13	.20	$\simeq .08$
Backward Looking	.4	.2	.4	0	0	0
Myopic	.4	.2	.16	.11	.13	$\simeq 0$

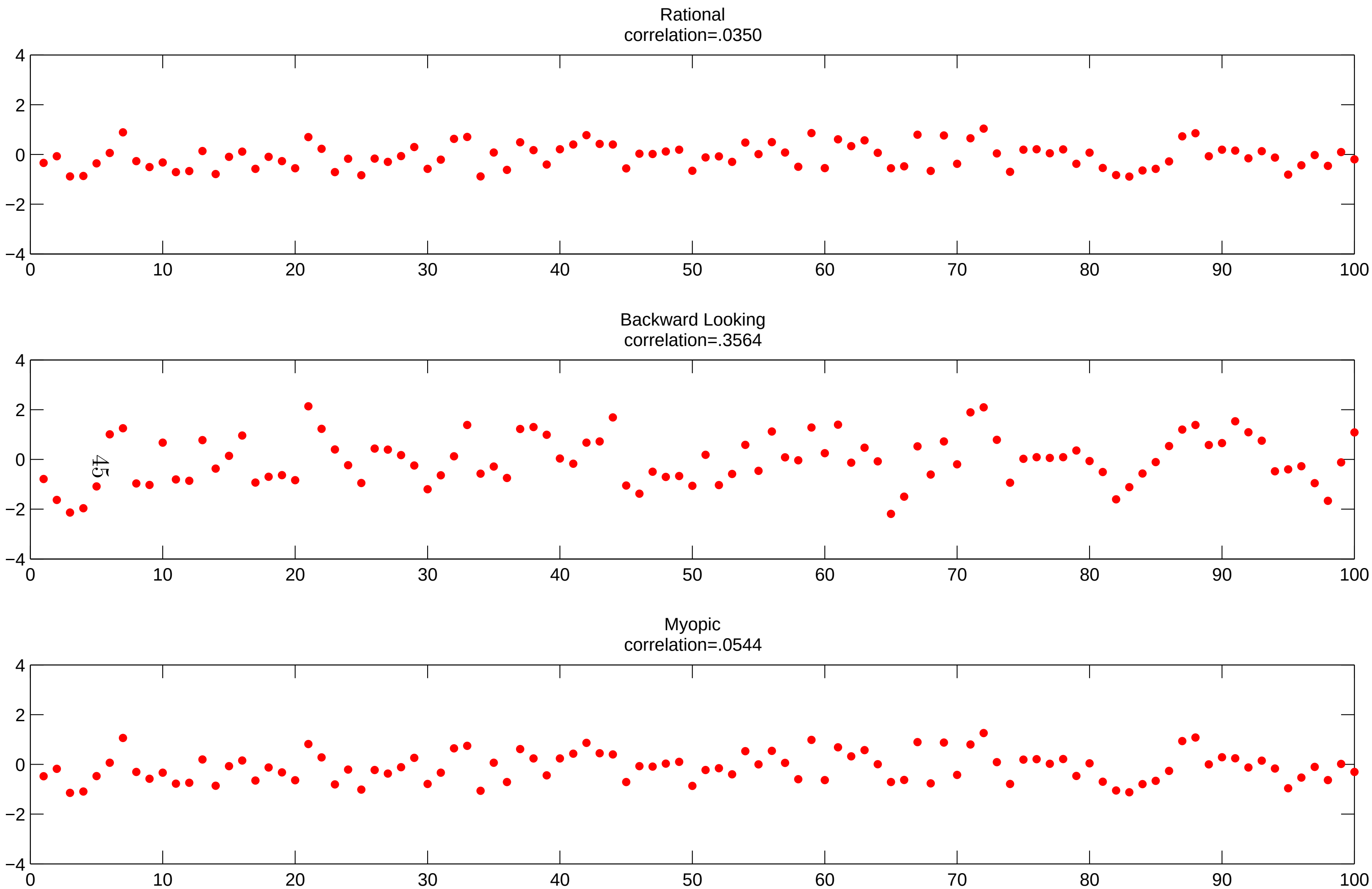
Table 4: $\alpha_1 = .4$, $\alpha_2 = .2$ and $\alpha_3 = .4$, $N = 5$



