Social Security and Risk Sharing*

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Abstract

In this paper we identify conditions under which the introduction of a pay-as-you-go social security system is ex-ante Pareto-improving in a stochastic overlapping generations economy with capital accumulation and land. We argue that these conditions are consistent with many calibrations of the model used in the literature. In our model financial markets are complete and competitive equilibria are interim Pareto efficient. Therefore, a welfare improvement can only be obtained if agents' welfare is evaluated ex ante, and arises from the possibility of inducing, through social security, an improved level of intergenerational risk sharing. We will also examine the optimal size of a given social security system as well as its optimal reform.

The analysis will be carried out in a relatively simple set-up, where the various effects of social security, on the prices of long-lived assets and the stock of capital, and hence on output, wages and risky rates of returns, can be clearly identified.

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1 Introduction

The pay-as-you-go social security system in the US was introduced as a tool to mitigate the effects of economic crises. In a special message to Congress accompanying the draft of the social security bill President Roosevelt said “No one can guarantee this country against the dangers of future depressions, but we can reduce those dangers. ... we can provide the means of mitigating their results. This plan for economic security is at once a measure of prevention and a measure of alleviation.” (see e.g. Kennedy (1999), page 270). In this paper we examine to what extent enhanced intergenerational risk sharing through a pay-as-you-go social security can alleviate the consequences of economic downturns. This idea dates back to at least Enders and Lapan (1982). More recently, it is used as an argument against privatization of social security. For example, Shiller (1999a) writes ”If risk management is to be really effective, it is most important that it help out in the most desperate situations, and this is what the US government’s social security, financed with income taxes, does.”

To properly evaluate whether a social security system allows to improve risk-sharing it is important to specify the welfare criterion that is used (and hence the market failures social security may address). If agents’ utility is evaluated at an interim stage, conditionally on the state at their birth, an improvement can only be obtained if some financial markets are missing, or the economy is dynamically inefficient. While one might argue that in reality crucial markets are missing (in particular annuity markets and markets for securities that pay contingent on idiosyncratic shocks), this source of inefficiency is not specific to economies with overlapping generations and other insurance schemes could be introduced which are Pareto-improving (in particular new financial assets, fully funded annuities etc.). Hence the presence of some missing markets might provide a justification for some government intervention but does not directly point to social security as an ideal instrument. Using an interim welfare criterion, several authors have examined the potential benefits of pay-as-you-go social security systems in realistically calibrated, dynamically efficient economies with missing markets (see e.g. Imrohoroglu et al. (1999) or Krueger and Kubler (2006)). They find that the negative effects of social security on the capital stock and wages clearly outweigh, quantitatively, any positive risk sharing effects of such a system.

However, if agents’ welfare is evaluated at an ex ante stage competitive equilibria in stochastic overlapping generation models are generally suboptimal, even when markets are complete, because agents are unable to trade to insure against the realization of the uncertainty at their birth. There must then be some transfers between generations which improve intergenerational risk sharing and constitute a Pareto-improvement. It is then particularly of interest to investigate under what conditions a pay-as-you-go social security system (or, more generally, one-sided transfers from the young to the old) is Pareto-improving according to an ex ante welfare criterion in economies where equilibria are interim Pareto efficient. In these economies the only possible source of an improvement is the imperfection in intergenerational risk sharing due to the limitations on trading imposed by the demographic structure.

In this paper we consider a class of overlapping generations economies where markets are complete, there is capital accumulation and land, an infinitely lived asset used in the production process together with labor and capital. The presence of land ensures that competitive equilibria are interim Pareto efficient. We show that, for a wide range of realistic specifications of the parameters of the economy, a pay-as-you-go social security system is ex-ante Pareto improving and we demonstrate that the effects on the equilibrium price of land play a crucial role in enhancing the welfare benefits of social security (in a slightly different framework, Diamond and Geanakoplos (2003) already point to the interaction of social security and land prices).

We consider two period overlapping generations economies with a single[1] agent per generation and stochastic shocks to aggregate production, and analyze three different pay-as-you-go systems: a defined contribution system, where transfers from the young are proportional to their income level, a defined benefits system, where transfers from the young to the old are state independent, and an ideal system, where any state contingent transfer from the young to the old is allowed. We decompose the effects of a social security scheme into: i) a direct transfer from the young to the old (the one prescribed by the scheme), ii) an indirect transfer (which may have positive or negative sign) induced by the general equilibrium effects of social security on the stock of capital, and hence on equilibrium wages and return to capital, and on the price of long lived assets, iii) a change in the level of total resources available for consumption (due to the change in the stock of capital). In the simple set-up of the economy considered, we are then able to identify several crucial conditions (primarily on the covariance between the shocks affecting the agents when young and old, on their risk aversion, and on the stochastic properties of the production shocks) under which these different components of the effects of a social security scheme have a positive effect on welfare.

We proceed in steps, by examining first the effects of the direct transfer specified by the different social security schemes. To this end we consider the special case where there is no production, nor land, so that there is no trade in equilibrium and there are no general equilibrium effects. We find that in order for the introduction of a defined benefits social security system to be Pareto-improving, we need[2] consumption (and hence income) of the old agents to be positively correlated with the consumption of the young and, at the same time, to exhibit a higher variability. Similar conditions must hold for a defined contribution system if risk aversion is sufficiently large. A weaker condition suffices for an ideal system to be improving: the existence of at least one state where the old are poorer than the

[1] Since we assume markets are complete, perfect risk sharing can be attained within each generation. We can abstract then from the heterogeneity within each generation so as to better focus on intergenerational risk sharing.

[2] It should be kept in mind that such conditions are derived for competitive equilibria which are interim efficient.

young (their discounted marginal utility for consumption exceeds the marginal utility when young).

Next, we consider the case where land is present, so that we do have a first general equilibrium effect given by the changes in the price of long-lived assets generated by the introduction of the policy. We find that the price of land tends to decrease, so that we have an indirect transfer from young to old agents of negative sign, which in interim efficient economies has typically a positive effect on the welfare of future generations. On this basis, we show that the presence of long-lived assets improves the case for social security, making the conditions for such policy to be welfare improving less stringent.

With production the introduction of social security in dynamic efficient economies tend to crowd out the investment in capital and hence to lower output level. Furthermore, the stochastic structure of the production shocks determine the correlation between wages, affecting the income when young, and return to capital, affecting the income when old. While we are able to identify some conditions on the properties of the production shocks under which the various social security schemes are improving, we show that in general it is rather difficult to find a rationale of pay-as-you-go social security in models with capital accumulation but without land.

On the basis of this analysis, we examine more general economies, where there is both land and capital. In this case we solve for equilibria numerically and show that for economies with somewhat realistic parameter specifications the introduction of a defined contributions as well as that of a defined benefits social security system is welfare improving. It turns out that, while the general equilibrium effect of social security on the equilibrium level of the stock of capital tends to decrease welfare, this effect is overcompensated by the effect of the change in the price of land. The sign (and size) of the direct effect is then crucial for determining if a (defined benefits or defined contributions) social security scheme is Pareto-improving. As argued above, this direct effect is positive if the consumption when old is positively correlated, but sufficiently more volatile, with the consumption when young.

In this set-up we also analyze the optimal size of the different social security systems and find that for various parameter specifications it is rather close to the estimated size of the US system. Finally, we examine the benefits of reforming an existing social security system to improve its risk-sharing properties. We find that large welfare gains can be obtained by reforming a pay-as-you-go defined contributions system making social security contributions state contingent, i.e. by moving in the direction of an ideal system. Such finding is in line with Shiller (1999b)'s observation that, the US system's risk sharing potentials seem limited in that the young's transfers to the current old do not depend on the wealth of the young relative to that of the old.

Starting with Gordon and Varian (1988) several papers have examined the scope for a pay-as-you-go social security system under an ex ante welfare criterion. Shiller (1999b), Ball and Mankiw (2001) and De Menil et al. (2005) approach this question by using a partial equilibrium analysis in that in their model there is no capital accumulation and no land, and agents have access to a risky storage technology. As argued in this paper, the general equilibrium effects induced by social security play a very important role in a proper assessment of the benefits of costs and benefits of social security.

Bohn (2003) (probably the closest to our analysis, see also Olovsson (2004)) compares the competitive equilibria with and without social security for a realistically calibrated version of an economy with capital accumulation. He shows that it is difficult to make an argument in favor of social security in such set-up. This result, however, depends crucially on the fact that Bohn abstracts from the presence of long-lived assets and restricts his attention to a specific form of production shocks. We should also point out that, without controlling for the fact that competitive equilibria without social security are interim efficient it is not possible to properly argue that any welfare gain is due to the improvements in intergenerational risk sharing induced by social security.

The effects of other forms of fiscal policy interventions on intergenerational risk sharing according to an ex ante welfare criterion have then been studied by various authors. Gale (1990) analyzes the efficient design of public debt, while Smetters (2004) examines the role of capital taxation.

The paper is organized as follows. In Section 2 we describe the class of economies under consideration and give conditions for interim and ex-ante optimality. In Section 3 we introduce various social security systems. Section 4 examines the direct effects of such systems. Section 5 focuses on the equilibrium effects on the price of land and Section 6 on the effects on capital accumulation. In Section 7 we analyze the effects of the alternative social security schemes for the economies under consideration, with production and land. In Section 8 we examine within the same framework the optimal size of a social security system as well as the welfare improving reforms of an existing system.

2 The OLG economy

We consider a stationary overlapping generations economy under uncertainty with 2 period - lived agents. Time runs from $t = 0$ to infinity. Each period a shock $s \in \mathcal{S} = \{1, ..., S\}$ realizes. Date-events, or nodes, are histories of shocks, and a specific date event at t is denoted by $s^t = (s_0...s_t)$. We collect all (finite) histories in an event tree Σ.

There are 4 commodities: capital, labor, a single consumption good, which is perishable, and a perfectly durable good, land.

As discussed in the introduction, our primary focus is on intergenerational risk sharing. We will abstract therefore from issues concerning intragenerational risk sharing by assuming there is one representative agent born at each date-event. The representative agent in each generation has a unit endowment of labor when young and zero when old as well as an endowment of the consumption good which depends on age and the current shock; for an agent born in node s^t it is given by $e^y(s^t) = e^y_{s_t}$ when young and $e^o(s^{t+1}) = e^o_{s_{t+1}}$ when old. At a given node s^t, we denote the consumption of the young agent by $c^y(s^t)$ and consumption of the old agent who was born at s^{t-1} by $c^o(s^t)$.

Agents' preferences are only defined over the consumption good and represented by the time-separable expected utility

$$U^{s^t}(c) = u(c^y(s^t)) + \beta \mathbb{E}_{s^t} v(c^o(s^{t+1}))$$

where $u(.)$ and $v(.)$ are increasing, concave and smooth functions. Agents supply their labor inelastically: $l_s^y = 1$ for all $s \in \mathcal{S}$. At the root node, s^0 there is one initial old agent with utility $U(c) = v(c^o(s^0))$.

At each date event s^t, there is a representative firm which produces the consumption good using capital k, labor l and land b as inputs. The firm's production function, f, is subject to stochastic shocks. Given any shock $s \in \mathcal{S}$, $f(k, l, b; s)$ is assumed to exhibit constant returns to scale in capital, labor and land. Capital is obtained from the consumption good in the previous period via a storage technology: more precisely, one unit of the consumption good at $t-1$ yields one unit of capital in each state at t. Land is perfectly durable and there is a fixed quantity of it, which we normalize to one.

After the shock is realized, the firm buys labor and capital and rents land from the households on the spot market so as to maximize spot-profits. We denote the price at s^t of capital by $r(s^t)$ (in terms of the consumption good whose price we normalize to 1), the price of labor by $w(s^t)$, and the rental price of land by $d(s^t)$, paid by producers to use land in the current production process.

At date 0 the initial old holds the entire amount of land as well as a given amount of capital, $k(s^{-1})$, to which we refer as the 'initial condition'. Land is then traded by consumers in the market and $q(s^t)$ denotes the price of land at s^t.

Given an initial condition $k(s^{-1})$, a feasible allocation is $((c^y(s^t), c^o(s^t)), k(s^t))$ such that

$$c^y(s^t) + c^o(s^t) + k(s^t) \leq e_{s^t} + f(k(s^{t-1}), 1, 1; s_t) \text{ for all } s^t,$$

where $e(s^t) = e^y_{s_t} + e^o_{s_t}$ denotes the agents' total endowment of the consumption good at s^t.

For simplicity we abstract from population growth or technological progress and assume the shocks to be i.i.d.; π_s denotes then the probability of shocks s occurring. Evidently, these are not innocuous assumptions when it comes to a quantitative analysis of social security. However, in this paper we focus largely on more qualitative issues.

