Sieve-based Empirical Likelihood under Semiparametric Conditional Moment Restrictions

Martin Burda*
Department of Economics, University of Pittsburgh

JOB MARKET PAPER

Updates available at www.pitt.edu/∼mab256

First draft: November 7, 2006
This version: January 13, 2007

Abstract

In this paper we propose a new Sieve-based Locally Weighted Conditional Empirical Likelihood (SLWCEL) estimator for models of conditional moment restrictions containing finite dimensional unknown parameters θ and infinite dimensional unknown functions h. The SLWCEL is a one-step information-theoretic alternative to the GMM-type sieve minimum distance estimators proposed by Ai and Chen (2003). We approximate h with a sieve and estimate both θ and h simultaneously conditional on exogenous regressors. Thus, the estimator permits dependence of h on endogenous regressors and θ. We establish consistency and convergence rates for the estimator and asymptotic normality for its parametric component of θ. The SLWCEL generalizes in two ways the Conditional Empirical Likelihood (CEL) of Kitamura, Tripathi and Ahn (2004) designed to supplant the classical GMM in parametric conditional moment restrictions models. First, we construct the CEL's dual global MD-objective function with a new weighting scheme that adapts to local inhomogeneities in the data. Second, we extend the resulting new estimator into the semiparametric environment defined by the presence of h. We show that the corresponding estimator of θ exhibits better finite-sample properties than found in the previous literature.

Keywords: Semi-/nonparametric conditional moment restrictions, empirical likelihood, sieve estimation, endogeneity.
JEL Classification: C13, C14, C20, C30.

---

*The author is grateful to Mehmet Caner, Nese Yildiz, Soiliou Daw Namoro, George-Levi Gayle and Taisuke Otsu for insightful comments and discussions. The author would also like to thank participants in seminars at the University of Pittsburgh, and the 16^{th} Annual Meeting of the Midwest Econometrics Group, Cincinnati, OH, October 2006. Address for Correspondence: Martin Burda, Department of Economics, University of Pittsburgh, 4521 Posvar Hall, 230 S. Bouquet St., Pittsburgh, PA 15260, USA. E-mail: mab256@pitt.edu; Phone: 412-600-7293; Fax: 412-648-1793.

1 Introduction

Moment restrictions frequently provide the basis for estimation and inference in economic problems. A general framework for analyzing economic data (Y, X) is to postulate conditional moment restrictions of the form

$$E\left[g\left(Z, \alpha_0\right)|X\right] = 0 \tag{1}$$

where $Z \equiv (Y', X_z')'$, Y is a vector of endogenous variables, X is a vector of conditioning variables (instruments), X_z is a subset of X, $g(\cdot)$ is a vector of functions known up a parameter α, and $F_{Y|X}$ is assumed unknown. The parameters of interest $\alpha_0 \equiv (\theta_0', h_0')'$ contain a vector of finite dimensional unknown parameters θ_0 and a vector of infinite dimensional unknown functions $h_0(\cdot) \equiv (h_{01}(\cdot), ..., h_{0q}(\cdot))'$. The inclusion of h_0 renders the condition (1) semiparametric, encompassing many important economic models. It includes for example the partially linear regression $g\left(Z, \alpha_0\right) = Y - X_1'\theta_0 - h_0(X_2)$ analyzed by Robinson (1988) and the index regression $g\left(Z, \alpha_0\right) = Y - h_0(X'\theta_0)$ studied by Powell et al. (1989) and Ichimura (1993).

Recently, Kitamura, Tripathi and Ahn (2004) analyzed the Conditional Empirical Likelihood (CEL)[1] based on a parametric counterpart of (1) (with θ only) that was shown to exhibit finite-sample properties superior to the Generalized Method of Moments. In this paper we first suggest a new Locally Weighted CEL (LWCEL) that fundamentally changes the form of CEL and further improves on it in terms of finite-sample properties. Then we extend the LWCEL to the semiparametric environment of model (1) proposing new Sieve-based Locally Weighted Conditional Empirical Likelihood (SLWCEL) estimator. The SLWCEL can be viewed as a one-step information-theoretic alternative to the GMM-type sieve minimum distance estimator analyzed by Ai and Chen (2003). In the remainder of the introduction we will elaborate on the heuristic origins of both estimators, and further analysis will follow thereafter.

1.1 Conditional Moments Based on θ_0

Without the unknown functions h_0, model (1) becomes the parametric model of conditional moment restrictions

$$E\left[g\left(Z, \theta_0\right)|X\right] = 0 \tag{2}$$

Typically, faced with the model (2) for estimation of θ_0, researchers would pick an arbitrary matrix-valued function $a(X)$ and estimate the unconditional moment model $E\left[a(X)g\left(Z, \theta_0\right)\right] = 0$ implied by (2) with an estimator such as the Generalized Method of Moments (GMM) (see e.g. Kitamura, 2006,

[1] A note on terminology: CEL is called "smoothed" and "sieve" empirical likelihood in KTA and Zhang and Gijbels (2003), respectively. Other types of smoothing have been introduced by Otsu (2003a) on moment restrictions in the quantile regression setting and hence KTA's original method is referred to as "conditional" empirical likelihood to avoid confusion. The CEL terminology was also adopted in Kitamura (2006).

p 26 for a discussion). This procedure is used under the presumption that the chosen instrument $a(X)$ identifies θ, which may not be true even if θ is identified in the conditional model (2) (Domínguez and Lobato, 2004). Moreover, the conversion to unconditional moments results in a loss of efficiency with respect to the information contained in (2). Chamberlain (1987) showed that such loss can be avoided by using the optimal IV estimator $a^*(X) = D'(X)V^{-1}(X)$ where $D(X) = E\left[\nabla_\theta g\left(Z, \theta_0\right)|X\right]$ and $V(X) = E\left[g\left(Z, \theta_0\right) g\left(Z, \theta_0\right)' |X\right]$. In practice, $a^*(X)$ can be estimated with a two-step procedure (Robinson, 1987; Newey, 1993). First an inefficient preliminary estimator $\widetilde{\theta}$ for θ_0 is obtained and the unknown functions $D(X)$ and $V(X)$ are estimated via a nonparametric regression of $\nabla_\theta g(Z, \widetilde{\theta})$ and $g(Z, \widetilde{\theta})g(Z, \widetilde{\theta})'$ on X. Second, the estimate of $a^*(X)$ is constructed with the estimates of $D(X)$ and $V(X)$ from the first step. However, as noted by Domínguez and Lobato (2004), the resulting moment condition $E\left[a^*(X)g\left(Z, \theta_0\right)\right] = 0$ may fail to identify θ while θ is identified under the original model (2). Moreover, satisfactory implementation of the nonparametric regression may require large samples thereby affecting the finite-sample performance of the feasible estimator of $a^*(X)$.

The methods typically employed for estimation of the unconditional model $E\left[a(X)g\left(Z, \theta_0\right)\right] = 0$ have also been subject to criticism. While the optimally-weighted two-step GMM (Hansen, 1982) is first-order asymptotically efficient, its finite sample properties have been reported as relatively poor. For example, a simulation study by Altonji and Segal (1996) documented a substantial small-sample bias of GMM when used to estimate covariance models. Other Monte Carlo experiments have shown that tests based on GMM often have true levels that differ greatly from their nominal levels when asymptotic critical values are used (Hall and Horowitz, 1996). Indeed, it has been widely recognized that the first-order asymptotic distribution of the GMM estimator provides a poor approximation to its finite-sample distribution (Ramalho, 2005).

A number of alternative estimators have been suggested to overcome this problem: Empirical Likelihood (EL) (Owen, 1988; Qin and Lawless, 1994; Imbens, 1997), the Continuous Updating Estimator (CUE) (Hansen et al., 1996) the Exponential Tilting Estimator (ET) (Kitamura and Stutzer, 1997; Imbens et al., 1998) and variations on these such as the Exponentially Tilted Empirical Likelihood (ETEL) (Schennach, 2006). The EL, CUE and ET belong to a common class of so-called Generalized Empirical Likelihood (GEL) estimators (Smith, 1997; Newey and Smith, 2004). These estimators circumvent the need for estimating a weight matrix in the two-step GMM procedure by directly minimizing an information-theory-based concept of closeness between the estimated distribution and the empirical distribution. A growing body of Monte Carlo evidence has revealed favorable finite-sample properties of the GEL estimators compared to GMM (see e.g. Ramalho, 2005, and references therein).

Recently, Newey and Smith (2004) showed analytically that while GMM and GEL share the same first-order asymptotic properties, their higher-order properties are different. Specifically, while the asymptotic bias of GMM often grows with the number of moment restrictions, the relatively smaller bias of EL does not. Moreover, after EL is bias corrected (using probabilities obtained from EL) it is higher-order efficient relative to other bias-corrected estimators.[2]

It is worth emphasizing that the GMM and GEL estimators mentioned so far are all based on *unconditional* moment restrictions burdened by the potential pitfalls described above. In addressing this problem, Kitamura, Tripathi and Ahn (2004) (henceforth KTA) recently developed a Conditional Empirical Likelihood (CEL) estimator that makes efficient use of the information contained in (2). Their one-step estimator achieves the semiparametric efficiency bound without explicitly estimating the optimal instruments. Similar analysis has been performed by Antoine et al. (2006) for the case of Euclidean conditional likelihood[3] and Smith (2003, 2005) for the GEL family of estimators.

As the first contribution of this paper, we propose a new form of the CEL estimator for models of conditional moment restrictions (2). Our estimator extends the one proposed by KTA. In particular, the CEL estimator can be expressed in a Minimum Distance (MD) dual representation which is informative about the underlying stochastic mechanism behind CEL. In previous forms of CEL the MD objective function consisted of a simple sum of local discrepancy measures. We propose a weighted sum that accounts for the overall relative importance of each local discrepancy measure in the MD objective function. In this way we utilize information about local inhomogeneities in the data that has not been previously exploited. An additional improvement in construction of each individual local discrepancy measure is gained by using flexible locally adaptive weights. Consequently, the new Locally Weighted CEL estimator (LWCEL) takes on a new form that differs from the currently available CEL format. In a Monte Carlo study we show that the LWCEL estimator exhibits better finite-sample properties than found in the previous literature. However, additional complications arise in the asymptotic analysis due to a newly introduced weighting term. An extension of LWCEL to the whole GEL family in the spirit of Smith (2003, 2005) is currently subject to our research and we plan to include it into further updates of this paper. Assessment of analytical higher-order properties along the lines of Newey and Smith (2004) remains beyond the scope of this paper.

1.2 Conditional Moments Based on (θ_0, h_0)

A semiparametric extension of (2) to model (1) is unquestionably desirable because economic theories seldom produce exact functional forms, and misspecifications in functional forms may lead to inconsistent parameter estimates. By specifying the model partially (i.e. including h_0 as part of the unknown parameters), the inconsistency problem can be alleviated. In general, semiparametric literature related to the model (1) has been growing rapidly (see e.g. Powell, 1994; Pagan and Ullah, 1999, for reviews). Most of the available results are derived using a plug-in procedure: first h_0 is

[2] Accordingly, the initial focus of this paper lies in EL as opposed to any other member of the GEL family of estimators.

[3] Antoine et al. (2006) show that the Euclidean empirical likelihood estimator coincides with the continuously updated GMM (CUE-GMM) as first proposed by Hansen et al. (1996).

estimated nonparametrically by $\widehat{h}$ and then θ_0 is estimated using a parametric method (e.g. GMM or GEL) with h_0 replaced by $\widehat{h}$. However, such plug-in estimators are not capable of handling models where the unknown functions h_0 depend on the endogenous variables Y, because in such models θ_0 affects h_0 as well. Thus, in models where h_0 depends on an endogenous regressor, h_0 and θ_0 need to be estimated simultaneously. There are very few results concerning simultaneous estimators. Earlier applications include a semiparametric censored regression estimator (Duncan, 1986) and a semi-nonparametric maximum likelihood estimator (Gallant and Nychka, 1987).

However, a general estimation method for the model (1) that permits dependence of h_0 on Y and θ_0 was not well analyzed until a recent work by Ai and Chen (2003). These authors proposed a GMM-type Sieve Minimum Distance (SMD) estimator of α_0 under (1), based on identification and consistency conditions derived by Newey and Powell (2003). Subsequent applications of the SMD estimator include Chen and Ludvigson (2006) in a habit-based asset pricing model (with unknown functional form of the habit) testing various hypotheses on stock return data, Blundell, Chen and Kristensen (2006) in a dynamic optimization model describing the allocation of total non-durable consumption expenditure, and Ai et al. (2006) investigating co-movement of commodity prices.

The first analysis that ventured into the realm of GEL-type estimators subject to conditional moment restrictions containing unknown functions is due to Otsu (2003b).[4] His shrinkage-type estimator is based on a penalized empirical log-likelihood ratio (PELR) which utilizes a penalty function $J(h)$ confining the minimization problem to a parameter space specified by the researcher. Usually, $J(h)$ is used to control some physical plausibility of h such as roughness of h. Otsu's (2003b) penalized likelihood method differs from sieve analysis and hence his treatment of asymptotics differs from ours.[5]

Otsu (2003b) suggests (in Remark 2.2) that it is also possible to use a deterministic sieve approximations, instead of the penalty function approach, resulting in a deterministic sieve empirical likelihood estimator (DSELE) that would also be, under suitable conditions, [first-order] asymptotically equivalent to the SMD of Ai and Chen (2003). Similar conjecture has been raised in Nishiyama et al. (2005) who noted the lack of theoretical justification for such procedure. Chen (2005, footnote 39) made the same type of conjecture in relation to the conditional parametric Euclidean empirical likelihood estimator of Antoine et al. (2006). However, despite calls for a theoretical justification of such procedures, no previous paper has performed the necessary theoretical analysis. Yet, in analogy to the parametric literature described above, developing a one-step simultaneous GEL-type sieve alternative to the two-step simultaneous SMD in the semiparametric case can lead to a similar type of improvement in terms of bias and higher-order efficiency and is therefore of great theoretical

[4] Up to date, the author has not been able to obtain a full copy of this paper. Only a cashed html version containing parts of the paper's text is publicly available.

[5] In the seminal paper by Shen (1997), penalized likelihood and the method of sieves are treated as two separate concepts. To achieve asymptotic normality, Otsu extends Theorem 2 of Shen (1997), whereas we extend Theorem 1 of Shen (1997) which is a separate result derived under different conditions from the former.

and practical interest.

As the second contribution of this paper, we extend the LWCEL estimator to the semiparametric environment defined by (1). We approximate h with a sieve and estimate θ_0 and h_0 simultaneously with LWCEL. We establish consistency of the resulting one-step estimator and asymptotic normality for its parametric component of θ. Our LWCEL under (1) can be viewed as a direct alternative to the SMD estimators. A Monte Carlo study comparing small sample properties of LWCEL with SMD is planned to be included in future updates of this paper. Analytical comparison of higher-order properties remains beyond the scope of this paper.

All of the simultaneous estimators mentioned above are based on the method of sieves (Grenander, 1981; Chen, 2005) where h_0 is estimated over a compact subspace that is dense in the full parameter space as sample size increases. This feature of sieves conveniently simplifies the infinite-dimensional model h_0 to its finite-dimensional counterpart suitable for estimation. Here we also adhere to the sieve methodology. However, the currently available relevant general theory papers dealing with sieve M-estimation (Wong and Severini, 1991; Shen and Wong, 1994; Shen, 1997; Chen and Shen, 1998) consider only one set of exogenous variables without endogenous regressors and hence we can not apply these results directly in our case. Therefore, in the asymptotic analysis we combine them with several results of Ai and Chen (2003) and our own new results necessitated by the specific nature of SLWCEL under (1). In particular, among other issues we derive an extension of Shen's (1997) theorem on asymptotic normality of general simultaneous sieve estimators for the case of endogenous regressors under strong conditions and then apply it to the SLWCEL case under weak primitive conditions.

The rest of the paper is organized as follows. In Section 2 we develop the new LWCEL estimator and its dual MD counterpart for conditional moment restrictions (2) containing a finite dimensional parameter $\theta \in \Theta \subseteq \mathbb{R}^{d_\theta}$ and contrast the LWCEL's finite sample properties to KTA's CEL. Section 3 extends the LWCEL to the semiparametric environment of model (1) containing both θ and a vector of infinite dimensional unknown functions $h(\cdot)$ in $\alpha \equiv (\theta', h')'$. In Section 4 we derive consistency of the Sieve-based LWCEL $\widehat{\alpha}_n$ under a general metric. In Section 5 we show that $\widehat{\alpha}_n$ converges to α_0 at the rate $n^{-1/4}$ under the Fisher metric, which is a sufficient rate result for asymptotic normality of SLWCEL's parametric component $\widehat{\theta}_n$ derived in Section 6. Section 7 presents the results of a small-scale pilot Monte Carlo simulation study and shows favorable performance of the LWCEL estimator $\widehat{\theta}_n$ compared to KTA's CEL. Section 8 concludes. All technical proofs are presented in the Appendices.

2 The Estimator

In-depth insights into the internal estimating mechanism of EL/CEL can be gained by examining their dual representation as Minimum Discrepancy (MD) estimators. In this Section, we first develop some intuition behind using discrepancy measures in construction of the MD representation of EL under *unconditional* moment restrictions

$$E\left[g\left(Z,\theta_0\right)\right]=0 \tag{3}$$

Building on this intuition we then review existing methods of localizing discrepancy measures for the MD representation of KTA's CEL under *conditional* moment restrictions (2) and the corresponding global objective function of CEL. Having established the background, we propose a new weighting scheme leading to the LWCEL under (2).

2.1 Unconditional Moment Restrictions

Empirical Likelihood falls into a class of estimators that admit a convenient representation as Minimum Discrepancy (MD) estimators (Corcoran, 1998):

$$\widehat{\theta}_{MD}=\arg\min_{\theta\in\Theta\pi_1,\ldots,\pi_n} I(q:\pi) \tag{4}$$

subject to moment and normalization restrictions

$$\sum\nolimits_{i=1}^{n}\pi_i g(z_i,\theta) = 0 \tag{5}$$

$$\sum\nolimits_{i=1}^{n}\pi_i = 1 \tag{6}$$

where (5) is a finite-sample counterpart of the model (3). The function $I(q:\pi)$ is a discrepancy[6] measure that quantifies the goodness-of-fit between two distributions π_i and q_i with support on the data.

A rich class of discrepancy measures commonly used in MD estimation are the Cressie-Read (CR) power-divergence statistics $I_\gamma(q:\pi)$ (Cressie and Read, 1984) indexed by a scalar γ. Let $Z_1,...,Z_n$ be independent random vectors in $\mathbb{R}^{d_Z}$ with common distribution function F_Z. Let $q=(q_1,...,q_n)$ and $\pi=(\pi_1,...,\pi_n)$ be two discrete probability distributions defined on the $(n-1)$ dimensional simplex $\Delta_n=\omega=\{(\omega_1,...,\omega_n):\omega_i\geq 0;\, i=1,...,n \text{ and } \sum_{i=1}^{n}\omega_i=1\}$. The power divergence for q and π is defined as

[6] The term "discrepancy" is used instead of "distance" because a discrepancy function in general need not be symmetric in its arguments (Corcoran, 1998).

$$I_\gamma(q:\pi) = \frac{1}{\gamma(\gamma+1)} \sum\nolimits_{i=1}^{n} q_i \left[\left(\frac{q_i}{\pi_i} \right)^{\gamma} - 1 \right] \quad (7)$$

In the MD context, π is the expected multinomial distribution of $Z_1, ..., Z_n$ with support on the sample, and q is the multinomial distribution of the observed data with the same support (Corcoran, 1998; Imbens et al., 1998). Since each data point has been observed exactly once, $q_i = 1/n$ and thus

$$I_\gamma(1/n:\pi) = \frac{1}{n\gamma(\gamma+1)} \sum\nolimits_{i=1}^{n} \left[(n\pi_i)^{-\gamma} - 1 \right] \quad (8)$$

The CR discrepancy measure contains the empirical, Kullback-Leibler, Euclidean and Hellinger discrepancies as special cases corresponding to $\gamma = 0, -1, -2,$ and $-\frac{1}{2}$, respectively (Corcoran, 1998). The cases $\gamma = 0$ and -1 are handled by taking limits. Thus, for EL,

$$I_0(1/n:\pi) = -n^{-1} \sum\nolimits_{i=1}^{n} \log(n\pi_i)$$

If only the data were observed, without assuming the constraints (3), then the discrepancy measure (7) would be minimized in (4) by a multinomial π that equals the multinomial q, that is the empirical distribution placing the mass $1/n$ at each point. However, in the class of MD estimators (4) under the constraints (3), the information on F_Z is contained not only in the data, but also in the theoretical constraints (3) that are assumed to hold for F_Z. Thus, the MD estimator (4) subject to (5) and (6) chooses the multinomial π that satisfies the constraints while keeping as close as possible to the information on F_Z conveyed by the data alone as expressed by q, the closeness being measured by $I_\gamma(q:\pi)$. The multinomial q acts as a data-determined benchmark for the MD minimization problem.

2.2 Conditional Moment Restrictions

The conditional moment model (2) provides restrictions on $F_{Z|X}$ whereby (2) is assumed to hold at each X. Such restrictions hold far more information than the model (3) which gives only the mean restriction over X: In constructing an estimator under (2) we can theoretically utilize an infinite number of restrictions on $F_{Z|X}$ at each $X = x$ as opposed to one restriction on F_Z under (3). While economic theory typically delivers (2), estimation techniques such as two-step efficient GMM of Hansen (1982) or EL of Qin and Lawless (1994) first convert (2) to (3) and then estimate (3) resulting in loss of information contained in (2) (Antoine et al., 2006, KTA).

Given a finite sample of data of size n in estimation, the theoretically infinite number of restrictions is typically reduced to n restrictions required to hold at the observed data values only. However, given each x_i only a single observation z_i from $F_{Z|X}$ is available; almost surely. This problem has been customarily handled in the econometric estimation literature by localization methods

(Tibshirani and Hastie, 1987). In the stream of literature most relevant to this paper, localization schemes have been used in the conditional moment context in LeBlanc and Crowley (1995), Zhang and Gijbels (2003), KTA for CEL, Antoine et al. (2006) for the CUE, Smith (2003, 2005) for GEL.

The general approach to localizing an MD estimator is to construct a localized version of the CR discrepancy measure $I_{\gamma i}(\cdot)$ at each x_i and then obtain a global MD estimator by summing $I_{\gamma i}(\cdot)$ over i subject to constraints imposed at each x_i. Information on $F_{Z|X}$ is inferred from the nearby observations if we assume that $F_{Z|X}$ is continuous with respect to X. In other words, in a neighborhood around x_i we approximate $F_{Z|X=x}$ by $F_{Z|X=x} \approx F_{Z|X=x_i}$. This implies that all the z_j with x_j lying in this neighborhood can be roughly viewed as observations from $F_{Z|X=x_i}$. Such "data augmentation" leads to the localized version of the CR discrepancy measure for each x_i

$$I_{i\gamma}(q_{ij} : \pi_{ij}) = \frac{1}{\gamma(\gamma+1)} \sum\nolimits_{j=1}^{n} q_{ij} \left[\left(\frac{q_{ij}}{\pi_{ij}} \right)^{\gamma} - 1 \right] \tag{9}$$

where π_{ij} are the implied probabilities to be estimated.