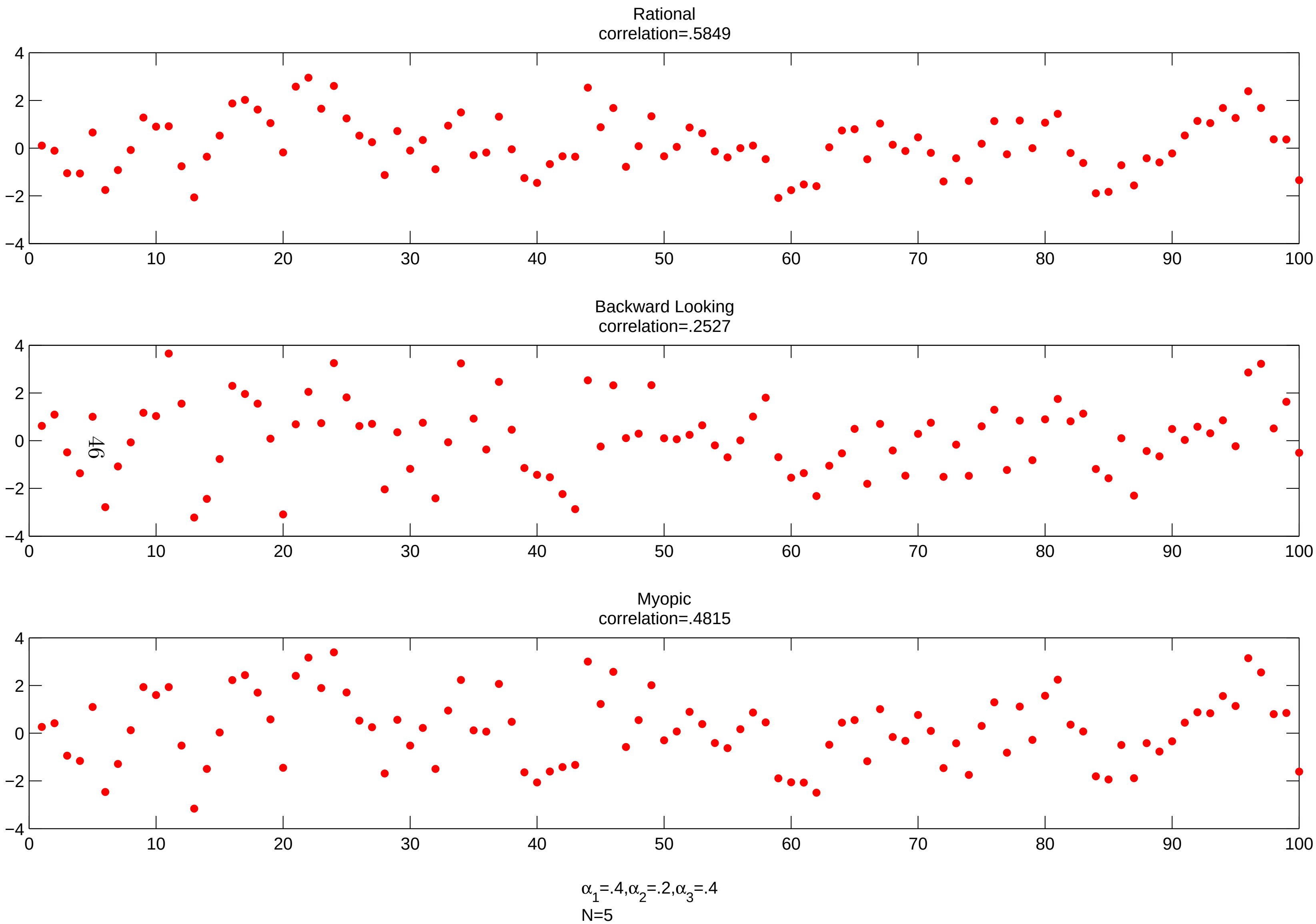
α_1=.4, α_2=.2, α_3=.4



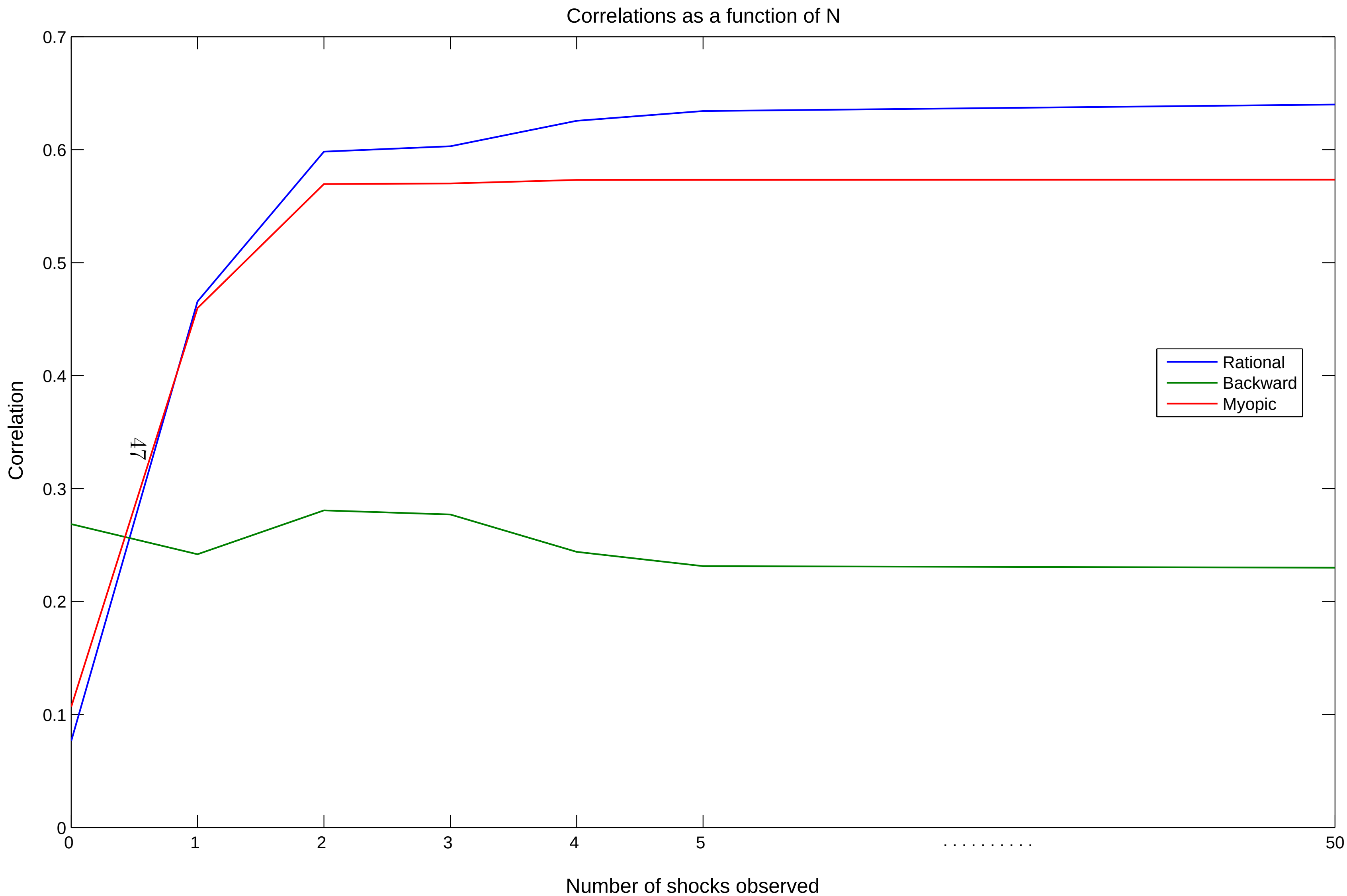
$\alpha_1=.05,\alpha_2=.95,\alpha_3=.05$



α_1=.2,α_2=.1,α_3=.7



Rational
correlation=.5849
Backward Looking
correlation=.2527
Myopic
correlation=.4815
4
2
0
−2
−4
0
10
20
30
40
50
60
70
80
90
100
α_1=.4,α_2=.2,α_3=.4
N=5


Correlations as a function of N
Correlation
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
. . . .
50
Number of shocks observed
Rational
Backward
Myopic

Identifiability

- We study the identifiability of the social interaction model in the dynamic economy. Naturally, the observation of equilibrium actions over time might help in identifying social interactions.

