# Epiphany in the Game of 21 

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#### Abstract

We suggest that performance in strategic settings depends on whether players realize that an optimal way to play may be feasible. We introduce a zero-sum game of perfect information, simple enough to allow computation of optimal play yet sufficiently complicated that most participants initially fail. The borderline solvability-by-humans makes it a suitable research tool for experimentally evaluating if play is affected by whether it dawns on a subject that an analytic solution may be possible. Our design includes a way to control for such insight. We also examine how this shapes subsequent learning towards optimization.


## JEL codes: C7, C9

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# Epiphany in the Game of 21 

Epiphany -<br>the sudden realisation or comprehension of the essence or meaning of something

[Wikipedia]

## I. Introduction

Understanding the workings of the human mind can be crucial to economists. Economic outcomes depend on behavior, and behavior is shaped by how people reason. In order to make reliable predictions it is useful to know the type of reasoning triggered by various situations and the effect on behaviour and outcomes.

Our paper contributes to the literature which seeks related insights through experimental games. ${ }^{1}$ We focus on a game we call The Game of 21; $G_{21}$ for short. One of us was introduced to it by a great-aunt, circa 1970. We have recently been told the game is sometimes also played in student bars in Australia, as a randomizing device to determine who buys the next round of beer. The rules: Two players, call them White and Green, take turns. White begins. To start off, he can choose either 1 or 2. Green observes this choice, then increments the "count" by adding one or two. That is, if White chooses 1 Green can follow up with 2 or 3 ; if White chooses 2 Green can follow up with 3 or 4 . White then observes Green's choice, and again increments the count by adding one or two. The game continues with the players thus taking turns, each player incrementing the count by one or two. The player who reaches 21 wins.

We now invite you, our dear reader, to answer a question before reading on:

## What would you do in this game?

[Don't flip page
'til you answer!]

[^1]$G_{21}$ features a second-mover advantage. Green actually has a dominant strategy that guarantees victory. ${ }^{2}$ Namely, at each stage of play, choose a multiple-ofthree $(=3,6, \ldots, 18,21)$. Furthermore, in any sub-game where a co-player has not chosen a multiple-of-three in the preceding stage, the player to move has a dominant strategy for that sub-game to play multiples-of-three from that point on.

Did you figure this out before turning the page? Our experience says many people don't grasp it, including "professional" conference audiences. Why? On further reflection one realizes there may be (at least) two reasons: First, a player may not realize the analytical nature of the problem; witness the beer-drinking students Down Under who viewed $G_{21}$ as a randomization device! Knowing the answer, this attitude seems puzzling. But bear in mind that most situations in life lack dominant strategies, e.g. when one needs to charm a lady at the airport counter who is about to charge for excess weight. Depending on a subject's associations, he or she may not think of the possibility that an optimal way to play $G_{21}$ could exist. ${ }^{3}$ Second, even if one realizes that logical analysis may hold the key, finding the answer may prove too difficult as it requires going through several steps-of-reasoning. ${ }^{4}$

Epiphany! That's what a player needs to master $G_{21}$. In fact, he needs two of them. First, he must realize the analytical nature of the problem, i.e., realize an

[^2]analytic solution may be possible. Call this epiphany-1. Second, he must discover and understand the dominant strategy. Call this epiphany-2. Our two main research hypotheses, described below, relate to these two forms of epiphany.

We face the task of controlling for and measuring epiphanies $1 \& 2$. Since these are cognitive concepts, not easy to observe directly, we need to come up with a design that allows us to draw insights based on observable data. The key feature of our design is to include a second game: The Game of $6, G_{6}$ for short, is played the same way as $G_{21}$ except whoever reaches 6 wins. Try $G_{6}$ on anyone, and they quickly figure out that they can win by picking 3 as Green. It seems natural, then, to posit that a person playing $G_{6}$ would carry the insight that an analytical solution is possible over to $G_{21}$, since $G_{6}$ and $G_{21}$ are so similar. $G_{6}$ induces epiphany-1.

Our design exploits this idea. We have two treatments. In the first subjects first play $G_{21}$ several times and then $G_{6}$ several times, in the other the order is reversed ( $G_{6}$ before $G_{21}$ ). We ask: will subjects playing $G_{21}$ in the latter treatment (presumed to have reached epiphany-1) play better or learn faster than subjects playing $G_{21}$ in the former treatment? That is, is reaching epiphany-1 an important part of learning and learning delay in $G_{21}$ ? This is our first research question.

Our second research question concerns whether and how subjects arrive at epiphany-2, given that they have reached epiphany-1. Is it the case that, over time and as subjects play more and more games, they learn gradually in the sense that they choose multiples-of-three at incrementally lower counts in $G_{21}$ (epiphany-2 by the backdoor)? Or could it be that subjects show no evidence of gradual learning before epiphany-2 occurs (learning with a leap)? We study the patterns, focusing on the data from the treatment where subjects play $G_{6}$ before $G_{21}$.

How does this approach add to the previous literature on strategic reasoning in games? We answer this question in section II, as we review related literature. Thereafter, section III describes our experimental design, section IV reports results regarding our two research questions (plus some), and section V concludes.

