

Differentiated Networks: Equilibrium and Efficiency*

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Abstract

We consider a model of price competition in a duopoly with product differentiation and network effects. The value of a good for a consumer is the sum of a common and an idiosyncratic component. The first captures the vertical dimension of quality, the second captures horizontal differentiation. Each consumer privately observes his own value for each good but cannot separate the common and the idiosyncratic component. Therefore, he has incomplete information about the value of the goods for the other consumers. After firms announce prices, consumers choose simultaneously which network to join, facing a coordination problem.

In the efficient allocation, both networks are active and the firm with the highest expected quality has the largest market share. To characterize the equilibrium allocation, we derive necessary and sufficient conditions for uniqueness of the equilibrium of the coordination game played by consumers for given prices. The equilibrium allocation differs from the efficient one for two reasons. First, the equilibrium allocation of consumers to the networks is too balanced, since consumers fail to internalize network externalities. Second, if access to the networks is priced by strategic firms, then the product with the highest expected quality is also the most expensive. This further reduces the asymmetry between market shares and therefore social welfare.

KEYWORDS: network externalities, product differentiation, price competition, coordination games, global games.

JEL CLASSIFICATION: D43, D62, L14, L15.

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1 Introduction

Many economic decisions, such as the purchase of a good by a population of consumers or the adoption of a standard of production by a set of firms, exhibit a specific form of strategic complementarity known as network externality. The set of consumers buying a specific product and the set of firms producing according to a given standard constitute virtual networks: the externality arises when the payoff of each player who belongs to a network is increasing in the total size of the network itself.

In most cases where two alternative network goods are available, they are strongly differentiated. Therefore, characterizing the social optimum is a non-trivial problem: the aggregate surplus from the network effect is maximized if all players join the same network, while the surplus from the intrinsic utility from consumption of the good is maximized if every player buys the product he likes the most.

This trade-off is exemplified by the case of PC and Macintosh. If there was only one standard, every computer owner could run all the existing software and easily exchange files and know-how with anybody else. On the other hand, one of the main reasons why the two standards do coexist is that the two types of computers have different strengths that appeal to the needs of different consumers.¹

Another example is the case of matrix programming languages. Matlab seems to be best suited to analyze time-series data, while Gauss seems to perform better when analyzing panel data.² At the same time, all programmers would benefit from the existence of a unique common language because they could exchange suggestions and pieces of code with a larger group of people.

In this paper, we investigate this trade-off between maximizing the aggregated network effect and letting consumers use their favorite product, and show that the market outcome is affected by two types of inefficiency.

We consider two cases, one where the networks are “sponsored” and one where the networks are “unsponsored”. In the network literature, this distinction refers to the cost an individual bears to join a network: a network is sponsored if the access to it is priced by a strategic player. While this is the case for most telecommunication networks, there are many cases of

¹This differentiation is both vertical and horizontal. On its website, Apple lists the top ten reasons to switch to a Mac. Among them, “reason number 2” is that a Mac doesn’t crash (clearly a claim of higher vertical quality) while “reason number 3” is the fact that Mac offers the best technology to store and play digital music (which is a feature that different consumers might value differently, depending on the specific use they make of their computers).

²See Rust (1993).

unsponsored networks as well. For example, by learning a new language, an individual can join a “communication network” that clearly exhibits network externalities. In this case, there is no centralized entity that owns the language and strategically sets a price for the access to the network. The “price” is simply the time spent studying the language.

In our model, we assume that two new, alternative network products are introduced and that consumers simultaneously choose which one to join. If the networks are sponsored, access to each network is priced by a strategic firm. If they are unsponsored, access is available at marginal cost. The goods are both vertically and horizontally differentiated but neither firms nor consumers can perfectly observe the vertical dimension of quality. More precisely, they observe a noisy public signal about the vertical quality of each good, and one of the two networks has a higher expected quality. Each consumer privately and perfectly observes his private value for each good but he cannot distinguish the common component (the objective quality of the good) from his idiosyncratic taste component. When choosing which network to join, each individual takes into account three elements: his private valuation for each good, the expected size of each network, and the cost he has to bear to join it.

Consider, for example, the introduction of two new, non-interconnected telecommunication networks, such as two softwares for videoconference. The assumption of noisy information about the vertical quality of each good captures the fact that the overall performance of such products critically depends on how well they interact with the complementary hardware and software, and this interaction cannot be perfectly tested before the product is introduced. The assumption of horizontal differentiation captures the idiosyncratic taste consumers might have for the more “recreational” features of these products, such as the graphic interface. Finally, the assumption that each consumer perfectly observes her private value for each good before making a purchase captures the fact that software developers typically make trial versions of their products available for free.

The main question addressed in this paper is how the equilibrium allocation of consumers across networks compares to the social optimum. We find that the optimal allocation of consumers across networks is asymmetric, with both networks having positive market shares and more than one half of the population joining the high-quality network. We then look at the allocation implemented by the market, distinguishing between the two cases of sponsored and unsponsored networks.

In the case of unsponsored networks, we identify a first source of inefficiency. Since consumers fail to internalize network effects, the equilibrium allocation is too balanced from a social point of view: the market share of the high-quality firm, though larger than one half, is still smaller than would be socially optimal. Looking at the case of sponsored networks,

we find that strategic pricing makes the inefficiency even worse: the high-quality firm has an intrinsic advantage that is reflected in an equilibrium price higher than the one charged by the competitor, and this in turn reduces her market share even more.

The paper also brings a methodological contribution to the literature on network markets. There is a problem with modelling the demand function for network goods. The game played by consumers choosing which network to join, for given prices, constitutes a coordination game and typically these games have multiple equilibria. Therefore, in a basic model of Bertrand competition between identical networks, the demand for each good is not a well-defined function of prices. We enrich the basic model by allowing for both horizontal and vertical differentiation. Consumers' choices are then a global game with correlated private values. This allows us to derive necessary and sufficient conditions on the information structure of the game for the demand function to be well-defined.

A number of papers analyze price competition between networks. De Palma and Leruth (1993) allow only for horizontal differentiation. Their emphasis is on the conditions for a unique equilibrium of the coordination game played by consumers, rather than the efficiency issues emphasized here. They prove that a large degree of differentiation is necessary and sufficient for uniqueness of the equilibrium in that context. This condition is similar to the condition we derive in the limit case of our model where all the players perfectly observe the vertical dimension of quality and the latter is identical for the two goods. On the other hand, Farrell and Katz (1998) assume only vertical differentiation, therefore their model allows for multiple sets of self-fulfilling expectations in the coordination game played by consumers. Our discussion of the optimal allocation of consumers between networks is also related to Farrell and Saloner (1986), who address the issue of whether complete standardization is efficient if consumers have heterogeneous preferences. Our work is also connected to the literature on duopolistic price competition with differentiated goods. For a discussion of this literature, see Tirole (1988).

In our paper, we model consumers' choice as a global game with correlated private values. Global games, first analyzed by Carlsson and van Damme (1993), have been typically applied to economic situations where players have common values, such as currency crises and regime switches. Morris and Shin (2004) analyze a private-value global game where two players choose between two alternative actions and each player privately observes his own payoff type which is the sum of a common component and an idiosyncratic component. They derive a condition on the information structure that is necessary and sufficient for a unique equilibrium in that context. The solution of the model is identical in the case of a continuum of players choosing between two actions (Morris and Shin (2003)). In our model, we apply the Morris-Shin condition

for uniqueness, and relate it to the degree of horizontal differentiation and the precision of the available information about the vertical dimension of quality.

The paper is organized as follows. In Section 2 we introduce the formal model. In Section 3 we characterize the socially optimal allocation of consumers to networks. In Section 4 we characterize the equilibrium allocation for both the case of unsponsored and sponsored networks and compare it to the efficient one. Section 5 concludes. All the proofs are relegated to the Appendix.

2 The Model

We assume that two indivisible network goods, a and b , become available to a population of consumers represented by a continuum of mass 1. The two goods are produced at the same, constant marginal cost c . If the networks are sponsored, each good is produced and sold by a firm who strategically chooses her price to maximize her profits:

$$\pi_j = (p^j - c) n^j$$

where p^j is the price charged by firm j (with $j = a, b$) and n^j is the number of units she sells. If the networks are unsponsored, there are no strategic firms and each good can be purchased at marginal cost.

Preferences exhibit network externalities: the utility that any consumer i derives from joining network j is increasing in n^j . More precisely, we assume:

$$U_i^j = x_i^j + n^j - p^j$$

where x_i^j represents the intrinsic value of good j for consumer i .

We further assume that the two goods are both vertically and horizontally differentiated. The intrinsic value of a good for a consumer is the sum of a common value component $\tilde{\theta}^j$, representing the vertical dimension of quality of brand j , and an idiosyncratic component $\tilde{\varepsilon}_i^j$, representing individual taste:

$$\tilde{x}_i^j \equiv \tilde{\theta}^j + \tilde{\varepsilon}_i^j.$$

We assume that each $\tilde{\theta}^j$ is a normal random variable:

$$\tilde{\theta}^j \sim N\left(y^j, \frac{2}{\alpha}\right).$$

The expected value of $\tilde{\theta}^j$, given by y^j , can be interpreted as a noisy public signal about $\tilde{\theta}^j$.

We model consumer heterogeneity assuming that each individual's idiosyncratic taste component is normally distributed around zero:

$$\tilde{\varepsilon}_i^j \sim N\left(0, \frac{2}{\beta}\right).$$

Finally, we assume that $\tilde{\theta}^a$, $\tilde{\theta}^b$ and each $\tilde{\varepsilon}_i^j$ are independently distributed. All the above distributional assumptions are common knowledge.

