Reputation Effects and Equilibrium Degeneracy in Continuous-Time Games*

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Abstract

We study a continuous-time reputation game between a large player and a population of small players in which the actions of the large player are imperfectly observable. We explore two versions of the game. In the complete information game, in which it is common knowledge that the large player is a strategic normal type, we show that *intertemporal incentives collapse*: irrespective of players' patience and signal informativeness, the set of equilibrium payoffs of the large player coincides with the convex hull of the set of static Nash equilibrium payoffs.

In the incomplete information game, the small players believe that the large player could be a strategic *normal type* or a *commitment type*, who plays the same action at all times. With this perturbation, nontrivial intertemporal incentives arise. In this two-type setting, we characterize the set of sequential equilibrium payoffs of the large player using an ordinary differential equation. Unlike in discrete time, in a large class of games in continuous time the sequential equilibrium is unique and Markov in the small players' belief for any prior.

1 Introduction.

In many economic environments a large player can benefit from committing to a course of actions to influence the behavior of a population of small players. A firm may wish to commit to fight potential entrants, to provide high quality to its customers, to honor implicit labor contracts, and to generate good returns to investors. Governments can benefit from commitment to a non-inflationary monetary policy, low capital taxation and efforts to fight corruption. Often the actions of the large player are imperfectly observable. For example, the quality of a firm's products may be a noisy outcome of a firm's hidden effort to maintain quality standards. The actual inflation rate can be a noisy signal of money supply.

We study a repeated game between a large player and a population of small players to gain insight behind the possibility of commitment in these situations. Our setting is a continuous-time analogue of the repeated game of Fudenberg and Levine (1992), hereafter FL, in which the public signals about the large player's actions are distorted by Brownian motion. We assume

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that there is a continuum of small players and that only the distribution of small players' actions is publicly observed, but not the actions of any individual small player. Hence, as in FL, the small players behave myopically in every equilibrium, acting to maximize their instantaneous expected payoffs.

First we consider the complete information version of this dynamic game, in which it is common knowledge that the large player is a strategic normal type. We find that, due to monitoring imperfection in a continuous-time setting with Brownian noise, the large player cannot achieve higher payoffs than in static Nash equilibria. This result does not hold in the discrete-time setting of FL. In a complete information discrete-time repeated game, payoffs above static Nash equilibrium can be attained in simple equilibria with two regimes: a commitment regime, where the large player's continuation payoff is greater than the static Nash equilibrium payoffs, and a punishment regime, where it is at least as low as a static Nash. In the commitment regime the large player chooses an action that differs from his myopic best response, which influences the actions of the small players in a positive way, for the fear of triggering the punishment regime if the signal about his actions turns out sufficiently negative.

We show that in a continuous-time setting such commitment to an action that differs from the large player's static best response becomes completely impossible. One way to explain this result is to borrow for intuition from Abreu, Milgrom, and Pearce (1991) who study discrete-time games in the limit as actions become more frequent. Translated into our setting, that intuition tells us that with frequent actions players see very little information per period and so the statistical tests that trigger punishments to support commitment give false positives too often. This effect is especially strong when information arrives continuously via a Brownian motion as shown in Sannikov and Skrzypacz (2006a) for games with frequent actions.[1] Directly in continuous time, we prove that the large player's payoff must depend continuously on information. *Both* rewards and punishments must be used to create incentives: if only punishments are used but not rewards, they would destroy too much value. However, in the best equilibrium the large player's incentives can be provided only via punishments. Therefore, that equilibrium cannot be better than the Nash equilibrium of a stage game.

The possibility of commitment reappears if the small players are uncertain about the type of the large player. Suppose that the the large player could be a *commitment type* who always plays the same action or a *normal type* who acts strategically. Then it may be attractive for the normal type to imitate the commitment type because the payoff of the normal type increases when he is perceived to be a commitment type with greater probability. In equilibrium this imitation is imperfect: if it were perfect, the public signal would not be informative about the large player's type, so imitation would have no value. The normal type obtains his maximal payoff when the population is certain that he is a commitment type. In this extreme case the population's beliefs would never change and the the normal type would "get away" with any action. This feature of the equilibrium is consistent with the fact that it is impossible to provide incentives to the normal type of the large player when his payoff is maximized.

We characterize equilibrium payoffs of the large player for *any* discount rate r using an ordinary differential equation. Unlike in discrete time, in many games of interest in continuous

[1] Also, see more recent and thorough studies of Fudenberg and Levine (2006) and Sannikov and Skrzypacz (2006b) into the differences between Poisson and Brownian information.

time sequential equilibrium for any prior is unique and Markov in the population's belief about the large player's type. In a Markov perfect equilibrium (see Maskin and Tirole (2001)) the population's belief completely determines all players' actions as well as the law, by which the population updates its belief about the large player's type from the public signals. The sequential equilibrium is unique and Markov if, roughly speaking, the stage game in which the large player's payoff is adjusted by reputational weights always has a unique Bayesian Nash equilibrium. Many games of interest satisfy these properties, creating a great potential for our results to be used in applications.

The reasons behind the Markov property of continuous-time sequential equilibria are connected with the reasons why non-trivial incentives cannot arise in a complete-information game. When the Nash equilibrium of the stage game is unique, then the only equilibrium of the complete-information repeated game is the repetition of Nash, which is, trivially, a Markov equilibrium. In our setting, continuous time precludes non-trivial incentives through rewards and punishments on top of the incentives naturally created through reputation.

In some continuous-time games, e.g. those that have more than one static Nash equilibrium, sequential equilibria are non-unique. For those games we characterize the upper and lower boundaries of the set of equilibrium payoffs of the large player as solutions to differential inclusions.

The incomplete information approach to reputations has its roots in the works of Kreps and Wilson (1982) and Milgrom and Roberts (1982), in their study of Selten's chain-store paradox, and of Kreps, Milgrom, Roberts, and Wilson (1982), in their analysis of cooperation in the finitely repeated Prisoner's Dilemma. Uncertainty over types, particularly over types that behave as automata committed to certain strategies, gives rise to phenomena that could not be explained by the equilibria of the underlying complete information games: entry deterrence in the Chain-store game and cooperation in (almost every period of) a finitely repeated Prisoner's Dilemma game.

Fudenberg and Levine (1992) study reputation effects in discounted discrete-time repeated games with imperfect monitoring played by a long-run player and a sequence of short-lived players. In their paper the short-run players also believe that, with positive probability, the long-run player is a type committed to a certain strategy. However, unlike the current paper, FL do not study the set of equilibrium payoffs for an arbitrary discount factor, but derive upper and lower bounds on the set of Nash equilibrium payoffs of the long-run player as the discount factor tends to one. When the set of commitment types is sufficiently rich and the monitoring technology satisfies an identification condition, the upper and lower bounds coincide with the long-run player's Stackelberg payoff, that is, the payoff he obtains from credibly committing to the strategy to which he would like to commit the most. A related paper, Faingold (2006), shows that the Fudenberg-Levine payoff bounds hold for a class of continuous-time games that includes the games we study in this paper. Faingold (2006) also shows that those bounds hold in discrete-time games with frequent actions uniformly in the length of the time period between actions.

We use methods related to those of Sannikov (2006a) and Sannikov (2006b) to derive the connection between the large player's incentives and the law of motion of the large player's continuation value, which forms a part of the recursive structure of our games. The other part comes from the evolution of the population's beliefs. The consistency and sequential rationality conditions for sequential equilibria of our games are formulated using these two variables. In many games of interest the equilibrium is unique and Markov, so the large player's continuation value is uniquely determined by the population's belief.

The paper is organized as follows. Section 2 presents our leading example. Section 3 introduces the continuous-time model. Section 4 provides a recursive characterization of public sequential equilibria. Section 5 examines the underlying complete information game. Section 6 provides the ODE characterization when equilibrium is unique. Section 7 extends the characterization to games with multiple equilibria. Section 8 concludes.

2 An Example: The Game of Quality Standards.

In this section we present an example of a repeated game with reputation from the class that we study in the paper. There is one large player, a service provider, and a unit mass of small players, consumers. At each moment of time $t \in [0, \infty)$ the service provider chooses his investment in quality $a_t \in [0, 1]$ and each customer i chooses a service level $b_t^i \in [0, 3]$. The service provider does not observe each customer individually, but sees only the average level of service $\bar{b}_t$ that the customers choose. Consumers do not see a_t either. Instead, they publicly observe the actual quality of the service dX_t, which is a noisy public signal of a_t.

$$dX_t = a_t(4 - \bar{b}_t)\, dt + (4 - \bar{b}_t)\, dZ_t,$$

where Z is a standard Brownian motion. The drift $a_t(4 - \bar{b}_t)$ is the expected quality flow at time t, and $4 - \bar{b}_t$ is the magnitude of the noise. The expected quality flow per customer deteriorates with greater usage. The noise is also decreasing with usage: the more customers use the service the better they learn its quality.

Consumer i pays the price equal to his service level b_t^i. The payoff of consumer i is

$$r \int_0^\infty e^{-rt}(b_t^i\, dX_t - b_t^i\, dt),$$

where $r > 0$ is a discount rate. Customers act to maximize their static payoff because the service provider only observes the aggregate usage.

The payoff of the service provider is

$$r \int_0^\infty e^{-rt}(\bar{b}_t - a_t)\, dt.$$

In the static Nash equilibrium of this game the service provider makes investment 0 in the quality of service, and the customers choose the service level of 0 too. As we show in Section 5, it turns out that in the repeated game without reputational effects, the only equilibrium is the repetition of the static Nash equilibrium. Without reputation, intertemporal incentives completely collapse and the large player's equilibrium payoff is 0.

However, if the large player was able to commit to invest a^* in quality and to credibly convey this commitment to the consumers, he would be able to influence the customers' decisions and get a better payoff. Then each consumer's choice b^i maximizes $b^i(a^*(4 - \bar{b}) - 1)$, and in equilibrium all customers would choose the same level $b^i = \max\{0, 4 - 1/a^*\}$. The service provider would

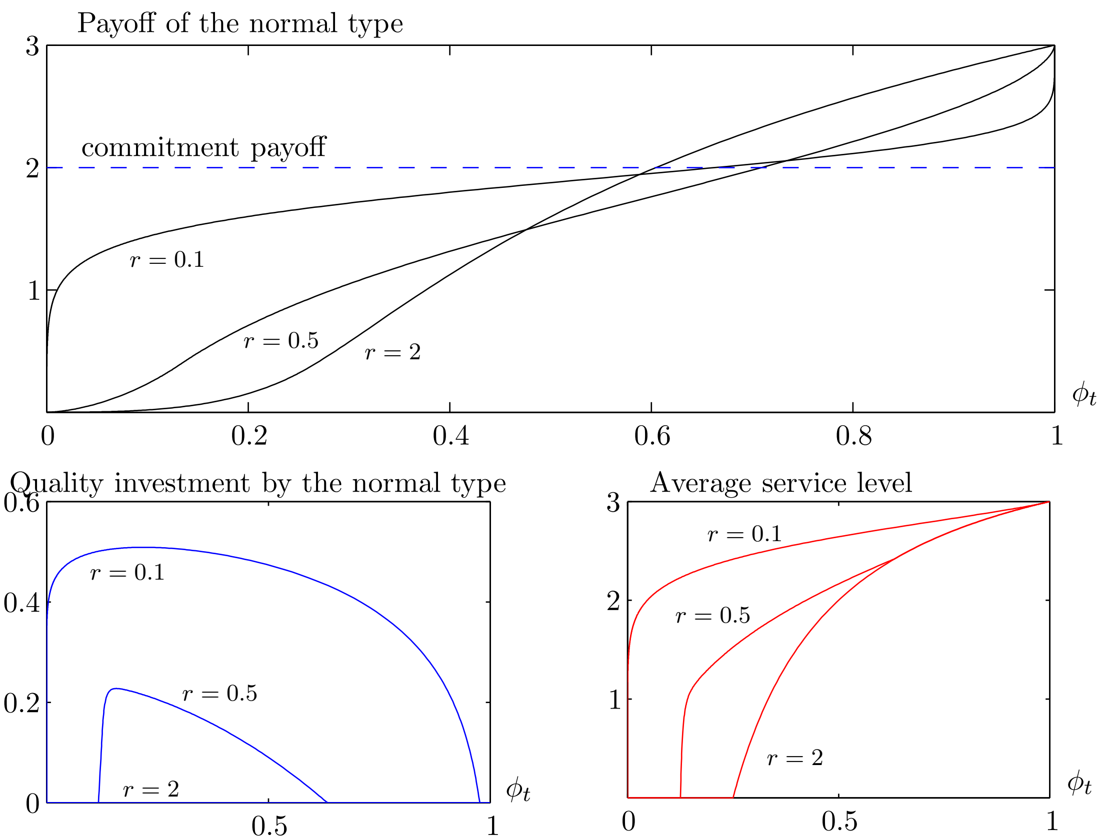

Figure 1: Equilibrium payoffs and actions in the game of quality standards.

earn $\max\{0, 4 - 1/a^*\} - a^*$. At $a^* = 1$ this function achieves it maximal value of 2, the large player's *commitment payoff*.

Following these observations, it is interesting to explore what happens in the game with reputation. That is, assume that at time 0 the consumers believe that with probability p the service provider is a *commitment type*, who always chooses investment $a^* = 1$, and with probability $1 - p$ he is a *normal type*, who chooses a_t to maximize his expected profit. What happens in equilibrium?

The top panel of Figure 1 shows the unique equilibrium payoff of the normal type as a function of the population's belief p for different discount rates r. In equilibrium the customers constantly update their belief ϕ_t, the probability assigned to the commitment type, using the public signal X_t. The equilibrium is Markov in ϕ_t, which uniquely determines the equilibrium actions of the normal type (bottom left panel) and the customers (bottom right panel).

Consistent with the asymptotic results in Faingold (2006), the computation shows that as $r \to 0$, the large player's payoff converges to his commitment payoff of 2. We also see from Figure 1 that the customer usage level b increases towards the commitment level of 3 as the discount rate r decreases towards 0. While the normal type chooses action 0 for all levels of ϕ_t when $r = 2$, as r is closer to 0, his action increases towards $a^* = 1$. However, the imitation of the commitment type by the normal type is never perfect, even for very low discount rates.

In this example for all discount rates r the equilibrium action of the normal type is exactly 0 near $\phi = 0$ and 1 and the population's action is 0 near $\phi = 0$ (not visible in Figure 1 for $r = 0.1$). The normal type of the large player imitates the commitment type only for intermediate levels of reputation.

3 The Game.

A *large player* participates in a dynamic game with a continuum of *small players* uniformly distributed on $I = [0, 1]$. At each time $t \in [0, \infty)$, the large player chooses an action $a_t \in A$ and each small player $i \in I$ chooses an action $b_t^i \in B$ based on their current information. Action spaces A and B are compact, convex subsets of a Euclidean space. The small players' moves are *anonymous*: at each time t, the large player observes the aggregate distribution $\bar{b}_t \in \Delta(B)$ of the small players' actions, but does not observe the action of any individual small player. There is *imperfect monitoring*: the large player's moves are not observable to the small players. Instead, the small players see a noisy public signal $(X_t)_{t \geq 0}$ that depends on the actions of the large player, the aggregate distribution of the small players' actions and noise. Specifically,

$$dX_t = \mu(a_t, \bar{b}_t)\, dt + \sigma(\bar{b}_t) \cdot dZ_t,$$

where (Z_t) is a d-dimensional Brownian motion, and the drift and the volatility of the signal are defined via continuously differentiable functions $\mu : A \times B \to \mathbb{R}^d$ and $\sigma : B \to \mathbb{R}^{d \times d}$, which are linearly extended to $A \times \Delta(B)$ and $\Delta(B)$ respectively.[2] For technical reasons, assume that there is $c > 0$ such that $|\sigma(b) \cdot y| \geq c|y|$, $\forall y \in \mathbb{R}^d$, $\forall b \in B$. Denote by $(\mathcal{F}_t)_{t \geq 0}$ the filtration generated by (X_t).

Our assumption that only the drift of X depends on the large player's action corresponds to the *constant support* assumption in discrete time. By Girsanov's Theorem the probability measures over the paths of two diffusion processes with the same volatility but different bounded drifts are *equivalent*, i.e., they have the same zero-probabilty events. Since the volatility of a continuous-time diffusion process is observable, we do not allow $\sigma(\bar{b})$ to depend on a.

Small players have identical preferences. The payoff of each small player depends only on his own action, the aggregate distribution of all small players' actions, and the sample path of the signal (X_t). A small player's payoff is

$$r \int_0^\infty e^{-rt} \, \big(u(b_t^i, \bar{b}_t)\, dt \; + \; v(b_t^i, \bar{b}_t) \cdot dX_t\big)$$

where $u : B \times B \to \mathbb{R}$ and $v : B \times B \to \mathbb{R}^d$ are continuously differentiable functions that are extended linearly to $B \times \Delta(B)$. Then the expected payoff flow of the small players $h : A \times B \times \Delta(B) \to \mathbb{R}$ is given by

$$h(a, b, \bar{b}) = u(b, \bar{b}) + v(b, \bar{b}) \cdot \mu(a, \bar{b}).$$

The small players' payoff functions are common knowledge.

[2]Functions μ and σ are extended to distributions over B by $\mu(a, \bar{b}) = \int_B \mu(a, b)\, \bar{b}(db)$ and $\sigma(\bar{b})\sigma(\bar{b})^\top = \int_B \sigma(b)\sigma(b)^\top \, \bar{b}(db)$.

The small players are uncertain about the type θ of the large player. At time 0 they believe that with probability $p \in [0,1]$ the large player is a *commitment type* ($\theta = \mathtt{c}$) and with probability $1-p$ he is a *normal type* ($\theta = \mathtt{n}$). The commitment type mechanically plays a fixed action $a^* \in A$ at all times. The normal type plays strategically to maximize his expected payoff. The payoff of the normal type of the large player is

$$r \int_0^\infty e^{-rt}\, g(a_t, \bar{b}_t)\, dt\,,$$

where the payoff flow is defined through a continuously differentiable function $g : A \times B \to \mathbb{R}$ that is extended linearly to $A \times \Delta(B)$.

In the dynamic game the small players update their beliefs about the type of the large player by Bayes rule from their observations of X. Denote by ϕ_t the probability that the small players assign to the large player being a commitment type at time $t \geq 0$.

A pure *public strategy* of the normal type of large player is a progressively measurable (with respect to $(\mathcal{F}_t)$) process $(a_t)_{t\geq 0}$ with values in A. Similarly, a pure public strategy of small player $i \in I$ is a progressively measurable process $(b^i_t)_{t\geq 0}$ with values in B. We assume that jointly the strategies of the small players and the aggregate distribution satisfy appropriate measurability properties.

