Long Persuasion Games[*]

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Abstract

This paper characterizes geometrically the sets of all Nash and perfect Bayesian equilibrium payoffs achievable with unmediated communication in persuasion games, i.e., games with an informed expert and an uninformed decisionmaker in which the expert's information is certifiable. The first equilibrium characterization is provided for unilateral persuasion games, and the second for multistage, bilateral persuasion games. As in Aumann and Hart (2003), we use the concepts of diconvexification and dimartingale. A leading example illustrates both geometric characterizations and shows how the expert, whatever his type, can increase his equilibrium payoff compared to all equilibria of the unilateral persuasion game by delaying information certification.

KEYWORDS: Belief consistency, cheap talk, diconvexification, dimartingale, disclosure of certifiable information, jointly controlled lotteries, long conversation, persuasion, sequential rationality, verifiable types.

JEL CLASSIFICATION: C72; D82.

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1 Introduction

As is now well known in the literature on cheap talk games (i.e., games with costless, non-binding, and unmediated communication), repeated communication generally allows to reach outcomes that cannot be implemented with unilateral or single-period communication, even if only one player is privately informed (see Aumann and Hart, 2003, Forges, 1984, 1990a, Krishna and Morgan, 2004, and Simon, 2002). In this paper we study this feature in "sender-receiver" communication games with partially verifiable types, also called persuasion games, in which the informed player (the expert, or "sender") has the ability to voluntarily certify partial or full information to the uninformed decisionmaker (the "receiver"). We characterize the sets of all Nash and perfect Bayesian equilibrium payoffs achievable with unmediated communication, by allowing players to talk for many periods. At each stage of this communication phase, the sender can certify part of his information.

This possibility of certifying information, in addition to make cheap talk claims, is justified by many concrete interactive decision situations. For example, players may present physical proofs such as documents, observable characteristics of a product, endowments or costs. Alternatively, in economic or legal interactions there may be labels, penalties for perjury, false advertising and warranty violations, or accounting principles that allow agents to submit substantive evidence of their information. Interesting phenomena similar to those obtained in the cheap talk case arise in games with strategic information certification. We show that several bilateral communication stages and delayed information certification allow to convey substantive information and lead to equilibrium outcomes that are not achievable when only one signalling stage is permitted. A leading example is analyzed in Section 2.

Our study is closely related to Aumann and Hart (2003) who characterized Nash equilibrium payoffs of long cheap talk games, i.e., the subset of communication equilibrium payoffs (Forges, 1986, 1990b; Myerson, 1982, 1986) that use only plain conversation. A communication equilibrium is a Nash equilibrium of an extension of the game allowing the players to communicate for several periods, with the help of a mediator, before they make their decisions. Here, we characterize the analog of that subset for certification equilibria (Forges and Koessler, 2005). A certification equilibrium is defined as a communication equilibrium, except that each player can also transmit reports from a type-dependent set, i.e., can send certified information into the communication system.

Our general model, presented in Section 3, is a one-side incomplete information game with an expert (the informed player) and a decision maker (the uninformed player). A common prior probability distribution first selects the expert's type in a finite set. The decision maker chooses his action without observing the expert's type. However, before the action phase, but after the expert learns his type, the players are able to directly communicate with each

other. The payoff of each player only depends on the expert's type and on the decision maker's action. Communication is assumed strategic, non-binding (no commitment and no contract are allowed), payoff-irrelevant, and unmediated. In addition, players are not able to observe private payoff-irrelevant signals ("private sunspots") and there is no extraneous noise in communication, which thus takes place "face-to-face". However, randomized strategies are allowed in both the communication and action phases.

Contrary to usual cheap talk games (Crawford and Sobel, 1982; Ben-Porath, 2003; Gerardi, 2004), our communication games allow the set of messages available to the expert to be type-dependent, which reflects the ability to certify information. We will assume that the expert has always the opportunity to remain silent, i.e., to send a meaningless message to the decision maker. Furthermore, to guarantee that our geometric characterization be sufficient for an equilibrium, we will require that players have access to a rich language and that information is fully certifiable. More precisely, we make the following assumption: for any set of types containing his real type, the expert has a sufficiently large set of messages allowing him to certify that his real type belongs to that set.

In the associated one-shot communication game the expert learns his type and sends a message to the decision maker, who then chooses an action. Such games are sometimes called persuasion or disclosure games (see, e.g., Milgrom, 1981; Milgrom and Roberts, 1986; Seidmann and Winter, 1997). To the best of our knowledge, this literature has always focused on one-shot information revelation with very specific assumptions on players' preferences, like single-peakedness, strict concavity and monotonicity. Our first result (Theorem 1) is a full characterization of Nash equilibrium payoffs of one-shot communication games with certifiable information. Roughly, equilibrium payoff vectors are obtained by convexifying the graph of the equilibrium payoff correspondence of the basic game without communication (the silent game), by keeping the payoff of the informed player constant and individually rational.

In a multistage communication game, the talking phase has an arbitrarily large number of periods. In each communication period *both* players simultaneously send a message. As in Hart (1985) and Aumann and Hart (2003), our equilibrium characterization makes use of the mathematical concepts of diconvexification and dimartingale. In Theorem 2 we characterize the set of all Nash equilibrium payoffs which can be achieved in a possibly very long multistage communication game. This characterization is in terms of starting points of dimartingales which converge to the graph of the equilibrium payoff correspondence of the silent game, and stay in an adapted set of individually rational payoffs for the informed player during the whole process. Individual rationality must indeed be formulated in a stage-dependent way in our model. This is the main difference with Aumann and Hart's (2003) characterization. Our representation can also be formulated by using the diconvexification operator. However, by contrast to Aumann and Hart (2003), the graph of the equilibrium payoff correspondence

of the multistage communication game is *not* the diconvexification of a given set.

In Theorem 3 we provide an analog of the first two theorems for perfect Bayesian equilibrium, thus restricting our attention to sequentially rational strategies and consistent beliefs. While the set of Nash equilibrium payoffs for persuasion games characterized in Theorem 2 includes the associate set for cheap talk games characterized in Aumann and Hart (2003), the set of perfect Bayesian equilibrium payoffs for persuasion games characterized in Theorem 3 has no inclusion relationship with the associated set for cheap talk games.

The paper is organized as follows. In the next section we present our leading example. Section 3 describes the model. Section 4 (Section 5, respectively) formulates the geometric characterizations of the Nash (perfect Bayesian, respectively) equilibrium payoffs, illustrates them through examples, and provides a more detailed comparison with Aumann and Hart (2003). We discuss extension to mediated persuasion in Section 6. Formal proofs of Theorem 1 (one-shot, unilateral persuasion), Theorem 2 (multistage, bilateral persuasion), and Theorem 3 (perfect Bayesian equilibrium) are provided in the Appendix.

2   An Example

In this section we study an example which motivates several aspects of our analysis. First, the example illustrates how by certifying their information players can reach equilibrium outcomes that cannot be achieved by any communication system with non-certifiable information. Second, the example shows that delayed information certification and multiple rounds of bilateral communication may be required to achieve some equilibrium payoffs, even if only one player has substantive information. Finally, the example provides instances in which equilibrium outcomes may or may not be Bayesian perfect.

Consider two players, player 1 (the expert) and player 2 (the decisionmaker), who are playing a strategic form game which depends on the true state of Nature, k_1 or k_2, each of probability $1/2$ (see Figure 1). Player 1 knows the true state of Nature but player 2 does not know the actual game being played. Player 2 must choose action j_1, j_2, j_3, j_4 or j_5, and player 1 has no choice. The expected payoff of player 2, as a function of his action and his belief $p \in [0, 1]$ about state k_1, is represented by Figure 2 on the next page (the thick lines denote his best-reply payoff).

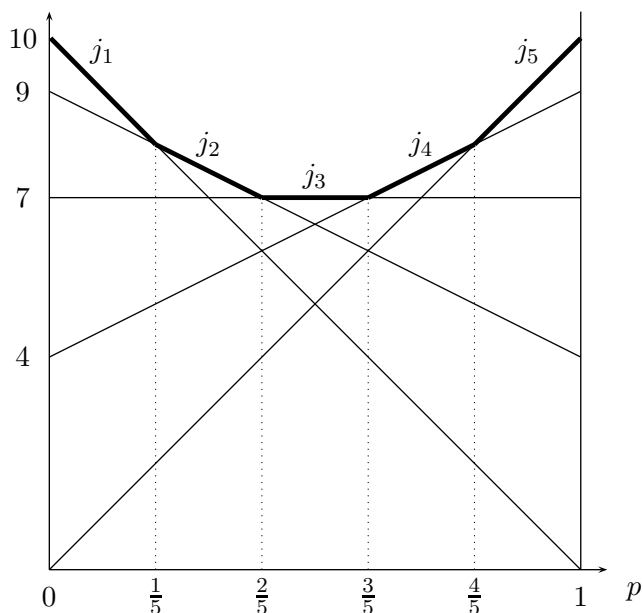|	j_1	j_2	j_3	j_4	j_5
k_1	5, 0	3, 4	0, 7	4, 9	2, 10
k_2	1, 10	3, 9	0, 7	5, 4	6, 0

Figure 1: Introductory example.

Figure 2: Player 2's expected payoffs (thin lines) and best-reply expected payoffs (thick lines) in the introductory example.

Without communication possibilities (in the "silent game"), the only equilibrium payoff is $(0, 7)$ since action j_3 yields the best expected payoff for player 2 given his prior belief $p = 1/2$. If, before player 2's decision, the players are able to talk to each other, but no information can be certified concerning the true state of Nature then, whatever the communication possibilities, the unique equilibrium payoff remains $(0, 7)$. Information transmission is not possible here because if player 2 chooses his action conditionally on the messages sent by player 1 then, whatever the true state of Nature, player 1 has always an incentive to use the messages he should have sent at the other state. In other words, information which is transmitted to player 2 is never credible, even if in every state it is to the advantage of both players that player 1 tells the truth to player 2, and that the latter believes him. Notice that allowing unboundedly long communication, or even adding a mediator, cannot help here: one can check that the unique communication equilibrium outcome is the equilibrium j_3 of the silent game.

Assume now that player 1 can voluntarily certify his information concerning the real state of Nature. That is, his informational reports are assumed truthful (the making of false statements is prohibited), but he may withhold his information since he is not required to make positive disclosures. Assume first that player 1 can only send a single message and that player 2 cannot send any message. More precisely, assume that player 1 can choose between two types of reports: either he certifies his information (he sends message $m = c^1$ if the real state is k_1 and message $m = c^2$ if the real state is k_2), or he certifies no information (he

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sends message $m = \overline{m}$ which is available whatever the true state). It is easy to see that full revelation of information is now an equilibrium, denoted by FRE: player 2 chooses action j_5 if player 1 reveals that the true state is k_1, he chooses j_1 if player 1 reveals that the true state is k_2, and chooses j_3 if player 1 reveals nothing. In such a situation, player 1 has no incentive not to reveal his information because his payoff would be zero instead of 2 in state k_1 and 1 in state k_2. Obviously, player 2 also behaves rationally because he chooses the best action for him in each state of Nature.

As in cheap talk games, the non-revealing outcome is also a Nash equilibrium, denoted by NRE, since player 2 can always ignore what player 1 says and choose action j_3. However, contrary to the fully revealing equilibrium, the non-revealing equilibrium is based on irrational choices off the equilibrium path since player 2 should not choose action j_3 when player 1 reveals him the true state of Nature (NRE is neither a perfect Bayesian equilibrium nor a subgame perfect equilibrium). Restrictions to credible (sequentially rational) moves off the equilibrium path are investigated in Section 5.

The two Nash equilibrium outcomes described above are *not* the only Nash equilibrium outcomes of the one-shot communication game with certifiable information. Indeed, if we allow player 1 to randomize, then there are two other *partially revealing* equilibria. One of them is better for player 1 than any of the previous pure strategy equilibria since it gives him a payoff of 2 whatever his type. In this equilibrium, denoted by PRE1, player 1 certifies his type (i.e., sends message c^1) with probability $1/3$ and remains silent (i.e., sends message $\overline{m}$) with probability $2/3$ in k_1, and he always remain silent in state k_2. Player 2's posterior beliefs are $\Pr(k_1 \mid \overline{m}) = \frac{\Pr(\overline{m}|k_1)\Pr(k_1)}{\Pr(\overline{m})} = \frac{2/6}{2/6+1/2} = 2/5$ and $\Pr(k_1 \mid c^1) = 1$, so he plays action j_5 when he receives message c^1 and is indifferent between j_2 and j_3 when he receives message $\overline{m}$. If he plays j_2 with probability $2/3$ and j_3 with probability $1/3$ after $\overline{m}$, and if he plays j_1 after the off-equilibrium message c^2 then player 1 has no incentive to deviate: in k_1 he gets a payoff of 2 if he sends message c^1 and also $(2/3) \times 3 + (1/3) \times 0 = 2$ if he sends message $\overline{m}$, so he is indifferent between the two messages; in k_2 he gets a payoff of 1 if he sends message c^2 and $(2/3) \times 3 + (1/3) \times 0 = 2$ if he sends message $\overline{m}$, so he strictly prefers to send message $\overline{m}$.

In the second partially revealing equilibrium with randomized certification, denoted by PRE2, player 1 always remains silent in state k_1; he certifies his type with probability $1/3$ and remains silent with probability $2/3$ in k_2. Player 2's posterior beliefs are $\Pr(k_1 \mid \overline{m}) = 3/5$ and $\Pr(k_1 \mid c^2) = 0$, so he plays action j_1 when he receives message c^2 and is indifferent between j_3 and j_4 when he receives message $\overline{m}$. If he plays j_3 with probability $4/5$ and j_4 with probability $1/5$ after message $\overline{m}$, and if he plays j_3 after the off-equilibrium message c^1 then it can be checked as before that player 1 has no incentive to deviate. Contrary to the previous partially revealing equilibrium, this equilibrium is based on irrational choices off the

equilibrium path since player 2 should not choose action j_3 when player 1 reveals him the true state of Nature (PRE2 is neither a perfect Bayesian equilibrium nor a subgame perfect equilibrium). Again, see Section 5 for Nash equilibrium refinements.

Now, we show that if players are able to talk to each other during several bilateral communication rounds and to delay information certification, then player 1 can reach even a higher equilibrium payoff of 3 whatever his type. This (perfect Bayesian) equilibrium can be achieved in three communication stages. In the first two communication stages there is no information certification, and in the last communication stage player 1 will certify his information to player 2 conditionally on what both players said in the previous communication stages.

In the first communication stage player 1 partially reveals (without certifying) his information by using a random communication strategy which transmits the correct information with probability 3/4 so as to leave some doubt in player 2's mind. That is, he sends message $m = a$ with probability 3/4 if the real state is k_1 and with probability 1/4 if the real state if k_2. Symmetrically, he sends message $m = b$ with probability 3/4 if the real state is k_2 and with probability 1/4 if the real state if k_1 (the labeling of these two messages is irrelevant but both messages a and b are cheap talk messages: they must be available to player 1 whatever his type). From Bayes' rule, player 2 will believe state k_1 with probability 3/4 if he receives message a and with probability 1/4 if he receives message b. Assume that player 2 chooses action j_2 whenever he receives message b. This choice is rational given his beliefs. Otherwise, when message a is sent, they agree on a jointly controlled $\frac{1}{2} - \frac{1}{2}$ lottery to reach the following compromise (this second communication stage conveys no substantive information, i.e., no information about the fundamentals of the game).[1] If head (H) occurs, then communication stops and thus player 1 chooses action j_4. On the contrary, if tail (T) occurs, then player 1 certifies his information in the last communication stage (he sends message c^k if the real state is k). Then, player 2 chooses action j_5 if c^1 is sent and action j_1 if c^2 is sent. Player 1 has no incentive to deviate if, for example, player 2 chooses action j_3 when player 1 deviates in the last communication stage by remaining silent. The whole communication and decision process in this equilibrium is summarized by Figure 3 (where "JCL" stands for "jointly controlled lottery"). Player 1's expected payoff is 3 whatever his type.