## II. Related literature

To see how we add to preceding literature, let us first describe a version of the classical guessing game: $N>2$ players simultaneously pick numbers in the range [0, 100]. Whoever is closest to $2 / 3$ of the average wins/splits a prize. The unique Nash equilibrium (also the result of iterated elimination of weakly dominated strategies) is for each player to pick 0 . However, in experiments, choices are all over, 0 s are rare, and 0s never win (unlike choices around 20); see e.g. Nagel (1995) and Camerer (2003, chapter 5).

This is sometimes taken to illustrate subjects' bounded reasoning abilities. High choices certainly make it clear that the players collectively do not manifest the degrees of mutual beliefs about mutual beliefs... about rational choices that might correspond to various rounds of iterated dominance. This does not, however, reveal much about any individual's ability to reason deeply. A smart and potentially deepreasoning individual should avoid the equilibrium strategy of 0 since most of the others choose high numbers! ${ }^{5}$

The game we study avoids such interpretational ambiguities. Playing the dominant multiples-of-three strategy is a best response regardless of beliefs about

[^3]others. Failure to choose a feasible multiple-of-three unambiguously indicates failure to work out the dominant strategy. Moreover, we can infer something regarding the number of steps-of-reasoning a subject is capable of by observing how early in the count of a game he starts choosing multiples-of-three (cf. footnote 4).

Another contribution of ours is best understood with reference to recent work on cognitive hierarchy or level-k models which can account for subjects' play in many experiments. The key idea, pioneered by Nagel (1995) and Stahl \& Wilson (1994, 1995), ${ }^{6}$ is that players are heterogeneous in terms of strategic sophistication. For example, level-0 players may choose randomly across all strategies. Level-1 players assume everyone else is level-0, and best respond; level-2 players assume everyone else is a level-1 player, and best respond; etc. ${ }^{7}$ Estimations of such models, for specific games, indicate a distribution of players concentrated around small but nonzero $k$ 's. Costa-Gomes \& Crawford (2006) report that "many subjects’ systematic deviations from equilibrium can be confidently attributed to non-equilibrium beliefs rather than irrationality" (p. 1767), thus describing data from games where subjects presumably succeeded in optimizing given their beliefs of others' strategies.

But this is in contrast with recent findings by Grosskopf \& Nagel (2008), on guessing games with $N=2$. Two players simultaneously pick numbers in the range of $[0,100]$ and whoever is closest to $2 / 3$ of the average number wins/splits a prize. The change from $N>2$ to $N=2$ alters the game's properties: a choice of 0 is now dominant. Student subjects as well as professional audiences at economics and psychology

[^4]conferences made choices that were not significantly different from the choices made in $N>2$ treatments. With $N=2,90 \%$ of the students and $63 \%$ of the professionals chose a dominated strategy! If one were to apply a level-k model with a distribution of players concentrated around small but non-zero $k$ 's it would suggest most players (all those for whom $k>0$ ) should choose 0 , at odds with Grosskopf \& Nagel's data.

Let's take stock. It seems that in some games it is easier for subjects to optimize than in others. We face the challenge of explaining how subjects calculate and learn what is in their best interest. This is a largely open research area, and we take early steps of exploration. ${ }^{8} G_{21}$ joins Grosskopf \& Nagel's $N=2$ games in having a dominant strategy which is non-obvious to compute. By its sequential structure, $G_{21}$ admits evaluation of how close subjects come to optimizing (cf. footnote 4), and our design allows for insights regarding learning. Our distinction between epiphanies 1 and 2 leads to new research questions which add structure to the approach.

After we started our project we learnt of work by Gneezy, Rustichini \& Vostroknutov (2007), involving similar games, conducted independently. $G_{21}$ features counting to 21 in increments of one or two; Gneezy et al have players count to 15 (or 17) with steps of one to three (or four). Some patterns of play accord well across studies, but research questions differ. Gneezy et al don't consider our key notion of epiphany-1; we don't explore response times which are central to them. Finally, McKinney \& Van Huyck $(2006,2007)$ study depth-of-strategic-reasoning related issues in Nim, an ancient game named in modern times by Bouton (1901-02). Again some features of play accord between studies, but $G_{21}$ and Nim are sufficiently

[^5]different that a direct comparison is difficult. ${ }^{9}$ McKinney \& Van Huyck also put more emphasis on identifying bounds of human reasoning and do not deal with epiphany-1.

## III. Design

Our subject pool was unusual. One of us (Martin) was teaching two sections of intermediate microeconomics. The course involved discussion of experimental methodology and results. To get the students excited about the topic, they were promised to get in-class experience of a "real" experiment, one generating data meant for publication. After some negotiation the Human Subjects Protection Program of the University of Arizona gave permission. ${ }^{10}$ Sessions were conducted at the Economic Science Laboratory. Since subjects were in class, we had no reason to make sure each was compensated for their time. We used a pay-a-random-subset-of-subjects approach as advocated by Bolle (1990): two subjects from each treatment were selected at random (one for $G_{21}$; one for $G_{6}$ ) and paid $\$ 5$ for each game won.

