Investment and Returns

Vito D. Gala*

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Abstract

This paper constructs a general equilibrium production economy with heterogeneous firms and irreversible investment that rationalizes the value premium. Firm investments play a central role in explaining the cross-sectional variation of stock returns. Profitable and fast growing "growth" firms have low expected returns because they provide "consumption insurance" to investors, especially in bad times. Countercyclical consumption volatility generates a larger value premium during recessions. Large firms grow more slowly, so the value premium is larger for small stocks. The model can replicate the failure of the unconditional CAPM and the relative success of the conditional CAPM and Fama and French (1993) factor model.

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1 Introduction

"Value" stocks with high book-to-market ratios have earned higher average returns than "growth" stocks with low book-to-market ratios (Fama and French (1992)). The Capital Asset Pricing Model (CAPM) fails to explain this pattern in average returns, as market betas have if anything an opposite pattern. These facts have motivated a large number of empirically successful asset pricing models that extend the CAPM in various ways (see Cochrane (2005) for a detailed survey). Despite their empirical success, however, little is known about the economic mechanism that generates risk, returns and firm characteristics from the underlying nature of investors' preferences and firms' technologies.

I develop a general equilibrium model that produces value and size effects. Firms are subject to aggregate and firm-specific productivity shocks. The firm-specific productivity shocks generate cross-sectional variation in firm characteristics such as market equity, book-to-market, investment and capital. Firms face adjustment costs and irreversibility in investment. The adjustment cost is lower for firms with low capital relative to the average firm capital. This specification implies that smaller firms invest more and grow faster, ceteris paribus, an assumption I verify empirically in Gala (2005b). This property makes the firm marginal and average q different, thus creating expected profitability and size effects in both firm investment and returns. Investor preferences are simple power utility. The model aggregates so that a single moment of the joint cross-sectional distribution of firm-specific productivity and capital is a sufficient state variable for aggregate quantities including the market risk premium.

I investigate the properties of the model through its analytic solution, and I simulate a calibrated version to study the model's ability to match facts in the data both quantitatively and qualitatively. The model captures the familiar features of the data. First, value firms with high book-to-market ratios and small firms with low market equity have higher average stock returns, and the value effect relating book-to-market ratios with returns is weaker for large firms. Second, the unconditional CAPM fails completely to capture this variation in average returns. Third, multifactor models such as the Fama-French (1993) three factor model and a conditional CAPM do capture the pattern of expected returns.

Most importantly, I relate patterns in the cross section of stock returns to the real side of the economy. In my model there are two aggregate state variables, the aggregate productivity shock and the state variable describing the cross-sectional distribution of capital stocks. Firms are then completely characterized by two additional variables, their firm-specific productivity shock and their level of capital. Thus, all firm-level variables including returns, book-to-market ratios, market value, investment and profitability are functions of firm-specific productivity and capital. As a result, I can characterize correlations between these observable variables. I find in the model and confirm in the data that firms with low book-to-market ratios have persistently high profitability and investment rates to go with their low stock returns. I also find that firms with low investment rates and small capital earn on average higher stock returns. These variables capture expected profitability and size effects similar to those captured by book-to-market and market equity. Expected profitability and size effects arise because at the firm-level marginal q differs from average q.

Risk premia derive ultimately from covariation with consumption. Investors bid up the prices of firms whose returns offer consumption insurance, and require a lower premium to hold their stocks.

A firm's ability to provide consumption insurance depends on its' ability to mitigate aggregate productivity shocks through investment in order to smooth dividends. Capital adjustment costs and irreversibility are the main impediments to such smoothing. During bad times – after a sequence of poor aggregate productivity shocks – "value" firms are more likely to face a binding investment irreversibility constraint. The firms are unprofitable; they would like to disinvest and sell off their capital stocks, but they cannot do so. If there is a further negative aggregate productivity shock, there is nothing they can do to mitigate a further decline in output and dividend. In contrast, growth firms are investing because they have persistently high profitability. They only face adjustment costs, smaller for small firms, to doing so. In the face of a negative aggregate productivity shock, they can easily lower investment and maintain their dividend in this high marginal utility state. Thus, the dividends of growth firms, and especially of small growth firms, will fall less than those of value firms when there is a fall in aggregate productivity. Value firms are riskier, and this difference is larger among small firms.

In good times, value firms are less likely to face a binding irreversibility constraint, so this difference between value and growth firm's ability to provide a smooth dividend stream is smaller in good times. Furthermore, the market price of aggregate productivity risk also rises in bad times. With more firms up against the irreversibility constraint, aggregate consumption growth becomes more volatile. Therefore the greater risk of value firms shows up in a conditional beta that is high in bad times when the market premium is high, but not necessarily in a high unconditional beta.

In simulations, I find that the unconditional CAPM fails to price book-to-market sorted portfolios – this theoretical possibility is quantitatively important. There is virtually no cross-sectional variation among unconditional market betas. The actual difference in average stock returns between the highest and the lowest book-to-market portfolios is 6.7%, but the unconditional CAPM only predicts -0.2%.

I find that multifactor models including a conditional CAPM and the Fama-French three factor model do quite well. In the model, there is one extra state variable that captures the cross-sectional distribution of firm capital, and thus all cross-sectional statistics as well as all conditioning information for aggregate returns depend on this variable along with aggregate productivity. The relative success of a conditional CAPM means that conditioning variables such as the dividend yield capture the information in these state variable for time-varying betas and market risk premiums. To interpret the success of the Fama-French model, I note that firms' market betas can be represented as an average of betas for assets in place and betas for growth options. The Fama-French factors, HML and SMB, provide good proxies to account for the covariation of each market betas' component and the market risk premium.

Comparison to the literature

My model builds on the work of Gomes, Kogan and Zhang (2003), who construct a multiple-firm general equilibrium model that links expected returns to size and book-to-market characteristics. My work differs from theirs along several dimensions. They model "projects", while I model "firms". In their economy, projects have ex-ante identical productivity, and once adopted, variation in the project-specific productivity only affects that project's capital. As in the standard Q - theory of investment, in my model variation in firm productivity affects current investment decisions and the entire stock of the firm's capital. As a consequence, my model not only can rationalize the positive relation between profitability and investment, but also makes the allocation of capital among firms with different productivities a new state variable for the dynamics of aggregate and firm-level variables.

Zhang (2005) also focuses on the relation between the value premium and firm investments using a neoclassical model with adjustment costs and costly reversibility. Zhang closes the model with an exogenous countercyclical market price of risk and solves the model numerically. My model is a full general equilibrium model, solved analytically, in which risk premia derive from investor's risk aversion and the equilibrium consumption stream. My model also goes beyond Zhang (2005) by replicating the failure of the CAPM and the relative success of alternative asset pricing models.

Santos and Veronesi (2005), construct a general equilibrium endowment economy – the dividend of the various firms is specified exogenously, but they include habit persistence preferences to make a closer link between the cross-section and aggregate puzzles. Their model also finds that a conditional CAPM and the Fama and French HML factor outperform the unconditional CAPM.

The number of papers exploring the implications of production and investment on the cross-section of stock returns has been growing rapidly. Additional contributions include Berk, Green, and Naik (1999), Gomes, Yaron and Zhang (2002), Xing (2002), Cooper (2003), Carlson, Fisher and Giammarino (2003), Gourio (2004), and Tuzel (2005).

2 The Economy

I consider an economy populated by a continuum of heterogeneous firms that produce a single nondurable consumption good (numeraire). Firms differ in the level of productivity and in the stock of physical capital they own. The flow of output can either be used for investment in physical capital or it can be paid out as dividends and consumed by households. Households are identical and derive income from accumulated financial wealth, which consists of riskless bonds in zero net supply and risky assets in positive net supply. The risky assets represent claims to firms' dividends. All agents are perfectly competitive in that they formulate optimal policies taking economy wide state variables as given. In the rest of the section, I describe the environment where the interaction of households and firms takes place, followed by the characterization of the equilibrium.

2.1 Firms

Firms are infinitely-lived and all-equity financed. Given their production and investment technologies, they formulate optimal investment policies to maximize the value of equity. I assume that the set of firms $\mathcal{F}$ is exogenously fixed and I use subscript i to index an individual firm.

2.1.1 Production

Consider a competitive firm that uses capital, K, to produce a nonstorable output flow, Y, according to a constant return to scale technology:

$$Y_{it} = \left(e^{a_t} x_{it} + f\right) K_{it} \tag{1}$$

where a and x denote economy wide and firm specific stochastic productivity, respectively. The parameter f represents a common time-invariant component of the firm marginal productivity of capital. Depending on its sign, f might be interpreted as a constant operating cost $(f < 0)$ or revenue $(f > 0)$ per unit of installed capital.

The productivity index a is common to all firms and evolves stochastically according to the process:

$$da_t = \kappa_a \left(\overline{a} - a_t\right) dt + \sigma_a dW_{at} \tag{2}$$

where W_a is a standard Brownian motion. This process is a standard linear mean reverting process with constant speed of mean reversion, $\kappa_a \in \mathbb{R}^{++}$, stationary mean, $\overline{a} \in \mathbb{R}^{++}$, and constant volatility, $\sigma_a \in \mathbb{R}^{++}$. The stochastic nature of the economy wide productivity introduces aggregate uncertainty, thus ensuring the existence of an ex-ante equity premium, which would otherwise equal zero.

The firm specific productivity x evolves according to the mean reverting square root process:

$$dx_{it} = \kappa_x \left(\overline{x} - x_{it}\right) dt + \sigma_x \sqrt{x_{it}} dW_{it} \tag{3}$$

where $\kappa_x \in \mathbb{R}^{++}$ denotes the speed of reversion to the stationary mean $\overline{x} \in \mathbb{R}^{++}$, and $\sigma_x \in \mathbb{R}^{++}$ the constant volatility loading on a scalar standard Brownian process W_i. Firm specific productivity shocks are idiosyncratic: they are independent of each other and of all other sources of randomness in the economy. The stationary distribution of the firm specific productivity is the same for all firms. Hence, differences in firms' productivity are driven by different paths in the Brownian innovations, W_i. This assumption has both an economically appealing interpretation and an analytic advantage. From an economic perspective, the firm specific productivity x captures the firm "competitive advantage". Given the transitory nature of x, the common stationary mean $\overline{x}$ guarantees that no single firm can sustain indefinitely its "competitive advantage", while the common speed of mean-reversion κ_x makes each firm equally able to preserve its short-term "competitive advantage". From an analytic point of view, it has the advantage that a law of large numbers (LLNs) for i.i.d. data can be applied to determine the conditional moments of the cross-sectional distribution of firm specific productivity.

The mean reversion property in (2) and (3) is important. At the aggregate level, economic growth is endogenously generated by accumulation of physical capital. Hence, mean-reversion in the stochastic productivity process, a, prevents the growth rate of output from exploding. At the firm level, mean reversion is necessary to obtain a non-degenerate equilibrium cross-sectional distribution of firms' productivity and capital. That is, it ensures that no firm will come to ever dominate the whole economy. This is consistent with the cross-sectional evidence on the negative relation between growth rates and size (e.g. Hall (1987), Evans (1987) and Gala (2005b)), as well as with the empirical evidence on the existence of a significant degree of heterogeneity and persistence in firms' productivity (e.g. Bartelsman and Doms (2000)).

2.1.2 Investment

The stock of capital K_i depreciates at a common fixed rate $0 \leq \delta \leq 1$ and it increases by undertaking gross investment at a rate I_i. Hence, the stock of capital accumulates according to the law of motion:

$$dK_{it} = (I_{it} - \delta K_{it})\, dt; \quad K_{it} \geq 0 \;\; \forall t. \tag{4}$$

Firms use an adjustment cost technology featuring investment irreversibility. In order to undertake gross investment of I_i units of capital, I_i units of output must be set aside to be installed as capital, together with $c(I_i, \cdot)$ units which are used during installation. Thus gross investment at a rate I_i has an opportunity cost of $I_i + c(I_i, \cdot)$ units of output. The adjustment cost $c(I_i, \cdot)$ has the following functional form:

$$c(I_i, K_i, k_i) = \alpha k_i^{\frac{1}{n-1}} \left(\frac{I_i - \widehat{I}_i}{K_i} \right)^{\frac{n}{n-1}} K_i \tag{5}$$

where $\alpha \in \mathbb{R}^{++}$ is the adjustment parameter and $n \in \{2, 4, 6, ...\}$ controls the degree of curvature in (5).[1] The adjustment cost function is strictly increasing and convex in firm investment I_i reflecting the fact that the more units of additional capital a firm attempts to incorporate into the existing one, the less effective those units are on the margin at expanding firm capacity. Only investment in excess of a minimum level $\widehat{I}_i$ requires additional costs in excess to the purchase costs of investment. The adjustment cost technology in (5) departs from the traditional formulation in that adjustment costs are scaled by the firm relative capital, defined as the ratio of the firm capital to aggregate (average) capital, $k_i \equiv K_i/K$, where $K \equiv \int_{i \in \mathcal{F}} K_i di$.[2]

The introduction of the scaling variable, k_i, implies an adjustment cost function with increasing return to scale in I_i and K_i, which makes firm growth less costly for firms with

[1] Although firm investment are irreversible, restricting the value of n to lie in the set of even integers, guarantees the existence of a one-to-one correspondence between investment and marginal q. Abel and Eberly (1997) use a similar adjustment cost specification in a model of firm investment decisions.

[2] Throughout the paper, I use the symbol $\int_{i \in \mathcal{F}} [\cdot]\, di$ to denote aggregation over the set of firms $\mathcal{F}$.

low capital relative to the average firm capital. This increasing return to scale property is consistent with the empirical evidence on the negative relation between firm growth and firm relative size as discussed in more details in Gala (2005b).

The linear homogeneity property of the adjustment cost function in I_i, K_i and K, is consistent with the independence of growth and size at the aggregate level. This last property favors scaling effects induced by the firm relative size, k_i, rather than firm size K_i. Therefore, the adjustment cost function (5) is jointly consistent with (i) the negative relation between growth and relative size at the firm-level, and (ii) the independence of growth and size at the aggregate level. While any increasing return to scale adjustment cost function can generate (i), not all of them can jointly generate (i) and (ii), unless the adjustment cost function exhibits increasing and constant return to scale at the firm and aggregate level, respectively.

The set of feasible investment policy is restricted to firm investment, I_i, in excess or equal to a minimum level, $\widehat{I}_i \geq 0$. The minimum level of investment, $\widehat{I}_i$, might be interpreted as a recurring investment necessary to keep the firm installed capital productive by partially replacing worn out equipment. The minimum level of investment is proportional to the firm capital, $\widehat{I}_i = \widehat{i} K_i$, where $0 \leq \widehat{i} < \delta$.[3] Given that excess investment is irreversible and $\widehat{i} < \delta$, firm capital can only decrease via depreciation and remains positive at all times. The restriction placed on the set of feasible investment policy not only ensures the positivity of firm capital, but also prevents firms from partitioning into smaller parts, which in turn guarantees that the competitive equilibrium is well-defined.

The adjustment cost function (5) has also an important analytic advantage: it reduces the number of economy wide state variables to be only the economy wide productivity a and the second moment of the (joint) cross-sectional distribution of firm specific productivity x_i and stock of capital K_i. Under the traditional adjustment cost formulation the number of state variables would be an infinite dimensional object, that is it would be necessary the knowledge of the entire (joint) cross-sectional distribution of firm specific productivity x_i and stock of capital K_i to compute aggregate quantities and prices. This formulation allows me to focus on an *exact* general equilibrium solution rather than resorting to *approximate* solutions.

Firms' equity represent claims on the stream of future dividends, which equal operating profits net of investment costs:

$$D_i = \left(e^a x_i + f\right) K_i - I_i - \alpha k_i^{\frac{1}{n-1}} \left(\frac{I_i - \widehat{I}_i}{K_i} \right)^{\frac{n}{n-1}} K_i. \tag{6}$$

Taking economy wide state variables as given, firms choose the optimal investment strategy so as to maximize the expected present value of future dividends.

[3]Restricting the set of feasible investment policy to nonnegative excess firm investments ($I_i - \widehat{I}_i \geq 0$) rather than to nonnegative firm investments ($I_i \geq 0$), while preserving the same model implications about the cross-section of investment and stock returns (as long as $\widehat{i} < \delta$), has the advantage of simplyfing the model solution as well as ensures the existence of well-defined investment growth rates.

2.2 Households

There is a continuum of identical infinitely lived households, who derive utility from the consumption flow of the nonstorable consumption good, C_t. The economic behavior of the entire household population can then be modeled as a single representative household, which I assume has standard time-separable preferences:

$$U \equiv \mathbf{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt \right] \tag{7}$$

where $\rho \in \mathbb{R}^{++}$ denotes the subjective discount rate and $\gamma \in \mathbb{R}^{++}$ is the Arrow-Pratt coefficient of relative risk-aversion. Households derive income only from accumulated financial wealth, W_t. Financial markets consist of a complete set of risky assets in positive net supply and a riskless bond in zero net supply. I assume that financial markets are perfect in that there are no frictions and no constraints on short sales or borrowing.

Taking the prices of all financial assets as given, the representative household chooses paths of consumption $\{C_t\}_{t\geq 0}$ to maximize its lifetime utility (7) subject to the budget constraint:

$$\mathbf{E}_t \left[\int_0^\infty \frac{\Lambda_{t+s}}{\Lambda_t} C_{t+s} ds \right] \leq W_t. \tag{8}$$

In a complete financial market the term Λ_{t+s}/Λ_t denotes the unique pricing kernel determining prices of all financial assets.

2.3 Equilibrium

With the description of the economic environment complete, the equilibrium of the model is characterized in two steps. First, I characterize the partial equilibrium optimal investment policy and the dynamics of the economy wide state variables. Second, I characterize the general equilibrium allocations and prices.

The optimality condition characterizing the equilibrium households' consumption policy implies a well-know relation between consumption and the pricing kernel:

$$\frac{\Lambda_{t+s}}{\Lambda_t} = e^{-\rho s} \left(\frac{C_{t+s}}{C_t} \right)^{-\gamma}. \tag{9}$$

The pricing kernel equals the marginal rate of intertemporal consumption substitution. In general equilibrium, the optimal aggregate consumption equals aggregate dividends as determined by the market clearing conditions.

In order to solve for the equilibrium, it is necessary to identify the state variables characterizing the dynamics of the aggregate state of the economy. As shown in the following propositions, the key aggregate state variables are the economy wide productivity a and the variable ω, which represents a capital-weighted average of the firm specific productivities:

$$\omega_t \equiv \int_{i\in F} x_{it}\frac{K_{it}}{K_t}di = \int_{i\in F} x_{it}k_{it}di. \tag{10}$$

This variable quantifies the conditional cross-sectional covariation between the firm specific productivity, x, and its relative capital, k. The persistence of the firm specific productivity and the path-dependent nature of the firm capital stock imply a nonzero endogenous cross-sectional covariation whenever firms condition their investments optimally on their current productivity. Given that the x's are cross-sectionally i.i.d., a law of large numbers for i.i.d. random variables implies that the cross-sectional distribution of firm specific productivity x is time-invariant and equals its steady-state distribution. Therefore, any change in ω reflects changes in the cross-sectional distribution of firm relative capital, k. I conjecture and verify that ω follows a singular diffusion process:

$$d\omega_t = \mu_\omega (a_t, \omega_t)\, dt. \tag{11}$$

whose drift is only a function of a and ω itself. Let I_t denote aggregate investment defined as $I_t \equiv \int_{i\in F} I_{it}di$. Given that firm capital depreciates at a common rate δ, it follows that the aggregate stock of capital, K_t, accumulates according to:

$$dK_t = (I_t - \delta K_t)\, dt; \;\; K_t \geq 0 \;\; \forall t. \tag{12}$$

Using the dynamics of firm and aggregate capital stocks, the firm relative capital k_i evolves according to:

$$dk_{it} = k_{it}\,(i_{it} - i_t)\, dt \tag{13}$$

where I conjecture and verify that the firm investment rate depends on x_i, k_i, a and ω, and the aggregate investment rate is only a function of the aggregate state variables a and ω. The conjecture that a and ω are the only relevant aggregate state variables implies that the pricing-kernel, Λ, evolves stochastically according to:

$$\frac{d\Lambda_t}{\Lambda_t} = -r\,(a_t, \omega_t)\, dt - \lambda\,(a_t, \omega_t)\, dW_{at} \tag{14}$$

where r and λ depend only on a and ω. As shown in the following propositions, all the relevant information about the aggregate state of the economy contained in the (joint) cross-sectional distribution of x and k can be sufficiently summarized by its second moment, ω. While solving for the equilibrium, I verify that (i) a and ω are the only state variables sufficient to describe the aggregate state of the economy; and (ii) the equilibrium dynamics of ω, k and Λ satisfy the conjectured laws of motion given in (11), (13) and (14), respectively.

The following proposition states the partial equilibrium optimal firm investment policy.

Proposition 1 (Optimal Investment Policy) *Given the dynamics of ω, k, and Λ described in (11), (13) and (14), the optimal firm investment policy $i_i^* \equiv I_i^*/K_i$ can be characterized as*

$$i_i^* = \widehat{\imath} + \left(\frac{n-1}{\alpha n}\right)^{n-1} (q_i - 1)^{n-1}\, k_i^{-1} \mathbf{1}_{\{q_i \geq 1\}} \tag{15}$$

with the firm marginal q given by

$$q_i \equiv q\left(a,\omega,x_i\right) = \overline{xq}\left(a,\omega\right) + \left[x_i - \overline{x}\right]\widehat{q}\left(a,\omega\right) \tag{16}$$

and $\overline{q}\left(a,\omega\right)$, $\widehat{q}\left(a,\omega\right)$ *defined as*

$$\overline{q}\left(a_t,\omega_t\right) = \mathbf{E}_t\left[\int_0^\infty e^{-\left(\delta-\widehat{i}\right)s}\frac{\Lambda_{t+s}}{\Lambda_t}\left[e^{a_{t+s}} + \overline{x}^{-1}\left(f-\widehat{i}\right)\right]ds\right] \tag{17}$$

$$\widehat{q}\left(a_t,\omega_t\right) = \mathbf{E}_t\left[\int_0^\infty e^{-\left(\kappa_x+\delta-\widehat{i}\right)s}\frac{\Lambda_{t+s}}{\Lambda_t}e^{a_{t+s}}ds\right]. \tag{18}$$

Proof: See Appendix.

The optimal investment policy in (15) originates from the optimality condition requiring a firm to invest till the marginal benefit of investment as measured by its marginal q equals its marginal opportunity cost.

The optimal investment rate in (15) can be decomposed into a minimum investment rate and an excess investment rate. The optimal firm investment rate equals its minimum level for binding irreversibility constraint, and exceeds the minimum investment rate, otherwise. The minimum investment rate, $\widehat{i}$, represents the firm investment commitment to partially replace worn out capital necessary to keep currently installed capital productive. The excess investment contributes to the firm growth of capital and depends on the firm marginal q_i, the firm relative size k_i and the model parameters α and n.

