A deprivation-based characterization of the Bonferroni index of inequality

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Abstract

We investigate several properties of the Bonferroni inequality indices, including their welfare theoretic interpretation. We also interpret and characterize the absolute Bonferroni index as the average of subgroup average depression indices, where to each income we associate a subgroup containing all persons whose incomes are not higher than this income. An aggregate depression index for a subgroup has been derived axiomatically as the sum of gaps between the subgroup highest income and all incomes not higher than that.

Key words: inequality, Bonferroni indices, welfare, transfers, depression, characterization.

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1. Introduction

Over the last thirty five years or so, the study of income inequality has become quite important for several reasons. However, there are indices which have not received much attention even though they have many advantages. One such index is the Bonferroni (1930) index, which is based on the comparison of the partial means and the general mean of an income distribution. One probable reason why this index has not been discussed much in the literature is that Bonferroni wrote his book in Italian. Among the very few English studies that investigated some properties of this index are those by Nygard and Sandstrom (1981), Giorgi (1984,1998), Tarsitano (1990), Giorgi and Mondani (1995), Aaberge(2000),Giorgi and Crescenzi (2001) and Chakravarty and Muliere (2003).

The Bonferroni index is a relative index, scaling incomes proportionally does not affect its value. It has a nice compromise property, when multiplied by the mean income it becomes an absolute index. The value of an absolute index remains unaltered when all incomes are translated, that is, augmented or diminished,by the same absolute amount. Conversely, by dividing the absolute Bonferroni index by the mean income we get its relative counterpart. In addition to possessing the respective compromise properties, the Bonferroni indices have some more interesting properties: (1) they are easy to compute, (2) they are continuous and bounded from below by zero , where this lower bound is achieved whenever incomes are equal, (3) they decrease under a rank preserving transfer of income from a rich to a poor and attach greater weight to transfers lower down the income scale, and (4) they implicitly define a common Bonferroni social welfare function ,which represents an ethical ordering of alternative distributions of income.

Knowing the properties of an inequality index is important since the answer one gets by the use of an index is in general dependent on the index. In this paper we investigate several properties of the Bonferroni indices, particularly, their relationship with the Gini indices, consistency with different types of income redistributive principles and correspondence with the Bonferroni social welfare function. A general class of social welfare functions, which has been discussed, among others, by Mehran(1976),Donaldson and Weymark(1980),Weymark(1981), Yaari(1987,1988), Aaberge(2000,2001) and Chatauneuf and Moyes(2006), is investigated further. It contains the Bonferroni, the Gini and the Donaldson-Weymark(1980) illfare ranked single-series Gini welfare functions as special cases.

We also analyze the absolute Bonferroni index from an alternative perspective, which argues that attitudes such as envy and depressions are important components of individual judgements so far as distributive justice is concerned. A person in subgroup *i* of persons with *i* lowest incomes may regard the subgroup highest income as his source of envy and suffer from depression on finding that he has a lower income. We present and discuss a number of properties that an aggregate index of depression in a subgroup should satisfy. The axioms proposed are sufficient to characterize a specific form of the index by means of a simple straightforward proof. The discussion makes the structure and the fundamental properties of the index quite transparent. The characterization is then extended to the entire population by aggregating a transformed version of subgroup indices. This summary index of depression for the population as a whole becomes the absolute Bonferroni inequality index.

The idea of interpreting inequality indices from such a perspective is not new. A person's feeling of depression about a higher income in the society can be measured by the shortfall of his income from the higher one and the average of all such depressions in all pair-wise comparisons becomes the Gini index (Sen, 1973).If the level of depression is proportional to the square of the difference in incomes, the resulting index of average depression becomes the squared coefficient of variation(Kakwani,1980).

Assuming that incomes are arranged in descending order, Donaldson and Weymark(1980) axiomatized a class of inequality indices characterized by a single parameter which contains the Gini index as a special case. They also axiomatized a similar class based on ascending order of incomes. The sum of two well-defined transformations of the Donaldson-Weymark families has been characterized by Tsui and Wang(2000) as a deprivation index using the concept of net marginal deprivation. According to net marginal deprivation a rank preserving increase in a person's income will generate two effects :(i) the feeling of deprivation among those poorer than him will increase, and (ii)his deprivation with respect to those richer than him decreases. It also bears some resemblance to the class of indices proposed by Berrebi and Silber(1981) , which is a mixture of the two Donaldson-Weymark families.

Following Runciman (1968) several researchers argued that the extent of deprivation felt by an individual is the sum of his income shortfalls from all persons richer than him, and attempted to discuss analytical properties of individual and aggregate deprivations , including their relationship with inequality indices and orderings. (See Yitzhaki, 1979, Hey and Lambert, 1980, Kakwani, 1984,Berrebi and Silber,1985,Chakravarty,1990,1997, Chakravarty and Mukherjee, 1999 , Chakravarty and Moyes, 2003 and Bossert and D'Ambrosio, 2006).

According to Temkin (1993) inequality can be viewed in terms of complaints of individuals located at disadvantaged positions in the income scale. A major case here is that the society highest income is the reference point for all and everybody except the richest has a legitimate complaint. Cowell and Ebert (2004) used this structure to derive a new class of inequality indices. The commonness between these studies and our framework is that all are based on different notions of envy, but the formulation we adopt is different from others.