- Graham-Hahn 2003 show that spatially correlated unobserved heterogeneity does not impede identifiability in a dynamic economy as long as the heterogeneity is not also temporally correlated - equation (14) on page 10.

- In Graham-Hahn 2003, the equilibrium configuration at any time t is generated by the policy functions obtained in the static economy (that is, agents' behavior is myopic), so that the evolution of the equilibrium configurations of actions over time represents essentially different independent realizations of the same stochastic process.

- The identifiability problem:
  - Consider the economy with complete information of preference shocks and past action. The policy function of agent a is:

$$x_t^a = \sum_{b>a} c_b x_t^b + c_0 x_{t-1}^a + \gamma \theta_t^a$$

  - Consider, similarly to the case of the static economy, an equivalent model in which each agent a's choice only depends on the realization of an unobservable factor u^a

$$x_t^a = \eta u_t^a,$$

    which is however spatially and temporally correlated,

$$u_t^a = \sum_{b>a} \rho_b u_t^b + \lambda u_{t-1}^a + \epsilon_t^a,$$

    for some i.i.d. zero-mean process $\epsilon_t = (\epsilon^a)_{a\in,t\geq 0}$.
  - With

$$\rho = \gamma, \ \eta = 1 - \gamma$$

    the economy with unobserved spatial correlation and no social interaction is observationally equivalent to the economy with social interaction, and identifiability fails.

- The identifiability results we obtained for the static economy can be extended to the dynamic economy, and hence identifiability obtains with incomplete information over preference shocks and/or with observable heterogeneity.

- Two different positive results:

  – Identifiability due to the dependence on own past actions.

    * Suppose the econometrician knows that agent a's action x_t^a depends on x_{t-1}^a, for instance because of a technological link, like e.g., adjustment costs.

    * In this case the policy function of agent a is:

    $$x_t^a = \eta u_t^a + \lambda x_{t-1}^a, \; u_t^a = \sum_{b>a} \rho_b u_t^b + \epsilon_t^a$$

    while, with correlated unobserved heterogeneity, his equilibrium action is determined by

    $$x_t^a = \sum_{b>a} \left(\rho_b x_t^b - \lambda \rho_b x_{t-1}^b \right) + \lambda x_{t-1}^a + \eta \epsilon_t^a.$$

    * Identifiability then obtains from the properties of the equilibrium configuration:

    $$E\left(x_t^a x_{t-1}^{a+1} \mid x_{t-1}^a \right) = 0$$

    in the economy with social interactions, while

    $$E\left(x_t^a x_{t-1}^{a+1} \mid x_{t-1}^a \right) > 0$$

    if equilibrium actions are determined by correlated unobserved heterogeneity.

- Identifiability due to the lack of stationarity of the policy function.
  - Suppose that the economy is truncated at some time T, $0 < T < \infty$.
  - In this case each agent's policy function is not stationary, while equilibrium actions determined by correlated unobserved heterogeneity, are necessarily stationary.
  - **T-Periods with no continuation** We look at the policy functions of truncated economies in the sense that agent 0 solves in period t, the following problem

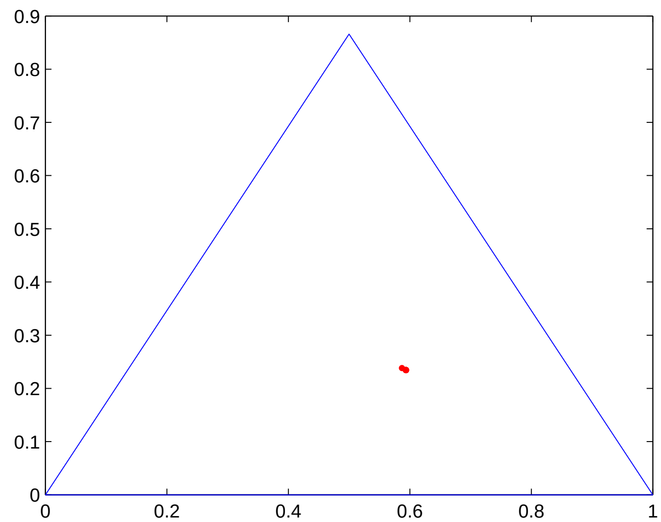
$$\max_{x_{t-1}^0 \in X} -\alpha_1 (x_t^0 - x_{t-1}^0)^2 - \alpha_2 (x_t^0 - \theta_t^0)^2$$
$$-\alpha_3 \int (x_t^0 - x_t^1)^2 \pi_{g^{T-t+1}}(Tx_{t-1}; dx_t^1)$$
$$+\beta V^{T-t}(x_t^0, \widehat{x}_t, \theta_{t+1}^N) \Pi_{g^{T-t+1}}(Tx_{t-1},; d\widehat{x}_t) \Pi_{b=1}^N \nu(d\theta_{t+1}^b)$$

  - where $t = 0, 1, \ldots, T$ and V^0 is the zero function and x_{-1} is given. By backward induction, we solve for the sequence of policy functions

