# Long-Run Productivity Risk: A New Hope for Production-Based Asset Pricing* (Technical Appendix)

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**Abstract**

This study examines the intertemporal distribution of productivity risk. Focusing on post-war US data, I show that the conditional mean of productivity growth is time-varying and extremely persistent. This generates uncertainty about the long-run perspectives of economic growth and affects asset prices. The data suggest that stock market prices are very sensitive to long-run news about productivity growth. After establishing this empirical link, I develop a production-based asset pricing model featuring long-run uncertainty about the productivity growth rate, convex adjustment costs, and recursive preferences à la Epstein-Zin. This model reproduces key features of both asset prices and macroeconomic quantities, including consumption, investment, and output. I also provide a detailed examination of the role of the intertemporal elasticity of substitution, relative risk aversion, and adjustment costs in this type of economy.

*Keywords:* Production, Long-Run Risk, Asset Pricing, Recursive Utility
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# Appendix

## A.1 Empirical Analysis: Short-Sample Properties

Consider the following model:

$$\begin{aligned}
\Delta a_{t+1} &= \mu + x_t + \sigma\epsilon_{a,t+1} \qquad \text{(A.1)}\\
x_t &= \rho x_{t-1} + \sigma_x \epsilon_{x,t}\\
\begin{bmatrix} \epsilon_{a,t+1} \\ \epsilon_{x,t+1} \end{bmatrix} &\sim iidN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).
\end{aligned}$$

Then the productivity growth rate has the following Wold representation (Hamilton (1994)):

$$\begin{aligned}
\Delta a_{t+1} &= \mu(1-\rho) + \rho\Delta a_t - b\epsilon_{a,t} + \sigma_a \epsilon_{a,t+1}, \qquad \text{(A.2)}\\
&\text{where}\\
b &= \rho K, 0 < K < 1\\
\epsilon_{a,t+1} &\sim iidN(0,1).
\end{aligned}$$

The scalar $K$ and the volatility $\sigma_a$ are both endogenous and depend on the ratio of the variance of the predictable component $x$ and the total variance of $\Delta a$. When the long-run component $x$ is not present, $K = 1$, implying that the AR root and the MA root are equal. In this case, these two roots perfectly cancel out each other and $\Delta a$ collapses to a simple *i.i.d.* process. When productivity growth instead has a long-run component, the two roots differ from each other, and the MA root is smaller than the AR root in population.

In table 1, I show that past information about productivity growth can explain at least 10% of the total variance of future productivity growth over an annual horizon. At the same time, however, the maximum likelihood point estimate of the AR root, $\widehat{\rho}$, is smaller than that one of the MA root, $\widehat{b}$. Such an empirical outcome might appear to contrast with the theoretical results derived above. In this section, however, I show that in reality this can simply be a small sample problem.

In order to study the small-sample properties of my estimator of the ARMA(1,1), I simulate model (A.1) at an annual frequency and I generate 1000 independent samples with 55 observations (the sample I use in the empirical part of this study is for the years 1948–2003). For the sake of simplicity I keep the volatility constant. In these simulations, I impose $\rho = .80$, a conservative figure that is consistent with the estimate obtained working with the data. I assume $\sigma = 2\%$ in order to match the historical variance of productivity growth. I impose $\sigma_x = .19\sigma$ so that the long-run component explains about 10% of the total variance of productivity growth.

For each sample, I estimate the model reported in eq. (A.2) through maximum likelihood. First of all, the point estimate of the MA root, $\widehat{b}$, is lerger than that of the RA root, $\widehat{\rho}$, in 476 samples. Furthermore, the point estimate of MA root is greater than one in 215 samples, implying that in small samples, the maximum likelihood estimates frequently do not satisfy the invertibility condition.

These results suggest that the model in eq. (A.1) can be considered a reasonable candidate to describe the long-run dynamics of productivity growth, even if in the data $\widehat{b} > \widehat{\rho}$.