Definition. *A* public sequential equilibrium *consists of a public strategy $(a_t)_{t\geq 0}$ of the normal type of large player, public strategies $(b^i_t)_{t\geq 0}$ of small players $i \in I$, and a progressively measurable belief process $(\phi_t)_{t\geq 0}$, such that at all times t and after all public histories:*

1. *the strategy of the normal type of large player maximizes his expected payoff*

$$E_t\left[r \int_0^\infty e^{-rt}\, g(a_t, \bar{b}_t)\, dt \mid \theta = \mathtt{n}\right]$$

2. *the strategy of each small player maximizes his expected payoff*

$$(1-\phi_t)\, E_t\left[r \int_0^\infty e^{-rt}\, h(a_t, b^i_t, \bar{b}_t)\, dt \mid \theta = \mathtt{n}\right] \;+\; \phi_t\, E_t\left[r \int_0^\infty e^{-rt}\, h(a^*, b^i_t, \bar{b}_t)\, dt \mid \theta = \mathtt{c}\right]$$

3. *the common prior is $\phi_0 = p$ and beliefs $(\phi_t)_{t>0}$ are determined by Bayes rule.*

A strategy profile that satisfies conditions 1 and 2 is called *sequentially rational.* A belief process (ϕ_t) that satisfies condition 3 is called *consistent.*

In the next section, Section 4, we explore these properties in detail and formalize them in our setting. We use this formalization in Section 5 to explore the game with prior $p = 0$, and in Section 6 to present a set of sufficient conditions under which the equilibrium for any prior is unique and Markov in the population's belief. For this case, we characterize the equilibrium payoffs of the normal type as well as the equilibrium strategies via an ordinary differential equation. In Section 7 we characterize equilibrium payoffs for the case when there may be multiple equilibria for any prior.

Remark 1. Although the aggregate distribution of small players' actions is publicly observable, our requirement that public strategies depend only on the sample paths of X is without loss of generality. In fact, for a given strategy profile, the public histories along which there are observations of $\bar{b}_t$ that differ from those on-the-path-of-play correspond to deviations by a positive measure of small players. Therefore our definition of public strategies does not alter the set of public sequential equilibrium outcomes.

Remark 2. All our results hold for public sequential equilibria in mixed strategies. A mixed public strategy of the large player is a random process $(\bar{a}_t)_{t\geq 0}$ progressively measurable with respect to $\mathcal{F}_t$ with values in $\Delta(A)$. The drift function μ should be extended linearly to $\Delta(A) \times \Delta(B)$ to allow for mixed strategies. Because there is a continuum of anonymous small players, the assumption that each of them plays a pure strategy is without loss of generality.

Remark 3. For both pure and mixed equilibria, the restriction to public strategies is without loss of generality in our games. For pure strategies, it is redundant to condition a player's current action on his private history, which is completely determined by the public history. For mixed strategies, the restriction to public strategies is without loss of generality in repeated games that have product structure, e.g. in our games.[3] Informally, to form a belief about his opponent's private histories, in a game with product structure a player can ignore his own past actions because they do not influence the signal about his opponent's actions. Formally, a mixed private strategy of the large player in our game is a random process (a_t) with values in A that is progressively measurable with respect to a filtration $\{\mathcal{G}_t\}$, which is generated by the public signals X and the large player's private randomization. For any private strategy of the large player, an equivalent mixed public strategies is defined by letting $\bar{a}_t$ be the conditional distribution of a_t given $\mathcal{F}_t$. Strategies a_t and $\bar{a}_t$ induce the same probability distributions over public signals and give the large player the same expected payoff (given $\mathcal{F}_t$).

4 The Structure of Sequential Equilibria

This section provides a characterization of public sequential equilibria of our game, which is summarized in Theorem 1. In equilibrium, the small players always choose a static best response given their belief about the large player's actions. The commitment type of the large player always chooses action a^*, while the normal type chooses his actions strategically taking into account his expected future payoff, which depends on the public signal X. The dynamic evolution of the small players' belief is also determined by X.

The equilibrium play has to satisfy two conditions: the beliefs must be consistent with the players' strategies, and the strategies must be sequentially rational given beliefs. For the consistency of beliefs, Proposition 1 presents equation (1) that describes how the small players' belief evolves with the public signal X. Sequential rationality of the normal type's strategy is verified by looking at the evolution of his continuation value W_t, the future expected payoff of the normal type given the history of public signals X up until time t. Proposition 2 presents a necessary and sufficient condition for the law of motion of a random process W, under which W is the continuation value of the normal type. Proposition 3 presents a condition for sequential rationality that is connected to the law of motion of W.

[3] In a game with product structure each public signal depends on the actions of only one large player.

Subsequent sections of our paper use this equilibrium characterization. Section 5 uses Theorem 1 to show that in the complete-information game, in which the population is certain that the type of the large player is normal, the set of the large player's public sequential equilibrium payoffs coincides with the convex hull of static Nash equilibrium payoffs. Section 6 analyzes a convenient class of games in which the public sequential equilibrium turn out to be unique and Markov in the population's belief. Section 7 characterizes the set of public sequential equilibrium payoffs of the large player generally.

We start with Proposition 1, which explains how the small players use Bayes rule to update their beliefs based on the observations of public signals.

Proposition 1 (Belief Consistency). *Fix a public strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ and a prior $p \in [0,1]$ on the commitment type. Belief process $(\phi_t)_{t\geq 0}$ is consistent with $(a_t, \bar{b}_t)_{t\geq 0}$ if, and only if, it satisfies equation*

$$d\phi_t = \gamma(a_t, \bar{b}_t, \phi_t) \cdot dZ_t^{\phi} \tag{1}$$

with initial condition $\phi_0 = p$, where

$$\begin{aligned} \gamma(a, \bar{b}, \phi) &= \phi(1-\phi)\sigma(\bar{b})^{-1}\left(\mu(a^*, \bar{b}) - \mu(a, \bar{b})\right), \quad &(2)\\ dZ_t^{\phi} &= \sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t, \bar{b}_t)\, dt), &(3)\\ \text{and} \quad \mu^{\phi}(a, \bar{b}) &= \phi\mu(a^*, \bar{b}) + (1-\phi)\mu(a, \bar{b}). &(4) \end{aligned}$$

In the equations in Proposition 1, (a_t) is the strategy that the normal type is supposed to follow. If the normal type deviates, his deviation affects only the drift of X, but not the other terms in equation (1).

Proof. The strategies of the two types of large player induce two different probability measures over the paths of the signal (X_t). From Girsanov's Theorem we can find the ratio ξ_t between the likelihood that a path $(X_s : s \in [0,t])$ arises for type $\mathtt{c}$ and the likelihood that it arises for type $\mathtt{n}$. This ratio is characterized by

$$d\xi_t = -\,\xi_t\, \rho_t \cdot dZ_s^{\mathtt{n}}, \quad \xi_0 = 1, \tag{5}$$

where $\rho_t = \sigma(\bar{b}_t)^{-1}\left(\mu(a^*, \bar{b}_t) - \mu(a_t, \bar{b}_t)\right)$ and $(Z_t^{\mathtt{n}})$ is a Brownian motion under the probability measure generated by type $\mathtt{n}$'s strategy.

Suppose that belief process (ϕ_t) is consistent with $(a_t, \bar{b}_t)_{t\geq 0}$. Then, by Bayes rule, the posterior after observing a path $(X_s : s \in [0,t])$ is

$$\phi_t = \frac{p\xi_t}{p\xi_t + (1-p)} \tag{6}$$

From Ito's formula,

$$\begin{aligned} d\phi_t &= \frac{p(1-p)}{(p\xi_t + (1-p))^2} d\xi_t - \frac{2p^2(1-p)}{(p\xi_t + (1-p))^3} \frac{\xi_t^2 \rho_t \cdot \rho_t}{2}\, dt \\ &= \phi_t(1-\phi_t)\rho_t \cdot dZ_t^{\mathtt{n}} - \phi_t^2(1-\phi_t)(\rho_t \cdot \rho_t)\, dt \qquad (7)\\ &= \phi_t(1-\phi_t)\rho_t \cdot dZ_t^{\phi} \end{aligned}$$

Conversely, suppose that (ϕ_t) is a process that satisfies equation (1) with initial condition $\phi_0 = p$. Define ξ_t using expression (6), i.e.,

$$\xi_t \;=\; \frac{1-p}{p}\frac{\phi_t}{1-\phi_t}.$$

By another application of Ito's formula, we conclude that (ξ_t) satisfies equation (5). This means that ξ_t is the ratio between the likelihood that a path $(X_s : s \in [0,t])$ arises for type c and the likelihood that it arises for type n. Hence, ϕ_t is determined by Bayes rule and the belief process is consistent. □

Coefficient γ in equation (1) is the volatility of beliefs: it reflects the speed with which the small players learn about the type of the large player. The definition of γ is important for the characterization of public sequential equilibria presented in Sections 6 and 7. The intuition behind equation (1) is as follows. If the small players are convinced about the type of the large player, then $\phi_t(1-\phi_t) = 0$, so they never change their beliefs. When $\phi_t \in (0,1)$ then $\gamma(a_t, \bar{b}_t, \phi_t)$ is larger, and learning is faster, when the noise $\sigma(\bar{b}_t)$ is smaller or the drifts produced by the two types differ more. From the small players' perspective, (Z_t^{ϕ}) is a Brownian motion and their belief (ϕ_t) is a martingale. From (7) we see that, conditional on the large player being the normal type, the drift of ϕ_t is non-positive: either the small players eventually learn they are facing the normal type, or the normal type eventually plays like the commitment type.

We now proceed to analyze the second important state descriptor of the interaction between the large and the small players, the continuation value of the normal type. A player's continuation value is his future expected payoff after a given public history for a given profile of continuation strategies. We derive how the large player's incentives arise from the law of motion of his continuation value. We will find that the large player's strategy is optimal if, and only if, a certain incentive compatibility condition holds at all times $t \geqslant 0$.

For a given strategy profile $S = (a_t, \bar{b}_t)_{t\geq 0}$, the *continuation value* $W_t(S)$ of the normal type of the large player is his expected payoff at time t when he plans to follow strategy (a_s) from time t onwards, i.e.

$$W_t(S) = E_t\left[r\int_t^{\infty} e^{-r(s-t)} g(a_s, \bar{b}_s)\, ds \mid \theta = \mathrm{n}\right] \tag{8}$$

Proposition 2 presents the law of motion of W_t.

Proposition 2 (Continuation Values)**.** *A bounded process $(W_t)_{t\geq 0}$ is the continuation value of the normal type under the public-strategy profile $S = (a_t, \bar{b}_t)_{t\geq 0}$ if, and only if, for some d-dimensional process β_t in $\mathcal{L}^*$, we have*

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\, dt \;+\; r\beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\, dt). \tag{9}$$

Representation (9) describes how $W_t(S)$, defined above, evolves with the public history. It is valid independently of the large player's actions until time t, which caused a given history $\{X_s, s \in [0,t]\}$ to realize. This fact is important in the proof of Proposition 3, which deals with incentives.

Proof. First, note that $W_t(S)$ is a bounded process by (8), and let us show that $W_t = W_t(S)$ satisfies (9) for some d-dimensional process β_t in $\mathcal{L}^*$. Denote by $V_t(S)$ the average discounted payoff of the normal type conditional on the public information at time t, i.e.,

$$V_t(S) = E_t\left[r\int_0^\infty e^{-rs} g(a_s, \bar{b}_s)\,ds \;\mid\; \theta = \mathtt{n}\right] = r\int_0^t e^{-rs} g(a_s, \bar{b}_s)\,ds + W_t(S) \tag{10}$$

Then V_t is a martingale when the large player is of normal type. By the Martingale Representation Theorem, there exists a d-dimensional process β_t in $\mathcal{L}^*$ such that

$$dV_t(S) = re^{-rt}\beta_t \cdot \sigma(\bar{b}_t) dZ_t^{\mathtt{n}} \tag{11}$$

where $dZ_t^{\mathtt{n}} = \sigma(\bar{b}_t)^{-1}(dX_t - \mu(a_t, \bar{b}_t)\,dt)$ is a Brownian motion from the point of view of the normal type of the large player.

Differentiating (10) with respect to time yields

$$dV_t(S) = re^{-rt} g(a_t, \bar{b}_t)\,dt \;-\; re^{-rt} W_t(S)\,dt \;+\; e^{-rt} dW_t(S) \tag{12}$$

Combining equations (11) and (12) yields (9).

Conversely, let us show if W_t is a bounded process that satisfies (9) then $W_t = W_t(S)$. When the large player is normal, the process

$$V_t = r\int_0^t e^{-rs} g(a_s, \bar{b}_s)\,ds + e^{-rt} W_t$$

is a martingale under the strategies $S = (a_t, \bar{b}_t)$ because $dV_t = re^{-rt}\beta_t \cdot \sigma(\bar{b}_t) dZ_t^{\mathtt{n}}$ by (9). Moreover, martingales V_t and $V_t(S)$ converge because both $e^{-rt}W_t$ and $e^{-rt}W_t(S)$ converge to 0. Therefore,

$$V_t = E_t[V_\infty] = E_t[V_\infty(S)] = V_t(S) \quad \Rightarrow \quad W_t = W_t(S)$$

for all t, as required. □

Next, we derive conditions for sequential rationality. The condition for the small players is straightforward: they maximize their static payoff because a deviation of an individual small player does not affect future equilibrium play. The situation of the normal type of large player is more complicated: he acts optimally if he maximizes the sum of the current payoff flow and the expected change in his continuation value.

Proposition 3 (Sequential Rationality)**.** *A public strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ is sequentially rational with respect to a belief process (ϕ_t) if, and only if, for all times $t \geq 0$ and after all public histories:*

$$a_t \;\in\; \arg\max_{a' \in A}\; g(a', \bar{b}_t) \;+\; \beta_t \cdot \mu(a', \bar{b}_t) \tag{13}$$

$$b \;\in\; \arg\max_{b' \in B}\; u(b', \bar{b}_t) + v(b', \bar{b}_t) \cdot \mu^{\phi_t}(a_t, \bar{b}_t), \quad \forall b \in \operatorname{supp} \bar{b}_t \tag{14}$$

Proof. Consider a strategy profile $(a_t, \bar{b}_t)$ and an alternative strategy $(\tilde{a}_t)$ of the normal type. Denote by W_t the continuation payoff of the normal type when he follows strategy (a_t) after time t, while the population follows $(\bar{b}_t)$. If the normal type of large player plays strategy $(\tilde{a}_t)$ up to time t and then switches back to (a_t), his expected payoff conditional on the public information at time t is given by

$$\tilde{V}_t = r \int_0^t e^{-rs} g(\tilde{a}_s, \bar{b}_s)\, ds \; + \; e^{-rt} W_t$$

By Proposition 2 and the expression above,

$$\begin{aligned} d\tilde{V}_t &= re^{-rt} \left(g(\tilde{a}_t, \bar{b}_t) - W_t \right) dt \; + \; e^{-rt} dW_t \\ &= re^{-rt} \left(\left(g(\tilde{a}_t, \bar{b}_t) - g(a_t, \bar{b}_t) \right) dt + \beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\, dt) \right) \end{aligned}$$

where the $\mathbb{R}^d$-valued process (β_t) is given by (9).

Hence the profile $(\tilde{a}_t, \bar{b}_t)$ yields the normal type expected payoff

$$\begin{aligned} \tilde{W}_0 &= E[\tilde{V}_\infty] \; = \; E\left[\tilde{V}_0 \; + \; \int_0^\infty d\tilde{V}_t \right] \\ &= W_0 \; + \; E\left[r \int_0^\infty e^{-rt} \left(g(\tilde{a}_t, \bar{b}_t) - g(a_t, \bar{b}_t) \; + \; \beta_t \cdot (\mu(\tilde{a}_t, \bar{b}_t) - \mu(a_t, \bar{b}_t) \right) dt \right] \end{aligned}$$

where the expectations are taken under the probability measure induced by $(\tilde{a}_t, \bar{b}_t)$, and so (X_t) has drift $\mu(\tilde{a}_t, \bar{b}_t)$.

Suppose that strategy profile $(a_t, \bar{b}_t)$ and belief process (ϕ_t) satisfy the incentive constraints (13) and (14). Then, for every $(\tilde{a}_t)$, one has $W_0 \geq \tilde{W}_0$, and the normal type is sequentially rational at time 0. By a similar argument, the normal type is sequentially rational at all times t, after all public histories. Also, note that small players are maximizing their instantaneous expected payoffs. Since the small players' actions are anonymous, no unilateral deviation by a small player can affect the future course of play. Therefore each small player is also sequentially rational.

Conversely, suppose that incentive constraint (13) fails. Choose a strategy $(\tilde{a}_t)$ such that $\tilde{a}_t$ attains the maximum in (13) for all $t \geq 0$. Then $\tilde{W}_0 > W_0$ and the large player is not sequentially rational. Likewise, if condition (14) fails, then a positive measure of small players is not maximizing their instantaneous expected payoffs. Since the small player's actions are anonymous, their strategies would not be sequentially rational. □

We can now summarize our characterization of sequential equilibria.

Theorem 1 (Sequential Equilibrium)**.** *A profile $(a_t, \bar{b}_t, \phi_t)$ is a public sequential equilibrium with continuation values (W_t) for the normal type if, and only if,*

1. *(W_t) is a bounded process that satisfies*

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\, dt \; + \; r\beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\, dt) \tag{15}$$

for some process $\beta \in \mathcal{L}^*$, *2. belief process* (ϕ_t) *follows*

$$d\phi_t = \gamma(a_t, \bar{b}_t, \phi_t)\, \sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t, \bar{b}_t)\, dt)\,, \quad \textit{and} \tag{16}$$

3. strategies $(a_t, \bar{b}_t)$ *satisfy the incentive constraints*

$$\begin{array}{c} a_t \in \arg\max_{a' \in A}\ g(a', \bar{b}_t) + \beta_t \mu(a', \bar{b}_t) \quad \text{and} \\ b \in \arg\max_{b' \in B} u(b', \bar{b}_t) + v(b', \bar{b}_t) \cdot \mu^{\phi_t}(a_t, \bar{b}_t)\,,\ \forall\, b \in \text{support}\, \bar{b}_t\,. \end{array} \tag{17}$$

for the process β *given by (15).*

Theorem 1 provides a characterization of public sequential equilibria which can be used to derive many of its properties. In Section 5 we apply Theorem 1 to the game with prior $p = 0$, the complete information game. In Sections 6 and 7 we characterize the entire correspondence $\mathcal{E} : [0,1] \rightrightarrows \mathbb{R}$ that maps a prior probability $p \in [0,1]$ on the commitment type into the set of public sequential equilibrium payoffs of the normal type in the game with prior p. Theorem 1 implies that $\mathcal{E}$ is the largest bounded correspondence such that a controlled process (W, ϕ), defined by (15) and (16), can be kept in Graph($\mathcal{E}$) by controls $(a_t, \bar{b}_t)$ and (β_t) that satisfy (17).[4]

4.1 Gradual revelation of the large player's type.

To finish this section, we apply Theorem 1 to show that Condition 1 below is necessary and sufficient for the reputation of the normal type to decay to 0 with probability 1 in any public sequential equilibrium (Proposition 4). Condition 1 states that in any Nash equilibrium of the static game with just the normal type, the large player cannot appear committed to action a^*.[5] Naturally, this condition plays an important role in Sections 6 and 7, where we characterize sequential equilibria with reputation.