Note that, unlike in the unconditional moment case (3), now the q_{ij} are not derived directly from observed data, since only one realization of the random vector z_j was actually observed at x_i. Rather, q_{ij} are inferred from neighboring observations. Since the data-determined q_{ij}s are used as a benchmark for minimization in the MD problem, the technique selected constructing these *inferred empirical probabilities* q_{ij} will inevitably influence the performance and outcome of the MD estimator in general. In the localization process, each q_{ij} measures the importance of the observation z_j to the augmented data at x_i. More precisely, for each x_i, observations with x_j that are "closer" to x_i are given more weight in assuming that they come from $F_{Z|X=x_i}$. The exact weights are captured by a weighting function w_{ij} calculated as the weight that x_j has in estimating $E_{F_{Z|X}}\left[g\left(Z,\theta\right)|X=x_i\right]$ with $\sum_{j=1}^{n} w_{ij} g\left(z_j, \theta\right)$, for a given θ. The w_{ij} can thus be regarded as nonparametric regression weights. The inferred empirical probabilities q_{ij} are then computed by normalizing the weights w_{ij} by

$$\sigma_i \equiv \sum\nolimits_{j=1}^{n} w_{ij} \tag{10}$$

so that q can satisfy the condition for a multinomial distribution $\sum_{i=1}^{n} q_{ij} = 1$, implying

$$q_{ij} = w_{ij}/\sigma_i \tag{11}$$

2.3 Existing Methods for MD Global Objective Function under Localization Schemes

In constructing the global MD objective function, the standard approach in the previous literature has been summing (9) over i, yielding the global MD objective function

$$\begin{aligned} M_\gamma(q,\pi) &= \sum\nolimits_{i=1}^{n} I_{i\gamma}(q_{ij}:\pi_{ij}) \\ &= \frac{1}{\gamma(\gamma+1)}\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=1}^{n} q_{ij}\left[\left(\frac{q_{ij}}{\pi_{ij}}\right)^{\gamma}-1\right] \end{aligned} \tag{12}$$

resulting in the localized MD estimator

$$\widehat{\theta}_{MDl} = \arg\min_{\theta\in\Theta,\pi} M_\gamma(q_{ij}:\pi_{ij}) \tag{13}$$

subject to moment and normalization restrictions

$$\sum_{j=1}^{n}\pi_{ij}g\left(z_j,\theta\right) = 0 \tag{14}$$

$$\sum_{j=1}^{n}\pi_{ij} = 1 \tag{15}$$

for each i. This form of the MD estimator was considered by Smith (2003) Antoine et al. (2006), and Smith (2005). In the case of CEL with $\gamma = 0$ analyzed in Zhang and Gijbels (2003) and KTA[7], (12) becomes

$$L(\pi_{ij},q_{ij}) = -\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}\ln\left(\frac{\pi_{ij}}{q_{ij}}\right)$$

In construction of the local discrepancy measures $I_{i\gamma}(q_{ij}:\pi_{ij})$, all previous literature, including KTA and Zhang and Gijbels (2003), use fixed-bandwidth kernel weights

$$w_{ij}^{k} = \frac{K\left((x_i-x_j)/b_n\right)}{\sum_{j=1}^{n}K\left((x_i-x_j)/b_n\right)} \tag{16}$$

Note that for (16) $\sigma_i = \sum_{j=1}^{n} w_{ij}^{k} = 1$ *by construction* and hence $w_{ij}^{k} = q_{ij}$. This, however, is a characteristic of the particular formulation of the weighting function (16) and does not hold in general. In the LWCEL proposed below, $\sigma_i \neq 1$ almost surely. This fact complicates the asymptotic analysis of the resulting estimator but plays crucial role in improving the finite-sample properties of LWCEL.

[7] Note that for $-1<\gamma<0$ we need to consider $-I_{\gamma i}(q_{ij}:\pi_{ij})$ to obtain a convex optimization problem. Also note that Kitamura et al. (2004) and Zhang and Gijbels (2003) use this form of localized likelihood but without explicitly stating the inferred empirical probabilities q_{ij}. Such omission is inconsequential for the special case of EL as the q_{ij}s cancel out of the first order condition of $I_{i\gamma=-1}(q_{ij}:\pi_{ij})$ with respect to π_{ij} due to the logarithmic functional form of $I_{i\gamma=-1}(q_{ij}:\pi_{ij})$ for EL. However, this does not hold for the CR class in general.

2.4 An Alternative Method

We will now propose a new weighting scheme for the construction of the MD objective function (and by duality CEL) addressing two separate features: (i) the weight σ_i of each localized discrepancy measure $I_{i\gamma}(q_{ij} : \pi_{ij})$ in the global sum, and (ii) the weights w_{ij} used in derivation of any given individual $I_{i\gamma}(q_{ij} : \pi_{ij})$. We will elaborate each feature in turn.

2.4.1 Global Weighting Scheme

In contrast to the simple sum (12), we argue that an appropriately weighted sum of $I_{i\gamma}(q_{ij} : \pi_{ij})$ in constructing the global MD objective function would utilize the information on $F_{Z|X}$ contained in the data more efficiently. Specifically, weighting each $I_{i\gamma}(q_{ij} : \pi_{ij})$ with σ_i defined in (10), i.e. the sum of the weights of individual observations, yields a global objective function for the MD minimization problem of the form

$$\begin{aligned} \widetilde{M}_\gamma(q,\pi) &= \sum\nolimits_{i=1}^{n} \sigma_i I_{i\gamma}(q_{ij} : \pi_{ij}) \\ &= \frac{1}{\gamma(\gamma+1)} \sum\nolimits_{i=1}^{n} \sigma_i \sum\nolimits_{j=1}^{n} q_{ij} \left[\left(\frac{q_{ij}}{\pi_{ij}} \right)^{\gamma} - 1 \right] \end{aligned} \tag{17}$$

Substituting for q_{ij} from (11) into (17) we obtain

$$\begin{aligned} \widetilde{M}_\gamma(q,\pi) &= \frac{1}{\gamma(\gamma+1)} \sum\nolimits_{i=1}^{n} \sigma_i \sum\nolimits_{j=1}^{n} \frac{w_{ij}}{\sigma_i} \left[\left(\frac{q_{ij}}{\pi_{ij}} \right)^{\gamma} - 1 \right] \\ &= \frac{1}{\gamma(\gamma+1)} \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{n} w_{ij} \left[\left(\frac{q_{ij}}{\pi_{ij}} \right)^{\gamma} - 1 \right] \end{aligned} \tag{18}$$

resulting in the MD estimator

$$\widehat{\theta}_{MDw} = \arg \min_{\theta \in \Theta, \pi} \widetilde{M}_\gamma(q_{ij} : \pi_{ij}) \tag{19}$$

subject to moment and normalization restrictions (14) and (15) for each i.

The factor σ_i reflects the importance that each individual $I_{i\gamma}(q_{ij} : \pi_{ij})$ exerts on the global sum (17). Take, for example, a data point (x_r, z_r) with many other observations x_j clustered closely around x_r. Under localization of $I_{r\gamma}(\cdot)$, each of such x_j is assigned a high weight w_{rj} because of their proximity to x_r. Such x_js can be credibly assumed to come from $F_{Z|X=x_r}$ and thus carry high informational content on $F_{Z|X=x_r}$. Accordingly, $\sigma_r = \sum_{j=1}^{n} w_{rj}$ will be high and the corresponding discrepancy measure $I_{r\gamma}(\cdot)$ is assigned a high importance in the global MD objective function (17). Take another data point (x_s, z_s), this time an outlier with neighboring points x_j far away. Here $\sigma_s = \sum_{j=1}^{n} w_{sj}$ will be small since each w_{sj} is small. It is doubtful that such x_js arrive from $F_{Z|X=x_s}$. The resulting local discrepancy measure $I_{s\gamma}(\cdot)$ is given a small importance in the global

MD sum (17) due to the relatively high uncertainty about $F_{Z|X=x_s}$.

Multiplying each $I_{i\gamma}(q_{ij} : \pi_{ij})$ with a constant σ_i does not change the meaning of any given individual $I_{i\gamma}(q_{ij} : \pi_{ij})$ which remain local discrepancy measures. However, our weighting now makes the importance of x_j for x_i comparable across i in the global sense of (18). The expression (18) can now be viewed as a weighted double sum of individual z_js augmented to x_i, for each i and j, with globally comparable weights. In contrast, under the simple sum of $I_{i\gamma}(q_{ij} : \pi_{ij})$ (12), the importance of each $I_{i\gamma}(q_{ij} : \pi_{ij})$ for the global MD function is set to unity, and the weight of x_j for x_r is not directly comparable to the weight of x_j for x_s.

At this point, we do not address optimality of the choice of the weight σ_i as defined in (10) for the global MD objective function. Showing optimality (in a sense) for this or perhaps other potential candidate weights is currently subject to our research. Nonetheless, different σ_i will not change the asymptotic analysis that is carried out in this paper for a generic stochastically bounded nonnegative σ_i. The current focus of this draft is breaking away from the $\sigma_i = 1$ paradigm of the previous literature and thus changing the form of the conditional GEL estimator, improving its finite sample performance.

2.4.2 Individual Weight Functions

The exact specification of the weighting function w_{ij} plays a crucial role for the performance and outcome of the estimator. The weights w_{ij}, when normalized to q_{ij}, in essence construct the counterpart of the empirical distribution estimate of $F_{Z|X}$ at $X = x_i$. For this purpose, given the information contained in the data and (2), we can use weights that are derived from estimating $E_{F_{Z|X}}\left[g\left(Z,\theta\right)|X = x_i\right]$ with $\sum_{j=1}^{n} w_{ij} g\left(z_j, \theta\right)$. Ideally, the regression estimator should be optimal in some sense, such as minimizing the Mean-Square Error (MSE) of the estimate.

We note that any w_{ij} that captures the information provided by the data about $F_{Z|X}$ more efficiently (in the MSE sense) than the currently used (16) will lead to the benchmark inferred empirical probabilities q_{ij} resulting in an estimate $\widehat{\pi}$ that is more representative of the underlying distribution $F_{Z|X}$ for each θ in (19), for a given finite sample $x_1, ..., x_n$. This, in turn, will lead to an estimate $\widehat{\theta}$ with a smaller MSE in (19). By virtue of the MD-GEL duality, the same argument holds for the GEL representations of the MD estimators.

Regression estimators that yield such w_{ij} have been analyzed well in the literature. For example, Müller and Stadtmüller (1987) or Brockmann et al. (1993) showed that a variable-bandwidth kernel estimator improves on its fixed-bandwidth counterpart in terms of Mean Integrated Square Error (MISE) by adapting locally to the density of the design points. Various methods of local polynomial modelling (Fan and Gijbels, 1996) or general series regression, including a host of function bases such as splines (Silverman, 1984) or wavelets (Donoho and Johnstone, 1995) share this localization feature and hence outperform (16) in terms of MISE.

2.5 Locally Weighted Conditional Empirical Likelihood

Despite their theoretical qualities, MD estimators are not attractive from the computational point of view. The MD optimization problems (13) and (19) each have dimension $n + \dim(\theta)$ which is larger than the sample size. For this reason, the GEL dual representation of MD estimators, derived via the method of Lagrange multipliers, is used in applications. As will be shown below on the special case of LWCEL, the GEL representation automates the selection of π_{ij}s as functions of θ and thus greatly reduces the dimensionality of the estimation problem. Nonetheless, the theoretical properties of the internal estimating mechanism are best studied in the MD form, justifying the MD discussion above.

The global objective MD function (18) with the newly proposed weighting scheme becomes in the case of LWCEL with $\gamma = 0$

$$L(\pi_{ij}, q_{ij}) = -\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \ln\left(\frac{\pi_{ij}}{q_{ij}}\right) \tag{20}$$

subject to (14) and (15)[8] holding for each i. The localized discrepancy measures $I_{i\gamma}(q_{ij} : \pi_{ij})$ thus become local empirical conditional log-likelihoods $l_i(\cdot) = \ln\left(\frac{\pi_{ij}}{q_{ij}}\right)$ with the global conditional log-likelihood $L(\cdot)$ being a weighted sum of $l_i(\cdot)$ over i. The LWCEL Lagrangian of the MD minimization problem (19) with the LWCEL objective function (20) becomes

$$L(\theta, \lambda, \mu, \pi) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \ln\left(\frac{\pi_{ij}}{w_{ij}}\right) - \sum_{i=1}^{n} \lambda_i' \sum_{j=1}^{n} \pi_{ij} g\left(z_j, \theta\right) - \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{n} \pi_{ij} - 1\right) \tag{21}$$

Taking first-order conditions yields

$$\frac{w_{ij}}{\widehat{\pi}_{ij}} = \widehat{\lambda}_i' g\left(z_j, \theta\right) + \widehat{\mu}_i \quad , \ \forall i, j \tag{22}$$

$$\sum_{j=1}^{n} \widehat{\pi}_{ij} g\left(z_j, \theta\right) = 0 \quad , \ \forall i \tag{23}$$

$$\sum_{j=1}^{n} \widehat{\pi}_{ij} = 1 \tag{24}$$

Summing (22) over j and using (23) yields, for each i,

$$\begin{aligned} \sigma_i &= \widehat{\lambda}_i' \sum_{j=1}^{n} \widehat{\pi}_{ij} g\left(z_j, \theta\right) + \widehat{\mu}_i \sum_{j=1}^{n} \widehat{\pi}_{ij} \\ \widehat{\mu}_i &= \sigma_i \end{aligned} \tag{25}$$

[8]Since the objective function depends on π_{ij} only through $\log(\pi_{ij}/q_{ij})$, the constraint $p_{ij} \geq 0$ does not bind.

Substituting (25) into (22) gives, for each i and j,

$$\widehat{\pi}_{ij} = \frac{w_{ij}}{\sigma_i + \lambda_i' g(z_j, \theta)} \tag{26}$$

Substituting (26) into the Lagrangian (21), and using (23) and (24), yields

$$L(\theta, \lambda) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln \left(\frac{1}{\sigma_i + \lambda_i' g(z_j, \theta)} \right) \tag{27}$$

Then the Locally Weighted Conditional Empirical Likelihood estimator with the new weighting scheme is defined as

$$\widehat{\theta}_{LWCEL} = \arg\max_{\theta \in \Theta} L(\theta, \lambda_i) \tag{28}$$

where λ_i solves[9]

$$\sum_{j=1}^{n} \frac{w_{ij} g(z_j, \theta)}{\sigma_i + \lambda_i' g(z_j, \theta)} = 0$$

obtained from (23) and (26). As discussed above, in general $\sigma_i \neq 1$. The presence of σ_i is the hallmark of LWCEL compared to the previous literature where, invariably, $\sigma_i = 1$. Given the expression (18), extension of (20) to the whole GEL family as considered in Smith (2005) appears relatively straightforward. Such generalization is currently subject to our research and we plan to include it into further updates of this paper.

The $\widehat{\theta}_{LWCEL}$ estimator defined in (28) is a special case of a corresponding estimator derived under semiparametric conditional moment restrictions in the next Section. For this reason, we will perform the asymptotic analysis pertaining to both estimators in the next Section.[10]

The MD estimator analyzed by Smith (2003, 2005) as well as the CEL estimator elaborated in KTA achieve the semiparametric efficiency lower bound (see Chamberlain, 1987). The weighting introduced for $\widehat{\theta}_{LWCEL}$ in this paper utilizes w_{ij} that improve on the fixed-bandwidth kernel weights (16) in finite samples in terms of MSE. We conclude that our new forms of the MD and CEL estimators exhibit first-order asymptotic equivalence in terms of consistency and asymptotic normality with the ones formulated in the previous literature, and hence also achieve the first-order asymptotic semiparametric efficiency lower bound. However, our $\widehat{\theta}_{LWCEL}$ improves on its previously analyzed forms in terms of finite sample performance.

[9] In line with KTA we adopt the notation λ_i as shorthand for $\lambda_i(x_i, \theta)$. When necessary, we explicitly write the full form to ensure that our arguments are unambiguous.

[10] Nonetheless, the author plans to include the full analysis of the finite-dimensional case either in future updates of this paper or in a separate paper dealing with the finite-dimensional case only.

3 Semiparametric Conditional Moment Restrictions

In this Section we extend the LWCEL estimator (27) to the semiparametric environment defined by (1). In doing so, we will use series estimation (see e.g. Newey, 1997) as a particular form of linear sieves in both approximating h and determining the weights w_{ij}. Series estimators are known to contain functional bases that are superior in terms of MSE criteria to fixed-bandwidth kernel estimators, especially in the presence of spatial inhomogeneities in the data (see e.g. Ramsey, 1999). Silverman (1984) showed that series estimators with spline basis functions behave approximately like the variable-bandwidth kernel estimator which improves on (16) in terms of MSE by the virtue of local adaptation. Another advantage of working with the LWCEL estimator based on series approximation is that truncation arguments in regions with small data density are not required in contrast to kernel weights.

3.1 Sieve-based Conditional Empirical Likelihood

The environment setup parallels the one of Newey and Powell (2003) and Ai and Chen (2003). Suppose that the observations $\{(Y_i, X_i) : i = 1, ..., n\}$ are drawn independently from the distribution of (Y, X) with support $\mathcal{Y}\times\mathcal{X}$, where $\mathcal{Y}$ is a subset of $\mathbb{R}^{d_Y}$ and $\mathcal{X}$ is a compact subset of $\mathbb{R}^{d_X}$. Suppose that the unknown distribution of (Y, X) satisfies the semiparametric conditional moment restrictions given by (1), where $g : \mathcal{Z} \times \mathcal{A} \to \mathbb{R}^{d_g}$ is a known mapping, up to an unknown vector of parameters, $\alpha_0 \equiv (\theta_0', h_0')' \in \mathcal{A} \equiv \Theta\times\mathcal{H}$, and $Z \equiv (Y', X_z')' \in \mathcal{Y}\times\mathcal{X}_Z \equiv \mathcal{Z} \subseteq \mathbb{R}^{d_Z}$ where $\mathcal{X}_Z \subseteq \mathcal{X}$. We assume that $\Theta \subseteq \mathbb{R}^{d_\theta}$ is compact with non-empty interior and that $\mathcal{H} \equiv \mathcal{H}^1 \times ... \times \mathcal{H}^{d_h}$ is a space of continuous functions. Since $\mathcal{H}$ is infinite-dimensional, in constructing a feasible estimator we follow the sieve literature (Grenander, 1981; Chen, 2005) by replacing $\mathcal{H}$ with a sieve space $\mathcal{H}_n \equiv \mathcal{H}_n^1 \times ... \times \mathcal{H}_n^{d_h}$ which is a computable and finite-dimensional compact parameter space that becomes dense in $\mathcal{H}$ as n increases.

Next, we introduce the series estimator used in the analysis (see Newey, 1997; Ai and Chen, 2003). For each $l = 1, ..., d_g$, and for a given α, let $\{p_{0j}(X),\ j = 1, 2, ...k_n\}$ denote a sequence of known basis functions (power series, splines, wavelets, etc.) and let $p^{k_n}(X) \equiv (p_{01}(X), ..., p_{0k_n}(X))'$. Let further $p^{k_n}(X)$ be a tensor-product linear sieve basis, which is a product of univariate sieves over d_X (see Ai and Chen, 2003, for details). Let $P = (p^{k_n}(x_1), ..., p^{k_n}(x_n))'$ be an $(n \times k_n)$ matrix. Consider the model (1) and denote the conditional mean function

$$\begin{aligned} m(X,\alpha) &\equiv E\left[g\left(Z,\alpha\right)|X\right] \\ &= \int g\left(Z,\alpha\right)dF_{Y|X} \end{aligned} \tag{29}$$

Let $\widehat{m}(X, \alpha) \equiv (\widehat{m}_1(X, \alpha), ..., \widehat{m}_{d_\rho}(X, \alpha))'$. A consistent nonparametric linear sieve estimator of

$m_l(X,\alpha)$ is given by

$$\widehat{m}_l(X,\alpha) = p^{k_n}(X)'\widehat{\kappa}_l$$

where h in $\alpha = (\theta', h')'$ is restricted to the sieve space $\mathcal{H}_n$ and $\widehat{\kappa}_l$ is an OLS estimate obtained by regressing $g_l(Y, X_z, \alpha)$ on $p^{k_n}(X)$,

$$\begin{aligned} \widehat{\kappa}_l &= (P'P)^{-1} P' g_l(Z,\alpha) \\ &= \sum_{j=1}^{n} p^{k_n}(x_j)' (P'P)^{-1} g_l(z_j,\alpha) \end{aligned} \tag{30}$$

and hence

$$\begin{aligned} \widehat{m}_l(x_i,\alpha) &= \widehat{E}_{Z|X}\left[g_l(Z,\alpha) | X = x_i\right] \\ &= p^{k_n}(x_i)'\widehat{\kappa}_l \\ &= \sum_{j=1}^{n} p^{k_n}(x_j)' (P'P)^{-1} p^{k_n}(x_i) g_l(z_j,\alpha) \\ &= \sum_{j=1}^{n} w_{ij} g_l(z_j,\alpha) \end{aligned}$$

after substituting from (30), $l = \{1, ..., d_g\}$. In the vector form

$$\widehat{m}(x_i,\alpha) = \sum_{j=1}^{n} w_{ij} g(z_j,\alpha)$$

The weights are given by

$$w_{ij} = p^{k_n}(x_j)' (P'P)^{-1} p^{k_n}(x_i) \tag{31}$$

and

$$\begin{aligned} \sigma_i &= \sum_{j=1}^{n} w_{ij} \\ &= \sum_{j=1}^{n} p^{k_n}(x_j)' (P'P)^{-1} p^{k_n}(x_i) \\ &= \mathbf{i}' P (P'P)^{-1} p^{k_n}(x_i) \end{aligned}$$

where $\mathbf{i}$ is a $(n \times 1)-$vector of ones.

We now turn to the derivation of LWCEL under (1). The Lagrangian[11] for the local semipara-

[11] As discussed above, omission of q_{ij} from the denominator of $\ln(\pi_{ij}/q_{ij})$ is inconsequential in the case of LWCEL.

metric EL estimator is

$$\max_{p_{ij}} \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \ln \pi_{ij} \quad \text{s.t.} \quad \pi_{ij} \geq 0, \quad \sum_{j=1}^{n} \pi_{ij} = 1, \quad \sum_{j=1}^{n} g\left(z_j, \alpha_n\right) \pi_{ij} = 0, \text{ for } i, j = 1, ..., n$$

where α_n is α restricted to the sieve space $\mathcal{A}_n$. Then,

$$\widehat{\pi}_{ij} = \frac{w_{ij}}{\sigma_i + \lambda_i' g\left(z_j, \alpha_n\right)} \tag{32}$$

and for each $\alpha_n \in \mathcal{A}_n$, λ_i solves

$$\sum_{j=1}^{n} \frac{w_{ij} g\left(z_j, \alpha_n\right)}{\sigma_i + \lambda_i' g\left(z_j, \alpha_n\right)} = 0 \tag{33}$$

The Sieve-based Locally Weighted Conditional Empirical Likelihood (SLWCEL) evaluated at α_n is defined as

$$L_{SLWCEL}(\alpha_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \ln \left\{ \frac{w_{ij}}{\sigma_i + \lambda_i' g\left(z_j, \alpha_n\right)} \right\}$$

where λ_i solves (33). The estimator of α_0 is defined as

$$\widehat{\alpha}_n = \arg \max_{\alpha_n \in \mathcal{A}_n} L_{SLWCEL}(\alpha_n) \tag{34}$$

Solving (34) is equivalent to solving

$$\widehat{\alpha}_n = \arg \max_{\alpha_n \in \mathcal{A}_n} G_n(\alpha_n) \tag{35}$$

where

$$G_n(\alpha_n) = -\frac{1}{n} \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \ln \left\{ \sigma_i + \lambda_i' g\left(z_j, \alpha_n\right) \right\} \tag{36}$$

Implementing our estimator is straightforward. One advantage of the sieve approach is that once $h \in \mathcal{H}$ is replaced by $h_n \in \mathcal{H}_n$, the estimation problem effectively becomes a parametric one. Commonly used statistical and econometric packages can then be used to compute the estimate. From (33) it follows that

$$\lambda_i = \arg \max_{\rho \in \mathbb{R}^{d_g}} \sum_{j=1}^{n} w_{ij} \ln \left\{ \sigma_i + \rho' g\left(z_j, \alpha_n\right) \right\} \tag{37}$$

This is a well-behaved optimization problem since the objective function is globally concave and can be solved by a Newton-Raphson numerical procedure. The outer loop (35) can be carried out using a numerical optimization procedure. For a relevant discussion of computational issues, see for example Kitamura (2006, section 8.1).

