MECHANISM DESIGN WITH EX-POST VERIFICATION AND LIMITED PUNISHMENTS

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Abstract. A principal has to give a job to one of several applicants. Each applicant values the job and has private information regarding the value to the principal of giving the job to him. The wage is exogenously fixed, but the principal will eventually learn the value of the applicant whom she hires after which she can impose a limited penalty on this applicant (e.g., fire him). If the number of applicants is high, the optimal hiring rule is a shortlisting procedure. Otherwise, the optimal hiring rule is a restricted bid auction, in which the applicants bid their value for the principal, possibly augmented by a shortlisting procedure for the applicants with minimal bids. From a methodological perspective, the paper provides a solution to a novel mechanism design problem without transfers that features a nontrivial interaction of incentive and feasibility constraints.

Keywords: mechanism design without transfers, matching with asymmetric information, stochastic mechanism, auction, feasibility constraint, shortlisting procedure

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1. Introduction

A principal has to allocate a job to one of several ex-ante identical applicants. Each applicant values the job and has private information (type) regarding his value to the principal. The wage is exogenously fixed. The principal will eventually learn the type of the applicant whom she hires, and she will be able to impose a penalty on this applicant if she is disappointed (e.g., fire him or deny a promotion). This is an allocation problem, in which the agents have private types; the type of the selected agent is verified ex-post; and limited penalties are possible.

In order to provide incentives for the agents to reveal their value to the principal, there should be a reward for revealing a low type. An optimal allocation rule maximizes the probability of selecting high types subject to promising low types a sufficient probability so that they do not want to lie.

If the penalty is a constant share of the agent's surplus, the optimal rule is particularly simple. When the number of agents is small, the optimal rule is an auction with two thresholds, in which the agents bid their types. A bid above the high threshold makes an agent into a "superstar," while a bid below the low threshold puts the agent on the waiting list. If there are superstars, the winner is selected among them at random. Otherwise, the winner is the highest bid among those between the thresholds. If all agents are on the waiting list, the winner is chosen randomly. The optimal rule bunches the types at the top and at the bottom but preserves competition in the middle of the support.

The distance between the thresholds decreases in the number of agents and vanishes at a finite number. The optimal rule becomes a shortlisting procedure, in which the agents report whether their types are above or below a bar and are shortlisted with certainty if "above" and only with some chance if "below". The winner is selected from the shortlist at random.

For more general penalties, the optimal rule is modified by putting the agents with the bids below the low threshold on the waiting list with different probabilities.

Methodologically, we study a mechanism design problem in which incentive compatibility and generalized Matthews-Border feasibility constraints are expressed in interim probabilities of selecting an agent. There are no transfers and the relevant incentive compatibility constraints are global rather than local. Hence, the standard envelope methods are not particularly useful. We solve the problem by splitting the feasibility and incentive compatibilty constraints in two subsets and solving two auxiliary maximization problems subject to different subsets of the constraints. We expect that this approach might prove useful for other mechanism design problems without transfers.

Ben-Porath, Dekel and Lipman (2014) (henceforth, BDL) study a similar model, with a key difference in assumptions about the verification technology. In BDL the principal can pay a cost and acquire information about the agents' types before making the allocative decision,[1] while in our model the type is revealed ex-post after the allocation decision is final. The tradeoff in BDL is between the allocative efficiency and the verification costs. In an optimal rule, report profiles are not verified iff all reports are below a cutoff. The optimal allocation is inefficient for low type profiles and efficient otherwise. By contrast, in our model the tradeoff is between the allocative efficiency and the incentives for low types to reveal themselves. In an optimal rule, the allocation is always inefficient and incentives for truthful reporting are provided by bunching at the top and at the bottom of the type distribution. We discuss other related literature in the conclusions.

2. Model

There is a principal who has to select one of $n \geq 2$ agents. The principal's payoff from a match with agent i is $x_i \in X \equiv [a, b]$, $a \geq 0$, where x_i is private to agent i. The values of x_i's are i.i.d. random draws, with continuously differentiable c.d.f. F on X, whose density f is positive almost everywhere on X. We use $\bar{x}$ to denote the type profile $(x_1, x_2, \ldots, x_n)$ and x_{-i} to denote the type profile excluding the type of agent i.

Each agent i makes a statement $y_i \in X$ about his type x_i, and the principal chooses an agent according to a specified rule. If an agent i is not selected, his payoff is 0. Otherwise, he obtains a payoff of $v(x_i) > 0$. In addition, we assume that if the agent is selected, the principal observes x_i and can impose a penalty $c(x_i) \geq 0$ on the agent.[2] Our primary interpretation of c is the upper bound on the expected penalty that can be imposed on the agent after his type has been verified.[3] Functions v and c are bounded and almost everywhere continuous on $[a, b]$. Note that $v - c$ can be non-monotonic.

Some of our assumptions are restrictive and should be relaxed in future work. The assumption that $a \geq 0$ implies that the principal would like to allocate the good regardless of the realized type profile. It is straightforward to relax, at the cost of elegance of the optimal allocation rule. The other two important assumptions are independence of the penalty from the report and that the verification technology can never misclassify a truthful report as a lie. The results will be affected if alternative assumptions are made; nevertheless, we hope that our approach can be used as a blueprint for analysis under such alternative assumptions.

[1]For other literature in which evidence that can be presented *before* an allocation decision, see, e.g., Townsend (1979), Grossman and Hart (1980), Grossman (1981), Milgrom (1981), Gale and Hellwig (1985), Green and Laffont (1986), Border and Sobel (1987), Postlewaite and Wettstein (1989), Mookherjee and Png (1989), Lipman and Seppi (1995), Seidmann and Winter (1997), Glazer and Rubinstein (2004, 2006, 2012, 2013), Forges and Koessler (2005), Bull and Watson (2007), Severinov and Deneckere (2006), Deneckere and Severinov (2008), Kartik, Ottaviani and Squintani (2007), Kartik (2009), Sher (2011), Sher and Vohra (2011), Ben-Porath and Lipman (2012), Dziuda (2012), and Kartik and Tercieux (2012).

[2]We could allow c to depend on the entire type profile, $c(x_1, x_2, \ldots, x_n)$, where n is the number of agents, without affecting any of the results. In that case, $c(x_i)$ should be thought of as the expected value of $c(x_1, x_2, \ldots, x_n)$ conditional on x_i.

[3]The assumption that x_i is verified with certainty can be relaxed; if $\alpha(x_i)$ is the probability that x_i is verified and $L(x_i)$ is the limit on i's liability, then set $c(x_i) = \alpha(x_i)L(x_i)$.

The principal has full commitment power and can choose any stochastic allocation rule that determines a probability of selecting each agent conditional on the report profile and the penalty conditional on the report profile and the type of the selected agent after it is verified ex-post. An allocation rule (p,ξ) associates with every profile of statements $\bar{y}=(y_1,\ldots,y_n)$ a probability distribution $p(\bar{y})$ over $\{1,2,\ldots,n\}$ and a family of functions $\xi_i(x_i,\bar{y})\in[0,1]$, $i=1,\ldots,n$, which determine the probability that agent i is penalized if he is selected given his type and the report profile. The allocation rule is common knowledge among the agents. The solution concept is Bayesian Nash.

By the revelation principle, it is sufficient to consider allocation rules in which truthful reporting constitutes an equilibrium. Since type x_i of the selected agent is verifiable, it is optimal to penalize the selected agent whenever he lies, $y_i\neq x_i$, and not to penalize him otherwise. Hence, we set $\xi_i(x_i,\bar{y})=0$ if $y_i=x_i$ and 1 otherwise and drop ξ in the description of the allocation rules. Thus, the payoff of agent i whose type is x_i and who reports y_i is equal to[4]

$$V_i(x_i,y_i)=\int_{x_{-i}\in X^{n-1}} p_i(y_i,x_{-i})(v(x_i)-\mathbf{1}_{y_i\neq x_i}c(x_i))\mathrm{d}\bar{F}_{-i}(x_{-i}).$$

The principal wishes to maximize the expected payoff

$$\max_p \int_{\bar{x}\in X^n}\sum_{i=1}^n x_i p_i(\bar{x})\mathrm{d}\bar{F}(\bar{x}) \tag{P$_0$}$$

subject to the incentive compatibility constraint

$$V_i(x_i,x_i)\geq V_i(x_i,y_i)\ \text{ for all } x_i,y_i\in X \text{ and all } i=1,\ldots,n. \tag{IC$_0$}$$

Denote by $h(x)$ the share of the surplus retained by a selected agent after deduction of the penalty, truncated at zero:

$$h(x)=\frac{\max\{v(x)-c(x),0\}}{v(x)},\quad x\in X.$$

Lemma 1. *Allocation rule p satisfies incentive compatibility constraint* (IC$_0$) *if and only if for every $i=1,\ldots,n$ there exists $r_i\in[0,1]$ such that for all $x_i\in X$*

$$r_i h(x_i)\leq\int_{x_{-i}\in X^{n-1}} p_i(x_i,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i})\leq r_i. \tag{1}$$

Proof. Each agent i's best deviation is the one that maximizes the probability of i being chosen, so (IC$_0$) is equivalent to

$$v(x_i)\int_{x_{-i}} p_i(x_i,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i})\geq(v(x_i)-c(x_i))\sup_{y_i\in X}\int_{x_{-i}} p_i(y_i,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i}).$$

Since the left-hand side is nonnegative, this inequality holds if and only if

$$\int_{x_{-i}} p_i(x_i,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i})\geq h(x_i)\sup_{y_i\in X}\int_{x_{-i}} p_i(y_i,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i}). \tag{2}$$

[4]Denote by $\bar{F}$ the joint c.d.f. of all n agents and by $\bar{F}_{-i}$ the joint c.d.f. of all agents except i. For each agent i, $p_i(y_i,y_{-i})$ stands for the probability of choosing i as a function of the profile of reports.