## A.2 Numerical methods

This section presents the model used to examine the link between productivity, asset prices and other macroeconomic fundamentals. To keep the analysis as simple as possible, I focus only on the representative agent consumption-saving problem and I keep constant the labor supply. The representative agent has preferences defined only by aggregate consumption:

$$\begin{aligned} U_t &= \left[(1-\delta)C_t^{1-\frac{1}{\Psi}} + \delta\left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\Psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\Psi}}} \\ 0 &\leq C_t \end{aligned}$$

The consumption good is produced according to the following constant returns-to-scale neoclassical production function:

$$Y_t = K_t^{\alpha}\left[A_t\overline{n}\right]^{1-\alpha},$$

where $K_t$ is the fixed stock of capital carried into date $t$, $\overline{n}$ is the fix labor input, and $A_t$ is an aggregate productivity shock. The productivity growth rate, $\Delta a_{t+1} \equiv \log(A_{t+1}/A_t)$, has a long-run risk component and evolves as described below:

$$\begin{aligned} \Delta a_{t+1} &= \mu + x_t + \sigma\epsilon_{a,t+1} \\ x_t &= \rho x_{t-1} + \sigma_x\epsilon_{x,t} \\ \begin{bmatrix} \epsilon_{a,t+1} \\ \epsilon_{x,t+1} \end{bmatrix} &\sim iidN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right). \end{aligned}$$

In this paper I want to study the role of uncertainty about the conditional mean of productivity growth. For this reason I assume that the volatility of the shocks to productivity is constant.

The resource constraint of this economy is:

$$C_t + I_t \leq Y_t$$

The capital stock evolves according to:

$$\begin{aligned} K_{t+1} &= (1-\delta_k)K_t + G\left(\frac{I_t}{K_t}\right)K_t \\ &\text{where} \\ G\Big(\frac{I_t}{K_t}\Big) &= \left[\frac{a_1}{1-\frac{1}{\tau}}\left(\frac{I_t}{K_t}\right)^{1-\frac{1}{\tau}} + a_2\right]. \end{aligned}$$

The rate of depreciation of capital is denoted by $\delta_k$ and the function $G(.)$ transforms investment in new capital.

Let us define the following stationary variables:

$$\{c_t, i_t, y_t, k_t, u_t\} \equiv \left\{\frac{C_t}{A_{t-1}}, \frac{I_t}{A_{t-1}}, \frac{Y_t}{A_{t-1}}, \frac{K_t}{A_{t-1}}, \frac{U_t}{A_{t-1}}\right\}. \tag{A.3}$$

Let $s_t \equiv [\Delta a_t, x_t, k_t]$ denote the vector of the states of the economy. Let $u(s)$ be the value of the planner's problem evaluated at the optimum for given state $s$. The planner's problem can be rewritten in the following recursive way:

$$\begin{aligned} u(s) &= \max_{c,k'}\left[(1-\delta)c^{1-\frac{1}{\Psi}} + \delta e^{(1-\frac{1}{\Psi})\Delta a}\left(E_s[u(s')^{1-\gamma}]^{\frac{1-\frac{1}{\Psi}}{1-\gamma}}\right)\right]^{\frac{1}{1-\frac{1}{\Psi}}} \\ &s.t. \\ c &\geq 0, k' \geq 0 \\ c+i &= y \equiv e^{(1-\alpha)\Delta a}k^{\alpha}\bar{n}^{(1-\alpha)} \\ k'e^{\Delta a} &= (1-\delta)k + G\left(\frac{i}{k}\right)k \\ x' &= \rho x + \sigma_x \epsilon_a' \\ \Delta a' &= \mu + x + \sigma\epsilon_a' \\ \epsilon_a' &\perp \epsilon_x' \end{aligned}$$

I solve this problem numerically by standard value function iterations. I program my algorithm in Fortran, Compaq Visualizer 6.a-professional, by HP.

The first thing I do is to discretize the state space. I use: (1) $N_a$ equidistant points for $\Delta a$ on the interval $[-2std[\Delta a] \ +2std[\Delta a]]$; (2) $N_x$ equidistant points for $x$ on the interval $[-2std[x] \ +2std[x]]$; (3) $N_k$ equidistant points for $k$ on the interval $[.1k^{ss} \ 1.9k^{ss}]$, where $k^{ss}$ is the value of capital at the deterministic steady-state. At this point I guess a value function $u(\Delta a, x, k)$ that is an array $N_a \times N_x \times N_k$. From now on, similarly to that done in Fortran, I use ':' to indicate all the elements in one dimension. So, for example, $u(:, x_i, k_l)$ indicates all the values of the utility function with respect to the state $\Delta a$, given $x = x_i$ and $k = k_l$. Now I am ready to start an iteration, I list below the required steps:

**STEP 1: Computing** $Q(\Delta a, x, k) \equiv E_s[u(\Delta a', x', k)^{1-\gamma}]$

I discretize a shock distributed according to a standard normal using $N_s$ points. Looking at the definition of $E_s[u^{'(1-\gamma)}]$, one can see that this object is in reality invariant with respect to $\Delta a$. The only two variables that have predictive power and really matter for the conditional expected value are $k$ and $x$. So, in order to accelerate this step, for every $(x_i, k_l)$ in the grid, I compute the following expression:

$$H(x_i, k_l) = \sum_{h=1}^{N_s} \sum_{f=1}^{N_s} u(\mu + x_i + \sigma\epsilon_h, \rho x + \sigma_x \epsilon_f, k_l)^{1-\gamma} \phi_f \phi_h$$

I compute $H(x_i, k_l)$ only $N_k \times N_x$ times and then impose $Q(:, x_i, k_l) = H(x_i, k_l)$. In order to compute $H(\cdot, \cdot)$ I have to be able to compute $u(\mu + x_i + \sigma\epsilon_h, \rho x + \sigma_x \epsilon_f, k_l)$. The initial guess gives information only on $u(\Delta a_j, x_i, k_l)$, so I need to be able to approximate and interpolate $u(\cdot, \cdot, k_l)$ on points that are outside the grid of the states. For a given $k_l$, $\log(u(:, :, k_l))$ gives me a set of points—over a two-dimensional surface—which I approximate using a tensor product of Chebyshev polynomials in $\Delta a$ and $x$. This surface is very close to being linear, so I do not need many polynomials to produce a good approximation. In particular, the number of polynomials is smaller than $N_{\Delta a} \times N_x$, and for this reason I compute the polynomials coefficients by a standard projection method (for every $k_l$ in the grid, I apply OLS on $\log(u(:, :, k_l))$, using as regressors all the polynomials I have in $\Delta a$ and $x$). Once I have the Chebyshev coefficients, I interpolate $\log(u)$ in correspondence of the generic points $(\mu + x_i + \sigma\epsilon_h, \rho x + \sigma_x \epsilon_f, k_l)$. After this, the computation of $H$ is straightforward.

**STEP 2: Interpolating** $Q(\Delta a, x, k)$ **with Respect to Capital.**

I want to solve the maximization problem by applying a standard Newton-algorithm to the control variable $k'$. In order to apply this algorithm I need to be able to evaluate the right-hand side of the recursion for every admissible value of the control variable $k'$. This means that I have to be able to evaluate $Q$ for every feasible value of capital $k'$. For every $x_i$ in my grid, I project through OLS-method $\log(H(x_i, :))$ on a constant, $\log(\overrightarrow{k})$ and $\log(\overrightarrow{k})^2$, where: $\overrightarrow{k}$ indicates the vector that collects the log-values of capital in my grid and $\log(\overrightarrow{k})^2$ is a vector collecting the square of log-values of capital. This transformation is very useful since $\log(H(x_i, :))$ is very close to being linear in log-capital and allows me to have very small interpolation residuals (in the order of 1.e-6, log units). For every $x_i$, I run this OLS interpolation and I get an OLS coefficients-vector that I denote as $\beta_i$. At this point, using the interpolation implies the following fast way to evaluate the expected value of future utility at $k'$: $H(x_i, k') = exp([1\ k'\ (k')^2]\beta_i)$.

**STEP 3: Maximizing**

Given the state $s = [\ \Delta a_j \quad x_i \quad k_l\ ]$, I solve the following maximization problem using the Fortran subroutine 'DUVMIF':

$$\begin{aligned}
\max_{c \geq 0} \quad & \left[(1-\delta)c^{1-\frac{1}{\Psi}} + \delta e^{(1-\frac{1}{\Psi})\Delta a_j}\left(exp([1\ k'\ (k')^2]\beta_i)^{\frac{1-\frac{1}{\Psi}}{1-\gamma}}\right)\right]^{\frac{1}{1-\frac{1}{\Psi}}} \\
& s.t. \\
i &= e^{(1-\alpha)\Delta a_j} k_l^{\alpha} \bar{n}^{(1-\alpha)} - c \\
k' &= \frac{(1-\delta) + G\left(\frac{i}{k_l}\right)}{e^{\Delta a_j}} k_l \\
k' &\geq 0
\end{aligned}$$

The subroutine also evaluates the objective function at the optimum and allows me to update the value function.