Condition 1. For any Nash equilibrium $(a^N, \bar{b}^N)$ of the static game with prior $p = 0$, $\mu(a^N, \bar{b}^N) \neq \mu(a^*, \bar{b}^N)$.

In discrete time, Cripps, Mailath, and Samuelson (2004) prove that the reputation of the normal type converges to 0 in any sequential equilibrium under conditions that are stronger than Condition 1. Among other things, they also require that the small players' best reply to the commitment action be strict. In discrete time, an analogue of Condition 1 alone would not be sufficient.[6]

[4] This means that there is no other bounded correspondence with this property whose graph contains the graph of $\mathcal{E}$ as a proper subset.

[5] Note that the action of the large player affects the small players' payoffs only through the drift of X.

[6] If the small players have two or more best reponses to the commitment action of the large player, then the discrete-time game with reputation may have an equilibrium in which the small players never learn the large player's type (even if an analogue of Condition 1 holds). For example, the normal type could have incentives to always take the commitment action if the public history determines his continuation payoff appropriately through the best response to the commitment action taken by the population. However, by an argument analogous to the proof of Theorem 2, the large player's incentives would collapse in such an equilibrium in continuous time.

Proposition 4. *If Condition 1 fails, then for any $p \in [0,1]$ the stage game has a Bayesian Nash equilibrium (BNE) in which the normal and the commitment types look the same to the population. The repetition of this BNE is a public sequential equilibrium of the repeated game with prior p, in which the population's belief stays constant.*

If Condition 1 holds, then in any public sequential equilibrium $\phi_t \to 0$ as $t \to \infty$ almost surely under the normal type.

Proposition 4 also implies that players never reach an absorbing state in any public sequential equilibrium if and only if Condition 1 holds. Players reach an absorbing state at time t if their actions as well as the population's beliefs remain fixed after that time. We know that in continuous-time games between two large players, equilibrium play sometimes *necessarily* reaches an absorbing state, as shown in Sannikov (2006b). This possibility requires special treatment in the characterization of equilibria in games between two large players.

Proof. If Condition 1 fails, then there is a static Nash equilibrium $(a^N, \bar{b}^N)$ of the complete-information game with $\mu(a^N, \bar{b}^N) = \mu(a^*, \bar{b}^N)$. It is easy to see that $(a^N, \bar{b}^N)$ is also a BNE of the stage game with any prior p. The repetition of this BNE is a public sequential equilibrim of the repeated game, in which the beliefs $\phi_t \in p$ remain constant. With these beliefs (16) and (17) hold, and $W_t = g(a^N, \bar{b}^N)$ for all t.

Conversely, if Condition 1 holds there is no BNE $(a, \bar{b})$ of the static game with prior $p > 0$ in which $\mu(a, \bar{b}) = \mu(a^*, \bar{b})$. Otherwise, $(a, \bar{b})$ would be a Nash equilibrium of the static game with prior $p = 0$, since the small players' payoffs depend on the actions of the large player only through the drift, a contradiction to Condition 1.

We present the rest of the proof in Appendix A, where we show that for some constants $C > 0$ and $M > 0$, in any sequential equilibrium at all times t either

(a) the absolute value of the volatility of ϕ_t is at least $C\phi_t(1-\phi_t)$ or

(b) the absolute value of the volatility of W_t is at least M.

To see this intuitively, note that if the volatility of ϕ_t at time t is 0, i.e. $\gamma(a_t, \bar{b}_t, \phi_t) = 0$, then $(a_t, \bar{b}_t)$ is *not* a BNE of the stage game by Condition 1. Then the incentive constraints (17) imply that $\beta_t \neq 0$. In Appendix A we rely on the fact that W_t is a bounded process to show that under conditions (a) and (b), ϕ_t eventually decays to 0 when the large player is normal. □

5 Equilibrium Degeneracy under Complete Information

In this section we examine the structure of the set of equilibrium payoffs of the large player in the complete information game ($p = 0$), that is, in the game in which it is common knowledge that the large player is the normal type.

Theorem 2. *Suppose the population of small players is convinced that the large player is the normal type, that is, $p = 0$. Then in any public sequential equilibrium the large player cannot achieve a payoff outside the convex hull of his stage-game Nash equilibrium payoff set, i.e.*

$$\mathcal{E}(0) = \text{co}\left\{ g(a, \bar{b}) \;:\; \begin{array}{l} a \in \arg\max_{a' \in A}\, g(a', \bar{b}) \\ b \in \arg\max_{b' \in B} u(b', \bar{b}) + v(b', \bar{b}) \cdot \mu(a, \bar{b}), \quad \forall b \in \text{support } \bar{b} \end{array} \right\}$$

Proof. Let $\bar{v}$ be the highest pure-strategy Nash equilibrium payoff of the large player in the static game. We show that it is impossible to achieve a payoff higher than $\bar{v}$ in any public equilibrium. (A proof for the lowest Nash equilibrium payoff is similar). Suppose there was a public equilibrium in which the large player's continuation value W_0 was greater than $\bar{v}$. By Proposition 3, for some progressively measurable process (β_t), the large player's continuation value must follow the SDE

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\, dt + r\beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\, dt),$$

where a_t maximizes $g(a', \bar{b}_t) + \beta_t \mu(a', \bar{b}_t)$. Denote $\bar{D} = W_0 - \bar{v}$. Let us show that as long as $W_t \geq \bar{v} + \bar{D}/2$, either the drift of W_t is greater than $r\bar{D}/4$ or the volatility of W_t is uniformly bounded away from 0. If $g(a_t, \bar{b}_t) < \bar{v} + \bar{D}/4$ then the drift of W_t is greater than $r\bar{D}/4$. If $g(a_t, \bar{b}_t) \geq \bar{v} + \bar{D}/4$, then Lemma 1, whose proof is in the Appendix, applies.

Lemma 1. *For any $\varepsilon > 0$ there exists $\delta > 0$ (independent of t or the sample path) such that $|\beta_t| \geq \delta$ whenever $g(a_t, \bar{b}_t) \geq \bar{v} + \varepsilon$.*

Therefore, W_t becomes arbitrarily large with positive probability, a contradiction. This completes the proof of Theorem 2. □

The intuition behind this result is as follows. In order to give incentives to the large player to take an action that results in a payoff better than static Nash, his continuation value must respond to the public signal X_t. When his continuation value reaches its upper bound, such incentives cannot be provided. In effect, if at the upper bound the large player's continuation value were sensitive to the public signal process (X_t), then with positive probability the continuation value would escape above this upper bound, which is not possible. Therefore, at the upper bound, continuation values cannot depend on the public signal and so, in the best equilibrium, the normal type must be playing a myopic best response.

While Theorem 2 does not hold in discrete time,[7] it is definitely *not* just a result of continuous-time technicalities. The large player's incentives to depart from a static best response become fragile when he is flexible to respond to public information quickly. The foundations of this result are similar to the deterioration of incentives due to the flexibility to respond to new information quickly in Abreu, Milgrom, and Pearce (1991) in a prisoners' dilemma with Poisson signals and, especially, in Sannikov and Skrzypacz (2006a) in a Cournot duopoly with Brownian signals.

Borrowing intuition from the latter paper, suppose that the large player must hold his action fixed for an interval of time of length $\Delta > 0$. Suppose that the large player's equilibrium incentives to take the Stackelberg action are created through a statistical test that triggers an equilibrium punishment if the signal is sufficiently bad. A profitable deviation has a gain on the order of Δ, the length of a time period. Therefore, such a deviation is prevented only if it increases the probability of triggering punishment by at least $O(\Delta)$. Sannikov and Skrzypacz (2006a) show that with Brownian signals, the log likelihood ratio for a test against any particular deviation is normally distributed. A deviation shifts the mean of this distribution by $O(\sqrt{\Delta})$. Then, a successful test against a deviation would generate a false positive with probability of $O(\sqrt{\Delta})$. This probability, which reflects the value destroyed in each period through punishments,

[7] Fudenberg and Levine (1994) show that equilibria with payoffs above static Nash often exist in discrete time, but they are always bounded from efficiency.

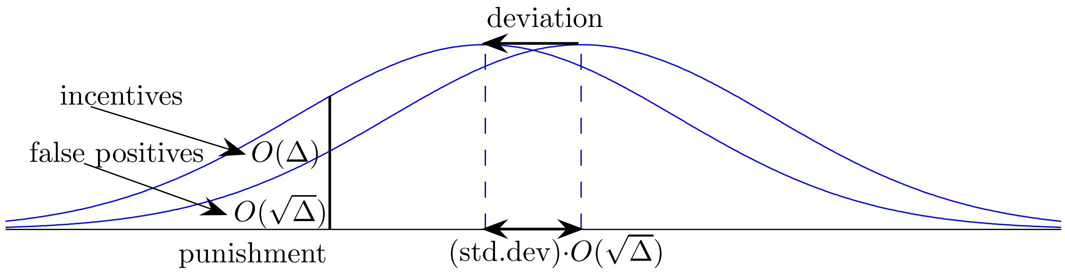

Figure 2: A statistical test to prevent a given deviation.

is disproportionately large for small Δ compared to the value created during a period of length Δ. This intuition implies that in equilibrium the large player cannot sustain payoffs above static Nash as $\Delta \to 0$. Figure 2 illustrates the densities of the log likelihood ratio under the 'recommended' action of the large player and a deviation, and the areas responsible for the large player's incentives and for false positives.

Apart from this statistical intuition, the analysis of the game in Sannikov and Skrzypacz (2006a), as well as in Abreu, Milgrom, and Pearce (1991), differ from ours. Those papers look at the game between two large players, either focusing on symmetric equilibria or assuming a failure of pairwise identifiability to derive their results. [8] In contrast, our result is proved directly in continuous time and for games from a different class, with small players but without any failure of identifiability.

Motivated by our result, Fudenberg and Levine (2006) recently did a very careful study, taking the period between actions Δ to 0 in the game between a large and a population of small players. They illustrate a number of differences between Poisson and Brownian signals, allowing the large player's action to affect not only the mean but also the variance of the Brownian signal.

6 Reputation Effects when the Equilibrium is Unique.

In many games of interest, including the game of quality standards of Section 2, for any prior $p \in (0,1)$ the public sequential equilibrium is unique and Markov in the population's belief. That is, the current belief ϕ_t uniquely determines the players' actions a_t and $\bar{b}_t$, and, consequently, the law by which the population's belief evolves

$$d\phi_t = \gamma(a_t, \bar{b}_t, \phi_t)\, dZ_t^{\phi} = -\frac{|\gamma(a_t, \bar{b}_t, \phi_t)|^2}{1-\phi_t} dt + \gamma(a_t, \bar{b}_t, \phi_t)\, dZ_t^{\mathtt{n}}\,. \tag{18}$$

where $dZ_t^{\mathtt{n}} = \sigma(\bar{b}_t)^{-1}(X_t - \mu(a_t, \bar{b}_t))$ is a Brownian motion under the strategy of the normal type. The continuation value of the normal type is also determined uniquely by the population's belief through a function $U : [0,1] \to \mathbb{R}$, which is illustrated in Figure 3 for our example of the quality commitment game. In this section we present a sufficient condition to guarantee these

[8]The assumption of pairwise identifiability, introduced to repeated games by Fudenberg, Levine, and Maskin (1994), states that deviations by different players can be statistically distinguished by looking at public signals.

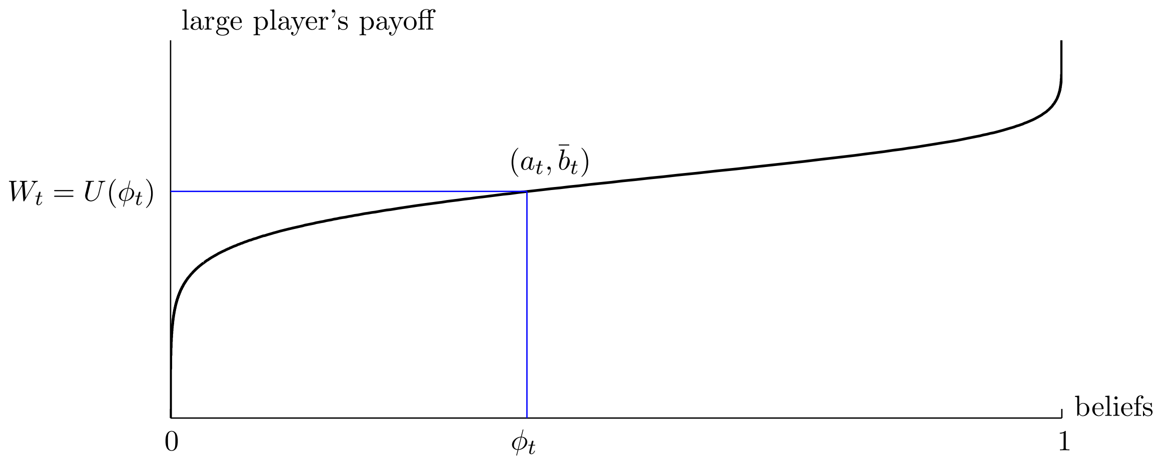

Figure 3: The large player's payoff in a Markov perfect equilibrium.

uniqueness and Markov properties, and characterize these equilibria by an ordinary differential equation.

To see the connection between beliefs, actions and the large player's continuation value in a Markov perfect equilibrium, note that by Ito's lemma the continuation value $W_t = U(\phi_t)$ of the normal type follows

$$dU(\phi_t) = |\gamma(a_t, \bar{b}_t, \phi_t)|^2 \left(\frac{U''(\phi_t)}{2} - \frac{U'(\phi_t)}{1-\phi_t} \right) dt + U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)\, dZ_t^{\mathtt{n}}. \tag{19}$$

At the same time, Proposition 2 gives the law of motion of $W_t = U(\phi_t)$

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\, dt \; + \; r\beta_t \sigma(\bar{b}_t)\, dZ_t^{\mathtt{n}}. \tag{20}$$

Matching drifts, we obtain a differential equation for the function U

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(a, \bar{b}))}{|\gamma(a, \bar{b}, \phi)|^2}\,, \tag{21}$$

where $(a, \bar{b})$ are the actions, yet to be determined, which correspond to the belief ϕ. We call equation (21) the *optimality equation.*

Matching volatilities, we find that $r\beta_t\sigma(\bar{b}_t) = U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)$. This condition determines β_t, which enters the player' incentive constraints from Proposition 3, which define the actions that correspond to belief ϕ_t.

Define the correspondence $\Psi(\phi, z)$ for $\phi \in [0,1]$ and $z \in \mathbb{R}$ by

$$\Psi(\phi, z) = \left\{ (a, \bar{b}) : \begin{array}{l} a \in \arg\max_{a' \in A} \; rg(a', \bar{b}) + z(\mu(a^*, \bar{b}) - \mu(a, \bar{b}))\sigma(\bar{b})^{-2}\mu(a', \bar{b}) \\ b \in \arg\max_{b' \in B} u(b', \bar{b}) + v(b', \bar{b}) \cdot \mu^{\phi}(a, \bar{b}), \quad \forall b \in \text{support}\, \bar{b} \end{array} \right\}$$

Assuming that $\Psi(\phi, z)$ is single-valued, the actions that enter equation (21), are given by $(a, \bar{b}) = \Psi(\phi, \phi(1-\phi)U'(\phi))$.

These simple properties of equilibria follow from the continuous-time formulation together with the assumption that Ψ is single-valued. As the reader may guess, this result follows from logic similar to that of the previous section. It is impossible to create incentives to sustain greater payoffs than those of the Markov perfect equilibrium. Informally, in a public sequential equilibrium that achieves the largest difference $W_0 - U(\phi_0)$ across all priors, the joint volatility of (ϕ_t, W_t) has to be parallel to the slope of U at $t = 0$, since $W_t - U(\phi_t)$ cannot increase for any realization of X. It follows that $r\beta_t\sigma(\bar{b}_t) = U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)$ at time 0. Since Ψ is a single-valued correspondence, the players' actions at time 0 have to be Markov, which leads to $W_t - U(\phi_t)$ having a positive drift at time 0, a contradiction.

In discrete-time reputation games equilibrium behavior is typically not determined by the population's posterior, and Markov perfect equilibria may not even exist. Generally, in many other classes of games, Markov perfect equilibria have proved simplifying and have been used extensively. In reputation games, the transition rule of the payoff-relevant state variable (the population's posterior) is endogenous. This renders existence results (for Markov perfect equilibrium) difficult to prove. Yet, in our continuous-time setting we are able to prove Theorem 3, the main result of this section[9]

Theorem 3 assumes Condition 1 from Section 4 and Condition 2:

Condition 2. Ψ is a nonempty, single-valued, Lipschitz-continuous correspondence that returns an atomic distribution of small players' actions for all $\phi \in [0, 1]$ and $z \in \mathbb{R}$.

Effectively, the correspondence $\Psi(\phi, z)$ returns the Bayesian Nash equilibria of an auxiliary static game in which the large player is a commitment type with probability ϕ and the payoffs of the normal type are perturbed by a reputational weight of z. In particular, with $\phi = z = 0$ Condition 2 implies that the stage game with a normal large player has a unique Nash equilibrium. Moreover, by Theorem 2, the complete information *dynamic game* also has a unique equilibrium, the repeated play of the static Nash.