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[1]A jointly controlled lottery is a mechanism that generates a uniform probability distribution on any finite set from private random communication strategies so that a unilateral deviation does not change the probability distribution. For example, a $\frac{1}{2} - \frac{1}{2}$ lottery can be generated as follows: each player chooses a message in $\{a, b\}$ at random, both players announce their choices simultaneously and the outcome is head (H) if the messages coincide and tail (T) otherwise.
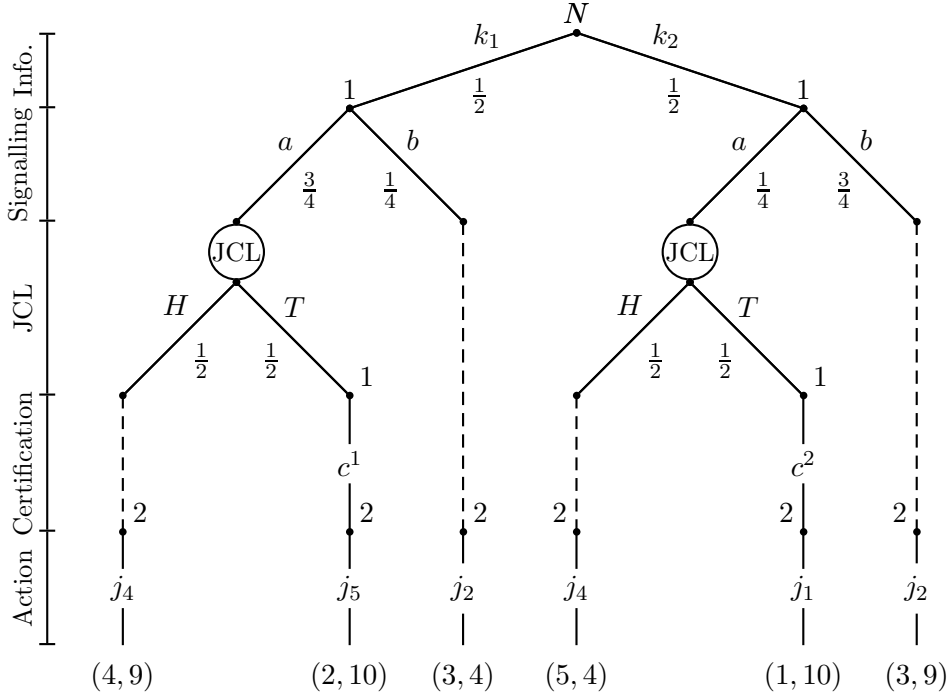
Figure 3: A perfect Bayesian equilibrium outcome for the introductory example.

3   Model

We consider two players: player 1 (the informed player, or expert) and player 2 (the uninformed decisionmaker (DM)). J ($|J| \geq 2$) is the finite action set of player 2 (player 1 has no action). K ($|K| \geq 2$) is the finite set of states (or types of player 1), with a common prior probability distribution $p = (p^1, \ldots, p^k, \ldots, p^K) \in \Delta(K)$. Let $\text{supp}[p] \equiv \{k \in K : p^k > 0\}$.[2] When player 2 chooses action $j \in J$ and the state is $k \in K$, the payoffs to player 1 and player 2 are $A^k(j)$ and $B^k(j)$, respectively.

3.1   Silent Game

The *silent game*, denoted by $\Gamma(p)$, consists of two phases. In the *information phase* a state $k \in K$ is picked at random according to the probability distribution p. Player 1 is perfectly informed about the true state k, while player 2 is not. In the *action phase*, player 2 chooses an action $j \in J$.

A strategy of player 2 in the silent game $\Gamma(p)$ is a mixed action $y \in \Delta(J)$. We extend payoff functions linearly to mixed actions: $A^k(y) = \sum_{j \in J} y(j) A^k(j)$ and $B^k(y) = \sum_{j \in J} y(j) B^k(j)$. The set of (Bayesian) Nash equilibria of the silent game $\Gamma(p)$ is the set of optimal mixed

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[2]We could assume w.l.o.g. that $p^k > 0$ for all $k \in K$ but in order to capture the games corresponding to an updating of the prior over K, we allow $p^k = 0$ for some k's.

actions for player 2 in the silent game $\Gamma(p)$. It is called the set of *non-revealing equilibria* at p, and is denoted by:

$$Y(p) \equiv \arg \max_{y \in \Delta(J)} \underbrace{\sum_{k \in K} p^k B^k(y)}_{p\,B(y)} = \left\{ y \in \Delta(J) : \sum_{k \in K} p^k B^k(y) \geq \sum_{k \in K} p^k B^k(j), \ \forall \, j \in J \right\}.$$

Remark 1 A pure action is always sufficient to maximize the decisionmaker's payoff. So, for all $j, j' \in \text{supp}[Y(p)]$ and $y \in \Delta(J)$ we have $p\,B(j) = p\,B(j') \geq p\,B(y)$. However, mixed actions will become useful once the action phase will be preceded by communication: (i) on the equilibrium path, to make player 1 indifferent between several messages, and (ii) off the equilibrium path, to punish player 1.

The resulting equilibrium payoffs are the $(K+1)$-dimensional vectors (a, β), where $a = (a^1, \ldots, a^K)$, $a^k = A^k(y)$ is the payoff of player 1 of type k, which is only relevant if $k \in \text{supp}[p]$, and the scalar $\beta = p\,B(y)$ is player 2's expected payoff (expectation over k). Let $\mathcal{E}(p)$ be the set of equilibrium payoffs of $\Gamma(p)$, also called the set of *non-revealing equilibrium payoffs* at p.[3] That is,

$$\mathcal{E}(p) \equiv \{ (a, \beta) \in \mathbb{R}^K \times \mathbb{R} : \exists \, y \in Y(p), \ a^k = A^k(y) \ \forall \, k \in \text{supp}[p], \ \beta = p\,B(y) \}.$$

3.2 Unilateral Persuasion Game

Here, we consider only direct (unmediated and noiseless) and unilateral communication, from player 1 to player 2. The finite set of messages available to player 1 is state-dependent and is denoted by $M(k)$ when his type is k. Let $M^1 = \bigcup_{k \in K} M(k)$ be the set of all messages that player 1 could send. The set $\bigcap_{k \in K} M(k)$ is the set of all cheap talk messages available to player 1, i.e., the set of all messages that player 1 can send whatever his type.

We assume that the set of cheap talk messages available to player 1 is nonempty. That is, there exists $\overline{m} \in M^1$ such that $M^{-1}(\overline{m}) = K$. This "right to remain silent" assumption will be needed for the "only if" part (from equilibrium to dimartingale) of our theorems. For the "if" part (from dimartingale to equilibrium), we will further assume that the message space and certifiability possibilities of the sender are sufficiently rich. That is, whatever his type k, and for each event $L \subseteq K$ containing k, player 1 can choose among a sufficiently large set of messages certifying that his real type is in L. Formally, we assume that

$$|\{ m \in M^1 : M^{-1}(m) = L \}| \geq |L| + 1, \ \text{ for all } L \subseteq K.$$

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[3]Our definition differs from Aumann and Hart's (2003) definition when the probability of some types vanishes. See Subsection 4.2 for a more detailed comparison.

Notice that this rich language and certifiability assumption implies the previous assumption that the set $\bigcap_{k \in K} M(k)$ is nonempty (simply take $L = K$). Assuming full certifiability only for singleton events $L = \{k\}$ would *not* be sufficient for the "if" part of the theorems.

The signalling game determined by Γ and p, denoted by $\Gamma_S(p)$, is obtained by adding a one-shot talking phase to the silent game $\Gamma(p)$ before the action phase but after the information phase. Therefore, this game corresponds to a standard persuasion game (Milgrom, 1981; Shin, 1994; Seidmann and Winter, 1997) and has three phases (see Figure 4).

Information phase	Talking phase	Action phase
Expert learns $k \in K$	Expert sends message $m^1 \in M(k)$	DM chooses action $j \in J$
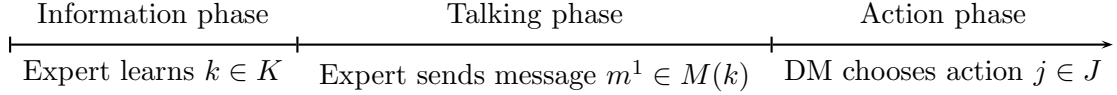
Figure 4: Unilateral persuasion (signalling) game $\Gamma_S(p)$.

The extensive form representation of the unilateral persuasion game with only two types, two cheap talk messages and one certificate for each type ($M(k) = \{a, b, c^k\}$, $k = k_1, k_2$) is given in Figure 5.
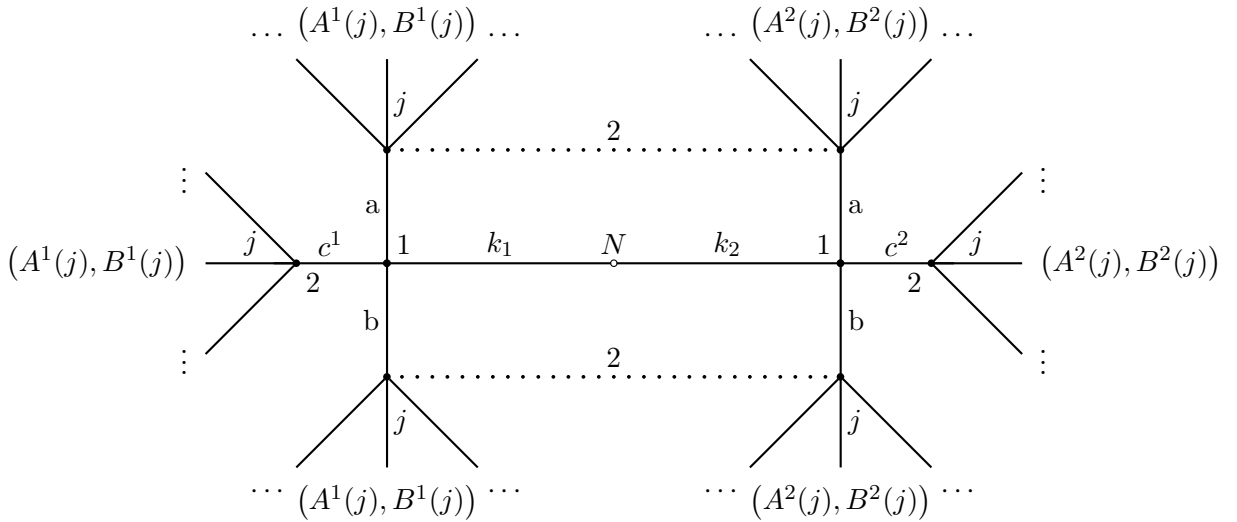
Figure 5: Extensive form of the unilateral persuasion game $\Gamma_S(p)$ with two types, two cheap talk messages and one certificate for each type ($M(k) = \{a, b, c^k\}$, $k = k_1, k_2$).

A strategy for player 1 in the unilateral persuasion game is a profile $\sigma = (\sigma^k)_{k \in K}$, with $\sigma^k \in \Delta(M(k))$ for all k. A strategy for player 2 is a function $\tau : M^1 \to \Delta(J)$. A pair of strategies (σ, τ) generates expected payoffs $(a^1_{\sigma,\tau}, \ldots, a^K_{\sigma,\tau})$ and $\beta_{\sigma,\tau}$ for player 1 and player 2, respectively. As usual, a (Bayesian) Nash equilibrium is a pair of strategies (σ, τ) satisfying $a^k_{\sigma,\tau} = \max_{\widetilde{\sigma}} a^k_{\widetilde{\sigma},\tau}$ for all $k \in \text{supp}[p]$ and $\beta_{\sigma,\tau} = \max_{\widetilde{\tau}} \beta_{\sigma,\widetilde{\tau}}$. Let $\mathcal{E}_S(p)$ be the set of Nash equilibrium payoffs of $\Gamma_S(p)$.

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3.3 Multistage, Bilateral Persuasion Game

We consider an arbitrarily large but finite number $n \geq 1$ of communication rounds. In each communication round $t = 1, \ldots, n$ each player can directly send a message to the other. As in the unilateral persuasion game, the finite set of messages available to player 1 is denoted by $M(k)$ when his type is k, $M^1 = \bigcup_{k \in K} M(k)$ is the set of all messages that player 1 could send, and $\bigcap_{k \in K} M(k) \neq \emptyset$ is the set of all cheap talk messages available to player 1. The finite set of messages available to player 2 is denoted by M^2, with $|M^2| \geq 2$.

As in the unilateral persuasion game we assume that $|\{m \in M^1 : M^{-1}(m) = L\}| \geq |L| + 1$ for all $L \subseteq K$. However, notice that in the multistage communication game it would be sufficient to have two cheap talk messages and that a combination of several certificates allows to certify any event $L \subseteq K$.[4] The above specific assumption on the richness of the message space is only for convenience.

The bilateral persuasion game with n communication stages, determined by Γ and p, is denoted by $\Gamma_n(p)$. It is obtained by adding a talking phase with n bilateral communication rounds to the silent game $\Gamma(p)$ before the action phase but after the information phase (see Figure 6). At each period $t = 1, \ldots, n$ of the *talking phase*, type $k \in K$ of player 1 sends a message $m_t^1 \in M(k)$ to player 2, and player 2 sends a message $m_t^2 \in M^2$ to player 1 (perfect monitoring). Messages are sent simultaneously.

Information phase	Talking phase ($n \geq 1$ rounds)	Action phase
Expert learns $k \in K$	Expert and DM send $(m_t^1, m_t^2) \in M(k) \times M^2$ $(t = 1, \ldots n)$	DM chooses $j \in J$
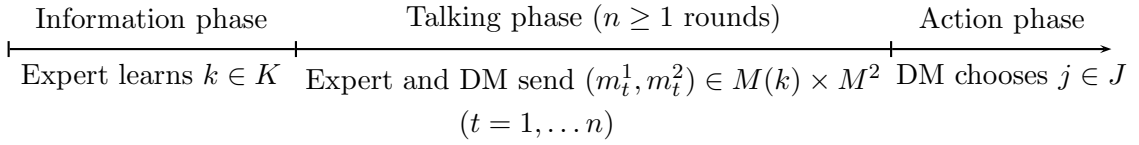
Figure 6: n-stage bilateral persuasion game $\Gamma_n(p)$.

A t-period *history*, $t = 0, 1, \ldots, n$, is a sequence consisting of t pairs of messages,

$$h_t = (m_1^1, m_1^2, \ldots, m_t^1, m_t^2) \in (M^1 \times M^2)^t.$$

The set of all t-period histories is denoted by $M_t = (M^1 \times M^2)^t$. A strategy[5] σ of player 1 in the n-period communication game $\Gamma_n(p)$ consists of a sequence of functions $\sigma_1, \ldots, \sigma_n$, where $\sigma_t = (\sigma_t^1, \ldots, \sigma_t^K)$ and $\sigma_t^k : M_{t-1} \to \Delta(M(k))$ for $k \in K$ and $t = 1, \ldots, n$. A strategy τ of player 2 consists of a sequence of functions $\tau_1, \ldots, \tau_n$, and a function τ_{n+1}, where $\tau_t : M_{t-1} \to \Delta(M^2)$ for $t = 1, \ldots, n$, and $\tau_{n+1} : M_n \to \Delta(J)$.

A pair of strategies (σ, τ) generates expected payoffs $a_{\sigma,\tau} = (a_{\sigma,\tau}^1, \ldots, a_{\sigma,\tau}^K)$ and $\beta_{\sigma,\tau}$ for

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[4]That is, it would be sufficient to assume that $|\bigcap_{k \in K} M(k)| \geq 2$, and $\forall \; k, \; \forall \; k' \neq k, \; \exists \; m \in M(k)$, $M^{-1}(m) = K \backslash \{k'\}$.

[5]We focus on finite games with perfect recall. Hence, by Kuhn's (1953) theorem behavioral strategies are without loss of generality.

player 1 and player 2, respectively. The set of (Bayesian) Nash equilibrium of the persuasion game $\Gamma_n(p)$ is denoted by $\mathcal{E}_n(p)$. Notice that $\mathcal{E}_S(p) \subseteq \mathcal{E}_n(p) \subseteq \mathcal{E}_{n+1}(p)$ for all $n \geq 1$. Let $\mathcal{E}_B(p) = \bigcup_{n \geq 1} \mathcal{E}_n(p)$ be the set of Nash equilibrium payoffs of all multistage, bilateral persuasion games determined by Γ and p.

4 Characterization of Nash Equilibrium Payoffs

4.1 Statement of the Results

Let H be the *graph* of the non-revealing equilibrium payoff correspondence, namely

$$H = \operatorname{gr} \mathcal{E} \equiv \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : (a, \beta) \in \mathcal{E}(p)\},$$

where $\mathcal{E}(p)$ has been defined in Subsection 3.1. Notice that the set $\mathcal{E}(p)$ is convex for all p. In other words, H is convex in (a, β) when p is kept constant. However, H need not be convex in (β, p) when a is kept constant.