Thus, (2) implies (1) by setting $r_i = \sup_{y_i \in X} \int_{x_{-i}} p_i(y_i, x_{-i}) \mathrm{d}\bar{F}_{-i}(x_{-i})$. Conversely, if (1) holds with some $r_i \in [0,1]$, then it also holds with $r_i' = \sup_{y_i \in X} \int_{x_{-i}} p_i(y_i, x_{-i}) \mathrm{d}\bar{F}_{-i}(x_{-i}) \leq r_i$, which implies (2). ∎

Problem in reduced form. We will approach problem (P_0) by formulating and solving its reduced form. Denote by $g_i(y)$ the probability that agent i is selected conditional on reporting y,

$$g_i(y) = \int_{x_{-i} \in X^{n-1}} p_i(y, x_{-i}) \mathrm{d}\bar{F}_{-i}(x_{-i}), \quad y \in X,$$

and define the *reduced-form allocation* $g : X \to [0, n]$, by

$$g(y) = \sum_{i=1}^{n} g_i(y), \quad y \in X.$$

We will now formulate the principal's problem in terms of g:

$$\text{(P)} \qquad \max_{g} \int_{x \in X} x g(x) \mathrm{d}F(x)$$

subject to the incentive compatibility constraint

$$\text{(IC)} \qquad v(x)g(x) \geq (v(x) - c(x)) \sup_{y \in X} g(y) \quad \text{for all } x \in X,$$

the feasibility condition (due to $\sum_i p_i(\bar{y}) = 1$ for all $\bar{y} \in X^n$)

$$(\mathrm{F}_0) \qquad \int_X g(x) \mathrm{d}F(x) = 1,$$

and a generalization of Matthews-Border feasibility criterion (Matthews 1984, Border 1991, Mierendorff 2011, Hart and Reny 2013) that guarantees existence of an allocation rule p that induces a given g (see Lemma 3 in the Appendix):

$$\text{(F)} \qquad \int_{\{x : g(x) \geq t\}} g(x) \mathrm{d}F(x) \leq 1 - \Big(F(\{x : g(x) < t\}) \Big)^n \quad \text{for all } t \in [0, n].$$

Lemma 2.
(i) If p is a solution of the principal's problem (P_0)*, then the reduced form of p is a solution of the reduced problem* (P)*.*

(ii) If g is a solution of the reduced problem (P)*, then it is the reduced form of some solution of the principal's problem* (P_0)*.*

Proof of Lemma 2. Observe that for every p and its reduced form g, objective functions in (P_0) and (P) are identical. We now verify that every solution of (P_0) is admissible for (P), and for every solution of (P) there is an admissible solution for (P_0).

Feasibility condition (F) is the criterion for existence of p that implements g. This condition is due to the following lemma, which is a generalization of Matthews-Border feasibility criterion (e.g., Border 1991, Proposition 3.1) to asymmetric mechanisms.

Let $\mathcal{Q}_n$ be the set of functions $q : X^n \to [0,1]^n$ such that $\sum q_i \leq 1$ and let λ be a measure on X. We say that $Q : X \to [0, n]$ is a reduced form of $q \in \mathcal{Q}_n$ if $Q(z) = \sum_i \int_{X^{n-1}} q_i(z, x_{-i}) \mathrm{d}\lambda^{n-1}(x_{-i})$ for all $z \in X$.

Lemma 3. *$Q : X \to [0,n]$ is the reduced form of some $q \in \mathcal{Q}_n$ if and only if*

$$\int_{\{x:Q(x)\geq z\}} Q(x)\mathrm{d}\lambda(x) \leq 1 - \Big(\lambda(\{x : Q(x) < z\})\Big)^n \quad \text{for all } z \in [0,n]. \tag{3}$$

Proof. Sufficiency is due to Proposition 3.1 in Border (1991) implying that if Q satisfies (3), then there exists a *symmetric* q whose reduced form is Q. To prove necessity, consider $q \in \mathcal{Q}_n$ and let Q be its reduced form. For every $t \in [0,n]$ denote $E_t = \{x \in X : Q(x) \geq t\}$. Then

$$\begin{aligned}
\int_{y\in E_t} Q(y)\mathrm{d}\lambda(y) &= \int_{y\in X}\left[\sum_{i=1}^n \int_{x_{-i}\in X^{n-1}} q_i(y,x_{-i})\mathrm{d}\lambda^{n-1}(x_{-i})\right]\mathbf{1}_{\{y\in E_t\}}\mathrm{d}\lambda(y)\\
&= \sum_{i=1}^n \left[\int_{(x_i,x_{-i})\in X^n} q_i(x_i,x_{-i})\mathbf{1}_{\{x_i\in E_t\}}\mathrm{d}\lambda^n(x_i,x_{-i})\right]\\
&\leq \sum_{i=1}^n \left[\int_{(x_i,x_{-i})\in X^n} q_i(x_i,x_{-i})\mathbf{1}_{\cup_j\{x_j\in E_t\}}\mathrm{d}\lambda^n(x_i,x_{-i})\right]\\
&= \int_{x\in X^n}\left(\sum_{i=1}^n q_i(x)\right)\mathbf{1}_{\cup_j\{x_j\in E_t\}}\mathrm{d}\lambda^n(x) \leq \int_{x\in X^n}\mathbf{1}_{\cup_j\{x_j\in E_t\}}\mathrm{d}\lambda^n(x)\\
&= 1 - \int_{x\in X^n}\mathbf{1}_{\cap_j\{x_j\in X\setminus E_t\}}\mathrm{d}\lambda^n(x) = 1 - \big(\lambda(X\setminus E_t)\big)^n.
\end{aligned}$$

■

Feasibility condition (F_0) is due to $\sum_i p_i(x) = 1$:

$$\begin{aligned}
\int_{y\in X} g(y)\mathrm{d}F(y) &= \int_{y\in X}\left[\sum_{i=1}^n \int_{x_{-i}\in X^{n-1}} p_i(y,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i})\right]\mathrm{d}F(y)\\
&= \int_{x\in X^n}\left(\sum_{i=1}^n p_i(x)\right)\mathrm{d}\bar{F}(x) = 1.
\end{aligned} \tag{4}$$

Let p be a solution of (P_0). Then its reduced form satisfies feasibility conditions (F) by Lemma 3 and (F_0) by (4). Incentive constraint (IC) is satisfied as well, since (IC_0) applies separately for each i and thus, in general, is stronger than (IC).

Conversely, let g be a solution of (P). Since g satisfies (F) and (F_0), by Proposition 3.1 in Border (1991) there exists a symmetric p whose reduced form is g. This p will satisfy incentive constraint (IC_0), since for symmetric mechanisms (IC) and (IC_0) are equivalent. ■

Problem (P) is interesting because of its constraints. First, incentive compatibility constraints (IC) are global rather than local as is often the case in mechanism design, especially in the environments with transfers. Second, feasibility constraint (F) is substantive and will bind at the optimum if and only if incentive compatibility constraint (IC) slacks, which is not the case in the classical mechanism design for allocation problems.

3. Optimal allocation rules in reduced form

We would like to maximize the value of g for high types and minimize it for low types subject to feasibility and incentive compatibility constraints. The novelty of our approach is that we fix a supremum value of g, denoted by r, and consider two auxiliary problems whose solutions are the maximal and the minimal trajectories of g that satisfy the constraints, respectively, and respect $r = sup_{y\in X} g(y)$. A candidate g_r is constructed by concatenating the two solutions. An optimal g is found by optimizing the principal's objective on the set of candidate solutions, parametrized by r.