Each consumer privately observes the vector

$$\mathbf{x}_i = (x_i^a, x_i^b) \in \mathbb{R}^2$$

which constitutes his type. We will denote a type profile for all consumers as

$$\mathbf{x} = \times_{i \in [0,1]} \mathbf{x}_i \in (\mathbb{R}^2)^{[0,1]}.$$

Notice that consumers have correlated private values: types are correlated, due to the presence of a common component, but each consumer's payoff does not depend directly on other consumers' types. We emphasize that a consumer can only observe his type but he cannot observe θ^j and ε_i^j separately. More precisely, neither firms nor consumers can perfectly observe the vertical quality of each good.

We can now formally define actions, strategies and payoff functions for the players. If the networks are sponsored, then the two firms and the population of consumers play a game of incomplete information in two stages. In the first stage, firms announce prices (p^a, p^b) simultaneously and noncooperatively. An action for firm j is a price $p^j \in \mathbb{R}$. The strategy space for the firms coincides with their action space and a strategy profile for the firms is a vector $\mathbf{p} = (p^a, p^b) \in \mathbb{R}^2$.

In the second stage of the game, consumers learn their types, observe prices and simultaneously choose a network. We assume complete market coverage and exclusivity: each consumer buys exactly one unit of one good, thus joining either network a or network b . Formally, an action for consumer i is $r_i \in \{a, b\}$, and an action profile for all the consumers is $\mathbf{r} \in \{a, b\}^{[0,1]}$. A pure strategy for a consumer is

$$s_i : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \{a, b\}$$

and a strategy profile for all consumers is $\mathbf{s} \equiv \times_{i \in [0,1]} s_i$. The size of network j when consumers play strategy profile \mathbf{s} and firms play strategy profile \mathbf{p} is $n^j(\mathbf{s}(\mathbf{x}, \mathbf{p}))$.

In the first stage, each firm maximizes her expected profits solving

$$\max_{p^j \in \mathbb{R}} \mathbb{E}_{\mathbf{x}} [\pi_j(n^j(\mathbf{s}(\mathbf{x}, \mathbf{p})), \mathbf{p})] = \max_{p^j \in \mathbb{R}} (p^j - c) \mathbb{E}_{\mathbf{x}} [n^j(\mathbf{s}(\mathbf{x}, \mathbf{p}))].$$

In the second stage, each consumer maximizes his expected net surplus solving

$$\begin{aligned} & \max_{j \in \{a,b\}} \mathbb{E}_{\mathbf{x}_{-i}} \left[U_i^j (\mathbf{x}_i, (\mathbf{s}(\mathbf{x}, \mathbf{p})), \mathbf{p}) \mid \mathbf{x}_i \right] = \\ & = \max_{j \in \{a,b\}} \left(x_i^j + \mathbb{E}_{\mathbf{x}_{-i}} [n^j (\mathbf{s}(\mathbf{x}, \mathbf{p})) \mid \mathbf{x}_i] - p^j \right). \end{aligned}$$

If networks are unsponsored, then both goods are priced at marginal cost and consumers play a static game of incomplete information equivalent to the second stage of the above game for the case $p^a = p^b = c$.

For notational convenience, we also define the following differences:

$$\begin{aligned} y & \equiv \frac{y^a - y^b}{2} \\ \tilde{\theta} & \equiv \frac{\tilde{\theta}^a - \tilde{\theta}^b}{2} \sim N \left(y, \frac{1}{\alpha} \right) \\ \tilde{\varepsilon}_i & \equiv \frac{\tilde{\varepsilon}_i^a - \tilde{\varepsilon}_i^b}{2} \sim N \left(0, \frac{1}{\beta} \right) \\ \tilde{x}_i & \equiv \frac{\tilde{x}_i^a - \tilde{x}_i^b}{2} \sim N \left(y, \frac{\alpha + \beta}{\alpha\beta} \right). \end{aligned}$$

The above variables can be easily interpreted. The random variable $\tilde{\theta}$ is an index of vertical differentiation. For positive realizations of $\tilde{\theta}$, good a has a higher objective quality than good b , and the opposite is true for negative values. The parameter y is a public signal about the difference in quality between the two goods. Without loss of generality, we will assume that $y \geq 0$ throughout the paper (i.e. we will assume that good a has a weakly higher expected quality than good b). The parameter α represents the precision of the public information about the difference in quality between the two goods. The random variable $\tilde{\varepsilon}_i$ captures the idiosyncratic differences in taste among consumers and β , the precision of its distribution, is an index of the amount of heterogeneity among consumers.³ More precisely, the smaller is β , the larger is the amount of horizontal differentiation between the two goods. The random variable \tilde{x}_i is a measure consumer i 's preference for good a , the good with the highest expected quality. For positive realizations of \tilde{x}_i , he prefers a to b . For negative realizations, his idiosyncratic preference for b is so strong that he prefers b to a . Notice that we use the notation \mathbf{x}_i for the vector (x_i^a, x_i^b) and the notation x_i for the difference $\frac{x_i^a - x_i^b}{2}$.

³Throughout the paper, we will assume that the law of large number holds for a continuum of independent variables, i.e. we will assume that the distribution of the idiosyncratic component in the population is the same as the distribution of any individual ε_i .

Finally, we define

$$p(p^a, p^b) \equiv \frac{p^a - p^b}{2}.$$

In what follows, we will use the simplified notation p to denote $p(p^a, p^b)$.

3 Efficient Allocation

In this section, we derive the ex-ante efficient allocation of consumers across networks. We choose to address the issue of efficiency from an ex-ante point of view because we believe that the right benchmark to compare to the allocation implemented by the market is the allocation that maximizes welfare given all, and only, the public information available on the market.⁴

First, we need to define the welfare criterion. Given our assumption of quasi-linear utility functions, aggregate welfare equals consumer gross surplus minus total cost, since prices are a transfer from consumers to firms. Also, since we assumed that the two firms have the same marginal cost and that the total number of units sold in the market is constant, we can ignore costs as we solve the welfare maximization problem. Finally, the only choice variable in the welfare maximization problem is the allocation of consumers between the two networks. Therefore, the welfare maximization problem is equivalent to the following problem⁵:

$$\begin{aligned} \max_{\mathcal{A}, \mathcal{B}} \mathbb{E}_{\mathbf{x}} [W(\mathcal{A}, \mathcal{B})] &= \max_{\mathcal{A}, \mathcal{B}} \mathbb{E}_{\mathbf{x}} \left[\int_{i \in \mathcal{A}} (x_i^a + n^a) di + \int_{i \in \mathcal{B}} (x_i^b + n^b) di \right] & (*) \\ \text{s.t. } \mathcal{A} \cup \mathcal{B} &= [0, 1] \\ \mathcal{A} \cap \mathcal{B} &= \emptyset. \end{aligned}$$

Next, we show that the optimal choice of the sets $(\mathcal{A}, \mathcal{B})$ can be characterized as a threshold allocation. By this we mean that a benevolent social planner with access to all the public information available on this market, and the power to assign each consumer to a network, would maximize social welfare by choosing a threshold t and allocating to network a those consumers with a preference for a larger than t and to network b those with a preference for a smaller than that. Formally, we define a threshold allocation as follows:

⁴For the concept of ex-ante efficiency, see Holmström and Myerson (1983).

⁵In the definition of problem (*) we are giving a heuristic description of the welfare criterion ignoring measurability issues. In the rest of the paper, all the allocations where we will evaluate welfare will be threshold allocations (see Def. 1). This fact, together with the assumption that the law of large number holds for a continuum of independent variables, will guarantee that welfare is well-defined for those allocations.

Definition 1

A *Threshold Allocation* is a couple $(\mathcal{A}(t), \mathcal{B}(t))$ such that $\exists t \in \mathbb{R}$ such that $\mathcal{A}(t) = \{i \in [0, 1] : x_i > t\}$ and $\mathcal{B}(t) = \{i \in [0, 1] : x_i \leq t\}$.

Lemma 1

The welfare maximizing allocation is a threshold allocation.

In what follows, we will denote by $(\mathcal{A}^*, \mathcal{B}^*)$ the solution to problem (*) and by t^* the corresponding threshold .

The intuition for the result presented in Lemma 1 is the following. Consider an arbitrary allocation of consumers to the networks. If this is not a threshold allocation, then there are at least two groups of consumers such that the first is allocated to network b , the second to network a , and those in the first group have a larger preference for a than those in the second group. Suppose we move a positive measure of consumers of the first group to network a and a set of consumers of the second group, with the same measure, to network b . Market shares stay constant, the gross surplus of the consumers we moved increases and that of any other consumer is unaffected. Therefore, total welfare increases and this proves that the initial allocation was not welfare maximizing.

In what follows, we will denote the welfare associated to a given threshold allocation as

$$\mathbb{E}_{\mathbf{x}} [W(t)] \equiv \mathbb{E}_{\mathbf{x}} [W(\mathcal{A}(t), \mathcal{B}(t))].$$

Given that the optimal allocation is associated to a threshold, the final step to solve the welfare maximization problem is to identify this optimal threshold. First, notice that the welfare function can be rewritten as the sum of three components:

$$\begin{aligned} \mathbb{E}_{\mathbf{x}} [W(t)] = \mathbb{E}_{\mathbf{x}} & \left[\left(\int_{i \in \mathcal{A}(t)} \theta^a di + \int_{i \in \mathcal{B}(t)} \theta^b di \right) + \right. & (1) \\ & + \left(\int_{i \in \mathcal{A}(t)} \varepsilon_i^a di + \int_{i \in \mathcal{B}(t)} \varepsilon_i^b di \right) + \\ & \left. + \left(\int_{i \in \mathcal{A}(t)} n^a di + \int_{i \in \mathcal{B}(t)} n^b di \right) \right]. \end{aligned}$$

The first component measures the aggregate surplus consumers derive from the vertical quality of the goods, the second measures the aggregate surplus derived from idiosyncratic taste and finally the third component measures the aggregate surplus derived from network effects. By

evaluating expression 1 for a given realization of $(\tilde{\theta}^a, \tilde{\theta}^b)$ and then taking the expectation, the welfare function can be finally be re-written as

$$\begin{aligned} \mathbb{E}_{\mathbf{x}} [W(t)] &= \mathbb{E}_{\theta^a, \theta^b} \left\{ \theta^a - 2\theta \Phi \left((t - \theta) \sqrt{\beta} \right) + \right. \\ &\quad \left. + \sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}} + \right. \\ &\quad \left. + 2 \left[\Phi \left((t - \theta) \sqrt{\beta} \right) \right]^2 - 2\Phi \left((t - \theta) \sqrt{\beta} \right) + 1 \right\} \end{aligned} \quad (2)$$

where $\Phi(\cdot)$ denotes the cdf of a standard normal random variable.⁶

Before we characterize the optimal threshold, it is worth to analyze in larger detail the three components of the welfare function.