After presenting the preliminaries, we discuss properties of the Bonferroni indices, including their welfare correspondence, in section 2. The characterization theorems are presented in section 3. Finally section 4 concludes.

2. Formal Framework and Properties

Consider a fixed homogeneous population $N = \{1, 2....., n\}$ of $n\,(n \geq 2)$ individuals. An income distribution in this population is represented by a non-negative illfare ranked vector $x = (x_1, x_2, ..., x_n)$, that is, $0 \leq x_1 \leq x_2 \leq ..... \leq x_n$. The set of all income distributions in the society is D^n. Let $S_i = \{1, 2,..., i\}$ be the subgroup of population with *i* lowest incomes $(x_1, ...x_i)$ in *x*. We write μ_i for the *ith* partial mean, that is, the mean income of S_i and μ for the population mean. 1^n will stand for the *n*-coordinated vector of ones. Given *n*, we denote the set $\{i/n \,|\, i = 0,1,......, n\}$ by *Q*.

The absolute Bonferroni index of inequality is defined as $B_A : D^n \to R^1$, where for all $x \in D^n$,

$$B_A = \mu - \frac{1}{n}\sum_{i=1}^{n}\mu_i$$

$$= \frac{1}{n}\sum_{i=1}^{n}x_i - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i}x_j \tag{1}$$

and R^1 is the real line. Thus, B_A is the amount by which the mean of the mean incomes of the subgroups S_i falls short of the population mean. Equivalently, it is the average of the absolute differences ($\mu - \mu_i$).

B_A is continuos and bounded from below by zero , where this lower bound is achieved if all the incomes are equally distributed. It is symmetric in the sense of its invariance under any permutation of incomes. (This property follows from the fact that we have defined B_A directly on an ordered distribution.) It satisfies the Pigou-Dalton condition, a postulate which requires inequality to reduce under a progressive transfer of income, an income transfer from a rich to a poor that does not change the relative positions of the donor and the recipient. Since B_A is symmetric, only rank preserving transfers are allowed under the Pigou-Dalton condition. In fact, B_A satisfies the principle of positional transfer sensitivity, a stronger redistributive criterion than the Pigou-Dalton condition(Aaberge,2000).According to the principle of positional transfer sensitivity a progressive income transfer between two individuals with a fixed difference in ranks will reduce inequality by a larger amount the lower the income of the donor is(Mehran, 1976 , Zoli, 1999 and Aaberge, 2000).An alternative to the principle of positional transfer sensitivity is Kolm's(1976) diminishing transfers principle, which demands that a progressive transfer with a fixed difference in income should be more equalizing at the lower end of the distribution.

It may now be worthwhile to compare B_A with the absolute Gini index $G_A : D^n \to R^1$, where for all $x \in D^n$,

$$G_A(x) = \mu - \frac{1}{n^2}\sum_{i=1}^{n}(2(n-i)+1)x_i\ . \tag{2}$$

G_A is a violator of the positional transfer sensitivity principle because it attaches equal weight to a given transfer irrespective of wherever it takes place , provided that it occurs between two persons with a fixed rank difference. However, it satisfies the Pigou-Dalton condition. But while G_A is population replication invariant, B_A is not. That is, if for any $x \in D^n$, if y is the k-fold replication of x, where $k \geq 2$ is any integer, then $G_A(y) = G_A(x)$ but $B_A(y) \neq B_A(x)$ .For instance, if $x = (1,2,3), y = (1,1,2,2,3,3)$,then $G_A(y) = G_A(x) = 4/9$, $B_A(x) = 1/2$ but $B_A(y) = 10.1/18$. (See also Tarsitano,1990.) 'This (population replication invariance) is clearly an important principle since we want to be able to use indices of inequality on sample distributions or on grouped data'(Donaldson and Weymark, 1980, p.72)..

The Kolm (1969) –Blackorby-Donaldson(1980) social welfare function corresponding to B_A is given by W_B, where for all $x \in D^n$,

$$W_B(x) = \mu - B_A(x)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i} x_j$$

$$= \sum_{i=1}^{n}\left(\frac{1}{n}\sum_{t=i}^{n}\frac{1}{t}\right)x_i . \qquad (3)$$

The corresponding social welfare function for the absolute Gini index is

$$W_G(x) = \mu - G_A(x) = \frac{1}{n^2}\sum_{i=1}^{n}(2(n-i)+1)x_i . \qquad (4)$$

W_B And W_G are continuous, increasing, linear homogeneous, unit translatable and strictly S-concave[1]. In terms of transfer, strict S-concavity means that welfare increases under a rank preserving progressive income transfer. Unit translatability of a welfare function means that a constant absolute translation of all incomes will translate welfare by the constant itself (see Blackorby and Donaldson, 1980)[2]. Since μ is unit translatable, unit translatability of W_B and W_G implies and is implied by translation invariance of B_A and G_A respectively. When efficiency considerations are absent (μ is fixed), an increase in B_A (G_A) is equivalent to a reduction W_B (W_G) and vice-versa.(See Blackorby and Donaldson,1978 ,Donaldson and Weymark,1980 and Chakravarty, 1988,1990, for further discussion on W_G .)

