The Economics of Contingent Re-Auctions*

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Abstract

We consider an auction environment where the object can be sold with usage restrictions that generate direct benefits to the seller but lower buyers' valuations. In this environment, sellers such as the FCC have used 'contingent re-auctions,' offering the restricted object with a reserve price, but re-auctioning it without the restrictions if the reserve is not met. We show that, in general, contingent re-auctions are neither efficient nor optimal for the seller. We propose an alternative, the 'exclusive-buyer mechanism,' which implements the efficient outcome in dominant strategies and, in certain environments, maximizes the seller's surplus across all feasible selling procedures.

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1 Introduction

Sellers sometimes have the ability to modify objects being offered for sale in ways that provide benefits to themselves or to the general public. For example, in U.S. Spectrum License Auction 73, which was held in January through March of 2008, the U.S. Federal Communications Commission (FCC) offered the licenses with substantial usage restrictions that the FCC viewed as in the public interest. The FCC decided to adopt a 'contingent re-auction' format, where it offered the restricted licenses first, with reserve prices, and committed to re-auction the licenses without many of the restrictions if the reserve prices were not met.

The recent attempted sale of the Italian airline Alitalia followed a similar pattern, although the procedure was not formalized to the same extent as that of the FCC. Alitalia was first put up for sale with a series of restrictions, such as limitations on the ability of the new owner to fire employees. These restrictions were perceived as desirable by the Italian government, but lowered the value of the airline to any bidder. After it became clear that no bidder was interested given the proposed conditions, the airline was put up for sale again with fewer restrictions.[1]

These are examples of auction environments with seller-benefiting restrictions. The seller has the ability to 'damage' the object for sale by imposing usage restrictions and receives a benefit B, in addition to the sale price, if and only if the object is sold in restricted form. Bidders value the unrestricted version of the object more than the restricted one.

Our paper contributes to the theoretical understanding of contingent re-auctions and proposes an alternative mechanism that can improve upon contingent re-auctions in terms of both social efficiency and expected seller surplus. Specifically, we characterize equilibrium behavior in contingent re-auctions. First, we analyze contingent re-auctions involving a pair of ascending-bid auctions, and second we analyze contingent re-auctions involving a pair of sealed-bid second-price auctions. We show that in both cases contingent re-auctions entail significant inefficiencies. We then identify an alternative auction mechanism, the 'exclusive-buyer mechanism,' whose parameters can always be chosen so that the mechanism induces the efficient outcome in dominant strategies.[2] In addition, we show that, in some environments, if

[1] Alitalia was put on sale on January 2007. A number of bidders initially expressed interest but they all dropped out by July 2007. The bidders cited "restrictive conditions imposed by the government and a lack of access to the airline's books. ... These conditions, when originally expressed ahead of the auction, included maintaining certain staff levels, continued operation of some routes and traffic rights regardless of profitability, preserving Alitalia's identity, and not selling certain Alitalia interests for three years." See Aude Lagorce, "Alitalia Still Hoping for Rescue," *MarketWatch*, September 12, 2007.

[2] In our model there are no informational or allocative externalities, i.e., bidders have private values and a buyer not receiving the object has no preference over which competitor receives the object. In environments with interdependent valuations (and possibly allocative externalities as well), Jehiel and Moldovanu (2001)

the parameters of an exclusive-buyer mechanism are chosen with the seller's surplus in mind, the exclusive-buyer mechanism improves upon contingent re-auctions in terms of seller's expected surplus.

An exclusive-buyer mechanism is an auction (either second price or ascending bid) with reserve price r for the exclusive right to choose between being awarded the restricted object at no additional cost and buying the unrestricted object for a fixed incremental payment p.[3] The unique weakly dominant strategy for each bidder is to bid its value for the right to choose.

When the reserve price is zero and the incremental payment p is set equal to the seller's benefit B, the outcome is efficient; that is, the object is sold in restricted form to a buyer with the highest value for the restricted object whenever the sum of the seller's benefit B plus the winner's value exceeds the highest value for the unrestricted object; otherwise the object is sold in unrestricted form to a buyer with the highest value for the unrestricted object. The efficiency properties of the exclusive-buyer mechanism extend to the case where there are multiple possible restrictions on the object, each valued differently by the buyers and each with a different benefit to the seller.

Because we assume that the buyers have private values, the efficient outcome in our model can also be induced by a Vickrey-Clarke-Groves (VCG) mechanism. However, as pointed out by Ausubel and Milgrom (2006) and Rothkopf (2007), Vickrey auctions have certain weaknesses. For example, they require bidders to reveal their true values, which they may not want to do if it weakens their bargaining position in future transactions.[4] In addition, as stated in Rothkopf (2007, p.195), "In government sales of extremely valuable assets, the political repercussions of revealing the gap between large offers and small revenue could be a dominant concern." An exclusive-buyer mechanism using an ascending-bid auction delivers efficiency without these weaknesses.

In an exclusive-buyer mechanism, the parameters r and p can also be chosen to maximize the expected seller surplus within the class of exclusive-buyer mechanisms. We show that when each buyer's value for the restricted object is a fixed percentage of its value for the unrestricted object and the seller's belief satisfies the usual mild regularity condition, the

show that it is generically impossible to implement efficient outcomes. See also Maskin (1992). Mezzetti (2004) shows that efficiency can be obtained with two-stage mechanisms in which payments can be conditioned on reports about the agents' allocation payoffs.

[3] The exclusive buyer mechanism is reminiscent of a right-to-choose auction, which is defined by Burguet (2007, p.167) as "a sequence of auction rounds where the winner of each chooses among the so far unsold goods." See also Gale and Hausch (1994), Burguet (2005), and Eliaz, Offerman, and Schotter (2008). In contrast, in the exclusive buyer mechanism, there is only one round and there is only one good for sale, but that good can be sold subject to restrictions.

[4] It is difficult for sellers such as the FCC to credibly commit to maintain the secrecy of bids because of the Freedom of Information Act.

seller-optimal exclusive-buyer mechanism is also globally optimal—it maximizes the seller's expected surplus across *all* feasible selling mechanisms.

Our results suggest that sellers such as the FCC can improve their auction design in environments where they view restrictions on the objects being sold as in the public interest. By switching from a contingent re-auction format to an exclusive-buyer mechanism, the FCC can always improve the efficiency of its auction and may be able to increase its expected surplus. In some cases, the *efficient* exclusive-buyer mechanism generates greater expected seller surplus than any contingent re-auction. In such cases, the efficient exclusive-buyer mechanism dominates contingent re-auctions in terms of both efficiency and seller surplus. In addition, the parameters of the exclusive-buyer mechanism, reserve r and incremental payment p, can be chosen to maximize any combination of efficiency and seller surplus, which may be valuable for a seller that is interested in both efficiency and its own surplus.[5]

We are not aware of any literature that considers contingent re-auctions directly, but there are several strands of related literature. First, on sequential auctions, Horstmann and LaCasse (1997) show that in common value environments the seller may choose not to sell an object, even if it receives bids above the announced reserve price, and then re-auction the item after some time in order to signal its private information about the value of the object. In the environment we consider, the seller has no private information. Cassady (1967), Ashenfelter (1989), and Porter (1995) indicate that goods that are not sold at an initial auction are often offered for sale again later, but in these cases it is the same items that are re-offered, not a modified version as in the cases we consider. McAdams and Schwarz (2007) show that a seller may benefit from being able to commit to a final round of offers. McAfee and Vincent (1997) consider a model in which a seller cannot commit not to re-auction an object if the announced reserve price is not met. They show that when the time between auctions goes to zero, the seller's expected revenues converge to that of a static auction with no reserve price, and they characterize the optimal dynamic reserve price policy of the seller. In our model, we assume that the seller can commit to a reserve price in the second auction, but we do not exclude the possibility that the reserve at the second auction is zero.

There is also a literature on auctions with resale by the winning buyer; see, e.g., Gupta and Lebrun (1999), Haile (2000, 2001, 2003), Zheng (2002), Garratt and Tröger (2005), Hafalir and Krishna (2007), Garratt, Tröger, and Zheng (2006), Lebrun (2007), and Pagnozzi

[5] The Communications Act of 1934 (as amended by the Telecom Act of 1996) gives the FCC the legal authority to auction spectrum licenses. The language of the Act suggests that efficiency concerns should dominate revenue concerns in the FCC's auction design choices. Section 309(j) of the Act states that one objective of the auctions is "recovery for the public of a portion of the value of the public spectrum," but it also states that "the Commission may not base a finding of public interest, convenience, and necessity solely or predominantly on the expectation of Federal revenues."

(2007). In our model, we assume no resale by buyers.

Mares and Swinkels (2008) consider a procurement environment in which the buyer receives an external benefit if a particular supplier is chosen to supply the object. They characterize the optimal mechanism and show that in many environments an appropriately chosen second-price mechanism dominates a first-price auction with a handicap. The external benefit to Mares and Swinkels' buyer is similar in flavor to our benefit B to the seller. However, in Mares and Swinkels, whether the benefit is received depends on the identity of the winner, and in our paper it does not—it depends on whether the object is allocated in restricted or unrestricted form. Because of this difference, the underlying analytics in the two papers are different. The relevant comparisons in Mares and Swinkels are among the optimal mechanism,[6] second-price mechanisms, and first-price mechanisms. The relevant comparisons in our paper are between contingent re-auctions and exclusive-buyer mechanisms.

In our general model, buyers have two-dimensional types. Each buyer has privately known values for both the unrestricted and restricted versions of the object. As shown in Wilson (1993), Armstrong (1996), Rochet and Choné (1998) and Manelli and Vincent (2007), results for mechanism design problems with multidimensional types can be difficult to obtain. A number of papers have contributed to the development of methods for such problems, including Rochet (1985), Matthews and Moore (1987), McAfee and McMillan (1988), Armstrong (1996), Rochet and Choné (1998) and Manelli and Vincent (2007), where the last three of these papers focus on the case of a multiproduct monopolist.

The problem of maximizing the seller surplus in the one-dimensional version of our model, which we consider in Section 5.2, is related to the classic paper by Mussa and Rosen (1978), which studies a model in which a monopolist offers a quality differentiated spectrum of the same good. Some differences between their model and ours are apparent: in their setup, higher quality versions of the good cost more to produce and there is no benefit to the seller associated with particular quality levels. There is however a way of relabeling variables that shows that the two models have a similar structure. One significant difference is that we have a single object for sale and more than one buyer, hence the seller's problem is not separable across buyers. In related work, Deneckere and McAfee (1996) shows that it can be optimal for a seller to offer both damaged and undamaged versions of a product. In our model, the seller can only offer one version of the product, either restricted or unrestricted, so there is no price discrimination motivation for restricting the use of the product.

The paper proceeds as follows. In Section 2, we describe in more detail the contingent

[6] In Mares and Swinkels, bidders' types are one-dimensional so the optimal mechanism can be derived using standard techniques. In our model, bidders' types are inherently two-dimensional because there are two forms of the object, restricted and unrestricted.

re-auction procedure used by the FCC. In Section 3, we describe our model. In Section 4, we use a simplified version of the model with one-dimensional buyers' types to illustrate the main ideas related to contingent re-auctions. In Section 5, we examine exclusive-buyer mechanisms, including the possibility of implementing the first-best or seller-optimal outcome with an exclusive-buyer mechanism. In Section 6, we provide detailed results for contingent re-auctions in the general model with two-dimensional types. Section 7 provides numerical calculations comparing the various mechanisms. Section 8 concludes with a discussion of implications for mechanism design in environments with seller-benefitting restrictions.

2 Contingent re-auction of spectrum licenses

FCC Auction 73 began on January 24, 2008, and ended on March 18, 2008. It was a large auction with 1,099 spectrum licenses in the 698–806 MHz band, which is referred to as the 700 MHz Band. The auction raised approximately $19 billion for the U.S. government (the largest auction in FCC history).

As usual for an FCC auction, the licenses were defined by their geographic scope and their location in the electromagnetic spectrum. The band plan for the 700 MHz auction defined five blocks of licenses: A, B, C, D, and E.[7] The D-block license was subject to conditions relating to a public/private partnership and provisions for a re-auction of that spectrum in the event its reserve was not met were not specified, so we focus on the other blocks.