While Condition 2 is fairly essential for the uniqueness result, Condition 1 is not. If Condition 2 holds by Condition 1 fails, then the characterization of Theorem 3 would not apply. Letting $(a^N, \bar{b}^N)$ denote the Nash equilibrium of the stage game, in which $\mu(a^N, \bar{b}^N) \neq \mu(a^*, \bar{b}^N)$, the dynamic game with prior p would have a unique public sequential equilibrium $(a_t = a^N, \bar{b}_t = \bar{b}^N, \phi_t = p)$, which is trivially Markov.[10]

Theorem 3. *Under Conditions 1 and 2, $\mathcal{E}(\phi)$ is a single-valued correspondence that coincides with the unique bounded solution of the optimality equation*

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\Psi(\phi, \phi(1-\phi)U'(\phi))))}{|\gamma(\Psi(\phi, \phi(1-\phi)U'(\phi)), \phi)|^2}. \tag{22}$$

At $p = 0$ and 1, $\mathcal{E}(\phi)$ satisfies the boundary conditions

$$\lim_{\phi \to p} U(\phi) = \mathcal{E}(p) = g(\Psi(p, 0)), \quad \text{and} \quad \lim_{\phi \to p} \phi(1-\phi)U'(\phi) = 0. \tag{23}$$

[9]We expect our methods to apply broadly to other continuous-time games, such as the Cournot competition with mean-reverting prices of Sannikov and Skrzypacz (2006a). In that model the market price is the payoff-relevant state variable.

[10]When Condition 1 fails but Condition 2 holds, by an argument similar to the proof of Theorem2 we can show that the large player cannot achieve any payoff other than $g(a^N, \bar{b}^N)$. Note that Theorem 1 implies that either $(a_t, \bar{b}_t) = (a^N, \bar{b}^N)$ or $|\beta_t| \neq 0$ at all times t.

For any prior $p \in (0,1)$ the unique public sequential equilibrium is a Markov perfect equilibrium in the population's belief. In this equilibrium, the players' actions at time t are given by

$$(a_t, \bar{b}_t) = \Psi(\phi_t, \phi_t(1-\phi_t)U'(\phi_t)), \tag{24}$$

the population's belief evolves according to

$$d\phi_t = \gamma(a_t, \bar{b}_t, \phi_t)\, \sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t, \bar{b}_t)\, dt), \tag{25}$$

and the continuation values of the normal type are given by $W_t = U(\phi_t)$.

Proof. Proposition 7 in the Appendix shows that under Conditions 1 and 2, there exists a unique continuous function $U : [0,1] \to \mathbb{R}$ that stays in the interval of feasible payoffs of the large player, satisfies equation (21) on $(0,1)$ and boundary conditions (36), which include (23).

We need to prove that for any prior $p \in (0,1)$ there are no public sequential equilibria with a payoff to the normal type different from $U(p)$, and that the unique equilibrium with value $U(p)$ satisfies the conditions of the theorem.

Let us show that for any prior $p \in (0,1)$, there are no equilibria with a payoff to the large player other than $U(p)$. Suppose, towards a contradiction, that for some $p \in [0,1]$, $(a_t, \bar{b}_t, \phi_t)$ is a public sequential equilibrium that yields the normal type a payoff of $W_0 \neq U(p)$. Without loss of generality, consider the case when $W_0 > U(p)$.

Then by Theorem 1, the population's equilibrium belief follows (18), the continuation value of the normal type follows (20) for some process (β_t), and equilibrium actions and beliefs satisfy the incentive constraints (17). Then, using (20) and (19), the process $D_t = W_t - U(\phi_t)$ has drift

$$\underbrace{rD_t + rU(\phi_t)}_{rW_t} - rg(a_t, \bar{b}_t) + |\gamma(a_t, \bar{b}_t, \phi_t)|^2 (\frac{U'(\phi_t)}{1-\phi_t} - \frac{U''(\phi_t)}{2}) \tag{26}$$

and volatility

$$r\beta_t \sigma(\bar{b}_t) - \gamma(a_t, \bar{b}_t, \phi_t)\, U'(\phi_t). \tag{27}$$

Lemma 14 in the Appendix shows that for any $\varepsilon > 0$ one can choose $\delta > 0$ such that for all $t \geq 0$, either

(a) the drift of D_t is greater than $rD_t - \varepsilon$ or

(b) the absolute value of the volatility of D_t is greater than δ

Here we provide a crude intuition behind Lemma 14. When the volatility of D_t is exactly 0, then $r\beta_t\sigma(\bar{b}_t) = \gamma(a_t, \bar{b}_t, \phi_t)\, U'(\phi_t)$, so

$$a_t \in \arg\max_{a' \in A}\ rg(a', \bar{b}_t) + \underbrace{U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)\sigma^{-1}(\bar{b}_t)}_{r\beta_t} \mu(a', \bar{b}_t)$$

$$b \in \arg\max_{b' \in B} u(b', \bar{b}_t) + v(b', \bar{b}_t) \cdot \mu^{\phi_t}(a_t, \bar{b}_t) \quad \forall\, b \in \text{support}\, \bar{b}_t$$

and $(a_t, \bar{b}_t) = \Psi(\phi_t, \phi_t(1-\phi_t)U'(\phi_t))$. Then by (21), the drift of D_t is exactly rD_t.

In order for the drift of D_t to be lower than rD_t, the volatility of D_t has to be different from zero. Lemma 14 in the Appendix presents a continuity argument to show that in order for the drift to be below $rD_t - \varepsilon$, the volatility of D_t has to be uniformly bounded away from 0.

By (a) and (b) above it follows that D_t would grow arbitrarily large with a positive probability, a contradiction since W_t and $U(\phi_t)$ are bounded processes. The contradiction shows that for any prior $p \in [0,1]$, there cannot be an equilibrium that yields the normal type a payoff larger than $U(p)$. In a similar way, it can be shown that no equilibrium yields a payoff below $U(p)$.

Next, let us construct an equilibrium for a given prior p with value $U(p)$ to the normal type of the large player. Let ϕ_t be a solution to the stochastic differential equation (25) with the actions defined by (24). Let us show that $(a_t, \bar{b}_t, \phi_t)$ is a public sequential equilibrium, in which the bounded process $W_t = U(\phi_t)$ is the large player's continuation value.

Proposition 1 immediately implies that the beliefs (ϕ_t) are consistent with the strategies $(a_t, \bar{b}_t)$. Moreover, since $W_t = U(\phi_t)$ is a bounded process with drift $r(W_t - g(a_t, \bar{b}_t))dt$ by (19) and (21), Proposition 2 implies that W_t is the continuation value of the normal type under the strategy profile $(a_t, \bar{b}_t)$. The process (β_t) associated with the representation of W_t in Proposition 2 is given by $r\beta_t\sigma(b_t) = U'(\phi_t)\gamma(a_t, b_t, \phi_t)$. To see that the public-strategy profile $(a_t, \bar{b}_t)$ is sequentially rational with respect to beliefs (ϕ_t), recall that $(a_t, \bar{b}_t) = \Psi(\phi_t, \phi_t(1-\phi_t)U'(\phi_t))$ and so[11]

$$\begin{aligned} a_t &= \arg\max\nolimits_{a' \in A} \; rg(a', b_t) + \underbrace{U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)\sigma^{-1}(\bar{b}_t)}_{r\beta_t} \mu(a', b_t) \\ \bar{b}_t &= \arg\max\nolimits_{b' \in B} u(b', \bar{b}_t) + v(b', \bar{b}_t) \cdot \mu^{\phi_t}(a_t, \bar{b}_t) \end{aligned} \tag{28}$$

From Proposition 3 it follows immediately that the strategy profile $(a_t, \bar{b}_t)$ is sequentially rational. We conclude that $(a_t, \bar{b}_t, \phi_t)$ is a public sequential equilibrium.

Let us show that the actions of the players are determined uniquely by the population's belief in any public sequential equilibrium $(a_t, \bar{b}_t, \phi_t)$ by (24). Let W_t be the continuation value of the normal type. We know that the pair (ϕ_t, W_t) must stay on the graph of U, because there are no public sequential equilibria with values other than $U(\phi_t)$ for any prior ϕ_t. Therefore, the volatility of $D_t = W_t - U(\phi_t)$ must be 0, i.e. $r\beta_t\sigma(\bar{b}_t) = U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)$. Then Propostion 3 implies that (28) holds and so $(a_t, \bar{b}_t) = \Psi(\phi_t, \phi_t(1-\phi_t)U'(\phi_t))$, as claimed. □

The game of quality standards of Section 2 satisfies Conditions 1 and 2, and so its equilibrium is unique and Markov for any prior. The correspondence $\Psi(\phi, z)$ for that game is given by

$$a = \begin{cases} 0 & \text{if } z \leq r, \\ 1 - r/z & \text{otherwise,} \end{cases} \quad \text{and} \quad b = \begin{cases} 0 \text{ if } \phi a^* + (1-\phi)a \leq 1/4, \\ 4 - 1/(\phi a^* + (1-\phi)a) \text{ otherwise.} \end{cases}$$

The example illustrates a number of properties that follow from Theorem 3:

(a) The players' actions, which are determined from the population's belief ϕ by $(a, \bar{b}) = \Psi(\phi, \phi(1-\phi)U'(\phi))$, vary continuously with ϕ. In particular, when the belief gets close

[11]Recall that Ψ is a single-valued correspondence that returns an atomic distribution of the small players' actions.

to 0, the actions converge to the static Nash equilibrium. Thus, there is no discontinuity for very small reputations, which is typical for infinitely repeated reputation games with perfect monitoring.

(b) The incentives of the normal type to imitate the commitment type are increasing in $\phi(1-\phi)U'(\phi)$. However, imitation is never perfect, which is true for all games that satisfy conditions 1 and 2. Indeed, since the actions are defined by (24), $(a_t = a^*, \bar{b}_t)$ would be a Bayesian Nash equilibrium of the stage game with prior ϕ_t if the normal type imitated the commitment type perfectly at time t. However, Condition 1 implies that the stage game does not have Bayesian Nash equilibria in which the normal type takes action a^*.

However, the actions of the players are not monotonic in beliefs. This is definitely so for the large player, whose actions converge to static best responses at $\phi = 0$ and 1. Although not visible in Figure 1 the small players' actions are also non-monotonic for some discount rates.[12] Nevertheless, the large player's equilibrium payoff U is monotonic in the population's belief in this example. This fact, which does not directly follow from Theorem 3, holds generally under additional mild conditions.

Lemma 2. *Suppose that the Bayesian Nash equilibrium payoff of the normal type is weakly increasing in the population's belief p. Then, the sequential equilibrium payoff $U(p)$ of the normal type is also weakly increasing in prior p.*

Proof. The Bayesian Nash equilibrium payoff of the normal type of the large player is given by $g(\Psi(\phi,0))$. Recall that $U(0) = g(\Psi(0,0))$ and $U(1) = g(\Psi(1,0))$.

Suppose U is not weakly increasing on $[0,1]$. Then U has a local maximum or a local minimum on $(0,1)$. Without loss of generality, consider a local minimum ϕ_0. Then $U'(\phi_0) = 0$, and we must have $U(\phi_0) \geq g(\Psi(\phi_0,0))$ because otherwise

$$U''(\phi_0) = \frac{2r(U(\phi_0) - g(\Psi(\phi_0,0)))}{|\gamma(\Psi(\phi_0,0),\phi_0)|^2} < 0.$$

Let ϕ_1 be the global maximum of U on $[0,\phi_0]$. Since ϕ_0 is a local minimum and $U(\phi_0) \geq U(0)$, $\phi_1 \neq 0, \phi_0$. Then $U'(\phi_0) = 0$ and

$$U(\phi_1) > U(\phi_0) \geq g(\Psi(\phi_0,0)) \geq g(\Psi(\phi_1,0)) \quad \Rightarrow$$

$$U''(\phi_1) = \frac{2r(U(\phi_1) - g(\Psi(\phi_1,0)))}{|\gamma(\Psi(\phi_1,0),\phi_1)|^2} > 0,$$

and so ϕ_1 cannot be a maximum, a contradiction. □

Remark. If the Bayesian Nash equilibrium payoff of the normal large player is increasing in the population's belief p, then the conclusion of Theorem 3 holds even if the correspondence $\Psi(\phi,z)$ is single-valued and Lipschitz-continuous only for $z \geq 0$.[13] Indeed, if construct a new

[12] For small discount rates r, not far from $\phi = 0$ the slope of U gets very high as it grows towards the commitment payoff. This can cause the normal type to get very close to imitating the commitment type, producing a peak in the small players' actions.

[13] Such a conclusion has practical value because under typical concavity assumptions, the large player's objective function in the definition of Ψ may become convex instead of concave for $z < 0$.

correspondence $\hat{\Psi}$ from Ψ by replacing values for $z < 0$ with a Lipschitz-continuous function, then the optimality equation with $\hat{\Psi}$ instead of Ψ would have a solution U with boundary conditions $U(0) = g(\Psi(0,0))$ and $U(1) = g(\Psi(1,0))$ by Theorem 3. By Lemma 7 this solution must be weakly decreasing, and therefore it satisfies the original equation with correspondence Ψ. Besides the construction of a solution, all other arguments of Theorem 3 apply to U.

7 General Characterization.

In this section we extend the characterization of Section 6 to environments in which multiple equilibria exist. When the correspondence $\Psi(\phi, z)$ is not single-valued, one should not expect $\mathcal{E}(p)$ to be single-valued either. Theorem 4 characterizes $\mathcal{E}(p)$ for the general case.

Throughout this section, we maintain Condition 1 but relax Condition 2 to

Condition 3. Ψ is a nonempty, compact-valued, upper hemi-continuous correspondence.

When Ψ is not single-valued, there may be many bounded functions that satisfy equation

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(a, \bar{b}))}{|\gamma(a, \bar{b}, \phi)|^2} \tag{29}$$

for some $(a, \bar{b}) \in \Psi(\phi, \phi(1-\phi)U'(\phi))$. The proof of Theorem 3 can be adapted to show for any such function and any prior p, there is a sequential equilibrium that achieves value $U(p)$ for the normal type. Therefore, it is natural to conjecture that the correspondence $\mathcal{E}(p)$ contains all values between its upper boundary, the largest solution of (29), and its lower boundary, the smallest solution of (29). The pair $(a, \bar{b}) \in \Psi(\phi, \phi(1-\phi)U'(\phi))$ would minimize the right-hand side of (29) for the largest solution, and maximize, for the smallest solution.

Unfortunately, the equation

$$U''(\phi) = H(\phi, U(\phi), U'(\phi)), \quad \text{where} \quad H(\phi, u, u') = \min_{(a,\bar{b}) \in \Psi(\phi, \phi(1-\phi)u')} \frac{2u'}{1-\phi} + \frac{2r(u - g(a, \bar{b}))}{|\gamma(a, \bar{b}, \phi)|^2}, \tag{30}$$

does not always have a solution in the classical sense. The reason is that Ψ is generally only upper hemi-continuous, but not continuous, and so the right-hand side of (30) fails to be Lipschitz-continuous. We call equation (30) *upper optimality equation.*

Due to this difficulty, we rely on a generalized notion of a solution, a *viscosity solution* (we define it below). We show that the upper boundary U of $\mathcal{E}(p)$ is the largest viscosity solution of (30), while the lower boundary L is the smallest solution of (30) with the minimum replaced by the maximum. If Ψ is single-valued in a neighborhood of $(\phi, \phi(1-\phi)U'(\phi))$ and H is Lipschitz-continuous in a neighborhood of $(\phi, U(\phi), U'(\phi)$, then the viscosity solution coincides with a classical solution of (30). Otherwise, we show that $U''(\phi)$, which exists almost everywhere since $U'(\phi)$ is absolutely continuous, can take any value between $H(\phi, U(\phi), U'(\phi))$ and $H^*(\phi, U(\phi), U'(\phi))$, where H^* denotes the upper semi-continuous envelope of H. H itself is lower semi-continuous, i.e. $H = H_*$.

Definition. *A bounded function* $U : (0,1) \to \mathbb{R}$ *is a* viscosity super-solution *equation (30) if for every* $\phi_0 \in (0,1)$ *and every* C^2 test function $V : (0,1) \to \mathbb{R}$,

$$U_*(\phi_0) = V(\phi_0) \text{ and } U_* \geq V \quad \Rightarrow \quad V''(\phi_0) \leq H^*(\phi, V(\phi_0), V'(\phi_0))$$

U *is a* viscosity sub-solution *if for every* $\phi_0 \in (0,1)$ *and every* C^2 *test function* $V : (0,1) \to \mathbb{R}$

$$U^*(\phi_0) = V(\phi_0) \text{ and } U^* \leq V \quad \Rightarrow \quad V''(\phi_0) \geq H_*(\phi, V(\phi_0), V'(\phi_0)).$$

A bounded function U *is a* viscosity solution *if it is both a super-solution and a sub-solution.*

Appendix D presents the details of our analysis, which we summarize here. Propositions 8 and 9 show that U, the upper boundary of $\mathcal{E}(p)$, is a bounded viscosity solution of the upper optimality equation. Lemma 17 shows that every bounded viscosity solution is a C^1 function with an absolutely continuous derivative (so that its second derivative exists almost everywhere). Finally, Proposition 10 shows that U is the largest viscosity solution of (30), and that

$$U''(\phi) \in [H(\phi, U(\phi), U'(\phi)),\ H^*(\phi, U(\phi), U'(\phi))]. \tag{31}$$

In particular, when H is continuous at $(\phi, U(\phi), U'(\phi))$ then U satisfies (30) in the classical sense.

We summarize our characterization in the following theorem.

Theorem 4. *Under Conditions 1 and 3, $\mathcal{E}$ is a nonempty, compact, convex-valued, upper hemi-continuous correspondence. The upper boundary U of $\mathcal{E}$ is a C^1 function with absolutely continuous derivative (so $U''(\phi)$ exists almost everywhere). Moreover, U is characterized as the maximal bounded function that satisfies the differential inclusion*

$$U''(\phi) \in [H(\phi, U(\phi), U'(\phi)),\ H^*(\phi, U(\phi), U'(\phi))], \tag{32}$$

where the lower semi-continuous function H is defined by (30) and H^* denotes the upper semi-continuous envelope of H. The lower boundary of $\mathcal{E}$ is determined analogously.

To see an example of such an equilibrium correspondence $\mathcal{E}(p)$, consider the following game, related to our example of quality commitment. Suppose that the large player, a service provider, chooses investment in quality $a_t \in [0,1]$, where $a^* = 1$ is the action of the commitment type, and each customer chooses a service level $b_t^i \in [0,2]$. The public signal about the large player's investment is

$$dX_t = a_t\, dt + dZ_t.$$

The large player's payoff flow is $(\bar{b}_t - a_t)\, dt$ and customer i receives payoff $b_t^i \bar{b}_t\, dX_t - b_t^i\, dt$. The customers' payoff functions capture positive network externalities: greater usage $\bar{b}_t$ of the service by other customers allows each individual costomer to enjoy the service more.