We had two treatments: in the $G_{21}$-then- $G_{6}$ treatment subjects first played $G_{21}$ five times and then $G_{6}$ five times, in the $G_{6}$-then- $G_{21}$ treatment the order was reversed. Subjects were not permitted to communicate with each other once the experiment had commenced, other than through selecting their choices of integers.

[^6]The $G_{21}$-then- $G_{6}$ treatment had 42 participants comprising seven groups of six subjects. Each subject received a "player ID" (A, B, C, D, E, or F), read through a "subject disclaimer form", and then got instructions with the rules of $G_{21}$. Each pair of members of each group played $G_{21}$ once, with new matches proceeding round-robin style with players alternating between White and Green positions. ${ }^{11}$ Play began once subjects had spent sufficient time studying the rules of the game. No hard time limit was imposed. Game sheets for each round were collected only after the last pair of players in that round had finished playing. We thus had seven groups, each featuring five rounds of play of $G_{21}$, with three games (each with two players) per round. After all of these $7 \times 5 \times 3=105$ games had terminated, ${ }^{12}$ with a winner determined for each, instructions describing the rules of $G_{6}$ were distributed, and round-robin play ensued as before, with another $7 \times 5 \times 3=105$ games.

The $G_{6}$-then- $G_{21}$ treatment had the same format, except the order of the games was reversed. We had 30 participants, producing 5 groups. We thus had $5 \times 5 \times 3=75$ games of $G_{6}$ and another $5 \times 5 \times 3=75$ games of $G_{21}$ in this treatment.

## IV. Results

Epiphany-1 is the insight that $G_{21}$ may be solvable by rational calculation, the dawning on a player that it may be that $\mathrm{s} / \mathrm{he}$ has a way of playing that guarantees a win. This is a cognitive concept which we can only study indirectly. We use $G_{6}$ as a tool to induce such insight in our subjects. The idea is that once subjects figure out that $G_{6}$ is "solvable", they will start thinking that $G_{21}$ may be solvable too since the

[^7]games have a similar structure. ${ }^{13}$ Hence, when we analyze our data we will assume that subjects in the $G_{6}$-then- $G_{21}$ treatment reach epiphany-1 before playing $G_{21}$, while subjects in the $G_{21}$-then- $G_{6}$ treatment may or may not have reached epiphany-1 before playing $G_{21}$. Conditional on that maintained assumption, we then test our two main hypotheses mentioned in the introduction.

This approach is meaningful only if two preliminary results hold:

PR(i) Most subjects playing five rounds of $G_{6}$ realize that $G_{6}$ may be solvable by rational calculation. (If they did not, our idea that such an insight extends to $G_{21}$ would lose its basis.)

PR(ii) Most subjects playing the Green position in $G_{21}$ for the first time do not immediately figure out that choosing the multiples-of-three is the best they can do. (If they did, then our conjecture that subjects in the $G_{21}$-then- $G_{6}$ treatment may not have reached epiphany-1 before playing $G_{21}$ would be vacuous.)

This section has three subsections: we establish preliminary results $\operatorname{PR}(i)$ and PR(ii) (IV.0) and then consider our two main research hypotheses (IV.1-2).

## IV. 0 Two Preliminary Results

$\operatorname{PR}(i)$ above is supported: most subjects playing five rounds of $G_{6}$ realize that $G_{6}$ may be solvable by rational calculation. Table 1 shows this with data from $G_{6}$ for both treatments. 167 of 180 Green players ( 93 percent) play perfect games - that is, their first move is 3 , and their second move is 6 , at which point they win. ${ }^{14}$

[^8]| Table 1: Perfect Play in $\mathbf{G}_{6}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Round | $\boldsymbol{G}_{6}$-then- $\mathrm{G}_{21}$ |  | $G_{21}$-then- $G_{6}$ |  | Pooled |  |
|  | Perfect Games | Percentage | Perfect Games | Percentage | Perfect Games | Percentage |
| 1 | 12/15 | 80 | 18/21 | 86 | 30/36 | 83 |
| 2 | 14/15 | 93 | 20/21 | 95 | 34/36 | 94 |
| 3 | 13/15 | 87 | 21/21 | 100 | 34/36 | 94 |
| 4 | 15/15 | 100 | 20/21 | 95 | 35/36 | 97 |
| 5 | 14/15 | 93 | 20/21 | 95 | 34/36 | 94 |
| All rounds | 68/75 | 91 | 99/105 | 94 | 167/180 | 93 |
| Notes: Columns 2, 4, and 6 ("Perfect Games") list the relative number of $G_{6}$ where Green played perfectly, for each round, by treatment and pooled. Columns 3, 5, and 7 provide the associated percentages. |  |  |  |  |  |  |

PR(ii) is supported too: most subjects playing the Green position in $G_{21}$ for the first time do not immediately figure out that choosing multiples-of-three is the best they can do. The evidence is in Table 2. Across treatments, in $G_{21}$, only 49 of 179 games $(27 \%)$ are played perfectly. ${ }^{15}$ The rates of perfect play are especially low in the early rounds of the $G_{21}$-then- $G_{6}$ treatment (e.g. 2 out of 20 , or $10 \%$, in round 1 ).