For blocks A, B, C and E, the FCC set block-specific aggregate reserve prices and attached significant performance requirements to the licenses. However, the FCC also ordered that if the reserve price for a block was not met, the block would be re-auctioned "as soon as possible" with less stringent requirements at the same reserve price.[8]

The performance requirements for the A, B, C, and E blocks included aggressive build-out requirements, asking license holders to provide service to a minimum percent of the geography covered by the A, B, and E-block licenses, and to a minimum percent of the

[7] The A-block licenses were 12 MHz licenses defined over 176 medium-sized geographic areas referred to as Economic Areas. The B-block licenses were 12 MHz licenses defined over 734 small geographic areas referred to as Cellular Market Areas. The C-block licenses were 22 MHz licenses defined over 12 large geographic areas referred to as Regional Economic Area Groups. In the C block, bidders were also able to submit package bids on three packages: the eight licenses covering the 50 U.S. states, the two Atlantic licenses (covering Puerto Rico, U.S. Virgin Islands, and the Gulf of Mexico), and the two Pacific licenses (covering Guam, Northern Mariana Islands, and American Samoa). The D-block was organized as a single 10 MHz nationwide license. The E-block licenses were 6 MHz licenses defined over the 176 Economic Areas.

[8] See the Second Report and Order (FCC 07-132) at paragraph 307. The reserve prices were: Block A, $1.81 billion; Block B, $1.38 billion; Block C, $4.64 billion; Block D, $1.33 billion; Block E, $0.90 billion. The FCC stated that "Because of the value-enhancing propagation characteristics and relatively unencumbered nature of the 700 MHz Band spectrum, we believe these are conservative estimates." (FCC Public Notice (DA 07-3415), paragraph 54)

population covered by the C-block licenses.[9] Through these build-out requirements, the FCC sought to promote service across as much of the geographic area of the country and to as much of the population as practicable, something that it viewed as being in the public interest. If the reserve prices were not met, the build-out requirements were to be relaxed.

In addition, specifically for the C–block licenses, the FCC required "licensees to allow customers, device manufacturers, third-party application developers, and others to use or develop the devices and applications of their choice, subject to certain conditions."[10] The FCC viewed this requirement of open platforms for devices and applications as being for the benefit of consumers.[11] In the event that the reserve price for the C block was not met, those licenses were to be offered without the open platform conditions.[12]

In the end, the reserve prices for the A, B, C, and E blocks were all met and so the re-auction was not invoked, although the FCC was prepared to do so if the reserves had not been met.

In the remainder of this paper, we analyze a contingent re-auction in the context of a theoretical model, and we contrast it with efficient and optimal mechanisms in an environment with seller-benefitting restrictions.

3 Model

The owner of a single object that can be sold either in restricted or unrestricted form faces a set $N \equiv \{1, ..., n\}$ of potential buyers. Buyer i's payoff is

$$u_i = h_i\, q_i^H + l_i\, q_i^L - m_i,$$

where l_i and h_i denote buyer i's privately known values for the object, in restricted and unrestricted form, q_i^L and q_i^H are the probabilities with which buyer i receives the restricted and unrestricted object, and m_i is buyer i's expected payment to the seller. Buyer i's type (l_i, h_i) is obtained as the realization of an independent random variable with cumulative

[9] The Second Report and Order (FCC 07-132) at paragraph 153 states that for the A, B, and E-block licenses, "licensees must provide signal coverage and offer service to: (1) at least 35 percent of the geographic area of their license within four years of the end of the DTV transition, and (2) at least 70 percent of the geographic area of their license at the end of the license term." For the C-block licenses, "licensees must provide signal coverage and offer service to: (1) at least 40 percent of the population of the license area within four years, and (2) at least 75 percent of the population of the license area by the end of the license term." (paragraph 162)

[10] Second Report and Order (FCC 07-132), paragraph 195.

[11] Second Report and Order (FCC 07-132), paragraph 201.

[12] Second Report and Order (FCC 07-132), paragraph 311. As discussed in paragraph 312, the band plan for the reauctioned C-block would also be modified.

distribution function F_i, density f_i, and support contained in the set

$$\left\{(l,h) \in \left[\underline{l},\bar{l}\right] \times \left[\underline{h},\bar{h}\right] \mid l \leq h\right\},$$

where $\underline{l}, \underline{h} \geq 0$.

The seller derives an extra benefit $B > 0$, in addition to the sale revenue, if and only if the object is sold in restricted form. The seller's costs of departing from the object, in either form, are normalized to zero. Thus, the seller's expected surplus is

$$u_0 = \sum_{i \in N} \left(m_i + B q_i^L\right).$$

We model the contingent re-auction procedure as follows. The seller first offers the *restricted* object for sale in a second-price or ascending-bid auction with reserve price r_1. If any bidder bids r_1 or above, the object is sold in restricted form. Otherwise the object is re-offered, this time in *unrestricted* form, in a second-price or ascending-bid auction with reserve price r_2. If no bidder bids r_2 or above at the second auction, the seller retains the object.

In referring to an ascending-bid auction, we need to be specific about the details of the auction format. A variety of formats are used in practice and in theory. For example, Milgrom and Weber (1982) describe a variant that they refer to as a "Japanese English Auction," where the price increases continuously until all but one bidder (irrevocably) cease to be active. Alternatively, one can assume that reentry is always possible, as is the case at many ascending-bid auctions used in practice.[13] Intermediate versions are possible as well.

Our results do not depend on whether a sealed-bid second-price auction or an ascending-bid auction is used in the second stage to sell the unrestricted object. In the first stage, a sealed-bid second-price auction is equivalent to an ascending-bid auction without reentry (with the same reserve price); but the outcome is different if an ascending-bid auction with reentry is used. Given this, to minimize repetition in what follows, we use the term *second-price auction* to mean either a sealed-bid second-price auction or an ascending-bid auction without reentry, and we use the term *ascending-bid auction* to mean an ascending-bid auction with reentry.

The format used by the FCC in Auction 73 had $r_1 = r_2$ and limited the ability of bidders to freely exit and enter the bidding through "activity" requirements.[14] However, because of

[13] See Izmalkov (2002) on ascending-bid auctions with reentry, and Marshall and Marx (2008) on differences in ascending-bid formats.

[14] In most of its auctions the FCC uses *activity rules*, which prevent bidders from waiting until the end of the auction before becoming active participants. To maintain their eligibility to bid, bidders must bid actively,

the large number of licenses that were offered and because activity was calculated across all licenses, bidders did have some ability to enter and exit the bidding on individual licenses.

We restrict attention to weakly undominated strategies. We say that a bidder 'bids x' to mean that: (i) at a *second-price* auction, the bidder bids x; and (ii) at an *ascending-bid* auction with reserve below x, the bidder starts by bidding the reserve price and remains active in the auction (i.e., continues to attempt to raise the bid if it is not the current high bidder) until the price reaches x. Whenever $l_i \geq r_1$, bidding l_i at the first auction dominates bidding any other value $x \geq r_1$. Similarly, whenever $h_i \geq r_2$, bidding h_i at the second auction dominates bidding any other value $x \geq r_2$. Thus, after ruling out dominated strategies, the conditional re-auction game becomes strategically equivalent to a game in which the strategy set of each bidder i consists of only two strategies: 'go,' i.e., bid l_i, and 'wait,' i.e., do not bid at the first auction unless some other bidder bids at or above r_1, in which case bid l_i if allowed (i.e., if the first auction is ascending-bid), and bid h_i at the second auction.

4 Illustration of ideas for contingent re-auctions

The fundamental issues emerging from our analyses of contingent re-auctions can be illustrated by analyzing a simple version of the model in which the bidders' privately known types are one-dimensional. Section 6 shows that the insights gained here apply more generally. Specifically, in this section, we assume that buyer i's value for the restricted object is given by $l_i = \alpha h_i$, where $\alpha \in (0, 1)$, and the values $h_1, ..., h_n$ are drawn independently from the same cumulative distribution function F with support $[\underline{h}, \bar{h}] \in \mathbf{R}_+$ and density f.

In this simple setting, any bidder with the highest value for the unrestricted object also has the highest value for the restricted object, i.e., $h_i \gtreqless h_j \Leftrightarrow l_i \gtreqless l_j$. This allows us to refer to the 'buyer with the highest value' unambiguously. Using standard notation, we let $h_{(j)}$ denote the j^{th}-highest order statistic among $h_1, ..., h_n$.

Efficiency requires that the object be allocated to a buyer with the highest value $h_{(1)}$, in restricted form, if

$$B + \alpha h_{(1)} > h_{(1)} \quad \Leftrightarrow \quad h_{(1)} < \frac{B}{1 - \alpha}, \tag{1}$$

and in unrestricted form if the opposite inequality holds.

For simplicity, in this section we assume that the reserve price at the second auction r_2 is zero. At the second auction bidder i will bid h_i. Therefore, whenever the second auction is reached, the object is sold (in unrestricted form) to a bidder with the highest value $h_{(1)}$ who pays the second-highest value $h_{(2)}$ and earns surplus $h_{(1)} - h_{(2)}$.

or have a standing high bid in every round. (FCC Auctions Glossary, http://wireless.fcc.gov/auctions/)

Now consider the first auction. Suppose first that an ascending-bid auction is used. If any bidder uses the 'go' strategy, i.e., initiates the bidding at the first auction, then a bidder with value $\alpha h_{(1)}$ will buy the object in restricted form, pay $\max\{r_1, \alpha h_{(2)}\}$, and earn $\alpha h_{(1)} - \max\{r_1, \alpha h_{(2)}\}$.

Thus, bidder i's payoffs from the 'go' and 'wait' strategies respectively, are

$$\pi_i^G \equiv \begin{cases} 0, & \text{if } h_i < h_{(1)} \\ \alpha h_i - \max\left\{r_1, \alpha h_{(2)}\right\} \leq \alpha\left(h_i - h_{(2)}\right), & \text{if } h_i = h_{(1)} \end{cases}$$

and

$$\pi_i^W \equiv \begin{cases} 0, & \text{if } h_i < h_{(1)} \\ \alpha\left(h_i - h_{(2)}\right), & \text{if } h_i = h_{(1)} \text{ and another bidder bids } b \geq r_1 \\ h_{(1)} - h_{(2)}, & \text{if } h_i = h_{(1)} \text{ and no bidder bids } b \geq r_1. \end{cases}$$

Comparing these expressions shows that $\pi_i^G \leq \pi_i^W$. Thus, not bidding at the first auction is the unique weakly dominant strategy for each bidder. We record this result in our first proposition.

Proposition 1 *In the unique equilibrium of the ascending-bid contingent re-auction, no bidder bids at the first auction, implying that the object is always sold in unrestricted form.*

Now suppose that a second-price auction is used in the first period. For sufficiently high values of the reserve price r_1, the equilibrium outcome is as in the the ascending-bid contingent re-auction—nobody bids at the first auction. For lower values of the reserve, the equilibrium behavior at the first auction involves (i) the lowest types not bidding, (ii) an interval of intermediate types randomizing between the 'go' and 'wait' strategies, and (iii) the highest types 'going' if r_1 is sufficiently low and 'waiting' otherwise.

Since the point of this section is to illustrate the main ideas, we only provide a formal characterization of equilibrium strategies for the case with two bidders. When $n > 2$, closed-form solutions for the mixing distributions are not generally available and are cumbersome to describe.

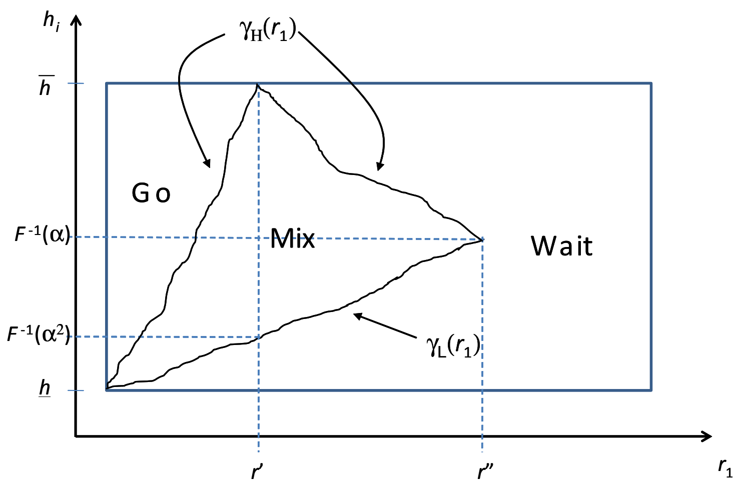
Whenever the reserve price r_1 exceeds the threshold r'' defined by

$$r'' \equiv \alpha E\left[h_i \mid h_i < F^{-1}(\alpha)\right],$$

nobody bids at the first auction. If instead $\alpha \underline{h} < r_1 < r''$, the equilibrium behavior at the first auction is characterized by a partition of the type interval $\left[\underline{h}, \overline{h}\right]$ into three subintervals, generically all of positive measure. The types in the bottom interval do not bid, the types in the middle interval randomize between waiting and bidding, and the types in the top interval bid or not, depending on whether r_1 is below or above the threshold r', defined by

$$r' \equiv \alpha E\left[h_i \mid h_i < F^{-1}\left(\alpha^2\right)\right].$$

Figure 1: Equilibrium strategies at the first auction of a second-price contingent re-auction as a function of the reserve price r_1 at the first auction and a bidder's type h_i



To complete the characterization, we define the boundaries of the region in which bidders mix between the 'go' and 'wait' strategies using functions $\gamma_L, \gamma_H : [\alpha\underline{h}, r''] \to \left[\underline{h}, \bar{h}\right]$ defined by

$$\frac{r_1}{\alpha} = E\left[h_i \mid h_i < \gamma_L\left(r_1\right)\right] \tag{2}$$

and

$$F\left(\gamma_H\left(r_1\right)\right) = \begin{cases} 1 + \alpha - \frac{1}{\alpha}F\left(\gamma_L\left(r_1\right)\right), & \text{if } r' < r_1 < r'' \\ \frac{1}{\alpha^2}F\left(\gamma_L\left(r_1\right)\right), & \text{if } \alpha\underline{h} \le r_1 < r'. \end{cases} \tag{3}$$

Both γ_L and γ_H are continuous, γ_L is always below γ_H. The function γ_L increases from $\gamma_L\left(\alpha\underline{h}\right) = \alpha\underline{h}$ to $\gamma_L\left(r''\right) = F^{-1}\left(\alpha\right)$. The function γ_H first increases, from $\gamma_H\left(\alpha\underline{h}\right) = \alpha\underline{h}$ to $\gamma_H\left(r'\right) = \bar{h}$, and then decreases to $\gamma_H\left(r''\right) = F^{-1}\left(\alpha\right)$. Figure 1 shows the graphs of these functions and indicates the equilibrium behavior for all relevant values of the reserve price r_1.