The unique Nash equilibrium of the stage game is $(0,0)$. The correspondence $\Psi(\phi, z)$ defines the action of the normal type uniquely by

$$a = \begin{cases} 0 & \text{if } z \leq r \\ 1 - r/z & \text{otherwise.} \end{cases} \tag{33}$$

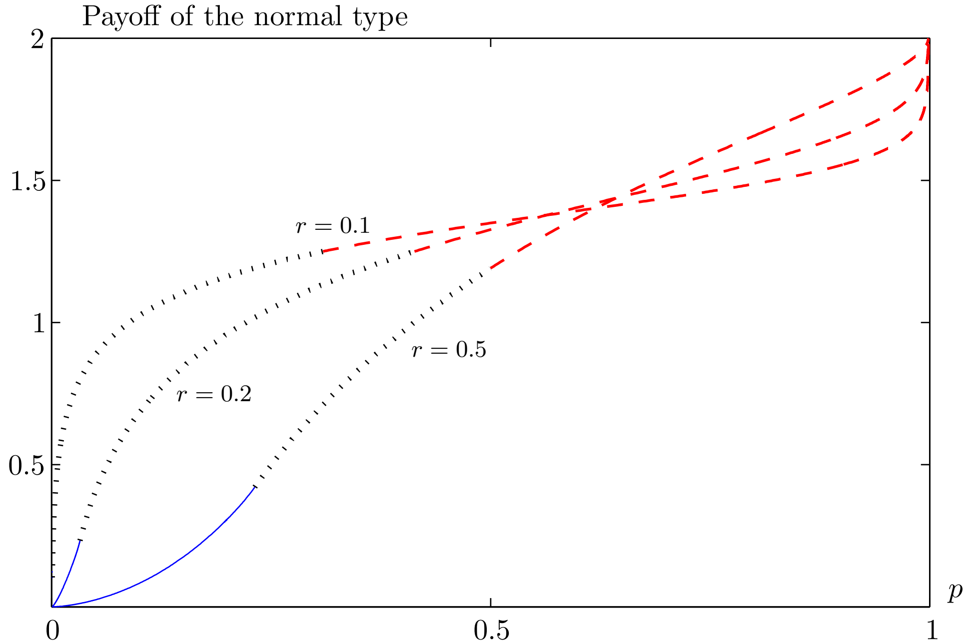

Figure 4: The upper boundary of $\mathcal{E}(p)$.

The customers' actions are uniquely $\bar{b} = 0$ only when $(1-\phi)a+\phi a^* < 1/2$. If $(1-\phi)a+\phi a^* \geq 1/2$ then the game among the customers, who face a coordination problem, has two pure equilibria with $\bar{b} = 0$ and $\bar{b} = 2$ (and one mixed equilibrium when $(1-\phi)a + \phi a^* > 1/2$). Thus, the correspondence $\Psi(\phi, z)$ is single-valued only on a subset of its domain.

How is this reflected in the equilibrium correspondence $\mathcal{E}(p)$? Figure 4 shows the upper boundary of $\mathcal{E}(p)$ for three discount rates $r = 0.1$, 0.2 and 0.5. The lower boundary for this example is always 0, because the game among the customers has an equilibrium with $\bar{b} = 0$.

For each discount rate, the upper boundary U is divided into three regions. In the region near 0, where the upper boundary is a solid line, the correspondence $\Psi(\phi, \phi(1-\phi)U'(\phi))$ is single-valued and U satisfies the upper optimality equation in the classical sense. In the region near 1, where the upper boundary is a dashed line, the correspondence Ψ is continuous and has three values (two pure and one mixed). There, U also satisfies the upper optimality equation with the population's action $\bar{b} = 2$. In middle region, where the upper boundary is a dotted line we have

$$U''(\phi) \in \left(\frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - 2 + a)}{|\gamma(a, 2, \phi)|^2}, \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - 0 + a)}{|\gamma(a, 0, \phi)|^2} \right),$$

where a is given by (33) and 0 and 2 are two values of $\bar{b}$ that the correspondence Ψ returns. In that range, the correspondence $\Psi(\phi, \phi(1-\phi)U'(\phi))$ is discontinuous in its arguments: if we lower $U(\phi)$ slightly the equilibrium among the customers with $\bar{b} = 2$ disappears. These properties of the upper boundary follow from the fact that it is the *largest* solution of the upper optimality equation.

A Appendix for Section 4

Assuming that Condition 1 holds, let us complete the proof of Proposition 4. We need the following lemma:

Lemma 3. *There exist $M > 0$ and $C > 0$ such that whenever $|\beta| \leq M$, and $(a, \bar{b}, \phi)$ satisfies the incentive constraints (17), we have*

$$|\gamma(a, \bar{b}, \phi)| \geq C\phi(1-\phi)\,.$$

Proof. Consider the set Φ of 4-tuples $(a, \bar{b}, \phi, \beta)$ such that the incentive constraints (17) hold and $\mu(a, \bar{b}) = \mu(a^*, \bar{b})$. Φ is a closed set that does not intersect the compact set $A \times \Delta(B) \times [0,1] \times \{0\}$, and therefore the distance $M' > 0$ between those two sets is positive. It follows that $|\beta| \geq M'$ for any $(a, \bar{b}, \phi, \beta) \in \Phi$.

Now, let $M = M'/2$. Let Φ' be the set of 4-tuples $(a, \bar{b}, \phi, \beta)$ such that the incentive constraints (17) hold and $|\beta| \leq M$. Φ' is a compact set, and so the continuous function $|\mu(a^*, \bar{b}) - \mu(a, \bar{b})|$ must reach a minimum C_1 on Φ'. We have $C_1 > 0$ because $|\beta| \geq 2M$ whenever $|\mu(a^*, \bar{b}) - \mu(a, \bar{b})| = 0$. Since for some $k > 0$, $|\sigma(\bar{b}) \cdot y| \leq k|y|$ for all y and $\bar{b}$, we have

$$|\gamma(a, \bar{b}, \phi)| \geq C\phi(1-\phi)$$

whenever $|\beta| \leq M$ and $(a, \bar{b}, \phi)$ satisfies the incentive constraints (17), where $C = C_1/k$. This concludes the proof of the lemma. □

Now, fix a public sequetial equilibrium $(a_t, \bar{b}_t, \phi_t)$ and $\varepsilon > 0$. Consider the function $f_1(W) = e^{K_1(W-\underline{g})}$. Then, by Ito's lemma, $f_1(W_t)$ has drift

$$K_1 e^{K_1(W-\underline{g})}(rW_t - g(a_t, \bar{b}_t)) + K_1^2/2 e^{K_1(W-\underline{g})} r^2 \beta_t^2,$$

which is always greater than or equal to

$$-K_1 e^{K_1(\bar{g}-\underline{g})} r(\bar{g} - \underline{g}),$$

and greater than or equal to

$$-K_1 e^{K_1(W-\underline{g})} r(\bar{g} - \underline{g}) + K_1^2/2 e^{K_1(W-\underline{g})} r^2 M^2 > 1$$

when $|\beta_t| \geq M$ (choosing K_1 sufficiently large).

Consider the function $f_2(\phi_t) = K_2(\phi_t^2 - 2\phi_t)$. We have

$$d\phi_t = -\frac{|\gamma(a_t, \bar{b}_t, \phi_t)|^2}{1-\phi_t}\,dt + \gamma(a_t, \bar{b}_t, \phi_t)\,dZ_t^{\mathrm{n}}$$

and so by Ito's lemma $f_2(\phi_t)$ has drift

$$-K_2 \frac{|\gamma(a_t, \bar{b}_t, \phi_t)|^2}{1-\phi_t}(2\phi_t - 2) + K_2 \frac{|\gamma(a_t, \bar{b}_t, \phi_t)|^2}{2} 2 = 3K_2 |\gamma(a_t, \bar{b}_t, \phi_t)|^2 \geq 0$$

When K_2 is sufficiently large, then the drift of $f_2(\phi_t)$ is greater than or equal to $K_1 e^{K_1(\bar{g}-\underline{g})} r(\bar{g} - \underline{g}) + 1$, whenever $\phi_t \in [\varepsilon, 1-\varepsilon]$ and $|\beta_t| \leq M$, so that $|\gamma(a, \bar{b}, \phi)| \geq C\phi(1-\phi)$ by Lemma 3.

It follows that until the stopping time τ when ϕ_t hits an endpoint of $[\varepsilon, 1-\varepsilon]$, the drift of $f_1(W_t) + f_2(\phi_t)$ is greater than or equal to 1.

But then for some constant K_3, since f_1 is bounded on $[\underline{g}, \bar{g}]$ and f_2 is bounded on $[\epsilon, 1-\epsilon]$, it follows that for all t

$$K_3 \geq E[f_1(W_{\min(\tau,t)}) + f_2(\phi_{\min(\tau,t)})] \geq f_1(W_0) + f_2(\phi_0) + \int_0^t \text{Prob}(\tau \geq s)\, ds$$

and so $\text{Prob}(\tau \geq s)$ must converge to 0 as $s \to \infty$.

But then ϕ_t must converge to 0 or 1 with probability 1, and it cannot be 1 with positive probability if the type is normal. This completes the proof of Proposition 4.

B Appendix for Section 5

This appendix supports Section 5.

Proof of Lemma 1. Pick any constant $M > 0$. Consider the set Φ_0 of triples $(a, b, \beta) \in A \times \Delta B \times [0, 1] \times \mathbb{R}^d$ that satisfy

$$a = \arg\max_{a' \in A}\ g(a', b), \quad b = \arg\max_{b' \in B} h(a, b', \bar{b}),\ \forall b \in \text{support}\, \bar{b}, \quad g(a, \bar{b}) \geq \bar{v} + \varepsilon \qquad (34)$$

and $|\beta| \leq M$.

The set Φ_0 is closed (since it is defined by weak inequalities) and bounded, and therefore compact. Therefore, the continuous function $|\beta|$ achieves its minimum δ on Φ_0, and $\delta > 0$ because of the condition $g(a, \bar{b}) \geq \bar{v} + \varepsilon$. It follows that $|\beta| \geq \min(M, \delta) > 0$ for any triple (a, b, β) that satisfies conditions (34). QED

C Appendix for Section 6

In this appendix we will maintain Conditions 1 and 2.

C.1 Existence of a bounded solution of the optimality equation.

In this subsection we prove the following Proposition.

Proposition 5. *The optimality equation has at least one solution that stays within the interval of all feasible payoffs of the large player on* $(0, 1)$.

The proof of Proposition 5 relies on several lemmas.

Lemma 4. *For all $\varepsilon > 0$ there exists $K > 0$ such that for all $\phi \in [0, 1]$ and $u' \in \mathbb{R}$,*

$$|u'||\gamma(a, \bar{b}, \phi)| \geqq K\,,$$

whenever $\phi(1-\phi)|u'| \geqq \varepsilon$ and $(a, \bar{b}) \in \Psi(\phi, \phi(1-\phi)u')$.

Proof. As shown in the proof of Proposition 4, given Condition 1 there is no Bayesian Nash equilibium $(a, \bar{b})$ of the static game with prior $p > 0$ in which $\mu(a, \bar{b}) = \mu(a^*, \bar{b})$.

If the statement of the lemma were false, there would exist a sequence $(a_n, \bar{b}_n, u'_n, \phi_n)$, with $(a_n, \bar{b}_n) \in \Psi(\phi_n, \phi_n(1-\phi_n)u'_n)$ and $\phi_n(1-\phi_n)|u'_n| \geq \varepsilon$ for all n, for which $|u'_n||\gamma(a_n, \bar{b}_n, \phi_n)|$ converged to 0. Let $(a, \bar{b}, \phi) \in A \times \Delta B \times [0, 1]$ denote the limit of a convergent subsequence. By upper hemi-continuity, $(a, \bar{b})$ is a BNE of the static game with prior ϕ. Hence, $\mu(a, \bar{b}) \neq \mu(a^*, \bar{b})$ and therefore $\liminf_n |u'_n||\gamma(a_n, \bar{b}_n, \phi_n)| \geqq \varepsilon|\sigma(b)^{-1}(\mu(a, \bar{b}) - \mu(a^*, \bar{b}))| > 0$, a contradiction. □

Lemma 5. *The solutions to the optimality equation exist locally for $\phi \in (0, 1)$ (that is, until a blowup point when $|U(\phi)|$ or $|U'(\phi)|$ become unboundedly large) and are unique and continuous in initial conditions.*

Proof. It follows from the standard theorem on existence, uniqueness and continuity of solutions to an ordinary differential equations in initial conditions, because the right hand side of the optimality equation is Lipschitz-continuous. Note that $\gamma(\Psi(\phi, \phi(1-\phi)U'(\phi)), \phi)$ does not reach 0 for any finite u' : if we had $\gamma(a, \bar{b}, \phi) = 0$ for $(a, \bar{b}) = \Psi(\phi, \phi(1-\phi)U'(\phi))$, then $(a, \bar{b})$ would be a BNE of the stage game with prior ϕ in which $\mu(a, \bar{b}) = \mu(a^*, \bar{b})$, a contradiction (see the proof of Lemma 4). □

Lemma 6. *Consider a solution $U(\phi)$ of the optimality equation. If there is a blowup at point $\phi_1 \in (0, 1)$ then both $|U(\phi)|$ and $|U'(\phi)|$ become unboundedly large near ϕ_1.*

Proof. By Lemma 4, there exists a constant $k > 0$ such that

$$|U'(\phi)||\gamma(\Psi(\phi, \phi(1-\phi)U'(\phi)), \phi)| \geq k > 0$$

in a neighborhood of ϕ_1, when $|U'(\phi)|$ is bounded away from 0. Suppose, towards a contradiction, that $U(\phi)$ is bounded from above by K near ϕ_1. Without loss of generality assume that $U'(\phi)$ (as opposed to $-U'(\phi)$) becomes arbitrarily large near ϕ_1, and that ϕ_1 is the right endpoint of the domain of the solution U. Then let us pick points $\phi_3 < \phi_2 < \phi_1$ such that $U'(\phi)$ stays positive on the interval (ϕ_3, ϕ_2) and $U'(\phi_2) - U'(\phi_3)$ is sufficiently large.

Consider the case when $U'(\phi)$ is monotonic on (ϕ_2, ϕ_3), and let us parameterize the interval (ϕ_3, ϕ_2) by $u' = U'(\phi)$. Denote

$$\xi(u') = \frac{dU(\phi)}{dU'(\phi)} = \frac{U'(\phi)}{U''(\phi)} > 0$$

Note that

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\Psi(\phi, \phi(1-\phi)U'(\phi))))}{|\gamma(\Psi(\phi, \phi(1-\phi)U'(\phi)), \phi)|^2} \leq k_1 U'(\phi) + k_2 U'(\phi)^2$$

for some constants k_1 and k_2 that depend on ϕ_1, K and the range of stage-game payoffs of the large player, so that $\xi(u') \geq 1/(k_1 + k_2 u')$.

Then

$$U(\phi_3) - U(\phi_2) = \int_{U'(\phi_2)}^{U'(\phi_3)} \xi(u')du' \geq \int_{U'(\phi_2)}^{U'(\phi_3)} 1/(k_1 + k_2 u')du' \quad (35)$$

This quantity grows arbitrarily large, leading to a contradiction, when $U'(\phi_3) - U'(\phi_2)$ gets large while $U'(\phi_2)$ stays fixed (this can be always guaranteed even if $U'(\phi)$ flips sign many times near ϕ_1.)

When $U'(\phi)$ is not monotonic on (ϕ_2, ϕ_3), a conclusion similar to (35) can be reached by splitting the integral into subintervals where $U'(\phi)$ is increasing (on which the bound (35) holds) and the rest of the subintervals (on which $U(\phi)$ is increasing). □

One consequence of Lemma 6 is that starting from any initial condition with $\phi_0 \in (0,1)$ the solution of the optimality equation exists until $\phi = 0$ and 1, or until $U(\phi)$ exits the range of feasible payoffs of the large player.

Lemma 7. *(Monotonicity) If two solutions U_1 and U_2 of the optimality equation satisfy $U_1(\phi_0) \leq U_2(\phi_0)$ and $U_1'(\phi_0) \leq U_2'(\phi_0)$ with at least one strict inequality, then $U_1(\phi) \leq U_2(\phi)$ and $U_1'(\phi) \leq U_2'(\phi)$ for all $\phi > \phi_0$ until the blowup point. Similarly, if $U_1(\phi_0) \leq U_2(\phi_0)$ and $U_1'(\phi_0) \geq U_2'(\phi_0)$ with at least one strict inequality, then $U_1(\phi) < U_2(\phi)$ and $U_1'(\phi) > U_2'(\phi)$ for all $\phi < \phi_0$ until the blowup point.*

Proof. Suppose that $U_1(\phi_0) \leq U_2(\phi_0)$ and $U_1'(\phi_0) < U_2'(\phi_0)$. If $U_1'(\phi) < U_2'(\phi)$ for all $\phi > \phi_0$ until the blowup point then we also have $U_1(\phi) < U_2(\phi)$ on that range. Otherwise, let

$$\phi_1 = \inf_{\phi \geq \phi_0} U_1'(\phi) \geq U_2'(\phi).$$

Then $U_1'(\phi_1) = U_2'(\phi_1)$ by continuity and $U_1(\phi_1) < U_2(\phi_1)$ since $U_1(\phi_0) \leq U_2(\phi_0)$ and $U_1(\phi) < U_2(\phi)$ on $[\phi_0, \phi_1)$. From the optimality equation, it follows that $U_1''(\phi_1) < U_2''(\phi_1) \Rightarrow U_1'(\phi_1 - \varepsilon) > U_2'(\phi_1 - \varepsilon)$ for sufficiently small ε, which contradicts the definition of ϕ_1.

For the case when $U_1(\phi_0) < U_2(\phi_0)$ and $U_1'(\phi_0) = U_2'(\phi_0)$ the optimality equation implies that $U_1''(\phi_0) < U_2''(\phi_0)$. Therefore, $U_1'(\phi) < U_2'(\phi)$ on $(\phi_0, \phi_0 + \varepsilon)$, and the argument proceeds as above.

The monotonicity argument for $\phi < \phi_0$ when $U_1(\phi_0) \leq U_2(\phi_0)$ and $U_1'(\phi_0) \geq U_2'(\phi_0)$ with at least one strict inequality is similar. □

Proof of Proposition 5. Denote by $[\underline{g}, \bar{g}]$ the interval of all feasible payoffs of the large player. Fix $\phi_0 \in (0,1)$.

(a) Note that for any if $|U'(\phi_0)|$ is sufficiently large then the solution U must exit the interval $[\underline{g}, \bar{g}]$ in a neighborhood of ϕ_0. This conclusion can be derived using an inequality similar to (35): $|U'(\phi)|$ cannot become small near ϕ_0 without a change in $U(\phi)$ of $\int_{|U'(\phi)|}^{|U'(\phi_0)|} 1/(k_1 + k_2|x|)dx$.