For every set of types $L \subseteq K$, let

$$\operatorname{INTIR}_L \equiv \{a \in \mathbb{R}^K : \exists\, \overline{y} \in \Delta(J),\ a^k \geq A^k(\overline{y}) \ \forall\, k \in L\},$$

be the set of payoffs that are interim individually rational for player 1 when we restrict the individual rationality constraint to a subset L of player 1's set of types. Remark that $\operatorname{INTIR}_L \subseteq \operatorname{INTIR}_{L'}$ whenever $L' \subseteq L$. Let I be the graph of the payoffs that are interim individually rational for player 1 in the silent game $\Gamma(p)$:

$$I \equiv \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : a \in \operatorname{INTIR}_{\operatorname{supp}[p]}\}.$$

As H, I is convex in (a, β) when p is kept constant, but not in p when a is kept constant. Obviously, every non-revealing equilibrium payoff is interim individually rational for player 1 so that $H \subseteq I$.

Let $H_1 \equiv \operatorname{conv}_a(H) \cap I$ be the set of expected payoffs obtained from H by convexifying in (β, p) when the payoff of player 1, a, is kept constant and is interim individually rational for player 1. Even if H is included in I, payoffs in $\operatorname{conv}_a(H)$ need not be interim individually rational for player 1, while this is clearly a necessary equilibrium condition. We thus have to require individual rationality explicitly in the definition of H_1.[6] It turns out that this requirement is also sufficient for the equilibrium characterization of the unilateral persuasion game.

---

[6] The restriction to $\operatorname{supp}[p]$ for individual rationality is irrelevant for the next theorem, but will be important in the multistage game.

Theorem 1 (Unilateral Persuasion) *The set $\mathcal{E}_S(p)$ of Nash equilibrium payoffs of the unilateral persuasion game $\Gamma_S(p)$ coincides with the p-section of H_1:*

$$\mathcal{E}_S(p) = H_1(p) \equiv \{(a, \beta) \in \mathbb{R}^K \times \mathbb{R} : (a, \beta, p) \in H_1\}.$$

In addition, any Nash equilibrium payoff of $\Gamma_S(p)$ can be obtained with at most $K+1$ messages.

Proof. See Subsection A.1. ∎

From the proof of the "if" part of the theorem (the construction of the sender's strategy), the following proposition is immediate:

Proposition 1 *Let $p^k > 0$ for all $k \in K$. Every equilibrium of the unilateral persuasion game $\Gamma_S(p)$ is outcome equivalent (i.e., it induces the same probability distribution over player 2's decision conditionally on k) to an equilibrium (σ, τ) with the following property:*

For all $m \in M^1$, if $\sigma^k(m) > 0$ for some $k \in K$, then $\sigma^{k'}(m) > 0$ for all $k' \in M^{-1}(m)$.

In particular, if a cheap talk message $\overline{m} \in \bigcap_{k \in K} M(k)$ is sent with strictly positive probability by player 1, then all types of player 1 send this message with strictly positive probability. The proposition says that in equilibrium, without loss of generality, if player 2's posterior about a certain type k of player 1 is null after some message m sent with strictly positive probability, then $k \notin M^{-1}(m)$, i.e., message m certifies that k is not realized. Hence, all types have strictly positive posterior probability after a cheap talk message (sent with strictly positive probability in equilibrium). Without using the geometric characterization of Theorem 1, the intuition of the proposition is as follows. Assume that type k' does not send a message m but could have sent it (i.e., $m \in M(k')$). Then, the types who send message m could have sent another message instead of m that certifies that k' is not realized, without changing player 2's posteriors and so without changing the equilibrium outcome.

In the next statement, we characterize the set of all equilibrium payoffs in all persuasion games with an arbitrary large but bounded number of bilateral communication rounds. The characterization states that all such equilibrium payoffs can be achieved in a canonical way, in which signalling and jointly controlled lotteries alternate. To state the result precisely, let us first consider the payoffs obtained as convex combinations of elements in H_1 with p fixed which are interim individually rational for player 1: $H_1^* = \operatorname{conv}_p(H_1) \cap I$. Since $H_1 \subseteq I$ and I is convex in (a, β) when p is fixed, $\operatorname{conv}_p(H_1) \subseteq I$ so that $H_1^* = \operatorname{conv}_p(H_1)$. We then proceed with H_1^* as we did above with H, namely convexifying in (p, β) keeping a constant and interim individually rational. This yields $H_{3/2} = \operatorname{conv}_a(H_1^*) \cap I$. Next, by convexifying in (a, β) at p fixed, we get $H_2 = \operatorname{conv}_p(H_{3/2}) = \operatorname{conv}_p(H_{3/2}) \cap I$. The p-section of the set

H_2 is the set of equilibrium payoffs of persuasion games with four canonical communication rounds: a jointly controlled lottery, a step of signalling, a second jointly controlled lottery, and a second step of signalling. Next, let H_3 be the set obtained from H_2 by convexifying in (β, p) when player 1's payoff a is fixed, and then by convexifying in (a, β) when player 2's belief p is fixed, with again the restriction that the payoff of player 1 is interim individually rational for the types with a strictly positive posterior. The p-section of the set H_3 is the set of equilibrium payoffs of persuasion games with six canonical communication rounds. The set H_n, $n \geq 2$, thus corresponds to $2n$ stages of canonical communication, in which signalling and jointly controlled lotteries alternate. We introduce a slight disymmetry in the definition of H_1, which captures a single stage of signalling for player 1. The limit of the increasing sequence H_1, H_2, ... constructed in this way is denoted by di-co $^{\mathrm{IR}}(H) \equiv \bigcup_{l \geq 1} H_l$ to recall the process of diconvexification used in the construction. Observe that, since I is not a di-convex set, di-co $^{\mathrm{IR}}(H)$ need not be di-convex (see the comparison with Aumann and Hart, 2003, in the next subsection). Points in di-co $^{\mathrm{IR}}(H)$ correspond to all equilibrium payoffs of bilateral persuasion games of bounded length. In the next theorem, the set di-co $^{\mathrm{IR}}(H)$ is expressed more elegantly as the set of starting points of particular martingales that converge to H.

Theorem 2 (Multistage, Bilateral Persuasion) *The set $\mathcal{E}_B(p)$ of all Nash equilibrium payoffs from all bilateral persuasion games $\Gamma_n(p)$, $n \geq 1$, coincides with the p-section of* di-co $^{\mathrm{IR}}(H)$*:*

$$\mathcal{E}_B(p) = H_B(p) \equiv \{(a, \beta) \in \mathbb{R}^K \times \mathbb{R} : (a, \beta, p) \in \text{di-co}^{\mathrm{IR}}(H)\}.$$

Equivalently, $(a, \beta) \in \mathcal{E}_B(p)$ if and only if there exists a martingale $\boldsymbol{z} = (\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$, with $\boldsymbol{z}_s = (\boldsymbol{a}_s, \boldsymbol{\beta}_s, \boldsymbol{p}_s) \in I$ for all $s = 0, 1, \ldots, N$, satisfying the following properties:

(D1) $\boldsymbol{z}_0 = (a, \beta, p)$. That is, the starting point (and expectation) of the martingale is the Nash equilibrium payoff under consideration.

(D2) $\boldsymbol{z}_N \in H$. That is, the martingale converges to the set of non-revealing equilibrium payoffs: $(\boldsymbol{a}_N, \boldsymbol{\beta}_N) \in \mathcal{E}(\boldsymbol{p}_N)$.

(D3) $\boldsymbol{a}_{s+1} = \boldsymbol{a}_s$ for all even s and $\boldsymbol{p}_{s+1} = \boldsymbol{p}_s$ for all odd s. That is, the martingale is a dimartingale.[7]

Proof. See Subsection A.2. ∎

Remark 2 Requiring $\boldsymbol{a}_N \in \mathrm{INTIR}_K$ guarantees $\boldsymbol{a}_s \in \mathrm{INTIR}_K \subseteq \mathrm{INTIR}_{\mathrm{supp}[\boldsymbol{p}_s]}$ for all s, but is a much too strong condition: it is easy to construct an example with an equilibrium

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[7]All statements involving random variables should be understood to hold for all states occurring with strictly positive probability.

payoff $(a, \beta) \in \mathcal{E}_B(p)$ but $\boldsymbol{a}_N \notin \text{INTIR}_K$. On the other hand, requiring $\boldsymbol{a}_0 \in \text{INTIR}_K$ is not sufficient. Indeed, one can easily construct a dimartingale with $a_0 \in \text{INTIR}_K$, $(\boldsymbol{a}_N, \boldsymbol{\beta}_N, \boldsymbol{p}_N) \in H$, but $(a, \beta) \notin \mathcal{E}_B(p)$ ($\boldsymbol{a}_s \notin \text{INTIR}_{\text{supp}[\boldsymbol{p}_s]}$ for some history at s). More generally, the condition $\boldsymbol{z}_s \in I$ is redundant at some stages s but not at all of them. For instance, if s is even, $\boldsymbol{a}_{s+1} = \boldsymbol{a}_s$, $\boldsymbol{a}_s \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_s]}$, and the fact that $\text{supp}[\boldsymbol{p}_{s+1}] \subseteq \text{supp}[\boldsymbol{p}_s]$ imply $\boldsymbol{a}_{s+1} \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_{s+1}]}$. But the converse is not true: one may have $\boldsymbol{a}_{s+1} \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_{s+1}]}$ without having $\boldsymbol{a}_s = \boldsymbol{a}_{s+1} \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_s]}$. If s is odd, $\boldsymbol{p}_{s+1} = \boldsymbol{p}_s$, $\boldsymbol{a}_{s+1} \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_{s+1}]}$ and the martingale property imply that $\boldsymbol{a}_s \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_s]}$. Again, the converse is not true. These properties explain why, starting from the end of the process in order to construct di-co $^{\text{IR}}(H)$, one had to intercept with I only when convexifying at a fixed.

Remark 3 In the previous statement, convex combinations of payoffs when p is fixed may involve irrational weights (i.e., in $\mathbb{R} \backslash \mathbb{Q}$). If the message sets M^1 and M^2 are finite, as assumed above, the standard use of jointly controlled lotteries only shows that a subset of $H_B(p)$, in which convex combinations of payoffs at p fixed have rational weights, is contained in $\mathcal{E}_B(p)$. However, $\mathcal{E}_B(p)$ is not necessarily included in that subset of $H_B(p)$: for instance, it may happen, at some point of the bilateral communication process, that player 2 alone performs a lottery (possibly with irrational weights) over two ways of pursuing the play which give him the same expected payoff. A full equivalence between $\mathcal{E}_B(p)$ and $H_B(p)$ can be obtained by allowing the cheap talk messages to lie in the unit interval (see, e.g., Gerardi, 2004) or by allowing the players to observe the outcome of a public lottery over a finite set at every stage (see also Subsection 4.2).

Remark 4 If there exists a worst outcome for player 1 (i.e., an action $j_w \in J$ such that $A^k(j_w) \leq A^k(j)$ for all $k \in K$ and $j \in J$), then the individual rationality conditions are automatically satisfied.

4.2 Comparison with Aumann and Hart (2003)

When some coordinates of p vanish, Aumann and Hart (2003) consider the *modified* equilibrium payoffs $\mathcal{E}^+(p)$ of the silent game $\Gamma(p)$, which is the same as $\mathcal{E}(p)$ except that when the probability of one of player 1's types vanishes, then the corresponding type of player 1 can only get *more* than his equilibrium payoff. That is, the set of modified non-revealing equilibrium payoffs is the set of all payoffs (a, β) such that there exits an equilibrium $y \in Y(p)$ of the silent game $\Gamma(p)$ satisfying

(i) $a^k \geq A^k(y)$, for all $k \in K$;

(ii) $a^k = A^k(y)$ if $p^k \neq 0$;

(iii) $\beta = \sum_{k \in K} p^k B^k(y)$.

The graph of the modified non-revealing equilibrium payoff correspondence is

$$G \equiv \text{gr}\, \mathcal{E}^+ \equiv \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : (a, \beta) \in \mathcal{E}^+(p)\}.$$

Here, we consider the more natural set of non-revealing equilibrium payoffs, $\mathcal{E}(p)$, in which it is understood that the types of player 1 which have probability zero can get *any* payoff (only conditions (ii) and (iii) above must be satisfied). Clearly, $\mathcal{E}^+(p) \subseteq \mathcal{E}(p)$ and if p has full support, both sets coincide.

Let di-co (G) be the smallest set which contains G and is convex in (a, β) (respectively (β, p)) when p (respectively a) is fixed. Aumann and Hart (2003) characterize the set of all equilibrium payoffs achieved with finitely many stages of bilateral cheap talk as the p-section of di-co (G).[8] This extremely elegant characterization relies on the identification of the modified set of non-revealing equilibrium payoffs $\mathcal{E}^+(p)$ for every non interior p, which ensures that *all* equilibrium conditions of player 1 can be written as equalities, namely captured by a dimartingale property. In this framework, player 1's expected payoff remains fully interim individually rational (in INTIR_K) all along the communication process. Indeed, at the end of the communication process, $(a_N, \beta_N, p_N) \in G$ so that $a_N \in \text{INTIR}_K$ by condition (i) above. It follows from the martingale property that $a_s \in \text{INTIR}_K$ for every s. Intuitively, $\mathcal{E}^+(p)$ reflects the strength of player 1's incentive compatibility conditions when types are not verifiable. Our starting set H corresponds to the *non*-modified graph of the non-revealing equilibrium payoff correspondence in the sense that we do not impose any condition on player 1's payoff when his type has zero probability. This captures the relative weakness of player 1's incentive compatibility conditions when types are verifiable. According to Theorem 2, $(a_N, \beta_N, p_N) \in H$, which only guarantees that $a_N \in \text{INTIR}_{\text{supp}[\boldsymbol{p}_N]}$. Indeed, if player 1 can send certificates in addition to cheap talk messages, some states of nature may be eliminated forever. Player 1's individual rationality conditions must thus be expressed relatively to the remaining possible states. The geometric properties of our final graph of equilibrium payoffs are not as transparent as in Aumann and Hart (2003) since, as observed above, di-co $^{\text{IR}}(H)$ is not necessarily convex in (β, p) when a is fixed. Obviously, this set is convex in (a, β) when p is fixed since the players can perform jointly controlled lotteries.

Another difference between this paper and Aumann and Hart (2003) is that they do not require the number of communication stages to be almost surely finite, i.e., they consider *any* converging dimartingale. We could reformulate Theorem 2 in the same way at the price of adding further technicalities in the proof (as in Aumann and Hart, 2003, Sections 4.2 and 8). This approach also entails conceptual difficulties, since it leads to assume that time has order

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[8]With the same restrictions as in Remark 3.

$\omega + 1$, namely that there is an infinite sequence of time periods, with an additional period *after* the whole sequence.[9] No game-theoretical example illustrates that such an infinite communication phase would enable the players to achieve relevant new equilibrium payoffs (see Aumann and Hart, 1986, for a mathematical example, and Krishna, 2005, for further discussion in the cheap talk case). Between the set $\mathcal{E}_B(p)$ characterized in Theorem 2 and the analog of the set considered in Aumann and Hart (2003), an interesting set consists of those equilibrium payoffs which are associated with a dimartingale which converges almost surely in a finite, but not necessarily uniformly bounded, number of stages. The examples of Forges (1984, 1990a) can be adapted to the current framework, so that the latter set may be larger than $\mathcal{E}_B(p)$. It can be characterized in the same way as in Theorem 2, again at the price of some technicalities.

4.3   Illustration of Theorem 1 (Unilateral Persuasion)

For the introductory example, the graph of the modified non-revealing equilibrium payoff correspondence, $G = \mathrm{gr}\,\mathcal{E}^+$, is represented on the (a^1, a^2)-coordinates by solid lines in Figure 7 on the following page. The graph of the non-revealing equilibrium payoff correspondence, $H = \mathrm{gr}\,\mathcal{E}$, is represented in the same figure by the solid and dashed lines. The sets G and H are also described in the second and third columns of Table 1 on page 19. Since all points at the north-east of $(0, 0)$ are interim individually rational for player 1, convexifying the set H by keeping a constant and interim individually rational yields three new points at $p = 1/2$: FRE, PRE1 and PRE2, which are exactly the three Nash equilibrium payoffs found in Section 2, in addition to the non-revealing equilibrium (NRE). Indeed, each of these points corresponds to two non-revealing equilibrium payoffs, at two different p's forming an interval that includes $p = 1/2$, giving the same payoff to player 1. Notice that, for example, the point PRE3 is not an equilibrium payoff for $p = 1/2$ because $1/2$ lies outside the interval $[3/5, 1]$.

4.4   Illustration of Theorem 2 (Multistage, Bilateral Persuasion)

The dimartingale corresponding to the equilibrium with three talking stages of the introductory example (see Figure 3 on page 8) is represented by Figure 8 on page 19. It leads to the point j_2 at $p = 1/2$ in Figure 7, which is not achievable at $p = 1/2$ with only one step of diconvexification.