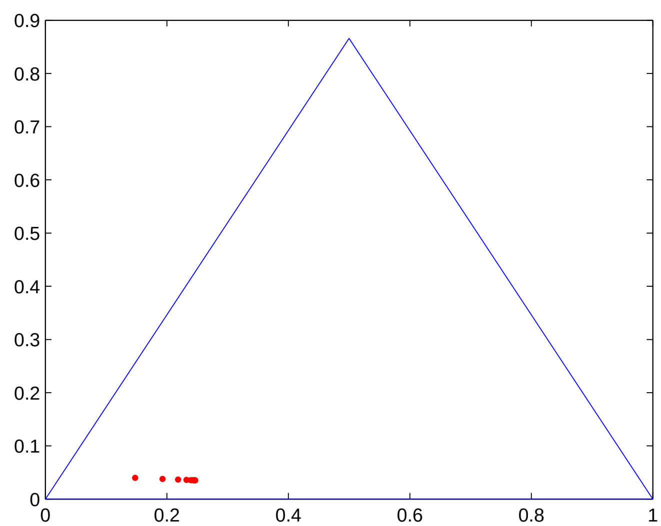
$$g^{T-t+1}(x_{t-1},\ {}_t\theta_N) = \sum_{i=0}^{\infty} c_i^t \, x_{t-1}^i + \sum_{j=0}^{N} d_j^t \, \theta_t^j + e^t \, \overline{\theta}$$

  for $t = 0, \ldots, T$.

- We represent the sequence of policy functions in the simplex with vertices $c_0, d_0, \sum_{i=1}^{\infty} c_i + \sum_{j=1}^{N} d_j + e$



$\alpha_1 = \alpha_2 = \alpha_3 = .33$



$\alpha_1 = .9, \alpha_2 = \alpha_3 = .05$

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6 Formal Analysis - incomplete information for simplicity

- Prior to his choice at time t, the agent $a \in \mathbb{A}$ observes the realization only of his own type θ_t^a. Agents to observe the entire action profile $\mathbf{x}_{t-1} = (x_\tau^a)_{a \in \mathbb{A}}$ of previous periods $\tau = t-1, t-2, \ldots$.

- **Definition 6.1** *A dynamic economy with local interactions is a tuple* $\mathcal{S} = (X, \Theta, u, \nu, \beta)$.

- We denote by $\mathbf{X} := \{\mathbf{x} = (x^a)_{a \in \mathbb{A}} : x^a \in X\}$ the space of all configurations of individual actions and let $\mathbf{X^0} := \{x = (x^a)_{a \geq 0}\}$.

- We study Markov perfect equilibria, in which the policy function of any agent at t will only depend on period $t-1$ actions. As in the static case, we shall focus on symmetric equilibria. Thus, the optimal action of an economic agent $a \in \mathbb{A}$ is determined by a choice function $g : \mathbf{X^0} \times \Theta \to X$ in the sense that

$$x_t^a = g(T^a x_{t-1}, \theta_t^a) \quad \text{where} \quad T^a x_{t-1} = \{x_{t-1}^b\}_{b \geq a}.$$

- In a symmetric situation, it is thus enough to analyze the optimization problem of a single reference agent, say of the agent $0 \in \mathbb{A}$. Given the action profile $x_{t-1} = (x_{t-1}^b)_{b \geq 0} \in \mathbf{X^0}$ of the agents $b \geq 0$ in the previous period and a continuous choice function $g : \mathbf{X^0} \times \Theta \to X$, the agent $a \geq 0$ takes as given his neighbor's current choice $g(T^a x_{t-1}, \theta_t^a)$. We denote by $\pi_g(T^a x_{t-1}; \cdot)$ the conditional law of the action x_t^a, given the previous configuration x_{t-1}, and so the choice function $g : \mathbf{X^0} \times \Theta \to X$ induces the Feller kernel

$$\Pi_g(x; \cdot) := \prod_{a=1}^{\infty} \pi_g(T^a x; \cdot). \tag{6}$$

- If the agent $0 \in \mathbb{A}$ believes that the agents $a > 0$ choose their actions according to g, the kernel Π_g describes the stochastic evolution of the process of individual states $\{(x_t^a)_{a>0}\}_{t \in \mathbb{N}}$. In this case, for any initial configuration of individual states $x \in \mathbf{X^0}$ and for each initial type θ_1^0, the optimization problem of the agent 0 is given by

$$\max_{\{x_t^0\}} \left\{ \int u(x_1^0, x^0, x_1^1, \theta_1^0) \pi_g(Tx; dx^1) + \sum_{t \geq 2} \beta^{t-1} \int u(x_t^0, x_{t-1}^0, x_t^1, \theta_t^0) \Pi_g^t(Tx; dx_t) \nu(d\theta_t^0) \right\}. \tag{7}$$

- The value function associated with this dynamic choice problem is defined by the fixed point of the functional equation

$$\begin{aligned} V_g(x_{t-1}, \theta_t^0) = V_g(x_{t-1}^0, Tx_{t-1}, \theta_t^0) &= \max_{x_t^0 \in X} \left\{ \int u\left(x_{t-1}^0, x_t^0, y_t^1, \theta_t^0\right) \pi_g(Tx_{t-1}; dy_t^1) \right. \\ &\quad \left. + \beta \int_{\mathbf{X^0} \times \Theta} V_g(x_t^0, \hat{x}_t, \theta^1) \Pi_g(Tx_{t-1}; d\hat{x}_t) \nu(d\theta^1) \right\}. \end{aligned} \tag{8}$$

- **Lemma 6.2** *Assume that the choice map g is continuous. Under our assumptions on the utility function u, the functional fixed point equation (8) has a unique bounded and continuous solution V_g on $\mathbf{X^0} \times \Theta$. Moreover, the map $V_g(\cdot, Tx_{t-1}, \theta_t^0)$ is strictly concave on X and there exists a unique continuous policy function $\hat{g}_g : \mathbf{X^0} \times \Theta \to X$ that satisfies*

$$\hat{g}_g\left(x_{t-1}, \theta_t^0\right) = \arg\max_{x_t^0 \in X} \left\{ \int u\left(x_{t-1}^0, x_t^0, y_t^1, \theta_t^0\right) \pi_g(Tx_{t-1}; dy_t^1) + \beta \int V_g(x_t^0, \hat{x}_t, \theta^1) \Pi_g(Tx_{t-1}; d\hat{x}_t) \nu(d\theta^1) \right\}. \qquad (9)$$

- **Definition 6.3** *A symmetric Markov perfect equilibrium of a dynamic economy with forward looking and locally interacting agents $\mathcal{S} = (X, \Theta, u, \nu, \beta)$, is a map $g^* : \mathbf{X^0} \times \Theta \to X$ such that*

$$g^*\left(x_{t-1}, \theta_t^0\right) = \arg\max_{x_t^0 \in X} \left\{ \int u\left(x_{t-1}^0, x_t^0, y_t^1, \theta_t^0\right) \pi_{g^*}(Tx_{t-1}; dy_t^1) + \beta \int V_{g^*}(x_t^0, \hat{x}_t, \theta^1) \Pi_{g^*}(Tx_{t-1}; d\hat{x}_t) \nu(d\theta^1) \right\}. \qquad (10)$$

- In order to state a general existence result for equilibria in dynamic random economies with forward looking interacting agents we need to introduce the notion of a correlation pattern.