For a given realization of $(\tilde{\theta}^a, \tilde{\theta}^b)$, the first component,

$$\theta^a - 2\theta \Phi \left((t - \theta) \sqrt{\beta} \right), \quad (3)$$

measures the aggregate surplus derived from the vertical quality of the goods. Expression 3 can be easily interpreted. If $t = -\infty$ it reduces to θ^a : all consumers are assigned to network a and a measure 1 of individuals enjoy quality θ^a . For a finite t , if we denote by $F(\cdot)$ the cdf of x_i , then only $1 - F(t)$ consumers are assigned to a , while the remaining $F(t)$ consumers are assigned to b and enjoy quality θ^b . Substituting to $F(t)$ its expression and using the variable θ defined in section 2, we get 3. If the realized quality difference θ is positive, it easy to see that 3 is maximized by $t = -\infty$, i.e. by an allocation where all consumers join the high quality network. (See Figure 1⁷).

⁶The derivation of 2 is included in the proof of Proposition 1.

⁷The main purpose of Figure 1 and Figure 2 is to describe the qualitative features of the components of consumer surplus associated with vertical and horizontal quality for a given positive θ that we fix equal to two. For this illustrative purpose, we abstract from the absolute values each of these functions takes for specific values of the parameters (α, β) and specific realizations of (θ^a, θ^b) . Therefore, we do not report the scale of the vertical axis.

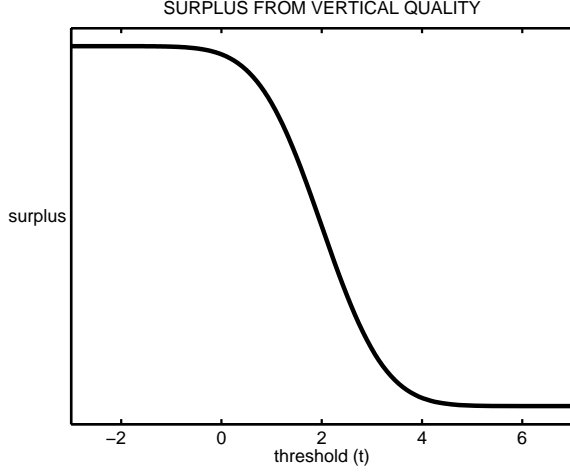


Figure 1: Aggregate surplus derived from the vertical quality of the goods for the case $\theta = 2$.

The second welfare component for a given realization of $(\tilde{\theta}^a, \tilde{\theta}^b)$,

$$\sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}}, \quad (4)$$

measures the portion of aggregate surplus due to consumers' idiosyncratic taste for the network to which they are allocated. Expression 4 is maximized at $t = \theta$. (See Figure. 2). Intuitively, this idiosyncratic surplus component is maximized when each consumer is assigned to the network for which he has an idiosyncratic preference. If θ were perfectly observable, this could be achieved by choosing $t = \theta$, so that every consumer with a positive ε_i would join a and the others would join b .

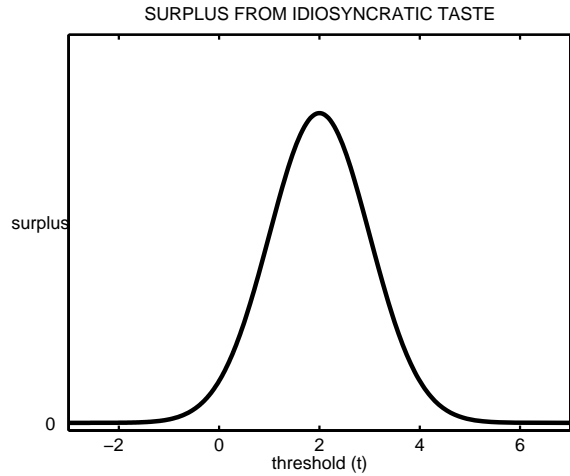


Figure 2: Aggregate surplus derived from idiosyncratic taste for the case $\theta = 2$.

Finally, for given $(\tilde{\theta}^a, \tilde{\theta}^b)$, the last welfare component,

$$2 \left[\Phi \left((t - \theta) \sqrt{\beta} \right) \right]^2 - 2\Phi \left((t - \theta) \sqrt{\beta} \right) + 1 \quad (5)$$

measures the aggregated network effect. Expression 5 can be easily interpreted. The total network effect is given by the sum of the squared market shares (all consumers in a network receive a positive effect measured by the size of the network itself). Denoting again by $F(\cdot)$ the cdf of x_i , for a given threshold t market shares are $n^a = 1 - F(t)$ and $n^b = F(t)$. Substituting to $F(t)$ its expression we get 5.

This component of consumer surplus is maximized by $t \in \{-\infty, +\infty\}$ since for any of those values all consumers join the same network and therefore each of them enjoys the largest possible network effect. (See Figure 3).

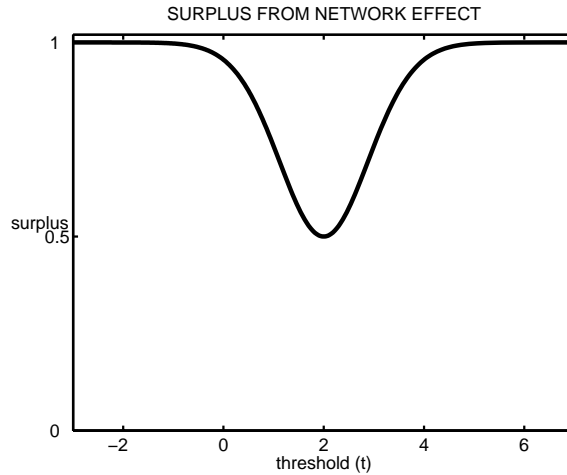


Figure 3: Aggregate surplus derived from the network effect for the case $\theta = 2$.

Given this decomposition, it appears that there is a trade-off between maximizing the first and the third component of welfare by assigning all consumers to the same network and maximizing the second component by splitting consumers equally. Proposition 1 characterizes the socially optimal allocation.

Proposition 1 (Ex-Ante Efficient Allocation)

Social welfare is maximized by an asymmetric threshold allocation such that

$$-1 < t^* < 0 < y.$$

As one would intuitively expect, $t^* < y$: the optimal allocation is such that the network with the largest expected quality has the largest market share.

Nonetheless, the tension between vertical differentiation and network effect on the one hand, and heterogeneity in consumers' taste on the other hand, results in an interior optimum. All the consumers with a positive x_i , who therefore have a private preference for a and who constitute more than one half of the population, and some of those with a small preference for b , should join the large network a . The minority of consumers with a strong preference for b should form another, much smaller network.

4 The Equilibrium Allocation

In this section, we characterize the allocations that prevail in equilibrium in the market, for the case of unsponsored and sponsored networks, in order to compare them to the efficient one that we characterized in Section 3. First, we will solve the coordination game consumers play when they choose which network to join, for given prices. Then, we will consider separately the case where prices are simply equal to the marginal cost and the case where they are set by strategic firms.

4.1 Consumer Coordination

The next result characterizes the equilibrium of the coordination game played by consumers for given price difference p and given expected quality difference y . This coordination game constitutes a global game with correlated private values, such as the one analyzed by Morris and Shin (2004). We can apply their methodology to characterize the equilibrium and identify the necessary and sufficient condition for its uniqueness.

Suppose consumers observe a quadruple $\{p^a, p^b, y^a, y^b\}$. Consider a type $\hat{\mathbf{x}}_i$ of consumer i with the following property: if a consumer has type $\hat{\mathbf{x}}_i$, and he believes that every consumer with a preference for a larger than his own will join a , and every consumer with a preference for a smaller than his own will choose b , then he is indifferent between the two networks. Formally, $\hat{\mathbf{x}}_i(p, y)$ is a solution to:

$$\begin{aligned} & (x_i^a + \Pr(x_{i'} > x_i) | \mathbf{x}_i - p^a) = \\ & = \left(x_i^b + \Pr(x_{i'} \leq x_i) | \mathbf{x}_i - p^b \right). \end{aligned} \tag{6}$$

Let $t(p, y)$ be the intrinsic preference for a of a consumer of type $\hat{\mathbf{x}}_i(p, y)$. That is:

$$t(p, y) \equiv \frac{\hat{x}_i^a(p, y) - \hat{x}_i^b(p, y)}{2}.$$

By rearranging 6, it can be shown that $t(p, y)$ is implicitly defined by the following equation:

$$t - \Phi[(t - y)z] + \frac{1}{2} - p = 0 \quad (7)$$

where

$$z = z(\alpha, \beta) \equiv \sqrt{\frac{\alpha^2 \beta}{(\alpha + \beta)(\alpha + 2\beta)}}. \quad (8)$$

The following proposition holds:

Proposition 2 (Equilibrium of the Coordination Game)

If $z \leq \sqrt{2\pi}$, the coordination game played by consumers after any price announcement (p^a, p^b) has the following unique Nash-equilibrium:

$$s_i^*(\mathbf{x}, \mathbf{p}) = \begin{cases} a & \text{if } x_i > t(p, y), \\ b & \text{if } x_i \leq t(p, y), \end{cases} \quad \forall i \in [0, 1].$$

where $t(p, y)$ satisfies

$$t(p, y) - \Phi[(t(p, y) - y)z] + \frac{1}{2} - p = 0.$$

The threshold $t(p, y)$ is strictly increasing in p and strictly decreasing in y .