We are now ready to provide the formal characterization of the equilibrium.

Proposition 2 *In a second-price contingent re-auction with two bidders, there is an equilibrium in which bidder i bids h_i at the second auction if it occurs and uses a strategy of 'go,'* *'mix,' or 'wait' at the first auction as follows:*

'go' if $r_1 \le r'$ and $h_i \ge \gamma_H(r_1)$;

'mix' ('wait' with prob. $\frac{\alpha}{1+\alpha}$ and 'go' with prob. $\frac{1}{1+\alpha}$) if $r_1 \le r''$ and $h_i \in [\gamma_L(r_1), \gamma_H(r_1)]$;

'wait' otherwise.

Proof. See the Appendix.

The main ideas in the proof are as follows. Suppose first that bidder i's opponent always waits. Then the type that is 'most tempted' to bid at the first auction is the one that would win the second auction with probability α, i.e., type $h_0 = F^{-1}(\alpha) \in (\underline{h}, \bar{h})$. The threshold r'' makes this type indifferent between buying the restricted object at price r'' and competing with the opponent at the second auction. Thus, not bidding at the first auction is an equilibrium if and only if $r_1 > r''$. For r_1 slightly below r'', there is a interval of types around h_0 that prefer bidding at the first auction if the opponent always waits. If instead the opponent also bids with positive probability, the incentive to bid decreases. For an appropriately chosen interval around h_0, both waiting and bidding are optimal. As r_1 decreases even further, the incentive to bid at the first auction becomes stronger for types with higher values, and when r_1 passes the lower threshold r', types near the top begin to prefer bidding at the first auction rather than waiting. The interval at the top increases as r_1 diminishes and covers the whole type interval when r_1 reaches the lowest value $\alpha\underline{h}$.

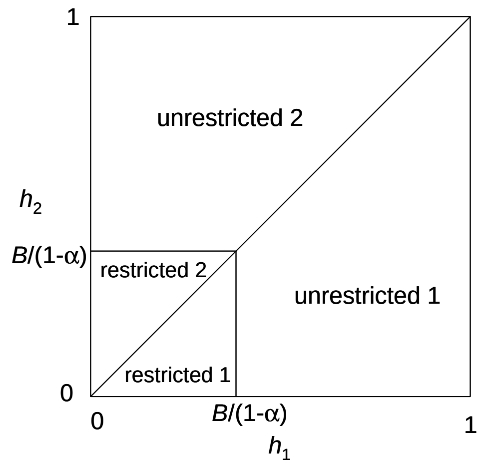
Given the equilibrium behavior in Proposition 2, one can show that the seller maximizes its revenue by choosing a reserve price of either zero or infinity.[15] That is, the seller maximizes its revenue by either holding a single auction for the restricted auction or a single auction for the unrestricted object. This implies that the revenue-maximizing second-price contingent re-auction is inefficient whenever B is such that both the restricted and unrestricted object should be sold with positive probability.

Even when a reserve price is chosen that allows for the possible sale of either the restricted or the unrestricted object, the inefficiencies associated with a second-price contingent re-auction are significant. To give a particular example, suppose $n = 2$ and $\alpha = \frac{3}{4}$, and let F be the uniform distribution on $[0, 1]$. The efficient outcome requires that the object be allocated in restricted form whenever both bidders' values for the unrestricted object are less than $\frac{B}{1-\alpha}$. For this two-bidder example, we can depict the efficient allocations as shown in Figure 2. Generally speaking, the object should be allocated in unrestricted whenever at least one bidder has a sufficiently high value for the unrestricted object, and it should

[15] One can show that the seller's expected revenue as a function of r_1 is convex at any local optimum in the relevant range.

be allocated in restricted form whenever both bidders have sufficiently low values for the unrestricted object.

Figure 2: Efficient allocations



Continuing with this example, if $r_1 = 0.15$, then one can show that $\gamma_L(r_1) = 0.40$ and $\gamma_H(r_1) = 0.71$. Using this and Proposition 2, we can depict the equilibrium allocations as shown in Figure 3.

Figure 3: Equilibrium allocations in contingent re-auctions
$(n = 2,\ l_i = \frac{3}{4}h_i,\ h_i \sim U[0,1],\ r_1 = 0.15)$

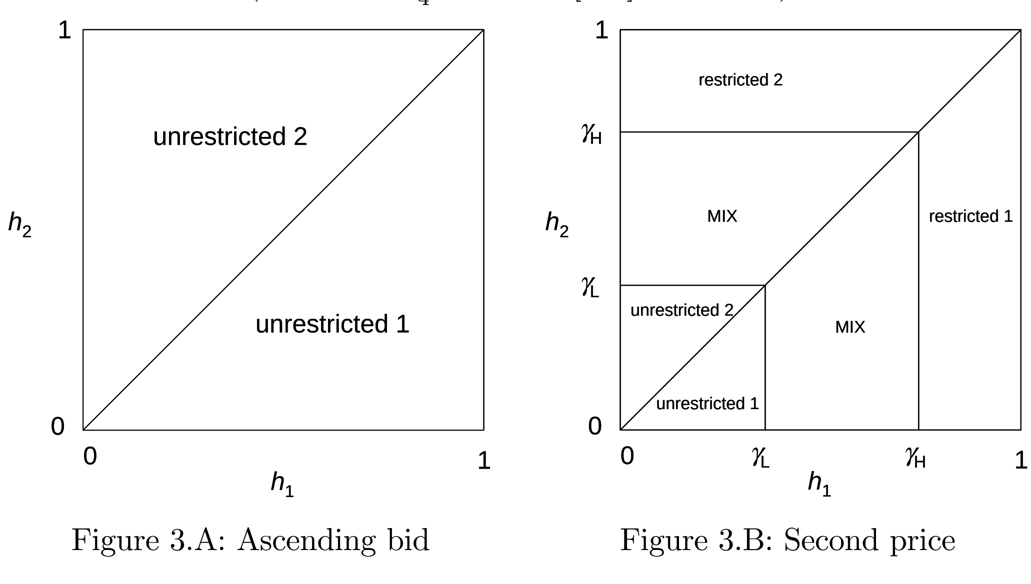


Figure 3.A: Ascending bid

Figure 3.B: Second price

It is striking how different the equilibrium allocations in Figure 3 are from the efficient allocation in Figure 2. In an ascending-bid contingent re-auction, the object is never allocated in restricted form, so the seller never receives the benefit B. In a second-price contingent re-auction with an intermediate reserve price, the object is allocated in restricted form whenever at least one of the bidders has a value for the unrestricted object greater than γ_H, and it is allocated in unrestricted form whenever both bidders have values for the unrestricted object less than γ_L. This is the reverse of what one would want for efficiency. Furthermore, bidders with values between γ_L and γ_H randomize over whether to enter the first auction or not, so when both bidders have values between γ_L and γ_H, it is possible for the restricted object to be allocated inefficiently, even conditional on the restriction being imposed (graphically, it is possible for bidder 1 to win the object 'above the diagonal' and bidder 2 to win it 'below the diagonal').

Regardless of whether an ascending-bid or second-price contingent re-auction is used, when efficiency requires that the object sometimes be allocated in restricted form and sometimes in unrestricted form, it is not possible to achieve the first-best with a contingent re-auction. Given this assessment of the contingent re-auction, it is worth considering whether some other mechanism can do better in terms of efficiency.

5 Exclusive-buyer mechanisms

An exclusive-buyer mechanism is an auction, with reserve price r, for the exclusive right to choose between being awarded the restricted object at no additional cost, and buying the unrestricted object for a fixed incremental payment p. Given the assumption of private values, our results are not affected by whether the auction is second-price or ascending-bid. In either format, the unique weakly dominant strategy for each bidder is to bid its value for the right to choose.

Because of the simplicity of the optimal strategies in an exclusive-buyer mechanism, we can analyze the mechanism in a two-dimensional type environment without difficulty. We begin by establishing the equilibrium strategies in the exclusive-buyer mechanism. Our results apply for exclusive-buyer mechanisms whose first-stage auctions use either a second-price or ascending-bid format.

If bidder i wins the first-stage auction, then in the second stage it chooses the restricted object if $l_i > h_i - p$ and the unrestricted object if $l_i < h_i - p$. The bidder is indifferent if $l_i = h_i - p$. Thus, bidder i receives value $\max\{l_i, h_i - p\}$ from winning the first-stage auction. It follows that there is an equilibrium of the first-stage auction in non-weakly-dominated strategies in which each bidder i bids $\max\{l_i, h_i - p\}$.

5.1 Efficiency

In the efficient allocation, buyer i receives the unrestricted object if

$$h_i > \max\{\max_{j\neq i} h_j, \max_j B + l_j\},$$

and buyer i receives the restricted object if

$$B + l_i > \max\{\max_j h_j, \max_{j\neq i} B + l_j\}.$$

This outcome can be implemented by an exclusive-buyer mechanism with no reserve price and an incremental payment for the unrestricted object equal to B.

To see this, note that bidder i has value $\max\{l_i, h_i - B\}$ from winning the first-stage auction. Thus, the winner of that auction will be the buyer with the maximal value of $\max\{l_i, h_i - B\}$. If buyer i is the winning bidder and $l_i > h_i - B$, then efficiency requires that buyer i receive the object in restricted form. If buyer i is the winning bidder and $l_i < h_i - B$, then efficiency requires that buyer i receive the object in unrestricted form. With an incremental price of B for the unrestricted object in the second stage of the exclusive-buyer mechanism, this is the outcome.

Proposition 3 *The efficient outcome can be implemented with an exclusive-buyer mechanism with no reserve price in the first stage and an incremental payment of B for the unrestricted object in the second stage.*

The efficient mechanism described in Proposition 3 is (ex post) outcome equivalent to a VCG mechanism (see Vickrey, 1961; Clarke, 1971; Groves, 1973; and Green and Laffont, 1977), but it is an indirect mechanism that can be implemented with a simple auction. Thus, this mechanism overcomes many of the implementation issues generally associated with VCG mechanisms (see Rothkopf, 2007). Implementation of the efficient mechanism requires only that the mechanism designer know B.

In addition, the exclusive-buyer mechanism can accommodate arbitrarily many possible restrictions. To see this, let $q \in [0, 1]$ denote the level of restriction, with $q = 0$ denoting no restriction and $q = 1$ the maximum restriction. Let $B(q)$ be the benefit to the seller when the version with restriction q is sold, where B is an increasing function of q. If the incremental price associated with restriction q is $p(q) = B(1) - B(q)$, then the outcome of the exclusive-buyer mechanism is efficient. The mechanism can accommodate either a discrete number or a continuum of possible restrictions.

Although VCG mechanisms can have low or zero revenues for the seller (Ausubel and Milgrom, 2006), as we show in Section 7, the *efficient* exclusive-buyer mechanism has greater expected seller surplus than the *optimal* contingent re-auction in some environments. In addition, as shown in Proposition 4, the exclusive-buyer mechanism can be optimal for the seller in some environments if the reserve price and the price for the unrestricted object are chosen appropriately.