| Table 2: Perfect Play in $\mathbf{G}_{21}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Round | $G_{6}$-then- ${ }_{21}$ |  | $\boldsymbol{G}_{21}$-then- $\mathrm{G}_{6}$ |  | Pooled |  |
|  | Perfect Games | Percentage | Perfect Games | Percentage | Perfect Games | Percentage |
| 1 | 3/15 | 20 | 2/20 | 10 | 5/35 | 14 |
| 2 | 5/15 | 33 | 3/21 | 14 | 8/36 | 22 |
| 3 | 6/15 | 40 | 4/21 | 19 | 10/36 | 28 |
| 4 | 5/15 | 33 | 7/21 | 33 | 12/36 | 33 |
| 5 | 8/15 | 53 | 6/21 | 29 | 14/36 | 39 |
| All rounds | 27/75 | 37 | 22/104 | 21 | 49/179 | 27 |
| Notes: Columns 2, 4, and 6 ("Perfect Games") list the relative number of $G_{21}$ where Green played perfectly, for each round, by treatment and pooled. Columns 3, 5, and 7 provide the associated percentages. |  |  |  |  |  |  |

One final comment about $\operatorname{PR}(i)$ and $\operatorname{PR}(i i)$ : In $G_{6}$ a player might stumble on his optimal strategy serendipitously - if he flipped a coin he would choose 3 with probability $1 / 2$ - and from there win almost for sure (only 2 of 169 Green players who

[^9]selected 3 failed to win in $G_{6}$ ). It's harder to stumble into the optimal strategy in $G_{21}$. We can, however, control for this potential confound if we simply count the number of Green players in $G_{21}$ who chose $3 ; 113$ out of 179 Green players did so. While this is significantly greater than expected from a coin-flip, it is significantly lower than the proportion of players who chose 3 in $G_{6}{ }^{16}$ The conclusion: more Green players in $G_{6}$ than in $G_{21}$ chose 3 because they figured out their dominant strategy.

## IV. 1 The Impact of Epiphany-1

In light of our support for $\operatorname{PR}(\mathrm{i})$ and $\mathrm{PR}(\mathrm{ii})$, we now proceed to consider our first main research hypothesis: Subjects playing $G_{6}$ figure out that an analytic solution is possible. It dawns on them that there may be an optimal way to play $G_{21}$ too. Even if they do not figure out the optimal pick-multiples-of-three strategy right away, on balance they will play $G_{21}$ better in the $G_{6}$-then- $G_{21}$ than in the $G_{21}$-then- $G_{6}$ treatment.

We approach this in a few complementary ways. First we ask: does epiphany1 facilitate epiphany-2? Recall from the introduction our terminology that a subject has reached epiphany-2 if he discovers and understands the dominant strategy in $G_{21}$. Like epiphany-1, this is a cognitive concept which we can only study indirectly. We compare frequencies of perfect play by Green players in $G_{21}$ across treatments. The idea is that if a subject plays $G_{21}$ perfectly this probably was no fluke; it is an indicator of epiphany-2. Table 3 records relevant data. 37 percent of Green players play $G_{21}$ perfectly in the $G_{6}$-then- $G_{21}$ treatment, compared to 21 percent in the $G_{21^{-}}$ then $-G_{6}$ treatment. This difference is significant at the $5 \%$ level $(Z$ statistic $=2.20)$.

[^10]| Table 3: Comparing play in $G_{21}$ across the two treatments |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| Measure | $G_{6}$-then- $G_{21}$ |  | $G_{21}$-then- $G_{6}$ |  |
|  | Raw count | Percentage | Raw count | Percentage |
| Perfect Play | 27/75 | 37 | 22/104 | 21 |
| Median Indicated Rationality | 6 |  | 12 |  |
| Moment of Epiphany-2 in |  |  |  |  |
| Round 1 | 5/30 | 17 | 3/42 | 7 |
| Round 2 | 4/30 | 13 | 3/42 | 7 |
| Round 3 | 2/30 | 7 | 2/42 | 5 |
| Round 4 | 2/30 | 7 | 3/42 | 7 |
| Round 5 | 1/30 | 3 | 4/42 | 10 |
| No moment of epiphany | 16/30 | 53 | 27/42 | 64 |
| Notes: The moment of epiphany-2 measure lists the number of subjects who reach epiphany-2 in that particular round as a proportion of all subjects playing in that treatment. |  |  |  |  |

The perfect play test refers to Green players only, and neither considers White players nor the dynamics as a subject plays five rounds of $G_{21}$. We next introduce a more dynamic new metric which considers all players: moment of epiphany-2. To help with the definition, we first introduce a notion of indicated rationality.

Definition 1: Consider i's choice $x<21$ in a given instance of $G_{21}$. This choice exhibits indicated rationality if $x$ is the smallest number such that: it is a multiple-ofthree and all $i$ 's subsequent choices in that game are all multiples-of-three.