(b) Also, note that if a solution U reaches the boundary of the region of feasible payoffs, it must exit the region and never reenter. Indeed, it is easy to see from the optimality equation that when $U'(\phi) = 0$, $U''(\phi) \geq 0$ if $U(\phi) \geq \bar{g}$, and $U''(\phi) \leq 0$ if $U(\phi) \leq \underline{g}$. Therefore, $U'(\phi)$ never changes its sign when $U(\phi)$ is outside $(\underline{g}, \bar{g})$.

(c) For a given level $U(\phi_0) = u$, consider solutions of the optimality equation for $\phi \leq \phi_0$ for different values of $U'(\phi_0)$. When $U'(\phi_0)$ is sufficiently large, the resulting solution will reach $\underline{g}$ at some point $\phi_1 \in (0, \phi_0)$ by (a). As $U'(\phi_0)$ decreases, ϕ_1 also decreases by Lemma 7, until for some value $U'(\phi_0) = L(u)$ the solution never reaches the lower boundary of the set of feasible payoffs for any $\phi_1 \in (0, \phi_0)$. Note that this solution never reaches the upper boundary of the set of feasible payoffs for any $\phi_1 \in (0, \phi_0)$: if it did, then the solution with slope $U'(\phi_0) = L(u) + \varepsilon$ would also reach the upper boundary by Lemma 5, and by (b) it would never reach the lower boundary. We conclude that the solution of the optimality equation with boundary conditions $U(\phi_0) = u$ and $U'(\phi_0) = L(u)$ stays within the range of feasible payoffs for all $\phi_1 \in (0, \phi_0)$.

(d) Similarly, define $R(u)$ as the smallest value of $U'(\phi_0)$ for which the resulting solution never reaches the largest feasible payoff of the large player at any $\phi \in (\phi_0, 1)$. Then the solution of the optimality equation with boundary conditions $U(\phi_0) = u$ and $U'(\phi_0) = R(u)$ stays within the range of feasible payoffs for all $\phi_1 \in (\phi_0, 1)$, by the same logic as in (c).

(e) Now, Lemma 7 implies that $L(u)$ is increasing in u and $R(u)$ is decreasing in u. Moreover, $L(\underline{g}) \leq 0 \leq L(\bar{g})$ and $R(\underline{g}) \geq 0 \geq R(\bar{g})$. Therefore, there exists a value of u for which $L(u) = R(u)$. The solution to the optimality equation with boundary conditions $U(\phi_0) = u$ and $U'(\phi_0) = L(u) = R(u)$ must stay within the interval of feasible payoffs for all $\phi \in (0, 1)$.

This completes the proof of Proposition 5. □

C.2 Regularity conditions at the boundary.

Proposition 6. *If U is a bounded solution of equation (21)on $(0, 1)$, then U satisfies the following* boundary conditions *at $p = 0, 1$:*

$$\lim_{\phi \to p} U(\phi) = g(\Psi(p, 0)) \,, \ \lim_{\phi \to p} \phi(1 - \phi) U'(\phi) = 0 \,, \ \lim_{\phi \to p} \phi^2 (1 - \phi)^2 U''(\phi) = 0 \,. \tag{36}$$

Proof. Direct from Lemmas 11, 12 and 13 below. Lemmas 8, 9 and 10 are intermediate steps. □

Lemma 8. *For all $M > 0$ there exists $C > 0$ such that*

$$|\gamma(a, \bar{b}, \phi)| \geqq C \, \phi(1 - \phi) \,,$$

whenever $\phi(1 - \phi)|u'| < M$ and $(a, \bar{b}) \in \Psi(\phi, \phi(1 - \phi)u')$.

Proof. Fix $M > 0$. By Lemma 4, for all $\varepsilon \in (0, M)$ there exists $K > 0$ such that

$$|\gamma(a, \bar{b}, \phi)| \geqq \frac{K}{|u'|} \geqq \frac{K}{M} \, \phi(1 - \phi)$$

whenever $\phi(1 - \phi)|u'| \in (\varepsilon, M)$ and $(a, \bar{b}) \in \Psi(\phi, \phi(1 - \phi)u')$.

Therefore, Lemma 8 can be false only if

$$\frac{|\gamma(a_n, \bar{b}_n, \phi_n)|}{\phi_n(1 - \phi_n)} = \sigma(\bar{b}_n)^{-1}(\mu(a^*, \bar{b}_n) - \mu(a_n, \bar{b}_n))$$

converges to 0 for some sequence $(a_n, \bar{b}_n, u'_n, \phi_n)$, with $(a_n, \bar{b}_n) \in \Psi(\phi_n, u'_n)$, $\phi_n \in (0, 1)$, and $\phi_n(1 - \phi_n)|u'_n| \to 0$. Let $(a, \bar{b}, \phi) \in A \times \Delta(B) \times [0, 1]$ denote the limit of a convergent subsequence. By upper hemi-continuity, $(a, \bar{b})$ is a BNE of the static game with prior ϕ. Hence, $\mu(a, \bar{b}) \neq \mu(a^*, \bar{b})$ and so $|\gamma(a_n, \bar{b}_n, \phi_n)|/(\phi_n(1 - \phi_n))$ cannot converge to 0, a contradiction. □

Lemma 9. *Let $U : (0,1) \to \mathbb{R}$ be a bounded, continuously differentiable function. Then*

$$\liminf_{\phi \to 0} \phi U'(\phi) \ \le \ 0 \ \le \ \limsup_{\phi \to 0} \phi U'(\phi)\,, \quad \text{and}$$

$$\liminf_{\phi \to 1} (1-\phi) U'(\phi) \ \le \ 0 \ \le \ \limsup_{\phi \to 1} (1-\phi) U'(\phi)\,.$$

Proof. Suppose, towards a contradiction, that $\liminf_{\phi \to 0} \phi U'(\phi) > 0$ (the case $\limsup_{\phi \to 0} \phi U'(\phi) < 0$ is analogous). Then for some $c > 0$ and $\bar{\phi} > 0$, for all $\phi \in (0, \bar{\phi}]$, $\phi U'(\phi) \ge c \Rightarrow U'(\phi) \ge c/\phi$. But then U cannot be bounded since the anti-derivative of $1/\phi$, $\log \phi$, tends to ∞ as $\phi \to 0$, a contradiction. The proof for the case $\phi \to 1$ is analogous. □

Lemma 10. *If $U : (0,1) \to \mathbb{R}$ is a bounded solution of the optimality equation, then U has bounded variation.*

Proof. Suppose there exists a bounded solution U of the optimality equation with unbounded variation near $p = 0$ (the case $p = 1$ is similar). Then let ϕ_n be a decreasing sequence of consecutive local maxima and minima of U, such that ϕ_n is a local maximum for n odd and a local minimum for n even.

Then for n odd we have $U'(\phi_n) = 0$ and $U''(\phi_n) \le 0$. From the optimality equation it follows that $g(\Psi(\phi_n, 0)) \ge U(\phi_n)$. Likewise, for n even we have $g(\Psi(\phi_n, 0)) \le U(\phi_n)$. Thus, the total variation of $g(\Psi(\phi, 0))$ on $(0, \phi_1]$ is no smaller than the total variation of U and therefore $g(\Psi(\phi, 0))$ has unbounded variation near zero. However, this is a contradiction, since $g(\Psi(\phi, 0))$ is Lipschitz continuous. □

Lemma 11. *If U is a bounded solution of the optimality equation, then $\lim_{\phi \to p} \phi(1-\phi) U'(\phi) = 0$ for $p \in \{0, 1\}$.*

Proof. Suppose, towards a contradiction, that $\phi U'(\phi) \nrightarrow 0$ as $\phi \to 0$. Then, by Lemma 9,

$$\liminf_{\phi \to 0} \phi U'(\phi) \ \le \ 0 \ \le \ \limsup_{\phi \to 0} \phi U'(\phi)\,,$$

with at least one strict inequality. Without loss of generality, assume $\limsup_{\phi \to 0} \phi U'(\phi) > 0$. Hence there exist constants $0 < k < K$, such that $\phi U'(\phi)$ crosses levels k and K infinitely many times in a neighborhood of 0.

By Lemma 8 there exists $C > 0$ such that

$$|\gamma(a, \bar{b}, \phi)| \ge C\,\phi\,,$$

whenever $\phi U'(\phi) \in (k, K)$ and $\phi \in (0, \frac{1}{2})$. Hence, by the optimality equation, we have

$$|U''(\phi)| \le \frac{L}{\phi^2}\,,$$

for some constant $L > 0$. This bound implies that for all $\phi \in (0, \frac{1}{2})$ with $\phi U'(\phi) \in (k, K)$, we have

$$|(\phi U'(\phi))'| \ \le \ |\phi U''(\phi)| + |U'(\phi)| \ = \ (1 + \frac{|\phi U''(\phi)|}{|U'(\phi)|})\,|U'(\phi)| \ \le \ (1 + \frac{L}{k})\,|U'(\phi)|\,,$$

which yields

$$|U'(\phi)| \geq \frac{|(\phi U'(\phi))'|}{1+L/k}\,.$$

It follows that on every interval where $\phi U'(\phi)$ crosses k and stays in (k, K) until crossing K, the total variation of U is at least $(K-k)/(1+L/k)$. Since this happens infinitely many times in a neighborhood of $\phi = 0$, function U must have unbounded variation in that neighborhood, a contradiction (by virtue of Lemma 10.)

The proof that $\lim_{\phi\to 1}(1-\phi)U'(\phi) = 0$ is analogous. □

Lemma 12. *If $U : (0,1) \to \mathbb{R}$ is a bounded solution of the Optimality equation, then for $p \in \{0,1\}$,*

$$\lim_{\phi\to p} U(\phi) = g(\Psi(p,0))\,.$$

Proof. First, by Lemma 10, U must have bounded variation and so the $\lim_{\phi\to p} U(\phi)$ exists. Consider $p = 0$ and assume, towards a contradiction, that $\lim_{\phi\to 0} U(\phi) = U_0 < g(a^N, b^N)$, where $(a^N, b^N) = \Psi(0,0)$ is the Nash equilibrium of the stage game (the proof for the reciprocal case is similar). By Lemma 11, $\lim_{\phi\to 0}\phi U'(\phi) = 0$, which implies that the function $\Psi(\phi, \phi(1-\phi)U'(\phi))$ is continuous at $\phi = 0$. Recall the optimality equation

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\Psi(\phi, \phi(1-\phi)U'(\phi)))}{|\gamma(\Psi(\phi,\phi(1-\phi)U'(\phi)),\phi)|^2} = \frac{2U'(\phi)}{1-\phi} + \frac{h(\phi)}{\phi^2},$$

where $h(\phi)$ is a continuous function that converges to

$$\frac{2r(U_0 - g(a^N, b^N))}{|\sigma(b^N)^{-1}(\mu(a^*, b^N) - \mu(a^N, b^N))|^2} < 0.$$

as $\phi \to 0$. Since $U'(\phi) = o(1/\phi)$ by Lemma 9, it follows that for some $\bar\phi > 0$, there exists a constant $K > 0$ such that

$$U''(\phi) < -\frac{K}{\phi^2}$$

for all $\phi \in (0, \bar\phi)$. But then U cannot be bounded since the second-order anti-derivative of $1/\phi^2$ ($-\log\phi$) tends to ∞ as $\phi \to 0$.

The proof for the case $p = 1$ is analogous. □

Lemma 13. *Let U be a solution of the Optimality Equation that satisfies:*

$$\lim_{\phi\to p} U(\phi) = g(\Psi(p,0)) \quad \text{and} \quad \lim_{\phi\to p}\phi(1-\phi)U'(\phi) = 0\,, \quad \text{for } p = 0 \text{ and } 1.$$

Then,

$$\lim_{\phi\to p}\phi^2(1-\phi)^2 U''(\phi) = 0\,.$$

Proof. Consider $p = 1$. Fix an arbitrary $M > 0$ and choose $\underline{\phi} \in (0,1)$ so that $(1-\phi)|U'(\phi)| < M$ for all $\phi \in (\underline{\phi}, 1)$. By Lemma 8 there exists $C > 0$ such that $|\gamma(\Psi(\phi, \phi(1-\phi)U'(\phi)), \phi)| \geq C(1-\phi)$ for all $\phi \in (\underline{\phi}, 1)$. Hence, by the optimality equation, we have for all $\phi \in (\underline{\phi}, 1)$:

$$\begin{aligned}(1-\phi)^2|U''(\phi)| &\leq 2(1-\phi)|U'(\phi)| + (1-\phi)^2\frac{2r|U(\phi) - g(\Psi(\phi,\phi(1-\phi)U'(\phi)))|}{|\gamma(\Psi(\phi,\phi(1-\phi)U'(\phi)),\phi)|^2}\\ &\leq 2(1-\phi)|U'(\phi)| + 2rC^{-2}|U(\phi) - g(\Psi(\phi,\phi(1-\phi)U'(\phi)))| \longrightarrow 0\,,\end{aligned}$$

as required. The case $p = 0$ is analogous. □

Proposition 7. *There exists a unique continuous function $U : [0,1] \to \mathbb{R}$ that stays in the interval of feasible payoffs of the large player, satisfies equation (21) on $(0,1)$ and conditions (36) at 0 and 1.*

Proof. Propositions 5 and 6 imply that there exists at least one such solution U. Suppose that V was another such solution. Assuming that $V(\phi) > U(\phi)$ for some ϕ, let $\phi_0 \in (0,1)$ be the point where the difference $V(\phi_0) - U(\phi_0)$ is maximized. But then by Lemma 7 the difference $V(\phi) - U(\phi)$ must be increasing for $\phi > \phi_0$, a contradiction. □

C.3 A uniform lower bound on volatility.

Lemma 14. *Let U be the unique bounded solution of the Optimality Equation and let*

$$d(a,\bar{b},\phi) = rU(\phi) - rg(a,\bar{b}) - \frac{|\gamma(a,b,\phi)|^2}{1-\phi}U'(\phi) - \frac{1}{2}|\gamma(a,b,\phi)|^2 U''(\phi), \tag{37}$$

and

$$f(a,\bar{b},\phi,\beta) = r\beta\sigma(\bar{b}) - \underbrace{\phi(1-\phi)\sigma(\bar{b})^{-1}(\mu(a^*,\bar{b}) - \mu(a,\bar{b}))}_{\gamma(a,b,\phi)}U'(\phi)\,. \tag{38}$$

For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(a,\bar{b},\phi,\beta)$ that satisfy

$$\begin{aligned} a &\in \arg\max_{a' \in A}\ rg(a',b) + r\beta\mu(a',b) \\ \bar{b} &\in \arg\max_{b' \in B}\ u(b',\bar{b}) + v(b',\bar{b}) \cdot \mu^{\phi}(a,\bar{b})\,, \quad \text{for all } \ b \in \operatorname{support}\bar{b}, \end{aligned} \tag{39}$$

either $d(a,\bar{b},\phi) > -\varepsilon$ or $f(a,\bar{b},\phi,\beta) \geq \delta$.

Proof. Since $\phi(1-\phi)U'(\phi)$ is bounded (by Lemma 11) and there exists $c > 0$ such that $|\sigma(\bar{b}) \cdot y| \geq c|y|$ for all $y \in \mathbb{R}^d$ and $\bar{b} \in \Delta B$, there exist constants $M > 0$ and $m > 0$ such that $|f(a,\bar{b},\phi,\beta)| > m$ for all $\beta \in \mathbb{R}^d$ with $|\beta| > M$.

Consider the set Φ of 4-tuples $(a,b,\phi,\beta) \in A \times \Delta B \times [0,1] \times \mathbb{R}^d$ with $|\beta| \leq M$, which satisfy (39) and $d(a,\bar{b},\phi) \leq -\varepsilon$. Since U satisfies the boundary conditions (36), $d(a,b,\phi)$ is a continuous function and the set Φ is a closed subset of the compact set

$$\{(a,b,\phi,\beta) \in A \times \Delta B \times [0,1] \times \mathbb{R}^d \ : \ |\beta| \leq M\},$$

and hence Φ is compact.[14]

Since U satisfies the boundary conditions (36), the function $|f(a,\bar{b},\phi,\beta)|$ is continuous. Hence, it achieves its minimum, δ, on Φ. We have $\delta > 0$, because, as we argued in the proof of Theorem 3, $d(a,\bar{b},\phi) = 0$ whenever $f(a,\bar{b},\phi,\beta) = 0$.

It follows that for all $(a,\bar{b},\phi,\beta)$ that satisfy (39), either $d(a,\bar{b},\phi) > -\varepsilon$ or $|f(a,\bar{b},\phi,\beta)| \geq \min(m,\delta)$. □

[14]Since B is compact, the set $\Delta(B)$ is compact in the topology of weak convergence of probability measures.

D Appendix for Section 7

Throughout this appendix, we will maintain Conditions 1 and 3.

Write U and L for the upper and lower boundaries of the correspondence $\mathcal{E}$ respectively, that is,

$$U(p) = \sup \mathcal{E}(p)\,, \quad L(p) = \inf \mathcal{E}(p)$$

for all $p \in [0,1]$.

Proposition 8. *The upper boundary $U : (0,1) \to \mathbb{R}$ is a viscosity sub-solution of the Upper Optimality equation.*

Proof. If U is not a sub-solution, there exists $q \in (0,1)$ and a $\mathcal{C}^2$-function $V : (0,1) \to \mathbb{R}$ such that $0 = (U^* - V)(q) > (U^* - V)(\phi)$ for all $\phi \in (0,1) \setminus \{q\}$, and

$$\frac{2V'(q)}{1-q} + \min_{(a,\bar{b}) \in \Psi(q, q(1-q)V'(q))} \frac{2r(U^*(q) - g(a,\bar{b}))}{|\gamma(a,\bar{b},q)|^2} \; - \; V''(q) \; > \; 0\,.$$

Since the left-hand side of the inequality above is l.s.c., there exist $\zeta, \varepsilon > 0$ such that

$$\frac{2V'(\phi)}{1-\phi} + \min_{(a,\bar{b}) \in \Psi(\phi, \phi(1-\phi)V'(\phi))} \frac{2r(V(\phi) - g(a,\bar{b}))}{|\gamma(a,\bar{b},\phi)|^2} - V''(\phi) \; > \; 2\,\zeta$$

for all $\phi \in [q - \varepsilon, q + \varepsilon] \subset (0,1)$.