Adding a jointly controlled lottery before a signalling stage allows a convexification by keeping p fixed. This leads to the graph $H_1^* = \mathrm{conv}_p(H_1)$ described on the a-coordinates in the fourth column of Table 1. For example, adding a jointly controlled lottery before a signalling stage at $p = 1/2$ leads to all convex combinations of equilibrium payoffs of the unilateral

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[9]Obviously, with a possibly infinite phase of communication, the problem mentioned in Remark 3 disappears.
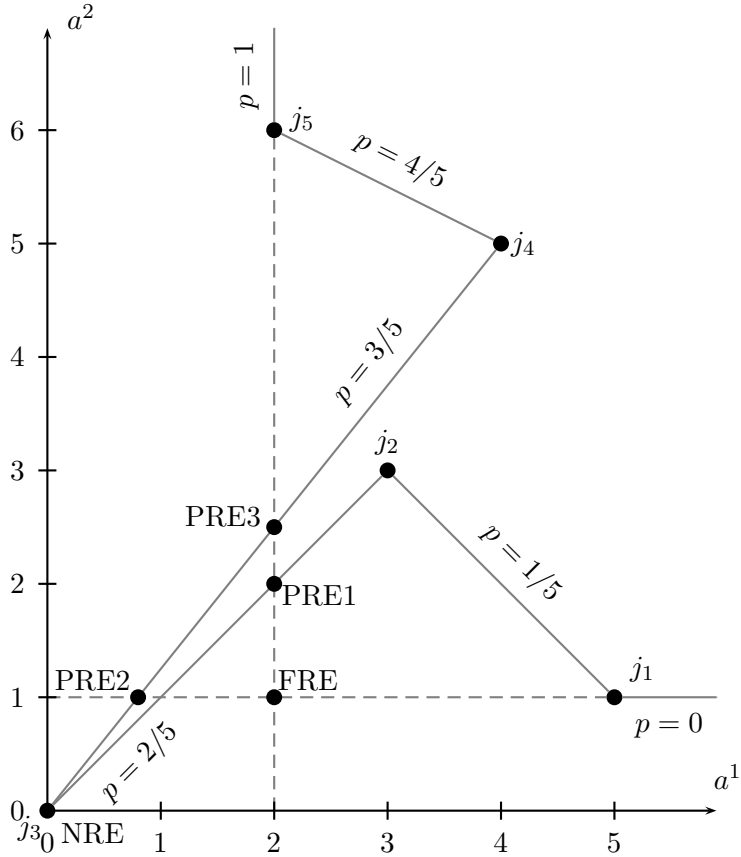
Figure 7: Modified non-revealing equilibrium payoffs (solid lines) and interim individually rational non-revealing equilibrium payoffs (solid and dashed lines) of the expert in the introductory example.

persuasion game, $[j_3, \text{FRE,PRE1,PRE2}]$. Adding a second signalling stage allows a second convexification by keeping a fixed. One can check that this does not yield new equilibrium payoffs, except for $p \in (2/5, 3/5)$. Indeed, for $p \in (2/5, 3/5)$ one can combine the sets $H_1^*(p') = [j_2, \text{PRE2}, \text{FRE}]$, $p' \in (1/5, 2/5)$, and $H_1^*(p'') = [j_4, \text{PRE3}, \text{FRE}]$, $p'' \in (3/5, 4/5)$, which leads to the payoffs in the triangle $[j_2, \text{PRE1,FRE}]$, which were not achievable at $p \in (2/5, 3/5)$ with only 2 canonical communication stages. Hence, for $p \in (2/5, 3/5)$, $H_2(p) = H_1^*(p) \cup [j_2, \text{PRE1,FRE}] = [j_3, \text{PRE2}, j_2, \text{FRE}]$. It is easy to verify that one cannot get new points after two steps of diconvexification in both directions, so $H_2 = H_n$ for all $n \geq 2$.

18
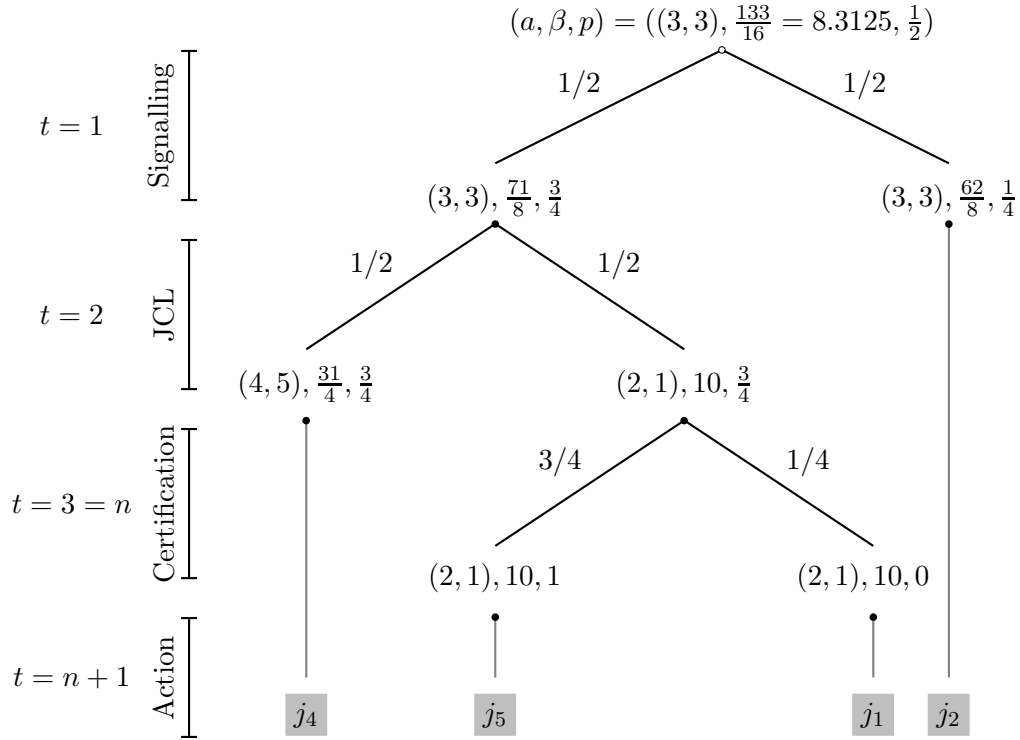
Figure 8: Dimartingale/diconvexification corresponding to the equilibrium with three talking stages in the introductory example.

p	G	H	$H_1^* = \operatorname{conv}_p(H_1)$	H_2
0	$(a^1 \geq 5, 1)$	$(a^1, 1)$	$\cdots$	$\cdots$
$(0, \frac{1}{5})$	j_1	j_1	$[j_1, \text{PRE2}]$	$\cdots$
$\frac{1}{5}$	$[j_1, j_2]$	$[j_1, j_2]$	$[j_1, j_2, \text{PRE2}]$	$\cdots$
$(\frac{1}{5}, \frac{2}{5})$	j_2	j_2	$[j_2, \text{PRE2}, \text{FRE}]$	$\cdots$
$\frac{2}{5}$	$[j_2, j_3]$	$[j_2, j_3]$	$[j_2, \text{PRE2}, j_3, \text{FRE}]$	$\cdots$
$(\frac{2}{5}, \frac{3}{5})$	j_3	j_3	$[j_3, \text{FRE}, \text{PRE1}, \text{PRE2}]$	$[j_3, \text{PRE2}, j_2, \text{FRE}]$
$\frac{3}{5}$	$[j_3, j_4]$	$[j_3, j_4]$	$[j_3, j_4, \text{FRE}]$	$\cdots$
$(\frac{3}{5}, \frac{4}{5})$	j_4	j_4	$[j_4, \text{PRE3}, \text{FRE}]$	$\cdots$
$\frac{4}{5}$	$[j_4, j_5]$	$[j_4, j_5]$	$[j_4, j_5, \text{FRE}]$	$\cdots$
$(\frac{4}{5}, 1)$	j_5	j_5	$[j_5, \text{FRE}]$	$\cdots$
1	$(2, a^2 \geq 6)$	$(2, a^2)$	$\cdots$	$\cdots$

Table 1: Diconvexification of the non-revealing equilibrium payoffs of the introductory example. "$\cdots$" means "as in the previous column".

5 Characterization of Perfect Bayesian Equilibrium Payoffs

In cheap talk games, it is well known that standard equilibrium refinements do not eliminate any Nash equilibrium outcome. On the contrary, when information is certifiable, the set of perfect Bayesian equilibrium outcomes is usually strictly included in the set of Nash equilibrium outcomes. In particular, the non-revealing equilibrium outcome is always a perfect Bayesian equilibrium outcome in cheap talk games, but not in persuasion games. This has already been observed in the literature on sender-receiver persuasion games (see, e.g., Milgrom, 1981, Milgrom and Roberts, 1986 and Seidmann and Winter, 1997) and, more generally, in Bayesian games with strategic information revelation (Okuno-Fujiwara et al., 1990). In these papers, it is shown that imposing sequential rationality restrictions off the equilibrium path is powerful enough to characterize a unique equilibrium outcome (typically, a fully revealing one) in some classes of games like monotonic sender-receiver games or linear n-player oligopoly games.

5.1 Examples

Consider again our introductory example, with the prior $p = 1/2$. At the non-revealing equilibrium (NRE), player 2 should choose action j_3 whatever the message sent by player 1. However, if player 1 sends a certificate c^k for type k, then in the second stage game player 2 is in a proper subgame in which action j_5 (action j_1, respectively) is strictly dominant if $k = k_1$ ($k = k_2$, respectively). Hence, the NRE is not subgame perfect, so it is not a perfect Bayesian equilibrium. The same conclusion holds at the second partially revealing equilibrium (PRE2), where a strictly dominated action is played in the proper subgame following the message c^1. On the contrary, all other Nash equilibrium outcomes of the unilateral persuasion game, as well as the 3-stage Nash equilibrium outcome depicted in Figure 3 on page 8, are perfect Bayesian equilibrium outcomes (see below for a precise definition).

In persuasion games, the set of subgame perfect equilibrium payoffs may not coincide with the set of perfect Bayesian equilibrium payoffs. To see this, consider the silent game of Figure 9. The optimal actions of the decisionmaker are

$$
Y(p) = \begin{cases} \{j_1\} & \text{if } p > 2/3, \\ \{j_2\} & \text{if } p < 2/3, \\ \Delta(\{j_1, j_2\}) & \text{if } p = 2/3. \end{cases}
$$

If $x > 3$ or $y > 1$, the unique Nash equilibrium of the persuasion game is non-revealing. Otherwise, there is a fully revealing Nash equilibrium. This can be seen in Figure 10 where the point FRE is interim individually rational if and only if it is at the north-east of (x, y).

$$
\begin{array}{c|ccc}
 & j_1 & j_2 & j_3 \\
\hline
k_1 & 3,1 & 4,0 & x,-1 \\
k_2 & 3,0 & 1,2 & y,-1 \\
\end{array}
\quad
\begin{array}{c}
p \\
\\
(1-p)
\end{array}
$$

Figure 9: Subgame perfect equilibrium vs. perfect Bayesian equilibrium.
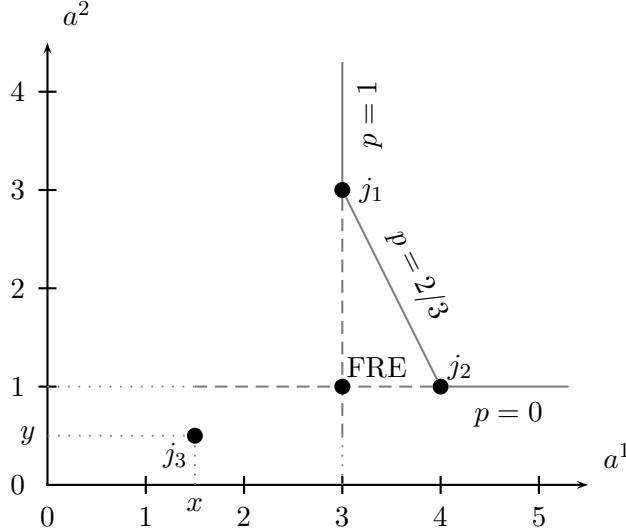
Figure 10: Modified (solid lines) and interim individually rational (solid and dashed lines) non-revealing equilibrium payoffs of the expert in the silent game of Figure 9.

The FRE is also subgame perfect since any proper subgame is reached with strictly positive probability. However, it is not a perfect Bayesian equilibrium because it is supported by action j_3 off the equilibrium path, but j_3 is not an optimal action for the decisionmaker whatever his belief about the expert's type.

5.2 Belief Consistency and Sequential Rationality

Roughly, a Nash equilibrium is a perfect Bayesian equilibrium if the strategy which is used by player 2 off the equilibrium path is optimal for player 2 for at least one belief over K consistent with the history of messages sent by player 1. Formally, let $P_{\sigma,\tau,p}$ be the probability distribution on $K \times M_n \times J$ generated by players' strategies and the prior probability distribution. For any type $k \in K$ and communication history $h_t = (m_1^1, m_1^2, \ldots, m_t^1, m_t^2) \in M_t$, denote player 2's conditional belief that player 1's type is k by $\mu(k \mid h_t)$, and let

$$
M_t^{-1}(h_t) \equiv \bigcap_{s=1}^{t} M^{-1}(m_s^1).
$$

Definition 1 A *perfect Bayesian equilibrium* of the n-stage persuasion game $\Gamma_n(p)$ is a pair

of strategy profiles and belief function $((\sigma, \tau), \mu)$ such that for every period t, history $h_t \in M_t$ and type $k \in K$:

(a) *Bayes' rule.* If $P_{\sigma,\tau,p}(h_t \mid l) > 0$ for some $l \in K$, then

$$\mu(k \mid h_t) = \frac{p^k P_{\sigma,\tau,p}(h_t \mid k)}{\sum_{l \in K} p^l P_{\sigma,\tau,p}(h_t \mid l)};$$

(b) *Player 1's sequential rationality.* σ maximizes player 1's expected payoff under (σ, τ) conditional on reaching h_t;

(c) *Player 2's sequential rationality.* τ maximizes player 2's expected payoff under μ and (σ, τ) conditional on reaching h_t;

(d) *Consistency with certification.* If $k \notin M_t^{-1}(h_t)$, then $\mu(k \mid h_t) = 0$;

(e) *"No signaling what you don't know".* For every communication history $\widetilde{h}_t$ that differs from h_t only in terms of player 2's messages (i.e., $h_t = (m_1^1, m_1^2, \ldots, m_t^1, m_t^2)$ and $\widetilde{h}_t = (m_1^1, \widetilde{m}_1^2, \ldots, m_t^1, \widetilde{m}_t^2)$), we have $\mu(k \mid h_t) = \mu(k \mid \widetilde{h}_t)$.

Conditions (a) to (c) are the weakest conditions for perfect Bayesian equilibrium (also called "weak sequential equilibrium"; see, e.g., Mas-Colell et al., 1995, Section 9.C). For histories occurring with strictly positive probability they imply that (σ, τ) is a Nash equilibrium. The sequential rationality condition for player 2 (condition (c)) also implies that $\tau_{n+1}(h_n) \in \arg\max_{y \in \Delta(J)} \sum_{k \in K} \mu(k \mid h_n) B^k(y) = Y(\mu(h_n))$, even for final communication histories h_n occurring with probability zero under (σ, τ).

Condition (d) is a belief consistency condition that is specific to the fact that player 1's message set is type-dependent, and is usual in the strategic information revelation literature (see, for example, Okuno-Fujiwara et al., 1990, condition (b) p. 29). It simply means that, even off the equilibrium path, player 2's belief about player 1's type should put strictly positive probability only on types that are able to send the observed sequences of messages $m_1^1, \ldots, m_t^1$. Finally, condition (e) is the "no signaling what you don't know" condition from Fudenberg and Tirole (1991), which means that player 2's belief about player 1's type should only be influenced by player 1's messages. The necessity of this condition for our equilibrium characterization in Theorem 3 is illustrated by an example in Subsection 5.4.