In what follows we consider first some special cases of this model, where capital is not productive, $(\partial f(k, l, b; s)/\partial k = \partial f(k, l, b; s)/\partial l = 0$ for all b, l, k and $s)$, in which case the economy reduces to a pure exchange one, and/or where land is not productive $(\partial f(k, l, b; s)/\partial b = 0$ for all b, l, k and $s)$, or equivalently where there is no land.

2.1 Optimality

As explained in the introduction, we distinguish between two welfare concepts. Given an initial condition $k(s^{-1})$, a feasible allocation $((c^y(s^t), c^o(s^t)), k(s^t))_{s^t \in \Sigma}$ is *conditionally*

Pareto optimal (CPO) if there is no other feasible allocation $((\tilde{c}^y(s^t), \tilde{c}^o(s^t)), \tilde{k}(s^t))_{s^t \in \Sigma}$, with

$$U^{s^t}(\tilde{c}^y(s^t), \tilde{c}^o(s^{t+1})) \geq U^{s^t}(c^y(s^t), c^o(s^{t+1})) \text{ for all } s^t, t$$

with the inequality holding strict for at least one s^t. Thus, in this notion agents' welfare is evaluated at the interim stage, after the realization of the uncertainty at the time of birth.

On the other hand, a feasible allocation $((c^y(s^t), c^o(s^t)), k(s^t))_{s^t \in \Sigma}$ is *ex ante Pareto-optimal*[3] if there is no other feasible allocation $((\tilde{c}^y(s^t), \tilde{c}^o(s^t)), \tilde{k}(s^t))_{s^t \in \Sigma}$, with

$$\mathbb{E}_0 U^{s^t}(\tilde{c}^y(s^t), \tilde{c}^o(s^{t+1})) \geq \mathbb{E}_0 U^{s^t}(c^y(s^t), c^o(s^{t+1})) \text{ for all } t = 0, 1, ...,$$

with the inequality holding strict for at least one t.

2.2 Competitive equilibria

A competitive equilibrium is a collection of choices for the households and firms such that households maximize utility, firms maximize spot profits and markets clear. It simplifies the characterization of equilibria to note that by market clearing, in equilibrium the firm will always buy the entire capital and labor and rent the entire amount of land from the households. Recalling that $l_s^y = 1$ for all s, an equilibrium is then characterized by the choices $\left\{c^y(s^t), c^o(s^t), k(s^t), b(s^t)\right\}_{s^t \in \Sigma}$, where $b(s^t)$ denotes the amount of land purchased by the young and $k(s^t)$ the amount of consumption good destined to capital at s^t, and prices $\left\{w(s^t), r(s^t), q(s^t), d(s^t)\right\}_{s^t \in \Sigma}$, such that:

i) at each s^t the young chooses $k(s^t), b(s^t)$ to maximize $u(c^y(s^t)) + \beta \mathbb{E}_{s^t} v(c^o(s^{t+1}))$ subject to

$$c^y(s^t) = e^y(s_t) + w(s^t) - q(s^t)b(s^t) - k(s^t)$$
$$c^o(s^{t+1}) = e^o(s_{t+1}) + \left(q(s^{t+1}) + d(s^{t+1})\right) b(s^t) + k(s^t) r(s^{t+1});$$

ii) firms maximize profits, i.e. using the market clearing for labor and capital,

$$r(s^t) = \frac{\partial f(k(s^{t-1}), 1, 1; s_t)}{\partial k}, \quad w(s^t) = \frac{\partial f(k(s^{t-1}), 1, 1; s_t)}{\partial l}, \quad d(s^t) = \frac{\partial f(k(s^{t-1}), 1, 1; s_t)}{\partial b};$$

iii) and the land market clears:

$$b(s^t) = 1.$$

The presence of an infinitely lived asset like land, yielding each period a dividend that is bounded away from zero ensures (see, e.g. Demange (2002)) that competitive equilibria are conditionally Pareto optimal, i.e. there is no possibility of welfare improvement conditionally on the state at birth of each generation. Hence the only possible source of inefficiency in the economy under consideration, when land is productive, is the fact that agents are unable to trade before they are born to ensure against the state at their birth.

[3] In the following we will sometimes drop the qualification 'ex ante'.

2.3 Stationary equilibria

The analysis is obviously much simpler when the competitive equilibrium is stationary in the strong sense that individual consumption only depends on the current shock, s, i.e.

$$(c^y(s^t), c^o(s^t)) = (c^y_{s_t}, c^o_{s_t}), \quad \forall s^t.$$

In such situations we can easily derive the conditions for CPO and ex ante optimality of competitive equilibria and study the welfare effects of stationary taxes and transfers.

Given our assumption that shocks are i.i.d., stationary equilibria always exist in the special case where capital and labor are not productive (pure exchange), whether or not land is productive. However, this is no longer true in the presence of production, in which case only the existence of ergodic Markov equilibria can be established under general conditions (see Wang (1993) for a proof of existence in an economy without land). We will consider some special cases where stationary equilibria still exist in the presence of production.

2.3.1 Conditional optimality

As shown by Aiyagari and Peled (1991), Chattopadhyay and Gottardi (1999) (see also Demange and Laroque (2000) for a model with production), a stationary equilibrium (in the above sense) is conditionally Pareto optimal if and only if the matrix of the representative agent's marginal rate of substitutions $\left\{ \frac{\beta \pi_{s'} v'(c^o_{s'})}{u'(c^y_s)};\ s, s' \in \mathcal{S} \right\}$ has a maximal eigenvalue less or equal than 1. In our set-up, the separability of the agent's utility function and the fact that shocks are i.i.d. imply that this matrix has always rank 1, and its largest eigenvalue is given by the sum of its diagonal elements. It then follows that a stationary equilibrium is CPO if and only if:

$$\beta \sum_{s \in \mathcal{S}} \pi_s \frac{v'(c^o_s)}{u'(c^y_s)} \leq 1, \tag{1}$$

a condition that can be readily verified. We can then say the economy is 'at the golden rule' if $\beta \sum_{s \in \mathcal{S}} \pi_s \frac{v'(c^o_s)}{u'(c^y_s)} = 1$.

It is useful to rewrite condition (1) as follows:

$$\text{cov}(\beta v'(c^o), \frac{1}{u'(c^y)}) + \mathbb{E}\left(\beta v'(c^o)\right) \mathbb{E}\left(\frac{1}{u'(c^y)}\right) \leq 1; \tag{2}$$

which implies then, by Jensen's inequality:

$$\text{cov}(\beta v'(c^o), \frac{1}{u'(c^y)}) + \mathbb{E}\left(\beta v'(c^o)\right) \left(\frac{1}{\mathbb{E}(u'(c^y))}\right) \leq 1. \tag{3}$$

The advantage of this formulation is that it clearly shows that CPO requires that at least one of the two following properties is satisfied: (i) on average the marginal utility of consumption when old is smaller than the marginal utility of consumption when young, (ii) the marginal utilities of consumption when old and the inverse of the marginal utility of consumption when young are negatively correlated (or equivalently, when c^y and c^o are co-monotonic,

c^y and c^o are positively correlated). While (i) is analogous to the condition for optimality found under certainty (agents should be on average richer when old than when young), (ii) identifies some specific properties of the allocation of risk within each generation.

2.3.2 Ex Ante Improving Transfers

At a stationary equilibrium the welfare, evaluated at the ex ante stage, is the same for all generations and given by:

$$\mathbb{E}_0 U^{s^t}(c^y(s^t), c^o(s^{t+1})) = \sum_{s \in \mathcal{S}} \pi_s (u(c_s^y) + \beta v(c_s^o)).$$

We derive a simple condition under which a stationary allocation cannot be improved by direct stationary transfers between young and old agents. Considering direct transfers is only a necessary condition for the ex ante optimality of an allocation, as a welfare improvement could be found by changing also the level of production and investment, as well as with nonstationary transfers. Since our focus is on the possibility that a stationary social security system is welfare improving, it is of particular interest to consider first the conditions where an improvement can be found only with stationary transfers.

Welfare improving stationary transfers do not exist if there isn't a vector $(T_s)_{s \in \mathcal{S}}$ such that an infinitesimal net transfer from the young to the old agents in the direction of $(T_s)_{s \in \mathcal{S}}$ has a (weakly) positive effect on the (ex ante) welfare of the representative generation:[4]

$$\sum_{s \in \mathcal{S}} \pi_s (-u'(c_s^y) + \beta v'(c_s^o)) T_s \geq 0 \tag{4}$$

as well as on the agents who are old when the transfers are introduced:

$$\sum_{s \in \mathcal{S}} \pi_s v'(c_s^o) T_s \geq 0. \tag{5}$$

with one at least of the two inequalities being strict.

Obviously, a vector $(T_s)_{s \in \mathcal{S}}$ satisfying conditions (4) and (5) exists if for some s, $(-u'(c_s^y) + \beta v'(c_s^o)) > 0$. Moreover, one can easily see that since the transfers are not restricted to be positive, an improvement is also possible if the vectors of the marginal utilities when young and old are are not collinear. Therefore focusing on direct transfers, one can characterize ex-ante efficiency as follows.

Proposition 1 *At any stationary allocation $\left(c_s^y, c_{s'}^o\right)_{s,s' \in \mathcal{S}}$, a necessary and sufficient condition for the non-existence of welfare improving stationary transfers is that the vectors $(u'(c_s^y))_{s \in \mathcal{S}}$ and $(v'(c_s^o)_{s \in \mathcal{S}})$ are collinear and that for all $s \in \mathcal{S}$, $\beta v'(c_s^o) \leq u'(c_s^y)$.*

[4]In the case of stationary Markov allocations a similar condition holds, though the expression is now evaluated with the invariant probability distribution.

Condition (4) can be rewritten as follows:

$$Cov\left\{\beta v'(c^o) - u'(c^y), T\right\} \geq \mathbb{E}\left(T\right)\left[\mathbb{E}\left(u'(c^y)\right) - \beta\mathbb{E}\left(v'(c^o)\right)\right] \tag{6}$$

Thus an improving transfer T should be characterized by a sufficiently high covariance with $v'(c^o)$ and a low covariance with $u'(c^y)$.

When ex ante welfare is considered the timing of the introduction of the transfers also plays a role. In particular, (5) applies to the case where, at the time in which the transfer scheme is introduced, the transfer starts operating at a given date *in all possible states*. We can understand this as describing a situation where the transfer scheme is announced one period in advance, say at the end of some date t, after some history s^t, and will be implemented starting from date $t+1$ at every successor node of s^t (hence there will be a transfer from the young to the old at date $t+1$ for each possible realization s of the uncertainty at date $t+1$).

If on the other hand the transfer scheme were not announced in advance, but began to operate at date $t+1$, when say the current shock is $\bar{s}$, the scheme would be welfare improving if the following conditions hold, in addition to (4):

$$\pi_{\bar{s}} v'(c^o_{\bar{s}}) T_{\bar{s}} \geq 0, \tag{7}$$

saying that the agents who are old at $t+1$ are not worse off, and:

$$-u'(c^y_{\bar{s}})T_{\bar{s}} + \sum_{s \in \mathcal{S}} \beta \pi_s v'(c^o_s)) T_s \geq 0. \tag{8}$$

stating that the agents who are young at $t+1$ are also not worse off. Note that (7) is equivalent to $T_{\bar{s}} \geq 0$, and given this it follows from (8) that (5) holds. The reverse however is not necessarily true. We conclude that the set of transfer schemes which are welfare improving when announced one period before their introduction includes the set of transfer schemes which are improving when they are introduced at the time of their announcement.

3 Social Security

We model social security as a system of non-negative transfers from the young to the old. In general the pattern of transfers is described by $(\tau(s^t))_{s^t \in \Sigma}$, where $\tau(s^t) \geq 0$. We let $\nu \geq 0$ denotes the size of the system, so that at any node s^t the current young transfers $\nu\tau(s^t)$ units of the consumption good to the current old.

Actual social security systems in most developed countries are characterized by the fact that neither the specification of taxes nor benefits seem to vary across states of the world. In various countries, like the US, a social security trust-fund stabilizes imbalances between benefits and contributions over the business cycle. We will abstract from this feature and not allow for the presence of a trust-fund, so as to focus on the pure intergenerational transfer component of social security systems. Evidently, this fact will lead us to underestimate the welfare benefits of social security systems.

In what follows, we will therefore restrict our attention to stationary social security systems that maintain budget balance in every period; i.e., current benefits coincide with current taxes and the specification of the transfer at each node s^t depends at most on the current state s_t, not on past history. We will consider three different kinds of stationary social security systems:

1. In the first case, the social security transfer is a suitably designed function of the current shock. The transfer pattern is thus given by $(\tau_s)_{s\in\mathcal{S}}$, where τ_s can be any non-negative number: in each state $s \in \mathcal{S}$ the current young makes then a transfer proportional to τ_s units of the consumption good to the current old. We will refer to this as an *'ideal' (stationary) social security system.*

2. In the second case the contributions paid by the young are proportional to their income. Since the latter may vary with the node, so will the level of the tax paid, but the tax rate is state invariant. That is, for all s^t, we have:

$$\tau(s^t) = e^y_{s_t} + w(s^t).$$

   We will refer to this as a '*defined contributions*' social security system since the social security tax-rate remains constant across states.