The firm marginal q measures the contribution of a marginal increase in the firm stock of capital to the market value of its equity. It is equal to the expected present value of the future stream of marginal operating profit net of the minimum investment expenditures accruing to the firm installed capital. Given the absence of arbitrage or equivalently the strictly positivity of the pricing-kernel Λ, the parameters' restriction $f \geq \widehat{i}$ suffices to ensure the positivity of the firm marginal q.[4] The factor $\exp\left(-\left(\delta-\widehat{i}\right)\right)$ in (17) and (18) captures the fact that productive capital effectively depreciates at a rate $\delta - \widehat{i}$ lower than its economic depreciation δ because of the firm minimum investment. The firm marginal q is increasing in the state variables a, ω and x reflecting the fact that high productivity values make the stock of capital more valuable to a firm. While the function $\overline{q}$ represents the component of the marginal q common to all firms, the function $\widehat{q}$ captures the sensitivity of the producer's marginal q to its firm specific productivity. In other words, it quantifies the extra contribution to the market value of firm capital attributable to a firm relative competitive advantage as measured by its firm specific productivity in excess to the market average, $x^i - \overline{x}$. Furthermore, $\widehat{q}$ is uniformly below $\overline{q}$, since it discounts a smaller stream of cash flows (provided $f \geq \widehat{i}$) at a higher rate.

While the size of the firm minimum investment is proportional to the firm capital, the size of excess investment is inversely proportional to the cost of capital adjustment.

[4]Given that the utility function in (7) satisfies the Inada conditions, the equilibrium aggregate consumption process is always strictly positive, which implies the absence of arbitrage.

This cost is increasing in the adjustment parameter α, the degree of convexity in the adjustment cost function controlled by n, and the firm relative capital k_i.

Equation (15) shows that, ceteris paribus, firm excess investment rate declines with firm relative capital a result consistent with the empirical evidence in Gala (2005b). The resulting negative relation between firm growth and relative size stems from the positive relation between the firm marginal cost of excess investment and its relative size.

This feature of the model emerges clearly if we interpret the factor $\left(\frac{n-1}{\alpha n}\right)^{n-1} k_i^{-1}$ in (15) as the adjustment time of the firm capital stock given one unit change in marginal q (e.g. Shapiro (1986) and Hall (2001)). For instance, in the case of quadratic adjustment costs ($n = 2$), if q rises by one unit, the firm investment rate will rise by $1/2\alpha k_i$. To cumulate to a unit increase, the flow must continue at this level for $2\alpha k_i$ periods. The lower the firm relative size the shorter the time it takes a firm to cumulate a unit increase in its investment rate. Thus, small firms can double their size in a shorter period of time relative to big firms, ceteris paribus.

The model also implies that small firms (low k^i) tend to growth faster than big firms (high k^i), especially during economic booms. This is due to the fact that a positive economy wide productivity shock a by increasing the marginal benefit of investment (marginal q is increasing in a) amplifies the negative relation of firm growth rates and relative size.

For firms with positive excess investment, the elasticity of investment rate with respect to relative size takes on values between minus one and zero. The importance of scale effects in firm investment rates can be controlled by the parameters $\widehat{i}$ and α. The higher their values the lower the elasticity of investment rate with respect to relative size.

2.3.1 Heterogeneity and Aggregation

The following proposition states the aggregate (average) quantities computed by aggregation of their firm-level counterparts. To compute aggregate quantities I appeal to a law of large numbers for a continuum of i.i.d. random variables.[5] According to the firm optimal investment policy in (15) a firm face a binding irreversibility constraint whenever its marginal q falls below one, or equivalently using equation (16), x_i falls below the threshold $\widetilde{x}_t \equiv \widetilde{x}\left(a_t, \omega_t\right) = \left[1 - \overline{x}\left(\overline{q}_t - \widehat{q}_t\right)\right]/\widehat{q}_t$, where $\widetilde{x} \in [0, 1/\widehat{q})$.

Proposition 2 (Aggregate Quantities) *Define $\theta \equiv 2\kappa_x/\sigma_x^2$ and $\upsilon \equiv 2\kappa_x\overline{x}/\sigma_x^2$. Then, the equilibrium aggregate output Y can be represented as*

$$Y \equiv \int_{i\in F} Y_i di = \left(e^a\omega + f\right)K, \tag{19}$$

[5] The techniques needed to prove the existence of a continuous-time i.i.d. process satisfying the strong law of large numbers were developed by Dobb (1953 chapter II), and an existence proof has been given in full by Judd (1985). Aside from technicalities, alternative models of law of large numbers for large economies have been formalized in Feldman and Gilles (1985), Uhlig (1990), Anderson (1991) and Green (1994).

the aggregate investment, I^*, can be characterized as

$$I^* \equiv \int_{i\in F} I_i^* di = \left[\widehat{i} + \left(\frac{n-1}{\alpha n}\right)^{n-1} g\left(a, \omega; n-1, 1\right)\right] K, \tag{20}$$

and similarly the aggregate dividend, D^*, can be written as

$$D^* \equiv \int_{i\in F} D_i^* di = Y - I^* - \alpha^{-(n-1)} \left(\frac{n-1}{n}\right)^{n} g\left(a, \omega; n, 1\right) K \tag{21}$$

where

$$g\left(a, \omega; m_1, m_2\right) \equiv \widehat{q}^{m_1} \sum_{k=0}^{m_1} \frac{\Gamma\left(m_1+1\right)\Gamma_U\left(k+\upsilon, \theta\widetilde{x}\right)}{\Gamma\left(m_1+1-k\right)\Gamma\left(k+m_2\right)\Gamma\left(\upsilon\right)} \left(-\widetilde{x}\right)^{m_1-k} \theta^{-k} \tag{22}$$

and Γ and Γ_U denote the gamma and the upper incomplete gamma function, respectively.

Proof: See Appendix.

As shown in (19), the stochastic component of the aggregate marginal productivity of capital can be represented as the product of two terms: the *exogenous* productivity index a, and the *endogenous* productivity index ω. This last one accounts for the distribution of capital among firms with different productivity.

Aggregate investment is proportional to aggregate capital, which implies that the growth of aggregate capital is scale independent, consistently with the empirical evidence in Gala (2005b). The aggregate investment rate is procyclical since it results from a capital-weighted average of a convex transformation of the firm marginal q.

Aggregate dividends are linearly homogeneous in aggregate capital. This makes the model equivalent to an "Ak" model of stochastic growth augmented by an adjustment cost technology. To ensure nonstochastic perpetual growth, it is sufficient to impose the condition that $(e^{\overline{a}}\overline{x} + f - \delta) > \rho$. If this condition were not satisfied, and instead $(e^{\overline{a}}\overline{x} + f - \delta) < \rho$, then the economy would shrink towards zero.

The following proposition characterizes the dynamics of the state variable ω representing the endogenous component of aggregate productivity.

Proposition 3 (Dynamics of ω) *The endogenous component of aggregate productivity, ω, evolves according to the singular stochastic process:*

$$d\omega = \left\{ \left[\kappa_x + i - \widehat{i}\right] \left(\overline{x} - \omega\right) + \left(i - \widehat{i}\right) \theta^{-1} \frac{g\left(a, \omega; n-1, 0\right)}{g\left(a, \omega; n-1, 1\right)} \right\} dt \tag{23}$$

with the function $g\left(\cdot\right)$ defined in (22).

Proof: See Appendix.

The irreversibility of firm investment prevents firm capital from being negative and thus ensures the positivity of ω. However, we can say more about the minimum attainable value of ω: the state variable ω is bounded from below by $\overline{x}$. This is due to the fact that at any time there is nonnegative cross-sectional covariation between the firm capital stock and its firm specific productivity, since firm gross investment endogenously increases with its productivity and the stock of capital depreciates at a common rate.

By the law of large numbers the cross-sectional distribution of firm specific productivity x is time-invariant and equals its steady-state distribution. Therefore, the state variable ω tracks the evolution of the capital allocation among firms with different productivity. When the capital is uniformly distributed across firms, which is when each firm has a capital stock equal to the aggregate (average) capital, ω is equal to the steady-state mean of the firm specific productivity $\overline{x}$. The higher the concentration of capital among more productive firms, the higher the value of ω.

The instantaneous change in the state variable ω is driven by two forces. First, the component in square brackets in (23) is positive and tend to pull ω back to its lower bound $\overline{x}$. This reverting force is offset by the second component in (23) which is always positive and increasing in the current value of ω. The first force is induced by the negative relation between firm growth and relative size: small firms tend to growth faster than big firms thus attenuating the cross-sectional dispersion in firm capital share. This effect becomes stronger during booms, when all firms benefit of the procyclical variation in their shadow value of capital. The second force tend always to increase ω and it is stronger the higher the cross-sectional dispersion of firm specific productivity and the better the current state of the economy. For a given firm relative size distribution, the higher the cross-sectional dispersion of firm specific productivity, the higher the value of ω, since more firms are concentrated on the right tail of the x distribution.

Indeed, the presence of these offsetting forces prevents the firm relative capital distribution from degenerating into a spike. Finally, the fact that ω is a sufficient statistic for the characterization of the aggregate state of the economy is the by-product of the linear production technology and the independence of firm excess investment from its capital stock.

2.3.2 Equilibrium Allocations

With the characterization of the optimal firm policies and aggregate quantities complete, I now state the definition of the competitive general equilibrium.

Definition 4 (Competitive Equilibrium) *A competitive general equilibrium is summarized by stochastic processes for the pricing kernel Λ, the optimal consumption policy C^*, and the optimal firm investment policy I_i^*, such that:(i) taking asset returns as given, the representative household maximizes its expected utility (7), subject to the budget constraint (8); (ii) taking the pricing kernel and aggregate capital as given, producers make investment decisions according to (15); (iii) consumption good market clears, $C^* = D^*$.*

The following proposition establishes the general equilibrium consumption and investment policies as the solution to a system of two partial differential equations and two algebraic equations.

Proposition 5 (Equilibrium Allocations) *The competitive equilibrium is characterized by the optimal firm investment policy* $i^*(a, \omega, x_i, k_i)$ *described in (15) - (16), and consumption policy* $C^*(a, \omega, K)$*, which satisfy:*

$$C^*(a, \omega, K) = c^*(a, \omega) K \tag{24}$$

and

$$c^* = e^a \omega + f - \widehat{i} - \left(\frac{n-1}{\alpha n}\right)^{n-1} \left[g(a, \omega; n-1, 1) + \left(\frac{n-1}{n}\right) g(a, \omega; n, 1)\right] \tag{25}$$

with the function $g(\cdot)$ *defined in (22), and*

$$\overline{q} = c^*(a, \omega)^\gamma \,\overline{\Phi}(a, \omega) \tag{26}$$

$$\widehat{q} = c^*(a, \omega)^\gamma \,\widehat{\Phi}(a, \omega) \tag{27}$$

where the functions $\overline{\Phi}(a, \omega)$ *and* $\widehat{\Phi}(a, \omega)$ *satisfy the partial differential equations (79) - (80) in Appendix.*

Proof: See Appendix.

The general equilibrium framework allows me to provide a consumption-technology based explanation of the behavior of investment and asset prices at both firm and aggregate levels.

The general equilibrium analysis differentiates my model from most of the existing literature, which instead proceed by keeping the pricing kernel entirely exogenous, thus separating the optimal investment decisions from the consumption allocation.

An exception in the existing literature is the general equilibrium analysis of Gomes, Kogan and Zhang (2003), who first proposed a general equilibrium framework for the analysis of the cross-section of asset returns. For the sake of analytical tractability, they assume that firm capital investment is ex-ante independent of current productivity. Although counterfactual, this assumption enables them to characterize the aggregate economy separately from the cross-section of firms. In my framework, the cross-section of firms is endogenous and affect the aggregate economy so that cross-sectional heterogeneity can indeed have asset pricing implications at both the aggregate and firm level.

2.3.3 Equilibrium Asset Prices

I now characterize the economy wide financial investment opportunity set, including both aggregate and firm level stock prices.

Financial Investment Opportunity Set The following proposition summarizes the results for the equilibrium values of the risk-free rate, r, and the market price of productivity risk, λ.

Proposition 6 (Financial Investment Opportunity Set) *The aggregate investment opportunity set is characterized by the equilibrium instantaneous risk-free rate:*

$$r(a,\omega) = \rho + \gamma \frac{\mathcal{D}^{a,\omega,K}[C^*]}{C^*} - \frac{1}{2}\gamma(\gamma+1)\sigma_a^2 \left[\frac{\partial_a C^*}{C^*}\right]^2 \tag{28}$$

and the equilibrium instantaneous market price of productivity risk:

$$\lambda(a,\omega) = \gamma\sigma_a \frac{\partial_a C^*}{C^*} \tag{29}$$

where $\mathcal{D}^{a,\omega,K}[\cdot]$ denotes the infinitesimal generator of the stochastic processes a, ω and K :

$$\mathcal{D}^{a,\omega,K}[M(\cdot)] = \kappa_a(\overline{a}-a)\partial_a M(\cdot) + \frac{\sigma_a^2}{2}\partial_{aa}^2 M(\cdot) + \mu_\omega(a,\omega)\partial_\omega M(\cdot) + (I^* - \delta K)\partial_K M(\cdot). \tag{30}$$

Proof: See Appendix.

As conjectured in (14), both the risk-free rate and the market price of productivity risk are only function of the economy wide productivity a and the state variable ω.

Aside from technicalities, the economic intuition behind equations (28) - (29) is quite simple. The risk-free rate is increasing in the subjective discount rate ρ, and the expected growth rate of aggregate consumption $\mathcal{D}[C^*]/C^*$. Those components reflect intertemporal substitution motives. The higher households' impatience and the expected consumption growth, the higher households' willingness to substitute future for current consumption. Then, households would like to borrow, driving up the equilibrium risk-free rate. The last term in (28) reflects precautionary savings motives. As aggregate uncertainty increases, households are more willing to save, driving down the equilibrium risk-free rate.

The market price of productivity risk represents the equilibrium premium per unit of risk that households require to hold the market portfolio and hence bear the systematic risk of aggregate consumption fluctuations. It is increasing in the household's coefficient of relative risk aversion γ, the volatility of economy wide productivity σ_a and the sensitivity of aggregate consumption to changes in economy wide productivity $\partial_a C^*/C^*$. The higher the aggregate consumption sensitivity to economy wide productivity shocks, the higher the uncertainty of the economy wide productivity and the more risk-averse the households, the higher the unit risk premium required to hold claims on aggregate consumption.

Individual Firm Asset Prices. Firms' equity represent claims on the dividends paid out to shareholders. The following proposition characterizes the equilibrium market value of an individual firm, V_i.

Proposition 7 (Equilibrium Firm-Level Asset Prices) *The market value of individual firm, V_i, is determined by*

$$V_i = \mathbf{E}_t \left[\int_0^{\infty} \frac{\Lambda_{t+s}}{\Lambda_t} D^i_{t+s} ds \right] = q(a, \omega, x_i) K_i + h(a, \omega, x_i) K \tag{31}$$

where the firm marginal q (i.e. $\partial V_i / \partial K_i$) is described in (16) and the firm marginal h (i.e. $\partial V_i / \partial K$) is determined by:

$$h(a, \omega, x_i) = c^*(a, \omega)^{\gamma} H(a, \omega, x_i) \tag{32}$$

where the function $H(a, \omega, x_i)$ satisfies the partial differential equation (88) provided in the Appendix.

Proof: See Appendix.

From equation (31), the market value of individual firms V_i is characterized as the sum of two components, $V_i = V_i^A + V_i^O$. The first component is the value of "assets in place": the present value of the firm future operating profits accruing to the firm stock of capital currently in place, and it is given by

$$V_i^A = q(a, \omega, x_i) K_i. \tag{33}$$

The firm marginal q described in (16) quantifies the present value of the firm future operating profits per unit of installed capital accounting for its effective economic depreciation.

The second component is the value of "growth opportunities": the present value of rents accruing to the firm from the adjustment technology, and it is determined by

$$V_i^O = h(a, \omega, x_i) K. \tag{34}$$

The function $h(a, \omega, x_i)$ represents the marginal contribution to the firm market value of a reduction in the capital adjustment cost. Specifically, the firm marginal h quantifies the marginal gain in the firm market value of a decrease in the firm relative size. The average capital in (34) arises because each firm benefits from the investment activity of other firms in the economy. That is, the value of "growth opportunities" represents the value of the option to use capital investments in order to take advantage of the current economic conditions. The higher the average capital in the economy, the lower the firm relative size, the lower the capital adjustment cost, and the more valuable this option.

In the absence of arbitrage, $h(a, \omega, x_i)$ is positive and increasing in the state variables a, ω and x_i reflecting the fact that a marginal decrease in the marginal cost of investment is more valuable the higher the firm productivity.

The non-homogeneity property in the firm cash flow translates one-to-one into the firm market value, thus creating a wedge between the firm marginal q (i.e. $\partial V_i/\partial K_i$) and the firm Tobin's Q (i.e. V_i/K_i): the value of growth opportunities per unit of installed capital, $(h_i K)/K_i$. The higher the firm profitability and the lower the firm relative size the larger this wedge.

Aggregate Market Value. The aggregate stock market value is defined as the price of a claim to aggregate consumption, which in equilibrium equals aggregate dividends. The law of one price and the absence of arbitrage ensure that its value can be computed by aggregating individual firm market values. Equivalently, it can be directly computed as the expected present value of future consumption streams. The following proposition characterizes the equilibrium stock market value, V.

Proposition 8 (Equilibrium Stock Market Value) *The stock market value, V, is determined by*

$$V = [q_m(a,\omega) + h_m(a,\omega)]\, K \tag{35}$$

where q_m denotes the average firm marginal q:

$$q_m(a,\omega) = \overline{xq}(a,\omega) + [\omega - \overline{x}]\, \widehat{q}(a,\omega)$$

with the functions $\overline{q}$ and $\widehat{q}$ described in (26) - (27), and the function h_m represents the average firm marginal h:

$$h_m(a,\omega) = c^*(a,\omega)^{\gamma}\, H_m(a,\omega) \tag{36}$$

where the function H_m satisfies the partial differential equation (94) provided in the Appendix.

Proof: See Appendix.

In line with the decomposition of individual firm market value, I characterize the aggregate stock market value V as the sum of the value of aggregate assets in place, V^A, and the value of aggregate growth opportunities, V^O. The value of aggregate assets in place is the present value of the aggregate operating profits accruing to the aggregate stock of capital currently in place, and it is given by

$$V^A \equiv \int_{i\in F} V_i^A di = q_m(a,\omega)\, K = [\overline{xq}(a,\omega) + [\omega - \overline{x}]\, \widehat{q}(a,\omega)]\, K. \tag{37}$$

The value of aggregate growth options is the present value of rents accruing to the economy as a whole (average firm) from the firm-level adjustment technology, and it is determined by

$$V_t^O \equiv \int_{i\in F} V_i^O di = h_m(a,\omega)\, K. \tag{38}$$

The marginal and Tobin's q of the average firm (aggregate economy) are identical. This equivalence stems from the homogeneity property of aggregate cash flow. The presence of scale effects induced by the firm relative size, while creating increasing return to scale in the firm cash flow, makes the cash flow of the average firm linearly homogeneous. Scale effects induced by firm size would counterfactually preserve the same increasing return to scale property both at the firm and aggregate level. Hence, the marginal q of the average firm exceeds the average firm marginal q, which causes aggregate under-investment relative to a social optimum and a Pareto inefficient allocations of resources.

2.3.4 Stock Returns and Conditional CAPM

With the characterization of the equilibrium aggregate and firm-level asset prices complete, I now describe the asset risk-return representation. From (35), the process for cumulative aggregate stock return can be represented as

$$dR = \frac{dV}{V} + \frac{D}{V}dt = \mu_R\left(a,\omega\right)dt + \sigma_R\left(a,\omega\right)dW_a \tag{39}$$

where the instantaneous aggregate stock market expected return $\mu_R\left(a,\omega\right)$ and volatility $\sigma_R\left(a,\omega\right)$ are respectively given by equation (97) and (98) in Appendix. Aggregate stock market returns vary because of innovations to the economy wide productivity, dW_a, which represents the only source of systematic risk in the economy.

Similarly, from (31), the cumulative stock return for a individual firm has the following factor representation:

$$dR_i = \frac{dV_i}{V_i} + \frac{D_i}{V_i}dt = \mu_{R_i}\left(a,\omega,x_i,k_i\right)dt + \sigma_{R_i,a}\left(a,\omega,x_i,k_i\right)dW_a + \sigma_{R_i,x}\left(a,\omega,x_i,k_i\right)dW_i \tag{40}$$

where the instantaneous stock expected return $\mu_{R_i}\left(a,\omega,x_i,k_i\right)$ and the volatility loadings on the economy wide productivity innovations $\sigma_{R_i,a}\left(a,\omega,x_i,k_i\right)$ and on the firm specific innovations $\sigma_{R_i,x}\left(a,\omega,x_i,k_i\right)$ are respectively given by equation (100), (101) and (102) in Appendix. Differently from the aggregate stock market returns, individual stock returns vary because of innovations to both the economy wide productivity, dW_a, and the firm specific productivity, dW_i, which represents the source of idiosyncratic risk in the economy.

The general equilibrium model implies conditional perfect correlation between the instantaneous aggregate stock market return and aggregate consumption growth (and hence the pricing-kernel). Given the single-factor nature of the model, where the only source of systematic risk is the aggregate productivity uncertainty, the cross-sectional distribution of expected returns is fully determined by the distribution of firm consumption or market betas. The risk-return relation of any traded asset in the economy can be characterized as a conditional Consumption-CAPM, or similarly as a conditional Capital Asset Pricing Model (CCAPM) since the aggregate market portfolio is instantaneously conditionally mean-variance efficient. The next proposition establishes the risk-return relation as CCAPM.

Proposition 9 (Conditional CAPM) *The instantaneous risk and expected return of individual firms can be characterized by a conditional CAPM:*

$$\mu_{R_i,t} = r_t + \beta_{it} \left[\mu_{R,t} - r_t \right] \tag{41}$$

with the conditional firm market beta given by

$$\beta_{it} = \frac{\partial \left[\ln \left(V_{it}/K_{it} \right) \right]}{\partial \left[\ln \left(V_t/K_t \right) \right]} = \frac{K_{it}}{V_{it}} \xi_t^{q_i} + \frac{K_t}{V_{it}} \xi_t^{h_i} \tag{42}$$

with $\xi_t^{q_i}$ and $\xi_t^{h_i}$ measuring the risk of firm assets in place and growth opportunities as described in Appendix.

Proof: See Appendix.