5.3 Statement of the Result

To get geometric characterizations like in Section 4 for perfect Bayesian equilibrium payoffs instead of Nash equilibrium payoffs we have to strengthen player 1's interim individual

rationality condition. More precisely, we must replace INTIR_L by

$$\text{INTIR}_L^{\text{PBE}} \equiv \{a \in \mathbb{R}^K : \forall\, X \subseteq L,\ \exists\, p_X \in \Delta(X) \text{ and } y_X \in Y(p_X),\ a^k \geq A^k(y_X)\ \forall\, k \in X\},$$

and define $\text{di-co}^{\text{PBE}}(H)$ as $\text{di-co}^{\text{IR}}(H)$ (see Section 4.1) by replacing I by $I^{\text{PBE}} \equiv \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : a \in \text{INTIR}_{\text{supp}[p]}^{\text{PBE}}\}$. This leads to the next theorem, which is the analog of Theorems 1 and 2 for perfect Bayesian equilibrium.

Theorem 3

(1) The set of all perfect Bayesian equilibrium payoffs of the unilateral persuasion game $\Gamma_S(p)$ coincides with the p-section of $\text{conv}_a(H) \cap I^{\text{PBE}}$.

(2) The set of all perfect Bayesian equilibrium payoffs from all bilateral persuasion games $\Gamma_n(p)$, $n \geq 1$, coincides with the p-section of $\text{di-co}^{\text{PBE}}(H)$. Equivalently, (a, β) is a perfect Bayesian equilibrium payoff of some bilateral persuasion game $\Gamma_n(p)$, $n \geq 1$, if and only if there exists a martingale $\boldsymbol{z} = (\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$, with $\boldsymbol{z}_s = (\boldsymbol{a}_s, \boldsymbol{\beta}_s, \boldsymbol{p}_s) \in I^{\text{PBE}}$ for all $s = 0, 1, \ldots, N$, satisfying properties (D1), (D2) and (D3) of Theorem 2.

Proof. See Subsection A.3. ∎

It is easy to see that subgame perfection is obtained as a special case when events X in the definition of $\text{INTIR}_L^{\text{PBE}}$ are reduced to singletons, i.e., by replacing $\text{INTIR}_L^{\text{PBE}}$ by

$$\text{INTIR}_L^{\text{SPE}} \equiv \{a \in \mathbb{R}^K : \forall\, k \in L,\ \exists\, y_k \in \arg\max_{y \in \Delta(J)} B^k(y),\ a^k \geq A^k(y_k)\}.$$

5.4 Illustration

In the introductory example, INTIR_K is the set of points at the north-east of $(0, 0)$ in Figure 7, and $\text{INTIR}_K^{\text{PBE}} = \text{INTIR}_K^{\text{SPE}} \subsetneq \text{INTIR}_K$ is the set of points at the north-east of $(2, 1)$. In the example of Figure 9, INTIR_K is the set of points at the north-east of (x, y) in Figure 10, $\text{INTIR}_K^{\text{SPE}} \subsetneq \text{INTIR}_K$ is the set of points at the north-east of $(3, 1)$, and $\text{INTIR}_K^{\text{PBE}} \subsetneq \text{INTIR}_K^{\text{SPE}}$ is the set of points at the north-east of the segment $[j_1, j_2]$.

The next example illustrates the importance in bilateral persuasion games of the belief consistency requirement "no signaling what you don't know". The silent game is given by Figure 11. In the unilateral persuasion game $\Gamma_S(p)$ with $p = (1/3, 1/3, 1/3)$, the non-revealing Nash equilibrium payoff $(3, 3, 3)$ for player 1 is not in $\text{INTIR}_K^{\text{PBE}}$, and is indeed not a perfect Bayesian equilibrium because if player 1 deviates by sending a message c^{12} such that $M^{-1}(c^{12}) = \{k_1, k_2\}$ then his payoff is strictly higher than 3 for k_1 or k_2 whatever the sequentially rational mixed action of player 2 (which must be in $Y(q)$ for some $q \in \Delta(\{k_1, k_2\})$). Now, consider the non-revealing Nash equilibrium of the 1-stage bilateral persuasion game $\Gamma_1(p)$ in which player 2 sends two messages a and b with probability $1/2$ each in the talking

phase, and plays action j_1 if a occurs and j_3 is b occurs after player 1 sends c^{12} off the equilibrium path. Given this strategy, player 1 has no incentive to deviate (he would get 2.5 instead of 3). Furthermore, player 2's strategy is sequentially rational with belief $\mu(k_1 \mid c^{12}, a) \leq 1/4$, $\mu(k_1 \mid c^{12}, b) \geq 3/4$ and $\mu(k_3 \mid c^{12}, \cdot) = 0$ off the equilibrium path. This belief satisfies consistency condition (d), but not (e).

	j_1	j_2	j_3	j_0
k_1	$5, 0$	$4, 3$	$0, 4$	$3, 0$
k_2	$0, 4$	$4, 3$	$5, 0$	$3, 0$
k_3	$0, -10$	$0, -10$	$0, -10$	$3, 0$

Figure 11: An example illustrating condition (e) ("no signaling what you don't know").

6 Mediated Persuasion

In this paper we assumed that communication between the expert and the decisionmaker takes place face-to-face. This excludes correlated extraneous signals and private recommendations. In particular, there is no uncertainty on the messages received by each party during the talking phase. If a mediator were available and if any form of costless communication were possible between the players, then the resulting set of Nash equilibrium outcomes would be the set of *certification equilibrium* outcomes introduced by Forges and Koessler (2005). Under the assumption of full certifiability made in the current paper, a single stage of mediated certification is sufficient and the set of certification equilibrium outcomes has a canonical representation characterized by a transition probability $q : K \to \Delta(J)$ and a punishment strategy $\overline{y} \in \Delta(J)$ satisfying the informational incentive constraint

$$\sum_{j \in J} q(j \mid k) A^k(j) \geq A^k(\overline{y}) \text{ for all } k \in K, \tag{1}$$

and the strategic incentive constraint

$$\sum_{k \in K} \Pr_q(k \mid j) B^k(j) \geq \sum_{k \in K} \Pr_q(k \mid j) B^k(j'), \quad \forall\, j \in \text{supp}[q],\ j' \in J. \tag{2}$$

Let $\mathcal{E}_M(p) \subseteq \mathbb{R}^K \times \mathbb{R}$ be the resulting set of mediated certification equilibrium payoffs. This set includes the set of Nash equilibrium payoffs achieved with face-to-face communication, so $\mathcal{E}(p) \subseteq \mathcal{E}_S(p) \subseteq \mathcal{E}_B(p) \subseteq \mathcal{E}_M(p)$, and all these inclusions may be strict.

The set of *communication equilibrium* outcomes (Myerson, 1982; Forges, 1986) is charac-

terized by recommendations satisfying (2) and (3):

$$\sum_{j \in J} q(j \mid k) A^k(j) \geq \sum_{j \in J} q(j \mid k') A^k(j) \text{ for all } k, k' \in K. \qquad (3)$$

Since condition (3) is a stronger requirement than (1), the set of certification equilibrium outcomes also includes the set of communication equilibrium outcomes.

The analysis is much more tractable when a mediator is available to help the players to communicate and to certify their information.[10] For example, the equilibrium outcome with three talking stages of the introductory example (see Figure 3 on page 8) can easily be implemented with the help of a mediator as follows. First, player 1 chooses whether to make a certifiable report to the mediator concerning the true state of the world. When there are only two types, player 1 has two possible reports in every state k: either he certifies his information by sending message c^k or he certifies nothing. Afterwards, the mediator gives a (random) recommendation of action to player 2 conditionally on the report of player 1. Denote respectively by $q(j \mid k)$ and $\overline{y}(j)$ the probabilities that the mediator recommends action j to player 2 when player 1 sends message c^k and $m \neq c^1, c^2$, respectively. The following recommendations mimic the equilibrium outcome:

$$q(j_4 \mid k_1) = q(j_5 \mid k_1) = 3/8 \qquad q(j_2 \mid k_1) = 1/4$$
$$q(j_1 \mid k_2) = q(j_4 \mid k_2) = 1/8 \qquad q(j_2 \mid k_2) = 3/4$$
$$\overline{y}(j_3) = 1.$$

If player 1 completely certifies his information and player 2 follows the recommendation of the mediator, then no player has an incentive to deviate. Indeed, player 1 never deviates since by certifying his information his payoff is always strictly positive, whereas by not certifying his information his payoff would be zero. From Bayes' rule, player 2's beliefs about the state of Nature given the recommendations of the mediator are $\Pr_q(k_1 \mid j_5) = 1$, $\Pr_q(k_1 \mid j_4) = 3/4$, $\Pr_q(k_1 \mid j_2) = 1/4$ and $\Pr_q(k_1 \mid j_1) = 0$, so the recommendations are optimal for him given his beliefs.

A   Appendix: Proofs

A.1   Proof of Theorem 1

We assume w.l.o.g. that $\text{supp}[p] = K$, so that $\mathcal{E}_S(p)$ can be characterized equivalently as the p-section of $\text{conv}_a(H) \cap \{(a, \beta, p) \in \mathbb{R}^K \times \mathbb{R} \times \Delta(K) : a \in \text{INTIR}_K\}$.

---

[10]In particular, certification equilibrium outcomes can be characterized in a canonical way for Bayesian games with any number of players, any information structure, and any assumption on certifiability possibilities.

A.1.1 From equilibrium to constrained convexification: $\mathcal{E}_S(p) \subseteq H_1(p)$

Let (σ, τ) be any Nash equilibrium of the unilateral persuasion game $\Gamma_S(p)$, where $p^k > 0$ for all $k \in K$, and let $(a, \beta) \in \mathcal{E}_S(p)$ be the associated equilibrium payoffs. We must show that (a, β, p) is in H_1, i.e., (a, β, p) can be obtained as a convex combination of points in $H = \operatorname{gr} \mathcal{E}$ by keeping a constant and interim individually rational ($a \in \operatorname{INTIR}_K$). Let $P = P_{\sigma, \tau, p}$ be the probability distribution on $\Omega = K \times M^1 \times J$ generated by players' strategies and the priors. So,

$$P(m) = \sum_{k \in K} p^k \, \sigma^k(m),$$

is the (ex ante) probability that player 1 sends message $m \in M^1$. Let $M^* = \{m \in M^1 : P(m) > 0\}$. For all $m \in M^*$, let

$$p_m^k = P(k \mid m) = \frac{p^k \, \sigma^k(m)}{P(m)},$$

be player 2's posterior about player 1's type after receiving message m, let $p_m = (p_m^k)_{k \in K}$, and let

$$\beta_m = \sum_{k \in K} p_m^k \, B^k(\tau(m)),$$

be the resulting expected payoff for player 2 when m is reached. Since $p^k = \sum_{m \in M^*} P(m) \, p_m^k$ for all $k \in K$ and $\beta = \sum_{m \in M^*} P(m) \, \beta_m$, we have

$$(a, \beta, p) = \sum_{m \in M^*} P(m) \, (a, \beta_m, p_m).$$

So, to show that (a, β, p) is a convex combination of points in H be keeping a constant it suffices to show that $(a, \beta_m, p_m) \in H$ for all $m \in M^*$, i.e., $(a, \beta_m) \in \mathcal{E}(p_m)$ for all $m \in M^*$. Player 2's equilibrium condition implies that $\tau(m) \in Y(p_m)$ for all $m \in M^*$, so condition (iii) in the definition of $\mathcal{E}(p_m)$ (see page 16) is satisfied for all $m \in M^*$. Player 1's equilibrium condition implies that $A^k(\tau(m)) = A^k(\tau(m'))$ whenever $\sigma^k(m) > 0$ and $\sigma^k(m') > 0$ (player 1 of type k should be indifferent between all messages that he sends with strictly positive probability), so

$$a^k = \sum_{m \in M^*} \sigma^k(m) \, A^k(\tau(m)) = A^k(\tau(m)),$$

for all m such that $\sigma^k(m) > 0$ (which is equivalent to $p_m^k > 0$ because $p^k > 0$), so condition (ii) in the definition of $\mathcal{E}(p_m)$ is also satisfied for all $m \in M^*$.

Remark 5 Notice that when $p_m^k = 0$ we may have $a^k < A^k(\tau(m))$ (because type k *cannot* send message m when $m \notin M(k)$), so when some coordinates of p_m vanish it is possible that

$(a, \beta_m, p_m) \notin G \equiv \operatorname{gr} \mathcal{E}^+$, contrary to the case of cheap talk (Aumann and Hart, 2003).

It remains to show that $a \in \operatorname{INTIR}_K$. Consider a message $\overline{m} \in \bigcap_{k \in K} M(k)$ (which exists by the "right to remain silent" assumption), and let $\overline{y} = \tau(\overline{m})$ ($\overline{m}$ may or may not be a message sent by player 1 with positive probability, so there may be no rationality condition on $\overline{y}$ for player 2 as long as no equilibrium refinement is introduced). By player 1's equilibrium condition, for all $k \in K$ and m such that $\sigma^k(m) > 0$ we have $a^k = A^k(\tau(m)) \geq A^k(\overline{y})$, which proves that $a \in \operatorname{INTIR}_K$. $\blacksquare$

A.1.2 From constrained convexification to equilibrium: $H_1(p) \subseteq \mathcal{E}_S(p)$

We start from (a, β, p), a convex combination of points in H by keeping a constant, with $a \in \operatorname{INTIR}_K$ and $p^k > 0$ for all $k \in K$, and we construct an equilibrium (σ, τ) of the unilateral persuasion game $\Gamma_S(p)$ with expected payoffs (a, β). Since $(a, \beta, p) \in \operatorname{conv}_a(H)$, we can write

$$(a, \beta, p) = \sum_{w \in W} \pi(w) \, (a, \beta_w, p_w),$$

with $\pi \in \Delta(W)$ and $(a, \beta_w, p_w) \in H$ for all $w \in W$. Without loss of generality we assume that π has full support. In addition, from Carathéodory's theorem we can let $|W| \leq K + 1$ since the dimension of $(\beta, p) \in \mathbb{R} \times \Delta(K)$ is equal to K. For all $w \in W$, we associate a set of types $\operatorname{supp}[p_w] \equiv \{k \in K : p_w^k > 0\}$ and a message $m_w \in M^1$ with $m_w \neq m_{w'}$ for $w \neq w'$, and $M^{-1}(m_w) = \operatorname{supp}[p_w]$. This is possible given our rich language and certifiability assumption.

Player 1's strategy σ. For all $k \in K$ and $w \in W$ define

$$\sigma^k(m_w) = \frac{\pi(w) \, p_w^k}{p^k} \quad (\text{and } \sigma^k(m) = 0 \text{ if } m \neq m_w \text{ for all } w \in W).$$

Player 2's strategy τ. Since by assumption $(a, \beta_w) \in \mathcal{E}(p_w)$, for all $w \in W$ we can define (see condition (ii) and (iii) of $\mathcal{E}(p_w)$),

$$y_w = \tau(m_w) \in Y(p_w) \quad \text{such that} \quad \begin{cases} a^k = A^k(\tau(m_w)) \text{ if } p_w^k > 0 \\ \beta_w = \sum_{k \in K} p_w^k \, B^k(\tau(m_w)). \end{cases}$$

For the other messages $m \neq m_w$, $w \in W$, since by definition $a \in \operatorname{INTIR}_K$, we can define

$$\tau(m) = \overline{y} \quad \text{such that} \quad a^k \geq A^k(\overline{y}) \text{ for all } k \in K.$$

Payoffs. We first verify that (a, β) is the payoff generated by the strategy profile (σ, τ) defined just before. Let $P = P_{\sigma, \tau, p}$ be the probability distribution on $\Omega = K \times M^1 \times J$

generated by those strategies and the prior, and let $E = E_{\sigma, \tau, p}$ be the associated expectation operator. First, we check that $P(m_w) = \pi(w)$ for all $w \in W$:

$$P(m_w) = \sum_{k \in K} p^k \, \sigma^k(m_w) = \sum_{k \in K} p^k \frac{\pi(w) \, p_w^k}{p^k} = \pi(w) \sum_{k \in K} p_w^k = \pi(w).$$

By construction, player 1's expected payoff when his type is k is given by

$$
\begin{aligned}
E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] &= \sum_{w \in W} P[\boldsymbol{m} = m_w \mid \boldsymbol{k} = k] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k, \boldsymbol{m} = m_w] \\
&= \sum_{w \in W} \sigma^k(m_w) \sum_{j \in J} \tau(m_w)(j) A^k(j) = \sum_{w \in W} \sigma^k(m_w) \, A^k(\tau(m_w)) = a^k,
\end{aligned}
$$

the last equality following from the construction of player 2's strategy: $A^k(\tau(m_w)) = a^k$ whenever $\sigma^k(m_w) > 0$ ($\Leftrightarrow p_w^k > 0$ because $p^k > 0$). Finally, player 2's expected payoff is

$$
\begin{aligned}
E[B^{\boldsymbol{k}}(\boldsymbol{j})] &= \sum_{k \in K} p^k E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] \\
&= \sum_{k \in K} p^k \sum_{w \in W} P[\boldsymbol{m} = m_w \mid \boldsymbol{k} = k] E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k, \boldsymbol{m} = m_w] \\
&= \sum_{k \in K} p^k \sum_{w \in W} \sigma^k(m_w) \sum_{j \in J} \tau(m_w)(j) B^k(j) = \sum_{k \in K} p^k \sum_{w \in W} \frac{\pi(w) p_w^k}{p^k} B^k(\tau(m_w)) \\
&= \sum_{w \in W} \pi(w) \sum_{k \in K} p_w^k B^k(\tau(m_w)) = \sum_{w \in W} \pi(w) \, \beta_w = \beta.
\end{aligned}
$$