Equivalently, for all $\phi \in [q - \varepsilon, q + \varepsilon]$ and $(a,\bar{b}) \in \Psi(\phi, \phi(1-\phi)V'(\phi))$,

$$rV(\phi) - rg(a,\bar{b}) + |\gamma(a,\bar{b},\phi)|^2 (\frac{V'(\phi)}{1-\phi} - \frac{V''(\phi)}{2}) \; > \; \zeta |\gamma(a,\bar{b},\phi)|^2\,,$$

which implies

$$rV(\phi) - rg(a,\bar{b}) + |\gamma(a,\bar{b},\phi)|^2 (\frac{V'(\phi)}{1-\phi} - \frac{V''(\phi)}{2}) \; > \; \zeta \underline{\gamma}^2 \; \equiv \; \kappa\,, \tag{40}$$

where $\underline{\gamma}$ denotes the minimum of $|\gamma(a,\bar{b},\phi)|$ over all $(a,\bar{b}) \in \Psi(\phi, \phi(1-\phi)V'(\phi))$ and $\phi \in [q - \varepsilon, q + \varepsilon]$. By Lemma 4, we have $\underline{\gamma} > 0$ and therefore $\kappa > 0$.

Since U^* is upper semi-continuous, we can choose $\delta \in (0, \frac{\kappa}{2r})$ small enough that for all $\phi \in [q - \varepsilon, q + \varepsilon]$,

$$V(\phi) - \delta \leqq U^*(\phi) \; \Rightarrow \; |\phi - q| < \frac{\varepsilon}{2}\,. \tag{41}$$

Since U is the upper boundary of the equilibrium correspondence and $V(q) = U^*(q)$, there exists a pair $(p, w) \in (q - \frac{\varepsilon}{2}, q + \frac{\varepsilon}{2}) \times \mathbb{R}$ such that $w \in \mathcal{E}(p)$ and $V(p) - \delta < w \leqq U(p)$.

Consider a public sequential equilibrium $(a_t, \bar{b}_t, \phi_t)$ of the game with prior p that yields the large player a payoff of w. Let (W_t) denote the large player's continuation payoff process. We will show that, for a finite stopping time τ, there is positive probability that $W_\tau > U^*(\phi_\tau)$, which yields a contradiction since U is the upper boundary.

Let $D_t = W_t - V_t$, where $V_t = V(\phi_t)$. By Itô's formula,

$$dV_t = |\gamma_t|^2 (\frac{V_t''}{2} - \frac{V_t'}{1-\phi_t})\, dt \; + \; \gamma_t V_t'\, dZ_t^{\mathrm{n}}\,,$$

where $V_t' = V'(\phi_t)$, $V_t'' = V''(\phi_t)$ and $\gamma_t = \gamma(a_t, \bar{b}_t, \phi_t)$. Therefore,

$$dD_t = (rD_t + rV_t - rg_t - |\gamma_t|^2(\frac{V_t''}{2} - \frac{V_t'}{1-\phi_t}))\, dt \; + \; (r\beta_t\sigma_t - \gamma_t V_t')\, dZ_t^{\mathrm{n}} \,,$$

where $g_t = g(a_t, \bar{b}_t)$, $\sigma_t = \sigma(\bar{b}_t)$ and $\beta \in \mathcal{L}^*$ is the random process of Proposition 2, that is,

$$dW_t = r(W_t - g_t)\, dt \; + \; r\beta_t\sigma_t\, dZ_t^{\mathrm{n}} \,.$$

Consider the stopping time $S = \inf\{t : D_t = -\delta\}$ and note that, since $V(p) - \delta < w$, we have $S > 0$.

Claim: There exists $\eta > 0$ such that for all $0 \leq t \leq S$, either the drift of D is greater than $\frac{\kappa}{2}$, or the norm of the volatility of D is greater than η.

Towards a proof of the claim, for each $(a, \bar{b}, \phi, \beta) \in A \times \Delta(B) \times [q-\varepsilon, q+\varepsilon] \times \mathbb{R}^d$ define:

$$\begin{aligned} d(a,\bar{b},\phi) &= rV(\phi) - rg(a,\bar{b}) - |\gamma(a,\bar{b},\phi)|^2(\frac{V''(\phi)}{2} - \frac{V'(\phi)}{1-\phi})\,, \\ f(a,\bar{b},\phi,\beta) &= r\beta\sigma(\bar{b}) - \gamma(a,\bar{b},\phi)V'(\phi)\,. \end{aligned}$$

Since V' is bounded on $[q-\varepsilon, q+\varepsilon]$ and there exists $c > 0$ such that $|\sigma(\bar{b})\cdot y| \geq c|y|$ for all $y \in \mathbb{R}^d$ and $\bar{b} \in \Delta B$, there exist constants $M, m > 0$ such that $|f(a,\bar{b},\phi,\beta)| > m$ whenever $|\beta| > M$.

Consider the set Φ of 4-tuples $(a, b, \phi, \beta) \in A \times \Delta B \times [q-\varepsilon, q+\varepsilon] \times \mathbb{R}^d$ with $|\beta| \leq M$ that satisfy:

$$\begin{aligned} a &\in \arg\max_{a' \in A} \; rg(a', b) + r\beta\mu(a', b) \\ \bar{b} &\in \arg\max_{b' \in B} \; u(b', \bar{b}) + v(b', \bar{b})\cdot\mu^{\phi}(a, \bar{b})\,, \quad \text{for all } \; b \in \operatorname{support}\bar{b}\,, \end{aligned}$$

and $d(a, \bar{b}, \phi) \leq \kappa$.

The continuous function $|f|$ achieves its minimum, η', on the compact set Φ. We have $\eta' > 0$, since otherwise $(a, \bar{b}) \in \Psi(\phi, \phi(1-\phi)V'(\phi))$ for some $(a, \bar{b}, \phi, \beta) \in \Phi$, which implies $d(a, \bar{b}, \phi) > \kappa$ by inequality (40), a contradiction.

In sum, we have shown that for all $0 \leq t \leq S$ such that $\phi_t \in [q-\varepsilon, q+\varepsilon]$, either:

$$\text{norm of the volatility of } D = |r\beta_t\sigma_t - \gamma_t V_t'| \geq \min\{\eta', m\} \equiv \eta\,, \; \text{ or}$$

$$\text{drift of } D = rD_t + d(a_t, \bar{b}_t, \phi_t) \geq -r\delta + \kappa > \frac{\kappa}{2}\,.$$

To conclude the proof of the claim, it remains to show that $S \leq T \equiv \inf\{t > 0 \,|\, \phi_t = q \pm \varepsilon\}$. In effect, since U is the upper boundary of the equilibrium correspondence, we have $W_T \leqq U^*(\phi_T)$ on $\{T < \infty\}$ and therefore, by (41), we have $D_T < -\delta$ almost surely on $\{T < \infty\}$. By the intermediate value theorem we have $S < T$ on $\{T < \infty\}$, which concludes the proof of the claim.

It follows directly from the claim that there is a finite stopping time τ such that $W_\tau - U^*(\phi_\tau) \geqq D_\tau > 0$ with positive probability, which is a contradiction since U is the upper boundary. □

The next lemma is an auxiliary result used in the proof of Proposition 9 below.

Lemma 15. *The correspondence $\mathcal{E}$ of public sequential equilibrium payoffs is convex-valued and has an arc-connected graph.*

Proof. We shall first prove that $\mathcal{E}$ is convex-valued. Fix $p \in (0,1)$, w^*, $w_* \in \mathcal{E}(p)$ (with $w^* > w_*$) and $v \in (w_*, w^*)$. We will prove that $v \in \mathcal{E}(p)$. Consider the set $\mathsf{V} = \{(\phi, w) \,|\, w = \alpha|\phi - p| + v\}$ where $\alpha > 0$ is chosen large enough so that $\alpha|\phi - p| + v > U(\phi)$ for all ϕ sufficiently close to 0 and 1. Let $(\phi_t, W_t)_{t \geq 0}$ be the belief / continuation value process of a public sequential equilibrium that yields the normal type a payoff of w^*. Let $\tau \equiv \inf\{t > 0 \,|\, (\phi_t, W_t) \in \mathsf{V}\}$. Proposition 6 implies that $\tau < \infty$ almost surely. If $\phi_\tau = p$ with probability 1, then $W_\tau = v$ and nothing remains to be shown. Otherwise, by the martingale property, we have $\phi_\tau < p$ with positive probability and $\phi_\tau > p$ also with positive probability. Hence, there exists a continuous curve $\mathcal{C} \subset \texttt{graph}\ \mathcal{E}$ with endpoints (p_1, w_1) and (p_2, w_2) such that $p_1 < p < p_2$, and for all $(\phi, w) \in \mathcal{C}$ we have $w > v$ and $\phi \in (p_1, p_2)$. Pick $0 < \varepsilon < p - p_1$. We will now construct a continuous curve $\mathcal{C}' \subset \texttt{graph}\,\mathcal{E}|_{(0,\,p_1+\varepsilon]}$ that has (p_1, w_1) as an endpoint and satisfies $\inf\{\phi \,|\, \exists w \text{ s.t. } (\phi, w) \in \mathcal{C}'\} = 0$. Fix a public sequential equilibrium of the dynamic game with prior p_1 that yields the normal type a payoff of w_1. Let $\mathsf{P}^{\mathtt{n}}$ denote the probability measure over the sample paths of X induced by the strategy of the normal type. By Proposition 6 we have $\phi_t \to 0$ $\mathsf{P}^{\mathtt{n}}$-almost surely. Moreover, since (ϕ_t) is a supermartingale under $\mathsf{P}^{\mathtt{n}}$, the maximal inequality for non-negative supermartingales yields:

$$\mathsf{P}^{\mathtt{n}}\left[\sup_{t \geq 0} \phi_t \leqq p_1 + \varepsilon\right] \;\geqq\; 1 - \frac{p_1}{p_1 + \varepsilon} > 0\,.$$

Choose a sample path $(\bar{\phi}_t, \bar{W}_t)$ with the property that $\bar{\phi}_t \to 0$ and $\bar{\phi}_t \leq p_1 + \varepsilon$ for all $t \geq 0$. Define the curve $\mathcal{C}'$ as the image of the sample path $t \mapsto (\bar{\phi}_t, \bar{W}_t)$. By a similar argument we can construct a continuous curve $\mathcal{C}'' \subset \texttt{graph}\,\mathcal{E}|_{[p_2-\varepsilon,1)}$ that has (p_2, w_2) as an endpoint and satisfies $\sup\{\phi \,|\, \exists w \text{ s.t. } (\phi, w) \in \mathcal{C}''\} = 1$.

Thus, we have constructed a continuous curve $\mathcal{C}^* \equiv \mathcal{C}' \cup \mathcal{C} \cup C'' \subset \texttt{graph}\,\mathcal{E}|_{(0,1)}$ that projects onto $(0,1)$ and satisfies $\inf\{w \,|\, (p, w) \in \mathcal{C}^*\} > v$. By a similar argument, there exists a continuous curve $\mathcal{C}_* \subset \texttt{graph}\,\mathcal{E}|_{(0,1)}$ that projects onto $(0,1)$ and satisfies $\sup\{w \,|\, (p, w) \in \mathcal{C}_*\} < v$.

Let $\phi \mapsto (\mathsf{a}(\phi), \bar{\mathsf{b}}(\phi))$ be a measurable selection from the correspondence of static Bayesian Nash equilibrium. Let (ϕ_t) be the unique weak solution of

$$d\phi_t = -\frac{|\gamma(\mathsf{a}(\phi_t), \bar{\mathsf{b}}(\phi_t), \phi_t)|^2}{1 - \phi_t} + \gamma(\mathsf{a}(\phi_t), \bar{\mathsf{b}}(\phi_t), \phi_t)\, dZ_t^n$$

with initial condition $\phi_0 = p$.[15] Let (W_t) be the unique solution of

$$dW_t = r(W_t - g(\mathsf{a}(\phi_t), \bar{\mathsf{b}}(\phi_t)))\, dt$$

with initial condition $W_0 = v$, up to the stopping time $T > 0$ when (ϕ_t, W_t) first hits either $\mathcal{C}^*$ or $\mathcal{C}_*$. Define a strategy profile $(a_t, \bar{b}_t)$ as follows: for $t < T$, $(a_t, \bar{b}_t) \equiv (\mathsf{a}(\phi_t), \bar{\mathsf{b}}(\phi_t))$; from $t = T$ onwards $(a_t, \bar{b}_t)$ follows an equilibrium of the game with prior ϕ_T. By Theorem 1 $(a_t, \bar{b}_t, \phi_t)$ is a sequential equilibrium of the game with prior p that yields the normal type a payoff of v. Hence $v \in \mathcal{E}(p)$, concluding the proof that $\mathcal{E}$ is convex-valued.

[15] Condition 1 ensures that $\gamma(\mathsf{a}(\phi_t), \bar{\mathsf{b}}(\phi_t), \phi_t)$ is bounded away from zero, hence standard results for existence/uniqueness of weak solutions apply.

We will now prove that the graph of $\mathcal{E}$ is arc-connected. Fix $p < q$, $v \in \mathcal{E}(p)$ and $w \in \mathcal{E}(q)$. Consider a sequential equilibrium of the game with prior q that yields the normal type a payoff of w. Since $\phi_t \to 0$, there exists a continuous curve $\mathcal{C} \subset \texttt{graph}\, \mathcal{E}$ with endpoints (q, w) and (p, v') for some $v' \in \mathcal{E}(p)$. Since $\mathcal{E}$ is convex-valued, the straight line $\mathcal{C}'$ connecting (p, v) to (p, v') is contained in the graph of $\mathcal{E}$. Hence $\mathcal{C} \cup \mathcal{C}'$ is a continuous curve connecting (q, w) and (p, v) that is contained in the graph of $\mathcal{E}$, concluding the proof that $\mathcal{E}$ has an arc-connected graph. □

Proposition 9. *The upper boundary $U : (0, 1) \to \mathbb{R}$ is a viscosity super-solution of the Upper Optimality equation.*

Proof. If U is not a super-solution, there exists $q \in (0, 1)$ and a $\mathcal{C}^2$-function $V : (0, 1) \to \mathbb{R}$ such that $0 = (U_* - V)(q) < (U_* - V)(\phi)$ for all $\phi \in (0, 1) \setminus \{q\}$, and

$$H^*(q, U_*(q), V'(q)) < V''(q)\,.$$

Therefore, there exist $\zeta, \varepsilon > 0$ such that for all $\phi \in [q - \varepsilon, q + \varepsilon] \subset (0, 1)$,

$$H(\phi, V(\phi), V'(\phi)) - V''(q) < -2\zeta\,,$$

which implies for all $\phi \in [q - \varepsilon, q + \varepsilon]$

$$rV(\phi) - \max_{(a,\bar{b}) \in \Psi(\phi, \phi(1-\phi)V'(\phi))} \{rg(a, \bar{b}) + |\gamma(a, \bar{b})|^2 (\frac{V'(\phi)}{1 - \phi} - \frac{V''(\phi)}{2})\} \;<\; -\zeta\underline{\gamma}^2 \equiv -\kappa\,, \qquad (42)$$

where $\underline{\gamma} > 0$ is the minimum of $|\gamma(a, \bar{b}, \phi)|$ over all $(a, \bar{b}) \in \Psi(\phi, \phi(1-\phi)V'(\phi))$ and $\phi \in [q-\varepsilon, q+\varepsilon]$.

Let $(\mathtt{a}(\cdot), \bar{\mathtt{b}}(\cdot))$ be a measurable selection

$$\phi \mapsto (\mathtt{a}(\phi), \bar{\mathtt{b}}(\phi)) \in \Psi\left(\phi,\, \phi(1 - \phi)V'(\phi)\right) \qquad (43)$$

such that

$$rV(\phi) - rg(\mathtt{a}(\phi),\, \bar{\mathtt{b}}(\phi)) - |\gamma(\mathtt{a}(\phi),\, \bar{\mathtt{b}}(\phi), \phi)|^2 (\frac{V''(\phi)}{2} - \frac{V'(\phi)}{1 - \phi}) < -\kappa\,, \qquad (44)$$

for all $\phi \in [q - \varepsilon, q + \varepsilon]$.

Since U_* is l.s.c., we can choose $\delta > 0$ small enough that for all $\phi \in [q - \varepsilon, q + \varepsilon]$,

$$V(\phi) + \delta \geqq U_*(\phi) \;\Rightarrow\; |\phi - q| < \frac{\varepsilon}{4}\,. \qquad (45)$$

Since $V(q) = U_*(q)$, there is some $p \in (q - \frac{\varepsilon}{4}, q + \frac{\varepsilon}{4})$ such that $V(p) + \delta > U(p)$.

Let (ϕ_t) be the unique weak solution of

$$d\phi_t = -\frac{|\gamma(\mathtt{a}(\phi_t), \bar{\mathtt{b}}(\phi_t), \phi_t)|^2}{1 - \phi_t}\, dt \;+\; \gamma(\mathtt{a}(\phi_t), \bar{\mathtt{b}}(\phi_t), \phi_t) \cdot dZ_t^{\mathtt{n}}\,, \quad \phi_0 = p\,, \qquad (46)$$

up to the first time it hits $q \pm \varepsilon$. [16]

[16] Existence follows from the fact that $V'(\phi)$ is bounded on $[q - \varepsilon, q + \varepsilon]$ and therefore γ is bounded away from zero (Lemma 8). Uniqueness is granted because process is one-dimensional (see Remark 4.32 on pg. 327 of Karatzas and Shreve.)

Consider the random process (W_t) that is the unique strong solution of

$$dW_t = r(W_t - g(\mathtt{a}(\phi_t), \bar{\mathtt{b}}(\phi_t)))\, dt + \gamma(\mathtt{a}(\phi_t), \bar{\mathtt{b}}(\phi_t), \phi_t)V'(\phi_t) \cdot dZ_t^{\mathtt{n}}\,, \quad W_0 = V(p) + \delta\,, \tag{47}$$

up to the positive stopping time T when ϕ_t hits $q \pm \varepsilon$ for the first time.[17]

By virtue of (45), there exist $v' \in \mathcal{E}(q - \frac{\varepsilon}{3})$ and $w' \in \mathcal{E}(q + \frac{\varepsilon}{3})$ such that $v' > V(q - \frac{\varepsilon}{3}) + \delta$ and $w' > V(q + \frac{\varepsilon}{3}) + \delta$. By Lemma 15, there exist $v \in \mathcal{E}(q - \frac{\varepsilon}{3})$ and $w \in \mathcal{E}(q + \frac{\varepsilon}{3})$, with $v \geq v'$ and $w \geq w'$, and a continuous path $(\tilde{\phi}, \tilde{u}) : [0,1] \to \mathtt{gr}\,\mathcal{E}|_{[q-\frac{\varepsilon}{2}, q+\frac{\varepsilon}{2}]}$, such that $(\tilde{\phi}(0), \tilde{u}(0)) = (q - \frac{\varepsilon}{3}, v)$ and $(\tilde{\phi}(1), \tilde{u}(1)) = (q + \frac{\varepsilon}{3}, w)$.