**STEP 4: Stopping Rule.**

Once finished with step 3, I measure the distance between the new value function, $u'$, and the initial one using a standard sup-norm. If $\frac{||(u')^{1-\frac{1}{\Psi}} - (u)^{1-\frac{1}{\Psi}}||}{1-\beta} < 1.e-8$, I stop. Otherwise I use the new value function as an initial guess and I go back to step 1.

## A.3 Spectrum

Given a stochastic process $X$, its spectral density evaluated at the frequency $f$ is:

$$S_X(f) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k cos(kf),$$

where

$$\gamma_k = cov(X_t, X_{t-k}).$$

Given the following moving average representation for $X$:

$$X_t = \sum_{j=0}^{\infty} \theta_j^X \epsilon_{t-j}^X,$$

the auto-covariances can be computed in the following way:

$$\gamma_k = \sum_{j=0}^{\infty} \theta_j^X \theta_{j-k}^X (\sigma_X)^2.$$

Notice also that given two *orthogonal* stochastic processes $X$ and $Y$, the spectrum of their sum is equal to the sum of their spectral densities:

$$S_{(X+Y)}(f) = S_X(f) + S_Y(f)$$

If we ignore nonlinear terms, the consumption growth rate obtained by the model can be represented as:

$$\Delta c_{t+1} - \overline{\Delta c} = \underbrace{\sum_{j=0}^{\infty} \theta_j^x \epsilon_{x,t-j}}_{\Delta c^{lrr}} + \underbrace{\sum_{j=0}^{\infty} \theta_j^a \epsilon_{a,t-j}}_{\Delta c^{srr}}.$$

The IRF produced by giving a one-pulse positive shock to the short-run component yields the sequence $\{\theta_j^a \sigma\}$, which allows me to compute the auto-covariances and the spectral density of $\Delta c^{lrr}$. The IRF produced by giving a one-pulse positive shock to the long-run component yields the sequence $\{\theta_j^x \sigma_x\}$, which allows me to compute the auto-covariances and the spectral density of $\Delta c^{srr}$. Since the long-run component news is orthogonal to the short-run news, the spectral density of the consumption growth is equal to the sum of the spectral densities of the consumption growth sub components that I denoted as $\Delta c^{srr}$ and $\Delta c^{lrr}$. Finally, I note the following:

(1) When I compute the monthly spectral density, for computational reasons I truncate the moving average and I consider only the first two hundred coefficients (i.e., I compute the IRF.s only for 200 periods).
(2) In order to have a consistent estimate of the spectrum of the quarterly consumption data I actually compute $S_X(f) = \frac{1}{2\pi}\sum_{k=1}^{8}\gamma_k cos(kf)(1-\frac{k}{9})$.
(3) When I compute the quarterly spectral density produced by the model, I use the same formula that I apply to the data: $S_X(f) = \frac{1}{2\pi}\sum_{k=1}^{8}\gamma_k cos(kf)(1-\frac{k}{9})$.

## A.4 Long-Horizon Variance

Let me define the long-run variance of the monthly productivity growth rate at the horizon $h$ as: $Var[\Delta a_{t+h|t}]/h$. Since I assume that the long-run and short-run news are uncorrelated, I can easily decompose the previous long-horizon variance in two subcomponents:

$$Var[\Delta a_{t+h|t}]/h = \underbrace{Var\left[\sum_{k=1}^{h-1} x_{t+k-1}\right]/h}_{Var_h^{lrr}(\Delta a)} + \underbrace{Var\left[\sum_{k=1}^{h} \epsilon_{a,t+k}\right]/h}_{Var_h^{srr}(\Delta a)}.$$

Since the short-run risk in productivity is $i.i.d.$, $Var_h^{srr}(\Delta a) = \sigma$. The long-run component is, instead, persistent over time and for this reason:

$$Var_h^{lrr}(\Delta a) = Var(x) + \frac{2}{j}\sum_{k=1}^{h-1}(h-k)\rho^k V(x).$$

If we ignore nonlinear terms, the consumption growth rate obtained in the production economy can be represented as:

$$\Delta c_{t+1} - \overline{\Delta c} = \underbrace{\sum_{j=0}^{\infty}\theta_j^x \epsilon_{x,t-j}}_{\Delta c^{lrr}} + \underbrace{\sum_{j=0}^{\infty}\theta_j^a \epsilon_{a,t-j}}_{\Delta c^{srr}}.$$