The decomposition of aggregate and firm-level asset prices into value of "assets in place" and value of "growth options" provides a convenient framework to relate individual firm and aggregate market values in that it unveils the relation between the riskiness of the firm value as measured by its market beta, β_{it}, and *observable* firm characteristics. Equation (42) provides a description of such a relation: market betas depend on the firm book-to-market (K_i/V_i), the firm size (V_i) and the quantities ξ^{q_i} and ξ^{h_i}.[6] The first term in the right hand side of equation (42) creates a positive relation between the firm book-to-market ratio and its market beta provided that $\xi^{q_i} > 0$. The term ξ^{q_i} denotes the firm specific elasticity of the firm marginal q (i.e. value of assets in place per unit of capital) with respect to changes in the aggregate market-to-book ratio. It measures the sensitivity of the firm marginal q and hence investments to changes in the aggregate state of the economy.

The second component in (42) determines a negative relation between the firm size and its market beta provided that $\xi^{h_i} > 0$. The term ξ^{h_i} denotes the firm specific elasticity of the firm growth option with respect to changes in the aggregate market-to-book ratio. It measures the sensitivity of the firm growth option to changes in the aggregate state of the economy. Furthermore, since both the elasticities ξ^{q_i} and ξ^{h_i} depend on the firm specific productivity, the relation between firm market betas, book-to-market and size is nonlinear. Since firm dividends are not homogeneous in firm capital, both fundamental firm characteristics, x_i and k_i, or market based firm characteristics, K_i/V_i and V_i, are needed to identify cross-sectional differences in expected returns.

The general equilibrium analysis provides a "consumption insurance" explanation for the relation between risk and expected returns. Specifically, given the investors' risk-aversion towards uncertain consumption stream and its preference for early consumption, a rational investor objective is "consumption-smoothing" over time and states of the economy. Therefore, the riskiness of a firm equity is directly linked to its ability to

[6] Since the produced consumption good in the economy can be used either for consumption or for investment as capital good, the price of new capital equals the price of the consumption good, which is normalized to one. Therefore, the firm stock of capital K, can be interpreted as its capital measured in historical costs (book value).

provide consumption insurance. The more able a firm is in this regard, the less risky is its equity: investors bid up the prices of those firms whose returns offer consumption insurance, and require a lower premium to hold their stocks.

A firm ability to provide consumption insurance depends on its' ability to use capital investment in response to shocks in the current state of the economy. Capital adjustment costs and irreversibility are the main impediments to the use of capital investment to smooth dividends. During bad times, unprofitable firms, value firms, would like to disinvest and sell off their capital stocks, but they cannot do so because face a binding investment irreversibility constraint. If there is a further negative aggregate productivity shock, there is nothing they can do to mitigate a further decline in output and dividend. In contrast, growth firms are investing because they have persistently high profitability. They only face adjustment costs to doing so. In the face of a negative aggregate productivity shock, they can lower investment and maintain their dividend in this high marginal utility state. Thus, the dividends of growth firms will fall less than those of value firms in response to an adverse aggregate productivity shock. As a result, value firms are riskier than growth firms.

This economic explanation of the value premium finds support in the empirical evidence provided by Xing and Zhang (2004), who find that the fundamentals of value firms such as earning, dividend and investment growth, are more adversely affected by negative business cycle shocks than those of growth firms.

2.3.5 The Relation between Marginal q and Tobin's Q

The relation between marginal q and Tobin's Q can be conveniently rewritten as a function of the expected returns earned on each component of the firm market value. The following proposition establishes this relation.

Proposition 10 (Marginal q and Tobin's Q) *A firm Tobin's Q can be related to its marginal q as*

$$\frac{V_i}{K_i} = q_i \frac{\left[\mu_{I_i} - \mu_{O_i}\right]}{\left[\mu_{R^i} - \mu_{O_i}\right]} \tag{43}$$

where μ_{I_i} and μ_{O_i} denote the instantaneous expected returns on firm investment (i.e. marginal q) and growth options, respectively given by:

$$\mu_{I_i} \equiv \frac{\mathcal{D}\left[q_i\right]}{q_i} + \frac{e^a x_i + f - \widehat{i}}{q_i} - \left(\delta - \widehat{i}\right) \tag{44}$$

$$\mu_{O_i} \equiv \frac{\mathcal{D}\left[h_i\right]}{h_i} + \frac{\eta\left(q_i - 1\right)^n}{h_i} \mathbf{1}_{\{q_i \geq 1\}} - \left(\delta - i\right). \tag{45}$$

The relation in (43) can be conveniently approximated around the unconditional mean stock return ($\overline{\mu}_R$) of a firm with equal marginal q and Tobin's Q as

$$\ln\left[\frac{V_i}{K_i}\right] \approx \ln\left[q_i\right] + \frac{\left[\mu_{I_i} - \mu_{R^i}\right]}{\overline{\mu}_R}. \tag{46}$$

Proof: See Appendix.

The relation (43) follows from the representation of a firm expected stock return as a weighted average of the returns on its assets in place and growth options. The expected return on firm investment equals the expected change in the firm marginal q (the market value of a unit of installed capital) plus the flow of marginal operating profit net of effective depreciation due to owning one additional unit of capital. The expected return on growth options is given by the expected change in the marginal contribution to the firm market value of a reduction in the capital adjustment cost.

The approximate relation (46) makes clear the relation between expected returns and firm valuation. If marginal q and Tobin's Q are identical then the expected return on firm investments equals the expected return on firm equity. This is consistent with Cochrane (1991) and Zhang (2005). However, if marginal q differs from Tobin's Q then the expected return on investment differs from the expected return on equity. Since the firm marginal q is never greater than the firm market value according to (31), it must be the case that the expected return on firm investment exceeds its cost capital. That is, undertaking investment projects whose expected returns are higher than the cost of capital creates value. This relation reconnect the economic Q-theory of investment to first principles of corporate finance.

Most of the existing literature in investment supports the equivalence of return on investment and return on firm equity based on the absence of arbitrage. This way of reasoning implicitly assumes the investor/consumer ability to directly trade in both claims to physical and financial capital (equity). Under this assumption, an arbitrage-free model of equilibrium investment would deliver the equivalence between returns on physical and financial capital. However, under this assumption the *technological* investment opportunity-set a firm faces when making investment decisions is equivalent to the *financial* investment opportunity set a market investor faces when making a decision on how to allocate its wealth among different financial assets.

My model offers an equilibrium model of investment that builds on the plausible assumption that the investor/consumer can directly trade only in claims to the firm financial capital (hence firm return on investment can differ from return on firm equity even in equilibrium), thus generating the existence of two (not necessarily identical) investment-opportunity sets: a *technological* investment-opportunities set faced by the firm, and a *financial* investment-opportunity set faced by the market investor. This explicitly recognizes a financial economic reason for the firm to exist, that is a firm can do something that market investors cannot do directly: a firm can access to a *technological* investment opportunity-set out of a single market investor/consumer reach.

3 Empirical Analysis

In this section I conduct a simulation study to evaluate the model's ability to reproduce the main empirical properties of firm investments and stock returns. The empirical analysis is based on a panel of firms drawn from the CRSP-COMPUSTAT merged database for the years 1962 through 2002. The description of the data is provided in the Appendix.

I restrict the values of the model parameters, γ, α, κ_a, $\overline{a}$, σ_a, κ_x and σ_x to approximately match the unconditional mean and standard deviations of consumption growth, real interest rate, equity premium, aggregate investment rate, the average volatility of stock returns and the average cross-sectional correlation between (the logarithms of) size and book-to-market. I simulate 100 artificial panels each with 200 firms and 5000 years. I calculate the return and quantity moments for each sample and then compute the cross-sample averages.

The values of the parameters used in the simulation study are as follows: the risk aversion coefficient, γ, 14; the time preference parameter, ρ, 0.01; the adjustment cost parameter, α, 2; the degree of curvature in the adjustment cost function, n, 2; the depreciation rate, δ, 0.13; the minimum investment rate, $\widehat{\imath}$, 0.12; the time-invariant component of productivity, f, 0.12; the long-run mean of the aggregate productivity variable, $\overline{a}$, -2.22; the rate of mean reversion of the productivity variable, κ_a, 0.27; the volatility of aggregate productivity, σ_a, 0.05; the long-run mean of the idiosyncratic productivity, $\overline{x}$, 1; the rate of mean reversion of the idiosyncratic productivity, κ_x, 0.15; and the volatility of the idiosyncratic productivity, σ_x, 0.27.

3.1 Aggregates

Although, the model ability to reproduce key features of aggregate data is not the main objective of the paper, it seems appropriate to ensure that the time series properties of stock returns are reasonable before proceeding to the analysis of their cross-sectional properties.

Table I compares the model implied unconditional moments of key aggregate variables with corresponding empirical estimates. The model captures well the historical level and volatility of the equity premium, while maintaining reasonably low values for the first two moments of the risk free rate and aggregate consumption growth. Given the power utility and the low historical volatility of aggregate consumption growth of about 2.5% for the sample starting in 1929 to 2004, this can be achieved with a sizeable value for the risk-aversion coefficient, 14 (Mehra and Prescott (1985)).

Although the model generates a plausible value for the equity premium volatility, the time separable nature of preferences implies that part of this variation is due to time-varying risk-free rate. In the model, economic growth occurs via capital accumulation, which implies that the average consumption growth of about 1% is approximately equal to the average net investment rate, $14\% - 13\%$.

Before examining the cross-sectional properties of stock returns, I investigate the model implications about aggregate stock returns predictability. Table II reports the results of predictability regressions of excess stock returns on log dividend-price ratio and log book-to-market ratio at annual frequency. Consistently with historical data, the coefficients of cumulative excess returns on log dividend-price ratio are positive: low prices relative to dividend imply high expected excess returns. The coefficients increase with the horizon, ranging from 0.85 to 5.29 over four and ten years. I found a similar pattern when using the log book-to-market as a predictor of excess stock returns. However, the log dividend-price has a superior predictability power especially at a shorter horizon. This is consistent with its higher persistence ranging from 0.28 to 0.08 versus the persistence of the log book-to-market from 0.19 to 0.03 over four and ten years, respectively.

The predictability of excess stock returns stems from the countercyclical property of the market price of aggregate productivity risk and the persistence in aggregate productivity. With time-separable utility, the market price of risk inherits its properties from the volatility of consumption growth. Figure 1 (Panel A) shows that the volatility of consumption growth is particularly high during bad times, consistently with the empirical findings in Kandel and Stambaugh (1990), Bansal and Yaron (2004), Bekaert and Liu (2004), and Lettau, Ludvigson, and Wachter (2005). This is the result of investment irreversibility. As shown in Figure 1 (Panel B), the investment irreversibility threshold is high during bad times. More firms are up against the investment irreversibility constraint and the economy has now less flexibility in using investment to insure consumption against further adverse productivity shocks. Thus, investors require a higher equity premium to bear the risk of more volatile consumption fluctuations.

3.2 The Cross-Section of Stock Returns

This section establishes the key quantitative results. After examining the relation between firm characteristics and stock returns in the next subsection, I analyze the performance of the CAPM and other asset pricing models. To facilitate the comparison with historical data I simulate 100 artificial panels each with 200 firms and 50 years of data. I then report cross-sample averages.

3.2.1 Stock Returns and Firm Characteristics

In the model cross-sectional differences in firm profitability and capital generate heterogeneity in market based firm characteristics such as market size and book-to-market ratio. The same differences in firm fundamentals are also responsible for the cross-sectional variation in investment rates and expected returns. In Table III and IV I compare summary statistics of the model with their empirical counterparts. I report average excess returns and firm characteristics for portfolios formed by a one-dimensional sort of stocks on firm book-to-market and market equity, respectively. Panel A shows summary statistics based on historical data, and Panel B those computed on the basis of simulated panels.

The pattern of excess stock returns and firm characteristics in the model matches the evidence well. Average returns fall from 13.41 % per year for the highest book-to-market portfolio to 6.67% for the lowest. The portfolio's Sharpe ratios share the same decreasing pattern, ranging from 0.58 to 0.28. The spread in average profitability between growth and value stocks is about 23%, which is close to its historical one, 27%. As expected, average investment rates correlate positively with profitability. The portfolios with higher book-to-market face binding excess investment irreversibility, which confirms their limited ability to provide adequate consumption insurance to investors.

The pattern of excess stock returns and firm characteristics for market equity-sorted portfolios is also consistent with historical data. Average returns and Sharpe ratios decrease with market equity, though the spread in returns between small and big size portfolios (about 4%) is lower than the historical one. Despite the positive relation between capital and market equity, investment rates also slightly increase from the smallest to the highest market equity portfolio because of the increase in profitability.

Table V shows the average excess returns and firm characteristics across 3 x 4 portfolios formed by a two-dimensional sort of stocks on firm market equity and book-to-market. Panel A shows summary statistics based on historical data, and Panel B those computed on simulated data. In the model and confirmed in the data, the size of the value premium varies with market equity: the value premium is larger for small stocks. In simulated data, the value premium declines from about 5% per year for the small size portfolios to about 3% for the big size portfolios, thus generating a difference close to the historical one of about 2.3%. The decline in the value premium is associated with a decrease in the investment rate spread both in simulated and historical data. This feature of the value premium arises in the model because capital adjustment costs are smaller for small firms, ceteris paribus.

This property of capital adjustment costs implies a negative relation between firm growth and capital, which is also present in historical data (Panel A). For any book-to-market percentile, despite the positive relation between profitability and market equity, investment rates decrease with market equity because firm capital increases.

To formally establish the relation between stock returns and firm characteristics, Table VI shows the results from the Fama-MacBeth regressions of excess stock returns on market-based firm characteristics such as book-to-market and market equity, and firm fundamentals such as investment rates and relative capital. Panel A reports statistics based on historical data, and Panel B those computed on the basis of simulated panels. All dependent variables are in logs.

The first two univariate regressions show that book-to-market and size appears to contain useful information about the cross-section of stock returns. The relation between returns and book-to-market is significantly positive, while the relation with market equity is significantly negative. Moreover, the slope coefficients are close to their historical counterparts.

When both market-based firm characteristics are used as dependent variables, the slope coefficients on market size and book-to-market are about -0.1% and 5.2%, respectively. Both coefficients are statistically significant at conventional levels. While the coefficient on book-to-market is in line with historical data, the market equity coefficient is lower. However, both in the model and historical data, book-to-market effects are economically more significant than size effects.

Adding the interaction term between market equity and book-to-market results in a negative coefficient of about -1.3% (-0.6% in the data), which confirms the empirically observed decline of the value premium with market equity.

In the second bivariate regression in simulated data (line 5), I run excess stock returns on investment rate and relative capital. The average slopes confirm the negative relation of average stock returns with both these variables. Both coefficients are more than two standard errors from zero. Controlling for a firm relative capital, doubling a firm investment rate decreases on average stock returns by about 4.2% per year. Similarly, a double of relative capital leads to a reduction in average returns of about 0.4% per year. The relative economic significance of investment rate and relative capital is similar in historical data. Moreover, the effect of market equity on average stock returns is of the same order of magnitude of that generated by a firm relative capital, both in simulated and historical data.

The correspondence between market-based firm characteristics and firm fundamentals is reinforced when I run regressions of stock returns on market size and investment rate (line 8). The economic and statistical significance of the average slopes remain unchanged. The same argument applies when I consider book-to-market and relative capital as dependent variables (line 6). Furthermore, including both book-to-market and investment rate (line 7), as well as market equity and relative capital (line 9), makes the average slopes statistically insignificant.

Thus, book-to-market and market equity on one side, and investment rate and relative size on the other side, capture similar expected profitability and size effects in average stock returns, in the model and in the data. Moreover, expected profitability effects as captured by book-to-market and investment rate are economically more important than size effects in explaining cross-sectional variation in average returns.

In the model, book-to-market and investment rate are related to expected returns because they proxy for firm profitability: firms with high book-to-market and low investment rate tend to be less profitable and therefore less valuable to investors looking for consumption insurance.

Figure 2 and 3 plot the average profitability and investment rate of value and growth portfolios for 11 years around portfolio formation and in the time series based on simulated and historical data, respectively. The figures show that book-to-market is associated with persistent differences in profitability and investment rates. Growth firms are on average more profitable and faster growing than value firms for five years before and after portfolio formation. The profitability of growth firms improves prior to portfolio formation and deteriorate thereafter. The opposite is true for value firms. Investment rates follow a similar pattern. Both patterns are driven by the mean-reverting behavior of the firm productivity and the endogeneity of firm investment. The persistent difference in profitability and investment rate between value and growth is also confirmed when examined chronologically. In sum, firm profitability and investment rate are what determines value or growth characteristics.

3.2.2 Asset Pricing Models

A central finding in the asset pricing literature is the failure of the CAPM to explain the cross-sectional differences in average stock returns. In my model, the CAPM does not hold, provided that there is enough covariation between the time-varying firm market betas and the expected equity premium. In this section, I investigate the extent to which the model is consistent with the failure of the CAPM and the success of alternative asset pricing models such as a conditional CAPM and the Fama and French (1993) model. I use as test assets the twelve book-to-market sorted portfolios, which provide a sizeable spread in average returns.

Table VII shows the results of time-series regressions of excess returns on each of the twelve portfolios on the excess returns on the market portfolio. I report the results based on historical and simulated data in Panel A and B, respectively. Each panel shows the intercepts, α, and the market betas, β_M, with their corresponding standard errors. Standard errors starred with an asterisk are statistically significant at the five percent level.

Both in historical and simulated data, the αs are large and statistically significant. Moreover, they share the same pattern: growth stocks have large negative alphas whereas value stocks have large positive ones. The market betas are all statistically significant. There is virtually no variation in market betas across portfolios, especially in simulated data. The failure of the CAPM can also be seen graphically in panel A of Figure 4 (simulated data) and Figure 5 (historical data), where I plot the model predicted vs. actual mean excess returns. In both cases, mean excess returns line up vertically rather than on the 45 degree line.

In line 2 and 6 of Table IX, I report the results from Fama-MacBeth cross-sectional regressions. The cross-sectional intercept is statistically significant in both historical and simulated data. The coefficient in historical data is a negative 14% and statistically significant at the ten percent level. In simulated data the coefficient is also wrong signed, -73%, and statistically significant. In contrast, line 1 of Table IX shows that, when using the model implied market beta as dependent variable, the intercept becomes insignificant, and the statistically significant coefficient only exceeds the average equity premium by less than 1%. Additionally, the adjusted R^2 reaches about 97%. Thus, the failure of the CAPM is due to its inability to account for the covariation between time-varying betas and market premium.

Given its superior predictability power, I use the observable log dividend-price ratio as conditioning variable in a conditional CAPM model. In simulated data (line 3), there is a substantial improvement in the fit with an adjusted R^2 of about 77% and all coefficients (including the intercept) are statistically significant as in historical data (line 7). However, the implied equity premium is wrong signed in both datasets. While performing better than its unconditional version, the conditional CAPM with the observable log dividend yield as conditioning variable can only partially capture the covariation of firm conditional betas and expected equity premium. Panel B of Figure 4 (simulated data) and Figure 5 (historical data) shows the conditional CAPM predicted vs. actual mean excess returns. In both simulated and historical data, the conditional CAPM does a better job in pricing the highest and lowest book-to-market portfolios than the CAPM. However, the overall fit is better in simulated data than in historical data.

Finally, I test the performance of a two factor model including the excess returns on the market and HML (MKT+HML), and the Fama-French model. In both simulated and historical data, including only the MKT and HML makes a considerable improvement over the CAPM and the conditional CAPM. This substantial improvement can be seen in line 4 and 8 of Table IX. Although larger than the one observed in the time-series, the average excess market return is now positive, and the average return on HML is in line with its time-series counterpart. Additionally, the adjusted R^2 rises substantially. The superior performance of the two factor model can also be seen graphically in Panel C of Figure 4 (simulated data) and Figure 5 (historical data), where mean excess returns now line up better on the 45 degree line.

The Fama and French (1993) model outperforms all the above mentioned asset pricing models. It makes the intercept statistically insignificant and the loading on the SMB and HML are close to their time-series averages of about 0.4% and 4.3% in simulated data. The implied size of the equity premium is about 3%, which is lower than its time-series counterpart, but it is statistically insignificant. The adjusted R^2 is about 94%. In historical data, the inclusion of SMB makes the size of the equity and value premia close to their historical average (about 7% and 5%, respectively), although the size premium becomes larger than its time-series counterpart (about 2.7%). The adjusted R^2 is 81%. The success of the Fama and French model can be better seen graphically. Panel D of Figure 4 (simulated data) and Figure 5 (historical data) shows that mean excess returns line up much better on the 45 degree line. The root mean squared alpha decreases from 2.55% per year (CAPM) to 0.91% (Fama and French model) in simulated data. Similarly, in historical data, the root mean squared alpha goes from 4.33% (CAPM) to 1.7% per year (Fama and French model).

To understand the success of the Fama-French model, I report in Table VIII (Panel A-B) the results from time-series regressions. Both in historical and simulated data, the alphas are lower by an order of magnitude relative to the CAPM alphas, and only few of them remain statistically significant in the model generated data. The market beta is flat across portfolios (a common finding in the asset pricing literature) and the loadings on SMB and HML share the same patterns. While only few loadings on SMB are significant, all HML loadings are significant and increasing in magnitude from growth to value.

To interpret the success of the Fama-French model, note that firms' market betas can be represented as a value-weighted average of betas for assets in place and betas for growth options. Interacting each of these betas with the expected excess returns on the market provides an alternative interpretation of the conditional CAPM relation. That is, the expected excess returns on any asset are proportional to the market price of aggregate assets in place risk and to the market price of aggregate growth option risk. While a weighted combination of the betas may not be significant in explaining the cross-section of stock returns, each risk source's beta may be significant since it carries a distinct price. Obviously, their sum equals the market price of aggregate productivity risk. While conditionally such a decomposition is redundant, it might be important unconditionally. Indeed, the Fama-French factors, SMB and HML, provide good proxies to account for the covariation of each market betas' component and the market risk premium.

From the market beta decomposition (42), firms with high book-to-market and similar market equity derive most of their riskiness from changes in the value of assets in place. Similarly, firms with small market equity and similar book-to-market derive most of their riskiness from changes in the value of growth options. Therefore, according to their definition, HML and SMB capture, within the model, the component of equity premium associated with the aggregate assets in place risk and the aggregate growth options risk, respectively. This is confirmed in Table VIII (Panel B) by the increasing pattern of the loadings on HML from growth to value. Similarly, the loadings on SMB increase from growth to value as their market equity decreases.

In Table X, I run Fama-MacBeth cross-sectional regressions of risk-adjusted returns (i.e. actual returns minus model predicted returns from the time-series regressions) on firm characteristics. In historical data, the conditional CAPM, the two factor model, and the Fama-French model drive out market-based firms characteristics, although the investment rate seems to capture part of cross-sectional variation in average returns unaccounted by these asset pricing models. However, the fact that the asset pricing models under consideration drive out market-based firm characteristics in historical data might be sample specific. Recently, Jagannathan and Wang (2005) find different evidence. This might be due to the more stringent sample selection criteria I use to construct my sample because of the inclusion of fundamental firm characteristics. In simulated data, part of the cross-sectional variation in average returns unaccounted by the conditional CAPM, is captured by the book-to-market ratio and the fundamental firm characteristics. In contrast, the two factor model and the Fama-French model drive out both book-to-market and market equity, although the firm investment rate has still some explanatory power as in the data.