4 Consistency

In this section we present some asymptotic results for the smoothed empirical likelihood estimator as defined in (34). The general approach follows closely the one developed in KTA. The following definitions, adopted from Ai and Chen (2003), are introduced:

Definition 4.1 *A real-valued measurable function* $g(Z, \alpha)$ *is* Hölder continuous *in* $\alpha \in \mathcal{A}$ *if there exist a constant* $\overline{\kappa} \in (0, 1]$ *and a measurable function* $c_2(Z)$ *with* $E\left[c_2(Z)^2|X\right]$ *bounded, such that* $|g(Z, \alpha_1) - g(Z, \alpha_2)| \leq c_2(Z) \|\alpha_1 - \alpha_2\|^{\overline{\kappa}}$ *for all* $Z \in \mathcal{Z}$*,* $\alpha_1, \alpha_2 \in \mathcal{A}$*.*

The Hölder space of smooth functions $\Lambda^{\overline{\gamma}}(\mathcal{X})$ of order $\overline{\gamma} > 0$ and the corresponding Hölder ball $\Lambda^{\overline{\gamma}}_c(\mathcal{X}) \equiv \{g \in \Lambda^{\overline{\gamma}}(\mathcal{X}) : \|g\|_{\Lambda^{\overline{\gamma}}} \leq c < \infty\}$ with radius c are defined in Ai and Chen (2003), p. 1800.

Definition 4.2 *A real-valued measurable function* $g(Z, \alpha)$ satisfies an envelope condition *over* $\alpha \in \mathcal{A}$ *if there exists a measurable function* $c_1(Z)$ *with* $E\left\{c_1(Z)^4\right\} < \infty$ *such that* $|g(Z, \alpha)| \leq c_1(Z)$ *for all* $Z \in \mathcal{Z}$ *and* $\alpha \in \mathcal{A}$*.*

The following Assumptions are made to facilitate the analysis:

Assumption 4.1 *For each* $\alpha \neq \alpha_0$ *there exists a set* $\mathcal{X}_\alpha$ *such that* $\Pr\{x \in \mathcal{X}_\alpha\}$*, and* $E\left[g\left(z, \alpha\right)|x\right] \neq 0$ *for every* $x \in \mathcal{X}_\alpha$*.*

Assumption 4.2 *(i) The data* $\{(Y_i, X_i)_{i=1}^n\}$ *are i.i.d.; (ii)* $\mathcal{X}$ *is compact with nonempty interior; (iii) the density of* X *is bounded and bounded away from zero.*

Assumption 4.3 *(i) The smallest and the largest eigenvalues of* $E\left[p^{k_n}(X) \times p^{k_n}(X)'\right]$ *are bounded and bounded away from zero for all* k_n*; (ii) for any* $g(\cdot)$ *with* $E\left[g(X)^2\right] < \infty$*, there exists* $p^{k_n}(X)'\kappa$ *such that* $E\left[\left\{g(X) - p^{k_n}(X)'\kappa\right\}^2\right] = o(1)$*.*

Assumption 4.4 *(i) There is a metric* $\|\cdot\|$ *such that* $\mathcal{A} \equiv \Theta \times \mathcal{H}$ *is compact under* $\|\cdot\|$*; (ii) for any* $\alpha \in \mathcal{A}$*, there exists* $\Pi_n\alpha \in \mathcal{A}_n \equiv \Theta \times \mathcal{H}_n$ *such that* $\|\Pi_n\alpha - \alpha\| = o(1)$*.*

Assumption 4.5 *(i)* $E\left[\left|g\left(Z, \alpha_0\right)\right|^2 |X\right]$ *is bounded; (ii)* $g\left(Z, \alpha\right)$ *is Hölder continuous in* $\alpha \in \mathcal{A}$*.*

Let $k_{1n} \equiv \dim(\mathcal{H}_n)$ denote the number of unknown sieve parameters in $h_n \in \mathcal{H}_n$.

Assumption 4.6 $k_{1n} \to \infty$*,* $k_n \to \infty$*,* $k_n/n \to 0$ *and* $d_g k_n \geq d_\theta + k_{1n}$*.*

Assumption 4.7 $E\|x\|^{1+\varrho} < \infty$ *for some* $\varrho < \infty$*.*

Assumption 4.8 $E\{\sup_{\alpha \in \mathcal{A}} \|g\left(Z, \alpha\right)\|^m\} < \infty$ *for some* $m \geq 8$*.*

Assumption 4.1 is Assumption 3.1 in KTA that guarantees identification of θ_0. Assumptions 4.2–4.6 are essentially the same conditions imposed in Newey and Powell (2003) and Ai and Chen (2003). Assumption 4.2 rules out time series observations. Assumptions 4.3–4.6 are typical conditions imposed for series (or linear sieve) estimation of conditional mean functions. Assumption 4.4(i) restricts the parameter space as well as the choice of the metric $\|\cdot\|$. It is a commonly imposed condition in the semiparametric econometrics literature, and is satisfied when the infinite-dimensional parameter space $\mathcal{H}$ consists of bounded and smooth functions (see Gallant and Nychka, 1987). Assumption 4.4(ii) is the definition of a sieve space. Assumption 4.5 is typically imposed on the residual function in the literature on parametric nonlinear estimation. Assumption 4.6 restricts the growth rate of the number of basis functions in the series approximation. Assumption 4.7 is Assumption 3.4(ii) in KTA, used in Lemma A.1. Assumption 4.8 is Assumption 3.2 in KTA used in Lemma A8.

The following Theorem provides a consistency result:

Theorem 4.1 *Let the Assumptions 4.1–4.7 hold. Then* $\|\widehat{\alpha}_n - \alpha_0\| = o_p(1)$.

The proof is derived in the Appendix. The proof proceeds along the lines of KTA. However, the fact that the sieve parameter space $\mathcal{H}_n$ grows dense in an infinite-dimensional space $\mathcal{H}$ now needs to be addressed. The inclusion of σ_i in the LWCEL objective function compared to KTA's CEL also complicates matters. We achieve some simplifications arising from not having to make use of truncation arguments for kernels. Since we are not dealing with kernels, unlike KTA we can not use Lemma B.1 of Ai (1997) to determine uniform convergence rates. For this purpose, we specialize Lemma A.1(A) of Ai and Chen (2003), derived for the combined space $\mathcal{X} \times \mathcal{A}$, to the space $\mathcal{X}$ only, with $g(z_j, \alpha)$ evaluated at a given fixed α. Since we do not have to account for growth restrictions on the parameter space in this Lemma, we are able to obtain faster convergence rate $\widetilde{\delta}_{1n}$ than Ai and Chen (2003).

5 Convergence Rates

Theorem 4.1 established consistency of $\widehat{\alpha}_n = (\widehat{\theta}_n, \widehat{h}_n)$ under a general metric $\|\cdot\|$ constrained only by Assumption 4.4(i). In order to ascertain asymptotic normality of $\widehat{\theta}_n$, one typically needs that $\widehat{\alpha}_n$ converge to α_0 at a rate faster than $n^{-1/4}$ (see e.g. Newey, 1994). As noted by Newey and Powell (2003), for model (1) where the unknown h_0 can depend on endogenous variables Y, it is generally difficult to obtain fast convergence rate under $\|\cdot\|$. Nonetheless, as demonstrated by Ai and Chen (2003), in simultaneous estimation of $(\widehat{\theta}_n, \widehat{h}_n)$ it is sufficient to show fast convergence rate of $\widehat{\alpha}_n = (\widehat{\theta}_n, \widehat{h}_n)$ for only a special case of $\|\cdot\|$ to derive asymptotic normality of $\widehat{\theta}_n$. Naturally, we will also follow this approach. However, since the objective function of the problem analyzed in Ai and Chen (2003) is different from ours, our metric also differs. While Ai and Chen (2003) used a quadratic form type metric, we perform the analysis under the Fisher metric $\|\cdot\|_F$ which is the natural choice for a likelihood-based scenario.

Some additional notation is necessary to introduce the Fisher metric. The properties of $\mathcal{A}$ and the notation for pathwise derivatives established in this paragraph borrows from Ai and Chen (2003). Suppose the parameter space $\mathcal{A}$ is connected in the sense that for any two points α_1, $\alpha_2 \in \mathcal{A}$ there exists a continuous path $\{\alpha(t) : t \in [0,1]\}$ in $\mathcal{A}$ such that $\alpha(0) = \alpha_1$ and $\alpha(1) = \alpha_2$. Also, suppose that $\mathcal{A}$ is convex at the true value α_0 in the sense that, for any $\alpha \in \mathcal{A}$, $(1-t)\alpha_0 + t\alpha \in$ for small $t > 0$. Furthermore, suppose that for almost all Z, $g(Z, (1-t)\alpha_0 + t\alpha)$ is continuously differentiable at $t = 0$. Denote the first pathwise derivative at the direction $[\alpha - \alpha_0]$ evaluated at α_0 by

$$\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha - \alpha_0] \equiv \left.\frac{dg(Z,(1-t)\alpha_0 + t\alpha)}{dt}\right|_{t=0} \quad \text{a.s. } Z$$

and for any α_1, $\alpha_2 \in \mathcal{A}$ denote

$$\begin{aligned} \frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1 - \alpha_2] &\equiv \frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1 - \alpha_0] - \frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_2 - \alpha_0] \\ \frac{dm(X,\alpha_0)}{d\alpha}[\alpha_1 - \alpha_2] &\equiv E\left\{ \left. \frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1 - \alpha_2] \right| X \right\} \end{aligned} \tag{38}$$

Furthermore, let

$$\varphi(X, Z, \alpha) \equiv \ln\left\{\sigma_x + \lambda'(X,\alpha) g(Z,\alpha)\right\} \tag{39}$$

$$\psi(X,\alpha) \equiv E\left[\varphi(X,Z,\alpha)\,|X\right] \tag{40}$$

where σ_x stands for σ_i evaluated at a generic $X = x$. For any α_1, $\alpha_2 \in \mathcal{A}$ the Fisher norm $\|\cdot\|_F$ (see e.g. Wong and Severini, 1991, p. 607) is defined[12] as

$$\|\alpha_1 - \alpha_2\|_F = \sqrt{E\left\{E\left[\left.\left(\frac{d\varphi(X,Z,\alpha_0)}{d\alpha}[\alpha_1 - \alpha_2]\right)' \frac{d\varphi(X,Z,\alpha_0)}{d\alpha}[\alpha_1 - \alpha_2]\right| X\right]\right\}} \tag{41}$$

Let $\overline{\mathbf{V}}$ denote the closure of the linear span of $\mathcal{A} - \{\alpha_0\}$ under the metric $\|\cdot\|_F$. Then $\left(\overline{\mathbf{V}}, \|\cdot\|_F\right)$ is a Hilbert space with the inner product

$$\langle v_1, v_2\rangle_F = \|v_1 - v_2\|_F^2$$

We will now show that our metric $\|\alpha_1 - \alpha_2\|_F$ is equivalent to a *conditional version* of the metric

[12] We use the inner product notation for the pathwise derivatives to explicitly account for the special case when $\alpha \equiv \theta \in \mathbb{R}^{d_\theta}$.

used in Ai and Chen (2003). Let

$$\begin{aligned} s(X,Z,\alpha) &\equiv \lambda'(\alpha,X)g\left(Z,\alpha\right) \\ \varpi\left(X,Z,\alpha\right) &\equiv \frac{d\varphi\left(X,Z,\alpha_0\right)}{ds(X,Z,\alpha)} \\ &= \frac{1}{\sigma_x + s(X,Z,\alpha)} \end{aligned}$$

where $s(X,Z,\alpha)$ and $\varpi\left(X,Z,\alpha\right)$ is scalars. Note that from the conditional moment restriction (1), under the expectation taken over Z conditional on X

$$\lambda(X,\alpha_0) = 0 \tag{42}$$

which means that the constraints on $F_{Y|X}$ imposed by (1) are satisfied with equality and the Lagrange multiplier $\lambda(X,\alpha_0)$ takes on the value 0. This is also apparent from Lemma A.8. We have

$$\begin{aligned} & E\left[\left(\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}[\alpha_1-\alpha_2]\right)'\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}[\alpha_1-\alpha_2]\middle| X\right] \\ = \; & E\left[\varpi\left(X,Z,\alpha_0\right)^2\left(\frac{ds(X,Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right)'\frac{ds(X,Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\middle| X\right] \\ = \; & E\left[\begin{array}{c} \varpi\left(X,Z,\alpha_0\right)^2\left(\lambda'(X,\alpha_0)\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]+g(Z,\alpha_0)\frac{d\lambda'(X,\alpha_0)}{d\alpha}\right)' \\ \times\left(\lambda'(X,\alpha_0)\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]+g(Z,\alpha_0)\frac{d\lambda'(X,\alpha_0)}{d\alpha}\right) \end{array}\middle| X\right] \\ = \; & A_1+A_2+A_3+A_4 \end{aligned} \tag{43}$$

where

$$\begin{aligned} A_1 &= E\left[\varpi\left(X,Z,\alpha_0\right)^2\left(\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right)'\lambda(X,\alpha_0)\lambda'(X,\alpha_0)\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\middle| X\right] \\ A_2 &= E\left[\varpi\left(X,Z,\alpha_0\right)^2\left(\frac{d\lambda(X,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right)'g(Z,\alpha_0)\lambda'(X,\alpha_0)\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\middle| X\right] \\ A_3 &= E\left[\varpi\left(X,Z,\alpha_0\right)^2\left(\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right)'\lambda'(X,\alpha_0)g(Z,\alpha_0)\frac{d\lambda'(X,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\middle| X\right] \\ A_4 &= E\left[\varpi\left(X,Z,\alpha_0\right)^2\left(\frac{d\lambda(X,Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right)'g(Z,\alpha_0)g'(Z,\alpha_0)\frac{d\lambda'(X,Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\middle| X\right] \end{aligned} \tag{44}$$

Using (42) yields $A_1 = A_2 = A_3 = 0$. By the definition of $\lambda(X,\alpha)$ in (37), $\lambda(X,\alpha)$ is a function of $g(Z,\alpha)$ which is a function of α. Moreover, $\lambda(X,\alpha)$ is a function of α *only* via $g(Z,\alpha)$. Hence, under the expectation taken over Z conditional on X

$$\frac{d\lambda(X,\alpha)}{d\alpha}[\alpha_1-\alpha_2] = \frac{d\lambda(X,\alpha)}{dg(Z,\alpha)}\frac{dg(Z,\alpha)}{d\alpha}[\alpha_1-\alpha_2] \tag{45}$$

In particular, under the expectation over Z conditional on X, $\lambda(X,\alpha)$ is defined implicitly as a function of $g(Z,\alpha)$ by the relation

$$F(\lambda,g) = E\left[\frac{g(Z,\alpha)}{\sigma_x+\lambda'(X,\alpha)g(Z,\alpha)}\middle| X\right] = 0$$

By the Implicit Function Theorem

$$
\begin{aligned}
\frac{d\lambda(X,\alpha)}{dg(Z,\alpha)} &= \frac{\partial F(\lambda,g)/\partial g(Z,\alpha)}{\partial F(\lambda,g)/\partial\lambda(X,\alpha)} \\
&= E\left[\left.\frac{(\sigma_x+\lambda'(X,\alpha)g(Z,\alpha)-\lambda'(X,\alpha)g(Z,\alpha))/\left(\sigma_x+\lambda'(X,\alpha)g(Z,\alpha)\right)^2}{-g(Z,\alpha)g'(Z,\alpha)/\left(\sigma_x+\lambda'(X,\alpha)g(Z,\alpha)\right)^2}\right| X\right] \\
&= -\sigma_x\left\{E\left[g(Z,\alpha)g'(Z,\alpha)|\, X\right]\right\}^{-1} \\
&= -\sigma_x\Sigma(X,\alpha)^{-1} \qquad (46)
\end{aligned}
$$

Substituting (46) into (45) we obtain

$$\frac{d\lambda(\alpha,X,Z)}{d\alpha}[\alpha_1-\alpha_2] = -\sigma_x\Sigma(X,\alpha)^{-1}\frac{dg(Z,\alpha)}{d\alpha}[\alpha_1-\alpha_2] \qquad (47)$$

Substituting (47) into (44) yields

$$A_4 = \sigma_x^2 E\left[\varpi\left(X,Z,\alpha_0\right)^2\left(\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right)' W_0(X,Z)^{-1}\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right| X\bigg]$$

where

$$W_0(X,Z)^{-1} \equiv \Sigma(X,\alpha_0)^{-1}g(Z,\alpha_0)g'(Z,\alpha_0)\Sigma(X,\alpha_0)^{-1}$$

Using (42) in $\varpi\left(X,Z,\alpha_0\right)$ results in

$$A_4 = E\left[\left(\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right)' W_0(X,Z)^{-1}\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right| X\bigg] \qquad (48)$$

Substituting (48) into (44) and (41) yields

$$\|\alpha_1-\alpha_2\|_F = \sqrt{E\left\{E\left[\left(\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right)' W_0(X,Z)^{-1}\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right| X\bigg]\right\}} \qquad (49)$$

The expression (49) can be viewed as a conditional version of the metric used in Ai and Chen (2003). In particular, if $\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]$ and $g(Z,\alpha_0)$ are independent conditional on X, then (49) reduces to $\sqrt{E\left\{\left(\frac{dm(X,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right)'\Sigma(X,\alpha_0)^{-1}\frac{dm(X,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right\}}$ which is the metric used in Ai and Chen (2003) with the efficient weighting matrix.

Note that by (42)

$$
\begin{aligned}
E\left[\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}[\alpha_1-\alpha_2]\right| X\right] &= \lambda'(X,\alpha_0)E\left[\left.\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]\right| X\right] \\
&\quad +\frac{d\lambda'(X,\alpha_0)}{d\alpha}[\alpha_1-\alpha_2]E\left[\left. g(Z,\alpha_0)\right| X\right] \\
&= 0
\end{aligned}
$$

and hence

$$
E\left[\left.\left(\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}[\alpha_1-\alpha_2]\right)'\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}[\alpha_1-\alpha_2]\right| X\right] = Var\left(\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}[\alpha_1-\alpha_2]\right| X\right)
$$

implying

$$
\begin{aligned}
\|\alpha_1-\alpha_2\|_F &= \sqrt{E\left\{Var\left(\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}[\alpha_1-\alpha_2]\right| X\right)\right\}} \\
\langle v,v\rangle_F &= E\left\{Var\left(\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}[v]\right| X\right)\right\}
\end{aligned}
$$

We will now introduce the conditions under which the desired convergence rates are derived.

Assumption 5.1 *(i) $\mathcal{A}$ is convex in α_0, and $g(Z,\alpha)$ is pathwise differentiable at α_0; (ii) for some $c_1, c_2 > 0$,*

$$
\begin{aligned}
c_1 E\left\{m(X,\alpha_n)'W_0(X)^{-1}m(X,\alpha_n)\right\} &\leq \|\alpha_n-\alpha_0\|_F^2 \\
&\leq c_2 E\left\{m(X,\alpha_n)'W_0(X)^{-1}m(X,\alpha_n)\right\}
\end{aligned}
$$

holds for all $\alpha_n \in \mathcal{A}_n$ with $\|\alpha_n-\alpha_0\| = o(1)$.

Assumption 5.2 *For any $\widetilde{g}(\cdot)$ in $\Lambda_c^{\overline{\gamma}}(\mathcal{X})$ with $\overline{\gamma} > d_x/2$, there exists $p^{k_n}(\cdot)'\kappa \in \Lambda_c^{\overline{\gamma}}(\mathcal{X})$ such that $\sup_{X\in\mathcal{X}}\left|\widetilde{g}(X)-p^{k_n}(X)'\kappa\right| = O(k_n^{-\overline{\gamma}/d_x})$, and $k_n^{-\overline{\gamma}/d_x} = o(n^{-1/4})$.*

Assumption 5.3 *(i) Each element of $g(Z,\alpha)$ satisfies an envelope condition in $\alpha_n \in \mathcal{A}_n$; (ii) each element of $m(X,\alpha) \in \Lambda_c^{\overline{\gamma}}(\mathcal{X})$ with $\overline{\gamma} > d_x/2$, for all $\alpha_n \in \mathcal{A}_n$.*

In line with Ai and Chen (2003), let $\xi_{0n} \equiv \sup_{X\in\mathcal{X}}\left\|p^{k_n}(X)\right\|_E$, which is nondecreasing in k_n. Denote $N(\delta,\mathcal{A}_n,\|\cdot\|)$ as the minimal number of radius δ covering balls of $\mathcal{A}_n$ under the $\|\cdot\|$ metric.

Assumption 5.4 $k_{1n}\times\ln n\times\xi_{0n}^2\times n^{-1/2} = o(1)$.

Assumption 5.5 $\ln\left[N(\varepsilon^{1/\kappa},\mathcal{A}_n,\|\cdot\|)\right] \leq$ *const.* $\times\, k_{1n}\times\ln(k_{1n}/\varepsilon)$.

Assumption 5.6 $\Sigma_0(X) \equiv Var\left[g(Z,\alpha_0)|X\right]$ *is positive definite for all $X \in \mathcal{X}$.*

The following result gives the convergence rate of the SLWCEL estimator under the Fisher metric. The proof is provided in the Appendix.

Theorem 5.1 *Under Assumptions 4.1 - 5.6, we have* $\|\widehat{\alpha}_n - \alpha_0\|_F = o_p(n^{-1/4})$.

6 Asymptotic Normality

To derive the asymptotic distribution of $\widehat{\theta}_n$, it suffices to derive the asymptotic distribution of $f(\widehat{\alpha}_n) \equiv \tau'\widehat{\theta}_n$ for any fixed non-zero $\tau \in R^{d_\theta}$. The difference $f(\widehat{\alpha}_n) - f(\alpha_0)$ is linked to the pathwise directional derivatives of the sample criterion function via the inner product involving a Riesz representor v^*. Application of a Central Limit Theorem for triangular arrays of functions indexed by a finite-dimensional parameter then shows the desired result. In this Section we introduce the necessary notation, compute the Riesz representor v^* and state the Theorem of $\sqrt{n}$-normality of $\widehat{\theta}_n$.