The equilibrium described by Proposition 2 is a symmetric equilibrium in switching strategies around the threshold $t(p, y)$: consumers with a preference for a larger than $t(p, y)$ join a and the others join b . Therefore, the equilibrium allocation is a threshold allocation. This will prove particularly useful as we compare the market allocation to the efficient one.

The two available pieces of public information, namely the public signal y about the relative quality of the two goods and the difference p between the two prices, affect the equilibrium threshold in a very intuitive way. For a given quality signal, if a becomes more expensive with respect to b , the threshold moves to the right, i.e. a loses some consumers to b .

For given prices, instead, if a 's expected quality advantage y increases, the threshold moves to the left, i.e. a gains some consumers.

To capture the intuition behind the uniqueness condition mentioned in the statement of Proposition 2, first notice that the above equilibrium in switching strategies is unique if there exists only one value of the threshold t that solves 7. In turn, 7 has a unique solution $t(p, y)$ if the second term is (almost) invariant with respect to t . That term represents the beliefs of a player who observes a private signal exactly equal to the threshold t . More precisely, it represents his expectation of the proportion of consumers with a private value x_i smaller

than his own, i.e. smaller than t . If this expectation is very sensitive to changes in t , than we say that the strategic uncertainty faced by a consumer is very sensitive to his type. As shown by Morris and Shin (2004), uniqueness of equilibrium in global games with correlated private values is guaranteed by a low degree of sensitivity of strategic uncertainty to a player's type. In particular, this low level of sensitivity can be achieved by one of two conditions, both summarized by the inequality $z \leq \sqrt{2\pi}$: either the private signals have to be very correlated or consumers preferences have to be very heterogeneous. In the first case, no matter what specific realization of \tilde{x}_i he observed, any consumer i knows $\tilde{x}_{i'}$ and \tilde{x}_i are highly correlated and therefore thinks that it is about as likely for any consumer i' to observe $x_{i'} > x_i$ or $x_{i'} < x_i$. In the second case, the high variance of the heterogeneous idiosyncratic component makes the private signals almost independent, therefore x_i is not very informative on $x_{i'}$ and so, once again, whatever the realization of \tilde{x}_i he observed consumer i thinks that it is about as likely for any consumer i' to observe $x_{i'} > x_i$ or $x_{i'} < x_i$.

Throughout the rest of the paper we will assume that the condition for uniqueness, $z \leq \sqrt{2\pi}$, is satisfied, focusing our attention on those network markets where the demand function is naturally well-defined. In other words, we will focus on markets where either there is a sufficient amount of horizontal differentiation or the information on the vertical quality of the goods is sufficiently noisy.

Before we consider the two cases of sponsored and unsponsored networks in more detail, we present one more result about the issue of consumer coordination.

Proposition 3 (Impact of Public Information)

Let $z \leq \sqrt{2\pi}$. For any couple (p, y) and for a given realization of θ , in equilibrium the networks realized market shares are

$$n^a = 1 - n^b = \Pr [x_i > t(p, y)] = 1 - \Phi \left[(t(p, y) - \theta) \sqrt{\beta} \right]. \tag{9}$$

Each firm's market share is strictly increasing in her expected quality.

This result has a natural interpretation. For a given θ , the market share of each network is given by the proportion of consumers with a value of x_i above or below the equilibrium threshold. After θ has been drawn, the distribution of x_i is uniquely determined, regardless of what its expected value y was. Therefore, if a network's market share is increasing in her *expected* quality, it has to be the case that the effect comes from a change in the threshold $t(p, y)$. The fact that $t(p, y)$ is actually affected by y might appear counterintuitive because this is a model with private values, where each consumer perfectly (and privately) observes the quality he would derive from the use of each good. Therefore, one might expect that after

observing his own x_i consumer i would discard the noisy information contained in its expected value y . Still, consumer i cares not only about his own x_i but also about the distribution of the private values of every other consumer in the market. This is true because of the presence of network effects: the signals observed by other consumers will affect their choices and i needs to take such choices into account because of the strategic complementarity. This is where y becomes relevant. The prior distribution of each $x_{i'}$ is centered around y . When consumer i observes x_i , and updates his belief about the distribution of $x_{i'}$, whatever the value of x_i he observed, the posterior expectation of $x_{i'}$ is increasing in y . Since in equilibrium consumers with a high $x_{i'}$ choose a , the higher y , the higher a 's expected market share from the point of view of consumer i , the higher his own convenience in choosing a .

4.2 Unsponsored Networks

We now consider the case where the two networks are unsponsored, in order to highlight the possible coordination failure arising on these markets when we abstract from the issue of strategic pricing. Without sponsors, consumers can join any network by paying a price equal to the marginal cost:

$$p^a = p^b = c, \quad \text{and} \quad p = 0.$$

The next proposition describes the equilibrium allocation and compares it to the efficient one.

Proposition 4 (Inefficiency with Unsponsored Networks)

With unsponsored networks, the equilibrium allocation is a threshold allocation with threshold

$$t^u \equiv t(0, y) \in (-0.5, 0).$$

Moreover,

$$t^u > t^* \quad \text{and} \quad \mathbb{E}_{\mathbf{x}} [W(t^u)] < \mathbb{E}_{\mathbf{x}} [W(t^*)].$$

The allocation implemented by the market if the networks are unsponsored shares some qualitative features with the efficient allocation: both are threshold allocations and in both cases, since the threshold is smaller than y , the network with higher expected quality has a market share larger than one half. Moreover, in both cases network a will be composed not only by those consumers with a positive value of x_i , but also by some consumers with a moderate private preference for b .

Nonetheless, the market does not implement the welfare maximizing allocation. In particular, the market allocation is more balanced than the efficient one. The source of this market

failure is the presence of network effects that consumers fail to internalize. To get an intuition for this result, it is useful to revise the decomposition of the welfare function that we presented in Section 3. For any realized level of quality of the two goods, social welfare is the sum of the gross surplus derived from the objective quality of the goods, gross surplus derived from idiosyncratic taste and, finally, gross surplus derived from network effects. The efficient threshold t^* is therefore the result of the compromise among different forces: from a social point of view, horizontal differentiation makes a symmetric allocation more desirable while vertical differentiation and the presence of network externalities make asymmetry more desirable. Since individual consumers do not internalize the network externality, society implements an allocation that is more symmetric than the efficient one.

4.3 Sponsored Networks

After identifying a source of inefficiency arising on network markets, we now ask the question of whether strategic pricing mitigates or aggravates this inefficiency. More precisely, we will now assume that the two networks are sponsored and that therefore access to each of them is priced by a strategic, profit-maximizing firm. Then, we will derive the equilibrium of the two-stage price competition game and compare the allocation of consumers induced in equilibrium to both the ex-ante efficient one and the one implemented by the market in the absence of strategic pricing.

Let

$$v \equiv v(\alpha, \beta) = \sqrt{\frac{\alpha\beta}{\alpha + \beta}}$$

denote the inverse of the standard error of \tilde{x}_i .

The next Lemma describes the demand functions for the two goods.

Lemma 2 (Expected Demand Functions)

Let $z \leq \sqrt{2\pi}$. The expected demand function for each network is well defined for any price couple (p^a, p^b) and it is given by

$$\mathbb{E}_{\mathbf{x}} [n^a] = 1 - \mathbb{E}_{\mathbf{x}} [n^b] = 1 - \Phi [(t(p, y) - y) v]$$

Moreover, for given prices each firm's expected market share is strictly increasing in the expected quality of her product.

The assumption $z \leq \sqrt{2\pi}$ guarantees that there is a unique equilibrium of the coordination game played by consumers in the second stage, therefore the demand function is well defined.

It can be interesting to compare the comparative statics result contained in Lemma 2 with that of Proposition 3. Proposition 3 looks at the market shares for a given realization of θ , and the only reason why they might be a function of y is through a change in $t(p, y)$. Lemma 2 instead, considers the expected market shares given the ex-ante information that is available to the firms when they choose prices: in this case, there are two mechanisms through which a network's expected size is increasing in its expected quality: one is the change in $t(p, y)$, the other is the shift in the distribution of x_i (which is centered around y).

We now have the tools to characterize the pure strategy subgame perfect equilibrium of the price competition game. Substituting the expected demand functions into the profit functions we can write the two firms' optimization problem at time 1 as

$$\begin{aligned}\max_{p^a \in \mathbb{R}} \mathbb{E}_{\mathbf{x}} [\pi_a] &= (p^a - c) [1 - \Phi [(t(p, y) - y)] v] \\ \max_{p^b \in \mathbb{R}} \mathbb{E}_{\mathbf{x}} [\pi_b] &= (p^b - c) \Phi [(t(p, y) - y)] v.\end{aligned}$$

Let p^s denote the value of p associated to the firms' equilibrium strategies and t^s denote $t(p^s, y)$. The following proposition holds:

Proposition 5 (Strategic Pricing Inefficiency)

If a pure strategy SPNE of the price competition game exists, then

$$p^s \in (0, y) \quad \text{and} \quad t^s \in (t^u, y).$$

Moreover,

$$t^* < t^u < t^s$$

and

$$\mathbb{E}_{\mathbf{x}} [W(t^s)] < \mathbb{E}_{\mathbf{x}} [W(t^u)] < \mathbb{E}_{\mathbf{x}} [W(t^*)].$$

As in many models of price competition with differentiated products, it is hard to prove the existence of an equilibrium in pure strategies for generic values of the parameters. A general sufficient condition for existence was derived by Caplin and Nalebuff (1991). Our model satisfies that condition for some limit cases. We have solved the model for a grid of values of the parameters (α, β) that satisfy the condition $z \leq \sqrt{2\pi}$ and in all those cases a pure strategy equilibrium exists and is unique.