Ausubel and Milgrom (2006) also raise the issue that seller revenues in a VCG can be non-monotonic in the set of bidders and amounts bid. This remains a possibility in the efficient exclusive-buyer mechanism. To see this, note that if the high bidder chooses the unrestricted object, the seller receives revenue from the first-stage auction and from the bidder's purchase of the unrestricted object. If a new bidder is added so that the new winner prefers the restricted object, the price paid at the first-stage auction necessarily increases, but the seller's overall surplus can decrease.

Finally, an exclusive-buyer mechanism for a single object is not vulnerable to collusion by a coalition of losing bidders nor is it vulnerable to the use of multiple bidding identities by a single bidder, issues that can arise in a multi-object VCG (Ausubel and Milgrom, 2006).

5.2 Optimality for the seller

While efficiency may be a key criterion for sellers such as the FCC, other sellers may be more focused on maximizing their own surplus. We now show that in the environment of Section 4 under standard regularity conditions, there exists an exclusive-buyer mechanism that maximizes the seller's expected surplus among *all* feasible selling procedures. Recall that in Section 4 we assume one-dimensional types, with $l_i = \alpha h_i$ for $\alpha \in (0,1)$ and h_i drawn from cumulative distribution function F with density f.

Let $v_H(h) \equiv h - \frac{1-F(h)}{f(h)}$ and $v_L(h) \equiv B + \alpha v_H(h)$. In the appendix as part of the proof of Proposition 4, we show that, using standard mechanism design techniques, the seller's problem boils down to maximizing

$$E_h \left[\sum\nolimits_{i=1}^{n} v_H(h_i)\ q_i^H(h) + v_L(h_i)\ q_i^L(h) \right] \tag{4}$$

subject to

$$\sum\nolimits_{i=1}^{n} \left[q_i^H(h) + q_i^L(h) \right] \leq 1 \qquad \forall h \in \left[\underline{h}, \bar{h} \right], \tag{5}$$

and

$$h_i \geq h_i' \quad \Rightarrow \quad A_i(h_i) \geq A_i(h_i'), \tag{6}$$

where $A_i(h_i) \equiv E_{h_{-i}} \left[q_i^H(h_i, h_{-i}) \right] + \alpha E_{h_{-i}} \left[q_i^L(h_i, h_{-i}) \right]$.

Imposing the usual regularity condition that v_H is strictly increasing, we have that v_L is also strictly increasing, and $0 < v_L'(h_i) = \alpha v_H'(h_i) < v_H'(h_i)$. Thus, v_L crosses v_H at most once from above, and the horizontal axis at most once from below. This implies that there exist two thresholds h_r and h_s such that $\underline{h} \leq h_r \leq h_s \leq \bar{h}$, and

$$w(h_i) \equiv \max\{0, v_L(h_i), v_H(h_i)\} = \begin{cases} v_H(h_i), & \text{if } h_i \in \left[h_s, \bar{h}\right]; \\ v_L(h_i), & \text{if } h_i \in [h_r, h_s]; \\ 0, & \text{if } h_i \in [\underline{h}, h_r]. \end{cases}$$

Setting

$$q_i^H(h) \equiv \begin{cases} 1, & \text{if } v_H(h_i) = w(h_i) > \max_{j \in N_{-i}}\{w(h_j)\} \\ 0, & \text{otherwise;} \end{cases} \tag{7}$$

and

$$q_i^L(h) \equiv \begin{cases} 1, & \text{if } v_L(h_i) = w(h_i) > \max_{j \in N_{-i}}\{w(h_j)\} \\ 0, & \text{otherwise;} \end{cases} \tag{8}$$

maximizes the expression in (4) subject to (5). The functions in (7) and (8) automatically satisfy the monotonicity condition in (6) and thus, together with any n-tuple of payment functions $m_1, ..., m_n$ consistent with (A.9) in the appendix and such that the lowest type of each buyer pays nothing, solve the seller's problem.

We now show that there is an exclusive-buyer mechanism that maximizes the expected seller surplus among all feasible selling procedures. Using the intuition from standard mechanism design, the object should be allocated to the buyer with the highest virtual valuation, provided this is positive. The same intuition carries over in our model, with the twist that the object is allocated in unrestricted form when the 'virtual unrestricted valuation' v^H is higher than the 'virtual restricted valuation' v^L. Thus, surplus maximization requires that the object is not allocated when v^H and v^L are both negative, is allocated in restricted form when $v^L > \max\left\{v^H, 0\right\}$, and is allocated in unrestricted form when $v^H > \max\left\{v^L, 0\right\}$.

Given our assumptions, $v^L < 0$ implies $v^H < -\frac{B}{\alpha} < 0$, so the interval over which both v^L and v^H are non-positive is $[\underline{h}, h_r]$. The interval over which $v^L \geq v^H$ is $[h_r, h_s]$, and the interval over which $v^H \geq v^L$ is $\left[h_s, \bar{h}\right]$.

Thus, an optimal mechanism should exclude buyers with $h < h_r$, assign the object in restricted form when the highest valuation buyer has $h_i \in (h_r, h_s)$, and assign the object in unrestricted form when the highest valuation buyer has $h_i > h_s$. This can be accomplished by the exclusive-buyer mechanism by appropriately choosing the first-stage reserve price and incremental payment.

Proposition 4 *An optimal mechanism for the seller is an exclusive-buyer mechanism with reserve price $r^* = \alpha h_r$ and incremental payment for the unrestricted object $p^* = (1-\alpha)h_s$.*

Proof. See the Appendix.

In the mechanism of Proposition 4, the optimal reserve price and the incremental price for the unrestricted object are independent of the number of buyers. Those parameters depend only on B and the distribution F. If B is zero, then $h_r = h_s$ and so the object is never allocated in restricted form. If B is sufficiently large, then $h_s = \bar{h}$ and so the object is never allocated in unrestricted form.

In the environment of Proposition 4, the optimal mechanism is not necessarily efficient; however, conditional on allocating the object, and conditional on the form of the object allocated, it is always allocated to the highest-valuing buyer.

5.3 Implementation

It would be straightforward for a seller such as the FCC to implement an exclusive-buyer mechanism. The FCC's existing auction software should be able to accommodate exclusive-buyer mechanisms being offered simultaneously for multiple licenses using their simultaneous multiple round auction format, although with multiple licenses and externalities across licenses, the outcome need not be efficient.

Given the FCC's familiarity with offering bidding credits (refunds), it could implement the second stage by offering a bidding credit to a bidder selecting a restricted license in the second stage, rather than charging an additional amount for the unrestricted license. The FCC would need to determine the bidding credit for the restricted licenses. The choice of bidding credit would need to balance the goals of efficiency and the recovery of value for the taxpayers. In contrast, in the contingent re-auction format, the FCC has to determine a reserve price to balance those same goals, but in the contingent re-auction format it may be that no choice of reserve price does a particularly good job on either dimension.

To maximize efficiency in the exclusive-buyer mechanism, the FCC would want to choose a bidding credit equal to the public value associated with the restriction. To maximize seller surplus, numerical examples we have examined suggest that one would want a smaller bidding credit, although the theory allows for the possibility that one might want a larger bidding credit. In balancing the goals of efficiency and revenue, the bidding credit would need to be some intermediate value. Since the expected surplus generated is continuous in the choice of bidding credit, small deviations from the optimal value would not have a large effect on the outcome.

An exclusive-buyer mechanism allows the FCC the option of specifying multiple possible restrictions with different associated bidding credits. For example, the FCC might offer a larger bidding credit to a bidder accepting a build-out requirement of 75% of the territory than to a bidder accepting a build-out requirement of only 60%.

6 General environments for contingent re-auctions

We now address contingent re-auctions in the context of two-dimensional types. We provide a characterization of equilibrium bidding in contingent re-auctions for general environments. For the results in this section, we assume that the value distributions $F_1, ..., F_n$ have common support

$$\left\{(l,h) \in \mathbb{R}_+^2 \mid l \in [\underline{l}, \bar{l}(h)],\ h \in [\underline{h}, \bar{h}]\right\},$$

where $\bar{l}(h) \leq h$ denotes the upper bound on a bidder's value for the restricted object when its value for the unrestricted object is h. We assume that the expectations stated below exist and that the function $\bar{l}(h)$ is differentiable and the derivative is bounded, i.e., $\frac{d\bar{l}}{dh} \leq K$ for each h, where K is a real number.

We start by observing that in any equilibrium with undominated strategies all bidders bid truthfully in the second auction, i.e., bidder i bids (up to) h_i. This is true both for the ascending-bid auction format and the second-price format. Letting $p_h(h_{-i}, r_2) = \max\{r_2, \max_{j \neq i} h_j\}$ denote the price paid by bidder i when it wins the second auction, we can write bidder i's payoff as

$$u_h^i(h_i, h_{-i}, r_2) = \max\{h_i - p_h(h_{-i}, r_2), 0\}.$$

The second round is reached only if all bidders adopt a strategy of 'wait' in the first round. In a second-price auction, this means not entering a bid at or above the reserve price. In an ascending-bid auction, this means not being the first to bid at or above the reserve price.

What is the expected surplus for a bidder of type (l_i, h_i) from adopting the 'wait' strategy? This will depend on how likely it is that other bidders also decide to 'wait' since it is only when all bidders decide to wait that the second stage is reached.

Let $W_j \subset \Theta_j$ be the set of types of bidder j that decide to wait in equilibrium. As long as $r_1 > \underline{l}$ this set will always be non-empty since, as a minimum, all types (l_i, h_i) with $l_i < r_1$ prefer to wait. Given a collection of sets $\{W_1, \ldots, W_n\}$, let $W_{-i} \equiv \Pi_{j \neq i} W_j$. The expected surplus from the second auction for type (l_i, h_i) of bidder i is

$$V_h^i(h_i \mid W_{-i}) = E_{l_{-i}, h_{-i}}\left[u_h(h_i, h_{-i}, r_2) \mid (l_{-i}, h_{-i}) \in W_{-i}\right] \ \Pr\left[(l_{-i}, h_{-i}) \in W_{-i}\right]. \qquad (9)$$

Note that V_h^i only depends on h_i, not on l_i, and is non-decreasing in h_i.

The expected payoff obtained in the first auction depends on the particular auction format that is adopted. However it is clear that, *conditional on participating in the first auction*, the only weakly dominant strategy is to bid l_i. It follows that the expected payoff depends only on l_i since the unrestricted object is never sold. Let $V_l^i(l_i \mid W_{-i})$ be the value of bidding in the first auction when the set of types W_{-i} is expected to 'wait.' Then it is clear that any equilibrium in undominated strategies is determined by a collection of subsets $\{W_1, \ldots, W_n\}$ such that types with $(l_i, h_i) \in W_i$ prefer to 'wait,' while types with $(l_i, h_i) \notin W_i$ prefer to 'go' at the first auction.

In other words, the strategy of a type (l_i, h_i) can be summarized in the simple decision of participating or not in the first auction. Intuitively, it is clear that the expected surplus from the first auction for type (l_i, h_i) of bidder i, which we denote $V_l^i(l_i \mid W_{-i})$, is increasing in l_i. Thus, in equilibrium, for each $h_i > r_2$ there is a threshold value $d_i(h_i)$ such that types with $l_i < d_i(h_i)$ prefer to suppress their bids and wait for the second auction, while types with $l_i > d_i(h_i)$ bid at or above the reserve price in the first auction. Thus, in every equilibrium the set W_i of types of agent i that adopt the 'wait' strategy must take the form

$$W_i = \{(l_i, h_i) \mid l_i \leq d_i(h_i)\} \tag{10}$$

for some function d_i, which we refer to as the delay threshold function.

Proposition 5 *In a contingent re-auction, any perfect Bayesian equilibrium in undominated strategies has the following structure: (i) bidder i bids h_i at the second auction if it occurs; (ii) for all $i \in N$, there exists a nondecreasing and continuous function $d_i^S : [\underline{h}, \bar{h}] \to [r_1, \bar{l}(\bar{h})]$ or $d_i^A : [\underline{h}, \bar{h}] \to [r_1, \bar{l}(\bar{h})]$ for a second-price or ascending-bid auction, respectively, such that in the first auction bidder i with type (l_i, h_i) uses the strategy of 'go' if $l_i > d_i(h_i)$ and 'wait' if $l_i < d_i(h_i)$.*

Proof. See the Appendix.

In general, the threshold functions d_i^S and d_i^A for each bidder i are different because the functions $V_l^i(l_i \mid W_{-i})$ are different depending on the auction format (we remind the reader that the value of $V_h^i(h_i \mid W_{-i})$ does not depend on the auction format). In a second-price auction, once the bids are submitted, no bidder can revise its behavior. Thus, a bidder that chooses to stay out of the first auction cannot react to others bidding above the reservation price r_1 and earns zero surplus. In contrast, an ascending-bid first-round auction (with reentry) allows any bidder to respond to others initiating the bidding in the first round.