4 Conclusion

In this paper, I show how the interaction between the economic behavior of utility-maximizing consumers and the behavior of value-maximizing firms can rationalize many empirical regularities in the cross-section of stock returns. Most importantly, this effort contributes to the understanding of how risk, returns and firm characteristics relate to the real side of the economy.

I construct a general equilibrium production economy with a continuum of heterogeneous firms and irreversible investment. The model aggregates so that aggregate productivity and a single moment of the joint cross-sectional distribution of firm-specific productivity and capital are sufficient state variables for the characterization of aggregate quantities and prices.

The dynamics of investors' demand for consumption insurance and irreversibility in firm investments play a key role in explaining value and size effects in stock returns and their relation to risk and firm fundamentals. The riskiness of firms' equity depends on their ability to supply consumption insurance. A firm can provide valuable consumption insurance if she can mitigate the effect of aggregate productivity shocks through investment in order to smooth dividends. Capital adjustment costs and investment irreversibility deprive unprofitable "small" and "value" firms of flexibility in cutting capital, causing them to be riskier than "big" and "growth" firms, especially in bad times when the aggregate consumption volatility (and the market price of productivity risk) is high.

The greater risk of small and value firms shows up in a conditional beta that is high in bad times when the market premium is high, but not necessarily in a high unconditional beta, thus explaining the failure of CAPM and relative success of a conditional CAPM and the Fama-French three factor model. These last two models are relatively successful because they better capture the covariation of firm conditional betas and the market risk premium.

5 Proofs and Technical Details

In this section I provide all the proofs and technical details. In the following, I omit the time subscript where unnecessary.

5.1 Proof of Proposition 1

Firms make investment decisions to maximize the expected present value of future dividends. Let $V_{it} \equiv V(a_t, \omega_t, x_{it}, k_{it}, K_{it})$ be the value function of the firm:

$$V_{it} = \max_{\left\{i_{it+s} \geq \widehat{i}: s \in \mathbb{R}^+\right\}} \mathbf{E}_t \left[\int_0^\infty \frac{\Lambda_{t+s}}{\Lambda_t} \left(e^{a_{t+s}} x_{it+s} + f - i_{it+s} - \alpha k_{it+s}^{\frac{1}{n-1}} \left(i_{it+s} - \widehat{i} \right)^{\frac{n}{n-1}} \right) K_{it+s} ds \right] \tag{47}$$

subject to the evolution of the economy wide productivity a in (2), the law of motion of the idiosyncratic productivity x_i in (3), the firm capital accumulation with its non-negativity constraint in (4), the evolution of the firm relative capital k_i in (13), and the conjectured dynamics of the equilibrium pricing-kernel Λ and the state variable ω described in (14) and (11), respectively.

Then, the firm value function V_i satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \max_{i_i \geq \widehat{i}} \left\{ \Lambda K_i \left[e^a x_i + f - i_i - \alpha k_i^{\frac{1}{n-1}} \left(i_i - \widehat{i} \right)^{\frac{n}{n-1}} \right] + \mathcal{D}\left[\Lambda V_i\right] \right\} \tag{48}$$

with $\mathcal{D}[\Lambda V_i]$ denoting the infinitesimal generator of the Markov processes a and x^i, and the singular processes K_i, k_i and ω, applied to the discounted firm value function ΛV_i, along with the transversality ("no bubble") condition:

$$\lim_{T\to\infty} \mathbf{E}_t \left[\left| \Lambda_{t+T} V_{it+T} \right| \right] = 0. \tag{49}$$

Conjecture that the value function takes the form:

$$V(a, \omega, x_i, k_i, K_i) = \left[q(a, \omega, x_i) + h(a, \omega, x_i) k_i^{-1} \right] K_i \tag{50}$$

Then, the HJB equation in (48) reads:

$$0 = \max_{i_i \geq \widehat{i}} \left\{ e^a x_i + f - i_i - \alpha k_i^{\frac{1}{n-1}} \left(i_i - \widehat{i} \right)^{\frac{n}{n-1}} + \frac{\mathcal{D}\left[\Lambda q_i K_i\right]}{\Lambda K_i} + \frac{\mathcal{D}\left[\Lambda h_i K\right]}{\Lambda K_i} \right\} \tag{51}$$

where

$$\begin{aligned} \mathcal{D}\left[\Lambda q_i K_i\right] &= \Lambda q_i \left(I_i - \delta K_i \right) + K^i \mathcal{D}\left[\Lambda q_i\right] \\ \mathcal{D}\left[\Lambda h_i K\right] &= \Lambda h_i \left(I - \delta K \right) + K \mathcal{D}\left[\Lambda h_i\right]. \end{aligned}$$

Rearranging terms in (51) leads to:

$$\begin{aligned}0 &= \max_{i_i \geq \widehat{\imath}} \left\{ [q_i - 1]\, i_i - \alpha k_i^{\frac{1}{n-1}} \left(i_i - \widehat{\imath} \right)^{\frac{n}{n-1}} \right\} \\ &\quad + e^{a} x_i + f - q_i \delta + \frac{\mathcal{D}[\Lambda q_i]}{\Lambda} + k_i^{-1} \left[h_i (i - \delta) + \frac{\mathcal{D}[\Lambda h_i]}{\Lambda} \right] \end{aligned} \quad (52)$$

Given that the maximand in (52) is strictly concave and everywhere differentiable in i_i, the first order condition uniquely determining the optimal investment policy i_i^* is given by

$$[q_i - 1] - \frac{\alpha n}{n-1} k_i^{\frac{1}{n-1}} \left(i_i - \widehat{\imath} \right)^{\frac{1}{n-1}} \leq 0 \quad (53)$$

along with the complementary slackness condition

$$\left[[q_i - 1] - \frac{\alpha n}{n-1} k_i^{\frac{1}{n-1}} \left(i_i - \widehat{\imath} \right)^{\frac{1}{n-1}} \right] \left(i_i - \widehat{\imath} \right) = 0. \quad (54)$$

According to equations (53) - (54) the firm optimal investment policy can be summarized as

$$i_i^* = \widehat{\imath} + \left(\frac{n-1}{\alpha n} \right)^{n-1} (q_i - 1)^{n-1} k_i^{-1} \mathbf{1}_{\{q_i \geq 1\}}. \quad (55)$$

When the irreversibility constraint is not binding, a firm equates the marginal cost of investment and its marginal benefit as measured by q_i. However, when the irreversibility constraint is binding, the optimal investment rate equals the minimum level $\widehat{\imath}$. Hence, evaluating equation (52) at the optimal investment policy (55) leads to:

$$\left[q_i \left(\delta - \widehat{\imath} \right) - e^{a} x_i - f + \widehat{\imath} - \frac{\mathcal{D}[\Lambda q_i]}{\Lambda} \right] = k_i^{-1} \left[\eta (q_i - 1)^n \mathbf{1}_{\{q_i \geq 1\}} + h_i (i - \delta) + \frac{\mathcal{D}[\Lambda h_i]}{\Lambda} \right] \quad (56)$$

where $\eta = (n-1)^{n-1} \alpha^{-(n-1)} n^{-n} > 0$. Since the left-hand side of equation (56) is independent of the value $k_i \in \mathbb{R}^+$, in order for (56) to hold for all $k_i \in \mathbb{R}^+$ the term in square brackets on the left-hand side must equal zero:

$$\Lambda \left(e^{a} x_i + f - \widehat{\imath} \right) - \Lambda q_i \left(\delta - \widehat{\imath} \right) + \mathcal{D}[\Lambda q_i] = 0 \quad (57)$$

and the right-hand side must also equal zero:

$$\Lambda \eta (q_i - 1)^n \mathbf{1}_{\{q_i \geq 1\}} - \Lambda h_i (\delta - i) + \mathcal{D}[\Lambda h_i] = 0. \quad (58)$$

The Feynman-Kac Theorem[7] implies that the partial differential equation (57) admits the following probabilistic solution for $q \in C^2 (\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$:

$$q(a_t, \omega_t, x_{it}) = \mathbf{E}_t \left[\int_0^\infty e^{-\left(\delta - \widehat{\imath}\right) s} \frac{\Lambda_{t+s}}{\Lambda_t} \left(e^{a_{t+s}} x_{it+s} + f - \widehat{\imath} \right) ds \right]. \quad (59)$$

[7] See, for example, Duffie (Appendix E, 2001), Karatzas and Shreve (Theorem 7.6, 1991), Krylov (Theorem 4, pag. 198, 1995), Yong and Zhou (Theorem 4.1-3, pag. 373-5,1999).

The probabilistic solution to the partial differential equation (59) can be further represented as

$$
\begin{aligned}
& q\left(a_t, \omega_t, x_{it}\right) \overset{(1)}{=} \int_0^{\infty} e^{-\left(\delta-\widehat{i}\right)s}\mathbf{E}_t\left[\frac{\Lambda_{t+s}}{\Lambda_t}e^{a_{t+s}}\right]\mathbf{E}_t\left[x_{it+s}\right]ds+\mathbf{E}_t\left[\int_0^{\infty} e^{-\left(\delta-\widehat{i}\right)s}\frac{\Lambda_{t+s}}{\Lambda_t}\left(f-\widehat{i}\right)ds\right] \\
& \overset{(2)}{=} \overline{x}\mathbf{E}_t\left[\int_0^{\infty} e^{-\left(\delta-\widehat{i}\right)s}\frac{\Lambda_{t+s}}{\Lambda_t}\left[e^{a_{t+s}}+\overline{x}^{-1}\left(f-\widehat{i}\right)\right]ds\right]+\left[x_{it}-\overline{x}\right]\mathbf{E}_t\left[\int_0^{\infty} e^{-\left(\kappa_x+\delta-\widehat{i}\right)s}\frac{\Lambda_{t+s}}{\Lambda_t}e^{a_{t+s}}ds\right] \\
& \overset{(3)}{=} \overline{xq}\left(a_t, \omega_t\right)+\left[x_{it}-\overline{x}\right]\widehat{q}\left(a_t, \omega_t\right)
\end{aligned}
\tag{60}
$$

where (1) follows from the application of Tonelli's Theorem and the independence of x_{it} from a_t and ω_t, (2) follows from the Strong Markov property of x_{it} and from $E\left[x_{it+s}|x_{it}\right]=\overline{x}+\left[x_{it}-\overline{x}\right]e^{-\kappa_x s}$ and (3) from

$$
\begin{aligned}
\overline{q}\left(a_t, \omega_t\right) &= \mathbf{E}_t\left[\int_0^{\infty} e^{-\left(\delta-\widehat{i}\right)s}\frac{\Lambda_{t+s}}{\Lambda_t}\left[e^{a_{t+s}}+\overline{x}^{-1}\left(f-\widehat{i}\right)\right]ds\right] \\
\widehat{q}\left(a_t, \omega_t\right) &= \mathbf{E}_t\left[\int_0^{\infty} e^{-\left(\kappa_x+\delta-\widehat{i}\right)s}\frac{\Lambda_{t+s}}{\Lambda_t}e^{a_{t+s}}ds\right].
\end{aligned}
$$

Given the absence of arbitrage or equivalently the strictly positivity of the pricing-kernel Λ inherited from the strictly positivity of aggregate consumption ensured by the Inada conditions, it is sufficient to restrict $f\geq\widehat{i}$ to ensures the positivity of the firm "marginal q".

Q.E.D.

5.2 Proof of Proposition 2

Let $\int_{i \in F} [\cdot]\, di$ denote the aggregation operator over firms, and define the aggregate (average) capital stock as $K \equiv \int_{i \in F} K_i di$. In order to facilitate the representation of aggregate quantities, let $g(a, \omega; m_1, m_2)$ denote a function of the state variables a and ω defined as

$$g(a, \omega; m_1, m_2) \equiv \widehat{q}^{m_1} \sum_{k=0}^{m_1} \frac{\Gamma(m_1+1)\, \Gamma_U(k+\upsilon, \theta\widetilde{x})}{\Gamma(m_1+1-k)\, \Gamma(k+m_2)\, \Gamma(\upsilon)} (-\widetilde{x})^{m_1-k}\, \theta^{-k} \tag{61}$$

where m_1 and m_2 represent constant parameters.

Aggregate output is defined as $Y \equiv \int_{i \in F} Y_i di$ and can be represented as

$$Y = \int_{i \in F} (e^a x_i K_i + f K_i)\, di \overset{(1)}{=} \left[e^a \int_{i \in F} x_i k_i di + f \right] K \overset{(2)}{=} (e^a \omega + f)\, K \tag{62}$$

where (1) follows from the definition of K and the firm relative capital $k_i \equiv K_i/K$, and (2) from the definition of the endogenous aggregate productivity component ω in (10).

Similarly, aggregate (average) investment is defined as $I \equiv \int_{i \in F} I_i^* di$ and can be characterized as follows:

$$\begin{aligned} I &\overset{(1)}{=} \int_{i \in F} \left[\widehat{i} K_i + \left(\frac{n-1}{\alpha n}\right)^{n-1} (q_i - 1)^{n-1} K \mathbf{1}_{\{q_i - 1 \geq 0\}} \right] di \\ &\overset{(2)}{=} \widehat{i} K + \left(\frac{n-1}{\alpha n}\right)^{n-1} \widehat{q}^{n-1} K \int_{i \in F} (x_i - \widetilde{x})^{n-1} \mathbf{1}_{\{x_i - \widetilde{x} \geq 0\}} di \\ &\overset{(3)}{=} \widehat{i} K + \left(\frac{n-1}{\alpha n}\right)^{n-1} \widehat{q}^{n-1} K \int_0^{\infty} (x - \widetilde{x})^{n-1} \mathbf{1}_{\{x - \widetilde{x} \geq 0\}} f_x(x; \theta, \upsilon)\, dx \\ &\overset{(4)}{=} \widehat{i} K + \left(\frac{n-1}{\alpha n}\right)^{n-1} \widehat{q}^{n-1} K \int_{\widetilde{x}^+}^{\infty} (x - \widetilde{x})^{n-1} \frac{\theta^\upsilon}{\Gamma(\upsilon)} x^{\upsilon-1} e^{-\theta x} dx \end{aligned} \tag{63}$$

where (1) follows from the firm optimal investment policy in (55), (2) from the definition of aggregate (average) capital $K \equiv \int_{i \in F} K_i di$ and from the fact that the firm marginal q can be rewritten as $q_i = 1 + \widehat{q}(x_i - \widetilde{x})$, where $\widetilde{x} \equiv [1 - \overline{x}(\overline{q} - \widehat{q})] / \widehat{q}$ denotes the investment irreversibility threshold with reference to the firm specific productivity and it is expressed in terms of the aggregate values $\overline{q}$ and $\widehat{q}$. The third equality follows from the Glivenko-Cantelli Theorem[8], according to which the cross-sectional distribution of the i.i.d. firm specific productivity x equals its stationary distribution $f_x(x; \theta, \upsilon)$, and (4) from the fact that the stationary distribution of the stochastic process x whose dynamics is given in (3) is a gamma distribution:[9]

$$f_x(x; \theta, \upsilon) = \frac{\theta^\upsilon}{\Gamma(\upsilon)} x^{\upsilon-1} e^{-\theta x} \mathbf{1}_{\{0 \leq x < \infty\}}; \quad \theta, \upsilon > 0 \tag{64}$$

[8]See, for example, Billingsley (Theorem 20.6, 1979) and Parthasarathy (Theorem II.7.1, 1967).

[9]See, for example, Cox, Ingersoll, Ross (1985).

with $\theta \equiv 2\kappa_x/\sigma_x^2$ and $\upsilon \equiv 2\kappa_x \overline{x}/\sigma_x^2$ ($\kappa_x, \overline{x} > 0$ and $\sigma_x \neq 0$). The value $\widetilde{x}^+$ in the lower limit of integration in (63) stands for $\max(0, \widetilde{x})$ and results from the product of the two indicator functions $\mathbf{1}_{\{x \geq \widetilde{x}\}} \times \mathbf{1}_{\{0 \leq x < \infty\}}$.

The integral in (63) can be further represented as:

$$
\begin{aligned}
&\int_{\widetilde{x}^+}^{\infty} \frac{\theta^\upsilon}{\Gamma(\upsilon)} (x-\widetilde{x})^{n-1} x^{\upsilon-1} e^{-\theta x} dx \overset{(1)}{=} \sum_{k=0}^{n-1} \frac{\Gamma(n)(-\widetilde{x})^{n-1-k}\theta^{-(k-1)}}{\Gamma(n-k)\Gamma(k+1)\Gamma(\upsilon)} \int_{\widetilde{x}^+}^{\infty} (\theta x)^{k+\upsilon-1} e^{-\theta x} dx \\
&\overset{(2)}{=} \sum_{k=0}^{n-1} \frac{\Gamma(n)(-\widetilde{x})^{n-1-k}\theta^{-k}}{\Gamma(n-k)\Gamma(k+1)\Gamma(\upsilon)} \int_{\theta\widetilde{x}^+}^{\infty} y^{k+\upsilon-1} e^{-y} dy \\
&\overset{(3)}{=} \sum_{k=0}^{n-1} \frac{\Gamma(n)\Gamma_U(k+\upsilon, \theta\widetilde{x}^+)}{\Gamma(n-k)\Gamma(k+1)\Gamma(\upsilon)} (-\widetilde{x})^{n-1-k}\theta^{-k} \qquad (65)
\end{aligned}
$$

where (1) follows from the Binomial Theorem $(x-\widetilde{x})^{n-1} = \sum_{k=0}^{n-1} \frac{\Gamma(n)x^k(-\widetilde{x})^{n-1-k}}{\Gamma(n-k)\Gamma(k+1)}$, (2) from the change of variable $y = \theta x$, and (3) from the definition of the upper incomplete gamma function $\Gamma_U(\alpha, z) \equiv \int_z^\infty x^{\alpha-1} e^{-x} dx$.

In order to ensure the existence of $q \in C^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$, I assume throughout the following analysis that the investment threshold never falls below zero, i.e. $\widetilde{x} \geq 0$. Under standard integrability conditions, it is sufficient to appropriately restrict the model parameters such that $\sup|\overline{q} - \widehat{q}| \leq 1/\overline{x}$ in order to meet this restriction. Furthermore, the strictly positivity of the firm marginal q implies that $\widetilde{x} < 1/\widehat{q}$. Hence, the existence of a strictly positive firm marginal $q \in C^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$ implies that the investment threshold $\widetilde{x} \in [0, 1/\widehat{q})$.

Therefore, aggregate investment can be characterized as:

$$
I = \left[\widehat{i} + \left(\frac{n-1}{\alpha n}\right)^{n-1} g(a, \omega; n-1, 1)\right] K \qquad (66)
$$

with the function g in (61) evaluated at $m_1 = n-1$ and $m_2 = 1$.

Aggregate (average) dividend is defined as $D \equiv \int_{i \in F} D_i^* di$ and can be characterized as follows:

$$
\begin{aligned}
D &\overset{(1)}{=} \int_{i \in F} \left[(e^a x_i + f) K_i - I_i - \alpha k_i^{\frac{1}{n-1}} \left(\frac{I_i - \widehat{I}_i}{K_i}\right)^{\frac{n}{n-1}} K_i\right] di \\
&\overset{(2)}{=} Y - I - \alpha \left(\frac{n-1}{\alpha n}\right)^n K \int_{i \in F} (q_i - 1)^n \mathbf{1}_{\{q_i \geq 1\}} di \\
&\overset{(3)}{=} Y - I - \alpha \left(\frac{n-1}{\alpha n}\right)^n K \widehat{q}^n \int_{i \in F} (x_i - \widetilde{x})^n \mathbf{1}_{\{x_i - \widetilde{x} \geq 0\}} di \\
&\overset{(4)}{=} Y - I - \alpha \left(\frac{n-1}{\alpha n}\right)^n K \widehat{q}^n \int_{\widetilde{x}}^{\infty} (x-\widetilde{x})^n \frac{\theta^\upsilon}{\Gamma(\upsilon)} x^{\upsilon-1} e^{-\theta x} dx \qquad (67)
\end{aligned}
$$

where (1) follows from the definition of firm dividends in (6), (2) from the firm optimal investment policy in (55) and definition of aggregate output and investment in (62) and (66), respectively. The third equality results from the fact that the firm marginal q can be rewritten as $q_i = 1 + \widehat{q}\,(x_i - \widetilde{x})$, where $\widetilde{x} \equiv [1 - \overline{x}\,(\overline{q} - \widehat{q})]\,/\widehat{q}$ denotes the investment irreversibility threshold. The last equality follows from the Glivenko-Cantelli Theorem, according to which the cross-sectional distribution of the i.i.d. firm specific productivity x equals its stationary distribution in (64), and the restriction on the investment threshold $\widetilde{x} \in [0, 1/\widehat{q})$.

The integral in (67) can be further represented as:

$$
\begin{aligned}
\int_{\widetilde{x}}^{\infty} (x-\widetilde{x})^n \frac{\theta^{\upsilon}}{\Gamma(\upsilon)} x^{\upsilon-1} e^{-\theta x} dx &\overset{(1)}{=} \sum_{k=0}^{n} \frac{\Gamma(n+1)(-\widetilde{x})^{n-k}\theta^{-(k-1)}}{\Gamma(n-k+1)\Gamma(k+1)\Gamma(\upsilon)} \int_{\widetilde{x}}^{\infty} (\theta x)^{k+\upsilon-1} e^{-\theta x} dx \\
&\overset{(2)}{=} \sum_{k=0}^{n} \frac{\Gamma(n+1)(-\widetilde{x})^{n-k}\theta^{-k}}{\Gamma(n-k+1)\Gamma(k+1)\Gamma(\upsilon)} \int_{\theta\widetilde{x}}^{\infty} (y)^{k+\upsilon-1} e^{-y} dy \\
&\overset{(3)}{=} \sum_{k=0}^{n} \frac{\Gamma(n+1)\Gamma_U(k+\upsilon, \theta\widetilde{x})}{\Gamma(n-k+1)\Gamma(k+1)\Gamma(\upsilon)} (-\widetilde{x})^{n-k}\theta^{-k} \qquad (68)
\end{aligned}
$$

where (1) follows from the Binomial Theorem $(x-\widetilde{x})^n = \sum_{k=0}^{n} \frac{\Gamma(n+1)x^k(-\widetilde{x})^{n-k}}{\Gamma(n-k+1)\Gamma(k+1)}$, (2) from the change of variable $y = \theta x$, and (3) from the definition of the upper incomplete gamma function $\Gamma_U(\alpha, z) \equiv \int_z^{\infty} x^{\alpha-1} e^{-x} dx$.