3. In the third case considered in this paper, benefits are state invariant. We call a social security system a '*defined benefits*' one if, for all $s \in \mathcal{S}$, $\tau_s = 1$.

Feldstein and Liebman (2001) characterize the US pay-as-you-go system as a defined benefits system and argue that some countries such as Sweden and Italy have defined contribution programs. The fact however that we require transfers to balance in each period constitutes, as we already argued, a departure from the features of the social security systems present in most industrial countries.

The ideal system is an idealization, which provides an important reference point by allowing us to identify the welfare maximizing stochastic structure of the transfers from young to old agents and can be contrasted with the welfare improving transfers $(T_s)_{s\in\mathcal{S}}$ discussed in the previous section, where no sign restriction was imposed on T_s. It also allows us to see how far existing defined benefit or contribution systems are from a welfare-maximizing system.

In the presence of a social security system $(\nu\tau(s^t))_{s^t\in\Sigma}$ a competitive equilibrium is again given by a collection of choices $\left\{c^y(s^t), c^o(s^t), k(s^t), b(s^t)\right\}_{s^t\in\Sigma}$ and prices $\left\{w(s^t), r(s^t), q(s^t), d(s^t)\right\}_{s^t\in\Sigma}$ such that households maximize utility, firms maximize spot profits and markets clear. The only difference with respect to conditions i)-iii) of Section 2.2 is the expression of the consumers' budget constraint, now given by:

$$c^y(s^t) = e^y(s_t) + w(s^t) - q(s^t)b(s^t) - k(s^t) - \nu\tau(s^t)$$
$$c^o(s^{t+1}) = e^o(s_{t+1}) + \left(q(s^{t+1}) + d(s^{t+1})\right) b(s^t) + k(s^t)r(s^{t+1}) + \nu\tau(s^{t+1}).$$

We intend to analyze the welfare effects of an infinitesimal increase of the scale of the social security system, $d\nu > 0$. We decompose these effects into those of the direct transfer prescribed by the social security scheme, $\tau(s^t)$, and the effects generated by the changes in equilibrium prices, i.e. by the general equilibrium effects of the change in the scheme.

In the next three Sections, we will consider some special cases where a stationary equilibrium exists. In these situations the transfers prescribed by the three types of stationary social security systems described above are also stationary. We show that the economy reaches a new stationary equilibrium in at most one period after the change in the social security system. Furthermore, the welfare effects are only given by the effects of the total transfers generated by the infinitesimal policy change, evaluated at the new stationary equilibrium (i.e. any change in available resources has no welfare effect). These transfers can be described by a vector $(T_s)_{s\in\mathcal{S}}$ where, in each state $s \in \mathcal{S}$, T_s is given by the sum of the direct transfer τ_s prescribed by the policy and the indirect transfer induced by the general equilibrium effects of the policy change. Once the pattern $(T_s)_{s\in\mathcal{S}}$ of the total transfers induced by an infinitesimal increase of the scale of a social security scheme $(\tau_s)_{s\in\mathcal{S}}$ is identified, we can use the analysis in the previous Section to ascertain whether or not such policy changes improve welfare (in the ex ante sense). For this, it suffices to verify whether conditions (4) and (5) are both satisfied for the values of $(T_s)_{s\in\mathcal{S}}$ we found.

We will derive the expressions of $(T_s)_{s\in\mathcal{S}}$ for the case where the effects of $d\nu$ are evaluated at $\nu = 0$, i.e. an infinitesimal amount of social security is introduced, starting from a situation without social security. It should be clear that the analysis can be immediately extended to the case where the policy change is evaluated at some $\nu > 0$.

4 Direct Effects of Social Security

We will consider first the case of pure exchange, without land (no factor is productive): there is a unique equilibrium, which is stationary, and given by autarky. While this case is almost trivial to analyze, it helps in identifying some of the main conditions needed for social security to be Pareto-improving. In this case in fact the total net transfer T_s induced by the policy in equilibrium in any state s coincides with the direct transfer τ_s prescribed by the policy. The welfare consequences of social security can then just be determined on the basis of the relationship between the direct transfers $(\tau_s)_{s\in S}$ and the stochastic pattern of the agents' marginal utility for consumption (in this case coinciding with their endowments). In the light of the discussion at the end of Section 2.3.2, we consider the case where the introduction of the social security scheme is announced one period in advance. Hence to determine whether the scheme is Pareto improving, we only have to verify that conditions

(4) and (5) are satisfied when T_s is replaced by τ_s, for all s:

$$\begin{aligned}\sum_{s\in\mathcal{S}}\pi_s(-u'(e^y_s)+\beta v'(e^o_s))\tau_s &> 0,\\ \sum_{s\in\mathcal{S}}\pi_s v'(c^o_s)\tau_s &\geq 0,\end{aligned}$$

(one of the two inequalities being strict).

Since the direct transfers τ_s from young to old agents prescribed by a social security scheme are required to be non-negative, the initial old are always at least weakly better off as a result of the introduction of the scheme $(\tau(s))_{s\in\mathcal{S}}$, i.e. the second of the two above inequalities is always satisfied.

4.1 Ideal social security system

All what is required for the existence of a Pareto improving ideal social security system is that the first of the two above condition holds. The circumstances under which this is possible are readily obtained:

PROPOSITION 2 *At an autarkic equilibrium, a Pareto improving ideal social security system exists if and only if there is at least one state $\bar{s}$ for which*

$$\beta v'(e^o_{\bar{s}}) > u'(e^y_{\bar{s}}).$$

Intuitively this is a very weak condition: it only requires the existence of one state, where the time (*but not* probability) discounted marginal utility of the old is larger than the marginal utility of the young, i.e. where we can say the old are 'poorer' than the young.

It is useful to contrast this condition with the necessary and sufficient condition (1) for the conditional optimality of the competitive equilibrium which, in the case of an autarkic equilibrium, reduces to:

$$\beta\sum_{s\in\mathcal{S}}\pi_s\frac{v'(e^o_s)}{u'(e^y_s)}\leq 1$$

Obviously, for a large set of economies for which the (autarkic) equilibrium is CPO we can find a Pareto-improving social security system. Conditional optimality, as we saw, requires that on average the old are 'richer' than the young, or alternatively that $\text{cov}(\beta v'(c^o), \frac{1}{u'(c^y)}) < 0$. As long as there is one shock $\bar{s}$ for which the old are 'poorer' than the young, social security can be Pareto-improving. The improvement can be attained with nonzero transfers from the young to the old only in one state, $\bar{s}$. Hence, for all $s \neq \bar{s}$ the young agents are better off conditionally on the state at birth; if the initial allocation is CPO the the agents born in state $\bar{s}$ must be worse of with social security.

While the analysis shows that an optimally designed social security system can be Pareto-improving for a large range of parameter values, actual social security systems are somewhere between defined benefits and defined contribution. We therefore examine now these two more realistic cases.

4.2 Defined benefits

When transfers are constant across all shocks, the simple analysis of the autarky case reveals one surprising necessary condition for a defined benefits social security system to be Pareto improving.

The necessary and sufficient condition for a defined benefits system to be Pareto improving, at an autarkic equilibrium, is again readily obtained from (4), setting $T_s = 1$ for all s:

$$\sum_{s \in \mathcal{S}} \pi_s \left(-u'(e^y_s) + \beta v'(e^o_s) \right) > 0. \tag{9}$$

Thus the average marginal utility of consumption has to be larger when old than when young. Recall that the necessary condition for (CPO) in Equation (3) implies that if this is the case, i.e. if $\mathbb{E}\left(\beta v'(c^o)\right)\left(\frac{1}{\mathbb{E}u'(c^y)}\right) \geq 1$, then $\text{cov}(\beta v'(c^o), \frac{1}{u'(c^y)}) < 0$ must hold for the allocation to be CPO.

In addition, we can rewrite (9) as:

$$\sum_{s \in \mathcal{S}} \pi_s u'(e^y_s) \left(-1 + \beta \frac{v'(e^o_s)}{u'(e^y_s)} \right) = \mathbb{E}\left(u'(e^y)\right) \mathbb{E}\left(-1 + \beta \frac{v'(e^o)}{u'(e^y)} \right) + Cov\left(u'(e^y), \beta \frac{v'(e^o)}{u'(e^y)} \right) > 0. \tag{10}$$

Since the necessary and sufficient condition for CPO, (1), requires that $\mathbb{E}\left(-1 + \beta \frac{v'(e^o)}{u'(e^y)}\right) \leq 0$, the second term in (10) has to be strictly positive for (9) to hold.

We have thus shown:

PROPOSITION 3 *At a conditionally Pareto optimal autarkic equilibrium, a defined benefits social security system can be Pareto improving only if:*

$$cov(\beta v'(e^o), \frac{1}{u'(e^y)}) < 0 < cov(u'(e^y), \beta \frac{v'(e^o)}{u'(e^y)}). \tag{11}$$

The first inequality in (11) says that a welfare improving defined benefits system can only be found when the marginal utility of the old and the inverse of the marginal utility of the young are *negatively* correlated. Hence, when the variables describing the endowment when young and when old are co-monotone, marginal utilities when young and when old must be positively correlated, i.e., in all states where the old are rich, the young must also be rich and vice versa! This may come a bit as a surprise as we might have expected that the margins for welfare improving transfers between young and old, enhancing risk sharing, would be greater when their income is negatively correlated. We should bear in mind though that we are limiting our attention here to deterministic transfers, so that mutual insurance cannot be properly achieved; moreover, the conditional optimality of the equilibrium imposes some restrictions on the pattern of the variability of consumption when young and when old[5].

[5] As we already noticed, the first inequality in (11) is in fact one of the two alternative necessary conditions for CPO we obtained from (3).

Furthermore, when utility is linear-concave or concave-linear (in that either $u(c) = c$ or $v(c) = c$), or when e^y or e^o are deterministic, this condition can never be satisfied, so that an improvement will never be possible with a defined benefit system.

The second inequality in (11) requires that $u'(e^y)$ and $\frac{v'(e^o)}{u'(e^y)}$ are positively correlated. Thus, when endowments when young and when old vary co-monotonically, we must have that not only $u'(e^y)$ and $v'(e^o)$ are positively correlated, as shown above, but that whenever $u'(e^y)$ increases, $v'(e^o)$ also increases, and more than $u'(e^y)$. Endowment when old must then vary in the same direction of the endowment when young, as we saw, but also have to exhibit a greater variability and/or the old must be more risk averse than the young. Given this feature, the fact that a deterministic transfer from the young to the old is welfare improving should not be surprising, since the old are bearing more risk than the young and it is beneficial for the young to provide them some insurance, even in the form of a deterministic transfer of income. Since adding a riskless stream of consumption to some risky level tends to decrease risk, there is a sense in which a defined benefits scheme shifts risk from the old to the young.

If the necessary condition (11) above is satisfied, it is indeed easy to construct examples where the introduction of a defined benefit system is Pareto-improving. In this regard, notice that at the golden rule the inequality on the right hand side of (11) is also a sufficient condition for the existence of an improving policy, so for any allocation close enough to the golden rule, an improvement is possible. Consider for example the case where $u(c) = v(c) = \log(c)$, $\beta = 1$, $S = 2$ and $\pi_1 = \pi_2 = 1/2$. If endowments are

$$(e^y(1), e^y(2)) = (1, 2), \quad (e^o(1), e^o(2)) = (0.75, 4)$$

the economy is conditionally optimal and (9) is satisfied. The introduction of a defined benefits social security system is then Pareto improving. The wealth (and hence the consumption) of young and old agents are clearly positively correlated, and exhibits a higher variability for the old than for the young agents.

4.3 Defined contributions

Under a defined contributions system the young pay, in each state, a constant fraction of their income, i.e. $\tau_s = e^y_s$ for all shocks $s \in \mathcal{S}$. Hence the necessary and sufficient condition for a defined contributions system to be welfare improving, at an autarkic equilibrium, is obtained from (4), setting $T_s = e^y_s$ for all s :

$$\sum_{s \in \mathcal{S}} \pi_s(-u'(e^y_s) + \beta v'(e^o_s))e^y_s > 0. \tag{12}$$

The existence of an improving policy requires in this case a condition on the joint pattern of the agent's endowment and its marginal utility, so that the elasticity of the agents' utility function will play a role.

