QUORUM AND TURNOUT IN REFERENDA

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Abstract. We provide a positive and normative analysis of referenda with a quorum limit. Voting and majority quorum are in practice the same: both always result in the same equilibria. The quorum often reduces the incentives to mobilize of the party supporting the status quo thereby reducing the turnout in equilibrium. A referendum which results in a turnout below some voting quorum may often have had a turnout above that quorum if only the quorum requirement were removed. Also, relative to a regime of no quorum, a quorum regime can indeed increase the chance that a referendum is passed and reduce the gains of the party supporting the status quo.

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"Last June, the church played a role in a referendum that sought to overturn parts of a restrictive law on in vitro fertilization[...]. To be valid, referendums in Italy need to attract the votes of more than half the electorate. Apparently fearing defeat, Cardinal Camillo Ruini called on Catholics to stay away so that the initiative would be thwarted with the help of the merely apathetic. His move was so blatantly tactical (and questionably democratic) that it prompted criticism from believers including Mr.Prodi and the leader of the National Alliance, Gianfranco Fini. But it worked. Only 26% of the electorate turned out to vote, so the legislation remained in force." The Economist, Dec. 10th 2005

1. Introduction

According to the Merriam-Webster, a quorum is the number (as a majority) of officers or members of a body that when duly assembled is legally competent to transact business. Typically, this is a majority of the people expected to be there, although many bodies may have a lower or higher quorum. In several countries the quorum requirement is applied to direct democracy institutions like popular referendums where the status quo can be replaced only if a majority of voters is in

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favor of it and also a certain minimum turnout of the electorate is met. This is intended to ensure that the result is representative of the will of the electorate and is analogous to the quorum required in a committee or legislature. One of the reasons for any quorum requirement is that, for the issue at stake in a referendum to be approved, it should be relevant enough, namely enough citizens should "care" about it. How much citizens "care" about an issue is measured by how many citizens vote on it on the referendum. The problem is that how many people turn out to vote depends on the mobilization effort of the parties. In other words, much citizens "care" about an issue is in fact endogenous.

Voters who are in favor of the status quo are able to use an obstructive strategy called, in the United States, quorum-busting. If a significant number of voters choose not to be present for the vote, the vote will fail due to lack of quorum, and the status quo will remain.

Many countries have referendum provisions with different quorum requirements. The following countries have a turnout quorum requirement: Italy (50%), Portugal (50%), Slovakia (50%), Taiwan (50%), Denmark (40%), Colombia (25%), Venezuela (25%). The following countries have referenda without any quorum requirement: France, Switzerland, Ireland, Spain (not binding referendum). And finally the following countries have a majority quorum requirement, namely, the reform party must have the majority and at list a certain turnout in order for the referendum to pass: Germany (25%), Scotland (40%), Sweden (50%), Belarus (50%), Latvia (50%). In the US there are referenda at the state level, some states have a quorum as a percentage of the turnout on the current election: Nebraska (35%), Massachusetts (30%), Mississippi (40%), Wyoming (50% of the votes of the proceeding general election) and some others do not, e.g. California and Texas.

The quote on the front page refers to a referendum on Stem Cell research in Italy in June 2005. The party against the proposal and in favor of the status quo had the advantage that if a quorum turnout of 50% was not reached than the status quo would prevail regardless of the result of the election. The strategy of the status quo party indeed was to encourage citizens more or less explicitly to go spend the weekend on the summer Italian beaches and forget about the vote and the issue. As it turned out, the strategy was successful and the status quo prevailed: the status quo party got only 12% of the votes but the total turnout was only 26%. This is surprising as Italy is known to have one of the highest turnouts in national elections (typically above 80% of eligible voters) compared to all other countries where voting is not mandatory. This begs the question of what would have happened for instance in that same Italian referendum if the quorum requirement was removed or lowered significantly? As a consequence of the increased mobilization effort of both parties possibly the turnout would have been higher than 50% and we would we have concluded that people indeed the issue was relevant enough. For instance, in June 2006 a referendum to amend the Italian constitution will be held, which will exceptionally not have any voting quorum requirement. It will be interesting to see if the turnout raises above 50% in this case.