Definition 2: Subject $i$ 's moment of epiphany-2 occurs in round $R \in\{1,2,3,4,5\}$ if $R$ is the earliest round such that $i$ has a choice $x$ that exhibits indicated rationality, and in any rounds $R \gg R$ all $i$ 's choices exhibit indicated rationality at the earliest available multiple-of-three. We additionally impose that $i$ has a moment of epiphany-2 only if she plays at least one round perfectly. (Note that a subject $i$ may have no moment of epiphany-2.)

Indicated rationality attempts to capture the number of steps-of-reasoning a subject displays in a particular game, while moment of epiphany tries to a capture the moment when a subject works out the dominant strategy, i.e., when she achieves epiphany-2. Both measures are imperfect. For example, a player may stumble onto the choices $15,18,21$ for no clever reason at all, and yet we would record 15 as the
indicated rational choice. ${ }^{17}$ Some imprecision seems unavoidable in any measure. We would be wary when using these notions to obtain measures of any individual's degree of rationality or insight. We therefore focus on statistical testing of aggregates, which can smooth out some of the noise due to the imprecision.

Definition 2 does not attempt to reflect players who reach epiphany-2 at some point during the final round of play; the rationale is that this seems more defensible than the assertion that someone who still made mistakes in the last round had nonetheless understood the dominant strategy.

Looking at play in $G_{21}$ across our two treatments, we first note that the median choice with indicated rationality in the $G_{6}$-then- $G_{21}$ treatment is 6 , while the median choice with indicated rationality in the $G_{21}$-then- $G_{6}$ treatment is 12 (see Table 3). In a sense, the median number of steps-of-reasoning in the $G_{21}$ games in the $G_{21}$-then- $G_{6}$ treatment is three (the steps involving 12, 15, 18, and 21; cf. footnote 3 ), while the median number of steps-of-reasoning in the $G_{21}$ games in the $G_{6}$-then- $G_{21}$ treatment is five (the steps being $6,9,12,15,18$, and 21). Thus, reaching epiphany- $1-$ as we maintain happens before $G_{21}$ is played in the $G_{6}$-then- $G_{21}$ treatment - seems to increase the median steps-of-reasoning achieved by subjects from three to five in $G_{21}$.

We next look at the distribution of the subjects' moment of epiphany-2 across the treatments. An important statistic here is the number of players who never achieve epiphany-2: 53 percent of players in the $G_{6}$-then- $G_{21}$ treatment and 64 percent in the $G_{21}$-then- $G_{6}$ treatment (see Table 3). These proportions are not significantly different $(Z=0.93)$ from each other, and seem to suggest that some players may never work out

[^11]the dominant strategy in $G_{21}$; there just may be too many steps-of-reasoning involved. This is despite many of these subjects successfully working out the dominant strategy in $G_{6}$ (either before or after playing $G_{21}$ ).

In addition, the data suggests that playing $G_{6}$ before playing $G_{21}$ may not help this group of subjects achieve epiphany-2. However, among the group of players who do achieve epiphany-2 (as indicated by our measure based on Definition 2), it appears that achieving epiphany- 1 does help some subjects achieve epiphany- 2 sooner. In the $G_{6}$-then- $G_{21}$ treatment, up to 37 percent of players ( $11 / 30$ ) achieve epiphany-2 by round 3, as opposed to 19 percent of players (8/42) who achieve epiphany-2 by round 3 in the $G_{21}$-then- $G_{6}$ treatment, a statistically significant difference ( $\mathrm{Z}=1.67$ ).

We noted earlier that the moment of epiphany- 2 measure has some shortcomings (e.g. footnote 13). While it shows that some particular player understood the dominant strategy by some moment, it does not exclude the possibility that it was understood before that moment. To view the data from yet another angle, we define epiphany-2 delay, which identifies the latest moment when epiphany-2 was demonstrably not achieved. To this end, we generate a scale $0 \rightarrow 105$ as follows: Each position in the first round of $\mathrm{G}_{21}$ is assigned $1 \rightarrow 21$; each position in round 2 is assigned $22 \rightarrow 42$ and so on up to $85 \rightarrow 105$ for the final round. We use this scale to record the last occasion when a player failed to choose an available multiple-of-three.

Definition 3: Subject $i$ 's epiphany-2 delay is an element of $\{0,1, \ldots, 105\}$ identifying the last occasion when $i$ fails to select a multiple-of-three when able to do so. ${ }^{18}$ If $i$ never misses such an opportunity, then we assign 0 as $i$ 's epiphany- 2 delay measure.

[^12]Comparing epiphany-2 delay in $\mathrm{G}_{21}$ across the two treatments, we find that in the $G_{6}$-then- $G_{21}$ treatment the mean epiphany-2 delay is 53.66 with a standard deviation of 36.53 . In $G_{21}$-then- $G_{6}$ the mean epiphany-2 delay is 68.93 with a standard deviation of 29.2. That is, on average the location of the last error subjects make in the $G_{6}$-then- $G_{21}$ treatment is 10 or 11 in the third round, while in the $G_{21}$-then- $G_{6}$ treatment it is 6 in the fourth round. The difference between the means across these two treatments is significant $(\mathrm{Z}=1.90)$ at the $5 \%$ level.