We can recover B_A from W_B and G_A from W_G using (3) and (4) respectively. Thus, these indices are exact in the sense that each of them implies and is implies by a social welfare function. From policy point of view, B_A (G_A) gives the per capita income that could be saved if society distributed incomes equally without any welfare loss, where welfare is measured by W_B (W_G). Each index is a measure of the total cost of per capita inequality in the sense that it tells us how much must be added in absolute terms to the income of every member in an *n*-person society to reach the same level of social welfare that would be achieved if everybody enjoyed the mean income of the current distribution, given that welfare evaluation is done with the respective welfare function

The relative Bonferroni index is defined by $B(x) = B_A(x)/\mu$, where $x \in D^n$ and $\mu > 0$ (see Nygard and Sandstrom, 1981). The Atkinson(1970)-Kolm(1969)-Sen(1973) social welfare function corresponding to *B* is given by

$$\mu(1 - B(x)) = W_B(x) .$$

Conversely, we can recover *B* from W_B using the above relation[3]. Thus, both B_A and *B* define a common social welfare function W_B. Linear homogeneity of W_B is necessary and sufficient for scale invariance of *B*. The index *B* determines the fraction of aggregate income that could be saved if the society distributed incomes equally without any welfare loss, where welfare is measured by W_B. Note that G_A is a compromise index as well- when divided by the mean income it becomes the well-known Gini index,

$$G(x) = 1 - \frac{1}{n^2 \mu} \sum_{i=1}^{n} (2(n-i)+1) x_i ,$$

which is a relative index. Clearly, we can relate G with W_G in the same way *B* has been related with W_B. Assuming that income follows a continuous type distribution, Aaberge(2000)showed that G (respectively, *B*) will satisfy the principle of diminishin transfers if F (respectively, log F) is strictly concave, where *F* is the distribution function. More generally, Aaberge(2000) showed that the moments of the Lorenz curve generate a conventional family of inequality indices which includes *G*. Relying on the diminishing transfers principle , it is demonstrated that these indices have transfer sensitivity property that depend on the shape of the income distribution.

With a given rank order of incomes, the Bonferroni and the Gini welfare functions W_B and W_G are linear in incomes. They are identical if $n = 2$.If we consider all continuous, increasing, strictly S-concave social welfare functions, which possess this restricted linearity condition, there does not seem to be any compelling reason to choose these two functions for special consideration. Equivalently, there seems to be no special status accorded to the weights $\left\{\sum_{t=i}^{n} 1/tn\right\}$ and $\{(2(n-i)+1)/n^2\}$ observed in (3) and (4). Therefore, it seems of interest to study a more general class of continuous, increasing, strictly S-concave welfare functions, which are linear in illfare ranked incomes. One possible such class is the class of rank dependent welfare functions

$$W_w(x) = \sum_{i=1}^{n} w_i x_i \, , \tag{5}$$

where $w = (w_1, w_2, ...w_n), w_i > 0$ for all i and $\sum_{i=1}^{n} w_i = 1, i = 1,2,...n$ (see Donaldson and Weymark, 1980,Weymark, 1981 and Yaari, 1987,1988). Thus, W_w is the weighted average of illfare ranked incomes. Positivity of w_i guarantees that W_w is increasing in individual incomes. This welfare function forms the basis of the following class of relative inequality indices :

$$I_w = 1 - \frac{W_w(x)}{\mu} \; . \tag{6}$$

Weymark(1981) referred to I_w as the generalized relative Gini index since it coincides with G if $w_i = (2(n-i)+1)/n^2$. (See also Mehran, 1976, for an earlier discussion.)Aaberge(2000) pointed out that B drops out as a member of I_w.The weight sequence $\{w_i\}$ for this particular case is $\left\{\sum_{t=i}^{n} 1/tn\right\}$.If we assume that $w_i = (i/n)^{\theta} - ((i-1)/n)^{\theta}$, where $0 < \theta < 1$, then I_w becomes the illfare ranked single – series relative Gini index I_{δ} (see Donaldson and Weymark,1980)[4].By defining appropriate preference relations on the set of Lorenz curves, Aaberge(2001) developed two alternative characterizations of Lorenz curve orderings. A complete characterization of G has also been obtained. Furthermore, axiomatic characterization of the extended Gini family(Donaldson and Weymark, 1980, Kakwani, 1980a and Yitzhaki, 1983) and an alternative generalized Gini family has been proposed.

We note that the weight sequences $\left\{\sum_{t=i}^{n} 1/tn\right\}$, $\left\{(2(n-i)+1)/n^2\right\}$ and $\left\{(i/n)^{\delta} - ((i-1)/n)^{\delta}\right\}$ for W_B , W_G and the single-series Gini welfare function $W_{\delta} = \mu(1 - I_{\delta})$ respectively, are decreasing in *i*. Decreasingness of $\{w_i\}$ is necessary and sufficient for the general welfare function W_w to be strictly S-concave(see Donaldson and Weymark , 1980 and Yaari,1988).Therefore, positivity of w_i along with its decreasingness ensures that W_w is increasing and strictly S-concave. Evidently, it is continuous as well. Mehran(1976) stated that I_w satisfies the principle of positional transfer sensitivity when the weights decrease with increasing intensity, that is, $w_i < w_{i+1}$ and $w_{i+1} - w_i < w_{i+2} - w_{i+1}$, where $i = 1,2,.....,n-2$ (see Aaberge,2000, for a formal proof) [5].