We separate the global incentive compatibility constraint in (IC) into two simpler constraints, one relevant for the minimal trajectory and another relevant for the maximal trajectory of g. Let $r = sup_{y\in X} g(y)$. Then, (IC) can be expressed as (c.f. Lemma 1)

$$g(x) \geq h(x)r, \quad x \in X, \tag{IC$_{\min}$}$$

$$g(x) \leq r, \qquad x \in X. \tag{IC$_{\max}$}$$

Note that feasibility constraint (F_0), $\int_X g(x)\mathrm{d}F(x) = 1$, implies that $r \in [1, \bar{r}]$,[5] where

$$\bar{r} = \left(\int_a^b h(x)\mathrm{d}F(x)\right)^{-1}.$$

For each $r \in [1, \bar{r}]$, we find the maximal and the minimal trajectory of g that satisfies feasibility and incentive compatibility constraints. The lowest trajectory minimizes the value of g and, hence, the relevant part of the incentive compatibility constraints is given by ($\mathrm{IC}_{\min}$). Let $G(x) := \int_a^x g(t)\mathrm{d}F(t)$ and consider the problem

$$\min_g \int_X G(x)\mathrm{d}F(x) \ \text{ s.t. } (\mathrm{IC}_{\min}) \text{ and} \tag{P$_{\min}$}$$

$$\int_{\{x:g(x)<t\}} g(x)\mathrm{d}F(x) \geq \left(F(\{x : g(x) < t\})\right)^n, \qquad t \in [0, n], \tag{F$'$}$$

Constraint (F$'$) is equivalent to feasibility constraint (F), but is expressed in terms of the complement sets. This will be convenient when we verify feasibility constraint after concatenating the minimal and the maximal trajectories of g.

Similarly, for the maximal trajectory of g the relevant part of the incentive constraint is ($\mathrm{IC}_{\max}$). Let $\bar{G}(x) := \int_x^b g(t)\mathrm{d}F(t)$ and consider the following problem

$$\max_g \int_X \bar{G}(x)\mathrm{d}F(x) \ \text{ s.t. } (\mathrm{IC}_{\max}) \text{ and (F).} \tag{P$_{\max}$}$$

Let $\underline{g}_r$ and $\overline{g}_r$ be solutions of problems ($\mathrm{P}_{\min}$) and ($\mathrm{P}_{\max}$), respectively. We construct a candidate $g_r^* : X \to [0, \bar{r}]$ as the concatenation of $\underline{g}_r$ and $\overline{g}_r$ at some $z \in X$:

$$g_r^*(x) = \begin{cases} \underline{g}_r(x), & x < z, \\ \overline{g}_r(x), & x \geq z, \end{cases} \tag{5}$$

[5]Indeed, if $r < 1$, then $g < 1$ and $\int_X g(x)\mathrm{d}F(x) < 1$. If $r > \bar{r}$, then $\int_X g(x)\mathrm{d}F(x) \geq \int_X rh(x)\mathrm{d}F(x) > 1$.

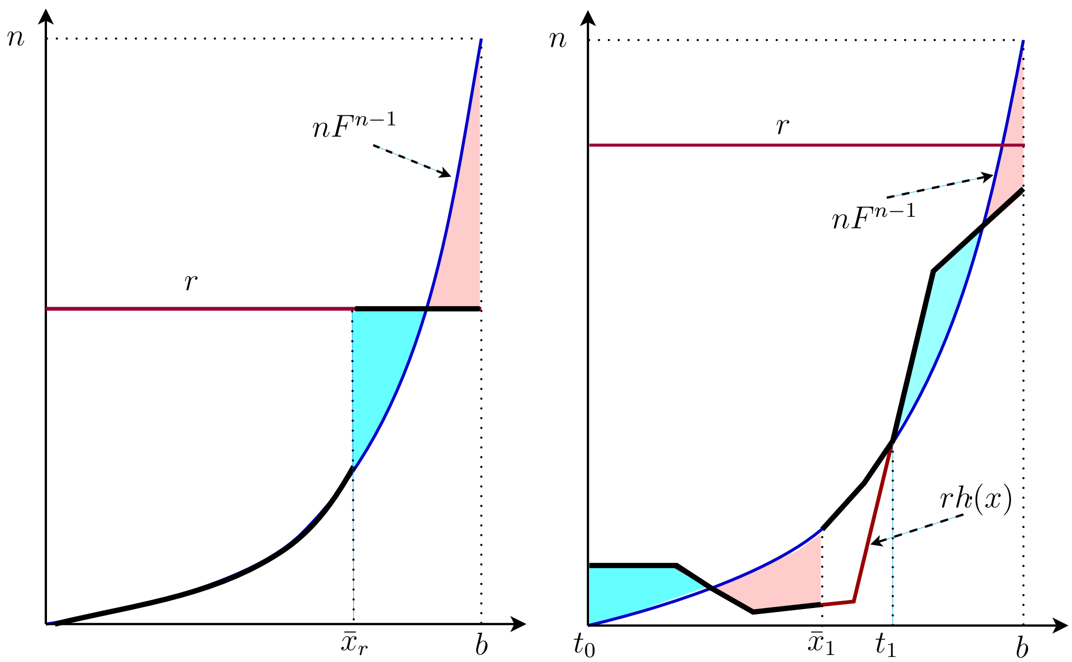

FIG. 1. Examples of solutions of $P_{\max}$ (left) and $P_{\min}$ (right).

For g_r^* to be feasible, it must satisfy (F_0) or, equivalently,

$$\int_a^z \underline{g}_r(x)\mathrm{d}F(x) + \int_z^b \overline{g}_r(x)\mathrm{d}F(x) = 1. \tag{6}$$

A solution of (6) exists.[6] We will show in the proof of Theorem 1 that the greatest solution of (6) induces feasible and incentive compatible g_r^*. Furthermore, g_r^* is unique: if there are multiple solutions z of (6) then the corresponding concatenated functions either violate incentive compatibility or induce same g_r^*.

Theorem 1. *Reduced-form allocation rule g^* is a solution of* (P) *if and only if $g^* = g_r^*$, where r solves*

$$\max_{r\in[1,\bar{r}]} \int_a^b x g_r^*(x)\mathrm{d}F(x),$$

and z is the greatest solution of (6).

The solutions $\underline{g}_r$ and $\overline{g}_r$ of problems ($P_{\min}$) and ($P_{\max}$) are illustrated by Fig. 1. The left diagram depicts $\overline{g}_r$.[7] The blue curve is $nF^{n-1}(x)$ and the red curve is r; the black curve depicts $\overline{g}_r(x)$. Starting from the right ($x = b$), the black line follows r so long as constraint (F) slacks. Down from point $\bar{x}_r$ constraint (F) is binding, and

[6] The value of $\int_a^z \underline{g}_r(x)\mathrm{d}F(x) + \int_z^b \overline{g}_r(x)\mathrm{d}F(x)$ is continuous in z and by (F′) and (F), $\int_a^b \overline{g}_r(x)\mathrm{d}F(x) \le 1 \le \int_a^b \underline{g}_r(x)\mathrm{d}F(x)$ for any $r \in [1,\bar{r}]$.

[7] Lemma 4 below characterizes $\overline{g}_r$.

the highest trajectory of $\bar{G}(x)$ that satisfies this constraint is exactly $1 - F^n(x)$. Since $\bar{G}(x) = \int_x^b \overline{g}_r(t)\mathrm{d}F(t)$, the solution $\overline{g}_r(t)$ is equal to $nF^{n-1}(t)$ for $t < \bar{x}_r$.

The right diagram on Fig. 1 depicts an example of $\underline{g}_r$. The blue curve is $nF^{n-1}(x)$ and the red curve is $rh(x)$; the black curve depicts $\underline{g}_r(x)$. Starting from the left ($x = a$), the black line follows $rh(x)$ up to the point $\bar{x}_1$ where blue area is equal to red area (so the feasibility constraint starts binding), and then jumps to $nF^{n-1}(x)$. Then, the black curve follows $nF^{n-1}(x)$ so long as it is above $rh(x)$. After the crossing point, t_1, the incentive constraint is binding again, and the black curve again follows $rh(x)$, etc.

Proof of Theorem 1. A necessary condition for G to be optimal is that it is maximal w.r.t. first-order stochastic dominance order (FOSD) on the set of c.d.f.s that satisfy (IC) and (F). We will prove that the set of FOSD maximal functions is $\{g_r^*\}_{r\in[1,\bar{r}]}$. Optimization on the set of these functions yields the solutions of (P).

We start by showing that $g_r^*(x) = r$ for all $x \geq z$. To prove this, we use of Lemma 4 stated in Section 5, which characterizes the solution ($\mathrm{P}_{\max}$).

By definition, (F) is satisfied with equality by $\overline{g}_r$ at $x = \bar{x}_r$. If (F$'$) is also satisfied with equality by $\underline{g}_r$ at $x = \bar{x}_r$, then (6) is satisfied with $z = \bar{x}_r$. Hence, the greatest solution of (6) is weakly higher than $\bar{x}_r$. If, on the other hand, (F$'$) is satisfied with strict inequality at $x = \bar{x}_r$, then the left hand side of (6) is less than one at $z = \bar{x}_r$, and is increasing in z, and has a unique solution on $(\bar{x}_r, b]$. By Lemma 4, $\overline{g}_r(x) = r$ for all $x \geq \bar{x}_r$ and, hence, $g_r^*(x) = r$ for all $x \geq z$ at any $z \geq \bar{x}_r$.

Furthermore, consider any solution z' of (6) such that $z' < \bar{x}_r$. Then, either ($\mathrm{IC}_{\min}$) is violated at some $x \geq z'$, in which case concatenation obtained at z' is not incentive compatible, or (F$'$) is satisfied with equality for all $x > z'$, so $\underline{g}_r(x) = nF^{n-1}(x)$ on $[z', \bar{x}_r]$. Also, by Lemma 4, $\overline{g}_r(x) = nF^{n-1}(x)$ on $[z', \bar{x}_r]$. Hence, concatenation at any $z \in [z', \bar{x}_r]$ produces the same g_r^* and, furthermore, $z = \bar{x}_r$ is the greatest solution of (6). Hence, all incentive compatible concatenations are identical.