Equilibrium condition for player 2. Next, we verify that τ is a best reply for player 2 to player 1's strategy σ. Since we have defined $\tau(m_w) \in Y(p_w)$ for all $w \in W$, and since the messages $(m_w)_{w \in W}$ are the only messages sent with strictly positive probability by player 1, it suffices to verify that p_w is the correct posterior belief of player 2 when he receives message m_w. This is immediately obtained by Bayes's rule given the definition of the strategy σ:

$$P[\boldsymbol{k} = k \mid \boldsymbol{m} = m_w] = \frac{P[\boldsymbol{m} = m_w \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{P[\boldsymbol{m} = m_w]} = \frac{\sigma^k(m_w) p^k}{\pi(w)} = p_w^k.$$

Equilibrium condition for player 1. Finally, we verify that σ^k is a best reply for player 1 of type k to player 2's strategy τ. Player 1 of type k sends each message m_w, $w \in W$, satisfying $p_w^k > 0$ ($\Leftrightarrow \sigma^k(m_w) > 0$ because $p^k > 0$) with strictly positive probability. By construction of player 2's strategy we have $A^k(\tau(m_w)) = a^k$ (see the previous paragraph "payoffs") for all such messages, so type k is indeed indifferent between all these messages. Next, remark that type k *cannot* send the other messages m_w satisfying $p_w^k = 0$ because such messages are such that $M^{-1}(m_w) = \text{supp}[p_w]$, with $k \notin \text{supp}[p_w]$, so $m_w \notin M(k)$. Finally, if player 1 sends a message off the equilibrium path, $\overline{m} \neq m_w$ for all $w \in W$ (so $P(\overline{m}) = 0$), then he gets

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$A^k(\tau(\overline{m})) = A^k(\overline{y}) \le a^k = A^k(\tau(m_w))$ for $\sigma^k(m_w) > 0$, so he does not deviate. ∎

A.2   Proof of Theorem 2

As in the proof of Theorem 1, we assume w.l.o.g. that $\mathrm{supp}[p] = K$.

A.2.1   From equilibrium to constrained dimartingale: $\mathcal{E}_B(p) \subseteq H_B(p)$

Except for the construction of player 1's sequence of virtual payoffs and the fact that we consider martingales that are bounded in length, this part of the proof is similar to the proof of Hart (1985) and Aumann and Hart (2003). Let (σ, τ) be any Nash equilibrium of the communication game $\Gamma_n(p)$ for some finite $n \ge 1$, where $p^k > 0$ for all $k \in K$, with payoffs $a = (a^1, \ldots, a^K) \in \mathbb{R}^K$ for player 1 and $\beta \in \mathbb{R}$ for player 2. We construct a sequence of random variables $\boldsymbol{z} = (\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$, with $N = 2n$, satisfying properties (D1) to (D3) of Theorem 2, the interim individual rationality conditions $\boldsymbol{z}_s \in I$ for all s, and the martingale property: $E[\boldsymbol{z}_{s+1} \mid \boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_s] = \boldsymbol{z}_s$, $s = 0, 1, \ldots, N$. We work on the probability space $\Omega = K \times M_n \times J$, where $M_n = (M^1 \times M^2)^n$. A realization $\omega = (k, m_1^1, m_1^2, \ldots, m_t^1, m_t^2, \ldots, m_n^1, m_n^2, j) \in \Omega$ consists in a type for player 1, a final communication history, and an action for player 2. All random variables (denoted in bold letters when there may be a risk of confusion) are defined on Ω. Let $P = P_{\sigma, \tau, p}$ be the probability distribution on Ω generated by players' strategies and the prior probability distribution on player 1's set of types, and let $E = E_{\sigma, \tau, p}$ be the corresponding expectation operator. For example, $P[\boldsymbol{k} = k] = p^k$ and $P[\boldsymbol{m}_t^1 = m \mid \boldsymbol{h}_{t-1} = h_{t-1}, \boldsymbol{k} = k] = \sigma_t^k(h_{t-1})(m)$.

For $s = 0, \ldots, N$ we construct a new "half-steps" random variable on Ω, $\boldsymbol{g}_s$, that corresponds to every history of talk, plus every history of talk followed by player 1's message in the next period. Formally,

$$g_s \equiv \begin{cases} h_t = (m_1^1, m_1^2, \ldots, m_t^1, m_t^2), & \text{if } s = 2t \text{ is even, } t = 0, \ldots, n \\ (h_t, m_{t+1}^1), & \text{if } s = 2t + 1 \text{ is odd, } t = 0, \ldots, n - 1. \end{cases}$$

So, $g_0 = h_0 = \emptyset$, $g_N = g_{2n} = h_n$, when s is even the last message in g_s is from player 2, and when s is odd the last message in g_s is from player 1. We consider this new random variable in order to have the dimartingale property (D3).

Sequence of posteriors $(\boldsymbol{p}_s)_{s=0,1,\ldots,N}$.   For each $k \in K$ and $s = 0, \ldots, N$, define

$$\boldsymbol{p}_s^k \equiv P[\boldsymbol{k} = k \mid \boldsymbol{g}_s],$$

and $\boldsymbol{p}_s = (\boldsymbol{p}_s^k)_{k \in K} \in \Delta(K)$.

Lemma 1 *The sequence* $(\boldsymbol{p}_s^k)_{s=0,\ldots,N}$ *is a (bounded) martingale satisfying*

(i) $\boldsymbol{p}_0 = p$;

(ii) $\boldsymbol{p}_{s+1} = \boldsymbol{p}_s$ *for all odd* s.

Proof. The martingale property is simply due to the fact that $(\boldsymbol{p}_s^k)_{s=0,\ldots,N}$ is a sequence of posteriors by conditioning on more and more information (it is adapted to the sequence of fields $(\mathcal{G}_s)_{s=0,\ldots,N}$ generated by $(\boldsymbol{g}_s)_{s=0,\ldots,N}$). (i) is immediate: $\boldsymbol{p}_0^k = P[\boldsymbol{k} = k \mid \boldsymbol{g}_0] = P[\boldsymbol{k} = k] = p^k$. To prove (ii), let $s = 2t + 1$ be an odd number. For each $k \in K$ we have

$$\boldsymbol{p}_{s+1}^k = P[\boldsymbol{k} = k \mid \boldsymbol{g}_{s+1}] = P[\boldsymbol{k} = k \mid \boldsymbol{h}_t, \boldsymbol{m}_{t+1}^1, \boldsymbol{m}_{t+1}^2] = P[\boldsymbol{k} = k \mid \boldsymbol{h}_t, \boldsymbol{m}_{t+1}^1] = \boldsymbol{p}_s^k,$$

the last but one equality following from the fact that, conditional on $(\boldsymbol{h}_t, \boldsymbol{m}_{t+1}^1)$, $\boldsymbol{m}_{t+1}^2$ and $\boldsymbol{k}$ are independent. ∎

Sequence of player 2's payoff $(\boldsymbol{\beta}_s)_{s=0,1,\ldots,N}$. For each $s = 0,\ldots,N$, define

$$\boldsymbol{\beta}_s \equiv E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s],$$

and let $\boldsymbol{y} = \tau_{n+1}(\boldsymbol{g}_N)$.

Lemma 2 *The sequence* $(\boldsymbol{\beta}_s)_{s=0,\ldots,N}$ *is a (bounded) martingale satisfying*

(i) $\boldsymbol{\beta}_0 = \beta$;

(ii) $\boldsymbol{\beta}_N = \sum_{k \in K} \boldsymbol{p}_N^k B^k(\boldsymbol{y})$, *with* $\boldsymbol{y} \in Y(\boldsymbol{p}_N)$.

Proof. The martingale property is due to the fact that $(\boldsymbol{\beta}_s)_{s=0,\ldots,N}$ is a sequence of conditional expectations of a fixed random variable by conditioning on more and more information. (i) is immediate by the definition of β: $\boldsymbol{\beta}_0 = E[B^{\boldsymbol{k}}(\boldsymbol{j})] = E\left[E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k}]\right] = \sum_{k \in K} p^k E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = \beta$. Next, we have

$$\boldsymbol{\beta}_N \equiv E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N] = E\left[E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N, \boldsymbol{k}]\right] = \sum_{k \in K} P[\boldsymbol{k} = k \mid \boldsymbol{g}_N] E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N, \boldsymbol{k} = k]$$

$$= \sum_{k \in K} \boldsymbol{p}_N^k E[B^k(\boldsymbol{j}) \mid \boldsymbol{g}_N] = \sum_{k \in K} \boldsymbol{p}_N^k B^k(\tau_{n+1}(\boldsymbol{g}_N)),$$

the last but one equality following from the fact that, conditional on $\boldsymbol{g}_N$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are independent.[11] The equilibrium condition of player 2 implies that $\boldsymbol{y} = \tau_{n+1}(\boldsymbol{g}_N) \in Y(\boldsymbol{p}_N)$. This completes the proof of the lemma. ∎

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[11] For the last equality, remember that we have extended B^k linearly to mixed actions.

At this stage, we have constructed $(\boldsymbol{p}_s)_{s=0,1,\ldots,N}$ and $(\boldsymbol{\beta}_s)_{s=0,1,\ldots,N}$ that have all the properties required by the theorem. It remains to construct an appropriate sequence of player 1's payoffs, which is more delicate.

Sequence of player 1's vector payoff $(\boldsymbol{a}_s^k)_{s=0,1,\ldots,N}$, $k \in K$. A first definition that could come to mind for the characterization of the sequence of player 1's payoffs is to simply take $E[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_s]$, which is always well defined. However, it is not relevant, in general, for type k (except when $s = N$). To see this, consider a very simple example with one unilateral communication period ($N = 1$), two types of equal probability ($K = \{k_1, k_2\}$, $p^1 = p^2 = 1/2$), and assume that in the first talking period type k_1 sends message m with probability one and type k_2 sends message $m' \neq m$ with probability one. After message m, player 2 chooses action j_1, and after message m' he chooses action j_2. Then, we would have $E[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_0] = (1/2)A^k(j_1) + (1/2)A^k(j_2)$, which is not meaningful for any type k.

A more meaningful definition of k's expected payoff is $E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k]$. Unfortunately, it is not well defined when $P[\boldsymbol{g}_s = g_s \mid \boldsymbol{k} = k] = 0$, and this can happen even when $P[\boldsymbol{g}_s = g_s] > 0$. This can be seen easily in the previous example, where $E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_1 = m', \boldsymbol{k} = k_1]$ is not well defined albeit $P[\boldsymbol{g}_1 = m'] = 1/2 > 0$.

Finally, it is worth noticing that the definition used by Aumann and Hart (2003) does not work in our setup. Indeed, they define the (highest) payoff that player 1 of type k can achieve against player 2's strategy τ after the history $\boldsymbol{g}_s$ as $\sup_{\tilde{\sigma}} E_{\tilde{\sigma}, \tau, p}[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_s]$, where the supremum is over all strategies $\tilde{\sigma}$ of player 1 such that $P_{\tilde{\sigma}, \tau, p}[\boldsymbol{g}_s \mid \boldsymbol{k} = k] > 0$. But this is not necessarily well defined in our setup even when $P[\boldsymbol{g}_s = g_s] > 0$ because a history g_s may contain a message (certificate) that *cannot* be sent by type k (for example, $g_1 = m \notin M(k)$).

Hence, we follow a different, and somehow simpler, approach. For each $k \in K$, we construct the sequence of type k's (virtual) payoff $(\boldsymbol{a}_s^k)_{s=0,1,\ldots,N}$ as follows. Let $\boldsymbol{a}_s^k = a_s^k(\boldsymbol{g}_s)$. When $P[\boldsymbol{g}_s = g_s \mid \boldsymbol{k} = k] > 0$, we define

$$a_s^k(g_s) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k],$$

which is unambiguously type k's expected payoff given the history g_s (and k). Clearly, for $s = 0$, $a_s^k(g_s)$ is always well defined: $a_0^k(g_0) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = a^k$. More generally, assume inductively that $a_s^k(g_s)$ is well defined, i.e., assume that $P[\boldsymbol{g}_s = g_s \mid \boldsymbol{k} = k] > 0$. If $s = 2t - 1$ is odd, then $g_{s+1} = (g_s, m_t^2)$, so $P[\boldsymbol{g}_{s+1} = g_{s+1} \mid \boldsymbol{k} = k] > 0$ when $P[\boldsymbol{m}_t^2 = m_t^2 \mid \boldsymbol{g}_s = g_s] > 0$, which implies that $a_{s+1}^k(g_{s+1})$ remains well defined. If $s = 2t$ is even, then we may have a problem to define $a_{s+1}^k(g_{s+1})$ because now it is player 1's message that is added to the history: $g_{s+1} = (g_s, m_{t+1}^1)$. Indeed, we may have $P[\boldsymbol{m}_{t+1}^1 = m_{t+1}^1 \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k] = \sigma_{t+1}^k(m_{t+1}^1 \mid h_t) = 0$ (even when $P[\boldsymbol{m}_{t+1}^1 = m_{t+1}^1 \mid \boldsymbol{g}_s = g_s] > 0$), so $P[\boldsymbol{g}_{s+1} = g_{s+1} \mid \boldsymbol{k} = k] = 0$. It that

situation, we let

$$a^k_{s+1}(g_s, m^1_{t+1}) = a^k_s(g_s).$$

First, notice that the equilibrium condition of player 1 implies $a^k_s(g_s) = a^k_{s+1}(g_s, m)$ for all m such that $\sigma^k_{t+1}(m \mid g_s) > 0$. Second notice that we will have the same problem in all histories following (g_s, m^1_{t+1}) (they have probability 0 conditional on k), so we fix more generally k's payoff for all these histories: $a^k_{s+l}(g_s, m^1_{t+1}, \ldots) = a^k_s(g_s)$, $l = 1, 2 \ldots$. This construction can be summarized formally as follows. For each $s = 0, \ldots, N$ and $k \in K$ define the random variable $\boldsymbol{f}^k_s$ as the longest subhistory of $\boldsymbol{g}_s$ satisfying $P[\boldsymbol{f}^k_s \mid \boldsymbol{k} = k] > 0$ (notice that this history necessarily ends with player 2's message, or is equal to $\boldsymbol{g}_s$), and let

$$\boldsymbol{a}^k_s = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{f}^k_s, \boldsymbol{k} = k].$$

This definition is equivalent to,

$$\boldsymbol{a}^k_s = \begin{cases} E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k], & \text{if } \boldsymbol{p}^k_s > 0 \\ \boldsymbol{a}^k_{\boldsymbol{r}}, & \text{if } \boldsymbol{p}^k_s = 0, \end{cases}$$

where $\boldsymbol{r}$ is a random variable (stopping time) which is equal to the largest r such that $\boldsymbol{p}^k_r > 0$.

Lemma 3 *For every $k \in K$, the sequence $(\boldsymbol{a}^k_s)_{s=0,\ldots,N}$ is a (bounded) martingale satisfying*

(i) $\boldsymbol{a}^k_0 = a^k$;

(ii) $\boldsymbol{a}^k_{s+1} = \boldsymbol{a}^k_s$ for all even s;

(iii) If $\boldsymbol{p}^k_N > 0$, then $\boldsymbol{a}^k_N = A^k(\boldsymbol{y})$, with $\boldsymbol{y} \in Y(\boldsymbol{p}_N)$.