The main qualitative features of the equilibrium of the price competition game are consistent with the standard results in duopolistic models of price competition with vertical differentiation⁸: in equilibrium, the firm selling the best product charges the highest price ($p^s > 0$)

⁸See, for example, Shaked and Sutton (1982).

but the difference in quality more than compensates the difference in prices ($y > p^s$), so that she also attracts a fraction of consumers larger than one half.⁹

What is relevant for efficiency, though, is *by how much* this market share exceeds one half. We have already shown that in the absence of strategic pricing the market share of firm a is too small, from a social point of view. According to Proposition 5, a 's market share is even smaller if the two networks are sponsored and this equilibrium provides even less welfare than the equilibrium with unsponsored networks. The reason why strategic pricing reduces welfare is quite straightforward: with vertical differentiation, the firm selling the best product has a natural advantage that is reflected in a higher equilibrium price. In turn, the fact that a is more expensive than b , *ceteris paribus*, shifts some consumers from a to b .

We have therefore identified two separate sources of inefficiency on network markets: the first is the presence of network externalities that consumers do not internalize, which determines a market allocation that is too balanced from a social point of view. The second, which arises only if networks are sponsored, is the presence of strategic pricing, that further reduces the asymmetry between network sizes.

5 Conclusions

We described the optimal allocation of consumers between two vertically and horizontally differentiated networks. We found that social welfare is maximized by an asymmetric allocation such that the network with the highest expected quality has the largest market share. We then compared this allocation with the ones implemented by the market if the networks are unsponsored or sponsored, respectively. Two sources of inefficiency emerged from the analysis: the failure to internalize network externalities and the presence of sponsors that strategically set prices for the products. Both these facts induce the market to implement an allocation of consumers that is too symmetric.

We analyzed efficiency from an ex-ante point of view. One possible alternative is to analyze efficiency from an ex-post point of view, assuming that a benevolent social planner can observe the realized distribution of private values in the population. It can be shown that also in that case the efficient allocation is a threshold allocation such that both networks are active and one of them has a larger market share than the other. The main difference with the ex-ante case is that the ex-post optimal threshold depends on the realization of θ . Also, in the ex-post efficient allocation the network with the largest market share is the one with the highest

⁹Note that in this model, the firm that has an advantage is not literally the firm “selling the best product” but is the firm that is expected to sell the best product.

realized quality, while in the ex-ante case it is the one with the highest *expected* quality.

Comparing the equilibrium allocation to the ex-post efficient allocation, the main result is that strategic pricing can increase welfare if the firm with the highest expected quality has the lowest realized quality and decrease welfare in the opposite case. The intuition for this result is the following. If the public signal is in favor of, say, firm a , in equilibrium she will charge a price higher than the competitor and ex-post this will negatively affect her market share (everything else equal). Therefore, if ex-post a is the best firm, strategic pricing shifts some consumers towards the worst firm, while if a is the worst firm, then strategic pricing shifts some consumers towards the best firm.

In our model, we assumed that consumer heterogeneity is unbounded while the utility derived from the network effect is bounded. The result that the ex-ante efficient threshold is an interior optimum holds under more general assumptions. For symmetric, bounded network effects, if the distribution of x_i is continuous and symmetric around the mean, then it is sufficient that the upper bound of the support of x_i is weakly larger than the individual surplus from being in a network of size one.

We also assumed complete market coverage. This assumption allowed us to abstract from the possible deadweight loss associated to strategic pricing and to focus only on the inefficiency arising from the difference in equilibrium prices.

Finally, we assumed that firms have access only to the public information available about the vertical quality of the products. A natural extension of the model would be to allow each firm to observe a private signal about the vertical quality of her product. This would add the issue of how firms use prices to signal their quality.

We leave for future research an extension of this model to a dynamic environment, where consumers choose sequentially and firms can adjust prices over time.

6 Appendix

The Appendix collects the proofs of all the results presented in the paper.

Proof of Lemma 1 The welfare function can be rewritten as

$$\begin{aligned} & \mathbb{E}_{\mathbf{x}} \left[\int_{i \in \mathcal{A}} (x_i^a + n^a) di + \int_{i \in \mathcal{B}} (x_i^b + n^b) di \right] = \\ &= \mathbb{E}_{\mathbf{x}} \left[\int_{i \in \mathcal{A}} n^a di + \int_{i \in \mathcal{A}} x_i^a di + \int_{i \in \mathcal{B}} n^b di + \int_{i \in \mathcal{B}} x_i^b di \right] = \\ &= \mathbb{E}_{\mathbf{x}} \left[[n^a]^2 + [n^b]^2 + \int_{i \in \mathcal{A}} x_i^a di + \int_{i \in \mathcal{B}} x_i^b di \right]. \end{aligned}$$

We will prove by contradiction that any allocation $(\mathcal{A}', \mathcal{B}')$ that is not a threshold allocation cannot maximize the welfare function. By construction, there exists no $X \in \mathbf{R}$ such that $\mathcal{B}' = \{i \in [0, 1] : x_i \leq X\}$ and $\mathcal{A}' = \{i \in [0, 1] : x_i > X\}$. Therefore, there exists at least a couple $(\mathcal{S}, \mathcal{T})$ such that $\mathcal{S} \subset \mathcal{A}$, $\mathcal{T} \subset \mathcal{B}$, $\int_{i \in \mathcal{S}} di = \int_{i \in \mathcal{T}} di$ and $x_i < x_{i'}$ for every (i, i') such that $i \in \mathcal{S}$ and $i' \in \mathcal{T}$. Consider now a different allocation $(\mathcal{A}'', \mathcal{B}'')$ such that $\mathcal{A}'' = (\mathcal{A}'/\mathcal{S}) \cup \mathcal{T}$ and $\mathcal{B}'' = (\mathcal{B}'/\mathcal{T}) \cup \mathcal{S}$. We will show that this allocation yields a larger welfare than $(\mathcal{A}', \mathcal{B}')$. Market shares are the same under both allocations, therefore the utility of every consumer in \mathcal{A}'/\mathcal{S} and in \mathcal{B}'/\mathcal{T} is the same. The utility of each individual $i \in \mathcal{S}$ is equal to $n^a + x_i^a$ under allocation $(\mathcal{A}', \mathcal{B}')$ and to $n^b + x_i^b$ under allocation $(\mathcal{A}'', \mathcal{B}'')$. Analogously, the utility of each consumer $i \in \mathcal{T}$ is $n^b + x_i^b$ under allocation $(\mathcal{A}', \mathcal{B}')$ and $n^a + x_i^a$ under allocation $(\mathcal{A}'', \mathcal{B}'')$. Therefore, the total change in welfare when moving from allocation $(\mathcal{A}', \mathcal{B}')$ to allocation $(\mathcal{A}'', \mathcal{B}'')$ is

$$\begin{aligned} & \mathbb{E}_{\mathbf{x}} \left[\int_{i \in \mathcal{S}} (n^b + x_i^b - n^a - x_i^a) di + \int_{i \in \mathcal{T}} (n^a + x_i^a - n^b - x_i^b) di \right] = \\ &= \mathbb{E}_{\mathbf{x}} \left[(n^b - n^a) \left(\int_{i \in \mathcal{S}} di - \int_{i \in \mathcal{T}} di \right) - \int_{i \in \mathcal{S}} 2x_i di + \int_{i \in \mathcal{T}} 2x_i di \right] = \\ &= 0 - \mathbb{E}_{\mathbf{x}} \left[\int_{i \in \mathcal{S}} 2x_i di + \int_{i \in \mathcal{T}} 2x_i di \right] > 0 \end{aligned}$$

where the last inequality holds because $x_i < x_{i'}$ for every (i, i') such that $i \in \mathcal{S}$ and $i' \in \mathcal{T}$. ■

Proof of Proposition 1 The proof of Proposition 1 is divided into 4 claims. In Claim 1 we derive expression 2 for the welfare function. In Claim 2 we prove that the welfare function has a global maximum and the maximum is attained at a point $t \in (-1, 1)$. In Claim 3 we prove that the optimum cannot be in the interval $t \in [0, y]$. For the case $y \geq 1$, this completes the proof. Claim 4 considers the case where $y < 1$ and shows that it cannot be the case that $t^* \in (y, 1)$. This concludes the proof.

Claim 1: Expression 2 represents the ex-ante welfare function associated to a generic threshold allocation.