The notion of transfer considered so far concerns only two persons. An alternative concept of transfer can be the one that involves the donor and more than one recipient. Chateauneuf and Moyes(2006) considered equalizing transformations of this type, which they called T-transformations ,and in each case they derived the sequence of equivalent operations needed to convert a dominated distribution into the dominating one, where the dominance criterion is defined according to some unambiguous rule. The following notion of egalitarianism is in line with one of these T-transformations.

Definition 1: Given $y \in D^n$, we say that *x* is obtained from *y* by an equally spread equitable transfer if

$$x_j = y_j - \delta \geq x_{j-1} \text{ for some } j > 1, \text{ for some } \delta > 0,$$

$$x_l = y_l + \frac{\delta}{k} \text{ for } 1 \leq l \leq k \leq j-1,$$

$$x_l = y_l \text{ for } l \in \{1,2,......,n\} - \{1,2,....k, j\}.$$

Thus, an **equally spread equitable transfer** (**EST**) is a rank preserving progressive transfer (of size δ >0) from some person (*j*) and it is equally shared by the set $\{1,2,...,k\}$of k worst off persons from among who are poorer than him. It may be noted that the recipients of the transfer need not be all persons poorer than the donor. Thus, an **EST** distributes the transfer among the recipients in a lexicographic manner in the sense that if there is only one recipient then he is the poorest person of the society. In case of more than one recipients, nobody can receive his appropriate share of the transfer unless all persons poorer than him have received their shares. If the donor is the richest person of the society, than one possible case is that all the remaining persons share the transfer equally. Clearly, because of its progressiveness, **EST** can be regarded as a condition for incorporating egalitarian bias into distributional judgements. A social welfare function will be called **lexicographically equity oriented (LEO)** if its value increases under an **EST**. Formally,

Definition 2: A social welfare function $W : D^n \to R^1$ is called lexicographically equity oriented if for all $y \in D^n$,$W(x) > W(y)$, where x is obtained from y by an equally spread equitable transfer.

In its general form. W_w will be **LEO** if and only if $\sum_{i=1}^{k} w_i / k > w_j$,where $k < j$ and $j > 1$ are arbitrary. Evidently this condition follows from decreasingness of $\{w_i\}$. To investigate the implication of **EST** on W_w in greater detail, we make a Donaldson-Weymark(1980) type structural assumption about w_i. More precisely, we assume that $w_i = H(i/n) - H((i-1)/n)$,where $H : Q \to R^1$ is positive, H (0)=0 and H (1)=1.(See equation (17) of Donaldson and Weymark, 1980.) For this specification of w_i, we denote W_w by W_H. Thus, for the three welfare functions W_B, W_G and W_δ, $H(i/n)$ will be $\sum_{j=1}^{i} \sum_{t=j}^{n} 1/tn$, $(i(2n-i))/n^2$ and $(i/n)^\theta$ respectively.

The following theorem identifies the weight sequence $\{H(i/n) - H((i-1)/n)\}$ for which the welfare function W_H increases under an **EST**.

Theorem 1: The welfare function W_H is lexicographically equity oriented if and only if $\frac{H(i/n)}{i/n}$ is decreasing in i/n.

Proof: Suppose that *x* is obtained from $y \in D^n$ by an **EST** of size δ >0 from person *j* to the first *k* worst off persons, where $k < j$ and $j > 1$ are arbitrary. W_H will be **LEO** if and only if

$$H(\frac{k}{n})\frac{\delta}{k} > (H(\frac{j}{n}) - H(\frac{j-1}{n}))\delta, \tag{7}$$

for all $k < j$. Dividing both sides of (7) by $1/n$ and canceling δ from both sides of the resulting expression, we get

$$\frac{H(k/n)}{k/n} > \frac{H(j/n) - H((j-1)/n)}{1/n}, \tag{8}$$

for all $k < j$. Suppose that $j = k+1$. Then (8) reduces to

$$\frac{H(k/n)}{k/n} > \frac{H((k+1)/n)}{(k+1)/n}, \tag{9}$$

for all $k = 1,2,....., n-1$. So (8) implies (9).

Now, from (9), for arbitrary $k < (j-1)$, we have

$$\frac{H(k/n)}{k/n} > \frac{H((j-1)/n)}{(j-1)/n} > \frac{H(j/n)}{j/n}. \tag{10}$$

Inequality (10) gives

$$\frac{H(k/n)}{k/n} > \frac{H((j-1)/n)}{(j-1)/n} > \frac{H(j/n) - H((j-1)/n)}{1/n},$$

from which we get (8). Hence (8) holds if and only if (9) holds. This completes the proof of the theorem. Δ

While Theorem 1 identifies the *H* function for which W_H becomes **LEO,** for its strict S-concavity we need decreasingness of $w_i = H(i/n) - H((i-1)/n)$ (Donaldson and Weymark, 1980 and Yaari, 1988). The following theorem shows the relationship between these two specific types of social welfare functions.

Theorem 2: If the welfare function W_H is strictly S-concave, then it is lexicographically equity oriented, but the converse is not true.