The coefficients $\{\theta_j^a, \theta_j^x\}$ can be recovered from the IRFs of the model as shown in the previous section. The auto-covariances generated by the short-run news can be computed in the following way:

$$\gamma_k^a \equiv \sum_{j=0}^{\infty}\theta_j^a\theta_{j-k}^a\sigma^2.$$

The implied long-run variance is computed as follows:

$$Var_h^{srr}(\Delta c) = \gamma_0^a + \frac{2}{j}\sum_{k=1}^{h-1}(h-k)\gamma_k^a.$$

Similarly, the auto-covariances generated by the long-run news can be computed as:

$$\gamma_k^x \equiv \sum_{j=0}^{\infty}\theta_j^x\theta_{j-k}^x(\sigma_x)^2.$$

The implied long-run variance is computed as follows:

$$Var_h^{lrr}(\Delta c) = \gamma_0^x + \frac{2}{j}\sum_{k=1}^{h-1}(h-k)\gamma_k^x.$$

For computational reasons I truncate the moving average and I consider only the first three hundred coefficients: $\{\theta_j^a, \theta_j^x\}_{j=1}^{300}$.

# References

[1] Hamilton, James D., 1994. Time Series Analysis. Princeton University Press, Princeton, NJ.

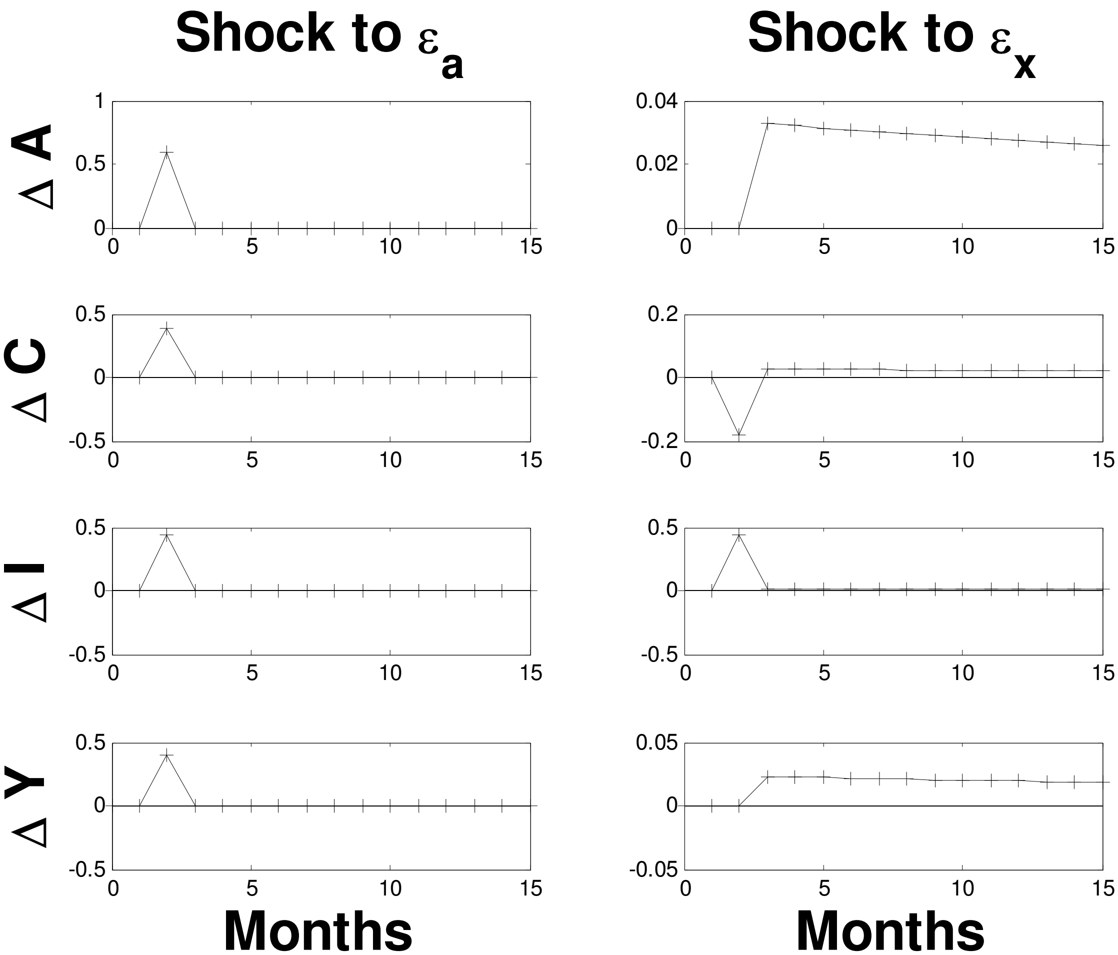


Fig. A 1 – Quantities Impulse Response Functions (IES=RRA$^{-1}$=2)

This figure shows monthly log-deviations from the steady state. Units are multiplied by 100. All the parameters are calibrated to the values reported in Table 6. The policy functions are computed numerically.

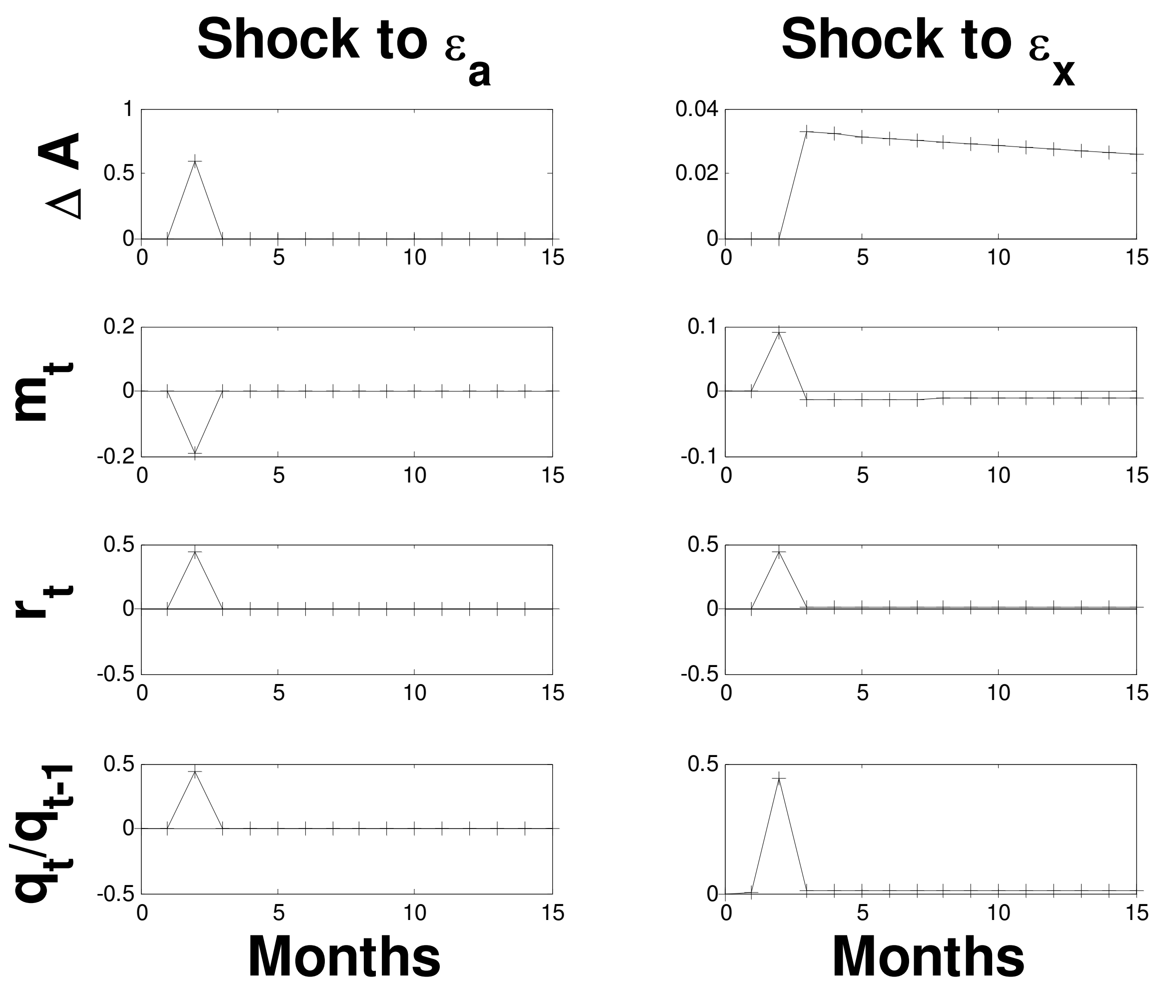


Fig. A 2 – Prices Impulse Response Functions (IES=RRA$^{-1}$=.8)

This figure shows monthly log-deviations from the steady state. Units are multiplied by 100. All the parameters are calibrated to the values reported in Table 6. The policy functions are computed numerically. Returns are not levered.