2. The Model

Consider a simple model of direct democracy where there are only two alternatives available: r ("reform") and s ("status quo"). The alternative that is implemented is s in the case that a total turnout threshold i.e. a quorum $q \in [0,1]$ is not met, while in the case that such a turnout quorum q is met, a simple majority rule decides the alternative implemented (ties are zero chance events).

The players of the game are two exogenously given parties supporting issues r and s. Slightly abusing notation, we will use the same symbol (e.g. s) to denote an issue and the party supporting that issue. There is a continuum of voters of measure 1, of which a random proportion $\widetilde{r} \in [0,1]$ supports issue r, while the proportion $1-\widetilde{r}$ supports issue s. We assume that, from the parties point of view, $\widetilde{r}$ is a random variable uniformly distribution on $[0,1]$. Each voter has a personal cost of voting $c \in [0,1]$ that is also drawn from a uniform distributed on $[0,1]$.

Parties decide simultaneously how many campaign funds or effort to spend to mobilize voters in order to win the referendum. Their objective functions are

$$\pi_r = BP - R$$
$$\pi_s = B(1-P) - S,$$

where P is the (endogenous) probability that alternative r is selected i.e. the referendum is approved, R (S) is the spending of the party supporting issue r (s) in order to mobilize voters, and $B > 0$ is the net benefit if the preferred alternative wins the referendum. The parameter B can also be interpreted as the importance of the issue at stake in the referendum. The effectiveness for mobilizing voters of some campaign spending x by a party is captured by a mobilization function $\rho(x) : \mathbb{R}_+ \to [0,1]$, which is continuous for $x \geq 0$, twice differentiable for $x > 0$ and satisfies the properties

$$\begin{aligned} \rho(x) &\geq 0, \qquad \rho'(x) > 0, \qquad \rho''(x) < 0 \\ \lim_{x \to 0} (x\rho'(x)) &= 0, \qquad \lim_{x \to 0} (x\rho''(x)) = 0 \end{aligned}$$

For given spending R of party r, a voter that supports issue r and has a voting cost equal to c votes for alternative r if and only if $\rho(R) > c$, (likewise, $\rho(S) > c$ for a supporter of issue s). Hence

$$\Pr(\rho(R) > c) = \rho(R),$$

and the vote shares for each party are

$$v_R = \widetilde{r}\rho(R), \qquad v_S = (1 - \widetilde{r})\rho(S).$$

P is the *joint* probability that the vote share of r is greater than the vote share of s *and* that the total turnout exceeds the quorum q. Therefore,

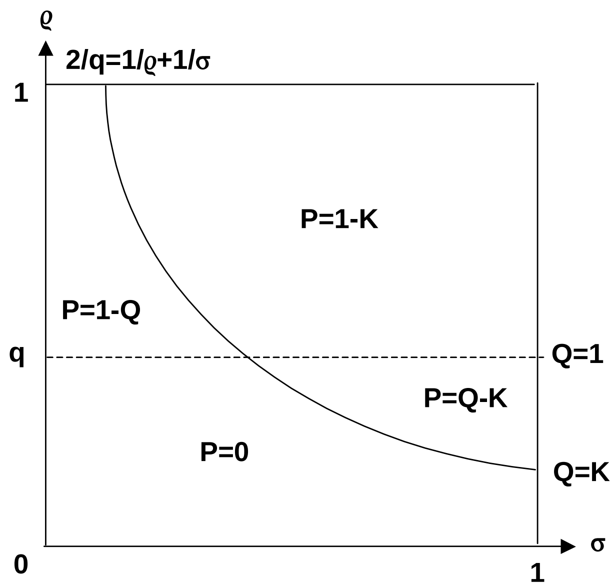
$$\begin{aligned} P &= \Pr\left((v_R \geq v)_S, (v_R + v_S \geq q)\right) \\ &= \Pr\left(\widetilde{r} \geq \frac{\rho(S)}{\rho(R) + \rho(S)}, (\rho(R) - \rho(S))\widetilde{r} \geq q - \rho(S)\right). \end{aligned}$$

By defining

$$Q = \frac{q - \rho(S)}{\rho(R) - \rho(S)}, \qquad K = \frac{\rho(S)}{\rho(R) + \rho(S)},$$

we can represent P as a function of $\rho \equiv \rho(R)$ and $\sigma \equiv \rho(S)$ for any given q. Q represent the concern for meeting the quorum requirement and K the concern for obtaining the majority of the votes.