Proof. To prove the martingale property we must show that $E[\boldsymbol{a}^k_{s+1} \mid \boldsymbol{g}_s] = \boldsymbol{a}^k_s$, for all $s = 0, 1, \ldots, N$. If $\boldsymbol{p}^k_{s+1} = 0$, then this property is immediate because by construction we have $\boldsymbol{a}^k_{s+1} = \boldsymbol{a}^k_s = \boldsymbol{a}^k_{\boldsymbol{r}}$, where $r \leq s$ is the largest number such that $\boldsymbol{p}^k_r > 0$. Now, consider the case $\boldsymbol{p}^k_{s+1} > 0$, and let $s = 2t - 1$ be odd (when s is even, the martingale property will follow from (ii)). Thus, $\boldsymbol{p}^k_s > 0$ and $\boldsymbol{g}_{s+1} = (\boldsymbol{g}_s, \boldsymbol{m}^2_t)$, which implies

$$\begin{cases} \boldsymbol{a}^k_{s+1} = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^2_t, \boldsymbol{k} = k] \\ \boldsymbol{a}^k_s = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k]. \end{cases}$$

So,

$$E[\boldsymbol{a}^k_{s+1} \mid \boldsymbol{g}_s] = \sum_{m \in \mathrm{supp}[\tau_t(\boldsymbol{g}_s)]} P[\boldsymbol{m}^2_t = m \mid \boldsymbol{g}_s] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^2_t = m, \boldsymbol{k} = k]$$

$$= \sum_{m \in \mathrm{supp}[\tau_t(\boldsymbol{g}_s)]} P[\boldsymbol{m}^2_t = m \mid \boldsymbol{g}_s, \boldsymbol{k} = k] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^2_t = m, \boldsymbol{k} = k]$$

$$= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k] = \boldsymbol{a}^k_s,$$

the second equality following from the fact that $\boldsymbol{m}^2_t$ and $\boldsymbol{k}$ are independent conditional on $\boldsymbol{g}_s$. This proves the martingale property for all odd s. Property (i) is immediate: $\boldsymbol{a}^k_0 = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = a^k$ by the definition of a^k. To prove (ii) let $s = 2t$ be even, so $\boldsymbol{g}_{s+1} = (\boldsymbol{g}_s, \boldsymbol{m}^1_{t+1})$. As before, when $\boldsymbol{p}^k_{s+1} = 0$ the property is immediate because $\boldsymbol{a}^k_{s+1} = \boldsymbol{a}^k_s = \boldsymbol{a}^k_r$, with $r \le s$. When $\boldsymbol{p}^k_{s+1} > 0$, then $\boldsymbol{p}^k_s > 0$ and $\boldsymbol{g}_{s+1} = (\boldsymbol{g}_s, \boldsymbol{m}^1_{t+1})$, so

$$\begin{cases} \boldsymbol{a}^k_{s+1} = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^1_{t+1}, \boldsymbol{k} = k] \\ \boldsymbol{a}^k_s = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{k} = k]. \end{cases}$$

In such a situation these two terms are equal by the equilibrium condition of player 1 since every message m^1_{t+1} player 1 of type k sends with strictly positive probability given $\boldsymbol{g}_s$ (and $\boldsymbol{k} = k$) should yield the same expected payoff to player 1 of type k:

$$\boldsymbol{a}^k_s = \sum_{m \in \mathrm{supp}[\sigma^k_{t+1}(\boldsymbol{g}_s)]} P[\boldsymbol{m}^1_{t+1} = m \mid \boldsymbol{g}_s, \boldsymbol{k} = k] E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^1_{t+1} = m, \boldsymbol{k} = k]$$

$$= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s, \boldsymbol{m}^1_{t+1} = m, \boldsymbol{k} = k], \quad \text{for all } m \in \mathrm{supp}[\sigma^k_{t+1}(\boldsymbol{g}_s)]$$

$$= \boldsymbol{a}^k_{s+1}.$$

Finally, to prove (iii), assume that $\boldsymbol{p}^k_N > 0$, so

$$\boldsymbol{a}^k_N = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_N, \boldsymbol{k} = k] = E[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_N]$$

$$= A^k(\tau_{n+1}(\boldsymbol{g}_N)) = A^k(\boldsymbol{y}), \text{ with } \boldsymbol{y} = \tau_{n+1}(\boldsymbol{g}_N) \in Y(\boldsymbol{p}_N),$$

the second equality following from the fact that $\boldsymbol{j}$ and $\boldsymbol{k}$ are independent conditional on $\boldsymbol{g}_N$, and the last from the equilibrium condition of player 2. ∎

Lemma 4 *For every $s = 0, 1, \ldots, N$ we have $\boldsymbol{a}_s \in \mathrm{INTIR}_{\mathrm{supp}[\boldsymbol{p}_s]}$.*

Proof. Let us fix a history g_s such that $P[\boldsymbol{g}_s = g_s] > 0$ and let $\mathrm{supp}[p_s] \subseteq K$, $\mathrm{supp}[p_s] \ne \emptyset$, be the set of types with a strictly positive posterior probability: $p^k_s = P[\boldsymbol{k} = k \mid \boldsymbol{g}_s = g_s] > 0$

for all $k \in \mathrm{supp}[p_s]$. We must show that there exists $\overline{y} \in \Delta(J)$ such that

$$E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k] \geq A^k(\overline{y}), \text{ for all } k \in \mathrm{supp}[p_s].$$

Player 1's equilibrium condition implies that, whatever his type $k \in \mathrm{supp}[p_s]$, if he sends the same message $\overline{m} \in \bigcap_{k \in K} M(k)$ in all upcoming periods $t' \geq \overline{t}$ (where $\overline{t} = (s+2)/2$ if s is even, and $\overline{t} = (s+3)/2$ if s is odd), then his expected payoff in the current period ($s/2$ if s is even, and $(s+1)/2$ if s is odd) is not increased, so

$$E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k] \geq E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = \overline{m} \ \forall \ t' \geq \overline{t}, \boldsymbol{k} = k], \text{ for all } k \in \mathrm{supp}[p_s].$$

The right hand side only depends on player 2's strategy and is thus well defined. As a consequence, given $\boldsymbol{g}_s = g_s$ and $\boldsymbol{m}_{t'}^1 = \overline{m} \ \forall \ t' \geq \overline{t}$, which specifies the sequence of all player 1's messages in the talking phase, $\boldsymbol{j}$ and $\boldsymbol{k}$ are independent. This implies

$$E\left[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = \overline{m} \ \forall \ t' \geq \overline{t}, \boldsymbol{k} = k\right] = E\left[A^k(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = \overline{m} \ \forall \ t' \geq \overline{t}\right]$$
$$= A^k\left(E\left[\tau_{n+1}(\boldsymbol{g}_N) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = \overline{m} \ \forall \ t' \geq \overline{t}\right]\right).$$

(Remember that we have extended linearly A^k to mixed actions.) Hence, by letting

$$\overline{y} = E\left[\tau_{n+1}(\boldsymbol{g}_N) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = \overline{m} \ \forall \ t' \geq \overline{t}\right],$$

which does not depend on k (it only depends on g_s and $\overline{m}$), we have completed the proof of the lemma. ∎

As we have already mentioned, $(\boldsymbol{p}_s)_{s=0,1,\ldots,N}$ and $(\boldsymbol{\beta}_s)_{s=0,1,\ldots,N}$ have all the properties required by Theorem 2 (see Lemma 1 and Lemma 2). By Lemma 3 and Lemma 4, the sequence $(\boldsymbol{a}_s)_{s=0,1,\ldots,N}$ also satisfies all the properties of the theorem. ∎

A.2.2 From constrained dimartingale to equilibrium: $H_B(p) \subseteq \mathcal{E}_B(p)$

Let $\boldsymbol{z} = (\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_N)$ be a martingale over some probability space $(F, \mathcal{F}, \pi)$ and (finite) sub σ-fields $(\mathcal{F}_t)_{t=1,\ldots,N}$, satisfying the properties of Theorem 2, with $p^k > 0$ for all $k \in K$, and $N = n$. We construct a Nash equilibrium (σ, τ) of the n-stage communication game $\Gamma_n(p)$ with expected payoffs (a, β). First, for convenience we define the martingale $\boldsymbol{z}$ on the nodes of a probability tree. We introduce a set W with $K+1$ elements, write F as W^N, and the atoms of $\mathcal{F}_t$ as elements g_t of W^t. We thus describe the martingale $\boldsymbol{z}$ as

$$\boldsymbol{z} = (z_t(\boldsymbol{g}_t))_{t=0,1,\ldots,n},$$

where for each $t = 0, 1, \ldots, n$, $\boldsymbol{g}_t \in W^t$, and

$$z_t(g_t) = (a_t(g_t), \beta_t(g_t), p_t(g_t)) = \sum_{w \in \mathrm{supp}[\pi(\cdot|g_t)]} \pi(w \mid g_t)\, z_{t+1}(g_t, w),$$

for all $g_t \in W^t$ satisfying $\pi(g_t) > 0$ (this is the martingale property). Notice that this implies $E[\boldsymbol{z}_t] = E[z_t(\boldsymbol{g}_t)] = \sum_{g_t \in W^t} \pi(g_t)\, z_t(g_t) = z_0$, $t = 0, 1, \ldots, n$. The properties of the martingale in Theorem 2 can be restated as follows:

(D1) $z_0(g_0) = z_0 = (a, \beta, p)$.

(D2) If $\pi(g_n) > 0$, then $(a_n(g_n), \beta_n(g_n)) \in \mathcal{E}(p_n(g_n))$.

(D3) $a_{t+1}(g_{t+1}) = a_t(g_t)$ for all even t and $p_{t+1}(g_{t+1}) = p_t(g_t)$ for all odd t, if $\pi(g_{t+1}) > 0$.

The interim individual rationality conditions for player 1 are restated as: for all $t = 0, 1, \ldots, n$, if $\pi(g_t) > 0$, then $a_t(g_t) \in \mathrm{INTIR}_{\mathrm{supp}[p_t(g_t)]}$.

In odd periods t, w_t is associated to a message $m_t^1 \in M^1$ of player 1 (player 2's message does not affect players' decisions at these periods), and in even periods t, w_t is directly associated to a jointly controlled lottery (possibly a series of jointly controlled lotteries), which is not explicitly formalized here.[12] Therefore, a history of messages h_n consists, with some abuse of notation, in a message $m_t^1 \in M^1$ of player 1 in each odd period t, and in a realization $w_t \in W$ of one or several jointly controlled lotteries in each even period t. Accordingly, in the remaining of the proof we only construct explicitly player 1's strategy σ_{t+1}^k, $k \in K$, when t is even, and player 2's strategy in the action phase, τ_{n+1}. The set of histories of the talking phase up to period t is

$$M_t = \begin{cases} (M^1 \times W)^{t/2} & \text{if } t \text{ is even,} \\ (M^1 \times W)^{(t-1)/2} \times W & \text{if } t \text{ is odd.} \end{cases}$$

To each sequence $g_t = (w_1, \ldots, w_t) \in W^t$ such that $\pi(g_t) > 0$ we associate a history $\phi_t(g_t) \in M_t$, with $\phi_t(g_t) \neq \phi_t(g_t')$ whenever $g_t \neq g_t'$, as follows:

$$\phi_t(g_t) = \phi_t(w_1, w_2, w_3, w_4 \ldots, w_t) = (m_1(w_1), w_2, m_3(g_3), w_4, \ldots),$$

where $g_r = (w_1, \ldots, w_r)$, $r < t$, is a subsequence of g_t, and for all odd t, $m_t(g_t) \in M^1$, $m_t(g_{t-1}, w_t) \neq m_t(g_{t-1}, w_t')$ whenever $w_t \neq w_t'$, and

$$M^{-1}(m_t(g_t)) = \mathrm{supp}[p_t(g_t)].$$

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[12]The technique is standard; see, e.g., Aumann and Maschler (1995) and Aumann and Hart (2003). Note that irrational probabilities might lead to infinitely many jointly controlled lotteries (recall Remark 3). For simplicity, the reader may simply consider w_t as a signal publicly observed in even periods.

Player 1's strategy σ. For each even period $t = 0, 2, 4, \ldots$, each sequence $g_t \in W^t$ with strictly positive probability and each type $k \in \text{supp}[p_t(g_t)]$ we construct player 1's local strategy $\sigma_{t+1}^k(\phi_t(g_t))$. For each $w \in \text{supp}[\pi(\cdot \mid g_t)]$, define

$$\sigma_{t+1}^k(m_{t+1}(g_t, w) \mid \phi_t(g_t)) = \frac{\pi(w \mid g_t) \, p_{t+1}^k(g_t, w)}{p_t^k(g_t)},$$

and $\sigma_{t+1}^k(m \mid \phi_t(g_t)) = 0$ if $m \neq m_{t+1}(g_t, w)$ for all $w \in W$.

Player 2's strategy τ. We construct the local strategy $\tau_{n+1}(h_n)$ of player 2 for each final history of talk $h_n \in M_n$, with *and* without strictly positive probability (players' strategies in the talking phase are irrelevant off the equilibrium path, but player 2's strategy in the action phase is very important even after 0-probability histories).

If $h_n = \phi_n(g_n)$ for some $g_n \in W^n$ such that $\pi(g_n) > 0$, then by the second property of the martingale assumed in the theorem, $(a_n(g_n), \beta_n(g_n)) \in \mathcal{E}(p_n(g_n))$, so we can define,

$$y(g_n) = \tau_{n+1}(h_n) \in Y(p_n(g_n)) \ \ \text{such that} \ \ \begin{cases} a_n^k(g_n) = A^k(y(g_n)) \text{ if } p_n^k(g_n) > 0 \\ \beta_n(g_n) = \sum_{k \in K} p_n^k(g_n) \, B^k(y(g_n)). \end{cases}$$

Otherwise, if $h_n \neq \phi_n(g_n)$ for all $g_n \in W^n$ such that $\pi(g_n) > 0$, then consider the longest subsequence $g_t = (w_1, w_2, \ldots, w_t)$ such that $h_t = \phi_t(g_t)$ and $\pi(g_t) > 0$ (note: t may be 0) and define

$$\tau_{n+1}(h_n) = \overline{y} \ \ \text{such that} \ \ a_t^k(g_t) \geq A^k(\overline{y}) \ \ \text{for all } k \in \text{supp}[p_t(g_t)].$$

This is possible by the individual rationality conditions of the martingale.

Next, we check that (σ, τ) generates the appropriate expected payoffs and constitutes a Nash equilibrium of $\Gamma_n(p)$. Let $P = P_{\sigma, \tau, p}$ be the probability distribution on $\Omega = K \times M_n \times J$ induced by (σ, τ) and p, and let $E = E_{\sigma, \tau, p}$ be the corresponding expectation operator.[13]

Lemma 5 *For all $t = 0, 1, \ldots, n$ and $g_t \in W^t$, $\pi(g_t) > 0$, we have:*

(i) $P[\boldsymbol{h}_t = \phi_t(g_t)] = \pi(g_t)$;

(ii) $P[\boldsymbol{k} = k \mid \boldsymbol{h}_t = \phi_t(g_t)] = p_t^k(g_t)$ *for all $k \in K$.*

Proof. By induction on t. For $t = 0$ property (ii) is immediate: $P[\boldsymbol{k} = k] = p^k = p_0^k(g_0)$. For $t = 1$:

---

[13]Since JCL are not formalized, P and E also depend on π for the realizations $w_t \in W$ of JCL (public signals) in even periods.

(i) $P[\boldsymbol{h}_1 = \phi_1(g_1)] = \sum_{k \in K} p^k P[\boldsymbol{h}_1 = \phi_1(g_1) \mid \boldsymbol{k} = k] = \sum_{k \in K} p^k \sigma_1^k(\phi_1(g_1))$

$$= \sum_{k \in K} p^k \sigma_1^k(m_1(g_1)) = \sum_{k \in K} p^k \frac{\pi(g_1)\, p_1^k(g_1)}{p^k}$$

$$= \pi(g_1) \sum_{k \in K} p_1^k(g_1) = \pi(g_1).$$

(ii) $P[\boldsymbol{k} = k \mid \boldsymbol{h}_1 = \phi_1(g_1)] = \dfrac{P[\boldsymbol{h}_1 = \phi_1(g_1) \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{P[\boldsymbol{h}_1 = \phi_1(g_1)]}$

$$= \frac{\sigma_1^k(m_1(g_1))\, p^k}{P[\boldsymbol{h}_1 = \phi_1(g_1)]} = \frac{\sigma_1^k(m_1(g_1))\, p^k}{\pi(g_1)} \quad \text{by (i) just above}$$

$$= \frac{\pi(g_1)\, p_1^k(g_1)}{p_0^k} \frac{p^k}{\pi(g_1)} = p_1^k(g_1).$$

Now assume that properties (i) and (ii) are satisfied at t, and let us check them at $t + 1$. We distinguish two cases: (a) t is odd, i.e., a JCL is added in $t+1$; (b) t is even, i.e., player 1's signal is added in $t + 1$. Case (a) is simpler because we can exploit the fact that the JCL does not depend on k. In the rest of the proof of the lemma, let $g_{t+1} = (g_t, w_{t+1}) \in W^{t+1}$.