Proceeding along similar lines to the previous subsection, we note first that the CPO condition (1), applied to the autarkic equilibrium allocation, can also be rewritten as

$$1 \geq \text{cov}(\beta v'(e^o)e^y, \frac{1}{u'(e^y)e^y}) + \mathbb{E}\left(\beta v'(e^o)e^y\right)\mathbb{E}\left(\frac{1}{u'(e^y)e^y}\right), \tag{13}$$

and implies, by Jensen's inequality, an expression analogous to (3) above:

$$1 \geq \text{cov}(\beta v'(e^o)e^y, \frac{1}{u'(e^y)e^y}) + \mathbb{E}\left(\beta v'(e^o)e^y\right)\left(\frac{1}{\mathbb{E}\left(u'(e^y)e^y\right)}\right), \tag{14}$$

where, if (12) holds, the second term is greater than one and hence the first one has to be negative.

Moreover, using the CPO condition (1) we obtain this other implication:

$$\text{cov}(u'(e^y)e^y, \beta\frac{v'(e^o)}{u'(e^y)}) \geq \mathbb{E}\left(e^y\beta v'(e^o)\right) - \mathbb{E}\left(u'(e^y)e^y\right), \tag{15}$$

whose term on the right hand side is positive, whenever the necessary and sufficient condition (12) for the policy to be improving is satisfied.

We have proved so the following:

Proposition 4 *A necessary condition for a defined contributions system to be Pareto improving, at a conditionally Pareto optimal autarkic equilibrium, is:*[6]

$$\mathit{cov}(\beta v'(e^o)e^y, \frac{1}{u'(e^y)e^y}) < 0 < \mathit{cov}(u'(e^y)e^y, \beta\frac{v'(e^o)}{u'(e^y)}). \tag{16}$$

Note that the condition does not rule out the possibility that in this case an improvement may be found even when the agents' utility function is linear - concave (in which case, e^y and e^o, if co-monotonic, have to be negatively correlated), or concave - linear, i.e. whatever the pattern of the risk aversion over the agents' lifetime, or if the endowment when old is deterministic. On the other hand, a defined contributions policy can never be welfare improving if the endowments when young are riskless.

Condition (16) is somewhat harder to interpret than the analogous condition we obtained in the case of defined benefits. It is useful to consider the special case where agents have the same constant relative risk aversion utility function when young and old, given by

$$u(c) = v(c) = \frac{c^{1-\sigma}}{1-\sigma}, \tag{17}$$

when the coefficient of risk aversion is $\sigma \neq 1$ and by $u(c) = v(c) = \log(c)$ when $\sigma = 1$. In this case, condition (16) simplifies to:

$$\text{cov}(\beta\frac{e^y}{(e^o)^\sigma}, \frac{1}{(e^y)^{1-\sigma}}) < 0 < \text{cov}((e^y)^{1-\sigma}, \beta\left(\frac{e^y}{e^o}\right)^\sigma). \tag{18}$$

[6] Again if we are at the golden rule, the inequality on the right hand side of (16) is also a sufficient condition for the existence of an improving defined contributions policy.

Note that when $\sigma = 1$, i.e. when consumers have logarithmic preferences, this inequality can never hold, thus a defined contributions social security system can never be Pareto-improving.

When $\sigma < 1$, an improvement is only possible, when the endowment of the young and the old vary co-monotonically, if in all states where the endowment when young is high (resp. low), the endowment when old is low (resp. high), i.e. the two are negatively correlated, or the endowment when old is also high but exhibits less variability than the endowment when young. Observe that, somewhat surprisingly, in this case an improvement is always possible if endowments when old are riskless, endowments when young are risky and the economy is sufficiently close to the golden rule. With social security, consumption when old becomes risky; however, the 'representative agent' is compensated for this by having less risk when young.

On the other hand, when $\sigma > 1$ the situation is somewhat analogous to the one we found in the case of defined benefits: an improvement is only possible (again under the comononicity assumption) if whenever the endowment when young is high the endowment when old is even higher, thus when endowments when young and old are *positively* correlated and the endowments when old fluctuate more than the endowments when young. Note that in this case an improvement is impossible if the endowments when old are riskless. Here the intuition from the defined benefits case carries over.

5 Effects on the Price of Long-lived Assets

As explained in the introduction, the analysis of the pure exchange case without land neglects many important general equilibrium effects of the introduction of a social security system. In this section we examine the effect of the introduction of social security on the price of long-lived assets such as land. Hence we maintain the restriction that capital and labor are not productive but suppose now that land is productive, and constitutes an infinitely-lived asset in unit net supply paying each period a dividend $d_s \equiv \partial f(0,1,1;s)/\partial b$ whenever shock s realizes.

In the presence of land a stationary equilibrium still exists both without and with a (stationary) social security system. We will again examine the effects on the stationary equilibrium of introducing an infinitesimal amount of social security, starting from a situation with zero social security transfers, $\nu = 0$. We still consider the case where the introduction of social security is announced at some date t, after some history s^t, and after all trades have taken place at that date, and will start being implemented from date $t+1$, at every successor node of s^t; at $t+1$ the price of land q varies and settles immediately at its new stationary equilibrium level. The correct timing of the introduction turns out to be even more crucial here.

FACT 1 *In the presence of land, announcing the social security policy one period before its* *introduction is always better than not announcing it; furthermore, if the policy is announced more than one period in advance, it can never be welfare improving.*

The argument is as follows. Suppose the policy were announced at date t and starts being implemented only at, say, $t+Z$, for $Z>1$. Then at date $t+Z-1$ the price of land will settle at its new stationary equilibrium level which, as we show below, will be lower. The prices at all intermediate dates (between $t+1$ and $t+Z-2$) may then also vary; whatever the direction in which they vary, since the price at $t+Z-1$ will be lower, we can say that for at least one generation the price, compared to the initial equilibrium, will be greater or equal when young and lower when old. The welfare of this generation will thus necessarily decrease, so that a welfare improvement cannot be attained in this case.

The net transfer T_s induced by the introduction of social security is now equal to the sum of the direct transfer τ_s prescribed by it and the indirect transfer induced by the change in the equilibrium price of land (the price effect). Since the total outstanding amount of land is 1, we have:

$$T_s = \tau_s + \frac{dq_s}{d\nu}.$$

To simplify the analysis, we will consider first the case in which agents' preferences are linear concave, i.e. $u(x)=x$, $v(x)$ concave. In this case, it is easy to solve explicitly for the equilibrium price of land:

$$q = \beta \sum_{s\in\mathcal{S}} \pi_s (q+d_s) v'(e_s^o + q + d_s + \nu\tau_s). \tag{19}$$

The price of land is constant across states and the price change is given by

$$\frac{dq}{d\nu} = -\frac{\beta\sum_{s\in\mathcal{S}} \pi_s\tau_s(q+d_s)v''(e_s^o+d_s+q)}{-1+\beta\sum_{s\in\mathcal{S}}\pi_s(v'(e_s^o+q+d_s)+(q+d_s)v''(e_s^o+q+d_s))}.$$

Hence the indirect transfer induced by the introduction of social security is deterministic. Moreover, since equilibria are always CPO, from (1) we get that the following must hold, $-1+\beta\sum_{s\in\mathcal{S}}\pi_s v'(c_s^0)\leq 0$, so that the price effect is always negative. Thus in the current set-up a social security transfer scheme might even make the initial old (i.e., the agents who are old at date t) worse off, since the transfer they receive from the scheme may be more than offset by the reduction in the value of their land holdings.

On the basis of the above, we can explicitly determine the total transfer generated by the policy. For any $\bar{s}\in\mathcal{S}$:

$$T_{\bar{s}} = \tau_{\bar{s}} + \frac{dq}{d\nu} = \frac{\tau_{\bar{s}}(1-\beta\sum_{s\in\mathcal{S}}\pi_s v'(c_s^o)) + \sum_{s\in\mathcal{S}}\pi_s(\tau_s-\tau_{\bar{s}})\beta v''(c_s^o)(q+d_s)}{1-\beta\sum_{s\in\mathcal{S}}\pi_s\left[v'(c_s^o)+(q+d_s)v''(c_s^o)\right]}, \tag{20}$$

where $c_s^o = e_s^o + q + d_s$ is the equilibrium consumption of the agents when old in the initial equilibrium. Substituting this expression into (5), and noting that its denominator is always positive (given CPO), and independent of $\bar{s}$, we find that the introduction of social security improves the initial old if and only if

$$\sum_{\bar{s}\in\mathcal{S}} \pi_{\bar{s}} v'(c^o_{\bar{s}}) \left(\tau_{\bar{s}}(1-\beta\sum_{s\in\mathcal{S}}\pi_s v'(c^o_s)) + \sum_{s\in\mathcal{S}}\pi_s\beta(\tau_s-\tau_{\bar{s}})v''(c^o_s)(q+d_s) \right) > 0 \tag{21}$$

Evidently, the old can be made better off by introducing social security uniformly across states. However, this will not help future generations.

If we then substitute (20) into (4), we obtain the following necessary and sufficient condition for the introduction of social security to improve all future generations:

$$\sum_{\bar{s}\in\mathcal{S}} \pi_{\bar{s}}(-1+\beta v'(c^o_{\bar{s}})) \left(\tau_{\bar{s}}(1-\beta\sum_{s\in\mathcal{S}}\pi_s v'(c^o_s)) + \sum_{s\in\mathcal{S}}\pi_s\beta(\tau_s-\tau_{\bar{s}})v''(c^o_s)(q+d_s) \right) > 0. \tag{22}$$

As we saw in the previous section, the necessary and sufficient condition for an optimally designed ideal social security system to be Pareto improving, at an autarkic equilibrium, with linear concave preferences, is that there is some state $\hat{s}$, for which: $\beta v'(c^o_{\hat{s}}) > 1$. We will show that, in the presence of land, this same condition also suffices for the existence of a welfare improving ideal system:

PROPOSITION 5 *In the presence of land, with linear-concave preferences, a welfare improving ideal social security system exists if in the equilibrium without social security there exists some shock* $\hat{s}$ *for which* $\beta v'(c^o_{\hat{s}}) > 1$.

The proofs of the propositions in this section can be found in the appendix.

In the presence of land, the condition stated in the above Proposition is only sufficient, no longer necessary for the existence of an improving ideal social security welfare scheme. This is because the indirect transfer generated by the policy is a negative transfer from the young to the old. Hence in this case it is possible to design social security schemes which have better insurance policies by implying a total transfer from the young to the old in the states where the young are rich, and a transfer from the old to the young when the latter are poor. As a consequence, a welfare improving scheme may exist even when, for all s, we have $\beta v'(c^o_s) \leq 1$. To see this consider the following simple example.

Suppose $v(c) = \log(c)$ and $e^o_s = 0$ for all $s \in \mathcal{S}$. From (19) we find that, in the absence of social security, $q = \beta$. It is clear that in this case, since $d_s \geq 0$ for all s, $\beta v'(c^o_s) = \frac{\beta}{\beta+d_s} \leq 1$ for all s. Consider then an ideal social security system with $\tau_{\bar{s}} = 1$ and $\tau_s = 0$ for all $s \neq \bar{s}$; we obtain,

$$\frac{\partial q}{\partial \nu} = -\frac{\pi_{\bar{s}}\beta}{\beta+d_{\bar{s}}},$$

and the condition for an improvement of a representative generation, equation (4), becomes

$$\sum_{s=1}^{S}\pi_s(1-\frac{\beta}{\beta+d_s})\frac{\pi_{\bar{s}}\beta}{\beta+d_{\bar{s}}} + \pi_{\bar{s}}(-1+\frac{\beta}{\beta+d_{\bar{s}}}) > 0.$$

This is obviously satisfied if $d_{\bar{s}}$ is sufficiently small compared to other dividends (in particular, if $d_{\bar{s}} = 0$ and $d_s > 0$ for all $s \neq \bar{s}$). The condition for the welfare of the initial old to increase, (5), becomes in this case:

$$\frac{\pi_{\bar{s}}}{\beta + d_{\bar{s}}} > \frac{\beta \pi_{\bar{s}}}{\beta + d_{\bar{s}}} \sum_{s=1}^{S} \frac{\pi_s}{\beta + d_s},$$

which is always true.

Since at CPO allocations the old tend to be richer than the young, we can conclude that the indirect transfer is in the 'right direction' and its negative sign, combined with the positive sign of the direct transfer, allows to generate a richer pattern of transfers between young and old and hence to make an improvement in intergenerational risk sharing easier.

5.1 Defined benefits and defined contributions

In the two cases which are a more realistic description of actually observed social security systems, defined benefits and defined contribution, the overall transfer amounts to a combination of the transfer prescribed by the policy and an indirect transfer like in a negative defined benefit system. We can then use our findings for the autarky case, when preferences are linear-concave, to show:

Proposition 6 *In the presence of land, with linear-concave preferences:*

- the introduction of a defined benefits social security system is never Pareto-improving;

- the introduction of a defined contributions scheme, on the other hand, will be Pareto-improving under weaker conditions (on the pattern of covariances of consumption) than under autarky.