To avoid repetition, we focus our analysis mostly on the ascending-bid format and then comment on the second-price format in Section 6.3.

6.1 Existence

From Proposition 5 we know that every equilibrium is characterized by a collection of functions $\{d_1, \ldots, d_n\}$; however, the proposition does not establish existence of an equilibrium. In this subsection, we focus on ascending-bid contingent re-auctions and prove that an equilibrium exists when bidders are symmetric. Although we do not have a proof for the case of asymmetric bidders, we have no reason to believe that equilibria would not exist in that case as well.

We say that the equilibrium is *symmetric* if there is a function d such that $d_i = d$ for each i. The next proposition establishes that a symmetric equilibrium exists when bidders are symmetric. For the purposes of this proposition, we assume that $\bar{l}(h)$ has a bounded derivative.

Proposition 6 *Assuming symmetric bidders, i.e., $F_1 = ... = F_n$, there exists a symmetric perfect Bayesian equilibrium of the ascending-bid contingent re-auction characterized by a nondecreasing and continuous function $d^A : \left[\underline{h}, \bar{h}\right] \to \left[r_1, \bar{l}\left(\bar{h}\right)\right]$.*

The proof of Proposition 6 is based on an application of Schauder's Fixed-Point Theorem.[16] The proof is technical in nature and so to conserve space we relegate that proof to the online appendix associated with this paper.

6.2 Strategic delay

We now turn to the characterization of the symmetric equilibria. One important question for the auction designer is the following. Suppose that a reserve price r_1 is imposed in the first auction. It is obvious that all types with $l_i < r_1$ will not bid in the first auction, but will wait for the second auction. However, it is not true that all types with $l_i > r_1$ are willing to bid in the first auction. The next proposition establishes that, in fact, in any equilibrium there are types with $l_i > r_1$ that prefer to wait for the second auction.

Proposition 7 *Assuming symmetric bidders, in every equilibrium there is an open set of types (l_i, h_i) with $l_i > r_1$ that do not bid in the first auction of an ascending-bid contingent re-auction unless some other bidder enters a bid of r_1 or more.*

[16] See Stokey, Lucas, and Prescott (1989) for a statement of the theorem.

Proof. See the Appendix.

The intuition for Proposition 7 is straightforward: if l_i is close to r_1, bidder i's expected surplus from the first auction is small relative to its expected surplus at the second auction. Thus, if l_i is close to r_1, bidder i prefers not to bid in the first auction. Of course, if some other bidder enters a bid of r_1 or more, a bidder's best reply is then to bid up to its value.

Proposition 7 implies that if the seller wants the object to be sold for values l_i above a threshold $\widehat{l}$, then the reserve price should be set at a level strictly less than $\widehat{l}$. It also implies that in every equilibrium there is 'excessive delay' in the sense that the second auction is reached with a probability strictly higher than $\Pr[\max_{i\in N} l_i \leq r_1]$.

It is worth pointing out that the result of Proposition 7 holds for any reserve price $r_1 > \underline{l}$. In fact, there is a kind of 'multiplier effect' similar to the one discussed in Brusco and Lopomo (2007). When the reservation price is $r_1 > \underline{l}$ all types with $l_i \in [\underline{l}, r_1]$ prefer not to bid in the first auction, thus making the probability of reaching the second auction strictly positive. This implies that types with $l_i = r_1 + \varepsilon$ also prefer to delay if ε is sufficiently small, since the expected gain in the first auction is small. In turn, this increases the probability that the second auction is reached, thus potentially convincing other types to delay the bid. What happens when $r_1 \to \underline{l}$ depends on the distribution of types, but using arguments similar to the ones in Brusco and Lopomo (2007), it is possible to produce examples in which

$$\lim_{r_1 \downarrow \underline{l}} \Pr\left[\text{second auction is reached} \mid r_1\right] > 0.$$

Since at $r_1 = \underline{l}$ there is no delay, the implication is that, under some conditions, the imposition of even a minimal reserve price may produce non-trivial delays.

In some situations delay can be significant. The extreme case is the one in which $d_i^A(h_i) = \max\left\{r_1, \bar{l}(h_i)\right\}$ for each h_i, so that the equilibrium involves delay with probability 1. For this to be an equilibrium it must be the case that every type (l_i, h_i) prefers to wait for the second auction rather than participate in the first. If all the other bidders 'wait,' then the payoff from waiting is simply $E_{h_{-i}}\left[\max\left\{h_i - p_h(h_{-i}, r_2), 0\right\}\right]$. Since for each h_i, the type l_i that receives the highest expected payoff from bidding in the first auction is $\bar{l}(h_i)$, the condition for having an equilibrium involving delay with probability 1 is that for each h_i,

$$E_{l_{-i}}\left[\max\left\{\bar{l}(h_i) - p_l(l_{-i}, r_1), 0\right\}\right] \leq E_{h_{-i}}\left[\max\left\{h_i - p_h(h_{-i}, r_2), 0\right\}\right]. \tag{11}$$

We record this result in the following proposition.

Proposition 8 *Assuming symmetric bidders, if condition* (11) *holds, then there exists an* *equilibrium of the ascending-bid contingent re-auction in which the bidders do not bid in the first auction, and bid up to their values in the second auction, i.e., delay occurs with probability one.*

In general, condition (11) is easier to satisfy if r_1 is large and r_2 is low; in fact it is trivially satisfied if $r_1 = \bar{l}\left(\bar{h}\right)$. When condition (11) is not satisfied, then any equilibrium must involve some bidding activity in the first auction.

We can further characterize the delay threshold function by noting that when $d^A(h_i) < \bar{l}(h)$, the type $\left(d^A(h_i), h_i\right)$ has to be indifferent between bidding in the first auction and waiting, so that

$$\begin{aligned} & E_{l_{-i}}\left[\max\left\{d^A(h_i) - p_l(l_{-i}, r_1), 0\right\} \mid l_j \leq d^A(h_j) \text{ each } j \neq i\right] \\ = \; & E_{h_{-i}}\left[\max\left\{h_i - p_h(h_{-i}, r_2), 0\right\} \mid l_j \leq d^A(h_j) \text{ each } j \neq i\right]. \end{aligned} \tag{12}$$

Using condition (12), we have the following characterization.

Proposition 9 *For an ascending-bid contingent re-auction with symmetric bidders, in any symmetric equilibrium the delay threshold function d^A has derivative*

$$d^{A\prime}(h_i) = \left(\frac{\int_{\underline{h}}^{h_i} F\left(d^A(x) \mid x\right) z(x)\, dx}{\int_{\underline{h}}^{h_i} F\left(d^A(x) \mid x\right) z(x)\, dx + \int_{h_i}^{\bar{h}} F\left(d^A(h_i) \mid x\right) z(x)\, dx}\right)^{n-1} \tag{13}$$

whenever $d^A(h_i) < \bar{l}(h_i)$.

Proof. See the Appendix.

Notice in particular that condition (13) implies $d^{A\prime}(h_i) < 1$ whenever $d^A(h_i) < \bar{l}(h_i)$. This result can be used to fully characterize the delay function in some special cases. For example, consider the case $\bar{l}(h_i) = h_i$ and suppose first that $r_1 < \bar{l}(\underline{h})$. Since any type $(l, \underline{h})$ earns nothing in the second auction, then all types with $l > r_1$ must prefer participation in the first auction, so that $d^A(\underline{h}) = r_1$. For $h_i > \underline{h}$ the delay function grows with slope strictly less than 1, while $\bar{l}(h_i)$ grows with slope 1. Thus, $d^A(h_i)$ never touches $\bar{l}(h_i)$, and we can characterize the function d^A by solving the integral equation (13) over the interval $\left[\underline{h}, \bar{h}\right]$ with initial condition $d^A(\underline{h}) = r_1$. On the other hand, suppose $r_1 \geq \bar{l}(\underline{h})$. In this case it must be that $d^A(\underline{h}) = \bar{l}(\underline{h})$. We then look for a value h^* such that $d^A(h_i) = \bar{l}(h_i)$ for $h_i \leq h^*$, and d^A is characterized by the integral equation (13) otherwise. The initial condition thus requires $V_h^*(h^*) = V_l^*\left(\bar{l}(h^*)\right)$, and a solution is obtained by solving (13) and this initial condition jointly in h^* and d^A, with $d^A(h) = \bar{l}(h)$ for $h \leq h^*$.

As another example, suppose $\bar{l}(h_i) = \bar{l}$ for each h_i (i.e., the support is rectangular). The only interesting case here is $r_1 < \bar{l}$. Thus, $d^A(\underline{h}) = r_1$. The function d^A must then increase with strictly positive slope until it reaches the value $\bar{l}$, and remain equal to $\bar{l}$ afterwards. In particular, suppose that we solve the integral equation (13) over the interval $[\underline{h}, \overline{h}]$ with initial condition $d^A(\underline{h}) = r_1$ and we find that $d^A(\overline{h}) < \bar{l}$. Then we have found the solution. Otherwise, there must be a value h^* such that d^A solves the integral equation (13) over the interval $[\underline{h}, h^*]$ with initial condition $d^A(\underline{h}) = r_1$, and it is equal to $\bar{l}$ when $h \geq h^*$.

6.3 Extension to a second-price format

The analysis for the second-price auction is similar to that for the ascending-bid auction, but the equilibrium bids in the first stage are different. The reason is that the value of not bidding in the first auction is lower because a bidder cannot respond if another bidder enters a bid at or above the reserve price. Because the waiting strategy has a lower payoff, we expect that fewer delays occur under a second-price format than under an ascending-bid format. However, we cannot conclude that in an ascending-bid auction we surely observe more delay than in a second-price auction. What complicates things is that an increased delay by other bidders typically increases both the 'wait' strategy and the value of the 'go' strategy for bidder i. The final equilibrium has to balance these effects, and it is not clear what the final result is.

There are however a couple of observations that we can make. The first relates to the existence of an equilibrium with total postponement. For the ascending-bid format we have seen that such an equilibrium is possible when condition (11) holds for each h_i. Assuming that all other bidders wait with probability 1, the expected value of waiting does not change under a second-price format, but the value of the first auction increases. In fact, if all the other players are expected not to participate in the first auction, bidding in the first auction gives the object to the bidder for sure at price r_1, for a surplus of $l_i - r_1$. Thus, the relevant condition for the existence of an equilibrium with total postponement is that for each h_i,

$$\bar{l}(h_i) - r_1 \leq E[\max\{h_i - p_h(h_{-i}, r_2), 0\}]. \tag{14}$$

Clearly, condition (14) is more difficult to satisfy than condition (11). Thus, equilibria with total postponement are less frequent with a second-price format than with an ascending-bid format.

The second observation is that, with a second-price format, as the number of bidders increases the equilibrium converges to one with 'sincere' bidding, i.e., one in which all bidders with $l_i > r_1$ bid in the first auction. This is proved in the following proposition.

Proposition 10 *Assume symmetric bidders and a symmetric equilibrium of the second-price contingent re-auction and let* d_n^S *be the delay threshold function when there are* n *bidders. Then* $\lim_{n\to\infty} d_n^S(h_i) = r_1$ *for each* h_i*.*

Proof. See the Appendix.

Proposition 10 shows that strategic delay is not an issue for a second-price contingent re-auction as the number of bidders grows large. A similar result does not hold for an ascending-bid contingent re-auction. The easiest way to grasp the intuition is to compare the conditions for the existence of an equilibrium with full postponement. For the second-price format, a quick inspection of condition (14) leads to the conclusion that the condition cannot be satisfied for large values of n, since the right-hand side goes to zero while the left-hand side is unaffected by n. For the ascending-bid auction, the relevant condition is (11). Looking at that condition it is easy to see that *both* sides converge to zero as n goes to infinity. Thus, whether or not the condition holds as n gets large depends on the probability distribution; in fact, it is possible to produce examples in which the equilibrium with total postponement exists for each n.

As a last remark, we observe that, as long as the reserve prices are chosen optimally, a contingent re-auction must do better than a single auction for the unrestricted or the restricted object. By choosing $r_1 = \bar{l}(\bar{h})$ and $r_2 = r^*$ the contingent re-auction yields the same allocation as a single auction for the unrestricted object with reserve price r^*. Similarly, by choosing $r_1 = r^*$ and $r_2 = \bar{h}$ the contingent re-auction yields the same allocation as a single auction for the restricted object with reserve price r^*. Thus, single auctions are just special cases of the contingent re-auction, which implies that it is always possible to do (weakly) better with a contingent re-auction than with a single auction.[17]

7 Comparisons

In this section, we provide some comparisons between the contingent re-auction and the exclusive-buyer mechanism. For comparison purposes, we focus on mechanisms that do not retain the object, i.e., contingent re-auctions where the reserve price in the second stage is $\underline{h}$ and exclusive-buyer mechanisms where the reserve price in the first stage is $\underline{l}$.