Proof: Strict S-concavity of W_H is equivalent to decreasingness of $H(i/n) - H((i-1)/n)$. Therefore, we have

$$H(\frac{1}{n}) - H(0) > H(\frac{2}{n}) - H(\frac{1}{n}), \tag{11}$$

from which, given $H(0) = 0$, it follows that

$$\frac{H(1/n)}{1/n} > \frac{H(2/n)}{2/n}. \tag{12}$$

Thus, decreasingness of $H(i/n)/(i/n)$ is true for the pair (1,2).

Now, suppose that the result is true for the consecutive integers (j-1) and j. That is, $H((j-1/n)/((j-1)/n)/> H(j/n)/(j/n)$. Equivalently,

$$H(\frac{j-1}{n}) > \frac{(j-1)}{j} H(\frac{j}{n}). \tag{13}$$

We will then show that it is true for j and (j+1) as well. Let us rewrite inequality $H(j/n) - H((j-1)/n) > H((j+1)/n) - H((j)/n)$ as

$$2H(\frac{j}{n}) > H(\frac{j+1}{n}) + H(\frac{j-1}{n}). \tag{14}$$

From (13) and (14), we get

$$2H\left(\frac{j}{n}\right) > H\left(\frac{j+1}{n}\right) + H\left(\frac{j-1}{n}\right) > H\left(\frac{j+1}{n}\right) + \frac{(j-1)}{j} H(\frac{j}{n}). \tag{15}$$

Now, given that the left hand side expression of (15) is greater than its right hand side expression, we subtract the second term of the latter from the former to deduce that

$$\frac{(j+1)H(j/n)}{j} > H(\frac{j+1}{n}). \tag{16}$$

We divide both sides of (16) by $1/n$ and rearrange terms to get

$$\frac{H(j/n)}{j/n} > \frac{H((j+1)/n)}{(j+1)/n}, \tag{17}$$

which is what we wanted to prove. Therefore, by the method of induction (17) is true for all $j = 1,2,.....,n-1$. Lexicographic equity orientation of W_H now follows from Theorem 1.

To see that the reverse implication is not true, define $H(i/n)$ as follows:

$$\begin{aligned} H(\frac{i}{n}) &= 0 \text{ if } \frac{i}{n} = 0, \\ &= \frac{i+c}{n}, \text{ where } 0 < c < 1, \text{ if } \frac{1}{n} \le \frac{i}{n} \le \frac{n-1}{n}, \qquad (18) \\ &= 1 \text{ if } \frac{i}{n} = 1. \end{aligned}$$

For the form of $H(i/n)$ given by (18), $H(i/n) - H((i-1)/n)$ is positive for all $i = 1,2,.....,n$. $H(i/n)/(i/n)$ is decreasing in (i/n), but $H(i/n) - H((i-1)/n)$ is not decreasing in $i, 1 \le i \le n$. The result now follows from the equivalence between strict S-concavity and decreasingness of $H(i/n) - H((i-1)/n)$ in $i, 1 \le i \le n$. This completes the proof of the theorem. Δ

Theorem 2 shows that in the context of W_H, **LEO** is a weaker condition than strict S-concavity. Thus, the Pigou-Dalton condition(under symmetry) entails the **EST** principle. Since W_B and W_G are strictly S-concave, from Theorem 2 it follows that they are LEO as well**.**

3. The Characterization Theorems

We begin this section by observing that B_A in (1) can be rewritten as

$$B_A(x) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{(x_i - x_j)}{i}. \qquad (19)$$

Now, any person *j* in subgroup *i* may feel depressed upon discovering that he has a lower income than the subgroup highest income x_i. Therefore, $(x_i - x_j)$ can be considered as a measure of *j's* depression in S_i. Then $\sum_{j=1}^{i} (x_i - x_j)/i$ is an indicator of average depression in S_i. Although *i* does not feel depressed in S_i, we include him in this expression for the sake of completeness. Since there are *n* subgroups of the type S_i, B_A is simply the average of subgroup average depressions.

A natural direction of investigation at this stage is to characterize the Bonferroni index axiomatically in an envy – deprivation framework. Such a characterization will enable us to understand the index in a more elaborate way through the axioms employed in the exercise. For this, we first have

Definition 3: For any income distribution $x \in D^n$, $d(x;S_i)$ denotes an index of aggregate depression of subgroup *i*.

We now introduce a number of axioms that *d* should satisfy. "The choice of axioms is always based on(subjective) value judgements"(Ebert and Moyes, 2000, p.263).

Focus (FOC): For all $x, y \in D^n$ if $x_j = y_j$ for all *j*, $1 \le j \le i$, then $d(x;S_i) = d(y;S_i)$.

FOC says that the depression index for subgroup *i* is independent of incomes of persons who are not in the subgroup. This is analogous to the poverty focus axiom, which says that a poverty index is independent of non -poor incomes.

Translation Invariance (TRI): For all $x \in D^n$, $d(x;S_i) = d(x + \alpha 1^n; S_i)$, where α is a scalar such that $x + \alpha 1^n \in D^n$.