For a given realization of $(\tilde{\theta}^a, \tilde{\theta}^b)$, take a generic threshold allocation $(\mathcal{A}(t), \mathcal{B}(t))$. Let $I(\cdot)$ denote the indicator function. The level of welfare associated to $(\mathcal{A}(t), \mathcal{B}(t))$ is

$$\begin{aligned}
& \mathbb{E}_{\mathbf{x}} \left[W(t) \mid \theta^a, \theta^b \right] = \\
&= \int_{-\infty}^{t-\theta} \theta^b f(\varepsilon_i) d\varepsilon_i + \int_{t-\theta}^{+\infty} \theta^a f(\varepsilon_i) d\varepsilon_i + \\
&+ \int_{-\infty}^{t-\theta} (1 - n^a) f(\varepsilon_i) d\varepsilon_i + \int_{t-\theta}^{+\infty} n^a f(\varepsilon_i) d\varepsilon_i + \\
&+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varepsilon_i^a I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \geq t - \theta\right) f(\varepsilon_i^a) f(\varepsilon_i^b) d\varepsilon_i^a d\varepsilon_i^b + \\
&+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varepsilon_i^b I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \leq t - \theta\right) f(\varepsilon_i^a) f(\varepsilon_i^b) d\varepsilon_i^a d\varepsilon_i^b = \\
&= \theta^b \int_{-\infty}^{t-\theta} f(\varepsilon_i) d\varepsilon_i + \theta^a \int_{t-\theta}^{+\infty} f(\varepsilon_i) d\varepsilon_i + \\
&+ (1 - n^a) \int_{-\infty}^{t-\theta} f(\varepsilon_i) d\varepsilon_i + n^a \int_{t-\theta}^{+\infty} f(\varepsilon_i) d\varepsilon_i + \\
&+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varepsilon_i^a I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \geq t - \theta\right) f(\varepsilon_i^a) f(\varepsilon_i^b) d\varepsilon_i^a d\varepsilon_i^b + \\
&+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varepsilon_i^b I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \leq t - \theta\right) f(\varepsilon_i^a) f(\varepsilon_i^b) d\varepsilon_i^a d\varepsilon_i^b = \\
&= \theta^b \Phi\left((t - \theta) \sqrt{\beta}\right) + \theta^a \left[1 - \Phi\left((t - \theta) \sqrt{\beta}\right)\right] + \\
&(1 - n^a) \Phi\left((t - \theta) \sqrt{\beta}\right) + n^a \left[1 - \Phi\left((t - \theta) \sqrt{\beta}\right)\right] + \\
&+ \int_{-\infty}^{+\infty} \left[\int_{\varepsilon_i^b + 2(t - \theta)}^{+\infty} \varepsilon_i^a f(\varepsilon_i^a) d\varepsilon_i^a \right] f(\varepsilon_i^b) d\varepsilon_i^b + \\
&+ \int_{-\infty}^{+\infty} \left[\int_{\varepsilon_i^a - 2(t - \theta)}^{+\infty} \varepsilon_i^b f(\varepsilon_i^b) d\varepsilon_i^b \right] f(\varepsilon_i^a) d\varepsilon_i^a =
\end{aligned}$$

$$\begin{aligned}
&= \theta^b \Phi \left((t - \theta) \sqrt{\beta} \right) + \theta^a \left[1 - \Phi \left((t - \theta) \sqrt{\beta} \right) \right] + \\
&\quad + \left[\Phi \left((t - \theta) \sqrt{\beta} \right) \right]^2 + \left[1 - \Phi \left((t - \theta) \sqrt{\beta} \right) \right]^2 + \\
&\quad + \int_{-\infty}^{+\infty} \left[\sqrt{\frac{2}{\beta}} \phi \left(\left(\varepsilon_i^b + 2(t - \theta) \right) \sqrt{\frac{\beta}{2}} \right) \right] f \left(\varepsilon_i^b \right) d\varepsilon_i^b + \\
&\quad + \int_{-\infty}^{+\infty} \left[\sqrt{\frac{2}{\beta}} \phi \left(\left(\varepsilon_i^a - 2(t - \theta) \right) \sqrt{\frac{\beta}{2}} \right) \right] f \left(\varepsilon_i^a \right) d\varepsilon_i^a = \\
&\quad = \Phi \left((t - \theta) \sqrt{\beta} \right) \left[\theta^b - \theta^a \right] + \theta^a + \\
&\quad + 2 \left[\Phi \left((t - \theta) \sqrt{\beta} \right) \right]^2 - 2\Phi \left((t - \theta) \sqrt{\beta} \right) + 1 + \\
&\quad \quad + \sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}}.
\end{aligned}$$

Therefore, the expected welfare as a function of a generic threshold t is:

$$\begin{aligned}
\mathbb{E}_{\mathbf{x}} [W(t)] &= \mathbb{E}_{\theta^a, \theta^b} \left\{ \theta^a - 2\theta \Phi \left((t - \theta) \sqrt{\beta} \right) + \right. \\
&\quad \left. + \sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}} + \right. \\
&\quad \left. + 2 \left[\Phi \left((t - \theta) \sqrt{\beta} \right) \right]^2 - 2\Phi \left((t - \theta) \sqrt{\beta} \right) + 1 \right\}.
\end{aligned}$$

Claim 2: The welfare function has a global maximum and the maximum is attained at a point $t \in (-1, 1)$.

Given Claim 1, the optimal threshold t^* solves the following problem:

$$\begin{aligned}
\max_t \mathbb{E}_{\mathbf{x}} [W(t)] &= \max_t \mathbb{E}_{\theta^a, \theta^b} \left\{ \left[\theta^a - 2\theta \Phi \left((t - \theta) \sqrt{\beta} \right) \right] + \right. \\
&\quad \left. + \left[\sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}} \right] + \right. \\
&\quad \left. + \left[2 \left[\Phi \left((t - \theta) \sqrt{\beta} \right) \right]^2 - 2\Phi \left((t - \theta) \sqrt{\beta} \right) + 1 \right] \right\}.
\end{aligned}$$

The welfare function is continuous and differentiable in t , and by Leibnitz' rule, its first derivative can be written as:

$$\begin{aligned}
&\frac{\partial}{\partial t} \mathbb{E}_{\mathbf{x}} [W(t)] = \\
&= \mathbb{E}_{\theta^a, \theta^b} \left[\left(2\Phi \left((t - \theta) \sqrt{\beta} \right) - 1 - \theta - t + \theta \right) \left(2\sqrt{\beta} \phi \left((t - \theta) \sqrt{\beta} \right) \right) \right] = \\
&= \mathbb{E}_{\theta^a, \theta^b} \left[\left(2\Phi \left((t - \theta) \sqrt{\beta} \right) - 1 - t \right) \left(2\sqrt{\beta} \phi \left((t - \theta) \sqrt{\beta} \right) \right) \right].
\end{aligned}$$

Notice that

$$2\sqrt{\beta}\phi\left((t-\theta)\sqrt{\beta}\right) > 0 \quad \forall\theta.$$

This implies that, since for every $t \leq -1$

$$2\Phi\left((t-\theta)\sqrt{\beta}\right) - 1 - t > 0 \quad \forall\theta$$

then it is true that for every $t \leq -1$

$$\mathbb{E}_{\theta^a, \theta^b} \left[\left(2\Phi\left((t-\theta)\sqrt{\beta}\right) - 1 - t \right) \left(2\sqrt{\beta}\phi\left((t-\theta)\sqrt{\beta}\right) \right) \right] > 0.$$

So the derivative is always positive for any $t \leq -1$.

Analogously, since for every $t \geq 1$ it is true that

$$2\Phi\left((t-\theta)\sqrt{\beta}\right) - 1 - t < 0 \quad \forall\theta,$$

then for every $t \geq 1$ it is also true that

$$\mathbb{E}_{\theta^a, \theta^b} \left[\left(2\Phi\left((t-\theta)\sqrt{\beta}\right) - 1 - t \right) \left(2\sqrt{\beta}\phi\left((t-\theta)\sqrt{\beta}\right) \right) \right] < 0,$$

i.e. the derivative is always negative for any $t \geq 1$.

If we restrict the domain of the welfare function to any set $[t_1, t_2]$ such that $(-1, 1) \subset [t_1, t_2]$, the function has a global maximum $t^* \in [t_1, t_2]$ by Weierstrass theorem. If then we extend the domain to \mathbb{R} , it is true that t^* is still the global maximum because $\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(-1)] \forall t < t_1$ and $\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(1)] \forall t > t_2$. By a similar argument, it has to be the case that $t^* \in (-1, 1) \subset [t_1, t_2]$ because

$\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(-1)] \forall t$ such that $t_1 < t \leq -1$ and $\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(1)] \forall t$ such that $1 \leq t < t_2$.

Claim 3 The optimum cannot be in the interval $t \in [0, y]$.

By the definition of a derivative,

$$\begin{aligned} \frac{\partial}{\partial t} \mathbb{E}_{\mathbf{x}}[W(t)] &= \lim_{\delta \rightarrow 0} \frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta} = \\ &= \lim_{\delta \rightarrow 0^+} \frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta}. \end{aligned}$$

Let $F(\cdot)$ denote the cdf of the random variable x_i .