Consider an environment with two symmetric bidders drawing their values (l, h) from the uniform distribution on $[0, 1] \times [3, 4]$. Given B, one can use numerical techniques to calculate

[17] It is important to remark that this is true only when the reserve prices are optimally chosen. The choice of optimal reserve prices in the contingent re-auction is quite complicated, as it requires knowledge not only of the distribution $F(l_i, h_i)$, but also of the equilibrium bidding functions in the first auction.

the seller's optimal reserve price for the first stage of the contingent re-auction. We restrict attention to reserve prices in the set $\{0, .05, ..., .95, 1\}$. The optimal first-stage reserve price is decreasing in B, so that that the object is more likely to be sold in restricted form when the benefit B from the restriction is larger.

The seller's surplus is bounded below by B since the seller could always auction the restricted object with zero reserve price. Thus, we focus on the incremental expected surplus above B that is generated by the various mechanisms. Table 1 shows that the exclusive-buyer mechanism generates more incremental surplus than the contingent re-auction, with an improvement of approximately 10% for benefits B that are in the middle of the range of the buyers' values for the unrestricted object.

Table 1: Comparisons of mechanisms that never retain the object for two symmetric buyers with (l, h) uniform on $[0, 1] \times [3, 4]$

B	Optimal first-stage reserve for an ascending-bid contingent re-auction	Optimal price for the unrestricted object in an exclusive buyer mechanism	Increase in incremental expected seller surplus above *B* from the optimal exclusive buyer mechanism relative to the optimal contingent re auction	Increase in incremental expected seller surplus above *B* from the efficient exclusive buyer mechanism relative to the optimal contingent re-auction
3	0.55	3.00	7.2%	7.2%
3.1	0.50	3.07	8.3%	8.1%
3.2	0.45	3.13	9.7%	8.9%
3.3	0.40	3.20	10.8%	9.1%
3.4	0.30	3.27	11.3%	8.9%
3.5	0.20	3.33	11.3%	8.2%
3.6	0	3.40	8.5%	5.2%
3.7	0	3.47	5.9%	2.7%
3.8	0	3.53	3.9%	1.4%
3.9	0	3.60	2.6%	1.0%
4	0	3.67	2.2%	1.4%

The optimal *r* is chosen from a grid of {0,.05,.1,…,.95,1}.
The delay threshold is calculated numerically and approximated with a third-degree polynomial.
Surpluses for the contingent re-auction are calculated using monte carlo simulation based on 20,000 draws done using the statistical software R. All standard errors are less than 0.00002. Surpluses for the exclusive buyer mechanism are calculated analytically.

As shown in Table 1, not only does the optimal exclusive-buyer mechanism provide the seller with greater expected seller surplus than the optimal contingent re-auction, but the *efficient* exclusive-buyer mechanism also provides the seller with greater expected seller surplus than the *optimal* contingent re-auction. (To see this in the table, note that the numbers in the last column are all positive.) Thus, in the environment considered here, the efficient exclusive-buyer mechanism performs better than the optimal contingent re-auction in terms of both efficiency and expected seller surplus.

8 Conclusion

Based on our results, we can offer some comments regarding contingent re-auctions that may improve their implementation, should sellers choose to use that mechanism. Specifically, by using a second-price auction or an ascending-bid auction with strict activity rules (limits on the ability to exit and enter the bidding), one can avoid the extreme equilibria involving no bids in the first auction that arise in contingent re-auctions using ascending-bid auctions that allow bidders to 'wait and see' before entering the bidding.

Loosely speaking, contingent re-auctions allocate the object in restricted form whenever at least one bidder has a sufficiently high value for the restricted object, whereas efficiency requires that the object be allocated in restricted form whenever all bidders have sufficiently low values for the unrestricted object. This suggests that in environments with correlation between bidders values for the restricted and unrestricted object the contingent re-auction will perform poorly in terms of efficiency. This intuition is borne out in our analysis.

Moving away from the contingent re-auction, we identify an easily implemented efficient mechanism for our general environment. It is an exclusive-buyer mechanism with no reserve price and an incremental price for the unrestricted object equal to the seller's benefit from the restriction. Thus, a seller interested in maximizing expected total surplus can do so with no knowledge of the number of bidders or the underlying type distributions and without asking winning bidders to reveal their valuations. The mechanism can be adapted easily to accommodate multiple possible restrictions. Furthermore, in our numerical example, the seller's expected surplus generated by the efficient exclusive-buyer mechanism actually increases slightly relative to the optimal contingent re-auction. Thus, at least in some cases, the seller can achieve an increase in efficiency and an increase it its own expected surplus simultaneously by switching from an optimal contingent re-auction to the efficient exclusive-buyer mechanism. In the example, such a switch also provides benefits to buyers, whose expected surplus increases.

When buyers' types are one-dimensional, the parameters of an exclusive-buyer mechanism can be chosen so that the mechanism maximizes the seller's expected surplus among all feasible selling procedures. With two-dimensional types, the usual technical complications arise. However, preliminary numerical results in Belloni, Lopomo, and Wang (2008) suggest that the exclusive-buyer mechanism in which the reserve price and incremental payment are chosen to maximize the seller's expected surplus performs well when compared with the seller's optimal mechanism.

In future work, we hope to expand and improve upon the results presented here. A particularly interesting extension would allow allocative externalities, where a buyer that does not receive the object might have preferences over which buyer does receive it and whether the object is allocated in restricted or unrestricted form.[18] Also, in our model some of the results for one-dimensional environments do not extend to the general multi-dimensional environment, so we hope to explore more generally when it is and is not appropriate to focus on one-dimensional environments in generally multi-dimensional problems.

To conclude, our primary recommendation is that sellers in environments with seller-benefitting restrictions consider using an exclusive-buyer mechanism, either one tailored to maximize efficiency or seller surplus, depending upon the seller's objectives.

A Appendix – Proofs

Proof of Proposition 2. We consider three cases based on the value of r_1.

Case 1: $r_1 > r''$ 'wait': Suppose first that bidder i's opponent follows the strategy given in the proposition and so always waits. Then bidder i's payoffs from 'waiting' and 'going' are, for all $h_i \in \left[\underline{h}, \overline{h}\right]$,

$$\Pi_i^W(h_i) = \int_{\underline{h}}^{h_i} (h_i - x)\, dF(x) \text{ and } \Pi_i^G(h_i) = \alpha h_i - r_1.$$

Because the difference $\Pi_i^W(h_i) - \Pi_i^G(h_i)$ is convex and its slope is

$$F(h_i) - \alpha = \begin{cases} 1 - \alpha > 0, & \text{if } h_i = \overline{h} \\ -\alpha < 0, & \text{if } h_i = \underline{h}, \end{cases}$$

it follows that $\Pi_i^W(h_i) - \Pi_i^G(h_i)$ is minimized at the point h_0 determined by the equality

$$\frac{d\Pi_i^W(h_i)}{dh_i} = \frac{d\Pi_i^G(h_i)}{dh_i} \quad \Leftrightarrow \quad \alpha = F(h_0).$$

Thus, type h_0 is the 'most tempted' to bid at the first auction, and no other type is willing to deviate if type h_0 has no incentive to do so; that is,

$$\Pi_i^G(h_0) \leq \Pi_i^W(h_0) \quad \Leftrightarrow \quad \alpha h_0 - r_1 \leq \int_{\underline{h}}^{h_0} F(x)\, dx.$$

[18] For a simple model in which one of the buyers values the restriction, see Brusco, Lopomo, and Marx (2008).

The last equality is equivalent to $r_1 \geq r''$. Thus, it is a best reply for bidder i also to follow the strategy given in the proposition.

Case 2: $\alpha\underline{h} < r_1 < r'$ 'wait-mix-go': Suppose bidder i's opponent behaves according to the 'wait-mix-go' strategy. Letting $h' \equiv \gamma_L(r_1)$ and $h'' \equiv \gamma_L(r_1)$, we can write bidder i's payoff functions, and their slopes, corresponding to the two strategies of 'wait' and 'go' as

	$\Pi_i^W(h_i)$	$\frac{d\Pi_i^W(h_i)}{dh_i}$
$h_i \in [\underline{h}, h']$	$\int_{\underline{h}}^{h_i}(h_i - x)dF(x)$	$F(h_i)$
$h_i \in (h', h'')$	$\int_{\underline{h}}^{h'}(h_i - x)dF(x) + \frac{\alpha}{1+\alpha}\int_{h'}^{h_i}(h_i - x)dF(x)$	$\frac{F(h')+\alpha F(h_i)}{1+\alpha}$
$h_i \in [h'', \overline{h}]$	$\int_{\underline{h}}^{h'}(h_i - x)dF(x) + \frac{\alpha}{1+\alpha}\int_{h'}^{h''}(h_i - x)dF(x)$	$\frac{F(h')+\alpha F(h'')}{1+\alpha}$

and

	$\Pi_i^G(h_i)$	$\frac{d\Pi_i^G(h_i)}{dh_i}$
$h_i \in [\underline{h}, h']$	$(\alpha h_i - r_1)\left[F(h') + \frac{\alpha}{1+\alpha}(F(h'') - F(h'))\right]$	$\frac{\alpha^2 F(h'')+\alpha F(h')}{1+\alpha}$
$h_i \in (h', h'')$	$(\alpha h_i - r_1)\frac{F(h')+\alpha F(h'')}{1+\alpha} + \frac{1}{1+\alpha}\int_{h'}^{h_i}(\alpha h_i - \alpha x)dF(x)$	$\frac{\alpha F(h_i)+\alpha^2 F(h'')}{1+\alpha}$
$h_i \in [h'', \overline{h}]$	$(\alpha h_i - r_1)\frac{F(h')+\alpha F(h'')}{1+\alpha} + \frac{1}{1+\alpha}\int_{h'}^{h''}(\alpha h_i - \alpha x)dF(x)$ $+\int_{h''}^{h_i}(\alpha h_i - \alpha x)dF(x)$	$\alpha F(h_i)$

Note that (2) and (3) together imply

$$\Pi_i^G(h') = \Pi_i^W(h'), \tag{A.1}$$

and (3) implies that $\forall h_i \in (h', h'')$,

$$\frac{d\Pi_i^G(h_i)}{dh_i} = \frac{d\Pi_i^W(h_i)}{dh_i}. \tag{A.2}$$

Equations (A.1) and (A.2) imply that $\Pi_i^G(h_i) = \Pi_i^W(h_i)$ for all $h_i \in [h', h'']$, hence mixing is optimal for all types in this interval. Moreover, the two payoff functions have the same slope at both cutoff types:

$$\frac{d\Pi_i^W(h')}{dh_i} = \frac{d\Pi_i^G(h')}{dh_i} = F(h') \qquad \text{and} \qquad \frac{d\Pi_i^W(h'')}{dh_i} = \frac{d\Pi_i^G(h'')}{dh_i} = \alpha F(h'').$$

For $h_i \in [\underline{h}, h']$, Π_i^G is linear, while Π_i^W is strictly convex. Thus, $\forall h_i \in [\underline{h}, h')$, $\Pi_i^W(h_i) > \Pi_i^G(h_i)$. Similarly, for $h_i \in [h'', \overline{h}]$, Π_i^G is strictly convex, while Π_i^W is linear, hence $\forall h_i \in$

$\left(h'', \overline{h}\right]$, $\Pi_i^W(h_i) < \Pi_i^G(h_i)$. Thus, the strategy described in the proposition is a best reply to itself.