TRI is essentially a value judgment assumption**.** It means that *d* remains unaltered under equal absolute changes in all incomes. That is, depression depends on absolute income differentials. It is comparable to invariance of absolute inequality indices (see, Blackorby and Donaldson, 1980).It is satisfied by the Yitzhaki(1979) local and global deprivation indices and the Cowell-Ebert(2004) complaint based inequality indices. Ebert and Moyes(2000) used this axiom to characterize the individual deprivation index suggested by Yitzhaki(1979). Under **TRI** dispersion around the mean may change. For instance**,** it regards the distribution (2,6,12) as identically depressed as (102,106,112) , but the dispersion around the mean for the latter is negligible. However, since people often feel depressed by looking at differences from higher incomes, **TRI** seems natural here. (See Kolm's(1976) discussion along this line.)

Linear Homogeneity (LIH): For all $x \in D^n, \lambda > 0$ $d(\lambda x;S_i) = \lambda d(x;S_i)$.

According to **LIH**, a proportional change in all incomes increases or decreases depression by the same proportion. Thus, the scale of incomes influences the index. If all incomes in the society are doubled so that it is becoming twice as rich, depression doubles too. Differences in living standard, as measured by the income gaps, are reflected in the index of depression. This property is shared by many absolute deprivation indices (see Chakravarty and Mukherjee, 1999; Ebert and Moyes, 2000, and Cowell and Ebert, 2002).

The next axiom is about depression difference in two consecutive subgroups.

Recursivity (REC): For all $x \in D^n$, $d(x;S_i) - d(x;S_{i-1}) = (i-1)f(x_{i-1}, x_i)$, where $i \geq 2$ and $f : D^2 \to R^1$.

Since $x_{i-1}(x_i)$ is the source of depression in $S_{i-1}(S_i)$ and since first (*i-2*) persons in the society are depressed in both S_{i-1} and S_i, we assume that the difference $d(x;S_i) - d(x;S_{i-1})$ depends on the two sources through some well – defined function *f* in an increasing manner and ignores the incomes of the commonly depressed (*i-2*) incomes. The simple formulation also shows the dependence of the difference increasingly on the number of persons who are depressed in S_i, which clearly includes the number that is depressed in S_{i-1}. **REC** is quite similar to a property of the Runciman – Yitzhaki – Kakwani index of individual deprivation. It says that the difference between the extents of deprivations felt by persons (*j-1*) and *j* depends directly on the product of the income difference ($x_j - x_{j-1}$) and (*n-j+1*), the number of persons about whom the worse off person (*j-1*) feels deprived.

Finally, the index is normalized by

Normalization (NOM): If $x \in D^n$ is of the form $(0, 0, 0, \ldots, 0, x_n)$, where $x_n > 0$, then $d(x;S_n) = (n-1)x_n$.

NOM says that in the particular case when only the richest person enjoys a positive income and all other persons have zero income, if we restrict attention to the largest subgroup, then the level of depression is given by the product of *(n-1),* the number of depressed persons and the only positive income x_n. Thus, our formulation shows that in this extreme case the amount of depression is increasingly related to the number of depressed persons and the income level about which their depression arises.

The axioms proposed above restrict an index of depression. They are consistent with one another (that is, they are not contradictory) and sufficient to characterize exactly one index.

Theorem 3: A depression index satisfies the axioms **FOC, TRI, LTH, REC** and **NOM** if and only if for all $x \in D^n$ it is identical to

$$d^*(x; S_i) = \sum_{j=1}^{i} (x_i - x_j) . \tag{20}$$

Proof: Assume that $d(x; S_i)$ satisfies the axioms listed in the theorem. By **FOC** we can rewrite $d(x; S_2)$ as $g(x_1, x_2; S_2)$, where g is translation invariant and linear homogenous in (x_1, x_2). Therefore, from **REC** it follows that

$$d(x; S_1) = g(x_1, x_2; S_2) - f(x_1, x_2) . \tag{21}$$

By **TRI** of g and f, the right hand side of (21) becomes $g(0, x_2 - x_1; S_2) - f(0, x_2 - x_1)$, which in view of **LIH** equals $(x_2 - x_1)\, g(0,1; S_2) - (x_2 - x_1) f(0,1)$. Thus, we have

$$d(x; S_1) = (x_2 - x_1)(g(0,1; S_2) - f(0,1)) . \tag{22}$$

By **FOC**, $d(x; S_1)$ is independent of x_2. Hence on the right hand side of (22) we can assume any value of x_2 satisfying the inequality $x_2 \geq x_1$. Therefore we can set $x_2 = x_1$ on the right hand side to get $d(x; S_1) = 0$.