Notice that

$$\begin{aligned} \mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)] &= F(t)[F(t+\delta) - F(t)] - [1 - F(t+\delta)][F(t+\delta) - F(t)] + \\ &+ [F(t+\delta) - F(t)][F(t+\delta) - 1 + F(t)] + \int_t^{t+\delta} -2x_i f(x_i) dx_i. \end{aligned}$$

Let

$$H(\delta) \equiv F(t)[F(t+\delta) - F(t)] - [1 - F(t+\delta)][F(t+\delta) - F(t)] + \\ + [F(t+\delta) - F(t)][F(t+\delta) - 1 + F(t)] - 2t[F(t+\delta) - F(t)].$$

Since

$$- \int_t^{t+\delta} 2x_i f(x_i) dx_i < -2t \int_t^{t+\delta} f(x_i) dx_i = -2t[F(t+\delta) - F(t)] \quad \forall \delta > 0,$$

then

$$\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)] < H(\delta) \quad \forall \delta > 0,$$

which implies

$$\frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta} < \frac{H(\delta)}{\delta} \quad \forall \delta > 0.$$

Taking the limit:

$$\begin{aligned} \frac{\partial}{\partial t} \mathbb{E}_{\mathbf{x}}[W(t)] &= \lim_{\delta \rightarrow 0^+} \frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta} \leq \lim_{\delta \rightarrow 0^+} \frac{H(\delta)}{\delta} = \\ &= \lim_{\delta \rightarrow 0^+} \frac{\{F(t)[F(t+\delta) - F(t)] - [1 - F(t+\delta)][F(t+\delta) - F(t)]\}}{\delta} + \\ &+ \lim_{\delta \rightarrow 0^+} \frac{\{[F(t+\delta) - F(t)][F(t+\delta) - 1 + F(t)] - 2t[F(t+\delta) - F(t)]\}}{\delta} = \\ &= \lim_{\delta \rightarrow 0^+} \frac{2\{[F(t+\delta) - F(t)][F(t) - 1 + F(t+\delta)] - t[F(t+\delta) - F(t)]\}}{\delta}. \end{aligned} \quad (10)$$

By l'Hôpital's rule, 10 is equal to:

$$\begin{aligned} 2 \lim_{\delta \rightarrow 0^+} \{f(t+\delta)[F(t) - 1 + F(t+\delta)] + [F(t+\delta) - F(t)]f(t+\delta) - tf(t+\delta)\} = \\ = 2f(t)\{2F(t) - 1 - t\} < 0 \end{aligned}$$

The last inequality holds because $y > 0$ and $t \in [0, y]$.

Claim 4. It cannot be the case that $t^* \in (y, 1)$.

We will prove Claim 4 by proving that for any $t > y$ there exists a $t' < y$ such that $\mathbb{E}_{\mathbf{x}}[W(t')] > \mathbb{E}_{\mathbf{x}}[W(t)]$.

Let $t \in (y, 1)$ and take $t' < y$ such that

$$|y - t'| = |y - t| \equiv \Delta.$$

It holds that

$$\begin{aligned} \mathbb{E}_{\mathbf{x}}[W(t')] - \mathbb{E}_{\mathbf{x}}[W(t)] = \\ = \mathbb{E}_{\theta^a, \theta^b} \left[\theta^a - 2\theta\Phi\left((t' - \theta)\sqrt{\beta}\right) \right] - \mathbb{E}_{\theta^a, \theta^b} \left[\theta^a - 2\theta\Phi\left((t - \theta)\sqrt{\beta}\right) \right] + \end{aligned}$$

$$\begin{aligned}
& +\mathbb{E}_{\theta^a, \theta^b} \left[\sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t'-\theta)^2\beta}{2}} \right] - \mathbb{E}_{\theta^a, \theta^b} \left[\sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}} \right] + \\
& +\mathbb{E}_{\theta^a, \theta^b} \left[\left[2 \left[\Phi \left((t' - \theta) \sqrt{\beta} \right) \right]^2 - 2\Phi \left((t' - \theta) \sqrt{\beta} \right) + 1 \right] \right] + \\
& -\mathbb{E}_{\theta^a, \theta^b} \left[\left[2 \left[\Phi \left((t - \theta) \sqrt{\beta} \right) \right]^2 - 2\Phi \left((t - \theta) \sqrt{\beta} \right) + 1 \right] \right] = \\
& = \mathbb{E}_\theta \left[-2\theta\Phi \left((t' - \theta) \sqrt{\beta} \right) \right] - \mathbb{E}_\theta \left[-2\theta\Phi \left((t - \theta) \sqrt{\beta} \right) \right].
\end{aligned}$$

The last equality holds because both

$$\mathbb{E}_{\theta^a, \theta^b} \left[\theta^a - 2\theta\Phi \left((t - \theta) \sqrt{\beta} \right) \right]$$

and

$$\mathbb{E}_{\theta^a, \theta^b} \left[\sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}} \right]$$

are symmetric with respect to y .

To establish the sign of

$$= \mathbb{E}_\theta \left[-2\theta\Phi \left((t' - \theta) \sqrt{\beta} \right) \right] - \mathbb{E}_\theta \left[-2\theta\Phi \left((t - \theta) \sqrt{\beta} \right) \right]$$

rewrite the expression as

$$\int_{-\infty}^{+\infty} 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta$$

where $g(\theta)$ denotes the pdf of the random variable θ .

Notice that

$$\begin{aligned}
& \int_{-\infty}^{+\infty} 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta = \\
& = \int_{-\infty}^0 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta + \\
& + \int_0^{2y} 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta + \\
& + \int_{2y}^{+\infty} 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta.
\end{aligned}$$

It holds that

$$\int_0^{2y} 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta > 0$$

since the integrand is nonnegative $\forall \theta \in [0, 2y]$ and strictly positive $\forall \theta \in (0, 2y]$.

Moreover,

$$\int_{-\infty}^0 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta \leq 0$$

because the integrand is nonpositive $\forall \theta \in (-\infty, 0]$ and

$$\int_0^{2y} 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta \geq 0$$

because the integrand is nonnegative $\forall \theta \in [2y, +\infty)$.

Next, we show that

$$\begin{aligned} & \int_{-\infty}^0 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta + \\ & + \int_0^{2y} 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta > 0. \end{aligned}$$

Take any $\theta \in [2y, +\infty)$ and denote $\theta - 2y = d > 0$.

For any such θ , there exists $\theta' \in (-\infty, 0]$ such that $\theta' = -d$. By symmetry of the distribution of θ , $g(\theta') = g(\theta)$.

Moreover,

$$\begin{aligned} & 2\theta' \left[\Phi \left((y + \Delta - \theta') \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta') \sqrt{\beta} \right) \right] + \\ & + 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] = \\ & = -2d \left[\Phi \left((y + d + \Delta) \sqrt{\beta} \right) - \Phi \left((y + d - \Delta) \sqrt{\beta} \right) \right] + \\ & + (4y + 2d) \left[\Phi \left((-y - d + \Delta) \sqrt{\beta} \right) - \Phi \left((-y - d - \Delta) \sqrt{\beta} \right) \right] = \\ & = 4y \left[\Phi \left((-y - d + \Delta) \sqrt{\beta} \right) - \Phi \left((-y - d - \Delta) \sqrt{\beta} \right) \right] > 0. \end{aligned}$$

Since $\forall \theta \in [2y, +\infty)$ there exists $\theta' \in (-\infty, 0]$ such that

$$\begin{aligned} & 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) + \\ & + 2\theta' \left[\Phi \left((y + \Delta - \theta') \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta') \sqrt{\beta} \right) \right] g(\theta') > 0, \end{aligned}$$

then

$$\begin{aligned} & \int_{-\infty}^0 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta + \\ & + \int_0^{2y} 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta \geq 0 \end{aligned}$$

and combining this inequality with

$$+ \int_0^{2y} 2\theta \left[\Phi \left((y + \Delta - \theta) \sqrt{\beta} \right) - \Phi \left((y - \Delta - \theta) \sqrt{\beta} \right) \right] g(\theta) d\theta > 0$$

we get

$$\mathbb{E}_\theta \left[-2\theta \Phi \left((t' - \theta) \sqrt{\beta} \right) \right] - \mathbb{E}_\theta \left[-2\theta \Phi \left((t - \theta) \sqrt{\beta} \right) \right] > 0.$$

This implies that if $0 < y \leq 1$ it cannot be the case that the welfare function attains its global maximum for $t \in (y, 1]$.

We conclude that both if $y \in (0, 1]$ and if $y > 1$ it has to be the case that the global maximizer of the welfare function, t^* , satisfies

$$-1 < t^* < 0 < y.$$

■

Proof of Proposition 2 The characterization of the equilibrium and derivation of the uniqueness condition follow directly from the Appendix of Morris and Shin (2004).

To evaluate the sign of $\frac{\partial t(p, y)}{\partial p}$, and $\frac{\partial t(p, y)}{\partial y}$, notice that since $t(p, y)$ has to satisfy expression 7, then:

$$\begin{aligned} \frac{\partial t(p, y)}{\partial p} &= - \frac{\frac{\partial [t - \Phi[(t-y)z] + \frac{1}{2} - p]}{\partial p}}{\frac{\partial [t - \Phi[(t-y)z] + \frac{1}{2} - p]}{\partial t}} = - \frac{-1}{1 - \phi[(t-y)z]z} \\ \frac{\partial t(p, y)}{\partial y} &= - \frac{\frac{\partial [t - \Phi[(t-y)z] + \frac{1}{2} - p]}{\partial y}}{\frac{\partial [t - \Phi[(t-y)z] + \frac{1}{2} - p]}{\partial t}} = - \frac{\phi[(t-y)z]z}{1 - \phi[(t-y)z]z}. \end{aligned}$$

When the uniqueness condition is satisfied, the denominators of the two expressions above are strictly positive, therefore

$$\begin{aligned} \frac{\partial t(p, y)}{\partial p} &> 0 \\ \frac{\partial t(p, y)}{\partial y} &< 0. \end{aligned}$$

■

Proof of Proposition 3 If consumers play the equilibrium strategy profile, then all the consumers who observe $x_i > t(p, y)$ and who therefore have an idiosyncratic taste component $\varepsilon_i > t(p, y) - \theta$, buy a and everyone else buys b .