Since $f(\alpha) \equiv \tau'\theta$ is a linear functional on $\overline{\mathbf{V}}$, it is bounded (i.e. continuous) if and only if

$$\sup_{0 \neq \alpha - \alpha_0 \in \overline{\mathbf{V}}} \frac{|f(\alpha) - f(\alpha_0)|}{\|\alpha - \alpha_0\|_F} < \infty$$

The Riesz Representation Theorem states that there exists a representor $v^* \in \overline{\mathbf{V}}$ satisfying

$$\|v^*\|_F \equiv \sup_{0 \neq \alpha - \alpha_0 \in \overline{\mathbf{V}}} \frac{|f(\alpha) - f(\alpha_0)|}{\|\alpha - \alpha_0\|_F} \tag{50}$$

and

$$f(\alpha) = f(\alpha_0) + \langle v^*, \alpha - \alpha_0 \rangle_F$$

Hence,

$$f(\widehat{\alpha}_n) - f(\alpha_0) = \langle v^*, \widehat{\alpha}_n - \alpha_0 \rangle_F$$

Let

$$\frac{dg(Z, \alpha_0)}{d\alpha}[\alpha - \alpha_0] \equiv \frac{dg(Z, \alpha_0)}{d\theta'}(\theta - \theta_0) + \frac{dg(Z, \alpha_0)}{dh}[h - h_0] \tag{51}$$

For any $h \in \overline{\mathcal{H}}$, there exists $w_j(\cdot) \in \overline{\mathcal{W}}$ for $j = 1, ..., d_\theta$ such that

$$h - h_0 = -(w_1, ..., w_{d_\theta})(\theta - \theta_0) = -w(\theta - \theta_0)$$

Define

$$\begin{aligned}\frac{dg(Z,\alpha_0)}{dh}[w] &\equiv \left(\frac{dg(Z,\alpha_0)}{dh}[w_1],...,\frac{dg(Z,\alpha_0)}{dh}[w_{d_\theta}]\right)\\ D_w(Z) &\equiv \frac{dg(Z,\alpha_0)}{d\theta'}-\frac{dg(Z,\alpha_0)}{dh}[w] \qquad (52)\end{aligned}$$

where $D_w(Z)$ is a $d_g\times d_\theta-$matrix valued function. Definitions (51) and (52) imply

$$\frac{dg(Z,\alpha_0)}{dh}[h-h_0]=-\frac{dg(Z,\alpha_0)}{dh}[w](\theta-\theta_0)$$

and hence

$$\begin{aligned}D_w(Z)(\theta-\theta_0) &= \frac{dg(Z,\alpha_0)}{d\theta'}(\theta-\theta_0)-\frac{dg(Z,\alpha_0)}{dh}[w](\theta-\theta_0)\\ &= \frac{dg(Z,\alpha_0)}{d\theta'}(\theta-\theta_0)+\frac{dg(Z,\alpha_0)}{dh}[h-h_0]\\ &= \frac{dg(Z,\alpha_0)}{d\alpha}[\alpha-\alpha_0] \qquad (53)\end{aligned}$$

By definition of $\|\cdot\|_F$ this implies

$$\begin{aligned}\|\alpha-\alpha_0\|_F^2 &= E\left\{E\left[\left(\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha-\alpha_0]\right)' W_0(Z,X)^{-1}\left(\frac{dg(Z,\alpha_0)}{d\alpha}[\alpha-\alpha_0]\right)\Big| X\right]\right\}\\ &= E\left\{E\left[(\theta-\theta_0)' D_w(Z)'W_0(Z,X)^{-1}D_w(Z)(\theta-\theta_0)\big| X\right]\right\} \qquad (54)\end{aligned}$$

Let $w^*=\left(w_1^*,...,w_{d_\theta}^*\right)$ be the solution to

$$\inf_{w_j\in\overline{\mathcal{W}},j=1,...,d_\theta} E\left\{E\left[(\theta-\theta_0)' D_w(Z)'W_0(Z,X)^{-1}D_w(Z)(\theta-\theta_0)\big| X\right]\right\} \qquad (55)$$

where "inf" is in positive semidefinite matrix sense. Using the definitions of w^*, $f(\alpha)$, (50) and (54)

$$\begin{aligned}\|v^*\|_F^2 &\equiv \sup_{0\neq\alpha-\alpha_0\in\overline{\mathbf{V}}}\frac{|f(\alpha)-f(\alpha_0)|^2}{\|\alpha-\alpha_0\|_F^2}\\ &= \frac{(\theta-\theta_0)'\tau\tau'(\theta-\theta_0)}{(\theta-\theta_0)'E\{E[D_w(Z)'W_0(Z,X)^{-1}D_w(Z)|X]\}(\theta-\theta_0)}\\ &= \tau'\left[E\left\{E\left[D_w(Z)'W_0(Z,X)^{-1}D_w(Z)\big| X\right]\right\}\right]^{-1}\tau \qquad (56)\end{aligned}$$

where $v^*\equiv(v_\theta^*,v_h^*)\in\overline{\mathbf{V}}$. By the definition of w^*, $v_h^*=-w^*\times v_\theta^*$. From this and (53) we have

$$\frac{dg(Z,\alpha_0)}{d\alpha}[v^*]=D_{w^*}(Z)v_\theta^* \qquad (57)$$

Let

$$v_\theta^* = \left[E\left\{E\left[\left. D_w(Z)'W_0(Z,X)^{-1}D_w(Z)\right| X\right]\right\}\right]^{-1}\tau \tag{58}$$

Substituting (58) into the definition of $\|\cdot\|_F^2$ in (41) via the expression for (57) yields

$$\begin{aligned}
\|v^*\|_F^2 &= E\left\{E\left[\left.\left(\frac{dg(Z,\alpha_0)}{d\alpha}[v^*]\right)' W_0(Z,X)^{-1}\left(\frac{dg(Z,\alpha_0)}{d\alpha}[v^*]\right)\right| X\right]\right\} \\
&= E\left\{E\left[\left.(D_{w^*}(Z)v_\theta^*)' W_0(Z,X)^{-1}(D_{w^*}(Z)v_\theta^*)\right| X\right]\right\} \\
&= v_\theta^{*\prime} E\left\{E\left[\left. D_{w^*}(Z)'W_0(Z,X)^{-1}D_{w^*}(Z)\right| X\right]\right\} v_\theta^* \\
&= \tau'\left[E\left\{E\left[\left. D_w(Z)'W_0(Z,X)^{-1}D_w(Z)\right| X\right]\right\}\right]^{-1} \\
&\quad \times E\left\{E\left[\left. D_{w^*}(Z)'W_0(Z,X)^{-1}D_{w^*}(Z)\right| X\right]\right\} \\
&\quad \times\left[E\left\{E\left[\left. D_w(Z)'W_0(Z,X)^{-1}D_w(Z)\right| X\right]\right\}\right]^{-1}\tau \\
&= \tau'\left[E\left\{E\left[\left. D_w(Z)'W_0(Z,X)^{-1}D_w(Z)\right| X\right]\right\}\right]^{-1}\tau
\end{aligned}$$

which matches (56) and thus validates (58) shown unique by the Riesz Representation Theorem.

The following additional conditions correspond to Assumptions 4.1-4.3 in Ai and Chen (2003) and are sufficient for the $\sqrt{n}$-normality of $\widehat{\theta}_n$:

Assumption 6.1 *(i)* $E\left\{E\left[\left. D_w(Z)'W_0(Z,X)^{-1}D_w(Z)\right| X\right]\right\}$ *is positive definite; (ii)* $\theta_0 \in int(\Theta)$; *(iii)* $\Sigma_0(X) \equiv Var[g(Z,\alpha_0)|X]$ *is positive definite for all* $X \in \mathcal{X}$.

Assumption 6.2 *There is a* $v_n^* = (v_\theta^*, -\Pi_n w^* \times v_\theta^*) \in \mathcal{A}_n - \alpha_0$ *such that* $\|v_n^* - v^*\|_F = O(n^{-1/4})$.

Following Ai and Chen (2003), let $\mathcal{N}_{0n} \equiv \{\alpha_n \in \mathcal{A}_n : \|\alpha_n - \alpha_0\| = o(1), \|\alpha_n - \alpha_0\|_F = o(n^{-1/4})\}$ and define $\mathcal{N}_0$ the same way with $\mathcal{A}_n$ replaced by $\mathcal{A}$. Also, for any $v \in \overline{\mathbf{V}}$, denote

$$\frac{dg(Z,\alpha)}{d\alpha}[v] \equiv \left.\frac{dg(Z,\alpha+tv)}{dt}\right|_{t=0} \quad \text{a.s. } Z$$

and

$$\frac{dm(Z,\alpha)}{d\alpha}[v] \equiv E\left\{\frac{dg(Z,\alpha)}{d\alpha}[v] \;\middle|\; X\right\} \quad \text{a.s. } Z$$

Assumption 6.3 *For all* $\alpha \in \mathcal{N}_0$*, the pathwise first derivative* $(dg(Z,\alpha(t))/d\alpha)[v]$ *exists a.s.* $Z \in \mathcal{Z}$. *Moreover, (i) each element of* $(dg(Z,\alpha(t))/d\alpha)[v_n^*]$ *satisfies the envelope condition and is Hölder continuous in* $\alpha \in \mathcal{N}_{0n}$*; (ii) each element of* $(dm(Z,\alpha(t))/d\alpha)[v_n^*]$ *is in* $\Lambda_c^\gamma(\mathcal{X})$, $\gamma > d_x/2$ *for all* $\alpha \in \mathcal{N}_0$.

The following result is proved in the Appendix.

Theorem 6.1 *Under Assumptions 4.1-4.8, 5.1-5.6 and 6.1-6.3,* $\sqrt{n}(\widehat{\theta}_n - \theta_0) \xrightarrow{d} N(0, \Omega)$ *where*

$$
\begin{aligned}
\Omega &= E\left[Var\left(\frac{d\varphi(X, Z, \alpha_0)}{dg(Z, \alpha)} D_{w^*}(Z)\middle| X\right)\right] \\
&= \left[E\left\{E\left[D_{w^*}(Z)' W_0(Z, X)^{-1} D_{w^*}(Z)\middle| X\right]\right\}\right]^{-1} \qquad (59)
\end{aligned}
$$

Note that if $D_w(Z)$ and $g(Z, \alpha_0)$ are independent conditional on X then the expression (59) reduces to the asymptotic variance-covariance formula (22) in Ai and Chen (2003) that is shown to be asymptotically efficient by these authors. A consistent estimator of Ω can be obtained in the following way: First estimate $W_0(x_i, z_j)^{-1}$ with

$$
\begin{aligned}
w_{ij} &= p^{k_n}(x_j)' (P'P)^{-1} p^{k_n}(x_i) \\
\widehat{\Sigma}(x_i, \widehat{\alpha}_n) &= \sum_{j=1}^{n} w_{ij} g(z_j, \widehat{\alpha}_n) g'(z_j, \widehat{\alpha}_n) \\
\widehat{W}_0(x_i, z_j)^{-1} &= \widehat{\Sigma}(x_i, \widehat{\alpha}_n)^{-1} g(z_j, \widehat{\alpha}_n) g'(z_j, \widehat{\alpha}_n) \widehat{\Sigma}(x_i, \widehat{\alpha}_n)^{-1} \qquad (60)
\end{aligned}
$$

Then for each $s = 1, ..., d_\theta$ estimate w_s^* with $\widehat{w}_s^*$ which is a solution to the minimization problem

$$
\begin{aligned}
\min_{w_s \in \mathcal{H}_n} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} &\left(\frac{dg(z_j, \widehat{\alpha}_n)}{d\theta_s} - \frac{dg(z_j, \widehat{\alpha}_n)}{dh}[w_s]\right)' \widehat{W}_0(z_j, x_i)^{-1} \\
&\times \left(\frac{dg(z_j, \widehat{\alpha}_n)}{d\theta_s} - \frac{dg(z_j, \widehat{\alpha}_n)}{dh}[w_s]\right)
\end{aligned}
$$

and let $\widehat{w}^* = (\widehat{w}_1^*, ..., \widehat{w}_{d_\theta}^*)$ implying

$$
\widehat{D}_{\widehat{w}^*}(z_j) = \frac{dg(z_j, \widehat{\alpha}_n)}{d\theta_s} - \frac{dg(z_j, \widehat{\alpha}_n)}{dh}[\widehat{w}^*] \qquad (61)
$$

Finally, use (60) and (61) in a finite-sample analog of (59) to obtain

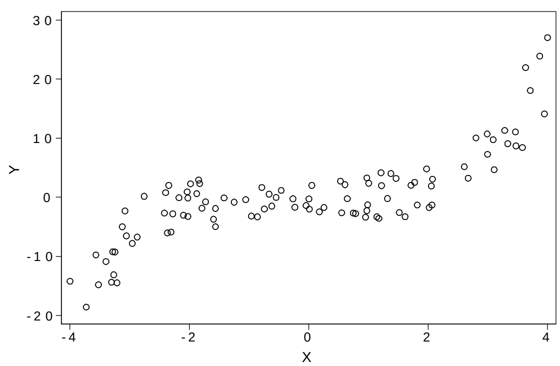
$$
\widehat{\Omega} = \left[\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w'_{ij} \widehat{D}_{\widehat{w}^*}(z_j)' \widehat{W}_0(x_i, z_j)^{-1} \widehat{D}_{\widehat{w}^*}(z_j)\right]^{-1}
$$

We note that for linear sieves computing $\widehat{w}_s^*$ does not require nonlinear optimization and thus the covariance estimator is easy to compute.

7 Simulation

To evaluate the finite sample performance of the estimator $\widehat{\theta}_{LWCEL}$ defined in (28) against KTA's $\widehat{\theta}_{CEL}$ we have conducted a small scale pilot Monte Carlo (MC) simulation study aimed at maximum simplicity of the simulation design. More extensive MC analysis assessing the performance of LWCEL and SLWCEL is currently being conducted and will be included in further updates of the paper. We set $Z = X$ and $Y = \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + e$ with heteroskedastic $e = 0.5u|X|$, $u = U(-5, 5)$. A random sample $N = 100$ of $X\tilde{}N(0, 2)$ was truncated at -1 and 1 and spread over the interval $[-4, 4]$ to avoid far outliers. The true parameter values were set at $\beta_1 = -0.2$, $\beta_2 = 0.1$, $\beta_3 = 0.3$. A typical data draw looks as follows:

Figure 1: Sample Simulated Data



In order to deal with possible negative arguments in the log function, we followed the approach suggested by Owen (2001) cited in Kitamura (2006) (p. 51): for a small number $\delta = 0.2$ we used the objective function

$$\log_* y = \begin{cases} \log(y) & \text{if } y > \delta \\ \log(\delta) - 1.5 + 2y/\delta - \delta^2/2\delta^2 & \text{if } y \leq \delta \end{cases}$$

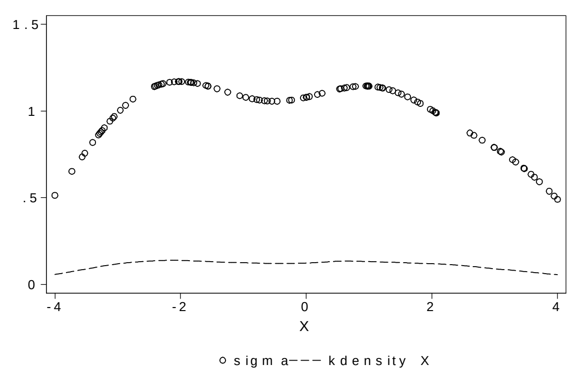
Indeed, the proportion of $y \leq \delta$ in the overall sample was 6.6×10^{-3} and 4.7×10^{-3} for $\widehat{\theta}_{LWCEL}$ and $\widehat{\theta}_{CEL}$, respectively. The Nadaraya-Watson kernel estimator (Pagan and Ullah, 1999, p.86) with the Gaussian kernel, employing the Silverman's rule of thumb for the bandwidth determination (Silverman, 1986, p.45), was used to calculate w_{ij} the case of $\widehat{\theta}_{CEL}$. Thus each i-th local conditional empirical likelihood of $\widehat{\theta}_{CEL}$ was normalized with its corresponding $\sum_{j=1}^{N} w_{ij}$ in the denominator of the Nadaraya-Watson kernel estimator. In contrast, the denominator of the Nadaraya-Watson kernel estimator was replaced with $n^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij}$ for the case of $\widehat{\theta}_{LWCEL}$. This is equivalent (up to a constant of proportionality) to weighting each i-th local conditional empirical likelihood of $\widehat{\theta}_{LWCEL}$ with σ_i as defined in (17). We compared bias, variance and mean-square error over 100 MC iterations on the three estimated coefficients β_1, β_2 and β_3. The results are as follows:

Table 1: Simulation Results

Criterion	*Estimate*	*CEL*	*LWCEL*
Bias	$\widehat{\beta}_1$	-9.100×10^{-2}	-8.619×10^{-2}
	$\widehat{\beta}_2$	1.436×10^{-2}	1.471×10^{-2}
	$\widehat{\beta}_3$	1.050×10^{-2}	9.416×10^{-3}
Variance	$\widehat{\beta}_1$	8.297×10^{-3}	6.189×10^{-3}
	$\widehat{\beta}_2$	2.474×10^{-3}	2.351×10^{-3}
	$\widehat{\beta}_3$	4.202×10^{-4}	3.916×10^{-4}
MSE	$\widehat{\beta}_1$	1.652×10^{-2}	1.362×10^{-2}
	$\widehat{\beta}_2$	2.681×10^{-3}	2.568×10^{-3}
	$\widehat{\beta}_3$	5.304×10^{-4}	4.802×10^{-4}

Both estimators performed relatively well under the simulation scenario which can be attributed to the relatively well-behaved nature of the data. Nonetheless, the $\widehat{\theta}_{LWCEL}$ improved on the $\widehat{\theta}_{CEL}$ in all cases, barring one bias term. The values of σ_i were also retained as an interesting byproduct of the $\widehat{\theta}_{LWCEL}$ estimation procedure, weighting individual local conditional empirical log likelihoods. Naturally, their magnitude follows the density of the data juxtaposed against σ_i in Figure 2:

Figure 2: Plot of σ_i against x_i



8 Conclusion

In this paper we propose a new form of the Conditional Empirical Likelihood (CEL), the Locally Weighted CEL (LWCEL) estimator for models of conditional moment restrictions that contain finite dimensional unknown parameters θ. This estimator extends the CEL analyzed by Kitamura et al. (2004). We construct the CEL dual MD global objective function with a weighting scheme containing two new features: the scheme accounts for the relative importance of each local discrepancy measure in the overall objective function, and adapts to local inhomogeneities in the data. In a Monte Carlo study, we show that the resulting estimator exhibits better finite-sample properties in the finite-dimensional case $E\left[g\left(Z,\theta_0\right)|X\right] = 0$ than found in the previous literature. We further extend the LWCEL estimator to the semiparametric environment defined by models of conditional moment restrictions $E\left[g\left(Z,\alpha_0\right)|X\right] = 0$ containing both θ and infinite dimensional unknown functions h. We establish consistency of the new estimator $\widehat{\alpha}_n$, convergence rates of $\widehat{\alpha}_n$ under the Fisher norm, and asymptotic normality of the finite-dimensional component $\widehat{\theta}_n$. The new Sieve-based LWCEL estimator (SLWCEL) is a direct alternative to the GMM-type sieve minimum distance estimators considered by Ai and Chen (2003) and Newey and Powell (2003). As shown by Newey and Smith (2004), GEL-type estimators, such as EL, outperform the GMM estimator in terms of higher-order properties in parametric models $E\left[g\left(Z,\theta_0\right)|X\right] = 0$. We conjecture that a similar type of improvements is likely to occur also in the semiparametric context of $E\left[g\left(Z,\alpha_0\right)|X\right] = 0$.

Appendix 1: Proofs of Main Results

Discussion of Consistency

In outlining our consistency proof, we follow the discussion as given by KTA and extend it to our case of infinite dimensional parameter space. For a standard extremum estimation procedure (for example via maximization), consistency can be shown by considering the sample objective function and its population counterpart and arguing in the following manner. Consider an arbitrary neighborhood of the true parameter value. Check that:

(A) Outside the neighborhood, the sample objective function is bounded away from the maximum of the population objective function achieved at the true parameter value, w.p.a. 1.

(B) The maximum of the sample objective function is by definition not smaller than its value at the true parameter value. The latter converges to the population objective function evaluated at the true value, due to the LLN.

By (A) and (B) the maximum of the sample objective function is unlikely to occur outside the (arbitrarily defined) neighborhood for large samples. This shows the consistency.

While Newey and Powell (2003) were able to recast their estimator as an argmin of a quadratic form delivering (A), in Chen (2005) (Theorem 3.1) (A) is assumed. In our problem, however, such approach cannot be applied directly. Specifically, showing (A) is problematic here, since the objective function G_n defined in (36) contains the Lagrange multiplier $\lambda(\alpha_n)$ which is endogenously determined at each α_n. Therefore, in our proof we follow the KTA approach binding G_n with a dominating function and then check (A) for the latter by comparing the convergence rates of G_n at α_0 and outside a $\delta-$neighborhood of α_0. The convergence rate of $G_n(\alpha_0)$ is a new result which differs from the one of KTA since the definition of our G_n contains an additional term σ_i arising from the use of a different weighting scheme and due to our estimator being based on series rather than kernel weights. In our proof, a Uniform Law of Large Numbers (ULLN) for the dominating function is used only for α_n outside the $\delta-$neighborhood of α_0.

Regarding the complications incurred by considering an infinite dimensional parameter space α, we note that our consistency proof differs from the ones used in Newey and Powell (2003) (Theorem 1) and Chen (2005) (Theorem 3.1) for M-estimators with α. Using a ULLN over the sieve space, these authors show that the sample objective function G_n and its expectation are, w.p.a 1, within a $\delta-$neighborhood of each other when evaluated at a parameter $\widetilde{\alpha}_n$ in the sieve space that converges to the true parameter value α_0. Existence of such parameter $\widetilde{\alpha}_n$ is guaranteed by the definition of the sieve space. This approach, however, would necessitate evaluating the convergence rates of $G_n(\widetilde{\alpha}_n)$ to its expectation which is problematic in our saddle-point case since it is difficult to capture the behavior of the endogenous $\lambda_i(\alpha)$ away from α_0. Recall that $\widehat{\alpha}_n$ is defined as maximizing $G_n(\alpha_n)$ over the sieve space $\mathcal{A}_n$ and thus using $G_n(\alpha)$, $\alpha \in \mathcal{A}$ for estimation purposes would yield an unfeasible estimator. Nonetheless, the function $g(z_j, \alpha)$ and hence the functions $G_n(\alpha)$ and $\Sigma_n(x_i, \alpha)$ can theoretically be evaluated at a generic parameter value $\alpha \in \mathcal{A}$ not restricted to the sieve space. Hence the asymptotic rate of convergence of $G_n(\alpha_0)$ at the true parameter value can be derived to facilitate asymptotic analysis.

Further Notation

Let us introduce some additional notation. Let $\|\cdot\|_E$ denote the Euclidean norm. Define

$$\begin{aligned} a_i &\equiv \sigma_i - 1 \\ &= \sum_{j=1}^{n} w_{ij} - 1 \\ &= \mathbf{i}' P \left(P'P\right)^{-1} p^{k_n}(x_i) - 1 \end{aligned}$$

For generic n vectors z and a vector x we drop the subscript i and use

$$a_x \equiv \mathbf{i}' P \left(P'P\right)^{-1} p^{k_n}(x) - 1 \tag{62}$$

Further define $B(\alpha_0, \delta)$ and $B_n(\alpha_0, \delta)$ as $\delta-$neighborhoods around α_0 with $B(\alpha_0, \delta) \subset \mathcal{A}$ and $B_n(\alpha_0, \delta) \subset \mathcal{A}_n$, respectively. Consider the function $\psi(X, \alpha)$ as defined in (40). Denote

$$\begin{aligned} \psi_n(x_i, \alpha) &\equiv \sum_{j=1}^{n} w_{ij} \varphi\left(x_i, z_j, \alpha\right) \\ &= \sum_{j=1}^{n} w_{ij} \ln\left\{\sigma_i + \lambda_i' g\left(z_j, \alpha\right)\right\} \end{aligned} \tag{63}$$

$$
\begin{aligned}
G_n(\alpha_n) &\equiv -\frac{1}{n}\sum_{i=1}^{n}\psi_n(x_i,\alpha) \\
&= -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\varphi\left(x_i,z_j,\alpha\right) \\
&= -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln\left\{\sigma_i+\lambda_i' g\left(z_j,\alpha_n\right)\right\} \qquad (64)
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_n(x_i,\alpha) &\equiv \sum_{j=1}^{n} w_{ij} g(z_j,\alpha) g'(z_j,\alpha) \qquad (65)\\
\Sigma(X,\alpha) &\equiv E_Z\left[\Sigma_n(X,\alpha)\right]
\end{aligned}
$$

and recall the definition of $\Sigma_0(X) \equiv Var[g(Z,\alpha_0)|X]$ in Assumption 6.1 (iii).

Main Proofs

Proof of Theorem 4.1. Following KTA, in the asymptotic analysis we will replace $\lambda_i(\alpha)$ by

$$
u\left(x_i,\alpha\right)=\frac{E\left[g\left(z,\alpha\right)|x_i\right]}{\left(1+\|E\left[g\left(z,\alpha\right)|x_i\right]\|\right)}
$$

For a constant $\widetilde{c}\in(0,1)$ define a sequence of truncation sets

$$
C_n=\left\{z:\sup_{\alpha\in\mathcal{A}}\left|a_x+u'\left(x,\alpha_n\right)g\left(z,\alpha_n\right)\right|\leq\widetilde{c}n^{1/m}\right\} \qquad (66)
$$

and let

$$
s_n\equiv n^{-1/m}\left[a_x+u'\left(x,\alpha_n\right)g\left(z,\alpha_n\right)\right]\mathbb{I}\left\{z\in C_n\right\} \qquad (67)
$$

Let

$$
\begin{aligned}
q_n\left(x,z,\alpha_n\right) &= -\log\left(1+n^{-1/m}\left[a_x+u'\left(x,\alpha_n\right)g\left(z,\alpha_n\right)\right]\mathbb{I}\left\{z\in C_n\right\}\right) \\
&= -\log\left(1+s_n\right)
\end{aligned}
$$

The modified objective function is

$$
Q_n(\alpha_n)=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} q_n\left(x_i,z_j,\alpha_n\right) \qquad (68)
$$

Note that

$$
G_n(\alpha_n)\leq Q_n(\alpha_n) \qquad (69)
$$

for all $\alpha_n\in\mathcal{A}_n$ by the optimality of λ_i.