Using the information $d(x; S_1) = 0$ in **REC** we get

$$\begin{aligned} d(x; S_2) &= f(x_1, x_2) \\ &= (x_2 - x_1) f(0,1) \\ &= k(x_2 - x_1) , \end{aligned} \tag{23}$$

where $k = f(0,1)$. Another application of **REC** gives

$$\begin{aligned} d(x; S_3) &= d(x; S_2) + 2 f(x_2, x_3) \\ &= k(x_2 - x_1) + 2k(x_3 - x_2) \\ &= k \sum_{j=1}^{3} (x_3 - x_j) . \end{aligned} \tag{24}$$

Continuing this way, we can show that for any *i*, $1 \le i \le n$,

$$d\,(x\,;S_i) = k\sum_{j=1}^{i}(x_i - x_j)\ . \qquad (25)$$

We note that in the extreme case described in **NOM**, the value of $d\,(x;S_n)$ given by (25) becomes $k\,(n-1)\,x_n$. But by the **NOM**, the value of the depression index in this case should be $(n-1)\,x_n$. Equating these two value of $d\,(x\,;S_n)$, we get *k = 1*. Substituting *k = 1* in (25), we note that $d\,(x;S_i)$ is identical with $d^*(x\,;S_i)$ in (20). The converse is obvious. Δ

The depression index characterized in theorem 4 is simply the sum of income gaps of all persons in S_i from the highest income x_i in. If x_i is taken as the poverty line in S_i, then $(x_i - x_j)$ is individual *j's* poverty gap and the depression index $d^*(x\,;S_i)$ gives the total amount of money necessary to put the persons in S_i at the poverty line. It is bounded between zero and $(i-1)x_i$,where the lower bound is achieved when all the incomes in S_i are equal, and the upper bound is attained in the extreme case when **NOM** is applied to S_i.Under rank preserving increments and reductions in x_i and $x_j\,(x_j < x_i)$ respectively, it is increasing in x_i and decreasing in x_j.

Essential to the construction of the index $d^*(x\,;S_i)$ are the reference group S_i and the reference income x_i in S_i, where $i = 1,2,....,n$. This may be contrasted with the simple Temkin (1993) structure where the only reference group is S_n and the reference income is x_n. In this structure the size of complaint experienced by person *i* is $(x_n - x_i)$ and hence $d^*(x\,;S_n)$ becomes the aggregate Temkin complaint. Cowell and Ebert (2002) derived a class of complaint based inequality indices by aggregating the individual complaints. A similar step for us is to develop a global depression index using the subgroup indices $d^*(x\,;S_i)$. We regard the overall depression in the society as a kind of social bad. Since there is a one-to-one correspondence between $d^*(x\,;S_i)$ and $d^*(x\,;S_i)/i$, in the rest of the paper we will use the average index $d^*(x\,;S_i)/i$ for our analysis.

For any $x \in D^n$, we will denote the society depression index is by $A(x_1,...,x_n)$ .Next, we assume that the index *A* can be identified with a real valued function of subgroup depression indices. Since for any $x \in D^n$, $d^*(x;S_1)=0$, a constant, we will not include it in this formulation. For the purpose at hand we write $e_i(x)$, or simply e_i, for $d^*(x;S_{i+1})/(i+1)$, where *i = 1,2,…,n –1*. Then our assumption can be formally stated as: there exists a real valued function *I* defined on R_+^{n-1} such that for all $(x_1,x_2,...,x_n)\in D^n$, the global depression index $A(x_1,x_2,...,x_n)$ can be written as $I(e_1(x),e_2(x),...,e_{n-1}(x))$, where R_+^{n-1} is the non- negative part of the (*n -1*) dimensional Euclidean space. This procedure, which Dutta, Pattanaik and Xu(2003),referred to as Procedure II, is adopted in many branches of economics. For instance, in welfare economics social utility is regarded as a function of individual utilities. Likewise, in the literature on human development, a functioning achievement index (e.g., the human development index) is assumed to depend on individual attainment indicators (seeUNDP,1990-2005 and Chakravarty, 2003).

We now propose some postulates for an arbitrary index *I*.

Additive Decomposability (ADD): For all $x, x' \in D^n$, $e(x),\ e(x') \in R_+^{n-1}$, $I(e(x)+e(x')) = I(e(x)) + I(e(x'))$.

Anonymity (ANY): For all $x \in D^n, e(x) \in R_+^{n-1}, I(e(x)) = I(e'(x))$, where $e'(x)$ is any permutation of *e(x).*

Strong Monotonicity (SMN): For all $x, x' \in D^n, e(x),\ e(x') \in R_+^{n-1}$, if $e_i(x) \ge e_i(x')$ for *i = 1, 2,…, n – 1*,with > for at least one *i*, then *I (e(x)) > I* $(e(x'))$.

Continuity (CON): *I* is a continuous function on R_+^{n-1}.

ADD says how to calculate depression when people derive income from two different sources. Suppose that there are two mutually exclusive and exhaustive sources of incomes, say wage and non – wage incomes. Let x_j^l be the income of person *j* from source l; where *j = 1, 2,….,n* and *l = 1, 2*. Since we have assumed at the outset that all income distributions are illfare ranked, ranks of individuals in the component distributions $x^l = (x_1^l, x_2^l, ...x_n^l) \in D^n$, where *l = 1,2* ,are the same. We then note that

$e_i(x) = e_i(x^1) + e_i(x^2)$, where $x = (x_1^1 + x_1^2, x_2^1 + x_2^2, ..., x_n^1 + x_n^2)$. **ADD** then demands that social depression based on sum of subgroup depressions calculated from component income distributions is the sum of social depressions derived from subgroup depressions for component distributions. Given that the ranks of the individuals in the component as well as in the original distributions are the same, it may be interesting to note that W_w satisfies a similar property in the sense that welfare from the aggregate distribution is the sum of welfares from component distributions. This property was used by Weymark(1981) to characterize the generalized Gini indices. Chakravarty(2003) used a similar source decomposability axiom to characterize a generalized form of the human development index. This property is, in fact, similar to the factor decomposition property of rank dependent inequality indices (See Kakwani,1980,Silber,1989, and Lambert,2001).