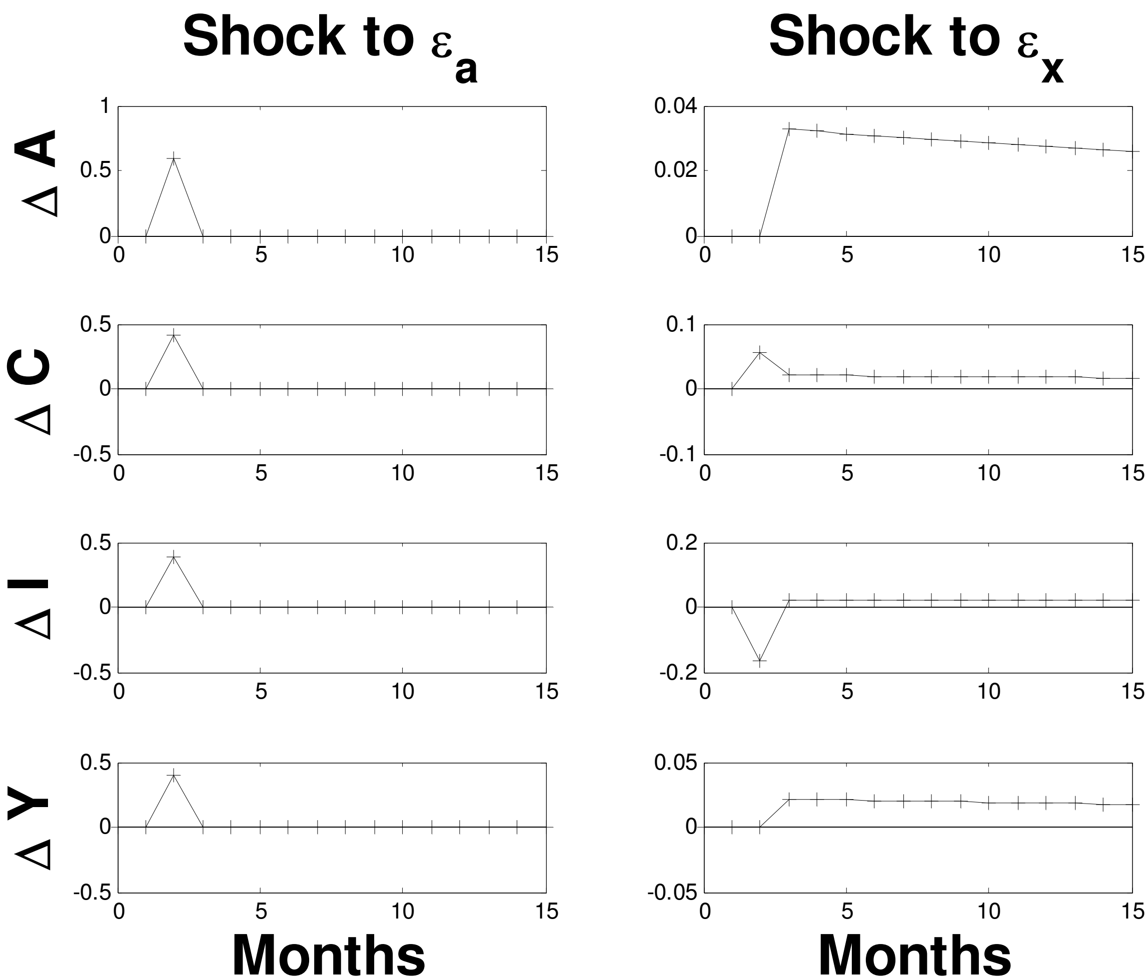


FIG. A 3 – QUANTITIES IMPULSE RESPONSE FUNCTIONS (IES=RRA$^{-1}$=.8)

This figure shows monthly log-deviations from the steady state. Units are multiplied by 100. All the parameters are calibrated to the values reported in Table 6. The policy functions are computed numerically.