Proposition 1. *$P(\sigma, \rho)$ takes the values shown in this figure*

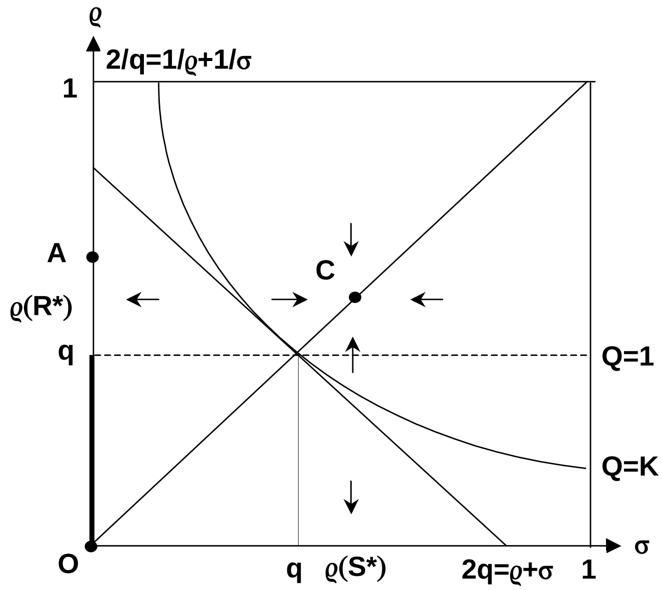


See the Appendix for the construction.

$P(\sigma,\rho)$ is continuous in its arguments on the whole space $(\sigma,\rho) \in [0,1]^2$. Above the curved line P depends on K, namely the party mobilization is high enough that the concern of winning is relevant. Below the curve line P is independent of K as the concern for obtaining the majority is secondary relative to the concern of meeting the quorum or not. In the benchmark case $q = 0$ the curved line on the picture collapses on the axes, and therefore on the whole space we have: $P = 1 - K$. As q increases the curved line moves up and right continuously up to $q = 1$ where it collapses to the point $(1,1)$ as P converges to zero. As it will become clear the regions in which the important strategic interplay between the two parties takes place are only two: the $P = 1 - K$ and the $P = 1 - Q$ regions.

3. Equilibria

We now characterize the Nash equilibria of this game for all values of the exogenous parameters (q, B). There are only three possible equilibria in pure strategies, which are represented in the picture below: two symmetric equilibria denoted by O and C, and an asymmetric equilibrium denoted by A.



3.1. **Symmetric Equilibrium.**

Proposition 2. *For* $q = 0$*, for any* $B > 0$ *there exists a unique equilibrium strategy profile* $C \equiv (S^*, R^*)$*. The equilibrium is symmetric and* $S^* = R^*$ *solves*

$$\gamma(R^*) \equiv \frac{\rho'(R^*)}{4\rho(R^*)} = B^{-1} \quad \Longleftrightarrow \quad R^* = \gamma^{-1}\left(B^{-1}\right)$$

For any $B > 0$*, the profile* (S^*, R^*) *is still an equilibrium if and only if*

$$q \leq \left[\left(\frac{1}{2} - B^{-1}\gamma^{-1}\left(B^{-1}\right)\right)\rho\left(\gamma^{-1}\left(B^{-1}\right)\right)\right] \equiv \underline{q}(B) \in \left(0, \frac{1}{2}\right)$$

The proof in the Appendix shows that (S^*, R^*) remains an equilibrium as long as q is so large to make the quorum busting deviation to $S = 0$ profitable for party s.

3.2. **Quorum Busting Equilibrium.** If the quorum requirement is too large $q > \underline{q}(B)$, then party s deviates to zero and the symmetric spending profile is no longer an equilibrium.