- **Definition 6.4** *For $C > 0$, let*

$$L_+^C := \{\mathbf{c} = (c_a)_{a \in \mathbb{N}} \ : \ c_a \geq 0, \ \sum_{a \in \mathbb{A}} c_a \leq C\}$$

denote the class of all non-negative sequences whose sum is bounded from above by C. A sequence $\mathbf{c} \in L_+^C$ will be called a correlation pattern with total impact C.

Each correlation pattern $\mathbf{c} \in L_+^C$ gives rise to a metric

$$d_{\mathbf{c}}(x, y) := \sum_{a \in \mathbb{N}} c_a |x^a - y^a|$$

that induces the product topology on $\mathbf{X^0}$. Thus, $(d_{\mathbf{c}}, \mathbf{X^0})$ is a compact metric space. In particular, the class

$$\mathrm{Lip}_{\mathbf{c}}^C := \{f : \mathbf{X^0} \to \mathbb{R} : |f(x) - f(y)| \leq d_{\mathbf{c}}(x, y)\}$$

of all functions $f : \mathbf{X^0} \to \mathbb{R}$ which are Lipschitz continuous with constant 1 with respect to the metric $d_{\mathbf{c}}$ is compact in the topology of uniform convergence.

Remark 6.5 *For a fixed $\theta^0 \in \Theta$, let $g(\cdot, \theta^0) \in \mathrm{Lip}_{\mathbf{c}}^C$ be the policy function of the agent $0 \in \mathbb{A}$. The constant c_a may be viewed as a measure for the total impact the current action x^a of the agent $a \geq 0$ has on the optimal action of agent $0 \in \mathbb{A}$. Since $C < \infty$, we have $\lim_{a \to \infty} c_a = 0$. Thus, the impact of an agent $a \in \mathbb{A}$ on the agent $0 \in \mathbb{A}$ tends to zero as $a \to \infty$. In this sense we consider economies with weak social interactions. The quantity C provides an upper bound for the total impact of the configuration $x = (x^a)_{a \geq 0}$ on the current choice of the agent $0 \in \mathbb{A}$.*

We are now going to formulate a general existence result for symmetric Markov perfect equilibria in dynamic economies with local interaction.

Theorem 6.6 *Assume that there exists $C < \infty$ such that the following holds:*

i. For any $\mathbf{c} \in L_+^C$, for all $\theta^0 \in \Theta$ and for each choice function $g(\cdot, \theta^0) \in \mathrm{Lip}_{\mathbf{c}}^C$, there exists $F(\mathbf{c}) \in L_+^C$ such that the unique policy function $\hat{g}_g(\cdot, \theta^0)$ which solves (9), is Lipschitz continuous with respect to the metric $d_{F(\mathbf{c})}$ uniformly in $\theta^0 \in \Theta$.

ii. The map $F : L_+^C \to L_+^C$ is continuous.

iii. We have $\lim_{n\to\infty} \|\hat{g}_{g_n}(\cdot, \theta^0) - \hat{g}_g(\cdot, \theta^0)\|_\infty = 0$ if $\lim_{n\to\infty} \|g_n - g\|_\infty = 0$.

Then the dynamic economy with local interactions has a symmetric Markov perfect equilibrium g^* and the function $g^*(\cdot, \theta^0)$ is Lipschitz continuous uniformly in θ^0.

Proof: Let us first show that, for any $C < \infty$, the convex set L_+^C may be viewed as a closed subset of $[0, C]^{\mathbb{N}}$ with respect to the product topology. To this end, let $\{\mathbf{c^n}\}_{n\in\mathbb{N}}$ be a sequence in L_+^C that converges to $\mathbf{c} = (c_a)_{a\geq 0} \in [0, C]^{\mathbb{N}}$ in the product topology. Clearly, $c_a \geq 0$ for all $a \in \mathbb{N}$, and so the sum $\widetilde{C} := \sum_{a\geq 0} c_a$ exists by monotone convergence. If $\widetilde{C} > C$, then there exists $b \in \mathbb{N}$ and $\epsilon > 0$ such that

$$C + \epsilon \leq \sum_{a=0}^{b} c_a = \sum_{a=0}^{b} \lim_{n\to\infty} c_a^n \leq \lim_{n\to\infty} \sum_{a\geq 0} c_a^n$$

which contradicts $\mathbf{c^n} \in L_+^C$.

Due to (i) and (iii) the operator $\hat{V}$ defined by (**??**) maps the compact and convex set $\mathrm{Lip}_{\mathbf{c}^*}^C$ continuously into itself and therefore does also have a fixed point g^*. □

- we study the asymptotic behavior of the process $\{\mathbf{x_t}\}_{t\in\mathbb{N}}$ in equilibrium. To this end, we denote by

$$\Pi_{g^*}(\mathbf{x}; \cdot) = \prod_{a\in\mathbb{A}} \pi_{g^*}(T^a x; \cdot)$$

the stochastic kernel on $\mathbf{X}$ induced by the policy function g^* and by $\Pi_{g^*}^t$, its t-fold iteration. Given an initial configuration $\mathbf{x} \in \mathbf{X}$, the measure $\Pi^t(\mathbf{x}; \cdot)$ describes the distribution of the configuration of individual states at time t. Let us introduce the vector $r^* = (r_a^*)_{a\in\mathbb{A}}$ with components

$$r_a^* := \sup\{\|\pi_{g^*}(x; \cdot) - \pi_{g^*}(y; \cdot)\| : x = y \text{ off } a\}. \tag{11}$$

Here, $\|\pi_{g^*}(x; \cdot) - \pi_{g^*}(y; \cdot)\|$ denotes the total variation of the signed measure $\pi_{g^*}(x; \cdot) - \pi_{g^*}(y; \cdot)$, and $x = y$ off a means that $x^b = y^b$ for all $b \neq a$. The next theorem gives sufficient conditions for convergence of the equilibrium process to a steady state. Its proof follows from a fundamental convergence theorem by Vaserstein 1969.