Therefore, aggregate dividend can be represented as:

$$
\frac{D}{K} = e^a \omega + f - \widehat{i} - \left(\frac{n-1}{\alpha n}\right)^{n-1} \left[g(a, \omega; n-1, 1) + \frac{n-1}{n} g(a, \omega; n, 1) \right] \qquad (69)
$$

where the function g is given in (61).

Q.E.D.

5.3 Proof of Proposition 3

The second moment of the joint cross-sectional distribution of x_i and k_i is defined as $\omega \equiv \int_{i \in F} x_i k_i di$. I now, derive its law of motion following two steps. First, I characterize the law of motion of the weighted firm specific productivity $x_i k_i$ as:

$$\begin{aligned} dx_i k_i &\overset{(1)}{=} k_i \left[\kappa_x \left(\overline{x} - x_i\right) dt + \sigma_x \sqrt{x_i} dW_i\right] + x_i k_i \left[i_i - i\right] dt \\ &\overset{(2)}{=} \left\{\kappa_x \left(\overline{x} - x_i\right) k_i + \left(\frac{n-1}{\alpha n}\right)^{n-1} \left[\widehat{q}^{n-1} \left(x_i - \widetilde{x}\right)^{n-1} x_i \mathbf{1}_{\{x_i \geq \widetilde{x}\}} - g\left(a, \omega; n-1, 1\right) x_i k_i\right]\right\} dt \\ &+ k_i \sigma_x \sqrt{x_i} dW_i \end{aligned} \tag{70}$$

where (1) follows from the application of Ito's Formula to $x_i k_i$ with the processes x_i and k_i evolving as in (3) and (13), respectively, and (2) from the optimal firm investment policy and aggregate investment rate in (55) and (66), respectively. Then, it follows that ω evolves according to:

$$\begin{aligned} d\omega &\overset{(1)}{=} \int_{i \in F} dx_i k_i di \\ &\overset{(2)}{=} \left\{\kappa_x \left(\overline{x} - \omega\right) + \left(\frac{n-1}{\alpha n}\right)^{n-1} \left[\widehat{q}^{n-1} \int_{i \in F} x_i \left(x_i - \widetilde{x}\right)^{n-1} \mathbf{1}_{\{x_i \geq \widetilde{x}\}} di - g\left(a, \omega; n-1, 1\right) \omega\right]\right\} dt \end{aligned} \tag{71}$$

where (1) follows from Fubini's Theorem under the assumption of joint measurability, (2) from the definition of $\omega \equiv \int_{i \in F} x_i k_i di$ and the fact that $\int_{i \in F} k_i di = 1$. The independence of $k_i \sqrt{x_i}$ and dW_i and the law of large numbers applied to dW_i's, which are cross-sectionally i.i.d. with zero mean and finite variance, ensures that $\sigma_x \int_{i \in F} k_i \sqrt{x_i} dW_i di = 0$.

The integral in (71) can be computed as:

$$\begin{aligned} \int_{i \in F} x_i \left(x_i - \widetilde{x}\right)^{n-1} \mathbf{1}_{\{x_i \geq \widetilde{x}\}} di &\overset{(1)}{=} \int_{\widetilde{x}}^{\infty} x \left(x - \widetilde{x}\right)^{n-1} \frac{\theta^{\upsilon}}{\Gamma(\upsilon)} x^{\upsilon - 1} e^{-\theta x} dx \\ &\overset{(2)}{=} \theta^{-1} \sum_{k=0}^{n-1} \frac{\Gamma(n) \left(-\widetilde{x}\right)^{n-1-k} \theta^{-k}}{\Gamma(n-k) \Gamma(k+1) \Gamma(\upsilon)} \int_{\theta \widetilde{x}}^{\infty} y^{k+\upsilon} e^{-y} dy \\ &\overset{(3)}{=} \theta^{-1} \sum_{k=0}^{n-1} \frac{\Gamma(n) \Gamma_U\left(k+\upsilon+1, \theta \widetilde{x}\right)}{\Gamma(n-k) \Gamma(k+1) \Gamma(\upsilon)} \left(-\widetilde{x}\right)^{n-1-k} \theta^{-k} \\ &\overset{(4)}{=} \theta^{-1} \sum_{k=0}^{n-1} \frac{\Gamma(n) \left(-\widetilde{x}\right)^{n-1-k} \theta^{-k}}{\Gamma(n-k) \Gamma(k+1) \Gamma(\upsilon)} \left[\left(k+\upsilon\right) \Gamma_U\left(k+\upsilon, \theta \widetilde{x}\right) + \left(\theta \widetilde{x}\right)^{k+\upsilon} e^{-\theta \widetilde{x}}\right] \\ &\overset{(5)}{=} \sum_{k=0}^{n-1} \frac{\Gamma(n) \Gamma_U\left(k+\upsilon, \theta \widetilde{x}\right)}{\Gamma(n-k) \Gamma(k+1) \Gamma(\upsilon)} \left(-\widetilde{x}\right)^{n-1-k} \theta^{-k} \left(\frac{k}{\theta} + \overline{x}\right) \end{aligned} \tag{72}$$

where (1) follows from the Glivenko-Cantelli Theorem, according to which the cross-sectional distribution of the i.i.d. firm specific productivity x equals its stationary distribution in (64), and the restriction on the investment threshold $\widetilde{x} \in [0, 1/\widehat{q})$. The second equality follows from the Binomial Theorem, $(x - \widetilde{x})^{n-1} = \sum_{k=0}^{n-1} \frac{\Gamma(n) x^k (-\widetilde{x})^{n-1-k}}{\Gamma(n-k)\Gamma(k+1)}$, and the change of variable $y = \theta x$, (3) from the definition of the upper incomplete gamma function $\Gamma_U(\alpha, z) \equiv \int_z^{\infty} x^{\alpha-1} e^{-x} dx$. The fourth equality results from the property of the upper incomplete gamma function (integration by parts), $\Gamma_U(k + \upsilon + 1, \theta\widetilde{x}) = (k + \upsilon)\Gamma_U(k + \upsilon, \theta\widetilde{x}) + (\theta\widetilde{x})^{k+\upsilon} e^{-\theta\widetilde{x}}$, and (5) from the fact that $\sum_{k=0}^{n-1} \frac{\Gamma(n)}{\Gamma(n-k)\Gamma(k+1)} (-1)^{n-1-k} = (1-1)^{n-1} = 0$ and $\upsilon/\theta = \overline{x}$.

Therefore, the endogenous component of aggregate productivity evolves according to:

$$
\begin{aligned}
d\omega &\overset{(1)}{=} \left\{ \left[\kappa_x + \left(\frac{n-1}{\alpha n} \right)^{n-1} g(a, \omega; n-1, 1) \right] (\overline{x} - \omega) + \left(\frac{n-1}{\alpha n} \right)^{n-1} \theta^{-1} g(a, \omega; n-1, 0) \right\} dt \\
&\overset{(2)}{=} \left\{ \left[\kappa_x + \left(i - \widehat{i} \right) \right] (\overline{x} - \omega) + \left(i - \widehat{i} \right) \theta^{-1} \frac{g(a, \omega; n-1, 0)}{g(a, \omega; n-1, 1)} \right\} dt \qquad (73)
\end{aligned}
$$

where (1) results from the property of the gamma function, $\Gamma(k+1) = k\Gamma(k)$, and the definition of the function g in (61), and (2) from the characterization of aggregate investment in (66).

Q.E.D.

5.4 Proof of Proposition 4

The market clearing condition for the consumption good requires aggregate consumption C^* to equal aggregate dividend D^*, hence equations (24) - (25) correspond to the aggregate dividend in (21).

The equilibrium prices and quantities depend on the firm marginal q as characterized in equations (16) - (18), which is now evaluated at the equilibrium pricing-kernel Λ. In equilibrium, the representative household intertemporal marginal rate of substitution between consumption at time $t+s$ and consumption at time t is given by:

$$\frac{\Lambda_{t+s}}{\Lambda_t} \overset{(1)}{=} e^{-\rho s}\left(\frac{C^*_{t+s}}{C^*_t}\right)^{-\gamma} \overset{(2)}{=} e^{-\rho s}\left(\frac{c^*\left(a_{t+s},\omega_{t+s}\right)}{c^*\left(a_t,\omega_t\right)}\right)^{-\gamma}\left(\frac{K_{t+s}}{K_t}\right)^{-\gamma}$$
$$\overset{(3)}{=} e^{-\int_t^{t+s}\left\{\rho+\gamma\left[\widehat{i}+\left(\frac{n-1}{\alpha n}\right)^{n-1} g(a_u,\omega_u;n-1,1)-\delta\right]\right\}du}\left(\frac{c^*\left(a_{t+s},\omega_{t+s}\right)}{c^*\left(a_t,\omega_t\right)}\right)^{-\gamma} \quad (74)$$

where (1) follows from the representative household's first-order optimality condition, (2) from the equilibrium consumption policy (24) - (25), and (3) from the dynamics of the aggregate stock of capital evaluated at the equilibrium aggregate investment in (66). From the characterization of $\overline{q}\left(a,\omega\right)$ and $\widehat{q}\left(a,\omega\right)$ in (17) - (18), after applying some straightforward algebra it follows that

$$\overline{q}_t = \mathbf{E}_t\left[\int_0^\infty e^{-\left(\delta-\widehat{i}\right)s}\frac{\Lambda_{t+s}}{\Lambda_t}\left[e^{a_{t+s}}+\overline{x}^{-1}\left(f-\widehat{i}\right)\right]ds\right] = c^*\left(a_t,\omega_t\right)^\gamma\,\overline{\Phi}\left(a_t,\omega_t\right) \quad (75)$$
$$\widehat{q}_t = \mathbf{E}_t\left[\int_0^\infty e^{-\left(\kappa_x+\delta-\widehat{i}\right)s}\frac{\Lambda_{t+s}}{\Lambda_t}e^{a_{t+s}}ds\right] = c^*\left(a_t,\omega_t\right)^\gamma\,\widehat{\Phi}\left(a_t,\omega_t\right) \quad (76)$$

where

$$\overline{\Phi}_t \equiv \mathbf{E}_t\left[\int_0^\infty e^{-\int_t^{t+s}\left\{\rho+(1-\gamma)\left(\delta-\widehat{i}\right)+\gamma\left(\frac{n-1}{\alpha n}\right)^{n-1}g(a_u,\omega_u;n-1,1)\right\}du}\frac{\left[e^{a_{t+s}}+\overline{x}^{-1}\left(f-\widehat{i}\right)\right]}{c^*\left(a_{t+s},\omega_{t+s}\right)^\gamma}ds\right] \quad (77)$$
$$\widehat{\Phi}_t \equiv \mathbf{E}_t\left[\int_0^\infty e^{-\int_t^{t+s}\left\{\rho+\kappa_x+(1-\gamma)\left(\delta-\widehat{i}\right)+\gamma\left(\frac{n-1}{\alpha n}\right)^{n-1}g(a_u,\omega_u;n-1,1)\right\}du}\frac{e^{a_{t+s}}}{c^*\left(a_{t+s},\omega_{t+s}\right)^\gamma}ds\right] \quad (78)$$

The Feynman-Kac Theorem implies that $\overline{\Phi}$, $\widehat{\Phi} \in C^2\left(\mathbb{R}\times\mathbb{R}^+\right)$ satisfy the following partial differential equations:

$$\left\{\rho+(1-\gamma)\left(\delta-\widehat{i}\right)+\gamma\left(\frac{n-1}{\alpha n}\right)^{n-1}g\left(a,\omega;n-1,1\right)\right\}\overline{\Phi}-\mathcal{D}\left[\overline{\Phi}\right] = \frac{e^a+\overline{x}^{-1}\left(f-\widehat{i}\right)}{c^*\left(a,\omega\right)^\gamma} \quad (79)$$
$$\left\{\rho+\kappa_x+(1-\gamma)\left(\delta-\widehat{i}\right)+\gamma\left(\frac{n-1}{\alpha n}\right)^{n-1}g\left(a,\omega;n-1,1\right)\right\}\widehat{\Phi}-\mathcal{D}\left[\widehat{\Phi}\right] = \frac{e^a}{c^*\left(a,\omega\right)^\gamma} \quad (80)$$

provided that standard integrability conditions are satisfied, i.e. $\overline{\Phi}$, $\widehat{\Phi} < \infty$. Notice that the existence of no arbitrage is ensured by the strictly positivity of the aggregate consumption process resulting from the fact that the marginal utility of consumption satisfies the Inada conditions. The stationarity and strictly positivity of the aggregate output-to-capital ratio and aggregate consumption-to-capital ratio imply that the aggregate investment rate is bounded. This in turn implies that $\overline{q}$ and $\widehat{q}$ are also bounded, since the aggregate investment rate is an increasing function of $\overline{q}$ and $\widehat{q}$.

Q.E.D.

5.5 Proof of Proposition 5

The equilibrium pricing kernel dynamics can be computed by applying Ito's Formula to the representative household marginal utility of consumption $\Lambda_t = e^{-\rho t}\left(C_t^*\right)^{-\gamma}$ as

$$\frac{d\Lambda}{\Lambda} = -\rho dt - \frac{\gamma}{C^*}dC^* + \frac{1}{2}\frac{\gamma\left(\gamma+1\right)}{\left(C^*\right)^2}\langle dC^*, dC^*\rangle = -r\left(a,\omega\right)dt - \lambda\left(a,\omega\right)dW_a \tag{81}$$

where

$$r\left(a,\omega\right) \equiv \rho + \gamma\frac{\mathcal{D}^{a,\omega,K}\left[C^*\left(a,\omega,K\right)\right]}{C^*\left(a,\omega,K\right)} - \frac{1}{2}\gamma\left(\gamma+1\right)\sigma_a^2\left[\frac{\partial_a C^*\left(a,\omega,K\right)}{C^*\left(a,\omega,K\right)}\right]^2 \tag{82}$$

$$\lambda\left(a,\omega\right) \equiv \gamma\sigma_a\frac{\partial_a C^*\left(a,\omega,K\right)}{C^*\left(a,\omega,K\right)}. \tag{83}$$

and

$$\frac{\mathcal{D}^{a,\omega,K}\left[C^*\left(a,\omega,K\right)\right]}{C^*\left(a,\omega,K\right)} \equiv \kappa_a\left(\overline{a}-a\right)\frac{\partial_a C^*\left(a,\omega,K\right)}{C^*\left(a,\omega,K\right)} + \frac{1}{2}\sigma_a^2\frac{\partial_{aa}^2 C^*\left(a,\omega,K\right)}{C^*\left(a,\omega,K\right)} + \mu_\omega\left(a_t,\omega_t\right)\frac{\partial_\omega C^*\left(a,\omega,K\right)}{C^*\left(a,\omega,K\right)} + \left(I^*-\delta K\right)\frac{\partial_K C^*\left(a,\omega,K\right)}{C^*\left(a,\omega,K\right)} \tag{84}$$

The independence of the risk-free rate and the market price of risk from the stock of aggregate capital follows from the linear homogeneous property of the aggregate consumption (24).

Q.E.D.

5.6 Proof of Proposition 6

The market value of firm equity can be represented as in (50). The firm marginal q can be characterized as in (16), with $\overline{q}$ and $\widehat{q}$ having the representation in (26) - (27), and satisfying the system of partial differential equations (79) - (80). The function $h(a, \omega, x_i)$ can be characterized as the probabilistic solution to the partial differential equation (58), which according to the Feynman-Kac Theorem admits the following representation for $h \in C^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$:

$$h(a_t, \omega_t, x_{it}) = \mathbf{E}_t \left[\int_0^\infty e^{-\int_t^{t+s}(\delta - i_u)du} \frac{\Lambda_{t+s}}{\Lambda_t} \eta (q_{it+s} - 1)^n \mathbf{1}_{\{q_{it+s} \geq 1\}} ds \right] \tag{85}$$

Notice that the strictly positivity of pricing-kernel Λ ensures the positivity of the function h. Evaluating (85) at the equilibrium pricing-kernel in (74), the function h can be represented as

$$h(a_t, \omega_t, x_{it}) = c^*(a_t, \omega_t)^\gamma H(a_t, \omega_t, x_{it}) \tag{86}$$

where

$$H_t \equiv \mathbf{E}_t \left[\int_0^\infty e^{-\int_t^{t+s}\{\rho + (1-\gamma)(\delta - i_u)\}du} \frac{\eta (q_{it+s} - 1)^n \mathbf{1}_{\{q_{it+s} \geq 1\}}}{c^*(a_{t+s}, \omega_{t+s})^\gamma} ds \right]. \tag{87}$$

The Feynman-Kac Theorem implies that $H \in C^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$ satisfies the following partial differential equation:

$$\left\{ \rho + (1-\gamma) \left(\delta - \widehat{i} - \left(\frac{n-1}{\alpha n} \right)^{n-1} g(a, \omega; n-1, 1) \right) \right\} H - \mathcal{D}^{a,\omega,x}[H] = \frac{\eta (q_i - 1)^n \mathbf{1}_{\{q_i \geq 1\}}}{c^*(a, \omega)^\gamma} \tag{88}$$

where $\mathcal{D}^{a,\omega,x}[H]$ denotes the infinitesimal generator of the stochastic processes a, ω and x, applied to the function H:

$$\mathcal{D}^{a,\omega,x}[H] = \kappa_a (\overline{a} - a) \partial_a H + \frac{\sigma_a^2}{2} \partial_{aa}^2 H + \mu_\omega (a, \omega) \partial_\omega H + \kappa_x (\overline{x} - x) \partial_x H + \frac{\sigma_x^2 x}{2} \partial_{xx}^2 H. \tag{89}$$

Q.E.D.

5.7 Proof of Proposition 7

The aggregate stock market value can be computed by aggregating individual firm market values as

$$\begin{aligned} V &= \int_{i\in F} V_i di \overset{(1)}{=} \int_{i\in F} q(a,\omega,x_i) K_i + h(a,\omega,x_i) K di \\ &\overset{(2)}{=} \left\{\overline{xq}(a,\omega) + [\omega - \overline{x}]\,\widehat{q}(a,\omega) + h_m(a,\omega)\right\} K \end{aligned} \tag{90}$$

where (1) follows from the firm market value representation in (31), and (2) from the definition of $K \equiv \int_{i\in F} K_i di$ and $\omega \equiv \int_{i\in F} x_i k_i di$. The function $h_m(a,\omega) \equiv \int_{i\in F} h(a,\omega,x_i)\, di$ can computed as

$$\begin{aligned} h_m(a,\omega) &\overset{(1)}{=} \mathbf{E}_t\left[\int_0^\infty e^{-\int_t^{t+s}(\delta-i_u)du}\frac{\Lambda_{t+s}}{\Lambda_t}\eta\left[\int_{i\in F}(q_{it+s}-1)^n \mathbf{1}_{\{q_{it+s}\geq 1\}}\right] dids\right] \\ &\overset{(2)}{=} \mathbf{E}_t\left[\int_0^\infty e^{-\int_t^{t+s}(\delta-i_u)du}\frac{\Lambda_{t+s}}{\Lambda_t}\eta\left[\widehat{q}_{t+s}^n\int_{i\in F}(x_{it+s}-\widetilde{x}_{t+s})^n \mathbf{1}_{\{x_{it+s}\geq\widetilde{x}_{t+s}\}}di\right] ds\right] \\ &\overset{(3)}{=} \mathbf{E}_t\left[\int_0^\infty e^{-\int_t^{t+s}(\delta-i_u)du}\frac{\Lambda_{t+s}}{\Lambda_t}\eta\left[\widehat{q}_{t+s}^n\int_{\widetilde{x}_{t+s}}^\infty(x_{t+s}-\widetilde{x}_{t+s})^n \frac{\theta^\upsilon}{\Gamma(\upsilon)}x_{t+s}^{\upsilon-1}e^{-\theta x_{t+s}}dx_{t+s}\right] ds\right] \\ &\overset{(4)}{=} \mathbf{E}_t\left[\int_0^\infty e^{-\int_t^{t+s}(\delta-i_u)du}\frac{\Lambda_{t+s}}{\Lambda_t}\eta g(a_{t+s},\omega_{t+s};n,1)\, ds\right] \end{aligned} \tag{91}$$

where (1) follows from the definition of $h_m(a,\omega,x_i)$ in (85) and Fubini's Theorem under the assumption of joint measurability, (2) from the representation of the firm marginal q as $q_i = 1+\widehat{q}(x_i-\widetilde{x})$. The third equality follows from the Glivenko-Cantelli Theorem, according to which the cross-sectional distribution of the i.i.d. firm specific productivity x equals its stationary distribution in (64), and the restriction on the investment threshold $\widetilde{x}\in[0,1/\widehat{q})$. The last equality follows from (68) and the definition of the function g in (61) evaluated at $m_1 = n$ and $m_2 = 1$.

Evaluating (91) at the equilibrium pricing-kernel in (74), the function h_m can be represented as

$$h_m(a_t,\omega_t) = c^*(a_t,\omega_t)^\gamma H_m(a_t,\omega_t) \tag{92}$$

where

$$H_{m,t} \equiv \mathbf{E}_t\left[\int_0^\infty e^{-\int_t^{t+s}\{\rho+(1-\gamma)(\delta-i_u)\}du}\frac{\eta g(a_{t+s},\omega_{t+s};n,1)}{c^*(a_{t+s},\omega_{t+s})^\gamma}ds\right] \tag{93}$$

The Feynman-Kac Theorem implies that $H_m \in C^2(\mathbb{R}\times\mathbb{R}^+)$ satisfies the following partial differential equation:

$$\left\{\rho+(1-\gamma)\left(\delta-\widehat{i}-\left(\frac{n-1}{\alpha n}\right)^{n-1} g(a,\omega;n-1,1)\right)\right\} H_m - \mathcal{D}^{a,\omega}[H_m] = \frac{\eta g(a,\omega;n,1)}{c^*(a,\omega)^\gamma} \tag{94}$$

where $\mathcal{D}^{a,\omega}[H_m]$ denotes the infinitesimal generator of the stochastic processes a and ω applied to the function H_m:

$$\mathcal{D}^{a,\omega}[H_m] = \kappa_a(\overline{a} - a)\,\partial_a H_m + \frac{\sigma_a^2}{2}\partial_{aa}^2 H_m + \mu_\omega(a,\omega)\,\partial_\omega H_m. \tag{95}$$

Q.E.D.