Denote by Λ the image of $(\tilde{\phi}, \tilde{u})$ and define

$$\tau \equiv \inf\{t \in [0,T] \,|\, (\phi_t, W_t) \in \Lambda\}\,.$$

We will now argue that $\tau < \infty$, that is, eventually the path (ϕ_t, W_t) must intersect the curve Λ. Let $\underline{\tilde{u}} = \min\{\tilde{u}(\ell) \,|\, \ell \in [0,1]\}$ and consider the stopping time

$$S \equiv \inf\{t : W_t = \underline{\tilde{u}} - 1\}\,.$$

We claim that $S \wedge T < \infty$. To prove the latter it suffices to show that W is unbounded from below on $\{T = \infty\}$. Let $D_t = W_t - V(\phi_t) - \delta$ for $0 \leqq t \leqq T$. By Itô's formula and equation (47),

$$\frac{dD_t}{dt} = rD_t + rV(\phi_t) - rg(\mathtt{a}(\phi_t),\, \bar{\mathtt{b}}(\phi_t)) - |\gamma(\mathtt{a}(\phi_t),\, \bar{\mathtt{b}}(\phi_t), \phi_t)|^2 \Big(\frac{V''(\phi_t)}{2} - \frac{V'(\phi_t)}{1-\phi_t}\Big)\,, \quad \text{for } 0 \leqq t \leqq T.$$

We have $D_0 = 0$ and, by virtue of (44),

$$\frac{dD_t}{dt} \leqq -\kappa < 0 \ \text{ almost surely on } \ \{T \geqq t\} \cap \{D_t \leqq 0\}\,. \tag{48}$$

Therefore D is unbounded from below on $\{T = \infty\}$. Since the continuous function V is bounded on $[q - \varepsilon, q + \varepsilon]$, we conclude that W is unbounded from below on $\{T = \infty\}$, demonstrating that $S \wedge T < \infty$.

To conclude that $\tau < \infty$ notice that

$$W_{S \wedge T} < \max\{\tilde{u}(\ell) \,|\, \ell \in [0,1],\, \tilde{\phi}(\ell) = \phi_{S\wedge T}\}\,,$$

and

$$W_0 > U(\phi_0) \geqq \max\{\tilde{u}(\ell) \,|\, \ell \in [0,1],\, \tilde{\phi}(\ell) = \phi_0\}\,,$$

and therefore (ϕ_0, W_0) and $(\phi_{S\wedge T}, W_{S\wedge T})$ lie on opposite sides of the continuous curve Λ. Since (ϕ_t, W_t) has continuous sample paths, it must intersect Λ at some time before $S \wedge T$. Since $S \wedge T < \infty$ we conclude that $\tau < \infty$, as claimed.

We now will construct a public sequential equilibrium of the game with prior p that yields the large player payoff $W_0 = V(p) + \delta$. Consider the strategy profile and belief process that coincides with $(a_t, \bar{b}_t, \phi_t)$ up to time τ, and follows a public sequential equilibrium of the game with prior ϕ_τ at all times after τ. Since $\tau < \infty$ almost surely, it follows from inclusion (43),

[17] Existence/uniqueness follows from the Lipschitz and linear growth conditions in W for fixed sample path, and the boundedness of $\gamma V'$ on $\{t \leqq T\}$.

equations (46) and (47), and Theorem 1 that the strategy profile / belief process constructed is a public sequential equilibrium of the game with prior p. By construction we have $W_0 \in \mathcal{E}(p)$, which is a contradiction since $W_0 = V(p) + \delta > U(p)$. This contradiction shows that U must be a super-solution of the Upper Optimality equation. $\square$

Lemma 16. *Every bounded viscosity solution of the upper optimality equation is locally Lipschitz continuous.*

Proof. En route to a contradiction, suppose U is a bounded viscosity solution that is not locally Lipschitz. That is, for some $p \in (0,1)$ and $\varepsilon \in (0, \frac{1}{2})$ satisfying $[p-2\varepsilon, p+2\varepsilon] \subset (0,1)$ the restriction of U to $[p-\varepsilon, p+\varepsilon]$ is not Lipschitz continuous. Let $M = \sup |U|$. By Lemmas 4 and 8 there exists $K > 0$ such that for all $(\phi, u, u') \in [p-2\varepsilon, p+2\varepsilon] \times [-M, M] \times \mathbb{R}$,

$$|H^*(\phi, u, u')| \leqq K(1 + |u'|^2), \tag{49}$$

Since the restriction of U to $[p-\varepsilon, p+\varepsilon]$ is not Lipschitz continuous, there exist ϕ_0, $\phi_1 \in [p-\varepsilon, p+\varepsilon]$ such that

$$\frac{|U_*(\phi_1) - U_*(\phi_0)|}{|\phi_1 - \phi_0|} \geqq \max\{1, e^{2M(4K+\frac{1}{\varepsilon})}\}. \tag{50}$$

Hereafter we will assume $\phi_1 > \phi_0$ and $U_*(\phi_1) > U_*(\phi_0)$. The proof for the reciprocal case is analogous and will be omitted.

Let $V : I \to \mathbb{R}$ be the solution of the differential equation

$$V''(\phi) = 2K(1 + V'(\phi)^2), \tag{51}$$

with initial conditions given by

$$V(\phi_1) = U_*(\phi_1) \text{ and } V'(\phi_1) = \frac{U_*(\phi_1) - U_*(\phi_0)}{\phi_1 - \phi_0}, \tag{52}$$

where the interval I is the maximal domain of V.

We claim V has the following two properties:

1. There exists $\phi^* \in I \cap (p-2\varepsilon, p+2\varepsilon)$ such that $V(\phi^*) = -M$ and $\phi^* < \phi_0$. In particular, $\phi_0 \in I$.

2. $V(\phi_0) > U_*(\phi_0)$.

We shall first prove property (a). For all $\phi \in I$ such that $V'(\phi) > 1$, we have $V''(\phi) < 4KV'(\phi)^2$ or, equivalently, $(\log V')'(\phi) < 4KV'(\phi)$, which yields

$$V(\hat{\phi}) - V(\tilde{\phi}) > \frac{1}{4K}(\log(V'(\hat{\phi})) - \log(V'(\tilde{\phi}))), \quad \forall \hat{\phi}, \tilde{\phi} \in I \text{ s.t. } V'(\hat{\phi}) > V'(\tilde{\phi}) > 1. \tag{53}$$

By (50) and (52), we have $\frac{1}{4K}\log(V'(\phi_1)) > 2M$, and therefore a unique $\tilde{\phi} \in I$ exists such that

$$\frac{1}{4K}(\log(V'(\phi_1)) - \log(V'(\tilde{\phi}))) = 2M. \tag{54}$$

Since $V'(\tilde{\phi}) > 1$, it follows from (53) that $V(\phi_1) - V(\tilde{\phi}) > 2M$ and so $V(\tilde{\phi}) < -M$. Since $V(\phi_1) > U(\phi_0) \geqq -M$, there exists some $\phi^* \in (\tilde{\phi}, \phi_1)$ such that $V(\phi^*) = -M$. Moreover ϕ^* must belong to $(p - 2\varepsilon, p + 2\varepsilon)$, because the convexity of V implies

$$\phi_1 - \phi^* < \frac{V(\phi_1) - V(\phi^*)}{V'(\phi^*)} < \frac{2M}{V'(\hat{\phi})} < \frac{2M}{\log(V'(\hat{\phi}))} = \frac{2M}{\log(V'(\phi_1)) - 8KM} < \varepsilon\,,$$

where the equality follows from (54) and the rightmost inequality follows from (50). Finally, we have $\phi^* < \phi_0$, otherwise the inequality $V(\phi^*) \leqq U_*(\phi_0)$ and the initial conditions (52) would imply

$$V'(\phi_1) = \frac{U_*(\phi_1) - U_*(\phi_0)}{\phi_1 - \phi_0} \leqq \frac{V(\phi_1) - V(\phi^*)}{\phi_1 - \phi^*}\,,$$

which would violate the strict convexity of V. This concludes the proof of property (a).

Turning to property (b), the strict convexity of V and the initial conditions (52) imply

$$\frac{U_*(\phi_1) - V(\phi_0)}{\phi_1 - \phi_0} = \frac{V(\phi_1) - V(\phi_0)}{\phi_1 - \phi_0} < V'(\phi_1) = \frac{U_*(\phi_1) - U_*(\phi_0)}{\phi_1 - \phi_0}\,,$$

and therefore $V(\phi_0) > U_*(\phi_0)$, as required.

Define

$$L = \max\left\{V(\phi) - U_*(\phi) \,|\, \phi \in [\phi^*, \phi_1]\right\}.$$

By property (b), we have $L > 0$. Let $\hat{\phi}$ be a point at which the maximum above is attained. Since $V(\phi_*) = -M$ and $V(\phi_1) = U_*(\phi_1)$, we have $\hat{\phi} \in (\phi^*, \phi_1)$ and therefore $V - L$ is a test function that satisfies

$$U_*(\hat{\phi}) = V(\hat{\phi}) - L\,,$$

and

$$U_*(\phi) \geqq V(\phi) - L \quad \text{for every } \phi \in (\phi^*, \phi_1)\,.$$

Since U is a viscosity supersolution,

$$V''(\hat{\phi}) \leqq H^*(\hat{\phi}, V(\hat{\phi}) - L, V'(\hat{\phi}))\,,$$

and hence, by (49),

$$V''(\hat{\phi}) \leqq K(1 + V'(\hat{\phi})^2) < 2K(1 + V'(\hat{\phi})^2))\,,$$

which is a contradiction, since by construction V satisfies equation (51). □

Lemma 17. *Every bounded viscosity solution of the upper optimality equation is continuously differentiable with absolutely continuous derivatives.*

Proof. Let $U : (0, 1) \to \mathbb{R}$ be a bounded solution of the upper optimality equation. By Lemma 16, U is locally Lipschitz and hence differentiable *almost* everywhere. We will now show that U is differentiable. Fix $\phi \in (0, 1)$. Since U is locally Lipschitz, there exist $\delta > 0$ and $k > 0$ such that for every $p \in (\phi - \delta, \phi + \delta)$ and every smooth test function $V : (\phi - \delta, \phi + \delta) \to \mathbb{R}$ satisfying $V(p) = U(p)$ and $V \geq U$ we have

$$|V'(p)| \leq k\,.$$

It follows from Lemma 8 that there exists some $M > 0$ such that

$$|H(p, U(p), V'(p))| \leq M$$

for every $p \in (\phi - \delta, \phi + \delta)$ and every smooth test function V satisfying $V \geq U$ and $V(p) = U(p)$.

Let us now show that for all $\varepsilon \in (0, \delta)$ and $\varepsilon' \in (0, \varepsilon)$

$$-M\varepsilon'(\varepsilon - \varepsilon') < U(\phi + \varepsilon') - \left(\frac{\varepsilon'}{\varepsilon}U(\phi + \varepsilon) + \frac{\varepsilon - \varepsilon'}{\varepsilon}U(\phi)\right) < M\varepsilon'(\varepsilon - \varepsilon'). \tag{55}$$

If not, for example if the second inequality fails, then we can choose $K > 0$ such that the C^2 function (a parabola)

$$f(\phi + \varepsilon') = \left(\frac{\varepsilon'}{\varepsilon}U(\phi + \varepsilon) + \frac{\varepsilon - \varepsilon'}{\varepsilon}U(\phi)\right) + M\varepsilon'(\varepsilon - \varepsilon') + K$$

is completely above $U(\phi + \varepsilon')$ except for a tangency point at $\varepsilon'' \in (0, \varepsilon)$. But this contradicts the fact that U is a viscosity subsolution, since $f''(\phi+\varepsilon'') = -2M < H(\phi+\varepsilon'', U(p+\varepsilon''), U'(\phi+\varepsilon''))$.

We conclude that the bounds in (55) are valid, and so for all $0 < \varepsilon' < \varepsilon < \delta$,

$$\left|\frac{U(\phi + \varepsilon') - U(\phi)}{\varepsilon'} - \frac{U(\phi + \varepsilon) - U(\phi)}{\varepsilon}\right| \leq M\varepsilon.$$

It follows that as ε converges to 0 from above,

$$\frac{U(\phi + \varepsilon) - U(\phi)}{\varepsilon}$$

converges to a limit $U'(\phi+)$. Similarly, if ε converges to 0 from below, the quotient above converges to a limit $U'(\phi-)$.

We claim that $U'(\phi+) = U'(\phi-)$. Otherwise, if for example $U'(\phi+) > U'(\phi-)$, then the function

$$f_1(\phi + \varepsilon') = U(\phi) + \varepsilon'\frac{U'(\phi-) + U'(\phi+)}{2} + M\varepsilon'^2$$

is below U in a neighborhood of ϕ, except for a tangency point at ϕ. But this leads to a contradiction, because $f_1''(\phi) = 2M > H(\phi, U(\phi), U'(\phi))$ and U is a super-solution. Therefore $U'(\phi+) = U'(\phi-)$ and we conclude that U is differentiable at every $\phi \in (0, 1)$.

We will now show that U' is locally Lipschitz. Fix $\phi \in (0, 1)$ and, arguing just as we did above, choose $\delta > 0$ and $M > 0$ so that

$$|H(p, U(p), V'(p))| \leq M$$

for every $p \in (\phi - \delta, \phi + \delta)$ and every smooth test function V satisfying $V(p) = U(p)$ and either $V \geq U$ or $V \leq U$. We affirm that for any $p \in (\phi - \delta, \phi + \delta)$ and $\varepsilon \in (0, \delta)$

$$|U'(p) - U'(p + \varepsilon)| \leq 2M\varepsilon.$$

If not, e.g. if $U'(p + \varepsilon) > U'(p) + 2M\varepsilon$ for some $p \in (\phi - \delta, \phi + \delta)$ and $\varepsilon \in (0, \delta)$, then the test function

$$f_2(p + \varepsilon') = \frac{\varepsilon'}{\varepsilon}U(p + \varepsilon) + \frac{\varepsilon - \varepsilon'}{\varepsilon}U(p) - M\varepsilon'(\varepsilon - \varepsilon')$$

must be above U at some $\varepsilon' \in (0, \varepsilon)$ (since $f_2'(p+\varepsilon) - f_2'(p) = 2M\varepsilon$.) Therefore, there exists a constant $K > 0$ such that $f_2(p+\varepsilon') - K$ stays below U for $\varepsilon' \in [0, \varepsilon]$, except for a tangency at some $\varepsilon'' \in (0, \varepsilon)$. But then

$$f_2''(\phi + \varepsilon'') = 2M > H(\phi + \varepsilon'', U(\phi + \varepsilon''), U'(\phi + \varepsilon'')),$$

contradicting the fact that U is a viscosity super-solution. □

Proposition 10. *The upper boundary U is a continuously differentiable function, with absolutely continuous derivatives. In addition, U is characterized as the maximal bounded solution of the following differential inclusion:*

$$U''(\phi) \in [H(\phi, U(\phi), U'(\phi)),\ H^*(\phi, U(\phi), U'(\phi))]\,. \tag{56}$$

Proof. First, note that by Propositions 8, 9 and 17, the upper boundary U is a differentiable function with absolutely continuous derivative that solves the differential inclusion (56).

If U is not a maximal solution, then there exists another bounded solution V of the differential inclusion (56) that is strictly above U at some $p \in (0,1)$. Choose $\varepsilon > 0$ such that $V(p) - \varepsilon > U(p)$. We will show that $V(p) - \varepsilon$ is the payoff of a public sequential equilibrium, which is a contradiction since U is the upper boundary.

From the inequality

$$V''(\phi) \geq H(\phi, V(\phi), V'(\phi)) \ \text{ a.e.}$$

it follows that a measurable selection $(\mathtt{a}(\phi), \bar{\mathtt{b}}(\phi)) \in \Psi(\phi, \phi(1-\phi)V'(\phi))$ exists such that

$$rV(\phi) - rg(\mathtt{a}(\phi), \bar{\mathtt{b}}(\phi), \phi) - |\gamma(\mathtt{a}(\phi), \bar{\mathtt{b}}(\phi), \phi)|^2 (\frac{V''(\phi)}{2} - \frac{V'(\phi)}{1-\phi}) \leq 0\,, \tag{57}$$

for almost every $\phi \in (0,1)$.

Let (ϕ_t) be the unique weak solution of

$$d\phi_t = -\frac{|\gamma(\mathtt{a}(\phi_t), \bar{\mathtt{b}}(\phi_t), \phi_t)|^2}{1-\phi_t} + \gamma(\mathtt{a}(\phi_t), \bar{\mathtt{b}}(\phi_t), \phi_t)\, dZ_t^{\mathtt{n}}\,,$$

with initial condition $\phi_0 = p$.

Let (W_t) be the unique strong solution of

$$dW_t = r(W_t - g(\mathtt{a}(\phi_t), \bar{\mathtt{b}}(\phi_t)))\, dt + V'(\phi_t)\gamma(\mathtt{a}(\phi_t), \bar{\mathtt{b}}(\phi_t), \phi_t)\, dZ_t^{\mathtt{n}}\,,$$

with initial condition $W_0 = V(p) - \varepsilon$.

Consider the process $D_t = W_t - V(\phi_t)$. It follows from Itô's formula for differentiable functions with absolutely continuous derivatives that:

$$\frac{dD_t}{dt} = rD_t + rV(\phi_t) - rg(\mathtt{a}(\phi_t), \bar{\mathtt{b}}(\phi_t), \phi_t) - |\gamma(\mathtt{a}(\phi_t), \bar{\mathtt{b}}(\phi_t), \phi_t)|^2 (\frac{V''(\phi_t)}{2} - \frac{V'(\phi_t)}{1-\phi_t})\,.$$

Therefore, by (57) we have

$$\frac{dD_t}{dt} \leq rD_t\,,$$

and since $D_0 = -\varepsilon < 0$ it follows that $W_t \searrow -\infty$.

Let τ be the first time that (ϕ_t, W_t) hits the graph of U. Consider a strategy profile / belief process that coincides with $(a_t, \bar{b}, \phi_t)$ up to time τ and, after that, follows a public sequential equilibrium of the game with prior ϕ_τ with value $U(\phi_\tau)$. It is immediate from Theorem 1 that the strategy profile / belief process constructed is a sequential equilibrium that yields the large player payoff $V(p) - \varepsilon > U(p)$, a contradiction. □

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