Theorem 6.7 *If $\sum_{a\in\mathbb{A}} r^a_{g^*} < 1$, then there exists a unique probability measure μ^* on the infinite configuration space $\mathbf{X}$ such that, for any initial configuration $\mathbf{x} \in \mathbf{X}$, the sequence $\Pi^t_{g^*}(\mathbf{x};\cdot)$ converges to μ^* in the topology of weak convergence for probability measures.*

6.1 Existence with Quadratic Utility and Incomplete Information

In this section we analyze an example with quadratic utility functions where the assumption of Theorem 6.6 and of Theorem 6.7 can indeed be verified.

Theorem 6.8 *Let $X = \Theta = [-1,1]$, and assume that $\mathbb{E}\theta^0_t = 0$, and that an agent $a \in \mathbb{A}$ only observes his own type θ^a. If the instantaneous utility function takes the quadratic form*

$$u\left(x^a_{t-1}, x^a_t, x^{a+1}_t, \theta^a_t\right) = -\alpha_1\left(x^a_{t-1} - x^a_t\right)^2 - \alpha_2\left(\theta^a_t - x^a_t\right)^2 - \alpha_3\left(x^{a+1}_t - x^a_t\right)^2 \quad (12)$$

for positive constants α_1, α_2 and α_3, then the economy has a symmetric Markov perfect equilibrium g^*. The policy function g^* can be chosen to be of the linear form

$$g^*(x,\theta^0) = c^*_0 x^0 + \gamma\theta^0 + \sum_{b\geq 1} c^*_b x^b$$

for some positive sequence $\mathbf{c}^* = (c^*_a)_{a\geq 0}$ and some constant $\gamma > 0$.

The proof of Theorem 6.8 will be carried out in several steps. In a first step, we are now going to establish the existence of an interior solution to an agent's optimization problem in an economy with quadratic utility functions.

Lemma 6.9 *Let $g : \mathbf{X^0} \times \Theta \to X$ be a continuous choice function for the agents $a > 0$. Under the assumptions of Theorem 6.8, the induced policy $\hat{g}_g$ function of the agent $0 \in \mathbb{A}$ is uniquely determined and*

$$\mathbb{P}\left(\hat{g}_g(x_{t-1},\theta^0_t) \in \{-1,1\} \text{ for some } t \in \mathbb{N}\right) = 0. \quad (13)$$

Thus, we have almost surely an interior solution.

Proof: The existence of a unique policy function follows from continuity of g along with the quadratic form of the utility functions using standard arguments from the theory of discounted dynamic programming. In order to prove (13), we put

$$\tau := \inf\left\{t > 0 : \ \hat{g}_g(x_{t-1},\theta^0_t) = 1\right\} \quad \text{and} \quad y_t := \hat{g}_g(x_{t-1},\theta^0_t).$$

It suffices to show that $\mathbb{P}[\tau < \infty] = 0$. Let us assume to the contrary that $\mathbb{P}[\tau < \infty] > 0$. In such a situation $y_\tau = 1$ is optimal and this means that

$$\begin{aligned} & -\alpha_1(1-y_{\tau-1})^2 - \beta\left(\alpha_1(1-y_{\tau+1})^2 + \alpha_2(1-x^1_\tau)^2 + \alpha_3(1-\theta^0_\tau)^2\right) \\ \geq \ & -\alpha_1(y-y_{\tau-1})^2 - \beta\left(\alpha_1(y-y_{\tau+1})^2 + \alpha_2(y-x^1_\tau)^2 + \alpha_3(y-\theta^0_\tau)^2\right) \end{aligned}$$

for all $y \in X$. Otherwise $y_\tau < 1$ would lead to a higher payoff. This, however, requires $\theta^0_\tau = y_{\tau-1} = y_{\tau+1} = 1$. This shows that $y_t = 1 = \theta^0_t$ for all $t \in \mathbb{N}$. This, of course, contradicts $\mathbb{E}\theta^0_t = 0$. Thus, $\mathbb{P}[\tau < \infty] = 0$. □

Let us now establish a first representation of the agents' policy function. To this end, we denote by $\mathcal{M}(\mathbf{X^0})$ the class of all probability measures on $\mathbf{X^0}$ equipped with the topology of weak convergence. The utility of the agent $0 \in \mathbb{A}$ at time $t \in \mathbb{N}$ depends on the actions x^a_t taken by the agents $a > 0$ only through his neighbor's expected action

$$z_t := \int y^1 \Pi_g(Tx_t; dy).$$

We may thus view the agent's dynamic problem as an optimization problem depending only on the stochastic sequence $\{\theta^0_t\}_{t\in\mathbb{N}}$ and on the deterministic sequence $\left\{\Pi^t_g(Tx;\cdot)\right\}_{t\in\mathbb{N}}$. In fact, in our present setting we can, for any initial configuration $x \in \mathbf{X^0}$, put $\mu(\cdot) := \Pi_g(Tx;\cdot)$ and rewrite his optimization (7) as

$$\max_{\{x^0_t\}_{t\in\mathbb{N}}} \left\{ U(x^0_1, x^0_0, \theta^0_1, \mu) + \sum_{t\geq 2} \beta^{t-1} \int U(x^0_t, x^0_{t-1}, \theta^0_t, \mu\Pi^t_g)\nu(d\theta^0_t) \right\} \tag{14}$$

where

$$U(x^0_1, x^0_0, \theta^0, \mu) := -\alpha_1(x^0_1 - x^0_0)^2 - \alpha_1(x^0_1 - \theta^0)^2 - \alpha_3 \int (x^0_1 - y^1)^2 \mu(dy).$$

This allows us to show that the agent's optimal action is given as a weighted sum of his present type, of his action taken in the previous period and of the expected future actions of his neighbor.