Then by the Taylor series expansion for logarithms

$$
\begin{aligned}
q_n(x,z,\alpha_n) &= -\log\left(1+s_n\right) \\
&= -s_n+\frac{\widetilde{s}_n^2}{2} \\
&= -s_n+\frac{s_n^2}{2(1-ts_n)} \\
&= -n^{-1/m}\left[a_x+u'\left(x,\alpha_n\right)g\left(z,\alpha_n\right)\right]\mathbb{I}\left\{z\in C_n\right\}+\frac{s_n^2}{2(1-ts_n)}
\end{aligned}
$$

$$
\begin{aligned}
&= n^{-1/m}\left[a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right] - n^{-1/m}\left[a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right] \\
&\quad -n^{-1/m}\left[a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right]\mathbb{I}\{z\in C_n\} + \frac{s_n^2}{2(1-ts_n)} \\
&= -n^{-1/m}\left[a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right] \\
&\quad +n^{-1/m}\left[a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right](1-\mathbb{I}\{z\in C_n\}) + \frac{s_n^2}{2(1-ts_n)} \\
&= -n^{-1/m}\left[a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right] + R_n(t,a_x,\alpha_n) \qquad (70)
\end{aligned}
$$

where

$$
\begin{aligned}
R_n(t,a_x,\alpha_n) &= n^{-1/m}\left[a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right](1-\mathbb{I}\{z\in C_n\}) \\
&\quad + \frac{n^{-2/m}\left[a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right]^2\mathbb{I}\{z\in C_n\}}{2(1-tn^{-1/m}\left[a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right]\mathbb{I}\{z\in C_n\})^2}
\end{aligned}
$$

Note that, by the triangle and Cauchy-Schwarz inequalities

$$
\begin{aligned}
|R_n(t,a_x,\alpha_n)| &\le n^{-1/m}\left[|a_x| + \left\|u'(x,\alpha_n)\right\|\left\|g(z,\alpha_n)\right\|\right](1-\mathbb{I}\{z\in C_n\}) \\
&\quad + \frac{n^{-2/m}\left[a_x^2 + 2\left\|a_x\right\|\left\|u'(x,\alpha_n)\right\|\left\|g(z,\alpha_n)\right\| + \left\|u'(x,\alpha_n)\right\|^2\left\|g(z,\alpha_n)\right\|^2\right]\mathbb{I}\{z\in C_n\}}{2(1-tn^{-1/m}\left[a_x + u'(x,\alpha_n)\, g_n(z,\alpha_n)\right])^2}
\end{aligned}
$$

and by $\left\|u'(x,\alpha_n)\right\| < 1$ we obtain

$$
\begin{aligned}
|R_n(t,a_x,\alpha_n)| &\le n^{-1/m}\left[|a_x| + \left\|g(z,\alpha_n)\right\|\right](1-\mathbb{I}\{z\in C_n\}) \\
&\quad + \frac{n^{-2/m}\left[a_x^2 + 2a_x\left\|g(z,\alpha_n)\right\| + \left\|g(z,\alpha_n)\right\|^2\right]}{2(1-tn^{-1/m}\left[a_x + u'(x,\alpha_n)\, g_n(z,\alpha_n)\right])^2}
\end{aligned}
$$

From (66) it follows that

$$
\begin{aligned}
\widetilde{c} &\ge n^{-1/m}\sup_{\alpha\in\mathcal{A}}\left|a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right| \\
&\ge n^{-1/m}\left|a_x + u'(x,\alpha_n)\, g(z,\alpha_n)\right| \\
&\ge tn^{-1/m}\left|a_x + u'(x,\alpha_n)\, g_n(z,\alpha_n)\right|
\end{aligned}
$$

and hence

$$
\begin{aligned}
|R_n(t,a_x,\alpha_n)| &\le n^{-1/m}\left[|a_x| + \left\|g(z,\alpha_n)\right\|\right](1-\mathbb{I}\{z\in C_n\}) \\
&\quad + \frac{n^{-2/m}\left[a_x^2 + 2a_x\left\|g(z,\alpha_n)\right\| + \left\|g(z,\alpha_n)\right\|^2\right]}{2(1-\widetilde{c})^2} \\
&= n^{-1/m}\left[|a_x| + \left\|g(z,\alpha_n)\right\|\right](1-\mathbb{I}\{z\in C_n\}) \\
&\quad + n^{-2/m}\frac{a_x^2}{2(1-\widetilde{c})^2} + \frac{n^{-2/m}\left[2a_x\left\|g(z,\alpha_n)\right\| + \left\|g(z,\alpha_n)\right\|^2\right]}{2(1-\widetilde{c})^2}
\end{aligned}
$$

taking sup over $\mathcal{A}$ we obtain

$$
\begin{aligned}
\sup_{\alpha\in\mathcal{A}}|R_n(t,a_x,\alpha_n)| &\le n^{-1/m}\left[|a_x| + \sup_{\alpha\in\mathcal{A}}\left\|g(z,\alpha_n)\right\|\right](1-\mathbb{I}\{z\in C_n\}) + n^{-2/m}\frac{a_x^2}{2(1-\widetilde{c})^2} \\
&\quad + \frac{n^{-2/m}\left[2a_x\sup_{\alpha\in\mathcal{A}}\left\|g(z,\alpha_n)\right\| + \sup_{\alpha\in\mathcal{A}}\left\|g(z,\alpha_n)\right\|^2\right]}{2(1-\widetilde{c})^2} \qquad (71)
\end{aligned}
$$

In view of (70) and (71) approximate $n^{1/m}Q_n(\alpha_n)$ by $n^{1/m}\overline{Q}_n(\alpha_n)$ where

$$
\overline{Q}_n(\alpha_n) = -\frac{1}{n^{1+1/m}}\sum_{i=1}^{n} u'(x_i,\alpha_n)\, E\left[g(z,\alpha_n)\,|x_i\right] \qquad (72)
$$

Lemma A.2 shows that

$$n^{1/m}Q_n(\alpha_n) = n^{1/m}\overline{Q}_n(\alpha_n) + o_p(1) \qquad \text{uniformly in } \alpha_n \in \mathcal{A}_n \tag{73}$$

Next, we will apply a uniform law of large numbers to $n^{1/m}\overline{Q}_n(\alpha)$ over the whole parameter space $\mathcal{A}$. Under Assumptions 4.4(i), 4.5, and 4.6 $E[g(z,\alpha)|x_i]$ is continuous in $\alpha \in \mathcal{A}$ by Corollary 4.2 of Newey (1991), and so is

$$-u'(x_i,\alpha)E[g(z,\alpha)|x_i] = -\frac{\|E[g(z,\alpha)|x_i]\|^2}{1+\|E[g(z,\alpha)|x_i]\|}$$

Under Assumption 4.5(i) $E\left[\sup_{\alpha\in\mathcal{A}}|-u'(x_i,\alpha)E[g(z,\alpha)|x_i]|\right] < \infty$. These, together with Assumption 4.4(i) satisfy the conditions of Lemma A2 of Newey and Powell (2003) implying the following uniform law of large numbers:

$$\sup_{\alpha\in\mathcal{A}}\left|n^{1/m}\overline{Q}_n(\alpha) - E\left[-u'(x_i,\alpha)E[g(z,\alpha)|x_i]\right]\right| = o_p(1) \tag{74}$$

where $-E[-u'(x_i,\alpha)E[g(z,\alpha)|x_i]]$ is continuous in $\mathcal{A}$. This function is bounded above by

$$-E\left[u'(x_i,\alpha)E[g(z,\alpha)|x_i]\right] \leq -E\left[\mathbb{I}\{x\in\mathcal{X}_{\mathcal{A}}\}\|E[g(z,\alpha)|x_i]\|^2/(1+\|E[g(z,\alpha)|x_i]\|)\right] \tag{75}$$

By Assumption 4.1, the right-hand side of this inequality is strictly negative at each $\alpha \neq \alpha_0$. Therefore, by continuity of $-E[u'(x_i,\alpha)E[g(z,\alpha)|x_i]]$ and compactness of $\mathcal{A}$, there exists a strictly positive number $H(\delta)$ such that

$$\sup_{\alpha\in\mathcal{A}\setminus B(\alpha_0,\delta)} E\left[-u'(x_i,\alpha)E[g(z,\alpha)|x_i]\right] \leq -H(\delta) \tag{76}$$

By (69), (73), and Assumption 4.4(ii) we have

$$\sup_{\alpha_n\in\mathcal{A}_n} n^{1/m}G_n(\alpha_n) \leq \sup_{\alpha_n\in\mathcal{A}_n} n^{1/m}Q_n(\alpha_n) = \sup_{\alpha_n\in\mathcal{A}_n} n^{1/m}\overline{Q}_n(\alpha_n) + o_p(1) \tag{77}$$

Together (77) with (76) and (74) imply that

$$\Pr\left\{\sup_{\alpha_n\in\mathcal{A}_n\setminus B_n(\alpha_0,\delta)} G_n(\alpha_n) > -n^{-1/m}H(\delta)\right\} < \delta/2 \qquad \text{eventually.} \tag{78}$$

Next, we evaluate G_n at the true value α_0 and show that $G_n(\alpha_0)$ converges to its expectation faster than $G_n(\alpha_n)$ with α_n outside a $\delta-$neighborhood of α_0 whose convergence rate is given in (78). Having established this fact the conclusion of the proof is then straightforward. This approach was taken by KTA for the finite-dimensional parameter θ and we extend it to the infinite-dimensional parameter α. Our way of deriving the rate of convergence of $G_n(\alpha_0)$ differs from KTA, though, because we do not make use of kernel-based results. Rather, based on the series literature, we derive a new result for the rate of convergence by specializing Lemma A.1(A) of Ai and Chen (2003) to our case.

Using Lemma A.4 and the fact

$$1 + a_i = \sum_{j=1}^n w_{ij} > 0 \quad \text{for each } i$$

we obtain

$$\begin{aligned}
G_n(\alpha_0) &= -\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n w_{ij}\log\left(1 + a_i + \lambda_i'(\alpha_0)g(z_j,\alpha_0)\right) \\
&\geq -\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n w_{ij}\left(a_i + \lambda_i'(\alpha_0)g(z_j,\alpha_0)\right) \\
&= -\frac{1}{n}\sum_{i=1}^n \lambda_i'(\alpha_0)\sum_{j=1}^n w_{ij}g(z_j,\alpha_0) \\
&\geq -\max_{1\leq i\leq n}\|\lambda_i(\alpha_0)\|\max_{1\leq i\leq n}\left\|\sum_{j=1}^n w_{ij}g(z_j,\alpha_0)\right\|
\end{aligned}$$

Then by Lemmas A.1 and A8,

$$\begin{aligned} G_n(\alpha_0) &= \left\{ o_p(\widetilde{\delta}_{1n}) + o_p\left(\frac{1}{n^{\varrho - 1/m}}\right) \right\}^2 \\ &= o_p(r_n^2) \end{aligned}$$

where

$$r_n \equiv o_p(\widetilde{\delta}_{1n}) + o_p\left(\frac{1}{n^{\varrho-1/m}}\right)$$

with $\widetilde{\delta}_{1n}$ defined in Lemma A.7 and ϱ defined in 4.7. Therefore, we have the following LLN

$$\Pr\left\{G_n(\alpha_0) < -r_n^2 H(\delta)\right\} < \delta/2 \qquad \text{eventually.} \tag{79}$$

Denote

$$\begin{aligned} \widehat{Q}_1(\alpha) &\equiv n^{1/m} G_n(\alpha) \\ \widehat{Q}_2(\alpha) &\equiv r_n^{-2} G_n(\alpha) \\ Q_1(\alpha) &\equiv -E\left[u'(x,\alpha)E\left[g(z,\alpha)|x\right]\right] \\ Q_2(\alpha) &\equiv E\widehat{Q}_2(\alpha) \end{aligned}$$

where the last expectation is taken with respect to the joint density of (Y, X). Under Assumptions 4.4(i), 4.5, and 4.6 $Q_2(\alpha)$ is continuous in $\alpha \in \mathcal{A}$ by Corollary 4.2 of Newey (1991). Note that since $n^{1/m} r_n^2 \to 0$ and $n^{1/m} G_n(\alpha) \leq 0$, by (74) and (77), w.p.a. 1,

$$\begin{aligned} r_n^{-2} &> n^{1/m} \\ \widehat{Q}_2(\alpha) &\leq \widehat{Q}_1(\alpha) \end{aligned} \tag{80}$$

If we retain $\lambda_i(\alpha)$ instead of $u(x, \alpha)$ in the definition of $Q_n(\alpha)$ in (68), using $\lambda_i(\alpha) = O_p(1)$ which follows from (37), we can derive an analog of $\overline{Q}_n(\alpha)$ in (72) as

$$\overline{Q}_{2n}(\alpha) = -\frac{1}{n^{1+1/m}} \sum_{i=1}^{n} \lambda_i'(\alpha) E\left[g\left(z,\alpha\right)|x_i\right]$$

By a corresponding analog of (73) and the moment restriction $E\left[g\left(z,\alpha_0\right)|x_i\right] = 0$ it follows that $\overline{Q}_{2n}(\alpha_0) = 0$ and $Q_2(\alpha_0) = 0$. Also, by (75) $Q_1(\widehat{\alpha}_n) < 0$ for each $\theta \neq \theta_0$ and thus

$$Q_1(\widehat{\alpha}_n) \leq 0 \tag{81}$$

Then, w.p.a. 1,

$$\begin{aligned} Q_1(\widehat{\alpha}_n) &\geq \widehat{Q}_1(\widehat{\alpha}_n) + H(\delta)/2 && (82) \\ &\geq \widehat{Q}_1(\alpha_0) + H(\delta)/2 && (83) \\ &\geq \widehat{Q}_2(\alpha_0) + H(\delta)/2 && (84) \\ &> Q_2(\alpha_0) + H(\delta) && (85) \\ &= H(\delta) && (86) \end{aligned}$$

where (82) holds by (74) and (77), (83) holds by the definition of $\widehat{\alpha}_n$, (84) by (80), (85) by LLN at α_0 (79), and (86) by $Q_2(\alpha_0) = 0$. By (81) and δ being arbitrary, taking $H(\delta) \to 0$,

$$\widehat{Q}_1(\widehat{\alpha}_n) \xrightarrow{p} 0$$

Then, using Assumption 4.4(ii), $\Pr\left(\left|\widehat{Q}_1(\widehat{\alpha}) - Q_2(\alpha_0)\right| \geq H(\delta)\right) \to 0$ and by (78) $\Pr\left(\widehat{\alpha}_n \in \mathcal{A}_n \backslash B_n\left(\theta_0, \delta\right)\right) \to 0$.

■

Proof of Theorem 5.1.

In deriving the convergence rates under the Fisher norm $\|\cdot\|_F$ we will proceed in a way that is similar to the proof of Theorem 3.1 in Ai and Chen (2003). Specifically, we will use their Lemma A.1 and Corollary A.1 that hold for a generic function $m(X,\alpha)$ and the Euclidean metric. However, since our objective function and metric differs from the ones used by these authors, we need to derive the counterparts of their Corollaries A.2 and B.1 for our case.

Recall the definition of $G_n(\alpha_n)$ in (64)

$$G_n(\alpha_n) \equiv -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln\left\{\sigma_i + \lambda_i' g\left(z_j,\alpha_n\right)\right\}$$

and define

$$\overline{G}_n(\alpha_n) \equiv -\frac{1}{n}\sum_{i=1}^{n} E\left[\ln\left\{\sigma_i + \lambda_i' g\left(z,\alpha_n\right)\right\}|x_i\right] \tag{87}$$

Let $\delta_{0n} = o(n^{-1/4})$ and denote $\alpha_{n0} = \Pi\alpha_0$ (the orthogonal projection of α_0 onto the sieve space).

$$P\left(\|\widehat{\alpha}_n - \alpha_0\|_F \geq \delta_{0n}\right) = P\left(\sup_{\left\{\|\widehat{\alpha}_n-\alpha_0\|_F\geq\delta_{0n},\alpha_n\in\mathcal{A}_n\right\}} G_n(\alpha_n) \geq G_n(\alpha_{n0})\right)$$

For the sake of brevity, let "AC" stand for "Ai and Chen (2003)" for the remainder of the proof. Note that Assumptions 3.1-3.2, 3.6-3.8 and 4.1(iii) in AC are equivalent to our Assumptions 4.2, 4.3, 5.2, 4.5, 4.6, 5.3-5.5 and 5.6, respectively. Assumption 3.3 in AC is implied by our Assumption 4.1 and the condition (1). The analog of AC's Assumption 3.4 for our $\Sigma_n(x_i,\alpha)$ defined in (65) is satisfied by AC's Corollary A.1(i). Thus Assumptions of AC's Lemma A.1 and Corollary A.1 are satisfied.

Lemma B.1 states the counterparts of their AC's Corollaries A.2 and B.1 for our case. We note that condition (A) of our consistency proof was shown to hold for $G_n(\alpha_n)$ in Theorem 4.1. Since $\widetilde{G}_n(\alpha_n) \leq G_n(\alpha_n)$, by (77) the condition also holds for $\widetilde{G}_n(\alpha_n)$. Thus the identification condition is satisfied. Satisfying Assumptions of Theorem 1 of Shen and Wong (1994) is also a necessary condition for AC's Theorem 3.1. Since the role of the pseudodistance in Theorem 1 of Shen and Wong (1994) is performed by our metric $\|\cdot\|_F^2$ in a way topologically equivalent to the AC's one, and the remaining AC's Assumptions hold as described above, this condition is also satisfied. Invocation of AC's Theorem 3.1, with their objective function and metric replaced with ours, completes the proof.

■

Proof of Theorem 6.1.

Substituting (58) into (57) yields

$$\frac{dg(Z,\alpha_0)}{d\alpha}[v^*] = D_{w^*}(Z)\left[E\left\{E\left[D_w(Z)'W_0(Z,X)^{-1}D_w(Z)\right|X\right]\right\}\right]^{-1}\tau \tag{88}$$

Note that by the chain rule

$$\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}[v^*] = \frac{d\varphi\left(X,Z,\alpha_0\right)}{dg\left(Z,\alpha\right)}\frac{dg\left(Z,\alpha_0\right)}{d\alpha}[v^*] \tag{89}$$

Using Lemma C.1 and (88) in (89), we obtain

$$\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}[v^*] = \frac{d\varphi\left(X,Z,\alpha_0\right)}{dg\left(Z,\alpha\right)}D_{w^*}(Z)\left[E\left\{E\left[D_{w^*}(Z)'W_0(Z,X)^{-1}D_{w^*}(Z)\right|X\right]\right\}\right]^{-1}\tau \tag{90}$$

We will now check the conditions for Theorem 7.1 in Appendix 3 that is an extension of Theorem 1 of Shen (1997) to our conditional case. Lemma C.2 shows that under our Assumptions, Conditions A is satisfied. Since $\{g\left(z,\alpha_n\right) : \alpha_n \in \mathcal{A}_n\} \subset \Lambda_c^{\gamma}(\mathcal{X})$, Condition B follows directly from Lemma B.1. Since $\|\widehat{\alpha}_n - \alpha_0\|_F = o_p(n^{-1/4})$, then $\delta_n = n^{-1/4}$ and hence for Condition C we require

$$\begin{aligned}\sup_{\{\alpha_n\in\mathcal{A}_n:\|\alpha_n-\alpha_0\|\leq\delta_n\}}\|\varepsilon_n u^* - \varepsilon_n u_n^*\| &= O_p(\delta_n^{-1}\varepsilon_n^2)\\ &= O_p(n^{-1/4})\end{aligned}$$

which is satisfied by Assumption 6.2. Condition D follows from the smoothness of $\frac{d\varphi\left(x_i, z_j, \alpha_0\right)}{d\alpha}[\alpha - \alpha_0]$ in $\mathcal{N}_{0n}$. Condition F is satisfied by the definition of $f\left(\widehat{\alpha}_n\right) \equiv \tau'\widehat{\theta}_n$, $\omega = 1$, and Assumption 6.2. Condition G is satisfied by Assumption 6.1.

By Theorem 7.1 in Appendix 3, for arbitrarily fixed $\tau \in \mathbb{R}^{d_\theta}$ with $|\tau| \neq 0$,

$$\sqrt{n}\tau'(\widehat{\theta}_n - \theta_0) \xrightarrow{d} N(0, \Sigma_{v^*})$$

where

$$\begin{aligned} \Sigma_{v^*} &\equiv E\left[Var\left(\left.\frac{d\varphi\left(X, Z, \alpha_0\right)}{d\alpha}\right| X\right)\right] \\ &= \tau'\Omega\tau \end{aligned} \tag{91}$$

and hence

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \xrightarrow{d} N(0, \Omega)$$

Using (90) in (91) we obtain

$$\begin{aligned} \Omega &= \left[E\left\{E\left[D_{w^*}(Z)'W_0(Z,X)^{-1}D_{w^*}(Z)\right| X\right]\right\}\right]^{-1} \\ &\quad \times E\left[Var\left(\left.\frac{d\varphi\left(X, Z, \alpha_0\right)}{dg\left(Z, \alpha\right)}D_{w^*}(Z)\right| X\right)\right] \\ &\quad \times \left[E\left\{E\left[D_{w^*}(Z)'W_0(Z,X)^{-1}D_{w^*}(Z)\right| X\right]\right\}\right]^{-1} \end{aligned} \tag{92}$$

Using Lemma C.1 and (92)

$$\Omega = \left[E\left\{E\left[D_{w^*}(Z)'W_0(Z,X)^{-1}D_{w^*}(Z)\right| X\right]\right\}\right]^{-1}$$

■

Appendix 2: Auxiliary Results

A. CONSISTENCY

Lemma A.1 (B.3) *Let Assumptions 4.5 and 4.7 hold. Then, pointwise for a given* $\alpha \in \mathcal{A}$,

$$\max_{1\leq i\leq n}\left\|\sum_{j=1}^{n} w_{ij}g\left(z_j,\alpha\right)-E\left[g\left(z,\alpha\right)|x_i\right]\right\| = o_p(\widetilde{\delta}_{1n})+o_p\left(\frac{1}{n^{\varrho-1/m}}\right)$$

where $\widetilde{\delta}_{1n}$ *is defined in Lemma A.7 and* ϱ *in Assumption 4.7.*

Proof. Decompose

$$\begin{aligned}\left\|\sum_{j=1}^{n} w_{ij}g\left(z_j,\alpha\right)-E\left[g\left(z,\alpha\right)|x_i\right]\right\| \quad &\leq \quad \max_{1\leq i\leq n}\left\|\sum_{j=1}^{n} w_{ij}g\left(z_j,\alpha\right)-E\left[g\left(z,\alpha\right)|x_i\right]\right\|\mathbb{I}_{i,n} \\ &+\max_{1\leq i\leq n}\left\|\sum_{j=1}^{n} w_{ij}g\left(z_j,\alpha\right)-E\left[g\left(z,\alpha\right)|x_i\right]\right\|\max_{1\leq i\leq n}\mathbb{I}_{i,n}^c\end{aligned}$$

Note that he results of Lemma D.3 and D.5 in KTA hold also for w_{ij} as defined in this paper. Therefore

$$\max_{1\leq i\leq n}\left\|\sum_{j=1}^{n} w_{ij}g\left(z_j,\alpha\right)-E\left[g\left(z,\alpha\right)|x_i\right]\right\|\max_{1\leq i\leq n}\mathbb{I}_{i,n}^c = o_p\left(\frac{1}{n^{\varrho-1/m}}\right)$$

Next, pick any $\epsilon>0$, $c_n\downarrow 0$, and observe that

$$\Pr\left\{\max_{1\leq i\leq n}\left\|\sum_{j=1}^{n} w_{ij}g\left(z_j,\alpha\right)-E\left[g\left(z,\alpha\right)|x_i\right]\right\|\mathbb{I}_{i,n}>\epsilon c_n\right\}\leq \Pr\left\{\sup_{X\in\mathcal{X}}\left\|\sum_{j=1}^{n} w_{ij}g\left(z_j,\alpha\right)-E\left[g\left(z,\alpha\right)|x_i\right]\right\|>\epsilon c_n\right\}$$

Using Lemma A.7,

$$\Pr\left\{\sup_{X\in\mathcal{X}}\left\|\sum_{j=1}^{n} w_{ij}g\left(z_j,\alpha\right)-E\left[g\left(z,\alpha\right)|x_i\right]\right\|>\epsilon c_n\right\}\leq\epsilon$$

if

$$c_n=\widetilde{\delta}_{1n}$$

where $\widetilde{\delta}_{1n}$ is defined in Lemma A.7. Hence

$$\max_{1\leq i\leq n}\left\|\sum_{j=1}^{n} w_{ij}g\left(z_j,\alpha\right)-E\left[g\left(z,\alpha\right)|x_i\right]\right\|\mathbb{I}_{i,n}=o_p\left(\widetilde{\delta}_{1n}\right)$$

and the desired result follows. ■

Lemma A.2 (B.8) *Let Assumptions 4.5 and 4.7 hold. Then*

$$\sup_{\alpha_n\in\mathcal{A}_n}\left|Q_n(\alpha_n)-\overline{Q}_n(\alpha_n)\right|=o_p(n^{-1/m})$$