We call $\left(0, \widehat{R}\right)$ the quorum busting asymmetric spending profile: party s spends zero because its optimal strategy is to win by trying to keep the total turnout below quorum, whereas party r spends a positive amount $\widehat{R}$ trying to mobilize enough supporters to push the turnout above quorum with some probability. For $q > 0$, a quorum busting spending profile $\left(0,\widehat{R}\right)$ is an equilibrium if and only if the following compatibility conditions for both parties hold simultaneously:

$$\pi_r\left(0,\widehat{R}\right) \geq \pi_r(0,0)$$
$$\pi_s\left(0,\widehat{R}\right) \geq \pi_s\left(\widehat{S},\widehat{R}\right)$$

where $\widehat{R}$ and $\widehat{S}$ are functions of (q,B) implicitly defined as

$$\begin{aligned}\widehat{R} &= \arg\max\left(B\left(1-\frac{q}{\rho\left(\widehat{R}\right)}\right)-\widehat{R}\right)\\ \widehat{S} &= \arg\max\left(B\frac{\rho\left(\widehat{S}\right)}{\rho\left(\widehat{R}\right)+\rho\left(\widehat{S}\right)}-\widehat{S}\right)\end{aligned}$$

is the best response of s to $\widehat{R}$ inside the $P=1-K$ region. Note that $\widehat{R}$ and hence $\widehat{S}$ depend on B but also on q. For any $B>0$ define the two thresholds

$$\begin{aligned}\overline{q}(B) &: \pi_r\left(0,\widehat{R}\right)=\pi_r(0,0)\\ \widehat{q}(B) &: \pi_s\left(0,\widehat{R}\right)=\pi_s\left(\widehat{S},\widehat{R}\right)\end{aligned}$$

Proposition 3. *The profile* $A\equiv\left(0,\widehat{R}\right)$ *is an equilibrium if and only if* $q\in\left[\widehat{q}(B),\overline{q}(B)\right]$.

As we show later, for any $B>0$ the thresholds $\left(\overline{q}(B),\widehat{q}(B)\right)$ are uniquely defined, yet this quorum busting equilibrium may not always exist, since for low B we may have $\widehat{q}(B)>\overline{q}(B)$.

3.3. **Zero Spending Equilibrium.** The zero spending profile $O\equiv(0,0)$ is an equilibrium if and only it is optimal for r to spend zero when s spends zero, that is

$$\pi_r(0,0)\geq\pi_r\left(0,\widehat{R}\right) \tag{1}$$

Proposition 4. *The zero spending profile* $(0,0)$ *is an equilibrium if and only if*

$$q\in\left[\overline{q}(B),1\right]$$

For $q=0$ this is never an equilibrium for any $B>0$, since the r party can spend $\varepsilon>0$ and increase his probability of winning from one half to one.

3.4. **Equilibrium Regions.** The three pure strategy equilibria never overlap so we never have multiple equilibria. In a region of the parameter space (q, B) no pure strategy equilibrium exists yet we have a natural mixed strategy equilibrium.

The objective functions of the parties $\pi_s(S, R)$ and $\pi_r(S, R)$ are continuous in the whole space and that, in the region where $P = 1 - K$, $\pi_s(S, R)$ is single peaked in S, and $\pi_r(S, R)$ is single peaked in R. For all $\rho < q$, $S = 0$ is a dominant strategy for s (boldfaced line the figure above) as s can guarantee that $P = 0$, i.e., s can win the referendum with probability one at no cost and the maximum possible payoff $\pi_s = B$. Hence, no strictly positive equilibrium spending profile can be in the interior of the $\rho < q$ region.