5.8 Proof of Proposition 8

The equilibrium cumulative aggregate stock return dynamics can be computed as $dR = \frac{dV}{V} + \frac{D}{V}dt$, where the aggregate stock market return dynamics $\frac{dV}{V}$ is obtained by applying Ito's Formula to the function $V(a, \omega, K)$ defined in (35). It follows that

$$dR = \mu_R(a,\omega)\,dt + \sigma_R(a,\omega)\,dW_a \tag{96}$$

whose drift and diffusion are given by

$$\mu_R(a,\omega) = \frac{V^A}{V}\frac{\mathcal{D}^{a,\omega,K}\left[V^A\right]}{V^A} + \frac{V^O}{V}\frac{\mathcal{D}^{a,\omega,K}\left[V^O\right]}{V^O} + \frac{D}{V} \tag{97}$$

$$\sigma_R(a,\omega) = \left[\frac{V^A}{V}\frac{\partial_a V^A}{V^A} + \frac{V^O}{V}\frac{\partial_a V^O}{V^O}\right]\sigma_a. \tag{98}$$

Similarly, from (31), the cumulative firm stock return evolves according to:

$$dR_i = \mu_{R_i}(a,\omega,x_i,k_i)\,dt + \sigma_{R_i,a}(a,\omega,x_i,k_i)\,dW_a + \sigma_{R_i,x}(a,\omega,x_i,k_i)\,dW_i \tag{99}$$

whose drift and diffusions are determined by

$$\mu_{R_i} = \frac{V_i^A}{V_i}\frac{\mathcal{D}^{a,\omega,x,K_i}\left[V_i^A\right]}{V_i^A} + \frac{V_i^O}{V_i}\frac{\mathcal{D}^{a,\omega,x,K}\left[V_i^O\right]}{V_i^O} + \frac{D_i}{V_i} \tag{100}$$

$$\sigma_{R_i,a} = \left[\frac{V_i^A}{V_i}\frac{\partial_a V_i^A}{V_i^A} + \frac{V_i^O}{V_i}\frac{\partial_a V_i^O}{V_i^O}\right]\sigma_a \tag{101}$$

$$\sigma_{R_i,x} = \left[\frac{V_i^A}{V_i}\frac{\partial_x V_i^A}{V_i^A} + \frac{V_i^O}{V_i}\frac{\partial_x V_i^O}{V_i^O}\right]\sigma_x\sqrt{x_i} \tag{102}$$

The optimality condition of the producer's optimization problem described by the HJB equation (48) implies that at the optimum the following relation must hold:

$$0 = \Lambda D_i + \mathcal{D}\left[\Lambda V_i\right]. \tag{103}$$

Rewriting the infinitesimal generator of the discounted firm value $\mathcal{D}[\Lambda V_i]$ as $\mathbf{E}_t[d\Lambda V_i]/dt$, and dividing both sides of equation (103) by ΛV_i yields the more familiar relation:

$$0 = \frac{D_i}{V_i}dt + \mathbf{E}_t\left[\frac{d\Lambda V_i}{\Lambda V_i}\right]. \tag{104}$$

A straightforward application of Ito's Formula to the discounted firm value ΛV_i leads to the fundamental asset pricing relation:

$$E_t[dR_i] = r_t dt - E_t\left[\frac{d\Lambda}{\Lambda}\frac{dV_i}{V_i}\right] \tag{105}$$

where $E_t[dR_i] = E_t\left[\frac{dV_i}{V_i}\right] + \frac{D_i}{V_i}dt$ denotes the cumulative stock expected return and $r_t = -\frac{1}{dt}E_t\left[\frac{d\Lambda}{\Lambda}\right]$ the instantaneous risk-free rate. The asset pricing relation (105) must hold for any return including the aggregate stock market return. From the aggregate stock market return dynamics (96) and the equilibrium pricing kernel dynamics (81) it follows that the aggregate stock market return is instantaneously perfectly conditionally correlated with the pricing-kernel, that is the aggregate market portfolio is conditionally mean-variance efficient. Therefore, standard asset pricing results imply that the risk-return trade-off of any traded asset admits a beta-representation[10], which takes the form of a conditional CAPM:

$$\mu_{R_i,t} = r_t + \beta_{it} \left[\mu_{R,t} - r_t \right] \tag{106}$$

where the instantaneous conditional market beta $\beta_{it} \equiv \frac{cov_t(dR_i,dR)}{var_t(dR)}$, and $\mu_{R_i,t}$ and $\mu_{R,t}$ represent the instantaneous expected return on firm i stock and aggregate market portfolio as characterized in (100) and (97), respectively.

The conditional market beta can then be decomposed as:

$$\beta_{it} \equiv \frac{cov_t\left(dR_{it}, dR_t\right)}{var_t\left(dR_t\right)} \overset{(1)}{=} \frac{\sigma_{R_i,a}}{\sigma_R} \overset{(2)}{=} \frac{\partial \left[\ln\left(V_{it}/K_{it}\right)\right]}{\partial \left[\ln\left(V_t/K_t\right)\right]} \overset{(3)}{=} \frac{K_{it}}{V_{it}} \xi_t^{q_i} + \frac{K_t}{V_{it}} \xi_t^{h_i} \tag{107}$$

where (1) follows from the characterization of stock returns in (96) and (99), (2) from the representation of $\sigma_{R_i,a} = \sigma_a \partial_a \left[\ln\left(V_{it}/K_{it}\right)\right]$ and $\sigma_R = \sigma_a \partial_a \left[\ln\left(V_t/K_t\right)\right]$, and (3) from the definition of $\xi_t^{q_i} \equiv \partial q\left(a_t, \omega_t, x_{it}\right)/\partial \ln\left(V_t/K_t\right)$ and $\xi_t^{h_i} \equiv \partial h\left(a_t, \omega_t, x_{it}\right)/\partial \ln\left(V_t/K_t\right)$.

Q.E.D.

5.9 Proof of Proposition 9

The relation between marginal q and Tobin's Q can be derived as follows. Equation (57) implies that the firm marginal q must satisfy the following PDE:

$$\frac{\left(e^a x_i + f - \widehat{i}\right)}{q_i} - \left(\delta - \widehat{i}\right) + \frac{\mathcal{D}\left[\Lambda\right]}{\Lambda} + \frac{\mathcal{D}\left[q_i\right]}{q_i} + \frac{\langle d\Lambda, dq_i \rangle}{\Lambda q_i} = 0 \tag{108}$$

and similarly equation (58) leads to the following PDE for the firm marginal value of growth opportunities:

$$\frac{\eta\left(q_i - 1\right)^n}{h_i} \mathbf{1}_{\{q_i \geq 1\}} - \left(\delta - i\right) + \frac{\mathcal{D}\left[\Lambda\right]}{\Lambda} + \frac{\mathcal{D}\left[h_i\right]}{h_i} + \frac{\langle d\Lambda, dh_i \rangle}{\Lambda h_i} = 0 \tag{109}$$

From the definition of the firm cum-dividend stock returns in (100), it follows that

$$\begin{aligned} \mu_{R_i} &\overset{(1)}{=} \frac{K_i}{V_i} q_i \left[\frac{\mathcal{D}\left[q_i\right]}{q_i} + \left(i_i - \delta\right)\right] + \left(1 - \frac{K_i}{V_i} q_i\right) \left[\frac{\mathcal{D}\left[h_i\right]}{h_i} + \left(i - \delta\right)\right] + \frac{D_i}{V_i} \\ &\overset{(2)}{=} \mu_{O_i} + \frac{K_i}{V_i} q_i \left[\mu_{I_i} - \mu_{O_i}\right] \end{aligned} \tag{110}$$

[10]See, for instance, Cochrane (2001, Chapter 6) and Duffie (2001, Section 6D).

where (1) follows from the representation of firm cum-dividend stock returns in (100) and the definition of the infinitesimal generator $\mathcal{D}[\cdot]$; and (2) from the characterization of a firm dividend yield as

$$\frac{D_i}{V_i} = \frac{K_i}{V_i} q_i \left[\frac{e^a x_i + f - \widehat{i}}{q_i} - \left(i_i - \widehat{i} \right) \right] + \left(1 - \frac{K_i}{V_i} q_i \right) \frac{\eta \left(q_i - 1 \right)^n}{h_i} \mathbf{1}_{\{q_i \geq 1\}} \tag{111}$$

and the following definitions

$$\mu_{I_i} \equiv \frac{\mathcal{D}[q_i]}{q_i} + \frac{e^a x_i + f - \widehat{i}}{q_i} - \left(\delta - \widehat{i} \right) \tag{112}$$

$$\mu_{O_i} \equiv \frac{\mathcal{D}[h_i]}{h_i} + \frac{\eta \left(q_i - 1 \right)^n}{h_i} \mathbf{1}_{\{q_i \geq 1\}} - \left(\delta - i \right). \tag{113}$$

Then, rearranging equation (110) leads to

$$\frac{V_i}{K_i} = q_i \frac{\left[\mu_{I_i} - \mu_{O_i} \right]}{\left[\mu_{R_i} - \mu_{O_i} \right]}. \tag{114}$$

Taking logs,

$$\ln \left[\frac{V_i}{K_i} \right] = \ln \left[q_i \right] + \ln \frac{\left[\mu_{I_i} - \mu_{O_i} \right]}{\left[\mu_{R_i} - \mu_{O_i} \right]} \tag{115}$$

and performing a first-order log-linear approximation of the second term in the right-hand side of equation (115) around $\overline{\mu}_I$, $\overline{\mu}_O$ and $\overline{\mu}_R$ leads to

$$\ln \frac{\left[\mu_{I_i} - \mu_{O_i} \right]}{\left[\mu_{R_i} - \mu_{O_i} \right]} \simeq \ln \frac{\left[\overline{\mu}_I - \overline{\mu}_O \right]}{\left[\overline{\mu}_R - \overline{\mu}_O \right]} + \frac{\left[\mu_{I_i} - \mu_{O_i} \right]}{\left[\overline{\mu}_I - \overline{\mu}_O \right]} - \frac{\left[\mu_{R_i} - \mu_{O_i} \right]}{\left[\overline{\mu}_R - \overline{\mu}_O \right]}. \tag{116}$$

If the approximation is made around the unconditional mean returns of a firm with marginal q equal to Tobin's Q, then $\overline{\mu}_I = \overline{\mu}_R$ and $\overline{\mu}_O = 0$. For instance, one such a firm could be a social planner or representative firm that internalizes the firm-level scale effects. This leads to the following approximate relation between marginal q and Tobin' Q :

$$\ln \left[\frac{V_i}{K_i} \right] \simeq \ln \left[q_i \right] + \frac{\mu_{I_i} - \mu_{R_i}}{\overline{\mu}_R}. \tag{117}$$

Q.E.D.

5.10 Computation of Competitive Equilibrium

I solve for the competitive equilibrium iteratively. I approximate the system of partial differential equations for $\overline{q}(a,\omega)$ and $\widehat{q}(a,\omega)$ upon discretizing the state-space of a and ω. Let $i=1,2,...,I$ and $j=1,2,...,J$ index the value of $a\in\mathbb{R}$ and $\omega\in\mathbb{R}^{++}$ on the two-dimensional state-space, respectively. At each node $i\times j$, I can rewrite the discretized system of algebraic equations (75) - (76) as

$$\overline{q}_{i,j} = (c_{i,j})^{\gamma}\,\overline{\Phi}_{i,j} \quad (118a)$$

$$\widehat{q}_{i,j} = (c_{i,j})^{\gamma}\,\widehat{\Phi}_{i,j} \quad (118b)$$

along with the system of partial differential equations (79) - (80) that $\overline{\Phi}_{i,j}$, $\widehat{\Phi}_{i,j}\in C^2\left(\mathbb{R}\times\mathbb{R}^{++}\right)$ must satisfy:

$$\left\{\rho+(1-\gamma)\left(\delta-\widehat{i}\right)+\gamma\left(\frac{n-1}{\alpha n}\right)^{n-1}g_{i,j}\right\}\overline{\Phi}_{i,j}-\widehat{\mathcal{D}}\left[\overline{\Phi}_{i,j}\right] = \frac{e^{a_i}+\overline{x}^{-1}\left(f-\widehat{i}\right)}{(c_{i,j})^{\gamma}} \quad (119)$$

$$\left\{\rho+\kappa_x+(1-\gamma)\left(\delta-\widehat{i}\right)+\gamma\left(\frac{n-1}{\alpha n}\right)^{n-1}g_{i,j}\right\}\widehat{\Phi}_{i,j}-\widehat{\mathcal{D}}\left[\widehat{\Phi}_{i,j}\right] = \frac{e^{a_i}}{(c_{i,j})^{\gamma}} \quad (120)$$

where $\widehat{\mathcal{D}}[\Phi_{i,j}]$ is the finite-difference approximation to the infinitesimal generator $\mathcal{D}[\Phi]$ evaluated at the node $i\times j$:

$$\widehat{\mathcal{D}}[\Phi_{i,j}] = \kappa_a(\overline{a}-a_i)[\partial_a\Phi]_{i,j}+\frac{1}{2}\sigma_a^2\left[\partial_{aa}^2\Phi\right]_{i,j}+\mu_\omega(a_i,\omega_j)[\partial_\omega\Phi]_{i,j}. \quad (121)$$

$$[\partial_a\Phi]_{i,j} = \frac{\Phi_{i+1,j}-\Phi_{i-1,j}}{2h_a};\ [\partial_\omega\Phi]_{i,j}=\frac{\Phi_{i,j+1}-\Phi_{i,j-1}}{2h_\omega};\ \left[\partial_{aa}^2\Phi\right]_{i,j}=\frac{\Phi_{i+1,j}-2\Phi_{i,j}+\Phi_{i-1,j}}{h_a^2} \quad (122)$$

with h_a and h_ω being the increments of a and ω on the discrete two-dimensional state-space. The approximated system of partial differential equations (119) - (120) can be rewritten for $i=2,...,I-1$ and $j=2,...,J-1$ as a system of linear algebraic equations:

$$\overline{A}_{i,j}\overline{\Phi}_{i,j}+B_i\overline{\Phi}_{i+1,j}+C_i\overline{\Phi}_{i-1,j}+D_{i,j}\overline{\Phi}_{i,j+1}+E_{i,j}\overline{\Phi}_{i,j-1} = \overline{F}_{i,j} \quad (123)$$

$$\widehat{A}_{i,j}\widehat{\Phi}_{i,j}+B_i\widehat{\Phi}_{i+1,j}+C_i\widehat{\Phi}_{i-1,j}+D_{i,j}\widehat{\Phi}_{i,j+1}+E_{i,j}\widehat{\Phi}_{i,j-1} = \widehat{F}_{i,j} \quad (124)$$

where

$$\overline{A}_{i,j} \equiv \left\{\rho+(1-\gamma)\left(\delta-\widehat{i}\right)+\gamma\left(\frac{n-1}{\alpha n}\right)^{n-1}g_{i,j}+\frac{\sigma_a^2}{h_a^2}\right\}$$

$$\widehat{A}_{i,j} \equiv \left\{\rho+\kappa_x+(1-\gamma)\left(\delta-\widehat{i}\right)+\gamma\left(\frac{n-1}{\alpha n}\right)^{n-1}g_{i,j}+\frac{\sigma_a^2}{h_a^2}\right\}$$

$$B_i \equiv -\left[\frac{\kappa_a(\overline{a}-a_i)}{2h_a}+\frac{\sigma_a^2}{2h_a^2}\right];\ C_i\equiv\left[\frac{\kappa_a(\overline{a}-a_i)}{2h_a}-\frac{\sigma_a^2}{2h_a^2}\right]$$

$$D_{i,j} \equiv -\frac{\mu_\omega(a_i,\omega_j)}{2h_\omega};\quad E_{i,j}\equiv\frac{\mu_\omega(a_i,\omega_j)}{2h_\omega}$$

$$\overline{F}_{i,j} \equiv \left[e^{a_i}+\overline{x}^{-1}\left(f-\widehat{i}\right)\right](c_{i,j})^{-\gamma};\quad \widehat{F}_{i,j}\equiv e^{a_i}(c_{i,j})^{-\gamma}.$$

Including the zero-gradient boundary conditions, we can rewrite equation (123) in matrix form as:

$$\overline{M}\overline{\Phi} = \overline{F} \tag{125}$$

where M is a $[(I-2)\times(J-2)]\times[(I-2)\times(J-2)]$-dimensional five-diagonal matrix, the column vector $\overline{\Phi}$ is structured as

$$\underset{[[(I-2)\times(J-2)]\times 1]}{\overline{\Phi}} = \begin{bmatrix} \overline{\Phi}_2 \\ \cdots \\ \overline{\Phi}_{J-1} \end{bmatrix}, \quad \underset{[(I-2)\times 1]}{\overline{\Phi}_j} = \begin{bmatrix} \overline{\Phi}_{2,j} \\ \cdots \\ \overline{\Phi}_{I-1,j} \end{bmatrix},$$

and the column vector $\overline{F}$ is

$$\underset{[[(I-2)\times(J-2)]\times 1]}{\overline{F}} = \begin{bmatrix} \overline{F}_2 \\ \cdots \\ \overline{F}_{J-1} \end{bmatrix}, \quad \underset{[(I-2)\times 1]}{\overline{F}_j} = \begin{bmatrix} \overline{F}_{2,j} \\ \cdots \\ \overline{F}_{I-1,j} \end{bmatrix}.$$

Similarly, we can rewrite equation (124) in matrix form as

$$\widehat{M}\widehat{\Phi} = \widehat{F} \tag{126}$$

where the matrix $\widehat{M}$ and the column vectors $\widehat{\Phi}$ and $\widehat{F}$ preserve the same structure and dimensionality as for (125). Then, I solve the system of linear equations (125) - (126) along with with equations (118a) - (118b) by using the following iterative procedure. At each iteration n, given candidate solutions for $\overline{q}^{(n)}$ and $\widehat{q}^{(n)}$, we can compute the corresponding value of $\overline{\Phi}^{(n)}$ and $\widehat{\Phi}^{(n)}$ as

$$\begin{aligned} \overline{\Phi}^{(n)} &= \left[\overline{M}\left(\overline{q}^{(n)}, \widehat{q}^{(n)}\right)\right]^{-1} \overline{F}\left(\overline{q}^{(n)}, \widehat{q}^{(n)}\right) \\ \widehat{\Phi}^{(n)} &= \left[\widehat{M}\left(\overline{q}^{(n)}, \widehat{q}^{(n)}\right)\right]^{-1} \widehat{F}\left(\overline{q}^{(n)}, \widehat{q}^{(n)}\right). \end{aligned}$$

With those values at hand, we can solve for the equilibrium $\overline{q}^{(n)}$ and $\widehat{q}^{(n)}$ by using the Newton-Raphson iterative procedure on the system:

$$\begin{bmatrix} \overline{q}_{i,j}^{(n+1)} \\ \widehat{q}_{i,j}^{(n+1)} \end{bmatrix} = \begin{bmatrix} \overline{q}_{i,j}^{(n)} \\ \widehat{q}_{i,j}^{(n)} \end{bmatrix} - \Delta\left[I_2 - J_{i,j}^{(n)}\right]^{-1} \begin{bmatrix} \overline{q}_{i,j}^{(n)} - \left(c_{i,j}^{(n)}\right)^{\gamma} \overline{\Phi}_{i,j}^{(n)} \\ \widehat{q}_{i,j}^{(n)} - \left(c_{i,j}^{(n)}\right)^{\gamma} \widehat{\Phi}_{i,j}^{(n)} \end{bmatrix}$$

where $J_{i,j}^{(n)}$ denotes the 2×2 Jacobian matrix evaluated at a_i and ω_j

$$J_{i,j}^{(n)} = \begin{bmatrix} \left\{J_{i,j}^{(n)}\right\}_{11} & \left\{J_{i,j}^{(n)}\right\}_{12} \\ \left\{J_{i,j}^{(n)}\right\}_{21} & \left\{J_{i,j}^{(n)}\right\}_{22} \end{bmatrix} = \gamma\left(c_{i,j}^{(n)}\right)^{\gamma-1} \begin{bmatrix} \overline{\Phi}_{i,j}^{(n)} \frac{\partial c_{i,j}^{(n)}}{\partial \overline{q}} & \overline{\Phi}_{i,j}^{(n)} \frac{\partial c_{i,j}^{(n)}}{\partial \widehat{q}} \\ \widehat{\Phi}_{i,j}^{(n)} \frac{\partial c_{i,j}^{(n)}}{\partial \overline{q}} & \widehat{\Phi}_{i,j}^{(n)} \frac{\partial c_{i,j}^{(n)}}{\partial \widehat{q}} \end{bmatrix}$$

with

$$
\begin{aligned}
\left\{J_{i,j}^{(n)}\right\}_{11} &= \frac{\partial\left(c_{i,j}^{(n)}\right)^{\gamma}\overline{\Phi}_{i,j}^{(n)}}{\partial\overline{q}} = \gamma\overline{\Phi}_{i,j}^{(n)}\left(c_{i,j}^{(n)}\right)^{\gamma-1}\frac{\partial c_{i,j}^{(n)}}{\partial\overline{q}} \\
\left\{J_{i,j}^{(n)}\right\}_{12} &= \frac{\partial\left(c_{i,j}^{(n)}\right)^{\gamma}\overline{\Phi}_{i,j}^{(n)}}{\partial\widehat{q}} = \gamma\overline{\Phi}_{i,j}^{(n)}\left(c_{i,j}^{(n)}\right)^{\gamma-1}\frac{\partial c_{i,j}^{(n)}}{\partial\widehat{q}} \\
\left\{J_{i,j}^{(n)}\right\}_{21} &= \frac{\partial\left(c_{i,j}^{(n)}\right)^{\gamma}\widehat{\Phi}_{i,j}^{(n)}}{\partial\overline{q}} = \gamma\widehat{\Phi}_{i,j}^{(n)}\left(c_{i,j}^{(n)}\right)^{\gamma-1}\frac{\partial c_{i,j}^{(n)}}{\partial\overline{q}} \\
\left\{J_{i,j}^{(n)}\right\}_{22} &= \frac{\partial\left(c_{i,j}^{(n)}\right)^{\gamma}\widehat{\Phi}_{i,j}^{(n)}}{\partial\widehat{q}} = \gamma\widehat{\Phi}_{i,j}^{(n)}\left(c_{i,j}^{(n)}\right)^{\gamma-1}\frac{\partial c_{i,j}^{(n)}}{\partial\widehat{q}}
\end{aligned}
$$

and the step-size $0 < \Delta \leq 1$ is adjusted to ensure convergence.

5.11 Data Description

The empirical analysis is based on an unbalanced panel of firms drawn from the CRSP-COMPUSTAT merged database for the years 1962 through 2002. These data include only publicly traded firms in NYSE, NASDAQ and AMEX. I study only December fiscal year-end firms to eliminate the problem caused by the use of overlapping observations. To be included in the sample I require a firm to have at least three years of valid observations. I ignore firms with negative accounting numbers for book equity, capital and investment. I trim the values of extreme observations at the 0.5th and 99.5th percentiles or I use logs (where possible) to reduce the impact of extreme values which are common for ratios in firm panels drawn from accounting data.