Proof. Substituting from (70) for $q_n\left(x_i,z_j,\alpha_n\right)$ we obtain

$$\begin{aligned}&n^{1/m}\sup_{\alpha_n\in\mathcal{A}_n}\left|\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}q_n\left(x_i,z_j,\alpha_n\right)+\frac{1}{n^{1+1/m}}\sum_{i=1}^{n}u'\left(x_i,\alpha_n\right)E\left[g\left(z,\alpha_n\right)|x_i\right]\right| \\ \leq\quad &n^{1/m}\sup_{\alpha_n\in\mathcal{A}_n}\left|\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left\{-n^{-1/m}\left[a_i+u'\left(x_i,\alpha_n\right)g\left(z_j,\alpha_n\right)\right]\right\}+\frac{1}{n^{1+1/m}}\sum_{i=1}^{n}u'\left(x_i,\alpha_n\right)E\left[g\left(z,\alpha_n\right)|x_i\right]\right| \\ &+n^{1/m}\sup_{\alpha_n\in\mathcal{A}_n}\left|\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}R_n(t,a_i,\alpha_n)\right|\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\alpha_n \in \mathcal{A}_n} \left| -\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} a_i + \frac{1}{n} \sum_{i=1}^{n} u'(x_i, \alpha) E\left[g(z, \alpha_n) | x_i\right] - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} u'(x_i, \alpha_n) g(z_j, \alpha_n) \right| \\
&\quad + n^{1/m} \sup_{\alpha_n \in \mathcal{A}_n} \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} R_n(t, a_i, \alpha_n) \right| \\
&\leq -\sup_{\alpha_n \in \mathcal{A}_n} \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} a_i \right| + \sup_{\alpha_n \in \mathcal{A}_n} \frac{1}{n} \sum_{i=1}^{n} \left\| E\left[g(z, \alpha_n) | x_i\right] - \sum_{j=1}^{n} w_{ij} g(z_j, \alpha_n) \right\| \\
&\quad + n^{1/m} \sup_{\alpha_n \in \mathcal{A}_n} \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} R_n(t, a_i, \alpha_n) \right|
\end{aligned}
$$

The first term drops out by Lemma A.4, the second term is $o_p(1)$ by Corollary A.1(i) in Ai and Chen (2003), p. 1824, and the third term is $o_p(1)$ by Lemma A.3. ■

Lemma A.3 *Let Assumptions 4.5 and 4.7 hold. Then*

$$
n^{1/m} \sup_{\alpha_n \in \mathcal{A}_n} \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} R_n(t, a_i, \alpha_n) \right| = o_p(1)
$$

Proof. Note that by (71)

$$
\begin{aligned}
& n^{1/m} \sup_{\alpha_n \in \mathcal{A}_n} \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} R_n(t, a_i, \alpha_n) \right| \\
&\leq \frac{1}{n^{1-1/m}} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \sup_{\alpha_n \in \mathcal{A}_n} |R_n(t, a_i, \alpha_n)| \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \left[|a_i| + \sup_{\alpha_n \in \mathcal{A}_n} \|g(z_j, \alpha_n)\| \right] (1 - \mathbb{I}\{z_j \in C_n\}) \\
&\quad + \frac{1}{n^{1+1/m}} \frac{1}{2(1-\widetilde{c})^2} \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} w_{ij} \\
&\quad + \frac{1}{n^{1+1/m}} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \frac{\left[2a_i \sup_{\alpha_n \in \mathcal{A}_n} \|g(z_j, \alpha_n)\| + \sup_{\alpha \in \mathcal{A}} \|g(z_j, \alpha_n)\|^2\right]}{2(1-\widetilde{c})^2} \\
&= D_1 + D_2 + D_3
\end{aligned}
$$

By Assumption 4.5(i) and 4.4(ii), $\sup_{\alpha_n \in \mathcal{A}_n} \|g(z, \alpha_n)\| < \infty$. By Lemma A.5 $|a_i| < \infty$ and hence by Lemma A.6

$$
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \left[|a_i| + \sup_{\alpha_n \in \mathcal{A}_n} \|g(z_j, \alpha_n)\| \right] = O_p(1).
$$

Since $\max_{1 \leq j \leq n} \mathbb{I}\{z_j \notin C_n\} = o_p(1)$, $D_1 = o_p(1)$. By Lemma A.6 $D_2 = o_p(1)$.

$$
\begin{aligned}
D_3 &= \frac{1}{n^{1+1/m}} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \frac{\left[2a_i \sup_{\alpha_n \in \mathcal{A}_n} \|g(z_j, \alpha_n)\| + \sup_{\alpha \in \mathcal{A}} \|g(z_j, \alpha_n)\|^2\right]}{2(1-\widetilde{c})^2} \\
&= \frac{1}{n^{1+1/m}(1-\widetilde{c})^2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} a_i + \frac{1}{n^{1+1/m}} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \frac{\sup_{\alpha_n \in \mathcal{A}_n} \|g(z_j, \alpha_n)\|^2}{2(1-\widetilde{c})^2}
\end{aligned}
$$

where the first part drops out by Lemma A.4 and the second part is $o_p(1)$ by Assumption 4.5(i), 4.4(ii) and Lemma A.6. ■

Lemma A.4 *Under Assumptions 4.3 and 4.4, for w_{ij} defined in (31) and a_i defined in (62), it holds that*

$$
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} a_i = 0
$$

Proof.

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}a_i &= \frac{1}{n}\sum_{i=1}^{n} a_i \sum_{j=1}^{n} w_{ij} \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{n} w_{ij}-1\right]\sum_{j=1}^{n} w_{ij} \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\mathbf{i}'P\left(P'P\right)^{-1}p^{k_n}(x_i)\mathbf{i}'P\left(P'P\right)^{-1}p^{k_n}(x_i)-\mathbf{i}'P\left(P'P\right)^{-1}p^{k_n}(x_i)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\mathbf{i}'P\left(P'P\right)^{-1}p^{k_n}(x_i)p^{k_n}(x_i)'\left(P'P\right)^{-1}P'\mathbf{i}-\mathbf{i}'P\left(P'P\right)^{-1}p^{k_n}(x_i)\right] \\
&= \mathbf{i}'P\left(P'P\right)^{-1}\left(P'P\right)\left(P'P\right)^{-1}P'\mathbf{i}-\frac{1}{n}\sum_{i=1}^{n}\mathbf{i}'P\left(P'P\right)^{-1}p^{k_n}(x_i) \\
&= \frac{1}{n}\mathbf{i}'P\left(P'P\right)^{-1}P'\mathbf{i}-\frac{1}{n}\mathbf{i}'P\left(P'P\right)^{-1}P'\mathbf{i} \\
&= 0
\end{aligned}$$

■

Lemma A.5 *Under Assumptions 4.3 and 4.4, for w_{ij} defined in (31),*

$$\sum_{j=1}^{n} w_{ij} = O(1)$$

for each $X \in \mathcal{X}$.

Proof. By Assumption 4.3, for any $E\left[\rho_l(Z,\alpha)|x_i\right]$ there exists $p^{k_n}(x_i)'\pi_l = \sum_{j=1}^{n} w_{ij}g_l(z_j,\alpha)$ such that

$$E\left[E\left[g_l(Z,\alpha)|x_i\right]-\sum_{j=1}^{n} w_{ij}g_l(z_j,\alpha)\right] = O(1)$$

The result follows by boundedness of $g_l(z_j,\alpha)$. ■

Lemma A.6 *Under Assumptions 4.3 and 4.4, for w_{ij} defined in (31),*

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} = O_p(1)$$

Proof. Follows directly from Lemma A.5. ■

Lemma A.7 *Let*

$$\begin{aligned}
\xi_{0n} &\equiv \sup_{X\in\mathcal{X}}\left\|p^{k_n}(X)\right\|_E \\
\xi_{1n} &\equiv \sup_{X\in\mathcal{X}}\left\|\frac{\partial p^{k_n}(X)}{\partial x'}\right\|_E
\end{aligned}$$

Let $\widetilde{g}:\mathcal{Z}\to\mathbb{R}$ denote a generic measurable function of the data $Z\in\mathcal{Z}$, evaluated at a given fixed parameter α. Define $\varepsilon(Z,\alpha)=\widetilde{g}(Z,\alpha)-E\left[\widetilde{g}(Z,\alpha)|X\right]$ and $\varepsilon(\alpha)=\left(\varepsilon(Z_1,\alpha),...,\varepsilon(Z_n,\alpha)\right)'$.

Suppose that Assumptions 4.2 and 4.3(i) and the following are satisfied:

(i) There exists a constant c_{1n} and a measurable function $c_1(Z):\mathcal{Z}\to[0,\infty)$ with $E[c_1(Z)^p]<\infty$ for some $p\geq 4$ such that $|\widetilde{g}(Z,\alpha)|\leq c_{1n}c_1(Z)$ for all $Z\in\mathcal{Z}$;

(ii) There exists a positive value $\widetilde{\delta}_{1n}=o_p(1)$ such that

$$\frac{n\widetilde{\delta}_{1n}^2}{\ln\left[\left(\frac{\xi_{1n}c_{1n}}{\delta_{1n}}\right)^{d_x}\right]\max\left\{\xi_{0n}^2c_{1n}^2,\xi_{0n}^{2+2/p}\delta_{1n}^{1-2/p}c_{1n}^{1+2/p}\right\}}\to\infty$$

Then

$$p^{k_n}(X)'(P'P)^{-1}P'\varepsilon(\alpha) = o_p(\delta_{1n})$$

uniformly over $X \in \mathcal{X}$.

Proof. This result specializes Lemma A.1(A) in Ai and Chen (2003), derived for the combined space $\mathcal{X} \times \mathcal{A}$ to the space $\mathcal{X}$ only, with $g(z_j, \alpha)$ evaluated at a given fixed α. Since we do not have to account for growth restrictions on the parameter space, we are able to obtain faster convergence rate δ_{1n} than Ai and Chen (2003).

Let c denote a generic constant that may have different values in different expressions. For any pair $X_1 \in \mathcal{X}$ and $X_2 \in \mathcal{X}$

$$\begin{aligned} & \left|p^{k_n}(X_1)'(P'P)^{-1}P'\varepsilon(\alpha) - p^{k_n}(X_2)'(P'P)^{-1}P'\varepsilon(\alpha)\right| \\ = & \left|\left[p^{k_n}(X_1) - p^{k_n}(X_2)\right]'(P'P)^{-1}P'\varepsilon(\alpha)\right| \end{aligned}$$

Note that

$$\left\|p^{k_n}(X_1)' - p^{k_n}(X_2)'\right\|_E^2 \leq \xi_{1n}^2 \left\|X_1 - X_2\right\|_E^2$$

It follows that

$$\left|\left[p^{k_n}(X_1) - p^{k_n}(X_2)\right]'(P'P)^{-1}P'\varepsilon(\alpha)\right| \leq \xi_{1n}^2 \left\|X_1 - X_2\right\|_E^2 \sqrt{\frac{1}{n\lambda_n}\sum_{i=1}^{n}\varepsilon\left(Z_i, \alpha\right)^2}$$

where λ_n denotes the smallest eigenvalues of $P'P/n$. Condition (i) implies

$$\frac{1}{n}\sum_{i=1}^{n}\varepsilon\left(Z_i, \alpha\right)^2 \leq \frac{c_{1n}^2}{n}\sum_{i=1}^{n}\left(c_1\left(Z_i\right) + E\left[c_1\left(Z_i\right)|X_i\right]\right)^2$$

Assumption 4.3(i) implies $\lambda_n = O_p(1)$. Applying the weak law of large numbers and $E\left\{\left(E\left[c_1\left(Z_i\right)|X_i\right]\right)^2\right\} \leq E\left\{c_1(Z)^2\right\}$, we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\left(c_1\left(Z_i\right) + E\left[c_1\left(Z_i\right)|X_i\right]\right)^2 = O_p(1)$$

Thus there exists a constant c such that

$$\Pr\left(\sqrt{\frac{1}{n\lambda_n}\sum_{i=1}^{n}\left(c_1\left(Z_i\right) + E\left[c_1\left(Z_i\right)|X_i\right]\right)^2} > c\right) < \eta$$

for sufficiently large n.

For any small ϵ partition $\mathcal{X}$ into b_n mutually exclusive subsets $\mathcal{X}_m$, $m = 1, ..., b_n$, where $X_1 \in \mathcal{X}_m$ and $X_2 \in \mathcal{X}_m$ imply $\left\|X_1 - X_2\right\|_E^2 \leq \epsilon\widetilde{\delta}_{1n}/(c_{1n}\xi_{1n}c)$. Then with probability approaching one we have

$$\left|p^{k_n}(X_1)'(P'P)^{-1}P'\varepsilon(\alpha) - p^{k_n}(X_2)'(P'P)^{-1}P'\varepsilon(\alpha)\right| \leq \epsilon\widetilde{\delta}_{1n}$$

Let X_m denote a fixed point in $\mathcal{X}_m$. For any X there exists an m such that $\left\|X_1 - X_2\right\|_E^2 \leq \epsilon\widetilde{\delta}_{1n}/(c_{1n}\xi_{1n}c)$. Then with probability approaching one

$$\sup_{X \in \mathcal{X}}\left|p^{k_n}(X)'(P'P)^{-1}P'\varepsilon(\alpha)\right| \leq \epsilon\widetilde{\delta}_{1n} + \max_m \left|p^{k_n}(X_m)'(P'P)^{-1}P'\varepsilon(\alpha)\right|$$

Hence

$$\begin{aligned} & \Pr\left(\sup_{X \in \mathcal{X}}\left|p^{k_n}(X)'(P'P)^{-1}P'\varepsilon(\alpha)\right| > 2\epsilon\widetilde{\delta}_{1n}\right) \\ < \quad & 2\eta + \Pr\left(\max_m \left|p^{k_n}(X_m)'(P'P)^{-1}P'\varepsilon(\alpha)\right| > 2\epsilon\widetilde{\delta}_{1n}\right) \end{aligned}$$

For some constant c, let

$$M_n = \left(\frac{c\xi_{0n}c_{1n}}{\delta_{1n}\epsilon\eta}\right)^{2/p}$$

Define $d_{in} = \mathbb{I}\{c_1(Z) \le M_n\}$. Define $g_1(Z_i,\alpha) = d_{in}g_1(Z_i,\alpha)$ and $g_2(Z_i,\alpha) = (1-d_{in})\,g_1(Z_i,\alpha)$. Define $\varepsilon_1(Z_i,\alpha)$ and $\varepsilon_2(Z_i,\alpha)$ accordingly. It follows that

$$\begin{aligned}
& \Pr\left(\max_m \left|p^{k_n}(X_m)'(P'P)^{-1}P'\varepsilon(\alpha)\right| > 2\epsilon\widetilde{\delta}_{1n}\right) \\
\le\; & \Pr\left(\max_m \left|p^{k_n}(X_m)'(P'P)^{-1}\sum_{i=1}^{n}\varepsilon_1(Z_i,\alpha)\right| > \epsilon\widetilde{\delta}_{1n}\right) \\
& +\Pr\left(\max_m \left|p^{k_n}(X_m)'(P'P)^{-1}\sum_{i=1}^{n}\varepsilon_2(Z_i,\alpha)\right| > \epsilon\widetilde{\delta}_{1n}\right) \\
\equiv\; & P_1 + P_2
\end{aligned}$$

Ai and Chen (2003) show that $P_2 \le \eta$, along with

$$\sigma_m^2 \equiv nE\left\{\left[p^{k_n}(X_m)'(P'P)^{-1}\sum_{i=1}^{n}p^{k_n}(X_i)\varepsilon_1(Z_i,\alpha)\right]^2\right\} = O(c_{1n}^2\xi_{0n}^2)$$

and

$$\left|p^{k_n}(X_m)'(P'P/n)^{-1}p^{k_n}(X_i)\varepsilon_1(Z_i,\alpha)\right| \le \frac{M_n\xi_{0n}^2c_{1n}}{\lambda_n}$$

Noting that

$$\begin{aligned}
& \Pr\left(\left|p^{k_n}(X_m)'(P'P)^{-1}\sum_{i=1}^{n}\varepsilon_1(Z_i,\alpha)\right| > \epsilon\widetilde{\delta}_{1n}\right) \\
=\; & E\left[\Pr\left(\left|p^{k_n}(X_m)'(P'P)^{-1}\sum_{i=1}^{n}\varepsilon_1(Z_i,\alpha)\right| > \epsilon\widetilde{\delta}_{1n} \mid X_1,...,X_n\right)\right]
\end{aligned}$$

Ai and Chen (2003) apply the Bernstein inequality for independent processes to obtain

$$\begin{aligned}
& \Pr\left(\left|p^{k_n}(X_m)'(P'P)^{-1}\sum_{i=1}^{n}\varepsilon_1(Z_i,\alpha)\right| > \epsilon\delta_{1n}\right) \\
\le\; & 2E\left[\exp\left(-n\varepsilon^2\widetilde{\delta}_{1n}^2/\left(c\sigma_m^2 + M_n\xi_{0n}^2c_{1n}^2\lambda_n^{-1}\varepsilon\widetilde{\delta}_{1n}\right)\right)\right]
\end{aligned}$$

where $E[\cdot]$ is taken with respect to the joint distribution of $(X_1,...,X_n)$. Hence

$$P_1 < 2b_nE\left[\exp\left(-n\varepsilon^2\widetilde{\delta}_{1n}^2/\left(c\sigma_m^2 + M_n\xi_{0n}^2c_{1n}^2\lambda_n^{-1}\varepsilon\widetilde{\delta}_{1n}\right)\right)\right]$$

which is arbitrarily small if

$$\frac{n\widetilde{\delta}_{1n}^2}{\max\left\{\xi_{0n}^2c_{1n}^2, M_n\xi_{0n}^2c_{1n}\widetilde{\delta}_{1n}\right\}} - \ln(b_n) \to \infty$$

Since $\mathcal{X}$ is a compact subset in $\mathbb{R}^d$, we have

$$b_n = O\left(\left(\frac{\widetilde{\delta}_{1n}}{c_{1n}\xi_{1n}}\right)^{-d_x}\right)$$

Substituting for M_n and b_n we obtain

$$\frac{n\widetilde{\delta}_{1n}^2}{\ln(b_n)\max\left\{\xi_{0n}^2 c_{1n}^2, M_n\xi_{0n}^2 c_{1n}\widetilde{\delta}_{1n}\right\}}$$
$$= O\left(\frac{n\widetilde{\delta}_{1n}^2}{\ln\left[\left(\frac{\widetilde{\delta}_{1n}}{c_{1n}\xi_{1n}}\right)^{-d_x}\right]\max\left\{\xi_{0n}^2 c_{1n}^2, \xi_{0n}^{2+2/p}\widetilde{\delta}_{1n}^{1-2/p}c_{1n}^{1+2/p}\right\}}\right)$$

Thus, for $P_1 < \eta$ for sufficiently large n by condition (ii). ■

Lemma A.8 (part of B.1) *Let Assumptions 4.2-4.6 and 4.8 hold. Let also* $n^{1/m}\widetilde{\delta}_{1n} \downarrow 0$ *and* $\rho > 2/m$ *where* $\widetilde{\delta}_{1n}$ *is defined in Lemma A.7 and* ϱ *in Assumption 4.7. Then*

$$\max_{1\le i\le n}\|\lambda_i(\alpha_0)\| = o_p(\widetilde{\delta}_{1n}) + o_p\left(\frac{1}{n^{\varrho-1/m}}\right) \tag{93}$$

This Lemma is analogous to Lemma B.1 of KTA. However, the analysis is somewhat complicated due to the extra term σ_i. Moreover, here we do not make use of results related to kernel estimation. Thus, for example, consistency of the variance-covariance matrix $\Sigma_n(x_i,\alpha_0)$ follows from series results of Ai and Chen (2003).

Proof. In this Lemma, we will use the F.O.C.s (24) and (26) that combine to

$$\begin{aligned}\sum_{j=1}^{n}\frac{w_{ij}}{1+a_i+\lambda_i' g(x_j,\alpha)} &= \sum_{j=1}^{n}\frac{w_{ij}}{\lambda_i' g(x_j,\alpha)+\sigma_i}\\ &= \sum_{j=1}^{n}\widehat{\pi}_{ij}\\ &= 1\end{aligned} \tag{94}$$

Let

$$\lambda_i(\alpha_0) = \rho_i\xi_i \tag{95}$$

where $\rho_i \ge 0$ and $\xi_i \in \mathbb{R}^{d_g}$. It holds that

$$\begin{aligned}\sum_{j=1}^{n} w_{ij}\frac{[a_i+\lambda_i'(\alpha_0)g(z_j,\alpha_0)]^2}{1+a_i+\lambda_i'(\alpha_0)g(z_j,\alpha_0)} &= a_i^2\sum_{j=1}^{n}\frac{w_{ij}}{1+a_i+\lambda_i'(\alpha_0)g(z_j,\alpha_0)} + \frac{2a_i\rho_i\sum_{j=1}^{n} w_{ij}\xi_i' g(z_j,\alpha_0)}{1+a_i+\lambda_i'(\alpha_0)g(z_j,\alpha_0)}\\ &\quad + \frac{\rho_i^2\xi_i'\Sigma_n(x_i,\alpha_0)\xi_i}{1+a_i+\lambda_i'(\alpha_0)g(z_j,\alpha_0)}\end{aligned} \tag{96}$$

For the first term of the RHS sum of (96), using (94), it holds that

$$\begin{aligned}a_i^2\sum_{j=1}^{n}\frac{w_{ij}}{1+a_i+\lambda_i'(\alpha_0)g(z_j,\alpha_0)} &= a_i^2\\ &= (\sigma_i-1)^2\\ &= \sigma_i^2-2\sigma_i+1\end{aligned} \tag{97}$$

Substituting (97) into (96) yields

$$\begin{aligned}\sum_{j=1}^{n} w_{ij}\frac{[a_i+\lambda_i'(\alpha_0)g(z_j,\alpha_0)]^2}{1+a_i+\lambda_i'(\alpha_0)g(z_j,\alpha_0)} &= \sigma_i^2-2\sigma_i+1+\frac{2a_i\rho_i\sum_{j=1}^{n} w_{ij}\xi_i' g(z_j,\alpha_0)}{1+a_i+\lambda_i'(\alpha_0)g(z_j,\alpha_0)}\\ &\quad + \frac{\rho_i^2\xi_i'\Sigma_n(x_i,\alpha_0)\xi_i}{1+a_i+\lambda_i'(\alpha_0)g(z_j,\alpha_0)}\end{aligned} \tag{98}$$

Note that for a generic constant c

$$\begin{aligned}\frac{c^2}{1+c} &= \frac{c^2}{1+c} + (1-c) - (1-c) \\ &= \frac{c^2}{1+c} + \frac{(1-c)(1+c)}{1+c} - (1-c) \\ &= \frac{c^2}{1+c} + \frac{1-c^2}{1+c} - (1-c) \\ &= \frac{1}{1+c} - 1 + c\end{aligned}$$

Using this fact, letting $c = a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0)$, we have

$$\begin{aligned}\sum_{j=1}^{n} w_{ij} \frac{[a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0)]^2}{1 + a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0)} &= \sum_{j=1}^{n} w_{ij} \left\{ \frac{1}{1 + a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0)} - 1 + a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0) \right\} \\ &= \sum_{j=1}^{n} \frac{w_{ij}}{1 + a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0)} - \sum_{j=1}^{n} w_{ij} + \sum_{j=1}^{n} w_{ij} a_i \\ &\quad + \sum_{j=1}^{n} w_{ij} \lambda_i'(\alpha_0) g(z_j, \alpha_0) \\ &= 1 - \sum_{j=1}^{n} w_{ij} + \sum_{j=1}^{n} w_{ij} a_i + \sum_{j=1}^{n} w_{ij} \lambda_i'(\alpha_0) g(z_j, \alpha_0) \qquad (99)\end{aligned}$$