Next, we show that for every $r \in [1, \bar{r}]$, g_r^* satisfies (IC), (F_0), and (F). Note that g_r^* satisfies (F_0) and (F) by construction. To prove that g_r^* satisfies (IC), we need to verify that $\underline{g}_r(x)$ satisfies ($\mathrm{IC}_{\max}$) for $x < z$ and $\overline{g}_r(x)$ satisfies ($\mathrm{IC}_{\min}$) for $x \geq z$. We have shown above that $\overline{g}_r(x) = r$ for all $x \geq z$, which trivially satisfies ($\mathrm{IC}_{\min}$).

To verify ($\mathrm{IC}_{\max}$), observe that for $x \leq z$ it must be that $\underline{g}_r(x) \leq r$, as otherwise z is not a solution of (6). Assume by contradiction that $\underline{g}_r(x') > r$ for some $x' \leq z$. Since $rh(x') < r$, constraint (F$'$) must be binding at x', implying $\underline{g}_r(x') = nF^{n-1}(x') \geq r$. However, we have shown above that either $z = \bar{x}_r$ or (F$'$) is not binding at z. We obtain the contradiction in the former case since $nF^{n-1}(x') < nF^{n-1}(\bar{x}_r) < r$, where the last inequality is by construction of $\bar{x}_r$. In the latter case, $\underline{g}_r(z) < r$, implying that $\underline{g}_r$ is decreasing somewhere on $[x', z]$, which is impossible by (F$'$) since (F$'$) is satisfied with equality at x'.

Finally, observe that c.d.f. $G(x) = \int_a^x g(t)\mathrm{d}F(t)$ is FOSD maximal subject to (IC) and (F) if and only if there exists $r \in [1, \bar{r}]$ such that $g = g_r^*$. Indeed, consider an arbitrary $\tilde{g}$ that satisfies (IC), (F), and (F_0), and let $\tilde{G} = \int_a^x \tilde{g}(t)\mathrm{d}F(t)$. Let $r = \sup_X \tilde{g}(x)$. Then $\tilde{g}$ satisfies ($\mathrm{IC}_{\min}$) and ($\mathrm{IC}_{\max}$) with this r. Consider now $G_r^*(x) = \int_a^x \underline{g}_r^*(t)\mathrm{d}F(t)$, where

g_r^* is a concatenation of $\underline{g}_r$ and $\overline{g}_r$ at the largest point z such that (6) holds. Since $\int_a^x \underline{g}_r(t)\mathrm{d}F(t)$ describes the lowest trajectory $G(x)$ that satisfies ($\mathrm{IC_{max}}$) and (F), we have for all $x \leq z$

$$G_r^*(x) = \int_a^x \underline{g}_r(t)\mathrm{d}F(t) \leq \int_a^x \tilde{g}(t)\mathrm{d}F(t) = \tilde{G}(x).$$

Also, since $\int_x^b \overline{g}_r(t)\mathrm{d}F(t)$ describes the highest trajectory $\bar{G}(x)$ that satisfies ($\mathrm{IC_{min}}$) and (F), and $\bar{g}_r(x) = r$ for all $x > z$, we have

$$1 - G_r^*(x) = \int_x^b \overline{g}_r(t)\mathrm{d}F(t) \geq \int_x^b \tilde{g}(t)\mathrm{d}F(t) = 1 - \tilde{G}(x).$$

Hence, G_r^* FOSD $\tilde{G}$. ■

4. Upper bound

There are two qualitatively distinct cases: when feasibility constraint (F) is binding at the optimum and when it is not. Recall that (F) is necessary and, together with ($\mathrm{F_0}$), sufficient for existence of allocation rule p that a given induces reduced-form g. This constraint becomes weaker as n increases and, eventually, permits all nondegenerate distributions as $n \to \infty$. On the other hand, incentive compatibility constraint (IC) does not depend on n. So when n is large enough, the shape of the solution is determined entirely by (IC) and is independent of n.

Let z^* be the unique[8] solution of

$$r^* \left(\int_a^{z^*} x h(x)\mathrm{d}F(x) + \int_{z^*}^b x\mathrm{d}F(x) \right) = z^*, \tag{7}$$

where r^* is the normalizing constant:

$$r^* = \left(\int_a^{z^*} h(x)\mathrm{d}F(x) + \int_{z^*}^b \mathrm{d}F(x) \right)^{-1}. \tag{8}$$

Theorem 2. *For every number of agents, in any allocation rule the principal's payoff is at most z^*. Moreover, if a solution of* (P) *achieves the payoff of z^*, then its reduced form must be (almost everywhere) equal to*

$$g^*(x) = \begin{cases} r^* h(x), & x \leq z^*, \\ r^*, & x \geq z^*. \end{cases} \tag{9}$$

We obtain (7)-(9) by solving (P) subject to (IC) and ($\mathrm{F_0}$), while ignoring constraint (F). It is evident that the relaxed problem does not depend on n. Since the principal's objective is linear, any solution is almost everywhere boundary. We will focus on the solution that is boundary everywhere. This solution is a cutoff rule that maximizes the probability of selecting agents with types above z^* subject to the constraint that the types below z^* are selected with high enough probability to provide incentives for

[8]To show uniqueness of z^*, rewrite (7) and (8) as $\int_a^{z^*}(z^* - x)h(x)\mathrm{d}F(x) = \int_{z^*}^b (x - z^*)\mathrm{d}F(x)$ and observe that the left-hand side is strictly increasing, while the right-hand side is strictly decreasing in z^*.

truthful reporting, $g(x) \geq h(x) \sup g$. So, the solution is given by (9), where $r^* = \sup g^*$ is a constant determined by (F_0):

$$\int_a^{z^*} r^* h(x)\mathrm{d}F(x) + \int_{z^*}^b r^* \mathrm{d}F(x) = 1,$$

that yields (8).

The incentive constraint thus pins down the distribution of selected types, $G^*(x) = \int_a^x g^*(s)\mathrm{d}F(s)$ and determines the best attainable expected payoff $\int_a^b x\mathrm{d}G^*(x)$ absent feasibility constraint (F). Equation (7) is equivalent to equation $z^* = \int_a^b x\mathrm{d}G^*(x)$.

Heuristically, if the expected value $x^* = \int_a^b x\mathrm{d}G^*(x)$ of the chosen agent is above the cutoff z^*, the principal can increase the cutoff, thus increasing the relative probability of selecting an average type and improving her payoff. If, on the other hand, $x^* < z^*$, the principal can decrease the cutoff and thus decrease the relative probability of selecting an average type x^*, improving her payoff. Hence, at the optimum we have $x^* = z^*$.

Proof of Theorem 2. To derive the upper bound on the principal's payoff we solve (P) subject to (IC) and (F_0), while ignoring constraint (F).

Solving the programs corresponding to ($\mathrm{P}_{\min}$) and ($\mathrm{P}_{\max}$) with the relaxed set of constraints that ignores (F) gives

$$\underline{g}_r(x) = rh(x) \quad \text{and} \quad \overline{g}_r(x) = r, \quad x \in X.$$

By the argument in the proof of Theorem 1, each concatenation of $\underline{g}_r$ and $\overline{g}_r$ at z,

$$g_z^* = \begin{cases} rh(x), & x \leq z, \\ r, & x > z, \end{cases}$$

with r satisfying $\int_a^z rh(x)\mathrm{d}F(x) + \int_z^b r\mathrm{d}F(x) = 1$ is FOSD maximal. Solving for r yields

$$r = \left(\int_a^z h(x)\mathrm{d}F(x) + \int_z^b \mathrm{d}F(x)\right)^{-1} = (H(z) + 1 - F(z))^{-1}, \tag{10}$$

where we denote

$$H(x) = \int_a^x h(t)\mathrm{d}F(t).$$

Substituting g_z^* and (10) into the principal's objective function yields

$$\max_{z \in X} \frac{1}{H(z) + 1 - F(z)} \left(\int_a^z xh(x)\mathrm{d}F(x) + \int_z^b x\mathrm{d}F(x)\right)$$

The first-order condition is equivalent to

$$z(h(z)-1)f(z)(H(z)+1-F(z)) - (h(z)-1)f(z)\left(\int_a^z xh(x)\mathrm{d}F(x) + \int_z^b x\mathrm{d}F(x)\right) = 0.$$

By assumption, $f(x) > 0$ and $h(x) < 1$, hence the above is equivalent to

$$\int_a^z xh(x)\mathrm{d}F(x) + \int_z^b x\mathrm{d}F(x) = z(H(z) + 1 - F(z)). \tag{11}$$

Observe that (10) and (11) are identical to (7) and (8) with $z^* = z$ and $r^* = r$, and thus g_z^* is precisely (9). First-order condition (11) is also sufficient by the argument provided in Footnote 8. ■

Attainment of the upper bound. Let $E_t = \{x : r^*h(x) \le t\} \cap [a, z^*]$. Denote by $\bar{n}$ the smallest number that satisfies

$$\int_{E_t} r^*h(x)\mathrm{d}F(x) \ge \Big(F(E_t)\Big)^{\bar{n}} \quad \text{for all } t \in [0, r^*]. \tag{12}$$

This is a condition on primitives: F and h determine z^* and r^* and, consequently, $\bar{n}$. It means that feasibility constraint (F) is not binding for every type x for which incentive constraint (IC) holds as equality, $g^*(x) = r^*h(x) < r^*$.