For the second part of the claim, note that

$$\begin{aligned}\frac{\partial n^a(\cdot)}{\partial y^a} &= \frac{\partial n^a}{\partial y} \frac{1}{2} = \left(\frac{1}{2}\right) \left\{ -\phi \left[(t(p, y) - \theta) \sqrt{\beta} \right] \sqrt{\beta} \left(\frac{\partial t}{\partial y} \right) \right\} = \\ &= \left(\frac{1}{2}\right) \left\{ -\phi \left[(t(p, y) - \theta) \sqrt{\beta} \right] \sqrt{\beta} \left(-\frac{\phi \left[(t-y) z \right] z}{1 - \phi \left[(t-y) z \right] z} \right) \right\} > 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial n^b(\cdot)}{\partial y^b} &= \frac{\partial n^b}{\partial y} \left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right) \left\{ \phi \left[(t(p, y) - \theta) \sqrt{\beta} \right] \sqrt{\beta} \left(\frac{\partial t}{\partial y} \right) \right\} = \\ &= \left(-\frac{1}{2}\right) \left\{ \phi \left[(t(p, y) - \theta) \sqrt{\beta} \right] \sqrt{\beta} \left(-\frac{\phi \left[(t-y) z \right] z}{1 - \phi \left[(t-y) z \right] z} \right) \right\} > 0.\end{aligned}$$

■

Proof of Proposition 4 For the first statement, notice that the equilibrium threshold for the case of unsponsored networks, $t^u \equiv t(0, y)$, satisfies

$$t^u - \Phi \left[(t^u - y) z \right] + \frac{1}{2} = 0. \quad (11)$$

Under the uniqueness condition, the lhs of 11 is monotonically increasing in t and t^u is well-defined. Since $\Phi[\cdot] \in [0, 1]$, for any $t \geq 0$ it holds that the lhs of 11 is strictly negative if $t \leq -\frac{1}{2}$. Therefore, it has to be the case that $t^u > \frac{1}{2}$. Moreover, for $y = 0$ the solution to 11 is $t^u = 0$, and since $t(p, y)$ is strictly decreasing in y , it has to be the case that $t^u \in (-\frac{1}{2}, 0)$.

For the second and the third statement, we know from Proposition 1 that $t^* \in (-1, 0)$. Next, we will prove that the welfare function is strictly decreasing in the interval $t \in [t^u, 0]$, which in turn implies that $t^* \in (-1, t^u)$ and that t^u cannot be a point where the function attains its global maximum.

Let $t = t^u$. From the proof of Proposition 1, we know:

$$\begin{aligned}\frac{\partial}{\partial t} \mathbb{E}_{\mathbf{x}} [W(t)] &= \lim_{\delta \rightarrow 0} \frac{\mathbb{E}_{\mathbf{x}} [W(t + \delta)] - \mathbb{E}_{\mathbf{x}} [W(t)]}{\delta} = \\ &= \lim_{\delta \rightarrow 0^+} \frac{\mathbb{E}_{\mathbf{x}} [W(t + \delta)] - \mathbb{E}_{\mathbf{x}} [W(t)]}{\delta} = \\ &= 2f(t^u) \{2F(t^u) - 1 - t^u\}\end{aligned} \quad (12)$$

We will show that

$$2f(t^u) \{2F(t^u) - 1 - t^u\} < 0$$

and that therefore

$$\frac{\partial}{\partial t} \mathbb{E}_{\mathbf{x}} [W(t)] < 0.$$

First, notice that 12 has the same sign as $2F(t^u) - 1 - t^u$, since $f(t^u) > 0$.

From the definition of t^u

$$t^u = \Phi[(t^u - y)z] - \frac{1}{2}.$$

By assumption, $F(t^u) = \Phi[(t^u - y)v]$ where

$$v \equiv v(\alpha, \beta) = \sqrt{\frac{\alpha\beta}{\alpha + \beta}}.$$

Therefore, we need to check the sign of

$$\begin{aligned} & 2\Phi[(t^u - y)v] - 1 - \Phi[(t^u - y)z] + \frac{1}{2} = \\ & = \Phi[(t^u - y)v] - \Phi[(t^u - y)z] + \Phi[(t^u - y)v] - \frac{1}{2}. \end{aligned}$$

Since $v > z$ and $t^u - y < 0$, $\Phi[(t^u - y)v] - \Phi[(t^u - y)z] < 0$ and $\Phi[(t^u - y)v] < \frac{1}{2}$, therefore 12 is strictly negative, and so is the derivative of the welfare function evaluated at t^u .

Finally, let $t \in (t^u, 0]$. We will prove that in this interval $\frac{\partial}{\partial t} \mathbb{E}_{\mathbf{x}}[W(t)] < 0$ as well. As shown above, it holds that

$$\begin{aligned} \frac{\partial}{\partial t} \mathbb{E}_{\mathbf{x}}[W(t)] < 0 &= \lim_{\delta \rightarrow 0^+} \frac{\mathbb{E}_{\mathbf{x}}[W(t + \delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta} \leq \lim_{\delta \rightarrow 0^+} \frac{H(\delta)}{\delta} = \\ &= 2f(t) \{2F(t) - 1 - t\} \end{aligned}$$

and that the last expression has the same sign as $2F(t) - 1 - t$. The function $2F(t) - 1 - t$ is the difference of the two strictly increasing functions $r(t) = 2F(t)$ and $s(t) = 1 + t$. This difference is positive for $t < -1$, negative for $t = t^u$ and negative in $t = 0$ (since $F(\cdot)$ is the cdf of x_i which is distributed normally around $y > 0$).

Since the slope of $s(t)$ is constant and the slope of $r(t)$ is increasing in the interval $t \in (-1, 0]$, it cannot be the case that their difference is nonnegative in any point $t \in (t^u, 0)$. Therefore, it holds that $2F(t) - 1 - t < 0$ in the interval $t \in (t^u, 0]$, which in turn implies that the welfare function is strictly decreasing in the interval $t \in [t^u, 0]$.

We can conclude that $t^* \in (-1, t^u)$ and that t^u cannot be a point where the function attains its global maximum. ■

Proof of Lemma 2 To calculate the expected market shares, it is sufficient to take the expectation over θ of the market shares expressed in Proposition 3. For the second part of

the claim, note that

$$\begin{aligned}\frac{\partial \mathbb{E}_{\mathbf{x}} [n^a]}{\partial y^a} &= \frac{\partial \mathbb{E}_{\mathbf{x}} [n^a]}{\partial y} \frac{1}{2} = \left(\frac{1}{2}\right) \left\{ -\phi [(t(p, y) - y) v] v \left(\frac{\partial t}{\partial y} - 1\right) \right\} = \\ &= \left(\frac{1}{2}\right) \left\{ -\phi [(t(p, y) - y) v] v \left(-\frac{\phi [(t-y) z] z}{1 - \phi [(t-y) z] z} - 1\right) \right\} > 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathbb{E}_{\mathbf{x}} [n^b]}{\partial y^b} &= \frac{\partial \mathbb{E}_{\mathbf{x}} [n^b]}{\partial y} \left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right) \left\{ \phi [(t(p, y) - y) v] v \left(\frac{\partial t}{\partial y} - 1\right) \right\} = \\ &= \left(-\frac{1}{2}\right) \left\{ \phi [(t(p, y) - y) v] v \left(-\frac{\phi [(t-y) z] z}{1 - \phi [(t-y) z] z} - 1\right) \right\} > 0.\end{aligned}$$

■

Proof of Proposition 5 First, we prove that if a pure strategy SPNE exists, then it has to be the case that $p^s \in (0, y)$.

The first order conditions of the firms optimization problem are

$$\begin{aligned}\frac{\partial \mathbb{E}_{\mathbf{x}} [\pi_a]}{\partial p^a} &= 1 - \Phi [(t(p, y) - y) v] - (p^a - c) \phi [(t(p, y) - y) v] \frac{\partial t}{\partial p} \frac{v}{2} = 0 \\ \frac{\partial \mathbb{E}_{\mathbf{x}} [\pi_b]}{\partial p^b} &= \Phi [(t(p, y) - y) v] - (p^b - c) \phi [(t(p, y) - y) v] \frac{\partial t}{\partial p} \frac{v}{2} = 0.\end{aligned}$$

Both must be satisfied in equilibrium. Therefore, it has to be the case that (p^a, p^b) and $p^s = \frac{p^a - p^b}{2}$ satisfy:

$$1 - 2\Phi [(t(p, y) - y) v] = (p^a - p^b) \phi [(t(p, y) - y) v] \frac{\partial t}{\partial p} \frac{v}{2}. \quad (13)$$

First, note that 13 can't be satisfied if $p^s < 0$ because that would imply that the lhs is positive and the rhs is negative. Also, it cannot be the case that $p^s = 0$ in equilibrium because in that case the lhs would be positive and the rhs would be equal to zero. Moreover, it cannot be that $p^s = y$ in equilibrium because in that case the lhs would be equal to zero and the right hand side would be positive. Finally, it cannot be that $p^s > y$ in equilibrium because in that case the lhs would be negative and the rhs would be positive. Therefore, if a pure strategy SPNE exists it has to be such that $p^s \in (0, y)$. Notice that both the lhs and the rhs of 13 are continuous functions of p , therefore $G(p^s) \equiv lhs - rhs$ is continuous as well. Moreover, $G(p)$ is positive at $p = 0$ and is negative at $p = y$, therefore it must have at least one zero in the interval $p \in (0, y)$.

Next, we prove that $t^s \in (t^u, y)$.

From Proposition 2, $t(p, y)$ is strictly increasing in p and since $p^s > 0$ then it has to be the case that $t^u \equiv t(0, y) < t(p^s, y) \equiv t^s$.

From the definition of $t(p, y)$, it follows that $t(y, y) = y$. Since $t(p, y)$ is strictly increasing in p and $p^s < y$, then it has to be the case that $t^s < y$.

Finally, we need to show that $\mathbb{E}_{\mathbf{x}}[W(t^u)] < \mathbb{E}_{\mathbf{x}}[W(t^s)]$. This result holds because, as we have shown in the proof of Proposition 4, $\mathbb{E}_{\mathbf{x}}[W(t)]$ is strictly decreasing in the interval $t \in [t^u, y)$ and $t^s \in (t^u, y)$. ■

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