5.2 General preferences

When the utility over consumption when young is no longer linear, the price effect of the introduction of social security will typically depend on the current shock. Its stochastic structure, and in particular its correlation with consumption when young and old, will then play an important role in determining whether or not the introduction of social security is welfare improving.

For the case of utility functions that are strictly concave both in consumption when young and old, one cannot obtain, in general, closed-form solutions for the price of land across states. However it is easy to see from the expression of the first order conditions,

$$q_s u'(e_s^y - q_s - \nu\tau_s) = \beta \sum_{s \in \mathcal{S}} \pi_s (q + d_s) \beta v'(e_s^o + q + d_s + \nu\tau_s). \qquad s \in \mathcal{S},$$

that with i.i.d. shocks the price of land is higher when consumption of the young is higher.

To study the exact properties of equilibria one has then to revert to numerical solutions. In all the examples we considered the price of land decreases after the introduction of social security, and the magnitude of the absolute change in the price is positively correlated with consumption when young. The overall effect of the introduction of social security is again a combination of the transfer (from the young to the old) prescribed by the policy and another transfer in the opposite direction, which is bigger when young agents' consumption is higher, i.e. somewhat analogous to the direct transfer of a negative defined contributions system. We can then use again the arguments developed above to determine when such transfers are improving.

To demonstrate that our findings obtained for the simpler economies studied in the previous sections carry over to more general environments, we consider one such example, where agents have a constant relative risk aversion utility function with coefficient of risk aversion $\sigma = 2$, $u(c) = v(c) = -c^{-1}$, and $\beta = 1$. There are 2 states with $\pi_1 = \pi_2 = 0.5$ and land's dividends are deterministic: $d_1 = d_2 = 0.05$. Let $e_1^y = 1$, $e_2^y = 2$, $e_1^o = 0.1$, $e_2^o = 1$. The first two columns of Table 1 below show the consumption allocation as well as the prices of land. Note that consumption when old and when young are positively correlated and consumption when old is more volatile.

In this economy both the introduction of a defined contributions and of a defined benefits PAYGO system are Pareto-improving. Consider first the introduction of a defined contributions system at the scale $\nu = 0.01$ (i.e. a social security tax of 1 percent of young agents' income e_s^y in each state s whose revenue is paid to the current old)[7]. In the third column of Table 1 we see the effect of this scheme on the equilibrium price of land: the price of land always decreases and the magnitude of its change is larger in state 2, when young agents' consumption is higher. The large drop in the land price in state 2 leads to a reversal of the sign of the transfer – the total transfer to the old induced by this policy is positive in state 1 and negative in state 2. Thus we have a transfer from the young to the old in state 1 (where the young are richer) and from the old to the young in state 2 (where the old are richer), with a clear improvement in intergenerational risk sharing. Even though the direct transfer to the old agents is positively correlated with the young's, as well as the old's, consumption, the total transfer is negatively correlated with it (the indirect transfer induced by the price effect proves then stronger than the direct transfer prescribed by the policy) - thus helping the old in hedging their risk. The total transfer is then also negatively correlated to consumption when young, so the young will face altogether more risk, but as we noticed their consumption was less volatile than that of the young to begin with. As a consequence, it can be verified that both the initial old and all future generations gain.

States	c^y	c^o	q	T(0.01) defined contrib.	T(0.01) defined benefit
1	0.635	0.515	0.365	0.001 ($\Delta q_1 = -0.009$)	0.001 ($\Delta q_1 = -0.009$)
2	1.037	2.013	0.967	-0.001 ($\Delta q_2 = -0.021$)	-0.004 ($\Delta q_2 = -0.014$)

Table 1: Social security with land.

Consider next the introduction of a defined benefits system, also at the scale $\nu = 0.01$,

[7] Unlike in the previous analysis, the change in policy is here discrete, though small.

characterized then by a tax $\tau_s = 0.01$ in each state s. We can see in the last column of Table 1 that, as in the case of defined contributions, the strong negative price effect in state 2 leads to a reversal of the sign of the total transfer to the old. Since the direct transfer is here constant and the size of the (negative) indirect transfer is again, in absolute value, positively correlated with the young's, as well as the old's, consumption, the stochastic properties of the total transfer are here unambiguously those of the price effect. The total transfer to the old is then smaller (in fact negative) when the old are richer. We have therefore an improvement in intergenerational risk sharing which increases the utility of all future generations. Finally, it is easy to verify that the welfare of the initial old also increases.

6 Effects on Capital and Output

The preceding analysis abstracted from one important negative effect of social security. If the equilibrium is conditionally Pareto optimal, and if the introduction of a pay-as-you-go social security system leads to a reduction in savings, the stock of capital and hence aggregate equilibrium output and consumption will be reduced for future generations. In this section we will explore, within a simple set-up, the interaction between these effects and those on risk sharing of social security. In addition, when the output is subject to productivity shocks, their properties contribute in an important way to the correlation between consumption when old and when young and to the stochastic structure of the indirect transfers generated by social security.

To better focus on the effects of social security on capital and output, we consider here the case where capital and labor are productive but there is no land and agents have no endowments of the consumption good. Since young agents supply inelastically their (unit) endowment of labor and land is not productive, we can write $f(k, s)$ to denote the firm's production function $f(k, 1, 1, s)$; its first and second derivatives with respect to k are then denoted by $f_k(k, s)$ and $f_{kk}(k, s)$, its derivative with respect to l by $f_l(k, s)$ and the cross derivative by $f_{kl}(k, s)$.

We consider again the case where agents have a quasi-linear (linear concave) utility function since under this condition a stationary equilibrium still exists[8], both without and with a (stationary) social security system. Given the stationarity of allocations, we can use many of the results obtained in Sections 2 and 4 above. A slight difference is that the transition to the new steady state is now not immediate but takes one period. If social security is introduced at some time t,[9] in that period the current old only receive the direct transfers, there is no additional transfer induced by general equilibrium effects. The current young agents have to pay the transfer but their wage does not change; when old, next period, they will receive the transfer prescribed by the policy and will also be affected by the change

[8]With production this is no longer true with general preferences.

[9]Whether or not the policy were previously announced does not matter in this case.

in the interest rate induced by the change in the stock of capital. The new steady state is then reached at $t+1$.

Let k denote the level of savings of an agent when young (or, equivalently, the amount invested in the firms' technology, which will yield the same amount of capital next period). Since shocks are i.i.d., the first order condition for the consumer of the representative generation at a stationary equilibrium is

$$-1+\beta\sum_{s\in\mathcal{S}}\pi_s f_k(k,s)v'(f_k(k,s)k+\nu\tau_s)=0. \tag{23}$$

We see from (23) that the agents' supply of capital is state invariant; as a consequence the stationary equilibrium is characterized by a constant amount of capital.

The effect on the equilibrium stock of capital of the introduction of an infinitesimal amount of social security is then:

$$k_\nu=\frac{\partial k}{\partial \nu}=-\frac{\sum_{s\in\mathcal{S}}\pi_s f_k(k,s)v''(c_s^o)\tau_s}{\sum_{s\in\mathcal{S}}\pi_s[f_{kk}(k,s)v'(c_s^o)+f_k(k,s)(f_{kk}(k,s)k+f_k(k,s))v''(c_s^o)]}.$$

where c_s^o is again the equilibrium level of consumption of the representative agent when old before the introduction of social security, now given by $c_s^o=f_k(k,s)k$. Observe that this effect is negative whenever $\frac{f_{kk}k}{f_k}\geq -1$, a condition that in what follows we will assume is always satisfied[10].

What is the effect of the change in k on the level of the young and old agents' consumption? For the young, since the equilibrium wage is given by f_l and $c_s^y=f_l(k,s)-k$, it is $(f_{lk}(k,s)-1)\,k_\nu$, while for the old it is $(f_k(k,s)+kf_{kk}(k,s))\,k_\nu$. The constant returns to scale property of the production function implies that $kf_{kk}=-f_{lk}$. Hence the total change in the amount of resources available for consumption of the agents when young is:

$$-\tau_s-k_\nu\left(kf_{kk}(k,s)+1\right), \tag{24}$$

and for the old it is:

$$\tau_s+k_\nu\left(kf_{kk}(k,s)+f_k(k,s)\right). \tag{25}$$

Note that in this case the changes do not add to zero, but to $k_\nu(f_k(k,s)-1)$, thus we do not only have a transfer but a change in available resources. Since, whenever competitive equilibria are CPO we have[11] $\mathbb{E}(f_k)>1$, we see that the introduction of social security, by reducing the stock of capital, also lowers the expected value of output and average consumption.

[10] The condition is equivalent to capital income, $f_k k$, being increasing in k and is always satisfied, for instance, by Cobb-Douglas and in fact by all CES production functions as long as the elasticity of substitution is not too small.

[11] Recalling the necessary and sufficient condition for the equilibrium to be CPO, with linear - concave preferences given by $1\geq\beta\sum_{s\in\mathcal{S}}\pi_s v'(c_s^o)$, and using the first order conditions for the agents' optimization problem, (23), we obtain $-Cov(\beta v'(c^o),f_k)\leq\mathbb{E}(f_k)-1$. Since $c^o=f_k k$, the variables $v'(c^o)$ and f_k are clearly comonotonic and negatively correlated, so we get $\mathbb{E}(f_k)>1$.

However, it is important to notice that, as long as the policy is introduced at an infinitesimal level, by the envelope theorem the welfare consequences of the induced change in output are zero. The first order conditions for an agent's optimum (23) imply that the effect on the agent's expected utility of increasing consumption when young by $d\nu$ and lowering it when old by $f_k(k,s)d\nu$ is zero. Thus, in evaluating the welfare consequences of the changes in the consumption when young and old given in (24) and (25), we can ignore the last term, so that the change in resources available to the young is exactly equal to the opposite of the change in resources available to the old. We can then say that the overall effect of the policy is a pure transfer effect from the young to the old, given by:

$$T_s = \tau_s + k_\nu k f_{kk}(k,s), \tag{26}$$

which is strictly positive for all s.

We should stress that the above property follows from the fact that we are considering the introduction of an infinitesimal amount of social security, starting from a situation where its level is zero. When on the other hand a discrete change in policy is considered, its effect is not simply that of a transfer among generations, but the welfare consequences of the change in the output level, and hence of the resources available for consumption, have also to be taken into account. This will become clear in the next Section.

On the basis of the above argument, equation (4) can still be used (now with T as in (26)) to evaluate whether or not the introduction of a social security scheme is welfare improving. We see from (26) that the indirect transfer induced by the policy has always a positive sign and varies with the state, according to the stochastic properties of f_{kk}. This is the opposite of what we found in the case of land, where the indirect transfer was negative and, with linear concave preferences, deterministic. The fact that the indirect transfer is always non-negative makes the possibility of an improvement harder (since the overall transfer will then also be non-negative in every state, which limits the possibilities of improving intergenerational risk sharing; moreover, at conditionally Pareto efficient allocations, since the old tend to consume more than the young it is easier to improve the welfare of the representative generation with a transfer from the old to the young).

The stochastic properties of the indirect transfer, which depend on how the technology shocks affect f_{kk}, also matter. To see this more precisely, notice first that the necessary and sufficient condition for the introduction of social security to improve the utility of the representative generation at the new stationary equilibrium is obtained by substituting (26) for T in equation (4):

$$\sum_{s\in\mathcal{S}} \pi_s(-1+\beta v'(c_s^o))\,(\tau_s + k_\nu k f_{kk}(k,s)) > 0. \tag{27}$$

As discussed above, the initial old cannot lose since they only obtain the direct transfer. For the initial young (i.e. born in the period social security is introduced, before the new steady state is reached) the analogous condition is obtained:

$$-\sum_{s\in\mathcal{S}}\pi_s\tau_s+\sum_{s\in\mathcal{S}}\pi_s\beta v'(c^o_s)\left(\tau_s+k_\nu k f_{kk}(k,s)\right)>0, \tag{28}$$

and is always satisfied whenever (27) holds, since k_ν is negative. Hence in the presence of production, to find an improvement it suffices to consider equation (27).

The fact that the total transfer T_s is always nonnegative implies:

PROPOSITION 7 *A necessary condition for a stationary (ideal) social security scheme to be welfare improving (when preferences are linear - concave) is that for at least one state $\bar{s}$*

$$\beta v'(c^o_{\bar{s}})>1.$$

While in the pure exchange case the above condition is also sufficient for the existence of an ideal welfare improving scheme, this is no longer true in the presence of production and capital, because the (positive) indirect transfer induced by the policy implies that it is now more difficult to fully control the risk sharing characteristics of the total transfer.