Market equity is price times shares outstanding. Price is from CRSP (if available) or COMPUSTAT (item 199), shares outstanding are from CRSP (if available) or COMPUSTAT (item 25). Book-equity is computed as the sum of stockholders' equity and deferred taxes and investment tax credit minus book value of preferred stock. Negative or zero book values are treated as missing. Stockholders' equity is COMPUSTAT item 216 (if available), or COMPUSTAT item 60 plus COMPUSTAT item 130, or COMPUSTAT item 6 minus COMPUSTAT item 181. Deferred taxes and investment tax credit is COMPUSTAT item 35. Book value of preferred stock is COMPUSTAT item 56 (if available), or COMPUSTAT item 10, or COMPUSTAT item 130. Investment is capital expenditure (COMPUSTAT item 128). Capital is net property, plant and equipment (COMPUSTAT item 8). Stock returns are calculated from the beginning of July to the end of June of the following year. Profitability (ROE) is the ratio of common equity income to the book value of common equity at the beginning of fiscal year. Common equity income is the sum of end-of-year earnings before extraordinary items (COMPUSTAT item 18) and depreciation (COMPUSTAT item 14). Relative capital, is the value of a firm net property, plant and equipment (COMPUSTAT item 8), divided by the cross-sectional average value of net property, plant and equipment of all firms. All aggregate series are obtained by aggregating firm-level data (for example, the aggregate investment rate is the ratio of the sum of firm investments over the sum of firm capital). Aggregate capital is the cross-sectional average value of net property, plant and equipment of all firms.

All variables are in real terms. I use the implicit price deflator for non residential investment to deflate investment and capital. All other variables are deflated using the personal consumption expenditures deflator. Both price indexes are obtained from NIPA.

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Table I: Aggregate Moments

	Data		Model	
	Mean	Std.	Mean	Std.
dC/C	2.14	2.54	1.05	3.71
I/K	17.75	3.66	14.07	1.40
R_f	0.75	3.97	2.22	5.39
$R_M - R_f$	7.74	20.4	7.92	22.15

Notes to Table I. The table reports unconditional mean and standard deviations of consumption growth, aggregate investment-to-capital ratio, risk-free rate and equity premium. The numbers reported in columns denoted "Data" are computed using annual data from NIPA and CRSP for the period 1929 - 2004. The aggregate investment-to-capital ratio is computed as capital weighted average of firm investment rates from COMPUSTAT for the period 1962 - 2002. All series are in real terms. See Appendix for more details. The numbers reported in columns denoted "Model" are based on 100 artificial panels each with 200 firms and 5000 years of data. I calculate returns and quantity moments for each sample and then report the cross-sample averages. All numbers are annual percentages.

Table II: Excess Market Returns Predictability

	A: Historical Data				B: Simulated Data			
Horizon (k)	4	6	8	10	4	6	8	10
	$\ln(D/V)$				$\ln(D/V)$			
b	0.36	0.67	1.22	1.71	0.85	1.58	2.90	5.29
$\sigma(b)$	(0.21)*	(0.32)**	(0.33)***	(0.40)***	(0.12)***	(0.45)***	(0.65)***	(0.78)***
$\overline{R}^2$	0.07	0.14	0.30	0.39	0.17	0.16	0.16	0.15
	$\ln(K/V)$				$\ln(K/V)$			
b	0.42	0.63	1.04	1.60	1.36	2.66	5.13	9.58
$\sigma(b)$	(0.24)*	(0.27)**	(0.26)***	(0.22)***	(3.05)	(2.89)	(2.86)*	(2.87)***
$\overline{R}^2$	0.10	0.17	0.35	0.56	0.13	0.12	0.12	0.12

Notes to Table II. This table reports the results of predictability regressions of excess market returns (R^{eM}) at the 4, 6, 8, 10 year horizon on the log dividend yield ($\ln(D/V)$) and log book-to-market ($\ln(K/V)$):

$$R^{eM}_{t,t+k} = a_k + b_k x_t + \varepsilon_{t+k} \quad \text{for} \quad k = 4, 6, 8, 10.$$

I report Hansen-Hodrick corrected standard errors (in parenthesis), $\sigma(b)$. Standard errors starred with one, two and three asterisks are statistically significant at the ten, five and one percent level, respectively. $\overline{R}^2$ denotes adjusted R^2. Panel A reports the results based on historical data from the CRSP-COMPUSTAT merged database. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. Panel B shows the results based on 100 artificial panels each with 50 years of data. I calculate returns and characteristics for each sample and then report the cross-sample averages of coefficients, standard errors, and adjusted R^2.

Table III: Properties of Portfolios Sorted on Book-to-Market

	Portfolio											
	1A	**1B**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10A**	**10B**
	A: Historical Data											
R_p^e	2.99	3.64	4.53	5.45	5.25	6.02	6.76	8.16	7.25	10.74	12.99	13.66
SR_p	0.12	0.19	0.27	0.31	0.30	0.35	0.41	0.47	0.44	0.62	0.65	0.58
K_p/V_p	0.21	0.29	0.42	0.54	0.66	0.79	0.92	1.08	1.29	1.65	2.05	4.58
V_p/V	2.40	2.12	1.54	1.33	1.04	0.96	0.76	0.72	0.64	0.49	0.33	0.19
I_p/K_p	32.91	28.63	26.03	23.34	21.38	19.69	18.00	15.84	14.09	13.51	12.23	9.99
ROE_p	0.27	0.20	0.17	0.15	0.12	0.11	0.10	0.09	0.08	0.06	0.05	0.00
K_p/K	0.46	0.64	0.72	0.96	0.97	1.10	1.07	1.25	1.32	1.18	0.95	0.81
	B: Simulated Data											
R_p^e	6.67	6.42	6.49	6.74	7.31	7.93	8.39	9.18	10.28	11.18	12.20	13.41
SR_p	0.28	0.27	0.28	0.29	0.32	0.34	0.36	0.40	0.45	0.49	0.53	0.58
K_p/V_p	0.78	0.94	1.04	1.19	1.42	1.54	1.67	1.98	2.23	2.44	2.83	2.95
V_p/V	1.96	1.70	1.38	1.17	0.95	0.80	0.73	0.64	0.54	0.47	0.42	0.33
I_p/K_p	21.52	18.58	16.33	14.64	13.14	12.38	12.14	12.02	12.00	12.00	12.00	12.00
ROE_p	0.27	0.24	0.21	0.18	0.15	0.13	0.11	0.10	0.08	0.06	0.05	0.04
K_p/K	1.46	1.47	1.37	1.31	1.25	1.19	1.18	1.15	1.12	1.10	1.08	0.96

Notes to Table III. This table reports time-series averages of portfolios characteristics formed yearly on the basis of ranked values of book-to-market. Portfolios 2-9 cover corresponding book-to-market deciles. The bottom and top two portfolios (1A, 1B, 10A and 10B) split the bottom and top deciles in half. The portfolio excess return (R_p^e) and investment-to-capital ratio (I_p/K_p) are in percentage terms. SR_p denotes the portfolio Sharpe Ratio. K_p/V_p and V_p/V are the portfolio book-to-market and relative market value (portfolio market value relative to the aggregate market value), respectively. ROE_p is the portfolio profitability computed as portfolio common equity income to beginning-of-year portfolio book value of equity. K_p/K is the portfolio relative capital (portfolio capital relative to the aggregate capital). The portfolio value of R_p^e is a value-weighted average of excess returns for all firms in the portfolio. The portfolio values of K_p/V_p, I_p/K_p and ROE_p are computed as ratios of sums of the corresponding values of each firm characteristic for all firms in the portfolio. Panel A reports statistics based on historical data from the CRSP-COMPUSTAT merged database. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years of data. I calculate returns and firm characteristics for each sample and then report the cross-sample averages.

Table IV: Properties of Portfolios Sorted on Size

	Portfolio											
	1A	**1B**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10A**	**10B**
	A: Historical Data											
R_p^e	16.37	12.38	10.57	9.70	8.84	8.36	7.22	8.58	6.88	6.14	5.59	3.77
SR_p	0.63	0.49	0.43	0.40	0.37	0.38	0.36	0.44	0.36	0.37	0.37	0.23
K_p/V_p	1.33	1.25	1.11	1.05	0.99	0.91	0.83	0.79	0.75	0.73	0.67	0.50
V_p/V	0.01	0.02	0.04	0.07	0.11	0.18	0.31	0.56	1.10	2.48	4.44	32.03
I_p/K_p	18.60	18.41	20.31	19.12	18.38	18.60	19.56	18.54	17.72	16.23	17.31	18.87
ROE_p	0.00	0.01	0.03	0.04	0.05	0.07	0.08	0.09	0.10	0.10	0.11	0.13
K_p/K	0.01	0.02	0.04	0.08	0.13	0.19	0.29	0.52	0.97	2.09	3.72	7.64
	B: Simulated Data											
R_p^e	10.37	10.00	9.73	9.41	9.01	8.66	8.44	8.01	7.52	7.16	6.75	6.28
SR_p	0.45	0.43	0.42	0.41	0.39	0.37	0.37	0.35	0.32	0.31	0.29	0.27
K_p/V_p	2.11	2.00	1.93	1.85	1.74	1.64	1.59	1.46	1.32	1.21	1.08	0.94
V_p/V	0.10	0.17	0.22	0.30	0.46	0.54	0.64	0.95	1.27	1.59	2.38	2.97
I_p/K_p	12.49	12.51	12.51	12.61	12.77	12.92	13.03	13.32	13.72	14.08	14.49	14.86
ROE_p	0.08	0.09	0.09	0.10	0.11	0.12	0.12	0.14	0.16	0.17	0.20	0.23
K_p/K	0.19	0.29	0.39	0.49	0.67	0.83	0.94	1.15	1.45	1.70	2.03	2.55

Notes to Table IV. This table reports time-series averages of portfolios characteristics formed yearly on the basis of ranked values of market equity. Portfolios 2-9 cover corresponding market equity deciles. The bottom and top two portfolios (1A, 1B, 10A and 10B) split the bottom and top deciles in half. The portfolio excess return (R_p^e) and investment-to-capital ratio (I_p/K_p) are in percentage terms. SR_p denotes the portfolio Sharpe Ratio. K_p/V_p and V_p/V are the portfolio book-to-market and relative market value (portfolio market value relative to the aggregate market value), respectively. ROE_p is the portfolio profitability computed as portfolio common equity income to beginning-of-year portfolio book value of equity. K_p/K is the portfolio relative capital (portfolio capital relative to the aggregate capital). The portfolio value of R_p^e is a value-weighted average of excess returns for all firms in the portfolio. The portfolio values of K_p/V_p, I_p/K_p and ROE_p are computed as ratios of sums of the corresponding values of each firm characteristic for all firms in the portfolio. Panel A reports statistics based on historical data from the CRSP-COMPUSTAT merged database. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years of data. I calculate returns and firm characteristics for each sample and then report the cross-sample averages.

Table V: Properties of Portfolios Sorted on Book-to-Market and Size

	A: Historical Data					B: Simulated Data				
	Book-to-Market					Book-to-Market				
Size	1Q	2	3	4Q	|4Q-1Q|	1Q	2	3	4Q	|4Q-1Q|
	R^e_p					R^e_p				
Small	5.83	11.40	11.41	13.94	8.11	6.49	7.66	9.21	11.52	5.03
2	4.27	7.39	8.74	11.47	7.20	6.88	7.46	9.05	11.25	4.38
Big	3.83	5.24	6.18	9.66	5.83	6.41	7.26	8.90	9.37	2.96
	I_p/K_p					I_p/K_p				
Small	33.87	26.48	20.56	16.98	16.89	25.43	14.15	12.07	12.00	13.43
2	33.33	22.88	17.30	13.12	20.21	18.11	13.08	12.04	12.00	6.11
Big	26.74	20.79	14.87	12.63	14.11	15.85	12.65	12.03	12.00	3.85
	ROE_p					ROE_p				
Small	3.98	5.43	4.23	0.39	3.59	17.68	13.95	9.63	6.11	11.57
2	16.60	11.79	8.25	3.36	13.23	19.92	14.57	9.95	6.40	13.52
Big	20.24	13.02	8.82	6.14	14.10	22.22	15.26	10.37	6.94	15.28
	K_p/K					K_p/K				
Small	0.01	0.02	0.03	0.04	0.03	0.25	0.33	0.42	0.53	0.28
2	0.13	0.28	0.44	0.60	0.47	0.72	0.88	1.05	1.22	0.50
Big	1.72	3.52	5.53	7.50	5.79	1.66	1.79	2.06	2.44	0.79

Notes to Table V. This table reports time-series averages of portfolios characteristics formed yearly on the basis of ranked values of market equity and book-to-market. In particular, each year stocks are allocated to three size groups based on the breakpoints for the bottom 20 percent, middle 60 percent and top 20 percent of the ranked values of market equity. Similarly, each year stocks are allocated in an independent sort to four book-to-market groups based on the breakpoints for the 20, 50, and 80 percent of the ranked values of book-to-market. The twelve portfolios are the intersection of the three size and the four book-to-market groups. The numbers reported in columns denoted |4Q-1Q| represent the absolute value of the difference in the values of a variable between the highest and lowest book-to-market portfolios. The portfolio excess return (R^e_p), investment-to-capital ratio (I_p/K_p) and profitability (ROE_p) are in percentage terms. ROE_p is the portfolio profitability computed as portfolio common equity income to beginning-of-year portfolio book value of equity. K_p/K is the portfolio relative capital (portfolio capital relative to the aggregate capital). The portfolio value of R^e_p is a value-weighted average of excess returns for all firms in the portfolio. The portfolio values of I_p/K_p and ROE_p are computed as ratios of sums of the corresponding values of each firm characteristic for all firms in the portfolio. Panel A reports statistics based on historical data from the CRSP-COMPUSTAT merged database. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years of data. I calculate returns and firm characteristics for each sample and then report the cross-sample averages.

Table VI: Excess Returns and Firm Characteristics

	$\ln(K_i/V_i)$	$\ln(V_i/V)$	$\ln(I_i/K_i)$	$\ln(K_i/K)$	$\ln(K_i/V_i) \times \ln(V_i/V)$
			A: Historical Data		
1	4.36				
	(1.02)***				
2		−1.42			
		(0.53)***			
3	3.14	−1.06			
	(1.17)***	(0.53)**			
4	2.67	−1.43			−0.60
	(1.31)**	(0.61)**			(0.30)**
5			−1.80	−1.08	
			(0.85)**	(0.53)**	
6	3.48			−0.96	
	(1.11)***			(0.54)**	
7	2.97		−0.58	−1.08	
	(1.07)***		(0.76)	(0.53)**	
8		−1.09	−1.69		
		(0.53)**	(0.76)**		
9		−1.06	−1.79	−0.08	
		(0.62)*	(0.76)**	(0.71)	
			B: Simulated Panel		
1	5.29				
	(0.27)***				
2		−1.28			
		(0.07)***			
3	5.19	−0.06			
	(0.30)***	(0.02)***			
4	4.48	−0.18			−1.25
	(0.27)***	(0.08)**			(0.07)***
5			−4.20	−0.37	
			(1.36)***	(0.11)***	
6	5.25			−0.06	
	(0.28)***			(0.02)**	
7	2.49		−3.05	−0.43	
	(1.09)**		(1.87)	(0.19)**	
8		−0.93	−3.88		
		(0.17)***	(1.41)***		
9		−2.57	−3.08	2.17	
		(1.54)*	(1.57)**	(2.02)	

Notes to Table VI. The table reports coefficients and standard errors of Fama-

MacBeth cross-sectional regressions. Coefficients and standard errors (in parenthesis) are in percentage terms. The dependent variable is individual firm excess stock return, and the independent variables are the (logarithms of) book-to-market (K_i/V_i), relative market equity (V_i/V), investment-to-capital ratio (I_i/K_i), relative capital (K_i/K), and the interaction between the (logarithm of) book-to-market and relative market equity. Standard errors are adjusted for heteroskedasticity and serial correlation using Newey-West formula with one lag. Standard errors starred with one, two and three asterisks are statistically significant at the ten, five and one percent level, respectively. Panel A reports statistics based on historical data from the CRSP-COMPUSTAT merged database. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years of data. I calculate returns and firm characteristics for each sample and then report cross-sample averages of coefficients and standard errors.

Table VII: CAPM - Time Series Regressions

	Portfolio											
	1A	**1B**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10A**	**10B**
	A: Historical Data											
α	−3.61	−1.68	−0.57	0.11	0.06	1.21	2.25	3.08	2.97	6.52	8.24	8.56
$\sigma(\alpha)$	(1.15)*	(0.86)*	(0.61)	(0.88)	(1.19)	(1.49)	(1.12)*	(1.22)*	(1.72)	(2.08)*	(2.36)*	(3.17)*
β_M	1.31	1.06	1.01	1.06	1.03	0.96	0.90	1.01	0.85	0.84	0.94	1.01
$\sigma(\beta_M)$	(0.19)*	(0.08)*	(0.06)*	(0.11)*	(0.06)*	(0.06)*	(0.09)*	(0.06)*	(0.07)*	(0.14)*	(0.12)*	(0.16)*
	B: Simulated Data											
α	−1.28	−1.57	−1.50	−1.23	−0.62	0.03	0.50	1.29	2.42	3.35	4.39	5.63
$\sigma(\alpha)$	(0.32)*	(0.20)*	(0.12)*	(0.07)*	(0.03)*	(0.05)	(0.07)*	(0.10)*	(0.16)*	(0.21)*	(0.27)*	(0.34)*
β_M	0.99	1.00	1.01	1.01	1.00	1.00	1.00	1.00	0.99	0.99	0.98	0.98
$\sigma(\beta_M)$	(0.03)*	(0.02)*	(0.01)*	(0.01)*	(0.00)*	(0.00)*	(0.01)*	(0.01)*	(0.01)*	(0.02)*	(0.02)*	(0.03)*

Notes to Table VII. The table reports summary statistics of time series regressions of book-to-market sorted portfolios' excess stock returns (R^{ep}_{t+1}) on the excess stock market returns (R^{eM}_{t+1}):

$$R^{ep}_{t+1} = \alpha^p + \beta^p_M R^{eM}_{t+1} + \varepsilon^p_{t+1} \quad \text{for} \quad p = 1, ..., 12.$$

The time-series intercepts, α, and standard errors, $\sigma(\alpha)$, are in percentage terms. The coefficients, β_M, denote CAPM - βs. Standard errors (in parenthesis) are adjusted for heteroskedasticity and serial correlation using Newey-West formula with one lag. Standard errors starred with one asterisk are statistically significant at the five percent level. Panel A reports statistics based on historical data from CRSP. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years of data. I calculate returns for each sample and then report cross-sample averages of regression coefficients and standard errors.

Table VIII: Fama and French Model - Time Series Regressions

	Portfolio											
	1A	1B	2	3	4	5	6	7	8	9	10A	10B
	A: Historical Data											
α	0.29	0.64	0.74	−0.71	−0.77	−0.55	−0.24	0.08	−0.97	2.18	2.96	4.23
$\sigma(\alpha)$	(1.95)	(1.40)	(0.65)	(0.90)	(1.27)	(1.38)	(1.29)	(0.95)	(1.35)	(1.28)	(1.91)	(3.16)
β_M	1.13	0.98	1.00	1.08	1.05	0.96	0.94	1.08	0.93	0.95	1.01	1.04
$\sigma(\beta_M)$	(0.18)*	(0.07)*	(0.05)*	(0.10)*	(0.06)*	(0.07)*	(0.07)*	(0.04)*	(0.07)*	(0.14)*	(0.10)*	(0.16)*
β_{SMB}	0.11	−0.05	−0.13	0.02	0.03	0.21	0.19	0.14	0.27	0.21	0.47	0.48
$\sigma(\beta_{SMB})$	(0.16)	(0.09)	(0.06)*	(0.06)	(0.08)	(0.09)*	(0.10)*	(0.07)*	(0.09)*	(0.08)*	(0.15)*	(0.16)*
β_{HML}	−0.64	−0.36	−0.18	0.13	0.12	0.24	0.36	0.45	0.57	0.65	0.74	0.59
$\sigma(\beta_{HML})$	(0.13)*	(0.13)*	(0.04)*	(0.08)	(0.10)	(0.09)*	(0.10)*	(0.05)*	(0.07)*	(0.09)*	(0.11)*	(0.16)*
$\overline{R}^2$	0.97	0.99	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.99	0.98	0.97
	B: Simulated Data											
α	2.67	0.89	−0.02	−0.45	−0.56	−0.45	−0.33	−0.12	0.12	0.28	0.41	0.71
$\sigma(\alpha)$	(0.53)*	(0.22)*	(0.12)	(0.07)*	(0.09)*	(0.09)*	(0.08)*	(0.09)	(0.10)	(0.15)	(0.22)	(0.31)*
β_M	0.97	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99
$\sigma(\beta_M)$	(0.01)*	(0.01)*	(0.00)*	(0.00)*	(0.00)*	(0.00)*	(0.00)*	(0.00)*	(0.00)*	(0.00)*	(0.00)*	(0.01)*
β_{SMB}	−0.29	−0.06	0.00	−0.05	−0.08	0.00	0.12	0.35	0.63	0.95	1.37	1.68
$\sigma(\beta_{SMB})$	(0.52)	(0.21)	(0.12)	(0.07)	(0.08)	(0.08)	(0.08)	(0.09)*	(0.10)*	(0.13)*	(0.21)*	(0.30)*
β_{HML}	−0.90	−0.58	−0.35	−0.18	0.00	0.12	0.19	0.30	0.48	0.62	0.79	0.98
$\sigma(\beta_{HML})$	(0.10)*	(0.04)*	(0.02)*	(0.01)*	(0.02)	(0.02)*	(0.02)*	(0.02)*	(0.02)*	(0.03)*	(0.04)*	(0.06)*
$\overline{R}^2$	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Notes to Table VIII. The table reports summary statistics of time series regressions of book-to-market sorted portfolios' excess stock returns (R^{ep}_{t+1}) on the excess stock market returns (R^{eM}_{t+1}), the returns on SMB (R^{SMB}_{t+1}) and HML (R^{HML}_{t+1}) :

$$R^{ep}_{t+1} = \alpha^p + \beta^p_M R^{eM}_{t+1} + \beta^p_{SMB} R^{SMB}_{t+1} + \beta^p_{HML} R^{HML}_{t+1} + \varepsilon^p_{t+1} \quad \text{for} \quad p = 1, ..., 12.$$

The time-series intercepts, α, and standard errors, $\sigma(\alpha)$, are in percentage terms. Standard errors (in parenthesis) are adjusted for heteroskedasticity and serial correlation using Newey-West formula with one lag. Standard errors starred with one asterisk are statistically significant at the five percent level. $\overline{R}^2$ denotes adjusted

R^2. Panel A reports statistics based on historical data from CRSP. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years of data. I calculate returns for each sample and then report cross-sample averages of regression coefficients, standard errors, and adjusted R^2.