Lemma 6.10 *Assume that the assumptions of Theorem 6.8 are satisfied. Given an action profile $x \in \mathbf{X^0}$ and a choice function $g : \mathbf{X^0} \times \Theta \to X$ for the agents $a > 0$, the policy function of agent $0 \in \mathbb{A}$ is of the linear form*

$$\hat{g}_g(x, \theta) = \gamma_1 x^0 + \gamma_2 \theta^0 + \sum_{t\geq 1} \delta_{t-1} \int y^1 \Pi^t_g(Tx; dy). \tag{15}$$

With $\lambda := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\beta$ the constants $\gamma_1, \gamma_2, \delta_0, \delta_1, \ldots$ are given by

$$\gamma_1 := \frac{\lambda - \sqrt{\lambda^2 - 4\alpha_1^2\beta}}{2\alpha_1\beta}, \quad \textit{and} \quad \gamma_2 := \frac{\alpha_2}{\lambda - \gamma_1\alpha_1\beta}, \tag{16}$$

and by

$$\delta_0 := \frac{\alpha_3}{\lambda - \gamma_1\alpha_1\beta} \quad \textit{and} \quad \delta_{t+1} = \frac{\alpha_1\beta}{\lambda - \gamma_1\alpha_1\beta}\delta_t \quad \textit{for } t \geq 1. \tag{17}$$

The constants in (16) and (17) do not depend on g and satisfy $\gamma_1 + \gamma_2 + \sum_{t\geq 0} \delta_t \leq 1$.

Proof: Let us fix an initial configuration $x = (x^a)_{a\geq 0}$ and put $\mu := \Pi_g(Tx;\cdot)$. The value function associated with the optimization problem (14) solves the functional fixed point equation

$$\begin{aligned} V_g(x^0_0, \theta^0_1, \mu) \quad &= \quad \max_{x^0_1 \in X} \left\{ -\alpha_1(x^0_0 - x^0_1)^2 - \alpha_2(\theta^0_1 - x^0_1)^2 - \alpha_3 \int (y^1 - x^0_1)^2 \mu(dy) \right. \\ &\qquad \left. + \beta \int V_g(x^0_1, \theta^0_2, \mu\Pi_g)\nu(d\theta^0_2) \right\}. \end{aligned} \tag{18}$$

In view of Lemma 6.9 the fixed point equation (18) has a unique solution $V_g^* : X \times \Theta \times \mathcal{M}(\mathbf{X^0}) \to \mathbb{R}$ and the agent's policy function $\hat{g}_g : X \times \Theta \times \mathcal{M}(\mathbf{X^0}) \to X$ is uniquely determined and the optimal solution is almost surely interior. Thus, the first order condition takes the form

$$-2\alpha_1(x_0^0 - x_1^0) - 2\alpha(\theta_1^0 - x_1^0)^2 - 2\alpha_3 \int (y^1 - x_1^0)\mu(dy) + \beta \int \frac{\partial}{\partial x_1^0} V_g^*(x_1^0, \theta_2^0, \mu\Pi_g)\nu(d\theta_2^0) = 0,$$

and the envelope theorem gives us

$$\frac{\partial}{\partial x_1^0} V(x_1^0, \theta_2^0, \mu\Pi_g) = -2\alpha_1(x_1^0 - x_2^0) = -2\alpha_1 \left(x_1^0 - \hat{g}_g\left(x_1^0, \theta_2^0, \mu\Pi_g\right)\right). \tag{19}$$

This yields

$$x_1^0 = \frac{1}{\alpha_1 + \alpha_2 + \alpha_3 + \beta_1\alpha_1} \left(\alpha_1 x_0^0 + \alpha_2\theta_1^0 + \alpha_3 \int y^1 \mu(dy) + \alpha_1\beta \; \hat{g}_g\left(x_1^0, \theta_2^0, \mu\Pi_g\right) \right). \tag{20}$$

Let us now assume that we have the following alternative representation for the optimal path $\{x_t^0\}_{t\in\mathbb{N}}$:

$$x_t^0 = \gamma_1 x_{t-1}^0 + \gamma_2\theta_t^0 + \sum_{i=0}^{\infty} \delta_i z_{t+i} \in (0,1) \tag{21}$$

where z_t denotes the expected action of the agent $a = 1$ at time t. Using $\mathbb{E}\theta_t^0 = 0$, it does then follow from the first order condition, from (19) and from (20) that

$$x_1^0 = \frac{1}{\alpha_1 + \alpha_2 + \alpha_3 + \beta_1\alpha_1} \left(\alpha_1 x_0^0 + \alpha_2\theta_1^0 + \alpha_3 \int y^1 \Pi_g(x; dy) + \alpha_1\beta\gamma_1 x_1^0 + \alpha_1\beta \sum_{i=0}^{\infty} z_{2+i} \right). \tag{22}$$

Now we need to find coefficients $\gamma_1, \gamma_2, \delta_0, \delta_1, \ldots$ such that the representations in (21) and in (22) coincide. This can be accomplished recursively and yields the constants in (16) and (17).[1]

Notice, however, that we have not yet shown that the sum of the coefficients is bounded from above by 1. In order to prove this, we consider the situation in which the agents maximize the discounted sum of their expected utilities over the periods $t \in \{0, 1, \ldots, \tau\}$ and denote by $g^\tau(x, \theta^0)$ the optimal action of the agent $0 \in \mathbb{A}$. Using a cumbersome, but rather straightforward induction argument along with an argument similar to the one given in the proof of Lemma 6.9 one can easily show that

$$g^\tau(x, \theta^0) = \gamma_1^\tau x^0 + \gamma_2^\tau \theta^0 + \sum_{i=1}^{\tau} \delta_{i-1}^\tau z_i.$$