Having determined in (26) the value of the total transfer associated to any social security scheme, by a very similar argument to the one of the proof of Proposition 4 we can show that a necessary condition for scheme $(\tau_s)_{s\in\mathcal{S}}$ to be welfare improving is:

$$\mathrm{cov}(\beta v'(f_k k)\left(\tau+k_\nu k f_{kk}\right),\frac{1}{\tau+k_\nu k f_{kk}})<0<\mathrm{cov}(\tau+k_\nu k f_{kk},\beta v'(f_k k)). \tag{29}$$

From (29) we see that:

PROPOSITION 8 *At a CPO equilibrium with production, when preferences are linear - concave:*

- an improving defined benefits social security system only exists if the production shocks are such that $(-f_{kk})$ and $v'(c^o)$ are positively correlated;

- an improving defined contributions system only exists if $(-f_{kk})$, or $w=f-f_k k$, is positively correlated with $v'(c^o)$.

When f_k and f_{kk} are co-monotonic, the above necessary condition for defined benefits to be improving is equivalent to the condition that $(-f_{kk})$ and $f_k k$ are negatively correlated. A similar property holds for defined contribution.

On this basis we can look at various alternative specification of the production function, and in particular of the form of the technology shocks. We examine first the case of TFP shocks, with a Cobb-Douglas production function with capital share $\alpha\in(0,1)$:

$$f(k,s)=\xi_s k^\alpha,\ \ s\in\mathcal{S}.$$

Note that in this case $f_k=\alpha k^{\alpha-1}\xi$, $-f_{kk}=\alpha(1-\alpha)k^{\alpha-2}\xi$ and $w=(1-\alpha)k^\alpha\xi$ are all perfectly and positively correlated, so the above necessary conditions are all violated, which implies that neither defined benefits nor defined contributions can ever be improving in this set-up. The fundamental problem lies in the fact that with TFP shocks the marginal utility when old and the total transfer-payment induced by a defined benefits or defined contributions scheme are negatively correlated: when the old are rich, the transfer-payment is high and vice-versa.

Alternatively, consider the case where technological shocks are given by a combination of shocks to the depreciation rate of capital and of TFP shocks:

$$f(k,s) = \xi_s k^\alpha + (1-\delta_s)k, \quad s \in \mathcal{S} \tag{30}$$

If ξ and $1-\delta$ are sufficiently negatively correlated, also $-f_{kk} = \alpha(1-\alpha)k^{\alpha-2}\xi$ and $f_k = \xi\alpha k^{\alpha-1} + 1 - \delta$ may be negatively correlated. So we can show that in this case we can have an improvement both with defined benefits and defined contributions. Consider for example $v(c) = \log(c)$, $\beta = 1$, $\pi_1 = 0.093$, $\alpha = 0.3$, $\xi_1 = 1.1$, $\xi_2 = 0.9$, $\delta_1 = 1$, $\delta_2 = 0$. The equilibrium values are reported in the following table:

States	c^o	w	$-f_{kk}$	T
1	0.33	0.77	0.231	1.230
2	1.27	0.63	0.189	1.188

TABLE 2: Defined Benefits SS with shocks to TFP and capital depreciation.

The consumption of the old agents c^o is now negatively correlated with the indirect transfer induced by the policy (which is proportional to $-f_{kk}$). So will be then the total transfer $(T_s)_{s\in\mathcal{S}}$, reported in the last column of the table, in the case of a defined benefits scheme; such a scheme is thus improving in this example. Notice that wages are negatively correlated with c^o and it can be shown that a defined contributions system will also be improving.

Another possible specification sometimes found in the literature but not examined further in this paper is

$$f(k,s) = k^{\alpha_s}, \text{ for } \alpha_s > 0, \ s \in \mathcal{S},$$

where shocks are to the factors' shares. In this case, if $\alpha_s \leq 0.5$ for all s, $f_k = \alpha k^{\alpha-1}$ is positively correlated with $-f_{kk} = \alpha(1-\alpha)k^{\alpha-2}$ but possibly negatively correlated with $w = (1-\alpha)k^\alpha$. Hence we can have an improvement with defined contributions but there cannot be one with defined benefits.

We assumed so far that the agents' utility is linear-concave. When on the other hand the utility is strictly concave both with respect to the consumption when young and when old, the stochastic structure of the production shocks also affects the correlation between the marginal utility of consumption when young and when old which, as we saw in Section 4, plays a crucial role for the welfare effects of social security. Since in such case equilibria are no longer strictly stationary, a proper examination of it is postponed to the next section.

However, it is still useful to relate the results in this section to the findings of Bohn (2003), who considers a situation where agents have logarithmic preferences, both over consumption when young and old, $u(c) = v(c) = \log(c)$ and that the production function is as in (30) but with deterministic depreciation. In this situation, equilibrium consumption values are $\widetilde{c}^y(s^{t+1}) = \chi(1-\alpha)\xi_{s_{t+1}} k(s^t)^\alpha$, $\widetilde{c}^o(s^{t+1}) = \alpha\xi_{s_{t+1}} k(s^t)^\alpha + (1-\delta)k(s^t)$ (where $1-\chi$ is the constant savings rate). These can be approximated abstracting from the variability of k, so that the second of the two necessary conditions derived in Proposition 3 for the direct effect of a defined benefit scheme to be welfare improving can be applied, yielding $\mathrm{cov}(\frac{1}{\chi(1-\alpha)\xi}, \frac{\frac{1}{2}(1-\alpha)\xi}{\alpha\xi+(1-\delta)k^{1-\alpha}}) > 0$, which can never hold. This shows that it is difficult to make a case for a defined benefits social security system in the framework considered by Bohn. The intuition behind this is that such system, as we argued, shifts risk from the old to the young. When income when old is less risky to begin with and risk aversion is the same when old and young, this can never be improving.

7 Pareto-improving introduction of social security

We now investigate the possibility of Pareto improving social security schemes in more realistic set-ups where there is production which uses labor, capital and land as inputs. The model is explained in detail in Section 2. As no stationary equilibrium exists for this model, we have to compute equilibria numerically (we describe the algorithm in the Appendix).

Since we consider economies with two period - lived agents (i.e. a period corresponds to 30 years) and without population or technology growth it is not sensible to properly calibrate the model to match historic prices and quantities. However, we still want to consider a specification of preferences and technology which is roughly consistent with the calibrations of stochastic OLG models in the existing literature (e.g. Bohn (2003), Smetters (2004) or, to some extent, Constantinides et al (2002)).

There are 4 i.i.d. shocks, $s = 1, ..., 4$, preferences are age-invariant: $u(.) = v(.)$ and exhibit constant relative risk aversion, of the form (17), with $\beta = 1$ and $\sigma = 2$. In order to be able to control the correlation of returns to capital and wages, we consider a specification of the production shocks as at the end of the previous Section, where there is stochastic depreciation, in addition to TFP shocks:

$$f(k, l, b; s) = \xi_s k^\alpha l^\gamma b^{1-\alpha-\gamma} + (1-\delta_s)k, \quad s = 1, \ldots, 4.$$

Consistently with the existing literature (e.g. Imrohogolu et al. (2002)) we consider $\alpha = 0.28$ and $\gamma = 0.69$, i.e. the land share is 3 percent, the capital share 28 percent. We fix the TFP shocks to be $\xi_1 = \xi_2 = 1.15$, $\xi_3 = \xi_4 = 0.85$ and the depreciation shocks to be $\delta_1 = \delta_3 = \bar{\delta} + \zeta$ and $\delta_2 = \delta_4 = \bar{\delta} - \zeta$. We set average depreciation $\bar{\delta}$ to equal 0.9. Given an average annual depreciation of 5 percent, this is a bit too low for a 30-year time-interval; however, as we will discuss below a literal interpretation of δ as depreciation is difficult in a model with two period - lived agents. The size of the TFP shocks is roughly consistent with what is usually assumed in the literature, and so is the resulting coefficient of variation of wages. In Section 7.4 below we discuss how sensitive our findings are with respect to the specification of the size of the TFP shocks and of the preference parameters.

Given our previous analysis, it is clear that the welfare implications of different social security schemes will crucially depend on the vector (π, ζ), i.e. on the size of the depreciation shocks and the correlation properties of TFP and depreciation shocks, since this will govern the pattern of the volatilities and covariance of consumption when old and consumption when young. In the following we will show how different values for this vector will result in different welfare implications. There is no clear-cut empirical guidance in the choice of these parameters. First, it not possible to obtain good estimates of prices or quantities for 30-year periods. Secondly, it is well known that it is impossible to match both the Sharpe ratio and the volatility of consumption in this model. Smetters (2004) who matches average returns (and considers a model very similar to ours) takes ζ to be around 5, which in turn leads to unrealistically high consumption volatility. Furthermore, allowing δ to take values larger than 1 makes it difficult to interpret it as actual depreciation. On the other hand Bohn (2003) examines the case where depreciation is non-stochastic and close to 1 (i.e. ζ is close to 0). In this paper we consider therefore a variety of different parameter specifications: three possible values for the size of the depreciation shock, $\zeta \in \{0, 1, 2\}$, and three specifications of the probabilities, $\pi \in \{(1/4, 1/4, 1/4, 1/4), (0, 1/2, 1/2, 0), (0, 1/2, 0, 1/2)\}$, describing the cases where TFP (ξ) and depreciation ($1-\delta$) shocks are, respectively independent, positively and negatively correlated. For each value of the 'variable parameters' (π, ζ), we compute the competitive equilibrium and evaluate the welfare effects of introducing different types of social security systems.

7.1 Equilibrium prices and allocations

In order to get a first idea of how the different specifications of the parameters (π, ζ) imply different patterns for equilibrium prices and allocations, we report in Table 3 the resulting summary statistics for average returns to capital, coefficient of variation of returns, coefficient of variation of aggregate consumption and wages and correlation of returns and wages. These values can then be compared to estimates from the literature. Smetters (2004) estimates the 'true' average return to capital to be 1056 percent (for a 30-year horizon, this corresponds to 8.5 percent p.a.), and the coefficient of variation to be 0.87. He also estimates the correlation between returns and wage-income to be 0.75. We will not match any of these numbers in our specifications below, but it is instructive to see how the different choices of the parameters could be judged more or less realistic, depending on the resulting pattern of the equilibrium values.

In the table and the rest of the analysis, we use π^1 to refer to $(1/4, 1/4, 1/4, 1/4)$, π^2 refers to $(0, 0.5, 0.5, 0)$ and π^3 to $(0.5, 0, 0, 0.5)$. Since for $\zeta = 0$ equilibria for these three cases are identical, we only report the results for $(0, \pi^1)$.

(ζ, π)	Avg return	coeffvar return	coeffvar wages	corr returns wages	coeffvar agg. cons
$(0, \pi^1)$	1.32	0.05	0.20	0.27	0.25
$(1, \pi^1)$	1.84	0.54	0.19	0.08	0.29
$(2, \pi^1)$	2.71	0.74	0.18	0.10	0.30
$(1, \pi^2)$	2.02	0.57	0.19	0.996	0.38
$(2, \pi^2)$	2.96	0.76	0.17	0.999	0.40
$(1, \pi^3)$	1.67	0.56	0.19	-0.9998	0.04
$(2, \pi^3)$	2.31	0.79	0.18	-0.9998	0.054

TABLE 3: Summary statistics of equilibrium values (no SS)

As we see from the table, in all specifications we are far from matching the average return to capital or its variation. However, it is also clear that higher values of ζ lead to more realistic values for these statistics. In this sense, higher values of ζ appear more realistic. At the same time, they lead to unrealistically high coefficients of variation in aggregate consumption. When probabilities are given by π^2 the correlation of returns and wages is excessively high, but this case is still interesting given the fact that part of the literature takes that correlation to be significantly positive. On the other hand, specification π^3 leads to very low variation in consumption and a correlation of -1, which is quite unrealistic. However, it allows to clearly see what role the correlation plays for the welfare effects of social security.

Given the analysis in the previous sections, it is also of interest to know the implications of the correlation of returns and wages for the correlation of consumption when young and consumption when old. We report its values below for the case where the depreciation shock takes its intermediate value:

$(1, \pi^1)$	$(1, \pi^2)$	$(1, \pi^3)$
0.4897	0.9856	-0.8374

7.2 Decomposing direct and indirect effects

Since we approximate equilibria numerically, we will consider a small but discrete change in the size of the social security system. As described in Sections 2 and 5 above, we assume that the social security policy starts operating after the end of a given period, at all possible direct successor nodes (i.e. for all realizations of the shocks). Its introduction is not anticipated at the previous date. We trace the effects of the introduction of the policy along the event tree. We report welfare gains and losses (in wealth equivalents – the exact computations of welfare changes is reported in the Appendix) for the current generation and for the next 6 generations. After 5-6 periods welfare changes seem to stabilize.

In order to understand the sources of these welfare changes for the generations far in the future, we consider a first order approximation of such changes and decompose it into changes induced by intergenerational transfers (in turn divided into direct and indirect transfers) and changes induced by the crowding-out of capital investment induced by social security. The fact that we consider a discrete variation from zero to positive social security contributions implies that the welfare effects of the changes in the output level can no longer be ignored.