Taking these various measures together, we conclude that although one or two achieve only marginal statistical significance, all the differences are in the predicted direction and most are strongly so. This suggests that prior experience with a simple game of suitable structure does indeed induce epiphany-1, which then raises the likelihood epiphany-2 will be achieved in a similar game of greater depth.

## IV. 2 Post-Epiphany-1 Learning

We have seen that it takes time for subjects to learn to play the dominant strategy in $G_{21}$. While some of that delay is due to the absence of epiphany-1, in this section we focus on how learning happens after subjects have reached epiphany-1. We study the patterns by looking at data from the $G_{21}$ games from the $G_{6}$-then- $G_{21}$ treatment. ${ }^{19}$

To consider learning, let us first look at how the indicated rationality measure evolves across rounds (see Table 4).

[^13]| Table 4: Median Indicated Rationality |  |
| :---: | :---: |
|  | $\mathbf{G}_{\mathbf{6}}$-then- $\mathbf{G}_{\mathbf{2 1}}$ |
| Round 1 | 15 |
| Round 2 | 12 |
| Round 3 | 6 |
| Round 4 | 4.5 |
| Round 5 | 3 |

After playing 5 rounds of $G_{6}$, thus presumably reaching epiphany-1, the median indicated rationality of subjects in round one is 15 . Interpretation: after realizing that an analytic solution is possible, at least 50 percent of subjects appear to work out two steps-of-reasoning (cf. footnote 3 ) in the first round of $G_{21}$. In round two, they seem to work out one more step, with the median indicated rationality falling to 12 . Then there is a jump, with median indicated rationality falling to 6 by round three, indicating five steps-of-reasoning by at least half of the subjects playing in this round. The median indicated rationality of 3 by round five reflects the fact that, by the beginning of that round, almost half of the subjects have reached epiphany-2, and have worked out the dominant strategy. It is interesting to note that two levels of reasoning would seem to lead to insight on how to play $G_{6}$ perfectly, and that the median subject playing a game for the first time seems to be able to reason out two steps, which may be why so many subjects ( 93 percent across the two treatments) play perfectly in $G_{6}$, although a number of the same subjects fail to achieve epiphany2 in $G_{21}$ even after five rounds.

While there appears to be a steady learning process as the rounds progress on the part of the median subject, there is a substantial degree of heterogeneity in how
quickly subjects learn to play the dominant strategy. ${ }^{20}$ Many seem never to learn how to play (as measured by their moment-of-epiphany measure), while a number of them appear to reach epiphany-2 at or near the beginning. ${ }^{21}$

We note an interesting difference between the two treatments in terms of learning when we confine ourselves just to those subjects who fail to reach epiphany2. It appears that in the $G_{21}$-then- $G_{6}$ treatment, these subjects nevertheless are making progress towards learning the dominant strategy; in round one their median indicated rationality is 18 and by round five it falls to 10.5 . On the other hand, in the $G_{6}$-then$G_{21}$ treatment there is no evidence of learning by this (smaller) group of players; in round one their median indicated rationality is 15 and by round five it remains at 15 .

## V. Conclusions

How do you defeat a Gordian knot? How do you make an egg stand on end? Wise men failed to come up with answers until Alexander the Great and Christopher Columbus came along. Their legends teach us about how fame and fortune may be the product of clever insights.

How do humans play games? We suggest that an adequate answer requires understanding how humans reach clever insights. Most of economic theory assumes that decision-makers best respond to their beliefs. Yet optimizing is often complicated and there are an abundance of related issues to explore: Do decision makers understand when problems admit analytical solutions? Does the answer to the

[^14]previous question depend on their life experiences? How efficient are humans in calculating solutions? What are the processes by which they learn to optimize?

We explore related issues in connection with a two-player zero-sum game of perfect information: $G_{21}$ is much simpler than chess, possible to figure out optimal play for, and yet sufficiently complicated that most humans do not figure it out at least at first. The borderline solvability-by-humans makes it suitable as a research tool for shedding light on questions like those in the previous paragraph.

To structure our examination of human insights in games, we introduce two key notions: epiphany- 1 is the dawning on a subject that an analytic solution may be possible in $G_{21}$, and epiphany-2 is the discovery and understanding of the nature of a dominant strategy in $G_{21}$. Much like Columbus may have been inspired by Alexander (who epiphanized eighteen centuries earlier), epiphany-1 may facilitate epiphany-2.

Epiphany-1 and epiphany-2 are cognitive notions referring to human psychology, not easy to observe directly. However, we propose an experimental design which allows us to derive testable predictions based on observables nevertheless. The key idea is to use a second game, $G_{6}$, which is simpler than $G_{21}$, and which serves as a tool for inducing epiphany-1.

Our conclusions, conditional on our maintained assumptions that prior play of $G_{6}$ induced epiphany-1, are that as we conjectured achieving epiphany-1 improves performance in $G_{21}$ (according to a variety of measures. Furthermore, we examine the nature of post-epiphany-1 learning. Here we do not have a preconceived hypothesis. It turns out that learning towards epiphany-2 is gradual to some degree in most subjects. However, subjects exhibit a lot of associated heterogeneity. Experience matters in possibly predictable ways, but there is a lot of individual variation.