Here, the coefficients satisfy the recursive relations

$$\gamma_i^\tau = \frac{\alpha_i}{\lambda_\tau} \quad (i = 1, 2), \quad \delta_0^\tau = \frac{\alpha_3}{\lambda_{\tau+1}}, \quad \delta_i^\tau = \frac{\alpha_1\beta}{\lambda_\tau}\delta_{i-1}^{\tau-1} \quad (i = 1, 2, \ldots) \quad \text{and} \quad \lambda_{\tau+1} = \lambda - \frac{\alpha_1^2\beta}{\lambda_\tau}$$

[1] The solution method for linear rational expectations models in [7] cannot be applied in a context, as ours, with an infinite dimensional state space.

with $\lambda^0 = \alpha_1 + \alpha_2 + \alpha_3$. This shows that $\gamma_i^\tau \to \gamma_i$ and $\delta_i^\tau \to \delta_i$ for all $i = 0, 1, 2, \dots$ as $\tau \to \infty$. Thus,

$$\gamma_1 + \gamma_2 + \sum_{i \geq 0} \delta_i \leq 1 \quad \text{because} \quad \gamma_1^\tau + \gamma_2^\tau + \sum_{i \geq 0} \delta_i^\tau \leq 1 \quad \text{for all } \tau.$$

□

Our representation (15) of the policy function does not yet allow us to apply Theorem 6.6. For this we need a representation of $\hat{g}_g$ in terms of the sequence $(x^a)_{a \geq 0}$. This, however, can be accomplished as follows: Let us fix a correlation pattern $\mathbf{c} = (c_a)_{a \geq 1} \in L_+^{1-\gamma_2}$ and assume for the moment that the choice function of the agents $a > 0$ takes the linear form

$$\tilde{g}(T^a x, \theta^a) = c_0 x^a + \gamma_2 \theta^a + \sum_{b \geq 1} c_b x^{a+b}. \tag{23}$$

In view of (15), we have $c_0 = \gamma_1$ and that continuous choice function $\tilde{g}$ induces a Feller kernel $\Pi_{\tilde{g}}$ on $\mathbf{X^0}$. Thus, it follows from (23) and from $\mathbb{E}\theta_t^0 = 0$ that

$$\int y^1 \Pi_{\tilde{g}}(x; dy) = \sum_{a \geq 0} c_a x^{a+1}.$$

Thus, the expected action of the agent $a = 1$ in the second period is given by

$$\int y^1 \Pi_{\tilde{g}}^2(x; dy) = \sum_{a_1 \geq 0} c_{a_1} \int y^{a+1} \Pi_{\tilde{g}}(x; dy) = \sum_{a_1 \geq 0} c_{a_1} \sum_{a_2 \geq 0} c_{a_2} x^{a_1 + a_2 + 1},$$

and a simple induction argument shows that

$$\int y^1 \Pi_{\tilde{g}}^t(x; dy) = \sum_{a_1 \geq 0} \left(c_{a_1} \sum_{a_2 \geq 0} \left(c_{a_2} \cdots c_{a_{t-1}} \sum_{a_t \geq 0} c_{a_t} x^{a_1 + \cdots + a_t + 1} \right) \cdots \right) \tag{24}$$

for all $t \in \mathbb{N}$. This yields the following alternative representation for our policy function:

$$\hat{g}_{\tilde{g}}(x, \theta) = \gamma_1 x^0 + \gamma_2 \theta + \sum_{b \geq 1} l_b x^b$$

where the strictly positive sequence $(l_b)_{b \geq 1}$ is given by

$$l_b = F_b(c_0, c_1, \dots, c_{b-1}) := \sum_{t \geq 1} \delta_{t-1} \left(\sum_{a_1 = 0}^{b-1} \left(c_{a_1} \sum_{a_2 = 0}^{b-1} c_{a_2} \cdots \right) \sum_{a_t = 0}^{b-1} c_{a_t} \right) \mathbf{1}_{\{\sum_{i=1}^t a_i = b-1\}}. \tag{25}$$

We are now ready to prove Theorem 6.8.

Proof of Theorem 6.8: Since $\hat{g}_{\tilde{g}}(x, \theta^0) \in X$, we have $\sum_{b \geq 1} l_b \leq 1 - \gamma_1 - \gamma_2$. Thus, the map F defined by

$$F(\mathbf{c}) := (F_b(\gamma_1, c_1, \dots c_{b-1}))_{b \geq 1} \tag{26}$$

maps the set $L_+^{1-\gamma_1-\gamma_2}$ into itself. Since F is continuous in the product topology, it has a fixed point $\mathbf{c}^* = (c_a^*)_{a\geq 1}$ and

$$l_b = F_b(\gamma_1, c_1^*, \ldots, c_{b-1}^*) = c_b^* \quad \text{for all } b \geq 1.$$

Thus, the assumptions of Theorem 6.6 are satisfied and this proves the assertion. □

We turn now to study the convergence to a unique steady state for the example economy with quadratic preferences. Consider the representation

$$g^*(x; \theta^0) = c_0^* x^0 + \gamma_2 \theta^0 + \sum_{a\geq 1} c_a^* x^a.$$

of the policy function g^*. For any two configurations $x, y \in \mathbf{X^0}$ which differ only at site $a \in \mathbb{A}$ we have

$$|g^*(x, \theta^0) - g^*(y, \theta^0)| \leq c_a^* |x^a - y^a|,$$

Thus, assuming that the taste shocks are uniformly distributed on $[-1, 1]$ we obtain

$$|\pi_{g^*}(x; A) - \pi_{g*}(y; A)| \leq 2c_a^*$$

for all $A \in \mathcal{B}([-1, 1])$, and so $\sum_{a\geq 0} r_{g^*}^a < 1$ if $\sum_{a\geq 0} c_a^* < \frac{1}{2}$. Hence in our quadratic case study, we obtain convergence to a steady state whenever α_1 is big enough and if α_3 is small enough, i.e., if the interaction between different agents is not too strong.