Corollary 1. *There exists an allocation rule that attains the payoff of z^* if and only if $n \ge \bar{n}$.*

Proof of Corollary 1. By Theorem 2, z^* can be achieved if and only if g^* given by (9) is feasible. Thus we need to verify that g^* satisfies (F) if and only if (12) holds.

Denote $E_t = \{x \in X : g^*(x) \le t\}$. Since the image of g^* is $\{r^*h(x) : x \in [0, z^*]\} \cup \{r^*\}$, we have $E_t = \{x : r^*h(x) \le t\}$ when $t \in \{r^*h(x) : x \in [0, z^*]\}$ and $E_t = X$ when $t = r^*$. So, for $g = g^*$, (F) is equivalent to:

$$\int_{x \in E_t} r^*h(x)\mathrm{d}F(x) \ge \big(F(E_t)\big)^{\bar{n}} \quad \text{for all } t \in \{r^*h(x) : x \in [0, z^*]\}, \tag{13}$$

and

$$\int_{x \in X} g^*(x)\mathrm{d}F(x) \ge 1. \tag{14}$$

Observe that (13) is equivalent to (12), while (14) is redundant by (F_0). ■

Condition (12) is not particularly elegant. Instead, one can use the following sufficient condition, which is simple and independent of F and z^*. Let $\tilde{n}$ the smallest number such that[9]

$$\frac{c(x)}{v(x)} \le 1 - \frac{1}{\tilde{n}} \quad \text{for all } x \in X. \tag{15}$$

Corollary 2. *There exists an allocation rule that attains the payoff of z^* if $n \ge \tilde{n}$.*

Proof of Corollary 2. We need to verify (12) under the assumption that $n \ge \tilde{n}$, which is equivalent to

$$h(x) \ge \frac{1}{n}, \quad x \in X. \tag{16}$$

Let $E_t = \{x \in X : r^*h(x) \le t\}$. Note that (12) is equivalent to

$$\int_{A_t} r^*h(x)\mathrm{d}F(x) \ge \Big(F(A_t)\Big)^{\bar{n}} \quad \text{for all } t \le r^* \max_{x \in [a, z^*]} h(x).$$

[9]Note that $\tilde{n}$ exists if and only if $\sup_{x \in X} c(x)/v(x) < 1$.

z Denote $F_t = F(E_t)$ and $H_t = \int_{x\in E_t} h(x)\mathrm{d}F(x)$, and denote $H^* = H([a,z^*])$ and $F^* = F([a,z^*])$. In these notations we have $r^* = (H^* + 1 - F^*)^{-1}$, and the above inequality is equivalent to

$$H_t \geq (H^* + 1 - F^*)F_t^n,$$

or

$$H_t(1 - F_t^n) \geq \Big(1 - F_t + (H^* - H_t) - (F^* - F_t)\Big)F_t^n.$$

Since $(H^* - H_t) - (F^* - F_t) \leq 0$ by $h(x) \leq 1$ and since $H_t \geq \frac{1}{n}F_t$ by (16), the above inequality holds if

$$\frac{1}{n}F_t(1 - F_t^n) \geq (1 - F_t)F_t^n.$$

This is true, since

$$\frac{1 - F_t^n}{F_t^{n-1}(1 - F_t)} = \frac{F^{-n} - 1}{F^{-1} - 1} = 1 + F^{-1} + F^{-2} + \ldots + F^{-(n-1)} \geq n.$$

■

Note that in some cases $\bar{n}$ and $\tilde{n}$ need not be very large. For example, $\tilde{n} \leq 2$ if $c(x) \leq \frac{1}{2}v(x)$ for all x, i.e., agents can be penalized by at most half of their gross payoff.

Remark 1. Theorem 2 implies that the optimal rule with $\bar{n}$ agents is weakly superior to any rule with $n > \bar{n}$ agents. That is, the value of competition is limited and expanding the pool of agents beyond $\bar{n}$ confers no benefit to the principal.

Implementation of the upper bound. Consider the following *shortlisting procedure.* Let each agent $i = 1, \ldots, n$ be short-listed with some probability $q(y_i)$ given report y_i. The rule chooses an agent from the shortlist with equal probability. If the shortlist is empty, then the choice is made at random, uniformly among all n agents.

Corollary 3. *Let $n \geq \bar{n}$. Then the shortlisting procedure with*

$$q(x) = \begin{cases} \frac{Kh(x)-1}{K-1}, & x < z^*, \\ 1, & x \geq z^*. \end{cases} \tag{17}$$

attains the upper bound z^*, where K is the unique solution of

$$\frac{(K-1)^{n-1}}{K^n} = \frac{(r^*-1)^{n-1}}{(r^*)^n}, \quad K > r^*. \tag{18}$$

Proof of Corollary 3. Consider q defined by (17). Let $Q = \int_X q(x)\mathrm{d}F(x)$ be the ex-ante probability to be short-listed, and let A and B be the expected probabilities to be chosen conditional on being shortlisted and conditional on not being short-listed, respectively:

$$A = \sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1}Q^{k-1}(1-Q)^{n-k} \quad \text{and} \quad B = \frac{1}{n}(1-Q)^{n-1}.$$

Then an agent's probability to be chosen conditional on reporting x is equal to $g_i(x) = q(x)A + (1 - q(x))B$. Set $K = A/B$ and evaluate

$$g(x) \equiv \sum_i g_i(x) = n\left(q(x)A + (1-q(x))B\right) = \begin{cases} nAh(x), & x < z^*, \\ nA, & x \geq z^*. \end{cases}$$

By Proposition 2, the shortlisting procedure achieves z^* if $g(x) = g^*(x)$ for all x, where g^* is given by (9). This holds if $nA = r^*$. Thus we need to verify that condition (18) implies $A = r^*/n$. We have

$$Q = \int_X q(x)\mathrm{d}F(x) = \frac{K}{K-1}\left(\int_a^{z^*} h(x)\mathrm{d}F(x) + \int_{z^*}^b \mathrm{d}F(x)\right) - \frac{1}{K-1} \tag{19}$$
$$= \frac{K}{K-1}\frac{1}{r^*} - \frac{1}{K-1} = \frac{K - r^*}{r^*(K-1)}, \quad \text{thus} \;\; 1 - Q = \frac{K(r^*-1)}{r^*(K-1)},$$

where we used (8). Also,

$$A = \sum_{k=1}^n \frac{1}{k}\frac{(n-1)!}{(k-1)!(n-k)!}Q^{k-1}(1-Q)^{n-k} = \frac{1}{nQ}\sum_{k=1}^n \frac{n!}{k!(n-k)!}Q^k(1-Q)^{n-k}$$
$$= \frac{1}{nQ}\left(1 - (1-Q)^n\right).$$

Substituting (19) into the above yields

$$A = \frac{r^*(K-1)}{n(K-r^*)}\left(1 - \left(\frac{K}{r^*}\right)^n\left(\frac{r^*-1}{K-1}\right)^n\right).$$

By (18),

$$A = \frac{r^*(K-1)}{n(K-r^*)}\left(1 - \frac{r^*-1}{K-1}\right) = \frac{r^*(K-1)}{n(K-r^*)}\frac{K-r^*}{K-1} = \frac{r^*}{n}.$$

∎

5. Small number of agents

If the number of agents is small, $n < \bar{n}$, then feasibility constraint (F) is binding at the optimum. The upper bound cannot be attained, and Theorem 2 is not applicable. To find an optimal allocation rule as described by Theorem 1, we solve problems ($\mathrm{P}_{\min}$) and ($\mathrm{P}_{\max}$) subject to both incentive compatibility and feasibility constraints.

The solution of ($\mathrm{P}_{\max}$) is easy.