Suppose social security is introduced immediately after some node $s^{t'}$. For all $t > t'$, denote the consumption in the equilibrium without social security by $c(s^t)$ and the consumption in the equilibrium with social security by $\tilde{c}(s^T)$. Denote the total transfer to the old (given by the sum of the direct and indirect effects of the policy) as $T^o(s^t) = \tilde{c}^o(s^t) - c^o(s^t)$ and the one from the young by $T^y(s^t) = c^y(s^t) - \tilde{c}^y(s^t)$. In the presence of capital, we also need to define the total change in the level of aggregate consumption, $L(s^t) = T^o(s^t) - T^y(s^t)$.

A first order approximation of the effects of the introduction of social security on the welfare of generation t is given by:

$$\mathbb{E}_{s^0}\left\{\left(-u'(c^y(s^t,s_{t+1})) + \beta v'(c^o(s^t,s_{t+1}))\right)T^y(s^t,s_{t+1})\right\} + \tag{31}$$

$$\mathbb{E}_{s^0}\left\{\beta u'(c^o(s^t,s_{t+1}))[T^o\left(s^t,s_{t+1}\right) - T^y\left(s^t,s_{t+1}\right)]\right\} + \tag{32}$$

$$\mathbb{E}_{s^0}\left\{-u'(c^y(s^t))T^y\left(s^t\right) + u'(c^y(s^t,s_{t+1}))T^y\left(s^t,s_{t+1}\right)\right\} \tag{33}$$

The last term, (33), captures the welfare effect due to non-stationarity, i.e. the difference between the effect on the young agents of generation t and the one on the agents who will be young at date $t+1$. Under the assumption that there exists a ergodic Markov equilibrium, as $t \to \infty$, this last term tends to zero because for any ergodic Markov process (x_t) we have that $\mathbb{E}_0 x_t - \mathbb{E}_0 x_{t+1} \to 0$

The second term, (32), captures the welfare effect of the change in the aggregate level of resources available for consumption as a result of the policy (crowding out effect):

$$C(t) = \mathbb{E}_{s^0}\left\{\beta u'(c^o(s^{t+1}))L(s^{t+1})\right\}.$$

Finally, the first term, (31), measures the welfare effect of the total transfer from the young to the old at date $t+1$; it is then analogous to the expression obtained in the case of stationary equilibria. It will be useful to decompose this term further: the total transfer from the young $T^y(s^{t+1})$ is in turn equal to $\tau(s^{t+1}) + \Delta q(s^{t+1}) + [\Delta k(s^{t+1}) - \Delta w(s^{t+1})]$, where $\tau(s^{t+1})$ is the direct transfer prescribed by the social security scheme, $\Delta q(s^{t+1})$ the transfer induced by the change in the price of land (i.e. it is the difference between the price of land with and without social security), and $[\Delta k(s^{t+1}) - \Delta w(s^{t+1})]$ is the transfer induced by the change in the stock of capital (i.e. by the changes in the equilibrium levels of wages and savings). As a consequence we obtain:

$$\mathbb{E}_{s^0}\left\{\left(-u'(c^y(s^t,s_{t+1})) + \beta v'(c^o(s^t,s_{t+1}))\right)T^y\left(s^t,s_{t+1}\right)\right\} = D_d(t) + D_q(t) + D_k(t)$$

with

$$\begin{aligned}
D_d(t) &= \mathbb{E}_{s^0}\left\{\left(-u'(c^y(s^t,s_{t+1}) + \beta v'(c^o(s^t,s_{t+1}))\right)\tau(s^{t+1})\right\} \\
D_q(t) &= \mathbb{E}_{s^0}\left\{\left(-u'(c^y(s^t,s_{t+1}) + \beta v'(c^o(s^t,s_{t+1})))\right)\Delta q(s^{t+1})\right\} \\
D_k(t) &= \mathbb{E}_{s^0}\left\{\left(-u'(c^y(s^t,s_{t+1})) + \beta v'(c^o(s^t,s_{t+1}))\right)(\Delta k(s^{t+1}) - \Delta w(s^{t+1})\right\}
\end{aligned}$$

These are the three effects we discussed in Sections 4, 5 and 6. In our computations, the changes are not infinitesimal and we cannot compute $D(t)$ and $C(t)$ as $t \to \infty$. Nevertheless, it is instructive to report the values of $D(t'+6)$ and $C(t'+6)$. It turns out that after 6 periods the changes in these numbers are very small; hence reporting these number allows us to relate the findings in this section to the results obtained in the previous sections.

7.3 Introduction of social security

We report in this section the welfare effects of the introduction of a small pay-as-you-go system. We consider first the case of a defined benefits system and then the one of defined contributions.

7.3.1 Defined benefits

We compute the equilibrium for $\nu = 0.01$ and compare it with the computed equilibrium for $\nu = 0$. As explained above, to verify that the change is Pareto-improving we compute the total welfare changes for the next 6 generations after the introduction of the policy. The following table reports, for generation $t'+6$, the total change in welfare as well as its decomposition into the various effects explained in the previous section (in order to make the table easier to read, all welfare changes are multiplied by 10^4):

(ζ, π)	total change	D_k	D_q	D_d	crowding out C
$(0, \pi^1)$	-5.5	0.58	4.57	-6.73	-3.78
$(1, \pi^1)$	0.18	-0.61	4.87	-4.03	-0.05
$(2, \pi^1)$	1.13	-0.56	4.05	-2.37	0.34
$(1, \pi^2)$	2.10	-1.29	3.29	-0.47	1.01
$(2, \pi^2)$	2.64	-0.87.	1.02	2.61	0.78
$(1, \pi^3)$	-5.1	0.92	6.01	-9.52	-2.77
$(2, \pi^3)$	-4.5	0.61	8.03	-11.99	-1.91

TABLE 4: Welfare effects of introducing a defined benefits system

In all the cases reported above where the total welfare change of generation $t'+6$ is positive we do have in fact a Pareto improvement, since both the current old and all generations in the transition also gain. For instance, for the parameter value $(1, \pi_1)$ the current old gain 9.4 and the next 6 generations gain $(0.2, 0.08, 0.14, 0.17, 0.18, 0.18)$. The results for the other cases are similar and not reported.

There are several features of the results which are worth commenting on. Both for π^1 and π^2, there is a range of parameters generally considered realistic for which the introduction of social security is Pareto-improving. This is consistent with our earlier analysis as these are the cases where consumption when old is more volatile than consumption when young and they are positively correlated. On the other hand, with no depreciation shocks, social security is never improving. The reason is that in this case the old almost always consume more than the young. That is, the marginal utility when old is only very rarely above the marginal utility when young, and even then the difference is small, so that transfers from the young to the old are not improving.

To better understand our findings it is useful to consider the various components of the welfare effects of social security, by looking at the other columns of the table above. Note first that, in all cases, the welfare effect of the change in the land price (D_q) is positive and relatively large. On the other hand the effect of the transfer induced by the change in the stock of capital (D_k) is negative, with the main exception being the case of negatively correlated shocks, π^3 (this is in line with our findings in the previous Section, since $-f_{kk}$ and f_k are negatively correlated only for π^3). However quantitatively this effect is relatively small. A crucial role in determining the sign of the total welfare change is played by the direct transfer D_d. As we saw in Section 4, this effect can only be positive if consumption when old and when young are positively correlated and when consumption when old is sufficiently volatile. This explains why a welfare improvement is easiest to obtain for π^2 and why it is impossible for π^3. It also explains why a sufficiently high size of the depreciation shock ζ is needed for a positive welfare effects.

Note also that, when ζ is sufficiently high, the crowding out effect is actually positive. While we find that it is always true that $\mathbb{E}_0(L(s^t))$ is negative, the covariance between $L(s^t)$ and $u'(c^o(s^t))$ is in fact likely to be positive. This can be seen by noting that $L(s^t)$ is approximately $\Delta k(s^{t-1})f_k(s^t) - \Delta k(s^t)$; since Δk is typically negative, in states where the old are relatively poor and $u'(c^o)$ is large, f_k is low and hence the term $\Delta k(s^{t-1})f_k(s^t)$ is also large.

7.3.2 Defined contributions

We consider now the effects of introducing a defined contribution system. Since the equilibrium wages lie around 2.5-3 in most of the examples considered, to make the size of the system which is introduced comparable to the one of the previous section, we set the contribution rate at 0.35 per cent, i.e. $\nu = 0.0035$. The following table reports again the effects for generation $t' + 6$:

(ζ, π)	total change	D_k	D_q	D_d	crowding out C
$(0, \pi^1)$	-6.95	0.79	5.62	-8.46	-4.64
$(1, \pi^1)$	-0.40	-0.36	5.76	-5.53	-0.31
$(2, \pi^1)$	0.35	-0.30	4.96	-4.23	0.33
$(1, \pi^2)$	0.72	-0.68	5.30	-4.05	0.31
$(2, \pi^2)$	1.01	-0.42	4.12	-2.71	0.33
$(1, \pi^3)$	-2.99	0.34	5.10	-6.88	-1.42
$(2, \pi^3)$	-2.32	0.24	6.21	-8.13	-1.01

TABLE 5: Welfare effects of introducing a defined contributions system

Just as in the previous section, in all cases where the total change is positive, the current old and all future generations gain from an introduction of social security. Comparing the table above with the previous one, we see that it is slightly more difficult to obtain an improvement than it was in the case of defined benefits. The pattern of the welfare changes as well as of their components across the different specifications is largely analogous to the defined benefit case.

7.4 Sensitivity analysis

We discuss in this section how sensitive our findings are with respect to the specification of the preference parameters σ (coefficient of relative risk aversion) and β (discount factor). We consider the values $\sigma \in \{0.5, 2, 4\}$ and $\beta = \{0.44, 1\}$. These values cover the ranges considered realistic in the literature.

The analysis above showed that the higher the depreciation shock ζ, the more likely it is that the introduction of a social security system is Pareto improving. Hence here, for any given specification of the preference parameters $(\sigma, \beta) \in \{0.5, 2, 4\} \times \{0.44, 1\}$ and the probabilities π^i, $i = 1, 2, 3$, we search for the smallest value of $\zeta \in \{0, 0.1, 0.2, \ldots, 2\}$ for which the introduction of social security constitutes a Pareto-improvement in the implied economy. If there is no improvement for $\zeta = 2$, we report "> 2". The following table shows the results for the different specifications of (σ, β) and π^1, π^2:

Defined benefits

$\pi \backslash (\sigma, \beta)$	(0.5,1)	(2,1)	(4,1)	(0.5,0.44)	(2,0.44	(4,0.44)
π^1	> 2	1.0	0.5	> 2	1.4	0.7
π^2	1.9	0.5	0.3	> 2	0.8	0.4

Defined contributions

$\pi \backslash (\sigma, \beta)$	(0.5,1)	(2,1)	(4,1)	(0.5,0.44)	(2,0.44	(4,0.44)
π^1	> 2	1.3	0.6	> 2	1.6	0.8
π^2	> 2	0.7	0.4	> 2	0.9	0.5

Table 6: Threshold of ζ for which SS is improving

The table shows that a higher degree of risk aversion always helps, as the set of values of ζ for which social security is improving expands, while a higher discount rate slightly hurts. The positive role of σ is not surprising: improvements in risk sharing have a larger welfare effect for higher σ. Furthermore, in line with what we found earlier, we see that with defined contributions it is a bit more difficult to have an improvement, but it seems that for higher risk aversions the difference becomes very small. Note in particular, that for a coefficient of relative risk aversion of 4 and π^2 an improvement is possible in an economy with only very modest depreciation shocks, where $\delta_s \in \{0.6, 1.2\}$ for all s.

For the specification of the probabilities given by π^3 we find that an improvement is not possible in any of the cases considered. For the case of low risk aversion this is somewhat surprising, since our previous analysis suggests that, for the case of defined contributions, a negative correlation between consumption when old and consumption when young should help. However, it should be pointed out that the condition found in Proposition 4 is only necessary, not sufficient for an improvement, and in the situation considered here it still turns out that the welfare effect of the direct transfer is significantly negative, which should also reflect the fact that the economy is in this case rather far from the golden rule. In fact, for $\zeta = 1$, we obtain $D_l = 10.07$, $D_k = 0.40$ and $D_d = -13.05$. As before, the direct effect is crucial for the overall welfare effect.

Lastly, we perform a sensitivity analysis with respect to the variance of the TFP shock, ξ. We consider the case $\xi_1 = \xi_2 = 1.08$, $\xi_3 = \xi_4 = 0.92$, where the TFP shock has a lower variance. We find that both with defined benefits and with defined contribution, it is a bit more difficult to obtain an improvement, but the differences are quantitatively quite small. For defined benefits, with $\sigma = 2$, $\beta = 1$ and π^2 an improvement is obtained for all $\zeta > 0.7$. The results for other specifications are very similar.