Proposition 5. *For all $B > 0$ the thresholds are uniquely defined and*

$$\underline{q}(B) < \widehat{q}(B), \quad \underline{q}(B) < \overline{q}(B)$$

Moreover, there exists a $\overline{B}$ such that

$$B \gtrless \overline{B} \iff \widehat{q}(B) \lesseqgtr \overline{q}(B)$$

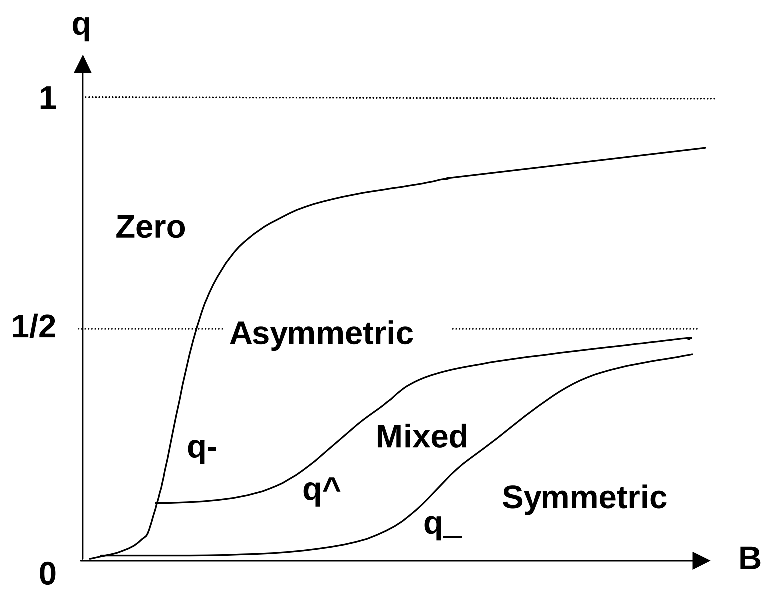
Summarizing, the equilibria for all parameter values (B, q) are

$$(S, R) = \begin{cases} C = (S^*, R^*) & \text{if } q \in \left[0, \underline{q}\right] \\ \left((\alpha, 1-\alpha), \widetilde{R}(q)\right) & \text{if } q \in \left(\underline{q}, \widehat{q}\right) \\ A = \left(0, \widehat{R}\right) & \text{if } q \in [\widehat{q}, \overline{q}] \\ O = (0, 0) & \text{if } q \in \left(\underline{q}, 1\right] \end{cases}$$

Proposition 6.

$$\begin{array}{llll} \dfrac{d\underline{q}}{dB} > 0, & \lim\limits_{B \to 0} \underline{q} = 0, & \lim\limits_{B \to \infty} \underline{q} \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right] \\ \dfrac{d\widehat{q}}{dB} > 0, & \lim\limits_{B \to 0} \widehat{q} = 0, & \lim\limits_{B \to \infty} \widehat{q} \in \left[\lim\limits_{B \to \infty} \underline{q}, \dfrac{1}{2}\right] \\ \dfrac{d\overline{q}}{dB} > 0, & \lim\limits_{B \to 0} \overline{q} = 0, & \lim\limits_{B \to \infty} \overline{q} \in \left[\dfrac{1}{2}, 1\right] \end{array}$$

Hence, the equilibrium regions have the boundaries as shown in the following picture.



The asymmetric equilibrium may not exist for low B, i.e. when $\widehat{q} > \overline{q}$ as the interval $[\widehat{q}, \overline{q}]$ disappears. Since $\underline{q} < \widehat{q}$, the interval $\left(\underline{q}, \widehat{q}\right)$ always exists so there is always a region of non-existence in pure strategies. There is a natural mixed strategy equilibrium in that region which smoothens the transition from the equilibrium in C and the equilibrium in A.

3.5. **Mixed Strategy Equilibrium.** For all $B > 0$ and for $q \in \left(\underline{q}(B), \widehat{q}(B)\right)$ define $\left(\widetilde{S}, \widetilde{R}\right)$ as

$$\begin{aligned}
\widetilde{S} &= \arg\max \left(B \frac{\rho(S)}{\rho(S) + \rho\left(\widetilde{R}\right)} - S \right) \\
\widetilde{R} &: \pi_s\left(0, \widetilde{R}\right) = \pi_s\left(\widetilde{S}, \widetilde{R}\right)
\end{aligned}$$