SMN says that of two distributions *x* and *y*, if for each subgroup, depression under *x* is not less than that under *y*, and for at least one subgroup, *x* has higher depression, then *x* will have more global depression than *y*. **SMN** is analogous to the strong Pareto principle, which demands that between two allocations *u* and *v*, if everybody finds *u* at least as good as *v* and at least one individual finds *u* better, then the *u* must be socially better than *v*. **ANY** means that depression remains unaltered under any reordering of subgroup depressions given by $e_i' s$. Thus, any characteristic other than subgroup depressions, e.g., names of the subgroups, is irrelevant to the measurement of global depression. Finally, according to **CON** minor change in subgroup depressions will lead to a minor change in the global index. Thus, a continuous depression index will not be oversensitive to minor observational errors in incomes.

The following theorem can now be presented.

Theorem 4: A global depression index satisfies **ADD, ANY, SMN** and **CON** if and only if it is a positive multiple of the absolute Bonferroni inequality index B_A.

Proof: For simplicity, let us write *e* for $e(x)$.Then **ADD**, which we can write explicitly as

$$I\,(e_1^1 + e_1^2, e_2^1 + e_2^2, ..., e_{n-1}^1 + e_{n-1}^2) = I\,(e_1^1, e_2^1, ..., e_{n-1}^1) + I\,(e_1^2, e_2^2, ..., e_{n-1}^2)\,, \qquad (26)$$

is a generalized Cauchy functional equation, of which the only continuous solution is

$$I\ (e_1,...,e_{n-1}) = \sum_{i=1}^{n-1} c_i\ e_i\ , \tag{27}$$

(see Aczel, 1966, p. 215). **ANY** implies that $c_i = c$ for all *i*. By **SMN**, *c* must be positive. Since *n* is fixed, we rewrite *c* as b/n, where $b > 0$. Therefore *I* in (27) becomes

$$\begin{aligned} I\ (e_1,...,e_{n-1}) &= \frac{b}{n}\sum_{i=1}^{n-1} e_i \\ &= \frac{b}{n}\sum_{i=1}^{n}\frac{d^*\ (x;S_i)}{i} \\ &= \frac{b}{n}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{(x_i - x_j)}{i}. \\ &= b\ B_A\ . \end{aligned} \tag{28}$$

This establishes the necessity part of the theorem. The sufficiency is easy to verify. Δ

The theorem proved above shows that the axioms are consistent: there is exactly one index satisfying all of them and it is the Bonferroni inequality index of the absolute type. The characterized index is a measure of social bad; it determines the aggregate depression in the society.

4. Conclusions

Although Bonferroni suggested an inequality index long time ago, it was not discussed much in the literature. We first discuss several properties of the relative and absolute versions of this index, including their relationship with the Gini indices and their welfare theoretic foundation. We then use an axiom system that corresponds to the type of assumptions made in the literature on the assessment of income distributions from the viewpoint of envy and depressions, and characterize the absolute form of the index. Thus, our discussion and characterization interpret the Bonferroni indices from alternative perspectives.

A plot of $d^*(x,S_i)/n$ against i/n, where i=0,1,…n, gives us the depression curve of *x*. For any $x \in D^n$, $d^*(x,S_i)/n$ can be written as (ix_i/n)-$GL(x,i/n)$,where

$GL(x, i/n) = \sum_{j=1}^{i} x_j / n\mu$ is the ordinate of the generalized Lorenz curve of *x* corresponding to the population proportion i/n .Thus, the generalized Lorenz curve of *x* has a negative monotonic relationship with its depression curve. It will certainly be worthwhile to develop an ordering based on the depression curve. But since in this paper we are mainly concerned with characterization, this is left as a future research program.

Notes

1. A function $g : D^n \to R^1$ is called S-concave if $g(Bx) \geq g(x)$ for all $x \in D^n$ and for all $n \times n$ bistochastic matrices *B*. An $n \times n$ non-negative matrix is called a bistochastic matrix if each of its rows and columns sums to one. For strictly S-concavity of g the weak inequality is to be replaced by a strict inequality whenever *Bx* is not a permutation of *x* .All strictly S-concave functions are symmetric.
2. Formally, a function $g : D^n \to R^1$ is called unit translatable if $g(x + \alpha 1^n) = g(x) + \alpha$, where α is any scalar such that $x + \alpha 1^n \in D^n$.
3. Strictly speaking, Banferroni suggested the use of $B' = (n/(n-1))B$ as an index of inequality. However, if we replace *B* by B' in the Bonferroni welfare function W_B , then it becomes independent of x_n , the highest income. Because of this undesirable feature of B' , here we use *B*, the Nygard-Sandstrom form of the Bonferroni index.
4. For further discussion on I_δ , see Lambert(1985,2001),Bossert(1990), Zoli(1999) and Aaberge(2000,2001).
5. Hey and Lambert(!980) ,Yaari(1987) and Ben Porath and Gilboa (1992) also provided normative the rank dependent social welfare functions.

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