By the definition of σ_i,

$$\begin{aligned}1 - \sum_{j=1}^{n} w_{ij} + a_i \sum_{j=1}^{n} w_{ij} &= 1 - \sigma_i + (\sigma_i - 1)\sigma_i \\ &= \sigma_i^2 - 2\sigma_i + 1 \qquad (100)\end{aligned}$$

Substituting (100) into (99) gives us

$$\sum_{j=1}^{n} w_{ij} \frac{[a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0)]^2}{1 + a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0)} = \sigma_i^2 - 2\sigma_i + 1 + \rho_i \sum_{j=1}^{n} w_{ij} \xi_i' g(z_j, \alpha_0) \qquad (101)$$

Combining (98) and (101) yields, after canceling $\sigma_i^2 - 2\sigma_i + 1$ from both sides,

$$\frac{2a_i \rho_i \sum_{j=1}^{n} w_{ij} \xi_i' g(z_j, \alpha_0)}{1 + a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0)} + \frac{\rho_i^2 \xi_i' \Sigma_n(x_i, \alpha_0) \xi_i}{1 + a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0)} = \rho_i \sum_{j=1}^{n} w_{ij} \xi_i' g(z_j, \alpha_0) \qquad (102)$$

Using Assumption 4.8, by Lemma D.2 in KTA,

$$\max_{1 \le j \le n} \|g(z_j, \alpha_0)\| = o_p(n^{1/m}) \qquad (103)$$

and this $o_p(n^{1/m})$ term does not depend on i, j, or $\alpha_n \in \mathcal{A}_n$. By (103) it holds that

$$0 \le 1 + a_i + \lambda_i'(\alpha_0) g(z_j, \alpha_0) \le 1 + a_i + \rho_i \|g(z_j, \alpha_0)\| = 1 + a_i + \rho_i o_p(n^{1/m}) \qquad (104)$$

Using (104) in (102) and canceling ρ_i yields

$$\frac{2a_i \sum_{j=1}^{n} w_{ij} \xi_i' g(z_j, \alpha_0)}{1 + a_i + \rho_i o_p(n^{1/m})} + \frac{\rho_i \xi_i' \Sigma_n(x_i, \alpha_0) \xi_i}{1 + a_i + \rho_i o_p(n^{1/m})} \le \sum_{j=1}^{n} w_{ij} \xi_i' g(z_j, \alpha_0) \qquad (105)$$

By Corollary D.1 of Ai and Chen (2003), $\Sigma_n(x_i, \alpha_0) = \Sigma(x_i, \alpha_0) + o_p(1)$ uniformly over $X \in \mathcal{X}$. Using the fact that $\xi_i' \Sigma(x_i, \alpha_0) \xi_i$ is bounded away from zero on $(x_i, \xi_i) \in \mathbb{R}^{d_X} \times \mathbb{R}^{d_g}$, we can divide (105) by

$\frac{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i}{1+a_i+\rho_i o_p(n^{1/m})}$ and rearrange terms to obtain

$$\begin{aligned}
\rho_i &\leq \left[1 + a_i + \rho_i o_p(n^{1/m})\right] \frac{\sum_{j=1}^n w_{ij}\xi_i' g\left(z_j, \alpha_0\right)}{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i} - 2a_i \frac{\sum_{j=1}^n w_{ij}\xi_i' g\left(z_j, \alpha_0\right)}{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i} \\
&= (1-a_i) \frac{\sum_{j=1}^n w_{ij}\xi_i' g\left(z_j, \alpha_0\right)}{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i} + \rho_i o_p(n^{1/m}) \frac{\sum_{j=1}^n w_{ij}\xi_i' g\left(z_j, \alpha_0\right)}{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i}
\end{aligned}$$

and hence

$$\begin{aligned}
\rho_i \left(1 - o_p(n^{1/m}) \frac{\sum_{j=1}^n w_{ij}\xi_i' g\left(z_j, \alpha_0\right)}{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i}\right) &\leq (1-a_i) \frac{\sum_{j=1}^n w_{ij}\xi_i' g\left(z_j, \alpha_0\right)}{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i} \\
\rho_i &\leq (1-a_i) \frac{\sum_{j=1}^n w_{ij}\xi_i' g\left(z_j, \alpha_0\right)}{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i} \\
&\quad \times \left(1 - o_p(n^{1/m}) \frac{\sum_{j=1}^n w_{ij}\xi_i' g\left(z_j, \alpha_0\right)}{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i}\right)^{-1} \qquad (106)
\end{aligned}$$

For the last term of the RHS of (106), using Lemma A.1 and $\|\xi_i'\| < \infty$ for all i, it holds that

$$\begin{aligned}
o_p(n^{1/m}) \frac{\sum_{j=1}^n w_{ij}\xi_i' g\left(z_j, \alpha_0\right)}{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i} &= o_p(n^{1/m}) \left\|\xi_i'\right\| \max_{1\leq i\leq n} \left\|\sum_{j=1}^n w_{ij} g\left(z_j, \alpha_0\right)\right\| \\
&= o_p(n^{1/m}) O(1) \left[o_p(\widetilde{\delta}_{1n}) + o_p\left(\frac{1}{n^{\varrho-1/m}}\right)\right] \\
&= o_p(n^{1/m}\widetilde{\delta}_{1n}) + o_p\left(\frac{1}{n^{\varrho-2/m}}\right) \qquad (107)
\end{aligned}$$

while for the first term of the RHS of (106), using also Lemma A.5,

$$\begin{aligned}
(1-a_i) \frac{\sum_{j=1}^n w_{ij}\xi_i' g\left(z_j, \alpha_0\right)}{\xi_i' \Sigma_n(x_i,\alpha_0)\xi_i} &= O(1) \left\|\xi_i'\right\| \max_{1\leq i\leq n} \left\|\sum_{j=1}^n w_{ij} g\left(z_j, \alpha_0\right)\right\| \\
&= O(1)O(1) \left[o_p(\widetilde{\delta}_{1n}) + o_p\left(\frac{1}{n^{\varrho-1/m}}\right)\right] \\
&= o_p(\widetilde{\delta}_{1n}) + o_p\left(\frac{1}{n^{\varrho-1/m}}\right) \qquad (108)
\end{aligned}$$

Under our assumptions, $n^{1/m}\widetilde{\delta}_{1n} \downarrow 0$ and $n^{-\varrho+2/m} \downarrow 0$ in (107). This used in (106) along with (108) and consistency of $\Sigma_n(x_i,\alpha_0)$, implies that

$$\max_{1\leq i\leq n} \|\rho_i\| = o_p(\widetilde{\delta}_{1n}) + o_p\left(\frac{1}{n^{\varrho-1/m}}\right)$$

which yields the desired result by the definition of ρ_i in (95). ∎

B: CONVERGENCE RATES

Lemma B.1 *Consider the functions* $G_n(\alpha_n)$ *and* $\overline{G}_n(\alpha_n)$ *defined in (64) and (87), respectively. Assumptions 4.1-4.3, 4.5, 4.6, 5.1-5.6 imply: (i)* $G_n(\alpha_n) - \overline{G}_n(\alpha_n) = o_p(n^{-1/4})$ *uniformly over* $\alpha_n \in \mathcal{A}_n$*; and (ii)* $G_n(\alpha_n) - G_n(\alpha_0) - \{\overline{G}_n(\alpha_n) - \overline{G}_n(\alpha_0)\} = o_p(\eta_n n^{-1/4})$ *uniformly over* $\alpha_n \in \mathcal{A}_n$ *with* $\|\alpha_n - \alpha_0\|_F \leq o(\eta_n)$*, where* $\eta_n = n^{-\tau}$ *with* $\tau \leq 1/4$.

Proof.

This Lemma shows the counterpart of AC's Corollary B.1 for our case. Since $\lambda_i(\alpha_n)$ solves

$$\sum_{j=1}^{n} \frac{w_{ij} g\left(z_j, \alpha_n\right)}{\sigma_i + \lambda_i' g\left(z_j, \alpha_n\right)} = 0 \tag{109}$$

denote by $\lambda_{i0}(\alpha_n)$ the solution to

$$E\left[\left.\frac{g\left(z_j, \alpha_n\right)}{\sigma_i + \lambda_i' g\left(z_j, \alpha_n\right)}\right| x_i\right] = 0$$

For the sake of brevity, let "VW" stand for "Van der Vaart and Wellner (1996)." Lemma A.5 and Assumption 4.5(i) suffice to satisfy the pointwise convergence condition of Lemma 3.3.5 (p. 311) in VW for the objective function (109). Note that $\{g(z, \alpha_n) : \alpha_n \in \mathcal{A}_n\} \subset \Lambda_c^{\overline{\gamma}}(\mathcal{X})$ and $\Lambda_c^{\overline{\gamma}}(\mathcal{X})$ is a Donsker class by Theorem 2.5.6 in VW. Since $\lambda_i(\alpha_n) \in R^{d_g}$, $\{\lambda_i(\alpha_n) : \alpha_n \in \mathcal{A}_n\}$ belongs to the Donsker class. By Example 2.10.8 (p. 192) in VW $\{\lambda_i' g(z, \alpha_n) : \alpha_n \in \mathcal{A}_n\}$ is Donsker. Since $0 < \sigma_i < \infty$ is a data-determined scalar by Lemma A.5, by Example 2.10.9 (p. 192) in VW (109) is Donsker in $\alpha_n \in \mathcal{A}_n$. Hence the Assumptions of Lemma 3.3.5 (p. 311) in VW are satisfied and we can invoke Theorem 3.3.1 (p. 310) in VW to conclude that $\|\lambda_i(\alpha_n) - \lambda_{i0}(\alpha_n)\|_E = O_p(n^{-1/2})$, uniformly over $\alpha_n \in \mathcal{A}_n$, for each i. Lemma A.1(A) of Ai and Chen (2003) (defining δ_{1n}) states that $\sum_{j=1}^{n} w_{ij} g(z_j, \alpha_n) - m(x_i, \alpha_n) = o_p(\delta_{1n})$ uniformly over $\mathcal{X} \times \mathcal{A}_n$. These two rate results for $\lambda_i(\alpha_n)$ and $g(z_j, \alpha_n)$, simple law of large numbers for σ_i and continuity of the log function satisfy the satisfy the pointwise convergence condition of Lemma 3.3.5 (p. 311) in VW for the objective function $G_n(\alpha_n)$. By Theorem 2.10.6 (p. 192) in VW $\{\ln[\sigma_i + \lambda_i' g(z_j, \alpha_n)] : \alpha_n \in \mathcal{A}_n\}$ is Donsker. By Lemma A.5, $0 < \sigma_i < \infty$ for each i and thus we can renormalize σ_i by dividing by $\sup_{1 \leq i \leq n} \sigma_i$ that guarantees $\sum_{i=1}^{n} \sigma_i < 1$. By Theorem 2.10.3 (p. 190) in VW

$$\begin{aligned}
\left|G_n(\alpha_n) - \overline{G}_n(\alpha_n)\right| &= \left|\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \ln\left\{\sigma_i + \lambda_i' g\left(z_j, \alpha_n\right)\right\} - \frac{1}{n}\sum_{i=1}^{n} E\left[\ln\left\{\sigma_i + \lambda_{i0}' g\left(z, \alpha_n\right)\right\} | x_i\right]\right| \\
&= O_p(n^{-1/2})
\end{aligned}$$

uniformly over $\alpha_n \in \mathcal{A}_n$, which shows the result (i) in this Lemma.

In order to show part (ii) of the proof, we first derive the counterpart of AC's Corollary A.2 that is a building block for their Corollary B.1 (ii). Note that since $m(X, \alpha_0) = 0$, $\|\alpha_n - \alpha_0\|_F = o_p(1)$ and AC's result $(i.1)$ of the proof of their Corollary A.2 holds also for our $\|m(X, \alpha)\|_E^2$, we only need to show the counterpart of their part $(i.2)$. We replace Assumption 3.9 of AC by our Assumption 5.1 which applies to our metric $\|\cdot\|_F$. This Assumption together with Lemma C.1 imply that $E\{\|m(X, \alpha)\|_E^2\}$ and $\|\alpha - \alpha_0\|_F^2$ are (topologically) equivalent. Then by Assumptions 4.1, 5.1, and 5.3(i), we have

$$E\left\{\left[\|m(X, \alpha)\|_E^2\right]^2\right\} \leq E\left\{\|m(X, \alpha)\|_E^2\right\} \times \left[\sup_{X, \alpha}\left\{\|m(X, \alpha)\|_E\right\}\right]^2 \leq const. \times \|\alpha_n - \alpha_0\|_F^2$$

satisfying part $(i.2)$. Part (ii) of AC's Corollary A.2 holds for our metric $\|\cdot\|_F$ by replacing their Assumption 3.9 with our Assumption 5.1. This, along with AC's Corollary A.1 shows (ii). ■

Lemma C.1 *Under Assumptions 4.1-5.6,*

$$
\begin{aligned}
& E\left[Var\left(\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{dg\left(Z,\alpha\right)}D_{w^*}(Z)\right| X\right)\right] \\
= & E\left\{E\left[\left. D_w(Z)'W_0(Z,X)^{-1}D_w(Z)\right| X\right]\right\} \\
= & E\left\{E\left[\left. D_w(Z)'\frac{d\varphi\left(X,Z,\alpha_0\right)}{dg(Z,\alpha)}\left(\frac{d\varphi\left(X,Z,\alpha_0\right)}{dg(Z,\alpha)}\right)' D_w(Z)\right| X\right]\right\}
\end{aligned}
$$

Proof. Using (53) and (51)

$$
\begin{aligned}
E\left[\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{dg\left(Z,\alpha\right)}D_{w^*}(Z)\right| X\right] &= E\left[\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{dg\left(Z,\alpha\right)}\frac{dg(Z,\alpha_0)}{d\alpha}\left[v^*\right]\right| X\right] \\
&= E\left[\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\alpha}\left[v^*\right]\right| X\right] \\
&= E\left[\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\theta'}\left(u_\theta^*-\theta_0\right)+\frac{d\varphi\left(X,Z,\alpha_0\right)}{dh}\left[u_h^*-h_0\right]\right| X\right] \\
&= E\left[\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{d\theta'}\right| X\right]\left(u_\theta^*-\theta_0\right)+E\left[\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{dh}\left[u_h^*-h_0\right]\right| X\right] \\
&= 0
\end{aligned}
$$

by the definition of α_0. Hence

$$
Var\left(\left.\frac{d\varphi\left(X,Z,\alpha_0\right)}{dg\left(Z,\alpha\right)}D_{w^*}(Z)\right| X\right) = E\left[\left. D_{w^*}(Z)'\frac{d\varphi\left(X,Z,\alpha_0\right)}{dg(Z,\alpha)}\left(\frac{d\varphi\left(X,Z,\alpha_0\right)}{dg(Z,\alpha)}\right)' D_{w^*}(Z)\right| X\right]
$$

Taking expectation over X yields the required result. ■

Lemma C.2 *Consider the notation for $v_n(\cdot)$ and $\widetilde{r}[\cdot]$ defined in Appendix 3. Then, under Assumptions 4.1-5.6,*

$$
n^{-1/2}v_n\left(\widetilde{r}[\alpha_n-\alpha_0,X,Y]-\widetilde{r}[P_n\alpha^*(a_n,\varepsilon_n)-\alpha_0,X,Y]\right)=o_p(n^{-1/4})
$$

Proof. This Lemma performs a similar function as Lemmas C.1 - C.3 in Ai and Chen (2003). By the definition of $v_n(\cdot)$ and $\widetilde{r}[\cdot]$,

$$
\begin{aligned}
& n^{-1/2}v_n\left(\widetilde{r}[\alpha_n-\alpha_0,X,Y]-\widetilde{r}[P_n\alpha^*(a_n,\varepsilon_n)-\alpha_0,X,Y]\right) \\
= & n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\begin{pmatrix} w_{ij}\left\{\widetilde{r}[\alpha_n-\alpha_0,x_i,y_j]-\widetilde{r}[P_n\alpha^*(a_n,\varepsilon_n)-\alpha_0,x_i,y_j]\right\} \\ -E\left\{\widetilde{r}[\alpha_n-\alpha_0,X,Y]-\widetilde{r}[P_n\alpha^*(a_n,\varepsilon_n)-\alpha_0,X,Y]\right\}\end{pmatrix} \\
= & A_1-A_2
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}w_{ij}\widetilde{r}[\alpha_n-\alpha_0,x_i,y_j]-E\widetilde{r}[\alpha_n-\alpha_0,X,Y] \\
A_2 &= n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}w_{ij}\widetilde{r}[\alpha_n+\varepsilon_n u_n^*-\alpha_0,x_i,y_j]-E\widetilde{r}[\alpha_n+\varepsilon_n u_n^*-\alpha_0,X,Y]
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= A_{11}-A_{12} \\
A_{11} &= n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}w_{ij}\varphi\left(x_i,z_j,\alpha\right)-E\varphi\left(z,x,\alpha\right) \\
A_{12} &= n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}w_{ij}\frac{d\varphi\left(x_i,z_j,\alpha_0\right)}{d\alpha}[\alpha-\alpha_0]-E\left\{\frac{d\varphi\left(x,z,\alpha_0\right)}{d\alpha}[\alpha-\alpha_0]\right\}
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= A_{21} - A_{22} \\
A_{21} &= n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\varphi\left(x, z, \alpha_n + \varepsilon_n u_n^*\right) - E\varphi\left(x, z, \alpha_n + \varepsilon_n u_n^*\right) \\
A_{22} &= n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\frac{d\varphi\left(x_i, z_j, \alpha_0\right)}{d\alpha}[\alpha_n + \varepsilon_n u_n^* - \alpha_0] - E\left\{\frac{d\varphi\left(x, z, \alpha_0\right)}{d\alpha}[\alpha_n + \varepsilon_n u_n^* - \alpha_0]\right\}
\end{aligned}
$$

The goal is to show $A_{11} - A_{12} - A_{21} + A_{22} = O_p(\varepsilon_n^2) = o_p(n^{-1/4})$. Note that $A_{11} = o_p(n^{-1/4})$ and $A_{21} = o_p(n^{-1/4})$ follows from parts A and B of AC's Lemma A.1. $A_{12} = o_p(n^{-1/4})$ and $A_{22} = o_p(n^{-1/4})$ follows from the rate results for A_{11} and A_{21}, respectively, and the continuous mapping theorem. ■

Appendix 3

In this Appendix we extend Theorem 1 of Shen (1997) to our conditional case.[13] Consider the setup as in Shen (1997), with the following modifications. Suppose that the observations $\{(X_i, Y_j) : i, j = 1, ..., n\}$ are drawn independently distributed according to density $p(\alpha_0, X_i, Y_j)$.

Define

$$K(\alpha_0, \alpha) = E_0 l(\alpha_0, X_i, Y_j) - E_0 l(\alpha, X_i, Y_j)$$

Let the empirical criterion be

$$L_n(\alpha) = n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} l(\alpha, X_i, Y_j)$$

where $l(\alpha, Y_j, X_i)$ is the criterion based on a single observation. Consider $l(\alpha, x, y)$ for which (*analog of Shen's (4.1)*)

$$\widetilde{r}[\alpha - \alpha_0, x, y] = l(\alpha, x, y) - l(\alpha_0, x, y) - l'_{\alpha_0}[\alpha - \alpha_0, x, y] \tag{S 4.1}$$

where $l'_{\alpha_0}[\alpha - \alpha_0, x, y]$ is defined as $\lim_{t \to 0}[l(a + t[\alpha - \alpha_0], x, y) - l(\alpha_0, x, y)]/t$. Denote $\widehat{\alpha}_n$ the maximizer of $L_n(\alpha_n)$ over $\alpha_n \in \mathcal{A}_n$. We estimate a real functional of $\widehat{\alpha}_n$ denoted as $f(\alpha)$. With $\widehat{\alpha}_n$ as defined, $f(\alpha)$ is estimated by a substitution estimate $f(\widehat{\alpha}_n)$. By the definition of $\widehat{\alpha}_n$, we have (*analog of Shen's (2.1)*)

$$L_n(\widehat{\alpha}_n) \geq \sup_{\alpha \in \mathcal{A}_n} L_n(\alpha_n) - O(\varepsilon_n^2) \tag{S 2.1}$$

where $\varepsilon_n^2 \to 0$ as $n \to \infty$. For any generic function $g(X, Y)$ let

$$\nu_n(g) = n^{-1} \sum_{i=1}^{n} n^{1/2} \left\{ \sum_{j=1}^{n} w_{ij} g(X_i, Y_j) - E\left[g(X, Y) | \, X = x_i \right] \right\}$$

be the empirical process induced by g. Let the convergence rate of the sieve estimate under $\|\cdot\|$ be $o_p(\delta_n)$ and let $\varepsilon_n^2 = o_p(n^{-1/2})$.

The following conditions are modified versions of Shen's 1997 (p. 2568) conditions:

Condition A (Stochastic Equicontinuity) *For $\widetilde{r}[\alpha - \alpha_0, x, y]$ defined in (S 4.1),*

$$\sup_{\{\alpha_n \in \mathcal{A}_n : \|\alpha_n - \alpha_0\| \leq \delta_n\}} n^{-1/2} \nu_n \left(\widetilde{r}[\alpha_n - \alpha_0, X, Y] - \widetilde{r}[\alpha_n + \varepsilon_n u_n^* - \alpha_0, X, Y] \right) = O_p(\varepsilon_n^2)$$

Condition B (Expectation of Criterion Difference)

$$\sup_{\{\alpha_n \in \mathcal{A}_n : \|\alpha_n - \alpha_0\| \leq \delta_n\}} \left[K\left(\alpha_0, \alpha_n + \varepsilon_n u_n^*\right) - K\left(\alpha_0, \alpha_n\right) \right] - \frac{1}{2} \left[\|\alpha_n + \varepsilon_n u^* - \alpha_0\|^2 - \|\alpha_n - \alpha_0\|^2 \right] = O_p(\varepsilon_n^2)$$

Condition C (Approximation Error)

$$\sup_{\{\alpha_n \in \mathcal{A}_n : \|\alpha_n - \alpha_0\| \leq \delta_n\}} \|\varepsilon_n u^* - \varepsilon_n u_n^*\| = O_p(\delta_n^{-1} \varepsilon_n^2)$$

In addition,

$$\sup_{\{\alpha_n \in \mathcal{A}_n : \|\alpha_n - \alpha_0\| \leq \delta_n\}} n^{-1/2} \nu_n \left(l'_{\alpha_0}[\varepsilon_n u^* - \varepsilon_n u_n^*, X, Y] \right) = O_p(\varepsilon_n^2)$$

Condition D (Gradient)

$$\sup_{\{\alpha_n \in \mathcal{A}_n : \|\alpha_n - \alpha_0\| \leq \delta_n\}} n^{-1/2} \nu_n \left(l'_{\alpha_0}[\alpha_n - \alpha_0, X, Y] \right) = O_p(\varepsilon_n)$$

Condition E (Smoothness)

Suppose the functional f has the following smoothness property: for any $\alpha_n \in \mathcal{A}_n$

$$\left| f_{\alpha_n} - f_{\alpha_0} - f'_{\alpha_0}[\alpha_n - \alpha_0] \right| \leq u_n \|\alpha_n - \alpha_0\|_F^{\omega} \tag{S 4.2}$$

[13] Measurability with respect to the underlying probability space is assumed throughout the paper and hence we do not distiguish outer expectation from the usual one.

as $\|\alpha_n - \alpha_0\|_F \to 0$ *where* ω *is the degree of smoothness of* $f'_{\alpha_0}[\alpha_n - \alpha_0]$ *at* α_0.

Condition F (Convergence Rates and Smoothness) $u_n \delta_n^{\omega} = O_p(n^{-1/2})$.

Condition G (Variance) $Var\left(l'_{\alpha_0}[v^*, X, Y]\right) < \infty$ *is positive definite for all* $X \in \mathcal{X}$, $y \in \mathcal{Y}$.