Lemma 7. *For given $B > 0$ and for all $q \in \left(\underline{q}, \widehat{q}\right)$ there exists a unique $\widetilde{R}(q) \in \left(R^*, \widehat{R}\right)$ such that the best response of party s is $S \in \{0, \widetilde{S} > 0\}$. Moreover*

$$\begin{aligned}
\widetilde{R}\left(\underline{q}\right) &= R^*, \quad \widetilde{R}\left(\widehat{q}\right) = \widehat{R} \\
\frac{\partial \widetilde{R}}{\partial q} &> 0, \quad \frac{\partial \widetilde{S}}{\partial q} < 0
\end{aligned}$$

Proposition 8. *For all $B > 0$ and $q \in \left(\underline{q}(B), \widehat{q}(B)\right)$, there is an equilibrium where r plays the pure strategy $\widetilde{\widetilde{R}}$ while s plays the mixed strategy*

$$S = \begin{cases} 0 & \text{with prob.} \quad \alpha \\ \widetilde{S} & \text{with prob.} \quad 1-\alpha \end{cases}$$

with

$$\alpha = \frac{B^{-1} - \frac{\rho'(\widetilde{R})\rho(\widetilde{S})}{\left(\rho(\widetilde{R})+\rho(\widetilde{S})\right)^2}}{\frac{q\rho'(\widetilde{R})}{\left(\rho(\widetilde{R})\right)^2} - \frac{\rho'(\widetilde{R})\rho(\widetilde{S})}{\left(\rho(\widetilde{R})+\rho(\widetilde{S})\right)^2}}$$

Moreover,

$$\begin{aligned} \alpha\left(\underline{q}\right) &= 0, \quad \alpha\left(\widehat{q}\right) = 1 \\ \frac{d\alpha}{dq} &= \frac{\partial\alpha}{\partial\widetilde{R}}\frac{\partial\widetilde{R}}{\partial q} + \frac{\partial\alpha}{\partial q} > 0??? \end{aligned}$$

So gradually and continuously we go from the pure strategy symmetric equilibrium (S^*, R^*) to the pure strategy asymmetric equilibrium $\left(0, \widehat{R}\right)$.

4. Expected Turnout

The expected turnout varies depending on which region of the parameter space we are in. For any given B we have:

- For $q \in \left(0, \underline{q},\right)$

  $$\mathrm{E}(T) = \rho(R^*) = \rho\left(\gamma^{-1}\left(B^{-1}\right)\right),$$

  it is constant for any given B, increasing in B, and always above q whenever the symmetric equilibrium exists (as depicted in Figure 2).

  If $q > \rho\left(\gamma^{-1}\left(\frac{1}{B}\right)\right)$ the symmetric equilibrium disappears because the turnout would not have been high enough to meet the quorum. However, if the quorum requirement is in the interval $q \in \left(\underline{q}, \rho\left(\gamma^{-1}\left(\frac{1}{B}\right)\right)\right)$, the symmetric equilibrium disappears even if the expected turnout generated by the symmetric spending profile would have been above the quorum requirement. Since $\underline{q} < 1/2$, Proposition 1 implies that if the turnout quorum requirement is set at the $q = 50\%$ level, the symmetric spending profile cannot be an equilibrium. This is clearly due to the assumption that there are no "strong partisan" voters. In a realistic and straightforward extension of the model we would have $\rho(0) > 0$ as some voters vote even if parties are not mobilizing. In this case, the symmetric equilibrium survives even for $q = 1/2$.

- For $q \in \left(\underline{q}, \widehat{q}\right)$

$$\mathrm{E}(T) = \frac{\rho\left(\widetilde{R}\right)}{2} + (1-\alpha(q))\frac{\rho\left(\widetilde{S}\right)}{2}$$

Conjecture 9. *The expected turnout in the mixed strategy equilibrium is below the turnout in the symmetric equilibrium*

$$E(T(q)) = \frac{\rho\left(\widetilde{R}\right)}{2} + (1-\alpha(q))\frac{\rho\left(\widetilde{S}\right)}{2} < \rho(R^*) = E\left(T\left(\underline{q}\right)\right)$$

Since $\frac{\rho(R(\widehat{q}))}{2} < \rho(R^*)$, $\lim_{q\to\underline{q}} \frac{\partial E(T)}{\partial q} < 0$, *and for* B *high enough* $\overline{q} \to \widehat{q}$, *we have that* $E(T) \le \rho(R^*)$.