Case 3: $r_1 \in (r', r'')$ 'wait-mix-wait': The proof for this case is similar to that for Case 2. Assuming bidder i's opponent behaves according to the 'wait-mix-wait' strategy, bidder i's payoff functions and their slopes corresponding to the strategies of 'wait' and 'go' are:

	$\Pi_i^W(h_i)$	$\frac{d\Pi_i^W(h_i)}{dh_i}$
$h_i \in [\underline{h}, h']$	$\int_{\underline{h}}^{h_i} (h_i - x)dF(x)$	$F(h_i)$
$h_i \in (h', h'')$	$\int_{\underline{h}}^{h'} (h_i - x)dF(x) + \frac{\alpha}{1+\alpha}\int_{h'}^{h_i} (h_i - x)dF(x)$	$\frac{F(h')+\alpha F(h_i)}{1+\alpha}$
$h_i \in \left[h'', \overline{h}\right]$	$\int_{\underline{h}}^{h'} (h_i - x)dF(x) + \frac{\alpha}{1+\alpha}\int_{h'}^{h''} (h_i - x)dF(x) + \int_{h''}^{h_i} (h_i - x)dF(x)$	$F(h_i) - \frac{F(h'')-F(h')}{1+\alpha}$

and

	$\Pi_i^G(h_i)$	$\frac{d\Pi_i^G(h_i)}{dh_i}$
$h_i \in [\underline{h}, h']$	$(\alpha h_i - r_1)\left(F(h') + 1 - F(h'') + \frac{\alpha}{1+\alpha}(F(h'') - F(h'))\right)$ $= (\alpha h_i - r_1)\frac{\alpha+F(h')+1-F(h'')}{\alpha+1}$	$\alpha - \frac{\alpha}{1+\alpha}(F(h'') - F(h'))$
$h_i \in (h', h'')$	$(\alpha h_i - r_1)\frac{\alpha+F(h')+1-F(h'')}{\alpha+1} + \frac{1}{1+\alpha}\int_{h'}^{h_i} (\alpha h_i - \alpha x)dF(x);$	$\alpha - \frac{\alpha}{1+\alpha}(F(h'') - F(h_i))$
$h_i \in \left[h'', \overline{h}\right]$	$(\alpha h_i - r_1)\frac{\alpha+F(h')+1-F(h'')}{\alpha+1} + \frac{1}{1+\alpha}\int_{h'}^{h''} (\alpha h_i - \alpha x)dF(x)$	α

It is routine to check that $\Pi_i^G(h_i) = \Pi_i^W(h_i)$ for all $h_i \in [h', h'']$. Since

$$\frac{d\Pi_i^W(h')}{dh_i} = \frac{d\Pi_i^G(h')}{dh_i} = F(h') \qquad \text{and} \qquad \frac{d\Pi_i^W(h'')}{dh_i} = \frac{d\Pi_i^G(h'')}{dh_i} = \alpha,$$

and Π_i^G is linear while Π_i^W is strictly convex on the both $(\underline{h}, h')$ and $\left(h'', \overline{h}\right)$, it follows that $\forall h_i \in [\underline{h}, h') \cup \left(h'', \overline{h}\right]$, $\Pi_i^W(h_i) > \Pi_i^G(h_i)$. Thus, the 'wait-mix-wait' strategy is a best reply to itself. ■

Proof of Proposition 4. When $l_i = \alpha h_i$, the type space is $\Theta \equiv \left[\underline{h}, \overline{h}\right]^n$. By the revelation principle (see Myerson, 1979), we restrict attention to the set of all direct mechanisms in which the buyers simply report their types and in which reporting truthfully is a Bayesian-Nash equilibrium. Formally, a direct mechanism consists of an *assignment* rule $q = \left(q_i^H, q_i^L\right)_{i=1}^n : \Theta \to \Delta^{2n}$, where Δ^{2n} denotes the simplex in $\mathbf{R}^{2n}$, and a payment rule

$m = (m_1, ...m_n) : \Theta \to \mathbf{R}^n$, specifying, for any profile of reported types $h = (h_1, ..., h_n) \in \Theta$, (i) the probabilities $q_i^H(h)$ and $q_i^L(h)$ that buyer i is awarded the object in unrestricted and restricted form, respectively, and (ii) its payment $m_i(h)$ to the seller. A mechanism (q, m) satisfies *incentive compatibility* if truth-telling is a Bayesian-Nash equilibrium, i.e., $\forall h_i', h_i \in V, \quad \forall i \in N,$

$$\widehat{U}_i(h_i, h_i) \geq \widehat{U}_i(h_i', h_i), \tag{IC}$$

and *individual rationality* if each buyer has no incentive to decline participation, i.e., $\forall h_i \in V, \quad \forall i \in N,$

$$\widehat{U}_i(h_i, h_i) \geq 0, \tag{IC}$$

where $\widehat{U}_i$ denotes buyer i's interim expected payoff function given by

$$\widehat{U}_i(h_i', h_i) \equiv E_{h_{-i}}\left[h_i q_i^H(h_i', h_{-i}) + \alpha h_i q_i^L(h_i', h_{-i}) - m_i(h_i', h_{-i})\right].$$

Standard arguments in mechanism design imply that (q, m) satisfies (IC) and (IR) if and only if for all $i \in N$,

$$U_i(\underline{h}) \geq 0, \tag{A.3}$$

and $\forall h_i \in \left[\underline{h}, \overline{h}\right]$,

$$U_i(h_i) = U_i(\underline{h}) + \int_{\underline{h}}^{h_i} \left(Q_i^H(x) + \alpha\, Q_i^L(x)\right) dx, \tag{A.4}$$

and

$$x \geq x' \quad \Rightarrow \quad Q_i^H(x) + \alpha Q_i^L(x) \geq Q_i^H(x') + \alpha Q_i^L(x'),$$

where $Q_i^x(h_i) \equiv E_{h_{-i}}[q_i^x(h_i, h_{-i})]$ for $x \in \{L, H\}$, $M_i(h_i) \equiv E_{h_{-i}}[m_i(h_i, h_{-i})]$, and

$$U_i(h_i) \equiv h_i Q_i^H(h_i) + \alpha h_i Q_i^L(h_i) - M_i(h_i). \tag{A.5}$$

Thus, buyer i's *ex ante* expected surplus can be written as

$$\begin{aligned}\int_{\underline{h}}^{\bar{h}} U_i(h_i) dF_i(h_i) &= U_i(\underline{h}) + \int_{\underline{h}}^{\bar{h}} \left(\int_{\underline{h}}^{h_i} \left(Q_i^H(x) + \alpha Q_i^L(x)\right) dx\right) dF(h_i) \\ &= U_i(\underline{h}) + \int_{\underline{h}}^{\bar{h}} \left[1 - F_i^H(h_i)\right]\left[Q_i^H(h_i) + \alpha Q_i^L(h_i)\right] dh_i,\end{aligned} \tag{A.6}$$

where the first equality uses (A.4) and the second uses integration by parts.

Setting $U_i(\underline{h}) = 0$ and using (A.5) and (A.6), we can write the seller's expected benefit generated *ex ante* by buyer i as follows:

$$\begin{aligned}\int_{\underline{h}}^{\bar{h}} \left(M_i(h_i) + BQ_i^L(h_i)\right) dF_i^H(h_i) &= \int_{\underline{h}}^{\bar{h}} \left[h_i Q_i^H(h_i) + (\alpha h_i + B) Q_i^L(h_i)\right] dF(h_i) \\ &\quad -\int_{\underline{h}}^{\bar{h}} \left[1 - F_i^H(h_i)\right] \left(Q_i^H(h_i) + \alpha Q_i^L(h_i)\right) dh_i \\ &= \int_{\underline{h}}^{\bar{h}} \left[v_i^H(h_i) Q_i^H(h_i) + v_i^L(h_i) Q_i^L(h_i)\right] dF(h_i).\end{aligned}$$

Finally, summing over all buyers, and 'unpacking' the interim Q's, we can write the seller's objective function as

$$\int_\Theta \sum_{i=1}^n \left[v_i^H(h_i) q_i^H(h) + v_i^L(h_i) q_i^L(h)\right] dF(h).$$

Maximizing pointwise, we find that the object should be given to the buyer with the highest virtual valuation:

$$q_i^H(h) = \chi_i^H(h) \equiv \begin{cases} 1, & \text{if } v_i^H(h_i) > \max\{0, v_i^L(h_i), v_j^H(h_j), v_j^L(h_j), j \neq i\} \\ 0, & \text{otherwise;} \end{cases} \tag{A.7}$$

$$q_i^L(h) = \chi_i^L(h) \equiv \begin{cases} 1, & \text{if } v_i^L(h_i) > \max\{0, \max_{j\neq i} v_j^L(h_j), \max_j v_j^H(h_j)\} \\ 0, & \text{otherwise.} \end{cases} \tag{A.8}$$

The functions $\left(\chi_i^H, \chi_i^L\right)$, together with any payment functions m_i that satisfy the envelope condition in (A.4), i.e.,

$$M_i(h_i) = h_i Q_i^H(h_i) + \alpha h_i Q_i^L(h_i) - \int_{\underline{h}_i}^{h_i} \left[Q_i^H(x) + \alpha Q_i^L(x)\right] dx, \tag{A.9}$$

constitute a solution if and only if the function

$$A_i^*(h_i) \equiv \int_{\Theta_{-i}} \left[\chi_i^H(h) + \alpha \chi_i^L(h)\right] dF_{-i}(h_{-i})$$

is nondecreasing for all $i \in N$. Given the assumption that F is regular, it follows that A_i is nondecreasing. In fact, these assumptions guarantee that the *ex post* assignment functions χ_i^H and χ_i^L defined in (A.7) and (A.8) are such that, for all $i \in N$, the function

$$a_i(h_i, h_{-i}) \equiv \chi_i^H(h_i, h_{-i}) + \alpha \chi_i^L(h_i, h_{-i})$$

is nondecreasing in h_i for all h_{-i}, guaranteeing that $A_i(h_i)$ is nondecreasing. Thus, the mechanism (χ, m^*), where

$$m_i^*(h_i, h_{-i}) \equiv h_i \chi_i^H(h_i, h_{-i}) + \alpha h_i \chi_i^L(h_i, h_{-i}) - \int_{\underline{h}_i}^{h_i} \left(\chi_i^H(z, h_{-i}) + \alpha \chi_i^L(z, h_{-i})\right) dz,$$

satisfies *ex-post* incentive compatibility and *ex-post* individually rationality and maximizes the seller's surplus among all interim incentive compatible and interim individually rational mechanisms.

In the exclusive-buyer mechanism described in the proposition, each buyer i bids $\max\{\alpha h_i, h_i - p^*\}$. Thus, buyer i bids αh_i if $h_i < h^*$ and $h_i - p^*$ if $h_i > h^*$.

The winner is a buyer with value $h_i = h_{(1)}$, as long as its bid is greater than r^*. If the winner has $h_{(1)} < h^*$, it chooses the restricted object, and if it has $h_{(1)} > h^*$, it chooses the unrestricted object. Given the definitions of h^r and h^*, one can show that the exclusive-buyer mechanism implements the allocation required of an optimal mechanism.

If buyer i submits the high bid in the first stage, and that bid is $\alpha h_i < \alpha h^*$, then buyer i pays $\max\{r^*, \max_{j\neq i} \alpha h_j\}$ and chooses the restricted object. If buyer i submits the high bid in the first stage and that bid is $h_i - p^* > \alpha h^*$, then buyer i pays $\max\{r^*, \max_{j\neq i} \max\{\alpha h_j, h_j - p^*\}\}$ and chooses the unrestricted object for an incremental payment of p^*. Thus, letting $y_i \equiv \max_{j\neq i} h_j$ and letting G be the distribution of y_i, buyer i's expected payment is

$$\begin{aligned} M^{EB}(h_i) &= \begin{cases} 0, & \text{if } h_i < h^r \\ E_{y_i}\left[\max\{r^*, \alpha y_i\} 1_{y_i < h_i}\right], & \text{if } h^r < h_i < h^* \\ E_{y_i}\left[\max\{r^*, \max\{\alpha y_i, y_i - p^*\}\} 1_{y_i < h_i}\right] + p^*, & \text{if } h^* < h_i \end{cases} \\ &= \begin{cases} 0, & \text{if } h_i < h^r \\ \alpha h_i G(h_i) - \int_{h^r}^{h_i} \alpha G(x) dx, & \text{if } h^r < h_i < h^* \\ h_i G(h_i) - \int_{h^r}^{h^*} \alpha G(x) dx - \int_{h^*}^{h_i} G(x) dx, & \text{if } h^* < h_i, \end{cases} \end{aligned}$$

which is the expected payment required in an optimal mechanism. ■

Proof of Proposition 5. Let $V_h^i(h_i \mid W_{-i})$ be as defined in (9) and observe that it is a nondecreasing function of h_i. Consider the ascending-bid format. If a type (l_i, h_i) bids r_1 in the first auction then it will trigger participation by all other players. Thus, the price paid in case of victory is $p_l(l_{-i}, r_1) = \max\{r_1, \max_{j\neq i} l_j\}$, which in turn implies that the expected payoff from being the first to bid at r_1 in the first auction is

$$V_l^i(l_i) = E_{l_{-i}}\left[\max\{l_i - p_l(l_{-i}, r_1), 0\}\right].$$

Notice that the payoff is independent of the set W that decides to delay.