Table IX: Asset Pricing Models: Fama-MacBeth Regressions

	Intercept	β	*MKT*	$\log(D/V) \times MKT$	*SMB*	*HML*	$\overline{R}^2$
				A: Simulated Data			
1	−0.18	0.09					0.97
	(0.13)	(0.04)**					
2	0.82		−0.73				0.54
	(0.12)***		(0.13)***				
3	0.46		−0.37	0.86			0.77
	(0.08)***		(0.08)***	(0.25)***			
4	−0.18		0.26			0.04	0.86
	(0.06)***		(0.07)***			(0.00)***	
5	0.04		0.03		0.01	0.03	0.94
	(0.06)		(0.06)		(0.00)***	(0.00)***	
				B: Historical Data			
6	0.22		−0.14				0.18
	(0.08)***		(0.08)*				
7	0.24		−0.17	0.61			0.25
	(0.08)***		(0.08)**	(0.26)**			
8	−0.10		0.15			0.07	0.77
	(0.06)		(0.07)**			(0.02)**	
9	−0.02		0.07		0.09	0.05	0.81
	(0.05)		(0.05)		(0.04)**	(0.02)**	

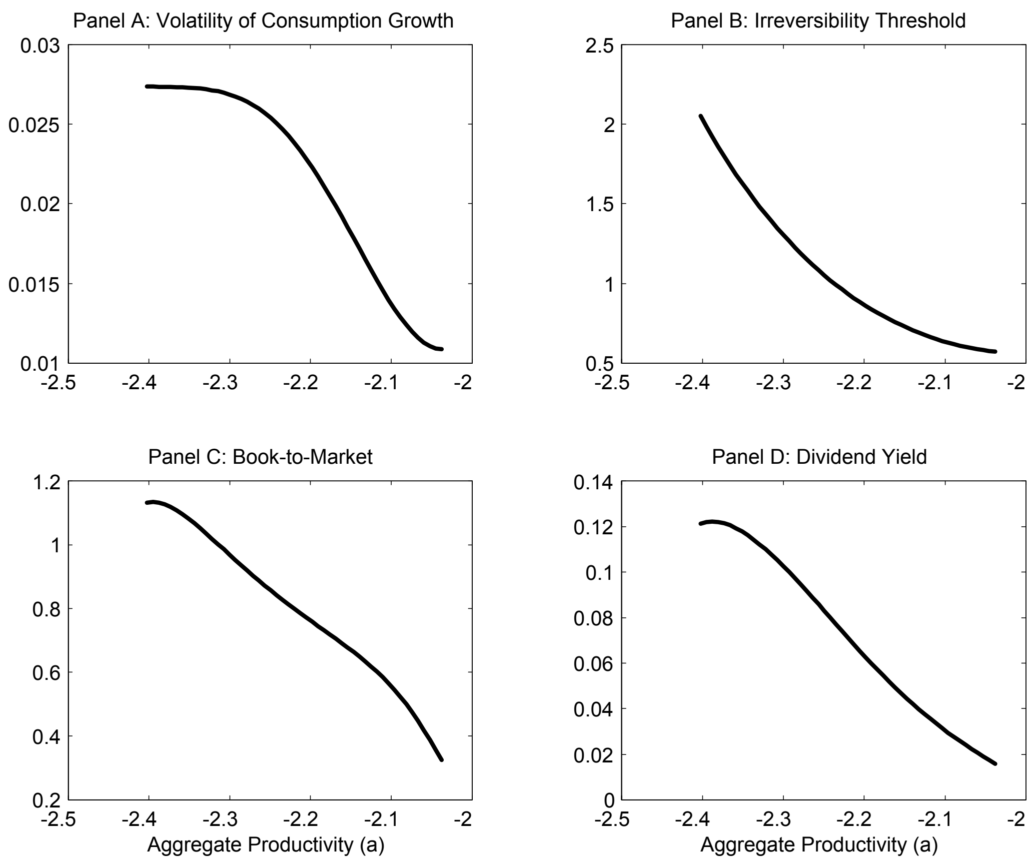
Notes to Table IX . The table reports summary statistics of Fama-MacBeth cross-sectional regressions. The dependent variable is excess stock return on book-to-market sorted portfolios, and the independent variables are a constant and betas estimated by time-series regression of excess returns on the factors. Standard errors (in parenthesis) are adjusted for heteroskedasticity, serial correlation (one lag) and sampling variation in estimated betas using GMM formulas. Standard errors starred with one, two and three asterisks are statistically significant at the ten, five and one percent level, respectively. $\overline{R}^2$ denotes adjusted R^2. Panel A shows the results based on 100 artificial panels each with 200 firms and 50 years of data. I calculate returns for each sample and then report cross-sample averages of coefficients, standard errors and adjusted R^2. Line 1, conditional CAPM regressions, where the independent variable is the model implied conditional β. Line 2, CAPM regressions, where MKT represents the average excess stock market return. Line 3, conditional CAPM regressions with the aggregate log dividend yield, $\log(D/V)$, as conditioning variable. Line 4, two-factor model (MKT and HML), where HML denotes the average return on the "high minus low" portfolio constructed as in Fama and French (1993). Line 5, Fama and French (1993) model, where SMB is the average return on the "small minus big" portfolio constructed as in Fama and French (1993). Panel B reports statistics based on historical data from CRSP. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix.

Table X: Risk-Adjusted Returns and Firm Characteristics

	Model	$\ln(K_p/V_p)$	$\ln(V_p/V)$	$\ln(I_p/K_p)$	$\ln(K_p/K)$
		A: Historical Data			
1	CCAPM	1.03	−2.77		
		(1.90)	(1.61)		
2				−10.49	−2.59
				(2.85)***	(1.21)*
3	2-FF	−1.98	−2.94		
		(1.68)	(1.68)		
4				−5.51	−2.59
				(1.93)**	(1.18)*
5	3-FF	−1.58	−2.64		
		(1.76)	(1.66)		
6				−4.84	−2.16
				(1.73)**	(1.26)
		B: Simulated Data			
7	CCAPM	4.57	−0.57		
		(0.47)***	(0.29)*		
8				−0.81	−4.06
				(0.06)***	(0.59)***
9	2-FF	−0.46	−0.58		
		(0.47)	(0.35)		
10				−0.24	−0.53
				(0.02)***	(0.25)*
11	3-FF	−1.22	−0.53		
		(0.63)*	(0.32)		
12				−0.08	0.16
				(0.02)***	(0.24)

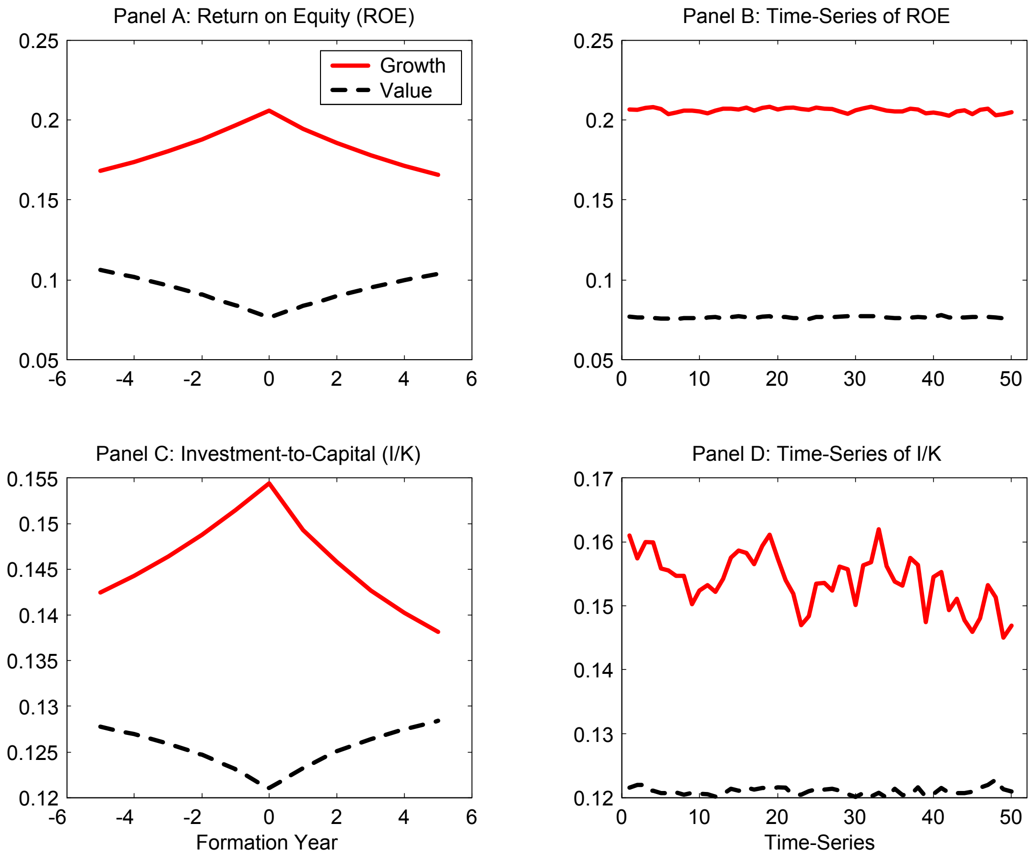
Notes to Table X. The table reports coefficients and standard errors of Fama-MacBeth cross-sectional regressions. Coefficients and standard errors (in parenthesis) are in percentage terms. The dependent variable is risk-adjusted return on book-to-market sorted portfolios, and the independent variables are the (logarithms of) book-to-market (K_p/V_p), relative market equity (V_p/V), investment-to-capital ratio (I_p/K_p), and relative capital (K_p/K). Risk-adjusted portfolio returns are computed as the difference between actual and model predicted portfolio returns. Standard errors are adjusted for heteroskedasticity, serial correlation (one lag) and sampling variation in estimated risk-adjusted returns using GMM formulas. Standard errors starred with one, two and three asterisks are statistically significant at the ten, five and one percent level, respectively. Panel A reports statistics based on historical data from CRSP. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix. Line 1-2, predicted portfolio returns are computed using the conditional CAPM with log dividend yield as conditioning variable in time-series regressions (CCAPM). Line 3-4, predicted portfolio returns are computed using the two factor (MKT and HML) model in time-series regressions (2-FF). Line 5-6, predicted portfolio returns are computed using the Fama and French (1993) three factor model in time-series regressions (3-FF). Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years of data. I calculate returns and portfolio characteristics for each sample and then report cross-sample averages of coefficients and standard errors.

Figure 1: Equilibrium Aggregate Variables



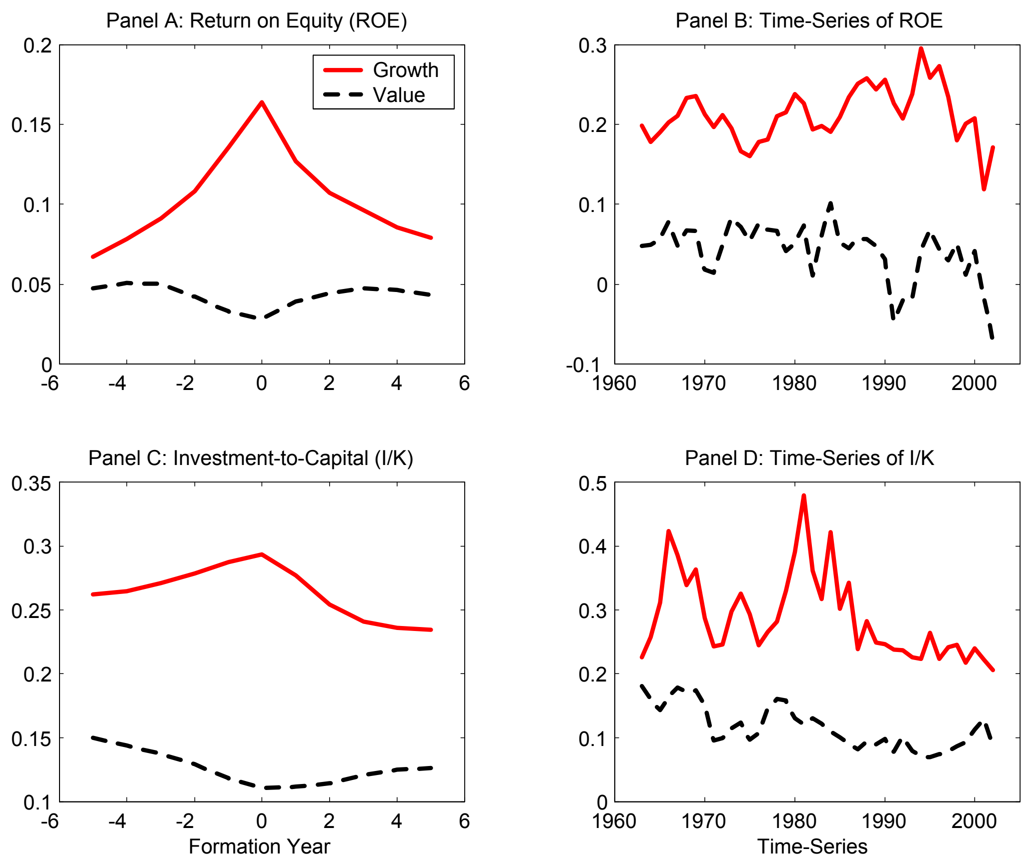
Notes to Figure 1. The figure plots some relevant aggregate variables in competitive equilibrium as a function of the aggregate productivity, a, and the average value of ω. Panel A: Volatility of consumption growth. Panel B: Investment irreversibility threshold, $\widetilde{x}$. Panel C: Aggregate book-to-market ratio. Panel D: Aggregate dividend yield.

Figure 2: Value vs. Growth in Simulated Data



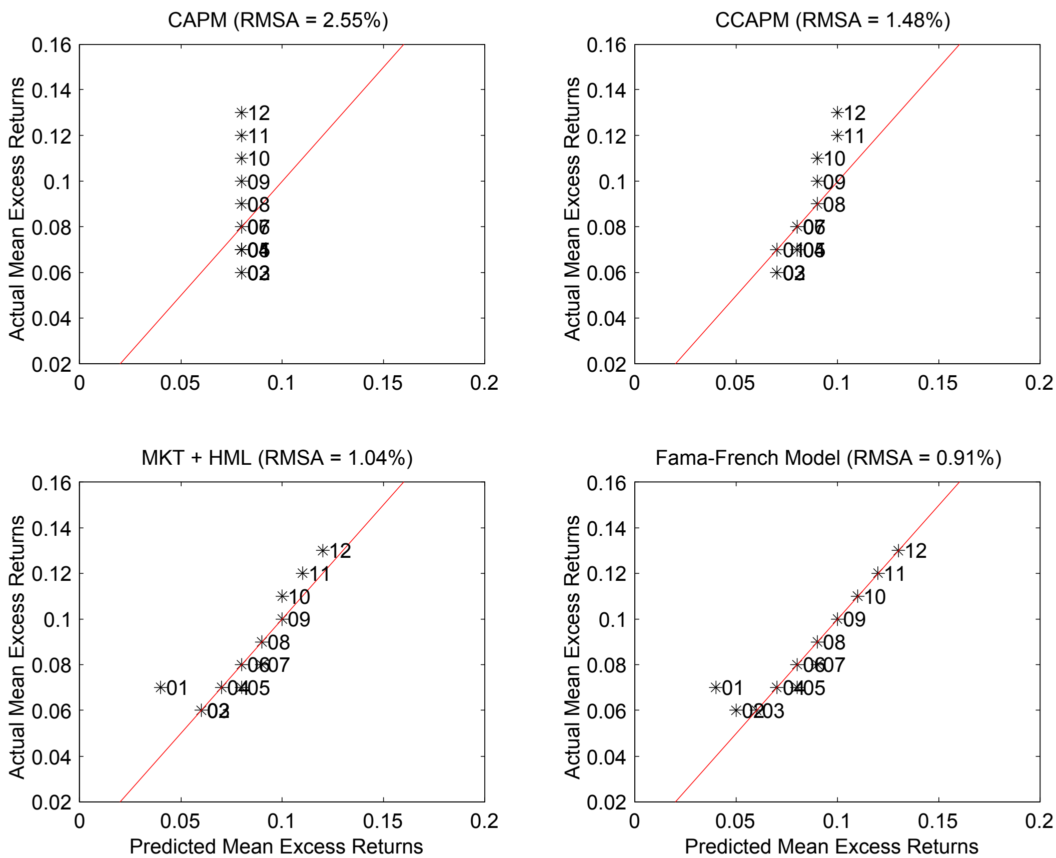
Notes to Figure 2. The figure illustrates the relation between profitability and investment-to-capital ratio for growth and value portfolios in simulated data.Growth (value) indicates the portfolio containing firms in the bottom (top) 20 percent of the values of book-to-market ratios. I measure profitability by return on equity (ROE) as $[\Delta K_t + D_t]/K_{t-1}$, where K_{t-1} denotes the book value of equity and D_t is the dividend payout. The profitability of a portfolio is defined as the sum of $[\Delta K_{it} + D_{it}]$ for all firms i in the portfolio divided by the sum of K_{it-1}. The investment-to-capital ratio of a portfolio is defined as the sum of I_{it} for all firms i in the portfolio divided by the sum of K_{it-1}. For each portfolio formation year t, the ratios of $[\Delta K_{t+k} + D_{t+k}]/K_{t+k-1}$ and I_{t+k}/K_{t+k-1} are calculated for year $t+k$, where $k = -5, ..., 5$. The ratio for year $t+k$ is then averaged across portfolio formation years. Panel A and C show the 11 - year evolution of profitability and investment-to-capital ratio for growth and value portfolios, respectively. Panel B and D show the time-series of profitability and investment-to-capital ratio for growth and value portfolios, respectively. The figure is based on 100 artificial panels each with 200 firms and 50 years of data. I calculate profitability and investment-to-capital ratio for value and growth portfolios for each sample, and then report cross-sample averages.

Figure 3: Value vs. Growth in Historical Data



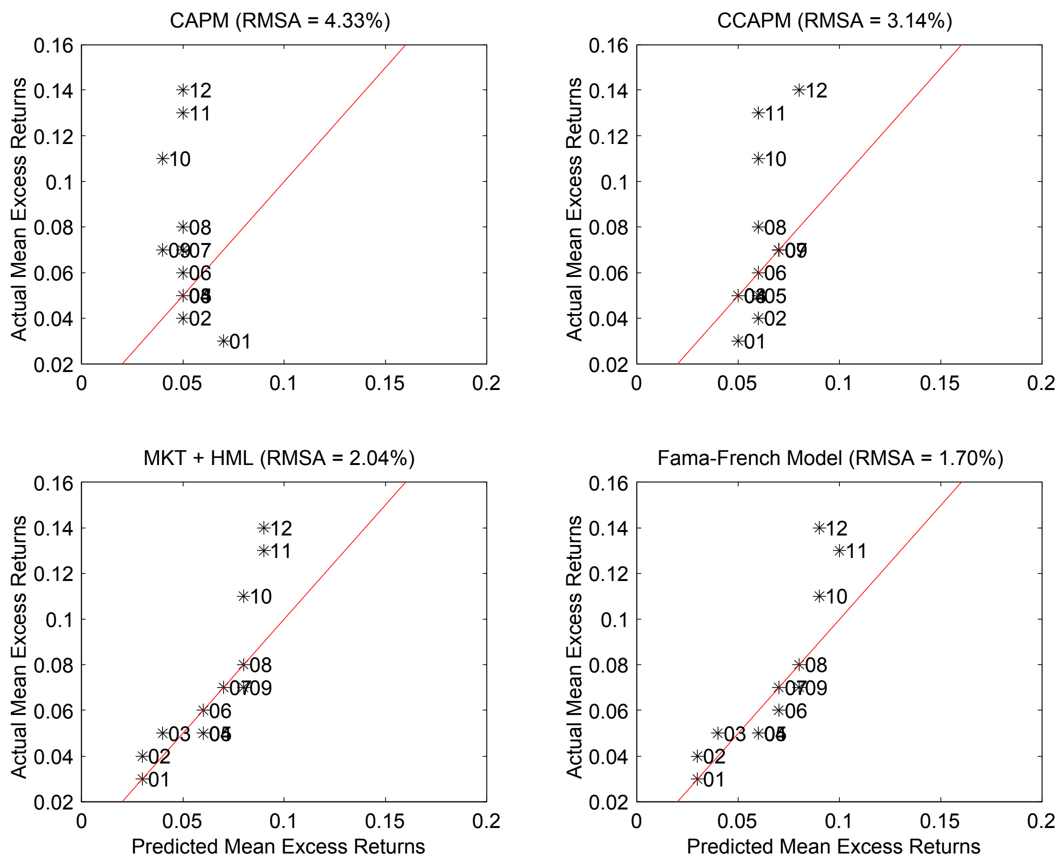
Notes to Figure 3. The figure illustrates the relation between profitability and investment-to-capital ratio for growth and value portfolios in historical data.Growth (value) indicates the portfolio containing firms in the bottom (top) 20 percent of the values of book-to-market ratios. I measure profitability by return on equity (ROE) as the ratio of common equity income for the fiscal year ending in calendar year t and the book value of equity for year $t-1$. The profitability of a portfolio is defined as the sum of common equity income for all firms in the portfolio divided by the sum of book value of equity. The investment-to-capital ratio of a portfolio is defined as the sum of capital expenditures for the fiscal year ending in calendar year t for all firms in the portfolio divided by the sum of net property, plant and equipment for year $t-1$. For each portfolio formation year t, the ROE_{t+k} and I_{t+k}/K_{t+k-1} are calculated for year $t+k$, where $k=-5,...,5$. The ratio for year $t+k$ is then averaged across portfolio formation years. Panel A and C show the 11 - year evolution of profitability and investment-to-capital ratio for growth and value portfolios, respectively. Panel B and D show their time-series dynamics. The figure is based on historical data from the CRSP-COMPUSTAT merged database for the period 1962 - 2002. More details are provided in Appendix.

Figure 4: Predicted vs. Actual Excess Returns in Simulated Data



Notes to Figure 4. The figure shows model predicted vs. actual annual mean excess returns on book-to-market sorted portfolios in simulated data. Panel A: CAPM. Panel B: conditional CAPM with log dividend yield as conditioning variable (CCAPM). Panel C: two factor model (MKT + HML). Panel D: Fama and French (1993) three factor model. RMSA is the root mean squared alpha. The figure is based on 100 artificial panels each with 200 firms and 50 years of data. I calculate portfolios returns for each sample and then report cross-sample averages.

Figure 5: Predicted vs. Actual Excess Returns in Historical Data



Notes to Figure 5. The figure shows model predicted vs. actual annual mean excess returns on book-to-market sorted portfolios in historical data. Panel A: CAPM. Panel B: conditional CAPM with log dividend yield as conditioning variable (CCAPM). Panel C: two factor model (MKT + HML). Panel D: Fama and French (1993) three factor model. RMSA is the root mean squared alpha. The figure is based on historical data from the CRSP-COMPUSTAT merged database for the period 1962 - 2002. More details are provided in Appendix.