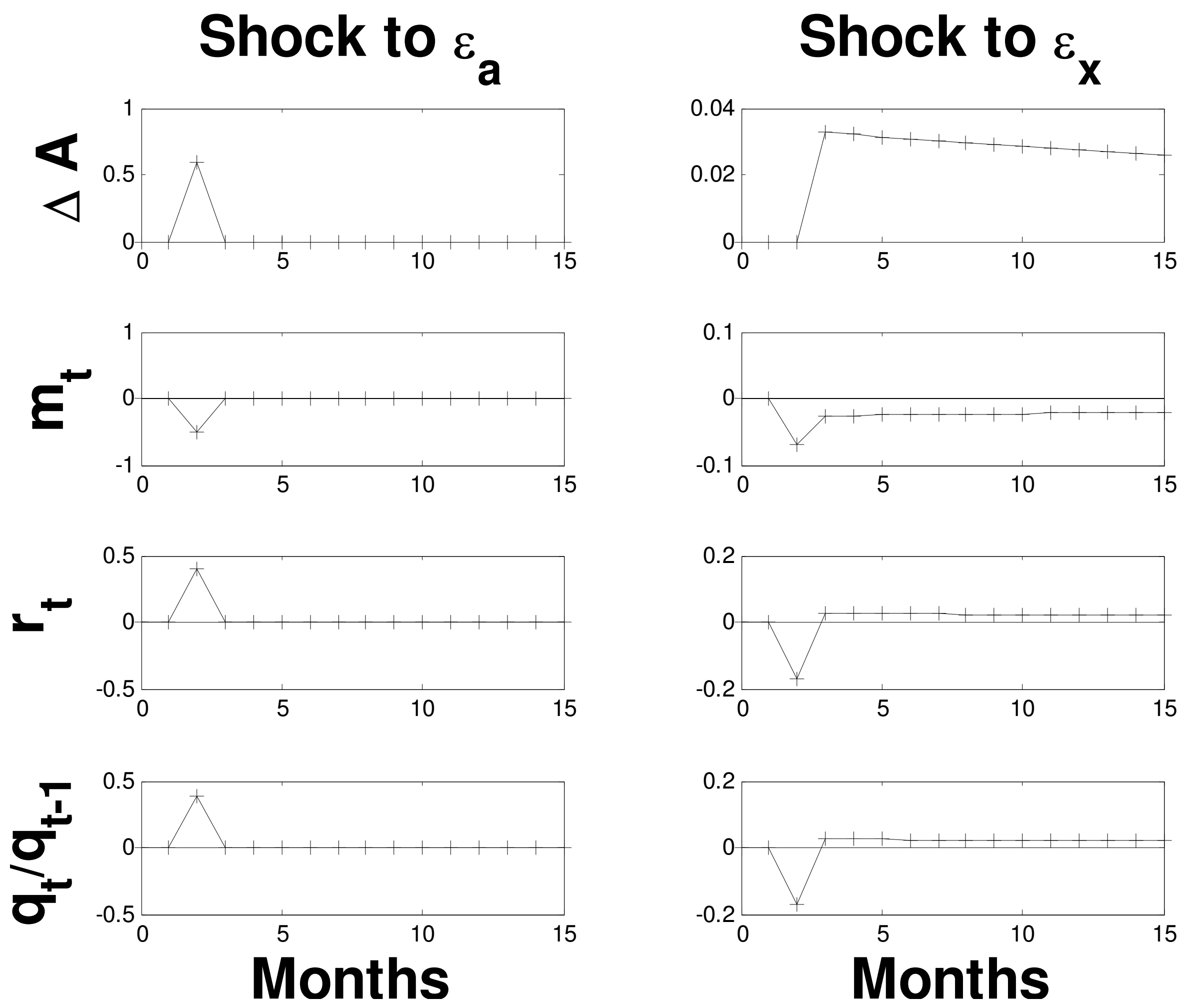


Fig. A 4 – Prices Impulse Response Functions (IES=RRA$^{-1}$=.8)

This figure shows monthly log-deviations from the steady state. Units are multiplied by 100. All the parameters are calibrated to the values reported in Table 6. The policy functions are computed numerically. Returns are not levered.