The expected payoff from delaying is

$$
\begin{aligned}
&D_i\left(l_i, h_i \mid W_{-i}\right) \\
&= E_{l_{-i}, h_{-i}}\left[\max\left\{l_i - p_l\left(l_{-i}, r_1\right), 0\right\} \mid \left(l_{-i}, h_{-i}\right) \notin W_{-i}\right] \Pr\left[\left(l_{-i}, h_{-i}\right) \notin W_{-i}\right] + V_h^i\left(h_i \mid W_{-i}\right).
\end{aligned}
$$

Thus, in equilibrium, the types that choose to delay must have $V_l^i\left(l_i\right) \leq D_i\left(l_i, h_i \mid W_{-i}\right)$. Let

$$
d_i^A\left(h_i\right) \equiv \sup\left\{l_i \in \left[0, \bar{l}(h_i)\right] \mid V_l^i\left(l_i\right) \leq D_i\left(l_i, h_i \mid W_{-i}\right)\right\}.
$$

After manipulations, $V_l^i\left(l_i\right) \leq D_i\left(l_i, h_i \mid W_{-i}\right)$ can be written as

$$
\begin{aligned}
&E_{l_{-i}, h_{-i}}\left[\max\left\{l_i - p\left(l_{-i}, r_1\right), 0\right\} \mid \left(l_{-i}, h_{-i}\right) \in W_{-i}\right] \\
&\leq E_{l_{-i}, h_{-i}}\left[\max\left\{h_i - p_i\left(h_{-i}, r_2\right), 0\right\} \mid \left(l_{-i}, h_{-i}\right) \in W_{-i}\right],
\end{aligned} \tag{A.10}
$$

so that the left-hand side depends on l_i only and the right-hand side depends on h_i only. Since the left-hand side of (A.10) is nondecreasing in l_i, it follows that for each h_i the types $\left(l_i', h_i\right)$ with $l_i' \in \left[0, d_i^A\left(h_i\right)\right]$ prefer to delay, while types $\left(l_i', h_i\right)$ with $l_i' > d_i^A\left(h_i\right)$ prefer to participate in the first auction. The function $d_i^A\left(h_i\right)$ is nondecreasing because the right-hand side of inequality (A.10) is nondecreasing, and it is continuous because the left-hand side of (A.10) is continuous in l_i and the right-hand side of (A.10) is continuous in h_i.

The proof for the second-price auction follows similar steps, so we will just sketch it. Let $b_j\left(l_j, h_j\right)$ be the bidding function used by bidder j in the first auction. Thus,

$$
W_j = \left\{\left(l_j, h_j\right) \mid b_j\left(l_j, h_j\right) < r_1\right\}.
$$

Bidding $l_i \geq r_1$ in the first auction yields

$$
V_l^i\left(l_i \mid W_{-i}\right) = E_{l_{-i}, h_{-i}}\left[s_i\left(l_i, h_{-i}, l_{-i}; b_{-i}\right)\right],
$$

where

$$
s_i\left(l_i, h_{-i}, l_{-i}; b_{-i}\right) \equiv \begin{cases} l_i - \max\left\{r_1, \max_{j \neq i} b_j\left(l_j, h_j\right)\right\}, & \text{if } \max_{j \neq i} b_j\left(l_j, h_j\right) < l_i \\ 0, & \text{otherwise.} \end{cases}
$$

In equilibrium, the types that decide to delay must be such that $V_l^i\left(l_i \mid W_{-i}\right) \leq V_h^i\left(h_i \mid W_{-i}\right)$, and at this point we can apply the same reasoning used for the ascending-bid format. ■

Proof of Proposition 7. Suppose not. By Proposition 5 the equilibrium bidding functions in the first auction must be such that $d_i^A\left(h_i\right) = r_1$ for each i and $h_i \in \left[\underline{h}, \overline{h}\right]$. Suppose that all bidders other than i use this strategy and consider the best response of type $(r_1+\varepsilon, h_i)$ of bidder i, with $h_i > r_2$. The expected payoff from bidding zero in the first auction is

$$V_h(h_i) = \int_{\underline{h}}^{h_i} \cdots \int_{\underline{h}}^{h_i} \int_{\underline{l}}^{r_1} \cdots \int_{\underline{l}}^{r_1} \left(h_i - \max\left\{r_2, \max_{j\neq i} h_j\right\}\right) \Pi_{j\neq i} f(l_j, h_j)\, dl_{-i} dh_{-i},$$

and the expected payoff from bidding $l_i \geq r_1$ in the first auction is

$$V_l^i(r_1+\varepsilon) \leq \int_{\underline{h}}^{\overline{h}} \cdots \int_{\underline{h}}^{\overline{h}} \int_{\underline{l}}^{r+\varepsilon} \cdots \int_{\underline{l}}^{r+\varepsilon} (r_1+\varepsilon-r_1)\, \Pi_{j\neq i} f(l_j, h_j)\, dl_{-i} dh_{-i} < \varepsilon.$$

For $h_i > r_2$, $V_h^i(h_i) > 0$, which implies that there is some value $\bar{\varepsilon}(h_i) > 0$ such that $V_l(r_1+\varepsilon) < V_h^i(h_i)$ for each $\varepsilon < \bar{\varepsilon}(h_i)$. This implies that it is not a best response for types (l_i, h_i) with $l_i \in (r_1, r_1 + \bar{\varepsilon}(h_i))$ to bid their value in the first auction. ■

Proof of Proposition 9. In this proof we assume an ascending-bid auction, so to improve readability, we use the notation d instead of d^A for the delay threshold. If we define

$$V_h^*(h_i) = \int_{\underline{h}}^{h_i} \cdots \int_{\underline{h}}^{h_i} \int_{\underline{l}}^{d(h_1)} \cdots \int_{\underline{l}}^{d(h_n)} \left(h_i - \max\left\{r_2, \max_{j\neq i} h_j\right\}\right) \Pi_{j\neq i} f(l_j, h_j)\, dl_{-i} dh_{-i}$$

and

$$\begin{aligned} V_l^*(l_i) &= \int_{\underline{h}}^{\overline{h}} \cdots \int_{\underline{h}}^{\overline{h}} \int_{\underline{l}}^{\min\{l_i, d(h_1)\}} \cdots \int_{\underline{l}}^{\min\{l_i, d(h_n)\}} \left(l_i - \max\left\{r_1, \max_{j\neq i} l_j\right\}\right) \\ &\quad \cdot \Pi_{j\neq i} f(l_j, h_j)\, dl_{-i} dh_{-i}, \end{aligned}$$

the condition (12) can be written as $V_l^*(d(h_i)) = V_h^*(h_i)$. If the delay threshold function d is differentiable at h_i and $d(h_i) < \bar{l}(h_i)$, then

$$\frac{dV_h^*(h_i)}{dh_i} = \frac{dV_l^*(d(h_i); d)}{dl_i} d'(h_i),$$

which implies

$$d'(h) = \frac{\frac{dV_h^*(h_i)}{dh_i}}{\frac{dV_l^*(d(h_i); d)}{dl_i}}. \tag{A.11}$$

Since types are independent across bidders we have

$$\begin{aligned}\frac{dV_h^*(h_i)}{dh_i} &= \int_{\underline{h}}^{h_i}\cdots\int_{\underline{h}}^{h_i}\int_{\underline{l}}^{d(h_1)}\cdots\int_{\underline{l}}^{d(h_n)}\Pi_{j\neq i}f(l_j,h_j)\,dl_{-i}dh_{-i} \\ &= \left(\int_{\underline{h}}^{h_i}F(d(x)\mid x)\,z(x)\,dx\right)^{n-1}\end{aligned}$$

and

$$\begin{aligned}\frac{dV_l^*(l_i;d)}{dl_i} &= \int_{\underline{h}}^{\overline{h}}\cdots\int_{\underline{h}}^{\overline{h}}\int_{\underline{l}}^{\min\{l_i,d(h_1)\}}\cdots\int_{\underline{l}}^{\min\{l_i,d(h_1)\}}\Pi_{j\neq i}f(l_j,h_j)\,dl_{-i}dh_{-i} \\ &= \left(\int_{\underline{h}}^{\overline{h}}F(\min\{l_i,d(x)\}\mid x)\,z(x)\,dx\right)^{n-1}.\end{aligned}$$

Thus, condition (A.11) can be written as

$$d'(h_i) = \left(\frac{\int_{\underline{h}}^{h_i}F(d(x)\mid x)\,z(x)\,dx}{\int_{\underline{h}}^{\overline{h}}F(\min\{d(h_i),d(x)\}\mid x)\,z(x)\,dx}\right)^{n-1},$$

and since the function d is nondecreasing this is equivalent to the expression in (13). ■

Proof of Proposition 10. In this proof we assume a second-price auction, so to improve readability, we use the notation d instead of d^S for the delay threshold. When n is sufficiently large, condition (14) cannot hold. Thus, in every equilibrium we have $d(h_i) < \bar{l}(h_i)$ for some h_i. Equation (13), which is obtained for the case of the ascending-bid format, shows that whenever $d(h_i) < \bar{l}(h_i)$ then $d'(h_i)$ is less than 1 and $d'(h_i)$ converges to zero as n goes to infinity. A similar argument can be made for the second price format. In that case we can show that the slope is given by

$$d_n'(h_i) = \left(\frac{\int_{\underline{h}}^{h_i}F(d(x)\mid x)\,z(x)\,dx}{\int_{\underline{h}}^{\overline{h}}F(\max\{d(h),d(x)\}\mid x)\,z(x)\,dx}\right)^{n-1}$$

so $\lim_{n\to+\infty} d_n'(h_i) = 0$. This in turn implies

$$\lim_{n\to+\infty}\int_{\underline{h}}^{h_i} d_n'(x)\,dx = 0.$$

If $r_1 \leq \bar{l}(\underline{h})$ then $\lim_{n\to+\infty} d_n(\underline{h}) = r_1$ and

$$\lim_{n\to+\infty} d(h_i) = \lim_{n\to+\infty} \left(d_n(\underline{h}) + \int_{\underline{h}}^{h_i} d_n'(x)\,dx \right) = r_1$$

for each h_i. If $r_1 > \bar{l}(\underline{h})$ then for each ε let n_ε be such that for every $n \geq n_\varepsilon$ the slope of $d_n' < \varepsilon$, and let $h_{n_\varepsilon}^*$ be the point at which the function $d(h_i)$ departs from $\bar{l}(h)$ and the slope of the function d_n becomes lower than ε. Notice further that, for each n, type $\left(\bar{l}(h_n^*), h_n^*\right)$ must be indifferent between bidding and not bidding in the first auction. The solution must therefore satisfy

$$\frac{V_h^*(h_n^*; d_n^*)}{V_l^*\left(\bar{l}(h_n^*); d_n^*\right)} = 1$$

for each n, which in turn implies that the limit as n grows to infinity must also be 1.

Let $h_\infty^* = \lim_{n\to\infty} h_n^*$. We want to show that $\lim_{n\to\infty} \bar{l}(h_n^*) = r_1$. Notice that we must have $\lim_{n\to\infty} d_n^*(h) = \bar{l}(h_\infty^*)$ for each h, we have

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{V_h^*(h_n^*; d_n^*)}{V_l^*\left(\bar{l}(h_n^*); d_n^*\right)} \\ &= \lim_{n\to\infty} \frac{\left(h_n^* - E^n\left[\max\left\{r_2, \max_{j\neq i} h_j\right\} \mid h_j < h_n^*, l_j < \bar{l}(h_n^*) \text{ each } j \neq i\right]\right)}{\left(\bar{l}(h_n^*) - r_1\right)} \\ &\quad \cdot \lim_{n\to\infty} \left(\frac{\Pr\left[h_j < h_n^*, l_j < \bar{l}(h_n^*)\right]}{\Pr\left[l_j < \bar{l}(h_n^*)\right]} \right)^{n-1}, \end{aligned} \tag{A.12}$$

where E^n denotes the expectation taken when there are n bidders. Here we have exploited the fact that, since the slope of d_n becomes arbitrarily close to zero after h_n^*, types $\bar{l}(h_n^*)$ wins only when all other types do not participate, i.e. $\Pr\left[l_j < \bar{l}(h_n^*)\right]$. Thus $V_l^*\left(\bar{l}(h_n^*); d_n^*\right)$ converges to $\left(\bar{l}(h_n^*) - r_1\right)\left(\Pr\left[l_j < \bar{l}(h_n^*)\right]\right)^{n-1}$.

Since the second limit is zero, the only way in which (A.12) can be satisfied is if

$$\lim_{n\to\infty} \frac{\left(h_n^* - E^n\left[\max\left\{r_2, \max_{j\neq i} h_j\right\} \mid h_j < h_n^*, l_j < \bar{l}(h_n^*) \text{ each } j \neq i\right]\right)}{\left(\bar{l}(h_n^*) - r_1\right)} = \infty,$$

but for this to be the case the denominator must converge to zero, thus establishing $\lim_{n\to\infty} \bar{l}(h_n^*) = r_1$. ■

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