Very little discourse in economics seems to be concerned with how human minds get primed to engage in rational thinking, and how insights are reached. ${ }^{22}$ More research is concerned with how players reason about others (see our section II and the references to the literature on level-k and cognitive hierarchy models). We suggest that these research goals are complementary, and that future work should keep both goals in mind.

While humans may have a language instinct with which to acquire proficiency in spoken language, strategic thinking, like written language, has to be learned the hard way. The connections between our findings and broader questions, such as why societies value schools, or how we may best structure teaching to foster insight and improve learning (e.g., begin with the simplest example of a concept), should be kept in mind although at the moment understanding such questions remains beyond our scope.

[^15]
## Appendix

Instructions, game sheets, guide cards for game matchings:
\{Subjects’ instructions were written on the same page as the game sheets. We explained verbally that movements for game matching should proceed according to schedule cards that we distributed. These assigned students to "tables" and explained who would act as the White/Green player. We indicate here the look of the instruction /game sheet for the game of twenty-one (the game of six was handled analogously) and the schedule card for one of a group's subjects.\}

## THE GAME OF 21

Welcome! The rules of the game are:
Each player takes turns playing the game, with the white player beginning. To begin, white can choose either the number 1 or the number 2, by circling one of them. The green player then plays by incrementing white's choice by 1 or by 2 . That is, if white had circled the number 1 , then green can choose either the number 2 or the number 3 . If, instead, white had chosen the number 2, then green can choose either the number 3 or the number 4 . Green uses a cross to mark his/her choice. The game continues with each player incrementing the other's choice by 1 or by 2 , until one player reaches 21 . The player who reaches 21 first wins.

WHITE PLAYER: Circle the number you choose in each round.
GREEN PLAYER: Use a cross to mark the number you choose in every round.
WHITE ALWAYS BEGINS. PLAYER WHO REACHES 21 FIRST WINS


WHITE PLAYER ID: $\qquad$
GREEN PLAYER ID: $\qquad$
WINNER ID:

## PLAYER B's SCHEDULE

| GAME 1: | Plays against A at Table 1. | Position: Green |
| :--- | :--- | :--- |
| GAME 2: | Plays against C at Table 3. | Position: White |
| GAME 3: | Plays against E at Table 2. | Position: Green |
| GAME 4: | Plays against F at Table 2. | Position: White |
| GAME 5: | Plays against D at Table 3. | Position: Green |

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[^1]:    ${ }^{1}$ For an entry, see the pioneering paper by Nagel (1995) which introduced so-called guessing games, the survey in chapter 6 in Camerer's (2003) book Behavioral Game Theory which covers many other games, and our further discussion and references in section II below.

[^2]:    ${ }^{2}$ The general insight that some player in $\mathrm{G}_{21}$ must have a dominant strategy can be gleaned (on a little reflection) by abstract principles (that $\mathrm{G}_{21}$ is a finite two-player zero-sum two-outcome games with perfect information) from Ewerhart (2000).
    ${ }^{3}$ Or consider $G_{21}$ with three rather than two players taking turns; no dominant strategy exists in this modified game. Is it really so obvious when to look for, or not conceive of, dominant strategies?
    ${ }^{4}$ First, and trivially, realize that a choice of 21 wins. Second, realize that if one chooses 18 then a win can be guaranteed. Third, realize that if one chooses 15 then one can similarly secure 18 , and so on. Ultimately, if one chooses 3 , and then a multiple-of-three in every subsequent move, then one can secure a win. According to the implicitly suggested metric, this calculation requires six steps of reasoning. Note that the described process resembles backward induction, but in fact is not backward induction since no reference is made to optimal subsequent co-player choices. The process considers each player $i$ in isolation and works backwards on $i$ 's nodes assigning an optimal choice only if this can be done regardless of subsequent opponent choices, and so exhibits non-existence except if a dominant strategy is uncovered for each subgame.

[^3]:    ${ }^{5}$ Camerer (2003, p.17) recognizes this confound and recounts how one player he knew to be very clever chose 18.1. Asking him later to explain his choice, he said he knew 0 was the equilibrium but believed his colleagues (all were Board members at Caltech) would only average two steps of reasoning and pick 25 . He optimized on that assumption, adding a little extra in case an odd high number were also chosen.

[^4]:    ${ }^{6}$ For further developments or applications, see Bosch-Domènech, Montalvo, Nagel \& Satorra (2002), Camerer, Ho \& Chong (2004), Costa-Gomes \& Crawford (2006), Costa-Gomes, Crawford \& Broseta (2001), Crawford (2003), Crawford, Gneezy \& Rottenstreich (2008), Crawford \& Iriberri (2007a, 2007b), Gneezy (2005), Ho, Camerer \& Weigelt (1998), Östling, Wang, Chou \& Camerer (2008), and Selten, Abbink, Buchta \& Sadrieh (2003).
    ${ }^{7}$ Some versions allow that level-k players best respond to some combination of players at level- $k$, for $k^{\prime}=0,1, \ldots, k-1$. See e.g. Camerer et al.