Lemma 4. *For every $r \in [1, \bar{r}]$, the solution of* ($\mathrm{P}_{\max}$) *is equal to*

$$\overline{g}_r(x) = \begin{cases} nF^{n-1}(x), & x \in [a, \bar{x}_r), \\ r, & x \in [\bar{x}_r, b], \end{cases}$$

where $\bar{x}_r < b$ is implicitly defined by

$$\int_{\bar{x}_r}^b r\mathrm{d}F(x) = 1 - F^n(\bar{x}_r). \tag{20}$$

Proof of Lemma 4. As $r \le \bar{r} < nF^{n-1}(b) = n$, there exists $\bar{x}_r$ such that feasibility constraint (F) does not bind while incentive constraint ($\mathrm{IC}_{\max}$) binds for $x \ge \bar{x}_r$, and the opposite is true for $x < \bar{x}_r$. Consequently, $\overline{g}_r(x) = r$ for $x \ge \bar{x}_r$, while $\overline{g}_r(x) = nF^{n-1}(x)$ for $x < \bar{x}_r$. The value of $\bar{x}_r$ is the unique solution of (20), which is the equation that guarantees that feasibility constraint binds at all $x \le \bar{x}_r$ and slacks at all $x > \bar{x}_r$. ■

The solution of ($\mathrm{P}_{\min}$) is more complex, as it involves function $h(x)$ in the constraints. To obtain tractable results, we make an assumption of "single-crossing" of incentive compatibility and feasibility conditions. Specifically, for every r there exists a threshold $\underline{x}_r$ such that for function $g(x) = rh(x)$ feasibility constraint (F′) is satisfied on interval $[a, z]$ for any z below the threshold and is violated for any z above the threshold.

Assumption 1 (Single-crossing property). *For every $r \in [1, \bar{r}]$ there exists $\underline{x}_r \in X$ such that*

$$\int_a^x rh(t)\mathrm{d}F(t) \ge F^n(x) \text{ if and only if } x \le \underline{x}_r, \tag{21}$$

and

$$\int_{E_t} rh(x)\mathrm{d}F(x) \ge F^n(E_t) \text{ for all } t \in [0, n], \tag{22}$$

where $E_t = \{x : rh(x) \le t\} \cap [a, \underline{x}_r]$.

The proposition below provides a sufficient condition for single crossing property to hold.

Lemma 5. *Assumption 1 holds if $h(F^{-1}(t))$ is weakly concave.*

Proof of Lemma 5. By concavity of $h(F^{-1}(t))$, for every $n \ge 2$, $h(F^{-1}(t)) - nt^{n-1}$ is concave (and strictly concave for $n > 2$). Hence, by monotonicity of F, for all $r \ge 0$

$$rh(x) - nF^{n-1}(x) \text{ is quasiconcave.}$$

Denote by $\tilde{x}$ the greatest solution of $rh(\tilde{x}) = nF^{n-1}(\tilde{x})$. Since $rh(0) \ge nF^{n-1}(0) = 0$ and $rh(1) \le nF^{n-1}(1) = n$ by $h(1) \le 1$ and $r \le n$, such a solution always exists, and moreover, $rh(t) - nF^{n-1}(t)$ is nonnegative on $[a, \tilde{x}]$ and negative on $(\tilde{x}, b]$. As a result,

$$\int_a^x (rh(t) - nF^{n-1}(t))\mathrm{d}F(t)$$

is positive and increasing for $x < \tilde{x}$ and strictly decreasing for $x > \tilde{x}$. Consequently, there exists an $\underline{x}_r$ that satisfies[10] (21).

Next, we show that (22) holds. If h is weakly increasing, then (22) is implied by (21). In the case of nonmonotonic h, the same argument applies after a measure-preserving monotonization of h, a "reordering" types in X in the ascending order in the image of

[10]Note for $n = 2$, $rh(x) - 2F(x)$ is *weakly* quasiconcave. So it is possible that there is an interval $[x', x'']$ of solutions of (24), but in that case $rh(x) = 2F(x)$ for all $x \in [x', x'']$, and hence every $\underline{x}_r \in [x', x'']$ defines the same function in (24).

h. Formally, a *measure-preserving monotonization of* h is a weakly increasing function $\tilde{h}: X \to [0,n]$ such that for every $t \in [0,1]$

$$\int_{\{x:h(x)\geq t\}} h(x)\mathrm{d}F(x) = \int_{\{x:\tilde{h}(x)\geq t\}} \tilde{h}(x)\mathrm{d}F(x).$$

Since $h \circ F^{-1}$ is concave by assumption, the monotonization procedure is straightforward. Fix r and define $\bar{t} = F(\underline{x}_r)$. Let $\bar{s} = \max_{t\in[0,\bar{t}]} rh(F^{-1}(t))$. For every $s \in [0,\bar{s}]$ define $E_s = \{t \in [0,\bar{t}] : h(F^{-1}(t)) \geq s\}$. Observe that $\sup E_s - \inf E_s$ is strictly decreasing and concave, due to concavity of $h \circ F^{-1}$. Let $\phi(t)$ be the inverse of $\bar{t} - \sup E_s + \inf E_s$. Thus, $\phi(t)$ is strictly increasing and concave, and hence $r\tilde{h}(x) - nF^{n-1}(x)$ is quasiconcave, where $\tilde{h}$ is implicitly defined by $\phi(t) = \tilde{h}(F(t))$, $t \in [0,\bar{t}]$. ■

Examples that satisfy Assumption 1 by verification of the condition in Lemma 5:

(a) Let the penalty be proportional to the value, $c(x) = \alpha v(x)$, $0 < \alpha < 1$. Then h is constant,

$$h(x) = 1 - \alpha, \quad x \in X.$$

(b) Suppose that the principal and the chosen agent share a unit surplus: the principal's payoff is x and the agent's payoff is $v(x) = 1 - x$. Let the penalty be constant, $c(x) = \bar{c} < 1$ and let $X = [a,b] \subseteq [0, 1-\bar{c}]$ (so that $v(x) \geq c(x)$ for all $x \in X$). Then

$$h(x) = 1 - \frac{\bar{c}}{1-x}, \quad x \in X,$$

and $h(F^{-1}(t))$ is concave, provided $f(x)(1-x)^2$ is weakly decreasing or, equivalently, $f'(x) < 2\frac{f(x)}{1-x}$ if f is differentiable.

(c) Suppose that the principal is benevolent and wishes to maximize the agents' surplus, that is, $v(x) = x$. Let the penalty be constant, $c(x) = \bar{c} < 1$, and let $X = [a,b] \subseteq [\bar{c}, 1]$ (so that $v(x) \geq c(x)$ for all $x \in X$). Then

$$h(x) = 1 - \frac{\bar{c}}{x}, \quad x \in X,$$

and $h(F^{-1}(t))$ is concave, provided $f(x)x^2$ is weakly increasing or, equivalently, $f'(x) \geq -2\frac{f(x)}{x}$ if f is differentiable.

Lemma 6. *Let Assumption 1 hold. Then for every $r \in [1,\bar{r}]$ the solution of problem* ($\mathrm{P}_{\min}$) *is equal to*

$$\underline{g}_r(x) = \begin{cases} rh(x), & x \in [a, \underline{x}_r], \\ nF^{n-1}(x), & x \in (\underline{x}_r, b], \end{cases} \tag{23}$$

where $\underline{x}_r > a$ is implicitly defined by

$$\int_a^{\underline{x}_r} rh(x)\mathrm{d}F(x) = F^n(\underline{x}_r). \tag{24}$$

Proof of Lemma 6. By Assumption 1 we have ($\mathrm{IC}_{\min}$) binding on $[a, \underline{x}_r]$ and (F′) binding on $(\underline{x}_r, b]$, where $\underline{x}_r$ defined by (23) is the threshold in (21). Consequently, $\underline{g}_r(x) = rh(x)$ on $[a, \underline{x}_r]$, while $\underline{g}_r(x) = nF^{n-1}(x)$ on $(\underline{x}_r, b]$. ■

Under Assumption 1, the solutions of ($\mathrm{P}_{\max}$) and ($\mathrm{P}_{\min}$) are given by Lemmata 4 and 6 whose concatenations are easily obtained using (F_0). This permits a clean characterization of the optimal rule.

Theorem 3. *Suppose that Assumption 1 holds and let $n < \bar{n}$. Then g^* is a solution of* (P) *if and only if*

$$g^*(x) = \begin{cases} rh(x), & x \leq \underline{x}_r, \\ nF^{n-1}(x), & \underline{x}_r < x \leq \bar{x}_r, \\ r, & x > \bar{x}_r, \end{cases} \tag{25}$$

where $\bar{x}_r$ and $\underline{x}_r$ are defined by (20) *and* (24)*, respectively, and $r \in [1, \bar{r}]$ is the solution of*

$$\int_a^{\underline{x}_r} (\underline{x}_r - x)h(x)\mathrm{d}F(x) = \int_{\bar{x}_r}^b (x - \bar{x}_r)\mathrm{d}F(x). \tag{26}$$

Equation (26) is the first-order condition w.r.t. r on the principal's expected payoff. The left-hand side is the marginal payoff loss due to assigning higher probability mass on types below $\underline{x}_r$, while the right-hand side is the marginal payoff gain due to assigning higher probability mass on types above $\overline{x}_r$. The reduced form allocation for types between $\underline{x}_r$ and $\overline{x}_r$ is determined by the highest ordered statistic, independent of r, and thus the expected payoff from these types is marginally constant w.r.t. r.