(a) (i) Since $t + 1$ is even we have:

$$P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] = P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w_{t+1})]$$

$$= P[\boldsymbol{h}_t = \phi_t(g_t)]\, P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w_{t+1}) \mid \boldsymbol{h}_t = \phi_t(g_t)]$$

$$= \pi(g_t)\, \pi(w_{t+1} \mid g_t), \quad \text{by property (i) at } t$$

$$= \pi(g_t, w_{t+1}) = \pi(g_{t+1}).$$

(a) (ii) Since $t + 1$ is even we have:

$$P[\boldsymbol{k} = k \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] = P[\boldsymbol{k} = k \mid \boldsymbol{h}_{t+1} = (\phi_t(g_t), w_{t+1})]$$

$$= P[\boldsymbol{k} = k \mid \boldsymbol{h}_t = \phi_t(g_t)] \text{ because } w_{t+1} \text{ and } k \text{ are independent}$$

$$= p_t^k(g_t) \text{ by property (ii) at } t$$

$$= p_{t+1}^k(g_{t+1}) \text{ by the third property of the martingale.}$$

(b) (i) Since $t + 1$ is odd we have:

$$
\begin{aligned}
P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] &= P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), m_{t+1}(g_{t+1})] \\
&= P[\boldsymbol{h}_t = \phi_t(g_t)] \, P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), m_{t+1}(g_{t+1})) \mid \boldsymbol{h}_t = \phi_t(g_t)] \\
&= \pi(g_t) \, P[\boldsymbol{m}_{t+1} = m_{t+1}(g_{t+1}) \mid \boldsymbol{h}_t = \phi_t(g_t)], \quad \text{by property (i) at } t \\
&= \pi(g_t) \sum_{k \in K} p_t^k(g_t) \, \sigma_{t+1}^k(m_{t+1}(g_{t+1}) \mid \phi_t(g_t)) \\
&= \pi(g_t) \sum_{k \in K} p_t^k(g_t) \frac{\pi(w_{t+1} \mid g_t) \, p_{t+1}^k(g_{t+1})}{p_t^k(g_t)} \\
&= \pi(g_t) \, \pi(w_{t+1} \mid g_t) = \pi(g_t, w_{t+1}) = \pi(g_{t+1}).
\end{aligned}
$$

(b) (ii) Since $t + 1$ is odd we have:

$$
\begin{aligned}
P[\boldsymbol{k} = k \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] &= \frac{P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1}) \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})]} \\
&= \frac{P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1}) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k] P[\boldsymbol{h}_t = \phi_t(g_t) \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{P[\boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})]} \\
&= \frac{P[\boldsymbol{m}_{t+1} = m_{t+1}(g_{t+1}) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k] P[\boldsymbol{h}_t = \phi_t(g_t) \mid \boldsymbol{k} = k] P[\boldsymbol{k} = k]}{\pi(g_{t+1})} \\
&= \frac{\sigma_{t+1}^k(m_{t+1}(g_{t+1}) \mid \phi_t(g_t)) P[\boldsymbol{h}_t = \phi_t(g_t)] P[\boldsymbol{k} = k \mid \boldsymbol{h}_t = \phi_t(g_t)]}{\pi(g_{t+1})},
\end{aligned}
$$

the last but one equality following from property (i) at $t + 1$, which has been checked just before. By properties (i) and (ii) at t this yields:

$$
\begin{aligned}
P[\boldsymbol{k} = k \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_{t+1})] &= \frac{\sigma_{t+1}^k(m_{t+1}(g_{t+1}) \mid \phi_t(g_t)) \pi(g_t) p_t^k(g_t)}{\pi(g_{t+1})} \\
&= \frac{\pi(w_{t+1} \mid g_t) \, p_{t+1}^k(g_{t+1})}{p_t^k(g_t)} \frac{p_t^k(g_t) \pi(g_t)}{\pi(g_{t+1})} = p_{t+1}^k(g_{t+1}).
\end{aligned}
$$

This completes the proof of Lemma 5. ∎

Lemma 6 *We have:*

(i) $E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] = a^k$ *for all* $k \in K$;

(ii) $E[B^{\boldsymbol{k}}(\boldsymbol{j})] = \beta$.

Proof. (i) We show by induction on t (starting from $t = n$) that, for $t = 0, 1, \ldots, n$,

$$
a_t^k(g_t) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k], \quad \forall \, k \in \text{supp}[p_t(g_t)]. \tag{4}
$$

In particular, for $t = 0$, this will lead to what we are required to prove:

$$a^k = a_0^k(g_0) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_0 = \phi_0(g_0), \boldsymbol{k} = k] = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k].$$

Let $t = n$. If $k \in \mathrm{supp}[p_n(g_n)]$, then, by the construction of player 2's strategy,

$$a_n^k(g_n) = A^k(\tau_{n+1}(\phi_n(g_n))) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_n = \phi_n(g_n), \boldsymbol{k} = k],$$

so property (4) is satisfied for $t = n$. Now assume that the property is satisfied at $t + 1$ and let us check it at t. Let $k \in \mathrm{supp}[p_t(g_t)]$. By the martingale property, we have

$$a_t^k(g_t) = \sum_{w \in \mathrm{supp}[\pi(\cdot|g_t)]} \pi(w \mid g_t) \, a_{t+1}^k(g_t, w).$$

We distinguish two cases: when t is odd and when t is even.

If t is odd. Then, $p_{t+1}(g_t, w) = p_t(g_t)$ for all $w \in \mathrm{supp}[\pi(\cdot \mid g_t)]$, which implies $\mathrm{supp}[p_{t+1}(g_t, w)] = \mathrm{supp}[p_t(g_t)]$, so $k \in \mathrm{supp}[p_{t+1}(g_t, w)]$ for all $w \in \mathrm{supp}[\pi(\cdot \mid g_t)]$. Therefore, by the induction hypothesis, for all $w \in \mathrm{supp}[\pi(\cdot \mid g_t)]$ we have

$$a_{t+1}^k(g_t, w) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k],$$

so $a_t^k(g_t) =$

$$\sum_{w \in \mathrm{supp}[\pi(\cdot|g_t)]} \pi(w \mid g_t) \, E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k]$$

$$= \sum_{w \in \mathrm{supp}[\pi(\cdot|g_t)]} P[\boldsymbol{h}_{t+1} = (\phi_t(g_t), w) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k] \, E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k]$$

$$= E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_t = \phi_t(g_t), \boldsymbol{k} = k].$$

If t is even. Then, $a_{t+1}^k(g_t, w) = a_t^k(g_t)$ for all $w \in \mathrm{supp}[\pi(\cdot \mid g_t)]$, which implies, by the induction hypothesis,

$$a_t^k(g_t) = E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{h}_{t+1} = \phi_{t+1}(g_t, w), \boldsymbol{k} = k],$$

for all w such that $p_{t+1}^k(g_t, w) > 0$. Hence, $a_t^k(g_t)$ is also equal to any average of the previous value, so we get property (4) at t.

(ii) Player 2's expected payoff is

$$
\begin{aligned}
E[B^{\boldsymbol{k}}(\boldsymbol{j})] &= \sum_{k \in K} p^k E[B^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{k} = k] \\
&= \sum_{k \in K} p^k \sum_{h_n \in M_n} P[\boldsymbol{h}_n = h_n \mid \boldsymbol{k} = k] \, E[B^k(\boldsymbol{j}) \mid \boldsymbol{k} = k, \boldsymbol{h}_n = h_n] \\
&= \sum_{k \in K} p^k \sum_{h_n \in M_n} P[\boldsymbol{h}_n = h_n \mid \boldsymbol{k} = k] \, B^k(\tau_{n+1}(h_n)) \\
&= \sum_{h_n \in M_n} P[\boldsymbol{h}_n = h_n] \sum_{k \in K} P[\boldsymbol{k} = k \mid \boldsymbol{h}_n = h_n] \, B^k(\tau_{n+1}(h_n)) \\
&= \sum_{g_n \in W^n} \pi(g_n) \sum_{k \in K} p_n^k(g_n) \, B^k(\tau_{n+1}(\phi_n(g_n))), \text{ by Lemma 5} \\
&= \sum_{g_n \in W^n} \pi(g_n)\beta_n(g_n), \text{ by the construction of player 2's strategy} \\
&= E[\boldsymbol{\beta}_n] = \beta_0 = \beta.
\end{aligned}
$$

This completes the proof of Lemma 6. ∎

Lemma 7 *The strategy τ of player 2 is a best reply to the strategy σ of player 1 in the n-stage communication game $\Gamma_n(p)$.*

Proof. Since $\tau_{n+1}(\phi_n(g_n)) \in Y(p_n(g_n))$ for $\pi(g_n) > 0$ it suffices to check that $p_n^k(g_n) = P[\boldsymbol{k} = k \mid \boldsymbol{h}_n = \phi_n(g_n)]$ for all $k \in K$. This has been proved in Lemma 5 (property (ii) with $t = n$). ∎

Lemma 8 *The strategy σ of player 1 is a best reply to the strategy τ of player 2 in the n-stage communication game $\Gamma_n(p)$.*

Proof. Fix t even, g_t such that $\pi(g_t) > 0$ and w such that $\pi(w \mid g_t) > 0$. Assume that player 1's type k is such that $p_t^k(g_t) > 0$. The strategy σ prescribes to send message $m_{t+1}(g_t, w)$ with probability $\sigma_{t+1}^k(m_{t+1}(g_t, w) \mid \phi_t(g_t)) > 0$ and any message which is not of the form $m_{t+1}(g_t, w)$ with probability 0. By construction, player 1 of type k is not able to send a message m of the form $m_{t+1}(g_t, w')$ with $p_{t+1}^k(g_t, w') = 0$, namely a message m that is sent along the equilibrium path but is not sent by type k. Furthermore, given the local strategy τ_{n+1} of player 2 constructed before, by the interim individually rational condition player 1 cannot profit from sending a message m off the equilibrium path, namely a message m not of the form $m_{t+1}(g_t, w')$. Finally, if from stage $t + 2$ on, player 1 follows the prescribed strategy σ, he cannot gain at stage $t + 1$ by sending $m_{t+1}(g_t, w)$ with a probability different from $\sigma_{t+1}^k(m_{t+1}(g_t, w) \mid \phi_t(g_t))$. Indeed, by the dimartingale property (D3) on page 35 and property (4) on page 38, he is indifferent between all the allowed messages. Hence,

by an induction argument, player 1 cannot gain by manipulating the probabilities of allowed messages. ∎

By Lemmas 6, 7 and 8, we have constructed the appropriate strategy profile. ∎

A.3   Proof of Theorem 3

We only give the proof for bilateral persuasion games. The unilateral case is similar and simpler.

A.3.1   From perfect Bayesian equilibrium to constrained dimartingale

The proof is as in Subsection A.2.1 except that we start from a PBE $((\sigma, \tau), \mu)$ and have to prove a stronger version of Lemma 4, namely $\boldsymbol{a}_s \in \text{INTIR}^{\text{PBE}}_{\text{supp}[\boldsymbol{p}_s]}$ for all $s = 0, 1, \dots, N$.

Fix any history g_s such that $P[\boldsymbol{g}_s = g_s] > 0$, any set of types $X \subseteq \text{supp}[p_s]$, where $p_s^k = P[\boldsymbol{k} = k \mid \boldsymbol{g}_s = g_s]$ for all $k \in K$, and assume that $s = 2t$ is even (the proof is similar when s is odd). We have to show that there exists $p_X \in \Delta(X)$ and $y_X \in Y(p_X)$ such that

$$E[A^{\boldsymbol{k}}(\boldsymbol{j}) \mid \boldsymbol{g}_s = g_s, \boldsymbol{k} = k] \geq A^k(y_X), \text{ for all } k \in X. \tag{5}$$

Let $m_X \in M^1$ be such that $M^{-1}(m_X) = X$, and consider any final communication history in which player 1 sends the message m_X in all periods $t' \geq t$ after history g_s. By belief consistency condition (d) of Definition 1, we have

$$\mu(g_s, m_X, m_{t+1}^2, m_X, m_{t+2}^2, \dots, m_X, m_n^2) \in \Delta(X),$$

for any sequence of messages $m_{t+1}^2 \in M^2$, $m_{t+2}^2 \in M^2$, ... Furthermore, by belief consistency condition (e) the above belief does not depend on the sequence of messages m_{t+1}^2, m_{t+2}^2, ... sent by player 2, so it is a constant, denoted by p_X, for any such sequence. Hence, by the sequential rationality condition (c) for player 2 we get

$$\tau_{n+1}(g_s, m_X, m_{t+1}^2, m_X, m_{t+2}^2, \dots, m_X, m_n^2) \in Y(p_X),$$

for all sequences of messages m_{t+1}^2, m_{t+2}^2, ... of player 2. Finally, the sequential rationality condition (b) for player 1 implies that inequality (5) is satisfied by letting

$$y_X \equiv E\left[\tau_{n+1}(\boldsymbol{g}_N) \mid \boldsymbol{g}_s = g_s, \boldsymbol{m}_{t'}^1 = m_X \ \forall \ t' \geq t+1\right] \in Y(p_X).$$

Proceeding in the same way for all $X \subseteq \text{supp}[p_s]$ we get $\boldsymbol{a}_s \in \text{INTIR}^{\text{PBE}}_{\text{supp}[\boldsymbol{p}_s]}$. ∎

A.3.2 From constrained dimartingale to perfect Bayesian equilibrium

Again, the proof follows the same lines as in Subsection A.2.2, except for the construction of player 2's strategy in the action phase. If $h_n = \phi_n(g_n)$ for some $g_n \in W^n$ such that $\pi(g_n) > 0$, $\tau_{n+1}(h_n)$ is defined as in the Nash equilibrium case, and player 2's belief is simply defined from Bayes' rule, $\mu(k \mid h_n) = p_n^k(g_n)$. Otherwise, if $h_n \neq \phi_n(g_n)$ for all $g_n \in W^n$ such that $\pi(g_n) > 0$, let $g_t = (w_1, w_2, \ldots, w_t)$ be the longest subsequence such that $h_t = \phi_t(g_t)$ and $\pi(g_t) > 0$. Since the martingale we start with satisfies $a_t(g_t) \in \text{INTIR}_{\text{supp}[p_t(g_t)]}^{\text{PBE}}$ when $\pi(g_t) > 0$, we can define

$$\tau_{n+1}(h_n) = y_X \in Y(p_X) \text{ such that } a_t^k(g_t) \geq A^k(y_X) \text{ for all } k \in X,$$

and $\mu(k \mid h_n) = p_X$ for some $p_X \in \Delta(X)$, where $X = M_n^{-1}(h_n) \subseteq \text{supp}[p_t(g_t)]$.

It remains to show the analogs of Lemma 7 and Lemma 8 for sequential rationality. Given the strategy σ of player 1 and since the belief function constructed above does not depend on player 2's messages, player 2's strategy in the communication phase, τ_t for $t \leq n$, has no impact on the outcome of the game. In the action phase, τ_{n+1} is sequentially rational given μ since by construction $\tau_{n+1}(h_n) \in Y(\mu(h_n))$ for any final communication history $h_n \in M_n$, along and off the equilibrium path.

Finally, let $h_t \in M_t$, t even, be some history such that $h_t = \phi_t(g_t)$ for some $g_t \in W^t$ with $\pi(g_t) > 0$. Player 1 does not deviate in period $t+1$ to messages along the equilibrium path by the same argument as in the Nash equilibrium case. If player 1 deviates in period $t+1$ to a message off the equilibrium path, $m_{t+1}^1 \neq m_{t+1}(g_t, w')$ for all $w' \in W$, his payoff becomes

$$A^k(\tau_{n+1}(h_t, m_{t+1}^1, w_{t+2}, m_{t+3}^1, w_{t+4}, \ldots)) = A^k(y_X) \leq a_t^k(g_t),$$

where $w_{t+2}, m_{t+3}^1, w_{t+4}, \ldots$ is some arbitrary sequence of messages, $a_t^k(g_t)$ is the payoff he gets when he does not deviate, and $X = M^{-1}(h_t, m_{t+1}^1, w_{t+2}, m_{t+3}^1, w_{t+4}, \ldots)$.

We have not specified what player 1 should do off the equilibrium path, for histories $h_t \neq \phi_t(g_t)$ for all $g_t \in W^t$, $\pi(g_t) > 0$. But the argument above for player 1 not to deviate does not depend on how player 1 behaves after period $t+1$ if he deviates to a message off the equilibrium path in period $t+1$, so it applies for any local strategy $\sigma_{t'}(h_t, m_{t+1}^1, w_{t+2}, m_{t+3}^1, \ldots)$, $t' \geq t+2$, sequentially rational or not. ∎

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