[^5]:    ${ }^{8}$ A different approach is developed by Johnson, Camerer, Sen \& Rymon (2002) who employ the 'Mouselab' system to study patterns of information search in alternating-offer shrinking pie games. They report that players tend not to backward induct; indeed a minority did not even glance at the pie sizes in later rounds, making backward induction impossible.

[^6]:    ${ }^{9}$ Several people suggested to us that $G_{21}$ is a version of Nim. However, it is straightforward to verify that no Nim game exists which has the same extensive game form as $G_{21}$. While $G_{21}$ and Nim both are finite two-player zero-sum two-outcome games with perfect information, in $G_{21}$ the root of any subgame (other than at the count of 20) has a binary choice set, a feature which cannot be preserved throughout any Nim game rich enough to allow as long paths of play as $G_{21}$ requires. Moreover, Bouton's (ingenious!) solution method, while similar to the pick-multiples-of-three solution of $G_{21}$ in the sense that it too produces a method by which a winning position ("safe combination") can be maintained through play, is very different in its details (which involve manipulations of binary scale of notation representations of positions) and does not apply to $G_{21}$.
    ${ }^{10}$ The issue was that research experiments usually occur outside of class, since participation is supposedly voluntary in a way which classes are not. We got around this by providing an alternative lecture (on theory regarding the involved games) for students who wished to opt out (no one did).

[^7]:    ${ }^{11}$ The Appendix contains instructions, game sheets, and the schedule-cards/protocol for matching pairs of players across rounds (which followed a so-called Howell movement, commonly used for conducting contract bridge-pairs tournaments).
    ${ }^{12}$ Our data analysis, however, is based on only 104 of these games. One pair of subjects (round 1, group 4, players E \& F) had not understood the instructions and played erroneously in their first round.

[^8]:    ${ }^{13}$ This need not mean that they figure out what the dominant strategy in $G_{21}$ is, only that they will realize that it may make sense to look for a dominant strategy in $G_{21}$.
    ${ }^{14}$ There is no significant difference in perfect play of $G_{6}$ between the $G_{6}$-then- $G_{21}$ and $G_{21}$-then- $G_{6}$ treatments.

[^9]:    ${ }^{15}$ There is a significant difference in perfect play of $G_{21}$ between the $G_{6}$-then- $G_{21}$ and $G_{21}$-then- $G_{6}$ treatments. This finding is central to our first main hypothesis, discussed further in the next subsection.

[^10]:    ${ }^{16}$ Also, the difference in the proportion of persons playing perfectly in all rounds of $G_{6}$ ( 93 percent) vs the proportion playing perfectly in all rounds of $G_{21}$ (27percent) is overwhelmingly significant ( Z statistic $=12.65$ ).

[^11]:    ${ }^{17}$ Similarly, the following example shows that there is imprecision associated also with Definition 2. Suppose a player in the White position has fully worked out the dominant strategy of playing multiples-of-three, but faces a Green player in that round who plays the dominant strategy, picking multiples-ofthree at each turn. Then the White player is denied an opportunity to pick a multiple-of-three in that round, and will have his moment of epiphany- 2 delayed by this measure.

[^12]:    ${ }^{18}$ Subject $i$ is able to choose a multiple-of-three every time his/her opponent has not played a multiple-of-three in his/her turn.

[^13]:    ${ }^{19}$ Note that using data from the $G_{21}$-then- $G_{6}$ would be inappropriate for answering the questions in this section since whatever delay in correct choices occurs may depend on the absence of epiphany- 1 in that treatment. However, we sometimes report the results from the $G_{21}$-then- $G_{6}$ treatment, mainly to contrast it with the $G_{6}$-then- $G_{21}$ treatment.

[^14]:    ${ }^{20}$ This result joins a wealth of research suggesting ways individuals differ, from the 'Big five' personality dimensions of extraversion, agreeableness, conscientiousness, neuroticism and openness to experience (John \& Srivastava, 1999) to psychological inclinations like sensitivity to emotional concerns (e.g. Krone 2003) or (of more relevance to us) level of thinking regarding the rationality and beliefs of others (e.g. many of the references cited in section II).
    ${ }^{21}$ Thirty percent of subjects have their moment of epiphany-2 by round 2 (see Table 3).

[^15]:    ${ }^{22}$ Although not game-theoretically anchored, discussions of entrepreneurial discovery and creativity have some of this flavour. See e.g. Hayek (1978/1984) and Kirzner (1985) for arm-chair reasoning, and Demmert \& Klein (2003) (D\&K) for a related out-door experiment. D\&K test whether the strength of financial incentives matters to whether subjects figure out the most efficient method (inverting of a plastic stool) for transferring water from spot A to spot B, thereby getting insights-by-analogy on a conjecture about entrepreneurial discovery by Hayek and Kirzner. D\&K use epiphany to refer to entrepreneurial discovery but their usage differs from ours as they exclude understanding that results from deliberate effort.