Proof of Theorem 3. By Theorem 1, the solution g^* is chosen among concatenations of $\underline{g}_r$ and $\overline{g}_r$. By Lemmata 4 and 6, g^* is given by (25) for some r if $\underline{x}_r < \overline{x}_r$ and by (9) if $\underline{x}_r \geq \overline{x}_r$. The latter case is ruled out by the assumption that $n < \bar{n}$: (12) is violated and the feasibility constraint must be binding for a positive measure of types. Hence, g^* is as in (25), and r is chosen to maximize the payoff of the principal:

$$\int_a^{\underline{x}_r} xrh(x)\mathrm{d}F(x) + \int_{\underline{x}_r}^{\overline{x}_r} xnF^{n-1}(x)\mathrm{d}F(x) + \int_{\overline{x}_r}^b xr\mathrm{d}F(x).$$

Taking the derivative w.r.t. r yields the first-order condition that is precisely (26). Since $\underline{x}_r$ is strictly increasing and $\overline{x}_r$ is strictly decreasing in r, the solution of (26) is unique. ■

Implementation of the optimal rule. Solution g^* can be implemented by a restricted-bid auction augmented by a shortlisting procedure. The principal asks each agent to report $y_i \in [\underline{x}_r, \overline{x}_r]$. Then, the agents are divided into three groups: regular candidates who report y_i strictly between $\underline{x}_r$ and $\bar{x}_r$, "superstars" who report $y_i = \bar{x}_r$, which is interpreted as "my type is equal to or above $\overline{x}_r$", and the "waiting list" candidates who report $y_i = \underline{x}_r$, which is interpreted as "my type is equal to or below $\underline{x}_r$". The principal chooses a superstar at random. If there are no superstars, the principal chooses a regular candidate with the highest report. If there are neither superstars nor regular candidates, the waiting-list candidates report their types to the principal, who then applies the shortlisting procedure q to all waiting-list candidates, where

$$q(x) = \frac{Kh(x) - 1}{K - 1}, \quad x \in [a, \underline{x}_r], \tag{27}$$

and K is the unique solution of

$$\frac{(K-1)^{n-1}}{K^n} = \frac{\left(\frac{F(\underline{x}_r)}{H(\underline{x}_r)} - 1\right)^{n-1}}{\left(\frac{F(\underline{x}_r)}{H(\underline{x}_r)}\right)^n}, \quad K > \frac{F(\underline{x}_r)}{H(\underline{x}_r)}. \tag{28}$$

After selection has been made, the principal penalizes the selected agent unless he has sent a truthful message (in particular, for $x > \overline{x}_r$, message $y = \overline{x}_r$ is considered truthful).

Corollary 4. *Let $n < \bar{n}$ and suppose that Assumption 1 holds. Then the restricted bid auction with shortlisting procedure q is optimal.*

The proof essentially repeats that of Corollary 3 and thus is omitted.

The interval $[\underline{x}_r, \bar{x}_r]$ where agents' types are fully separated shrinks as n increases and disappears as n approaches $\bar{n}$.

Corollary 5. *Suppose that Assumption 1 holds. Then, at the solution of* (P), *difference $\bar{x}_r - \underline{x}_r$ is decreasing in n; moreover, $\underline{x}_r \to z^*$ and $\bar{x}_r \to z^*$ as $n \to \bar{n}$.*

The proof is straightforward by definition of $\underline{x}_r$ and $\bar{x}_r$, and (26).

Example (Proportional penalty). Consider the example with the penalty proportional to the value, $c(x) = \alpha v(x)$. In this case, the shortlisting procedure in the optimal rule becomes degenerate (see (27)), and the optimal rule reduces to the *simple restricted bid auction* that allows the agents to report $y_i \in [\underline{x}_r, \bar{x}_r]$ and then selects an agent with the highest report, splitting ties randomly, and penalizes the selected agent unless he makes the report closest to the truth.

Corollary 6. *If $n < \bar{n}$ and $\frac{c(x)}{v(x)}$ is constant, then the simple restricted bid auction is optimal.*

If $n \geq \overline{n}$, the optimal rule becomes a two bid auction with limited participation: it dismisses a fixed number of agents, asks the remaining agents to report whether their type is above or below a threshold, and chooses at random any agent whose reported type is above the threshold if there is at least one such report, and any agent at random otherwise.

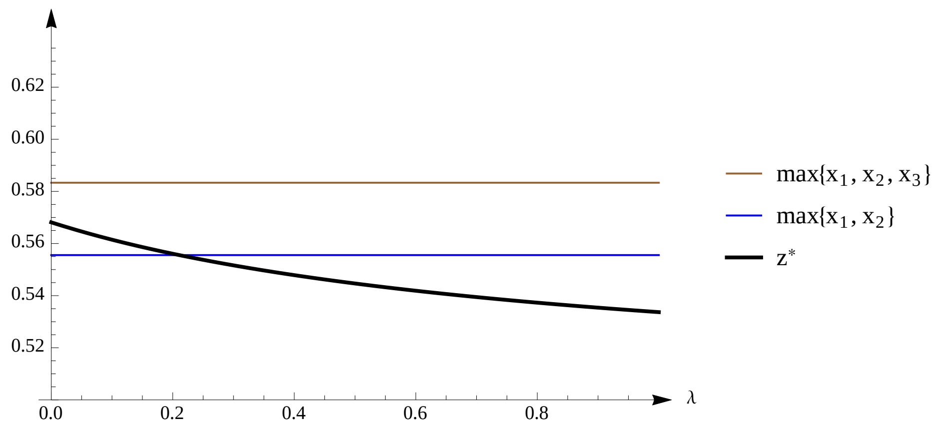

FIG. 2. Plot of z^* as a function of $\lambda \in [0,1]$ with $c = 1/3$.

6. Example

In this section we find the optimal reduced-form allocation rule g^* for an environment with constant penalty $c(x) = c \in (0, 1/2)$. We assume that the agent's payoff is equal to

$$v_\lambda(x) = (1-\lambda)x + \lambda(1-x), \quad \lambda \in [0,1].$$

Hence, for $\lambda = 1/2$, the agent's payoff $v(x) = 1/2$ is type independent; for $\lambda = 1$ the parties are splitting the unit surplus, $v(x) = 1-x$; for $\lambda = 0$, the parties' payoffs are aligned, $v(x) = x$ (c.f. examples (a), (b), and (c) after Assumption 1). We set $X = [c, 1-c]$, so $h_\lambda(x) = 1 - \frac{c}{v_\lambda(x)} \geq 0$ for all $x \in X$, and assume that F is uniform, $F(x) = \frac{x-c}{1-2c}$. Note that Assumption 1 is satisfied for any $\lambda \in [0,1]$ and any $c \in (0, 1/2)$.

By (7) and (8), and Theorem 2, the upper bound satisfies

$$\begin{aligned} z^* - \frac{1}{2} &= \frac{c}{1-2c}\left(z^* - c\right)^2 \quad \text{if } \lambda = \tfrac{1}{2}, \text{ and} \\ z^* - \frac{1}{2} &= \frac{c}{1-2c}\left(z^* I(z^*) - \frac{z^* - c - \lambda I(z^*)}{1-2\lambda}\right) \quad \text{if } \lambda \neq \tfrac{1}{2}, \end{aligned}$$

where $I(z^*) = \int_c^{z^*} \frac{1}{v_\lambda(x)} \mathrm{d}x = \frac{\ln v_\lambda(z^*) - \ln v_\lambda(c)}{1-2\lambda}$.

There is no closed-form solution for $\lambda \neq \frac{1}{2}$. The black curve on Fig. 2 shows the numeric solution for the upper bound on the principal's expected payoff as λ varies from 0 to 1 and $c = 1/3$. For comparison to the complete information case, the blue and brown curves show the expected values of the highest ordered statistics for $n = 2$ and $n = 3$, respectively.

Observe that

$$z^* = \left(\int_c^{z^*} h_\lambda(x) \frac{\mathrm{d}x}{1-2c} + \int_{z^*}^{1-c} \frac{\mathrm{d}x}{1-2c}\right)^{-1} = \left(1 - \frac{c}{1-2c} I(z^*)\right)^{-1}.$$

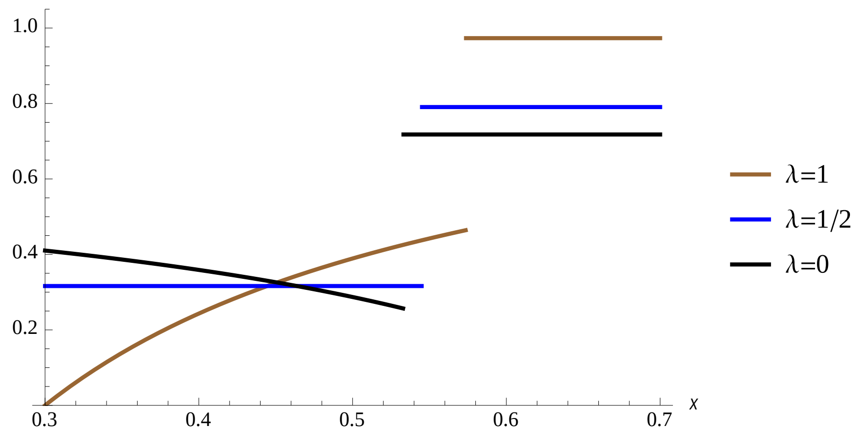

FIG. 3. Plots of $g_i^*(x)$ for $\lambda \in \{0, \frac{1}{2}, 1\}$ with $n = 2$ and $c = 1/3$.