Theorem 7.1 *Let the Conditions A-G hold. Then for the approximate substitution sieve estimate defined in (S 2.1),*

$$n^{-1/2}(f(\widehat{\alpha}_n) - f(\alpha_0)) \xrightarrow{d} N(0, E\left[Var\left(l'_{\alpha_0}[v^*, Y]\right)|X\right])$$

Proof of Theorem 7.1. Rearrange (S 4.1) as

$$l(\alpha, x, y) = \widetilde{r}[\alpha - \alpha_0, x, y] + l(\alpha_0, x, y) + l'_{\alpha_0}[\alpha - \alpha_0, x, y]$$

Subtract from (S 4.1) its expectation (under $P(\theta_0, X_i, Y_j)$ denoted by E_0), for a given (X_i, Y_j) to obtain

$$\begin{aligned} l(\alpha, x_i, y_j) - E_0 l(\alpha, x_i, y_j) &= l(\alpha, x_i, y_j) - E_0 l(\alpha, x_i, y_j) \\ &+ l'_{\alpha_0}[\alpha - \alpha_0, x_i, y_j] - E_0 l'_{\alpha_0}[\alpha - \alpha_0, x_i, y_j] \\ &+ \widetilde{r}[\alpha - \alpha_0, x_i, y_j] - E_0 \widetilde{r}[\alpha - \alpha_0, x_i, y_j] \end{aligned}$$

rearrange

$$\begin{aligned} l(\alpha, x_i, y_j) &= l(\alpha, x_i, y_j) - [E_0 l(\alpha, x_i, y_j) - E_0 l(\alpha, x_i, y_j)] \\ &+ l'_{\alpha_0}[\alpha - \alpha_0, x_i, y_j] - E_0 l'_{\alpha_0}[\alpha - \alpha_0, x_i, y_j] \\ &+ \widetilde{r}[\alpha - \alpha_0, x_i, y_j] - E_0 \widetilde{r}[\alpha - \alpha_0, x_i, y_j] \end{aligned}$$

take a weighted average over i, j with weights w_{ij}

$$\begin{aligned} n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} l(\alpha, x_i, y_j) &= n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} l(\alpha_0, x_i, y_j) \\ &- n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left[E_0 l(\alpha_0, x_i, y_j) - E_0 l(\alpha, x_i, y_j)\right] \\ &+ n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left(l'_{\alpha_0}[\alpha - \alpha_0, x_i, y_j] - E_0 l'_{\alpha_0}[\alpha - \alpha_0, x_i, y_j]\right) \\ &+ n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left(\widetilde{r}[\alpha - \alpha_0, x_i, y_j] - E_0 \widetilde{r}[\alpha - \alpha_0, x_i, y_j]\right) \end{aligned}$$

and hence using the notation above, for any $P_n\alpha_n \in \{P_n\alpha_n \in \mathcal{A}_n : \|P_n\alpha_n - \alpha_0\| \leq \delta_n\}$, we have

$$\begin{aligned} L_n(P_n\alpha_n) &= L_n(a_0) - K(\alpha_0, P_n\alpha_n) \\ &+ n^{-1/2}\nu_n(l'_{\theta_0}[P_n\alpha_n - \alpha_0, X, Y]) \\ &+ n^{-1/2}\nu_n(r[P_n\alpha_n - \alpha_0, X, Y]) \end{aligned} \quad \text{(S 9.1)}$$

Substituting $P_n\alpha_n$ by $\widehat{\alpha}_n$ here above, we obtain

$$\begin{aligned} L_n(\widehat{\alpha}_n) &= L_n(a_0) - K(\alpha_0, \widehat{\alpha}_n) \\ &+ n^{-1/2}\nu_n(l'_{\theta_0}[\widehat{\alpha}_n - \alpha_0, X, Y]) \\ &+ n^{-1/2}\nu_n(r[\widehat{\alpha}_n - \alpha_0, X, Y]) \end{aligned} \quad \text{(S 9.2)}$$

Subtracting (S 9.2) from (S 9.1) and substituting α_n by $\alpha^*(\widehat{\alpha}_n, \varepsilon_n)$ in (S 9.1), we have

$$\begin{aligned} & L_n(P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n)) - L_n(\widehat{\alpha}_n) \\ = \; & L_n(\alpha_0) - L_n(\alpha_0) \\ & -K(\theta_0, P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n) + K(\alpha_0, \widehat{\alpha}_n) \\ & +n^{-1/2}\nu_n(l'_{\alpha_0}[P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n) - \alpha_0, X, Y]) - n^{-1/2}\nu_n(l'_{\alpha_0}[\widehat{\alpha}_n - \alpha_0, X, Y]) \\ & +n^{-1/2}\nu_n(r[P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n) - \alpha_0, X, Y]) - n^{-1/2}\nu_n(r[\widehat{\alpha}_n - \alpha_0, X, Y]) \end{aligned}$$

which yields

$$\begin{aligned} L_n(\widehat{\alpha}_n) \;=\; & L_n(P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n)) \\ & - [K(\alpha_0, \widehat{\alpha}_n) - K(\theta_0, P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n)] \\ & +n^{-1/2}\nu_n(l'_{\alpha_0}[\widehat{\alpha}_n - P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n), X, Y]) \\ & +n^{-1/2}\nu_n(r[\widehat{\alpha}_n - P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n), X, Y]) \end{aligned}$$

By Condition A (second line of the following)

$$\begin{aligned} & n^{-1/2}\nu_n(r[P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n) - \alpha_0, X, Y]) - n^{-1/2}\nu_n(r[\widehat{\alpha}_n - \alpha_0, X, Y]) \\ = \; & n^{-1/2}\nu_n(r[\widehat{\alpha}_n - P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n), X, Y]) \\ = \; & O_p(\varepsilon_n^2) \end{aligned}$$

Using Condition B on the difference in Ks, we obtain

$$\begin{aligned} L_n(\widehat{\alpha}_n) \;=\; & L_n(P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n)) - \frac{1}{2}\left[\|\widehat{\alpha}_n - \alpha_0\|^2 - \|P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n) - \alpha_0\|^2\right] \\ & +n^{-1/2}\nu_n(l'_{\alpha_0}[\widehat{\alpha}_n - P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n), X, Y]) \\ & +O_p(\varepsilon_n^2) \end{aligned}$$

By Condition C (applicable to the second line)

$$\|P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n) - \alpha^*(\widehat{\alpha}_n, \varepsilon_n)\| = O(\delta_n^{-1}\varepsilon_n^2)$$

Hence, using (S 2.1) we have

$$\begin{aligned} -O(\varepsilon_n^2) \;\leq\; & -\frac{1}{2}\left[\|\widehat{\alpha}_n - \alpha_0\|^2 - \|P_n\alpha^*(\widehat{\alpha}_n, \varepsilon_n) - \alpha_0\|^2\right] \\ & +n^{-1/2}\nu_n(l'_{\alpha_0}[\widehat{\alpha}_n - \alpha^*(\widehat{\alpha}_n, \varepsilon_n), X, Y]) \\ & +O_p(\varepsilon_n^2) \end{aligned} \qquad \text{(S 9.3)}$$

We will use the relation

$$\begin{aligned} \widehat{\alpha}_n - \alpha^*(\widehat{\alpha}_n, \varepsilon_n) \;=\; & \widehat{\alpha}_n - \widehat{\alpha}_n + \varepsilon_n\widehat{\alpha}_n - \varepsilon_n u^* - \varepsilon_n\alpha_0 \\ =\; & -\varepsilon_n\left(u^* - (\widehat{\alpha}_n - \alpha_0)\right) \end{aligned}$$

in $\nu_n(l'_{\alpha_0}[\widehat{\alpha}_n - \alpha^*(\widehat{\alpha}_n, \varepsilon_n), X, Y])$ to get $-\nu_n(l'_{\alpha_0}[\varepsilon_n\left(u^* - (\widehat{\alpha}_n - \alpha_0)\right), X, Y])$.

In (S 9.3) we have

$$
\begin{aligned}
\left\| P_n a^*(\widehat{a}_n, \varepsilon_n) - a_0 \right\|^2 &= \left\| P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \alpha^*(\widehat{a}_n, \varepsilon_n) + \alpha^*(\widehat{a}_n, \varepsilon_n) - \theta_0 \right\|^2 \\
&= \left\| P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \alpha^*(\widehat{a}_n, \varepsilon_n) + (1-\varepsilon_n)(\widehat{\alpha}_n - \alpha_0) + \varepsilon_n u^* \right\|^2 \\
&\leq \left\| (1-\varepsilon_n)(\widehat{\alpha}_n - \alpha_0) \right\| \left\| P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \alpha^*(\widehat{a}_n, \varepsilon_n) + \varepsilon_n u^* \right\| \\
&\leq \left\| (1-\varepsilon_n)(\widehat{\alpha}_n - \alpha_0) \right\| \left\| P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \alpha^*(\widehat{a}_n, \varepsilon_n) \right\| \\
&\quad + \left\| (1-\varepsilon_n)(\widehat{\alpha}_n - \alpha_0) \right\| \left\| \varepsilon_n u^* \right\| \\
&= (1-\varepsilon_n) \left\| (\widehat{\alpha}_n - \alpha_0) \right\| \left\| P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \alpha^*(\widehat{a}_n, \varepsilon_n) \right\| \\
&\quad + (1-\varepsilon_n) \left\langle \widehat{\alpha}_n - \alpha_0, \varepsilon_n u^* \right\rangle
\end{aligned}
$$

We multiply $\|\widehat{a}_n - \alpha_0\|$ by the factor

$$
\begin{aligned}
1 - (1-\varepsilon_n)^2 &= 1 - (1 - 2\varepsilon_n + \varepsilon_n^2) \\
&= 2\varepsilon_n - \varepsilon_n^2
\end{aligned}
$$

which is a positive fraction that preserves the inequality. We also multiply $\|P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \theta_0\|^2$ by 2 which also preserves the inequality. Hence we obtain

$$
\begin{aligned}
-O(\varepsilon_n^2) \leq\ & -\frac{1}{2}\left[1 - (1-\varepsilon_n)^2\right] \|\widehat{\alpha}_n - \alpha_0\|^2 \\
& + (1-\varepsilon_n) \left\| (\widehat{\alpha}_n - \alpha_0) \right\| \left\| P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \alpha^*(\widehat{a}_n, \varepsilon_n) \right\| \\
& + (1-\varepsilon_n) \left\langle \widehat{\alpha}_n - \alpha_0, \varepsilon_n u^* \right\rangle \\
& - n^{-1/2} \nu_n (l'_{\alpha_0}[\varepsilon_n (u^* - (\widehat{\alpha}_n - \alpha_0)), X, Y]) \\
& + O_p(\varepsilon_n^2)
\end{aligned}
$$

Adding $\varepsilon_n \left\| (\widehat{\alpha}_n - \alpha_0) \right\| \left\| P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \alpha^*(\widehat{a}_n, \varepsilon_n) \right\|$ still preserves the inequality. For the first line, $\varepsilon_n^2 \|\widehat{\alpha}_n - \alpha_0\|^2 = O_p(\varepsilon_n^2)$. Hence

$$
\begin{aligned}
-O(\varepsilon_n^2) \leq\ & -\varepsilon_n \|\widehat{\alpha}_n - \alpha_0\|^2 + \left\| (\widehat{\alpha}_n - \alpha_0) \right\| \left\| P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \alpha^*(\widehat{a}_n, \varepsilon_n) \right\| \\
& + (1-\varepsilon_n) \left\langle \widehat{\alpha}_n - \alpha_0, \varepsilon_n u^* \right\rangle - n^{-1/2} \nu_n (l'_{\alpha_0}[\varepsilon_n (u^* - (\widehat{\alpha}_n - \alpha_0)), X, Y]) + O_p(\varepsilon_n^2)
\end{aligned}
$$

Note that

$$
\begin{aligned}
-\varepsilon_n \|\widehat{\alpha}_n - \alpha_0\|^2 &= O_p(\varepsilon_n) o_p(\delta^2) \\
&= o_p(\delta^2)
\end{aligned}
$$

By Condition C

$$
\left\| P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \alpha^*(\widehat{a}_n, \varepsilon_n) \right\| = O_p(\delta^{-1} \varepsilon_n^2)
$$

since

$$
\|\widehat{\alpha}_n - \alpha_0\| = o_p(\delta)
$$

then

$$
\begin{aligned}
\|\widehat{\alpha}_n - \alpha_0\| \left\| P_n \alpha^*(\widehat{a}_n, \varepsilon_n) - \alpha^*(\widehat{a}_n, \varepsilon_n) \right\| &= o_p(\delta) O_p(\delta^{-1} \varepsilon_n^2) \\
&= o_p(\varepsilon_n^2)
\end{aligned}
$$

and using Conditions C and D

$$
n^{-1/2} \nu_n (l'_{\alpha_0}[\varepsilon_n (u^* - (\widehat{\alpha}_n - \alpha_0)), X, Y]) = n^{-1/2} \nu_n (l'_{\alpha_0}[u^*, X, Y]) + O_p(\varepsilon_n^2) + O_p(\varepsilon_n^2)
$$

Hence

$$
-(1-\varepsilon_n) \left\langle \widehat{\alpha}_n - \alpha_0, u^* \right\rangle + n^{-1/2} \nu_n (l'_{\alpha_0}[u^*, X, Y]) = o_p(n^{-1/2}) \qquad \text{(S 9.4)}
$$

This gives, together with the inequality in (S 9.4) with u^* replaced by $-u^*$,

$$
\left| \left\langle \widehat{\alpha}_n - \alpha_0, u^* \right\rangle - n^{-1/2} \nu_n (l'_{\alpha_0}[u^*, X, Y]) \right| = o_p(n^{-1/2})
$$

so

$$\langle \widehat{\alpha}_n - \alpha_0, v^* \rangle = n^{-1/2} \nu_n (l'_{\alpha_0}[v^*, X, Y]) + o_p(n^{-1/2})$$

Hence, by (S 4.2)

$$\begin{aligned} f_{\alpha_n} - f_{\alpha_0} &= f'_{\alpha_0}[\alpha_n - \alpha_0] + o_p(u_n \|\alpha_n - \alpha_0\|_F^{\omega}) \\ &= \langle \widehat{\alpha}_n - \alpha_0, v^* \rangle + o_p(n^{-1/2}) \\ &= n^{-1/2} \nu_n (l'_{\alpha_0}[u^*, X, Y]) + o_p(n^{-1/2}) \\ &= n^{-1} \sum_{i=1}^{n} n^{1/2} \left\{ \sum_{j=1}^{n} w_{ij} l'_{\alpha_0}[u^*, X_i, Y_j] - E\left[l'_{\alpha_0}[u^*, X, Y] \middle| X = x_i \right] \right\} \end{aligned}$$

The result then follows from the Central Limit Theorem (CLT) for triangular arrays (Proposition) in Andrews (1994, p. 2251). Note that the conditions of the Proposition are satisfied under our assumptions. In particular, $\Theta \subseteq \mathbb{R}^{d_\theta}$ is compact, finite-dimensional convergence of $n^{1/2} \sum_{j=1}^{n} w_{ij} l'_{\alpha_0}[u^*, X_i, Y_j] - E\left[l'_{\alpha_0}[u^*, X, Y] \middle| X = x_i \right]$ holds for each x_i due to the classical Lindeberg-Levy CLT, and Condition A satisfies the stochastic equicontinuity requirement of the Proposition.

■

References

Ai, C. (1997). A semiparametric maximum likelihood estimator. *Econometrica 65*, 933–963.

Ai, C., A. Chatrath, and F. Song (2006). On the comovement of commodity prices. *American Journal of Agricultural Economics 88*(3), 574–588.

Ai, C. and X. Chen (2003). Efficient estimation of models with conditional moment restrictions containing unknown functions. *Econometrica 71*(6), 1795–1843.

Altonji, J. G. and L. M. Segal (1996, July). Small-sample bias in gmm estimation of covariance structures. *Journal of Business & Economic Statistics 14*(3), 353–66.

Andrews, D. W. K. (1994, May). Empirical process methods in econometrics. In R. F. Engle and D. McFadden (Eds.), *Handbook of Econometrics*, Volume 4 of *Handbook of Econometrics*, Chapter 37, pp. 2248–2294. Elsevier.

Antoine, B., H. Bonnal, and E. Renault (2006). On the efficient use of the informational content of estimating equations: Implied probabilities and maximum euclidean likelihood. forthcoming in the *Journal of Econometrics*.

Blundell, R., X. Chen, and D. Kristensen (2006). Semi-nonparametric iv estimation of shape invariant engel curves. working paper, New York University.

Brockmann, M., T. Gasser, and E. Herrmann (1993). Locally adaptive bandwidth choice for kernel regression estimators. *Journal of the American Statistical Association 88*(424), 1302–1309.

Chamberlain, G. (1987). Asymptotic efficiency in estimation with conditional moment restrictions. *Journal of Econometrics 34*, 305–334.

Chen, X. (2005). Large sample sieve estimation of semi-nonparametric models. Technical report, Department of Economics, New York University.

Chen, X. and S. Ludvigson (2006). Land of addicts? an empirical investigation of habit-based asset pricing models. Technical report, Department of Economics, New York University.

Chen, X. and X. Shen (1998, March). Sieve extremum estimates for weakly dependent data. *Econometrica 66*(2), 289–314.

Corcoran, S. A. (1998). Bartlett adjustment of empirical discrepancy statistics. *Biometrika 85*(4), 967–972.

Cressie, N. and T. R. C. Read (1984). Multinomial goodness-of-fit tests. *Journal of the Royal Statistical Society, Series B 46*(3), 440–464.

Domínguez, M. A. and I. N. Lobato (2004). Consistent estimation of models defined by conditional moment restrictions. *Econometrica 72*(5), 1601–1615.

Donoho, D. L. and I. M. Johnstone (1995). Adapting to unknown smoothness via wavelet shrinkage. *Journal of the American Statistical Association 90*(432), 1200–1224.

Duncan, G. M. (1986, June). A semi-parametric censored regression estimator. *Journal of Econometrics 32*(1), 5–34.

Fan, J. and I. Gijbels (1996). *Local Polynomial Modelling and Its Applications* (First ed.). London, UK: Chapman & Hall.

Gallant, A. R. and D. W. Nychka (1987). Semi-nonparametric maximum likelihood estimation. *Econometrica 55*(2), 363–390.

Grenander, U. (1981). *Abstract Inference*. New York: Wiley.

Hall, P. and J. L. Horowitz (1996, July). Bootstrap critical values for tests based on generalized-method-of-moments estimators. *Econometrica 64*(4), 891–916.

Hansen, L. P. (1982). Large sample properties of generalized method of moments estimators. *Econometrica 50*(4), 1029–54.

Hansen, L. P., J. Heaton, and A. Yaron (1996, July). Finite-sample properties of some alternative gmm estimators. *Journal of Business & Economic Statistics 14*(3), 262–80.

Ichimura, H. (1993). Semiparametric least squares (sls) and weighted sls estimation of single-index models. *Econometrica 58*, 71–120.

Imbens, G. H., R. Spady, and P. Johnson (1998). Information theoretic approaches to inference in moment condition models. *Econometrica 66*, 333–357.

Imbens, G. W. (1997, July). One-step estimators for over-identified generalized method of moments models. *Review of Economic Studies 64*(3), 359–83.

Kitamura, Y. (2006). Empirical likelihood methods in econometrics: Theory and practice. Cowles Foundation Discussion Paper No. 1569.

Kitamura, Y. and M. Stutzer (1997). An information-theoretic alternative to generalized method of moments estimation. *Econometrica 65*(4), 861–874.

Kitamura, Y., G. Tripathi, and H. Ahn (2004). Empirical likelihood-based inference in conditional moment restriction models. *Econometrica 72*(6), 1667–1714.

LeBlanc, M. and J. Crowley (1995). Semiparametric regression functionals. *Journal of the American Statistical Association 90*, 95–105.

Müller, H.-G. and U. Stadtmüller (1987). Variable bandwidth kernel estimators of regression curves. *Annals of Statistics 15*(1), 182–201.

Newey, W. K. (1991). Uniform convergence in probability and stochastic equicontinuity. *Econometrica 59*(4), 1161–1167.

Newey, W. K. (1993). Efficient estimation of models with conditional moment restrictions. In G. Maddala, C. Rao, and H. Vinod (Eds.), *Handbook of Statistics*, Volume 11, pp. 2111–2245. Amsterdam: Elsevier.

Newey, W. K. (1994). The asymptotic variance of semiparametric estimators. *Econometrica 62*(6), 1349–82.

Newey, W. K. (1997). Convergence rates and asymptotic normality for series estimators. *Journal of Econometrics 79*(1), 147–168.

Newey, W. K. and J. L. Powell (2003). Instrumental variable estimation of nonparametric models. *Econometrica 71*(5), 1565–1578.

Newey, W. K. and R. J. Smith (2004). Higher order properties of gmm and generalized empirical likelihood estimators. *Econometrica 72*(1), 219–255.

Nishiyama, Y., Q. Liu, and N. Sueishi (2005, December). Semiparametric estimators for conditional moment restrictions containing nonparametric functions: Comparison of gmm and empirical likelihood procedures. In A. Zerger and R. Argent (Eds.), *MODSIM 2005 International Congress on Modelling and Simulation*, pp. 170–176. Modelling and Simulation Society of Australia and New Zealand. ISBN: 0-9758400-2-9.

Otsu, T. (2003a). Empirical likelihood for quantile regression. Manuscript.

Otsu, T. (2003b). Penalized empirical likelihood estimation of conditional moment restriction models with unknown functions. Department of Economics, University of Wisconsin-Madison.

Owen, A. (1988). Empirical likelihood ratio confidence intervals for a single functional. *Biometrika 75*(2), 237–49.

Owen, A. (2001). *Empirical Likelihood*. Chapman and Hall/CRC.

Pagan, A. and A. Ullah (1999). *Nonparametric Econometrics*. Cambridge University Press.

Powell, J. L. (1994, May). Estimation of semiparametric models. In R. F. Engle and D. McFadden (Eds.), *Handbook of Econometrics*, Volume 4 of *Handbook of Econometrics*, Chapter 41, pp. 2443–2521. Elsevier.

Powell, J. L., J. H. Stock, and T. M. Stoker (1989). Semiparametric estimation of index coefficients. *Econometrica 57*(6), 1403–30.

Qin, J. and J. Lawless (1994). Empirical likelihood and general estimating equations. *The Annals of Statistics 22*(1), 300–325.

Ramalho, J. (2005). Small sample bias of alternative estimation methods for moment condition models: Monte carlo evidence for covariance structures. *Studies in Nonlinear Dynamics & Econometrics 9*(1).

Ramsey, J. B. (1999). The contribution of wavelets to the analysis of economic and financial data. *Phil. Trans. R. Soc. Lond. A 357*, 2593–2606.

Robinson, P. M. (1987). Asymptotically efficient estimation in the presence of heteroskedasticity of unknown form. *Econometrica 55*, 875–891.

Robinson, P. M. (1988). Root-n-consistent semiparametric regression. *Econometrica 56*(4), 931–54.

Schennach, S. M. (2006). Point estimation with exponentially tilted empirical likelihood. Technical report, University of Chicago. forthcoming in the Annals of Statistics.

Shen, X. (1997). On methods of sieves and penalization. *The Annals of Statistics 25*(6), 2555–2591.

Shen, X. and W. H. Wong (1994). Convergence rate of sieve estimates. *The Annals of Statistics 22*(2), 580–615.

Silverman, B. (1986). *Density Estimation for Statistics and Data Analysis*. London: Chapman and Hall.

Silverman, B. W. (1984). Spline smoothing: The equivalent variable kernel method. *Annals of Statistics 12*(3), 898–916.

Smith, R. J. (1997, March). Alternative semi-parametric likelihood approaches to generalised method of moments estimation. *Economic Journal 107*(441), 503–19.

Smith, R. J. (2003). Local gel estimation with conditional moment restrictions. Technical report, University of Warwick.

Smith, R. J. (2005). Local gel methods for conditional moment restrictions. Cemmap working paper cwp15/05, University of Warwick.

Tibshirani, R. and T. Hastie (1987). Local likelihood estimation. *Journal of the American Statistical Association 82*(398), 559–567.

Van der Vaart, A. W. and J. A. Wellner (1996). *Weak Convergence and Empirical Processes*. Springer-Verlag New York, Inc.

Wong, W. H. and T. A. Severini (1991). On maximum likelihood estimation in infinite dimensional parameter spaces. *The Annals of Statistics 19*(2), 603–632.

Zhang, J. and I. Gijbels (2003). Sieve empirical likelihood and extensions of the generalized least squares. *Scandinavian Journal of Statistics 30*, 1–24.