8 Pareto improving reform of social security

We now consider a situation where a social security system is already in place. In this case we investigate welfare improving changes in the system, both in its size and its nature, and hence discuss the optimal size and optimal reform of the system. In this section, we focus on the benchmark case of the preference parameters: $\sigma = 2$, $\beta = 1$.

8.1 Optimal size of the system

The actual size of the system in the US lies around 12 percent. By most accounts this is viewed as too large a system. However, it seems that for some of our parameterizations above, this proves actually to be the 'right size'.

To investigate the optimal size of a social security system, we suppose the economy starts with a social security system of size ν and study for which values of ν it is Pareto-improving to further increase the size of the system. We see that the larger the initial size ν of the system the harder it is that an increase in ν is welfare improving. We search then for all $\nu \in \{0, 0.01, 0.02, \ldots, 1\}$ for the largest value of ν for which a (small) increase in the size of the social security system is Pareto-improving. In the second line of the following table we report, for different parameter specifications, the level ν of the tax rate in a defined contributions system such that at $\nu - 0.005$ an increase by 0.005 of the tax rate is still slightly Pareto-improving, but at ν an increase by 0.005 makes future generations worse off. In the first line of the table we report then the corresponding values for a defined benefits system; in that case, to make the numbers comparable, the size ν of the system (i.e. the level of the contributions paid) is written as percentage of average wages.

	π^1, $\zeta = 2$	π^2, $\zeta = 1$	π^2, $\zeta = 2$
Defined benefits	0.11	0.12	0.21
Defined contributions	0.08	0.09	0.19

Table 7: Optimal size of the SS system

We see from the table that for a variety of specifications of the parameters a social security system that is roughly the size of the current US system seems about optimal.

8.2 Risk-sharing reforms

As argued above, a large defined contributions system as the one observed in many industrial countries appears to be at, if not above, the optimal size of such system. As a consequence, welfare improvements cannot be attained by further increasing the size of the system. The question then arises if a reform of such a system can lead to a Pareto-improvement. It is clear that this is only possible if the transfer to the old is increased in at least one state, given that it should be decreased in others. Given our theoretical results in Sections 2 and 4 above, a candidate reform would be to decrease the transfers in states where the old are relatively rich while increasing them (to compensate the current old) in states where the old are relatively poor.

Consider the case where $\zeta = 1$, $\pi = \pi^2$ and a defined contributions system is present with tax rate $\nu = 0.10$. Hence from table 7 we see that the size of the system is slightly too large for the economy considered. In this situation, however, quantitatively large welfare gains can still be obtained by reforming the system, in particular by making the tax rate state dependent. If the system is changed to the following one, $\tau(s^t) = 0$ for $s_t = 2$ and $\tau(s^t) = 0.2w(s^t)$ for $s_t = 3$, i.e. if for the bad return state the payroll tax is increased to 20 percent while in the good return state it is decreased to zero, the welfare gains for the current old and all future generations *in percent* are given by

$$(2.9, 2.6, 2.6, 3.1, 3.2, 3.2).$$

That is, the representative future generation gains 3.2 percent in wealth equivalence, while the current old, although they paid the full tax of 10 percent when they were young, gain in expected value 2.9 percent through the change in the risk-sharing characteristics of the system.

9 Conclusion

The idea that a pay-as-you-go social security system can lead to enhanced intergenerational risk sharing has been formalized in various papers (see the literature review in the introduction). However, it is also well known that the general equilibrium effects of social security lead to lower capital formation and hence lower consumption for future generations if the economy is dynamically efficient. In most quantitative studies this second effect seems to overcompensate any beneficial effects of enhanced risk sharing.

We show that the presence of a durable good like land as an additional factor of production mitigates the crowding out effect and that intergenerational risk-sharing provides a normative justification of a pay-as-you-go social security system even if one takes into account the effects on the capital stock and equilibrium prices and if markets are complete. It is crucial to note that in our framework social security is only desirable under an ex-ante welfare criterion: in the economies considered competitive equilibria are in fact always interim efficient, the only possibility for an improvement is then due to agents' inability to trade before their birth, which prevents the attainment of efficient intergenerational risk sharing.[12]

Appendix

Details on computations

In Sections 7 and 8 we seek to find an admissible range for the capital stock $\Theta \subset \mathbb{R}_{++}$ as well as functions from the current shock and the beginning of period capital stock to land prices and investments, $\rho_q : \Theta \times \mathcal{S} \to \mathbb{R}_+$, $\rho_k : \Theta \times \mathcal{S} \to \Theta$ such that for all shocks $\bar{s} \in \mathcal{S}$ and all $k_- \in \Theta$ the following inequalities hold for small $\epsilon \geq 0$

$$\begin{array}{c} \| -1 + \beta \sum_{s\in\mathcal{S}} \pi_s f_k(\rho_k(k,\bar{s}),1,1;s)\frac{v'(c_s^o)}{u'(c^y)} \| < \epsilon \\ \| -\rho_q(k,\bar{s}) + \beta \sum_{s\in\mathcal{S}} \pi_s(\rho_q(\rho_k(k,\bar{s}),s) + f_b(\rho_k(k,\bar{s}),1,1;s))\frac{v'(c_s^0)}{u'(c^y)} \| < \epsilon, \end{array} \tag{A.1}$$

where $c^y = f_l(k,1,1;\bar{s}) - \rho_k(k,\bar{s}) - \rho_q(k,\bar{s})$ and $c_s^o = \rho_k(k,\bar{s}) f_k(\rho_k(k,\bar{s}),1,1;s) + \rho_q(\rho_k(k,\bar{s}),s)$, for all $s \in \mathcal{S}$; $f_b(\rho_k(k,\bar{s}),1,1;s)) \equiv \frac{\partial f(\rho_k(k,\bar{s}),1,1;s)}{\partial b}$. The terms on the left hand side of $(A.1)$ constitute the first order conditions for a competitive equilibrium without social security; suitably modified expressions hold in the case where a social security system is present.

We use a collocation algorithm as described for example in Krueger and Kubler (2004) to approximate these functions numerically. For this, we write ρ_k and ρ_q as cubic splines (i.e. piece-wise cubic polynomials) with 200 collocation points. We solve for the unknown spline coefficients using time-iteration, i.e. given an approximation for ρ_k and ρ_q next period, ρ_k^N and ρ_q^N, we solve for optimal choices and prices the current period on a grid of 200 points and interpolate the solution to obtain new functions ρ_k^{N+1} and ρ_q^{N+1}. This procedure is repeated until for some $\bar{N}$,

$$\|\rho_q^{\bar{N}} - \rho_q^{\bar{N}-1}\|_\infty + \|\rho_k^{\bar{N}} - \rho_k^{\bar{N}-1}\|_\infty < 10^{-10}.$$

With the candidate function $\rho_q^{\bar{N}}$ and $\rho_k^{\bar{N}}$, we determine the error in the above system of equations. If $\epsilon < 10^{-5}$, we accept this as an approximate solution and report equilibrium prices and welfare levels for this approximation.

[12] Under an interim criterion the presence of land tends to decrease rather than increase the scope for social security because it provides an important tool for self-insurance.

Welfare computations

As already said in Section 7.2, we suppose that social security is introduced, unanticipated, at some time $t'+1$ for all possible realizations of the shocks, $1, ..., 4$, i.e. at nodes $(s^{t'}, 1), ..., (s^{t'}, 4)$. Let $\tilde{c}^y, \tilde{c}^o$ denote the equilibrium consumption levels in the economy with social security and c^y, c^o the consumption levels in the economy without social security. Given our assumption of CRRA utility functions, the welfare change for the initial old, in wealth equivalent terms (i.e. the percentage change Δ in consumption, uniform across all states, making agents indifferent between $\tilde{c}^o$ and $c^o(1+\Delta)$) is given by $\left(\frac{\sum_{s\in\mathcal{S}}\pi_s(\tilde{c}^o(s^{t'},s))^{1-\sigma}}{\sum_{s\in\mathcal{S}}\pi_s(c^0(s^{t'},s))^{1-\sigma}}\right)^{\frac{1}{1-\sigma}} - 1$. The welfare change for a generation born T generations after the introduction of social security $T = 0, 1, ...$ is then given by

$$\left(\frac{\sum_{s^{t'+T}\succ s^{t'}}\pi(s^{t'+1+T}|s^{t'})((\tilde{c}^y(s^{t'+1+T}))^{1-\sigma}+\beta\sum_{s\in\mathcal{S}}\pi_s(\tilde{c}^o(s^{t'+1+T},s))^{1-\sigma})}{\sum_{s^{t'+T}\succ s^{t'}}\pi(s^{t'+1+T}|s^{t'})((c^y(s^{t'+1+T}))^{1-\sigma}+\beta\sum_{s\in\mathcal{S}}\pi_s(c^o(s^{t'+1+T},s))^{1-\sigma})}\right)^{\frac{1}{1-\sigma}} - 1.$$

Proofs

Proof of Proposition 5. In Proposition 2 we showed that, under the same condition of this Proposition, the following transfer scheme is always welfare improving, at an autarkic equilibrium: $\tau_{\hat{s}} > 0$, $\tau_s = 0$ for all $s \neq \hat{s}$. We will show that, in the presence of land, it is always possible to design an ideal social security system for which the set of total transfers is exactly the same (and hence will be Pareto improving).

To make the notation a bit simpler we will assume that for all other $s \neq \hat{s}$, $\beta v'(c^o_s) < 1$ (it should become clear that this is an innocuous assumption).

It is immediate to see that we can always find some $\delta < 1$ such that

$$\delta(1-\beta\sum_{s\in\mathcal{S}}v'(c^o_s)) + \pi_{\hat{s}}(1-\delta)\beta v''(c^o_{\hat{s}})(q+d_{\hat{s}}) = 0.$$

Consider then a stationary social security system characterized by $\tau_{\hat{s}} = 1$ and $\tau_s = \delta$ for all $s \neq \hat{s}$. By substituting the previous expression in (20), we find that this system will induce the following total transfers across generations:

$$T_s = \tau_s + \frac{dq}{d\nu} = \frac{\delta(1-\beta\sum_{s\in\mathcal{S}}\pi_s v'(c^o_s)) + \pi_{\hat{s}}(\delta-1)\beta v''(c^o_{\hat{s}})(q+d_{\hat{s}})}{1-\beta\sum_{s\in\mathcal{S}}\pi_s\left[v'(c^o_s)+(q+d_s)v''(c^o_s)\right]}, \text{ for } s \neq \hat{s}.$$

which, given the above specification of δ, equals zero, and

$$T_{\hat{s}} = \tau_{\hat{s}} + \frac{dq}{d\nu} = \frac{(1-\beta\sum_{s\in\mathcal{S}}\pi_s v'(c^o_s)) + \sum_{s\neq\hat{s}}\pi_s(\delta-1)\beta v''(c^o_s)(q+d_s)}{1-\beta\sum_{s\in\mathcal{S}}\pi_s\left[v'(c^o_s)+(q+d_s)v''(c^o_s)\right]} > 0$$

where the sign follows from the conditional optimality of the equilibrium and the fact that $\delta < 1$. Under the condition $\beta v'(c^o_{\hat{s}}) > 1$, this policy will clearly improve future generations, as well as the initial old, i.e. satisfy (22) and (21). ■

Proof of Proposition 6. If τ_s is constant across all shocks s, since the price effect $\frac{dq}{d\nu}$ is also, as we saw, independent of s, so will be the total transfer induced by the policy. From (20) we see in fact that, when $\tau_s = \tau$ for all s:

$$T_s = \tau_s + \frac{dq}{d\nu} = \frac{\tau(1 - \beta \sum_{s \in \mathcal{S}} \pi_s v'(c_s^o))}{1 - \beta \sum_{s \in \mathcal{S}} \pi_s \left[v'(c_s^o) + (q + d_s) v''(c_s^o)\right]},$$

which is constant across s. Since in Proposition 3 we showed that, when agents have linear - concave preferences, a deterministic transfer from the young to the old can never be welfare improving, this implies that a defined benefits system can never be welfare improving in this case.

In the case of a defined contributions system, on the other hand, the total transfer induced by the policy is

$$T_s = e_s^y + \frac{dq}{d\nu} = e_s^y + \frac{\beta \sum_{s \in \mathcal{S}} \pi_s e_s^y (q + d_s) v''(e_s^o + d_s + q)}{1 - \beta \sum_{s \in \mathcal{S}} \pi_s \left[v'(e_s^o + q + d_s) + (q + d_s) v''(e_s^o + q + d_s)\right]}.$$

Since the second term is always negative and independent of s, we can say the total effect of a defined contributions system with land is analogous to that of a combination of a defined contributions and a negative defined benefits scheme under autarky. By comparing Equations (8) and (1), we see that the latter has always (at least with linear - concave preferences) a positive effect on the welfare of the representative generation. Hence the conditions (on the parameters of the economy) under which a defined contributions system is Pareto improving are weaker than the ones we found under autarky. ■

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