If the upper bound z^* is achievable, the optimal rule

$$g^*(x) = \begin{cases} r^*\left(1 - \frac{c}{(1-\lambda)x + \lambda(1-x)}\right), & x \le z^*, \\ r^*, & x \ge z^* \end{cases} \tag{29}$$

is numerically computable for each $c \in (0, 1/2)$ and each $\lambda \in [0, 1]$. Fig. 3 shows the ex-ante probability $g_i^*(x) = \frac{1}{n} g^*(x)$ that an agent is selected conditional on her type x, for $c = 1/3$ and $n = 2$ and three values of λ that correspond to examples (a), (b), and (c) after Assumption 1.

We now determine the value of n required to achieve z^*. Sufficient condition (15) reduces to

$$n \ge \tilde{n} = \max\left\{\frac{\lambda}{\lambda - c}, \frac{1-\lambda}{1-\lambda-c}\right\}.$$

Necessary and sufficient condition (12) reduces to

$$1 - r^*\left(1 - \frac{z^* - c}{1 - 2c}\right) = \left(\frac{z^* - c}{1 - 2c}\right)^{\bar{n}},$$

from which $\bar{n}$ can be computed numerically. For $c = 1/3$, the value of $\tilde{n}$ is given by the blue curve on Fig. 4 (left) and the value of $\bar{n}$ is given by the black curve on Fig. 4 (right).

Note that $\bar{n}$ is substantially lower than $\tilde{n}$ for all λ, so condition (12) is tighter than condition (15). This is especially important for $\lambda < c$ and $\lambda > 1 - c$ where $\tilde{n} = \infty$, so condition (15) is uninformative.

We thus established that the upper bound z^* can be attained by optimal allocation rule (29) for all $\lambda \in [0, 1]$ if $n \ge 3$ and for $\lambda \ge \bar{\lambda} \approx 0.807$ if $n = 2$. It remains to solve the problem for $n = 2$ and $\lambda < \bar{\lambda}$. By Proposition 3, we need to find $\overline{x}_r$ from (20) and

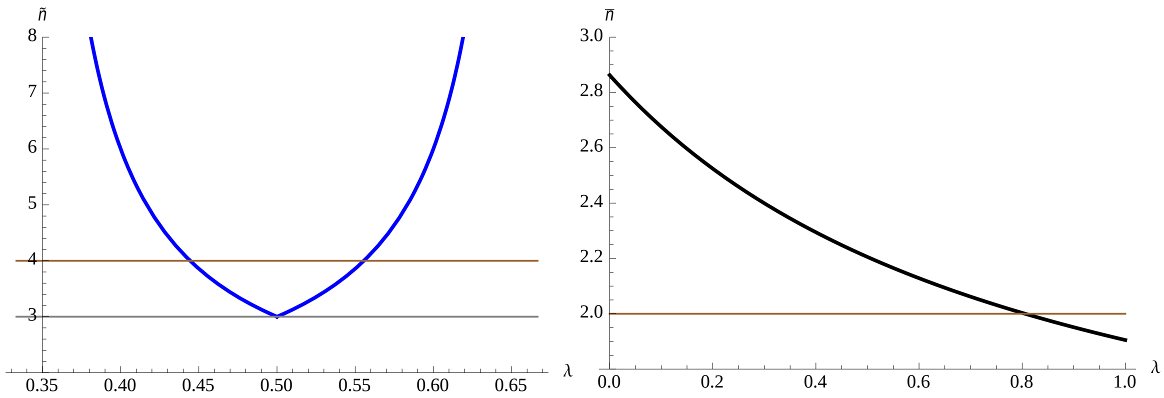

FIG. 4. Plots of $\tilde{n}$ and $\bar{n}$ as functions of $\lambda \in [0, 1]$ with $c = 1/3$.

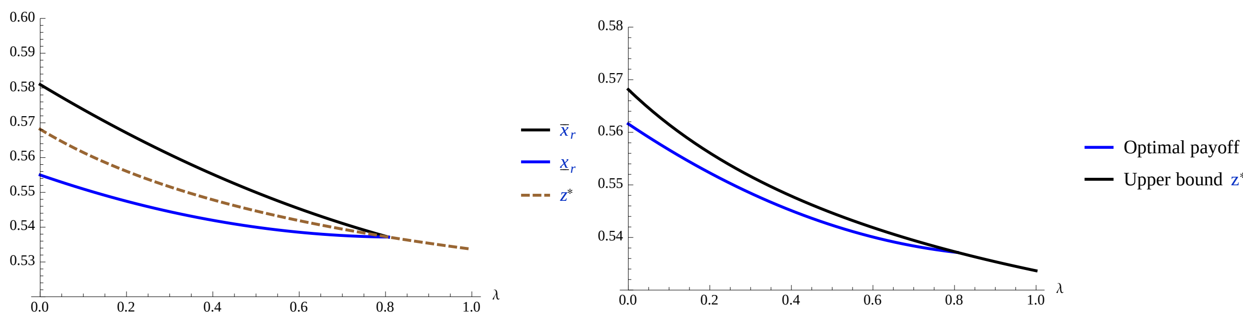

FIG. 5. Optimal rule when the upper bound is not achieved.

$\underline{x}_r$ from (24), and satisfy (26). The solution of (20) is straightforward,

$$\overline{x}_r = c + (1 - 2c)(r - 1). \tag{30}$$

Equation (24) can be simplified as

$$r\left(1 - \frac{I(\underline{x}_r)}{\underline{x}_r - c}\right) = \frac{\underline{x}_r - c}{1 - 2c}, \tag{31}$$

Finally, equation (26) reduces to

$$\frac{(\underline{x}_r - c)^2}{2(1 - 2c)} - \frac{c}{2(1 - 2c)}\left(\underline{x}_r I(\underline{x}_r) - \frac{\underline{x}_r - c - \lambda I(\underline{x}_r)}{1 - 2\lambda}\right) = \frac{(1 - c - \overline{x}_r)^2}{2(1 - 2c)}. \tag{32}$$

Solving the system of equations (30), (31), and (32) for $(r, \underline{x}_r, \overline{x}_r)$ numerically yields all the components of the optimal allocation rule (25).

Fig. 5 (left) depicts $\underline{x}_r$, and $\overline{x}_r$ at optimal r for $c = 1/3$. The competitive range $[\underline{x}_r, \overline{x}_r]$ of the optimal allocation is greater when the principal's and agents' preferences are more aligned (small λ), but shrinks as the principal's bias grows (λ increases) and eventually vanishes at $\lambda = \bar{\lambda}$.

Fig. 5 (right) shows that the optimal expected payoff of the principal (blue curve) decreases as the principal's and agents' preferences become less aligned (λ increases), though slower than the upper bound z^* (black curve) does. Note that for $\lambda > \bar{\lambda}$ the upper bound is achieved, so the two curves coincide.

7. Related literature

In our model, the utility is not transferable. Optimal mechanism design with transfers that can depend on ex-post information has been studied in, e.g., Mezzetti (2004), DeMarzo, Kremer and Skrzypacz (2005), Eraslan and Yimaz (2007), Dang, Gorton and Holmström (2013), Deb and Mishra (2013), and Ekmekci, Kos and Vohra (2013). This literature is surveyed in Skrzypacz (2013).

In our model, there are restricted ex-post transfers (penalties). Burguet, Ganuza and Hauk (2012) and Decarolis (2014) study contract design with limited liability where, similarly to our model, agents with low values are given rents to stop them bidding too aggressively to win the contract.[11] Bar and Gordon (forthcoming) study an allocation problem with ex-post verifiable types and non-negative transfers, in which the allocation might be inefficient due to incentives to save on the subsidies paid to the agents. By contrast, in our model the principal does not collect the penalties imposed on the agents.

There is a literature on mechanism design with partial transfers in which the agents' information is non-verifiable. In Chakravarty and Kaplan (2013) and Condorelli (2012b), a benevolent principal would like to allocate an object to the agent with the highest valuation, and the agents signal their private types by exerting socially wasteful effort. Condorelli (2012b) studies a general model with heterogeneous objects and agents and characterizes optimal allocation rules where a socially wasteful cost is a part of mechanism design. Chakravarty and Kaplan (2013) restrict attention to homogeneous objects and agents, and consider environments in which socially wasteful cost has two components: an exogenously given type and a component controlled by the principal. In particular, they demonstrate conditions under which, surprisingly, the uniform lottery is optimal.[12] Che, Gale and Kim (2013) consider a problem of efficient allocation of a resource to budget constrained agents and show that a random allocation with resale can outperform competitive market allocation. In an allocation problem in which the private and the social values of the agents' are private information, Condorelli (2012a) characterizes conditions under which optimal mechanism is stochastic and does not employ payments.

Appendix

[11] Similar forces are at play in Mookherjee and Png (1989) who solve for the optimal punishment schedule for crimes when punishments are bounded.

[12] See also McAfee and McMillan (1992), Hartline and Roughgarden (2008), Yoon (2011) for environments without transfers and money burning. In addition, money burning is studied in Ambrus and Egorov (2012) in the context of a delegation model.

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