Conjecture 10. *The expected turnout is decreasing for* $q \in \left(\underline{q}, \widehat{q}\right)$

$$\begin{aligned} \frac{dE(T)}{dq} &= \frac{\partial \widetilde{R}}{\partial q}\frac{\partial E(T)}{\partial \widetilde{R}} + \frac{\partial E(T)}{\partial q} < 0 \\ &= \frac{1}{2}\left(\frac{\partial \widetilde{R}}{\partial q}\frac{\partial}{\partial \widetilde{R}}\left(\rho\left(\widetilde{R}\right) + (1-\alpha)\rho\left(\widetilde{S}\right)\right) - \frac{\partial \alpha}{\partial q}\rho\left(\widetilde{S}\right)\right) \end{aligned}$$

- For $q \in (\widehat{q}, \overline{q})$

$$\mathrm{E}(T) = \frac{\rho\left(\widehat{R}\right)}{2} < \frac{1}{2}$$

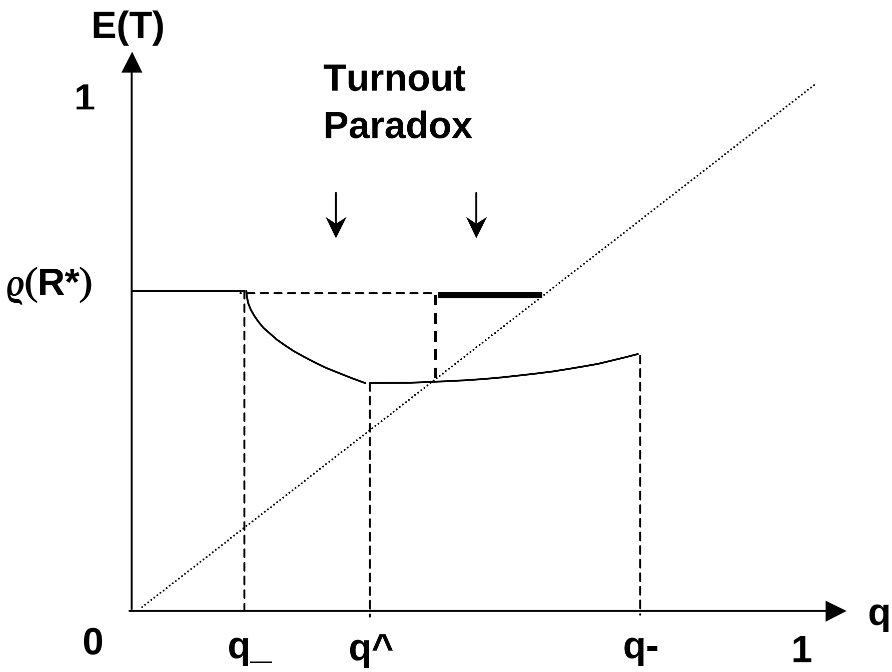
it is increasing.

$$\begin{aligned} \frac{\rho(R(\widehat{q}))}{2} &> q, \qquad \frac{\rho\left(\widehat{R}\right)}{2} < q \quad \text{for } B > \overline{B} \\ \text{where } \overline{q}\left(\overline{B}\right) &= \frac{1}{2} \end{aligned}$$

It might be the case, that the introduction of a quorum requirement motivated by the idea of validating the referendum results only if the public interest is high enough, ends up generating in equilibrium less interest for the issue. In order to make this statement more precise, we need to characterize the other two possible equilibria of the game.

- For $q \in (\overline{q}, 1)$

$$\mathrm{E}(T) = 0$$



The expected turnout is continuos for all $q \neq \overline{q}$. So

$$\lim_{q \to \underline{q}} E(T) = \rho(R^*), \quad \lim_{q \to \widehat{q}} E(T) = \frac{\rho(R(\widehat{q}))}{2}$$

Moreover,

$$E(T) > q, \qquad q \in (0, \widehat{q})$$

since in this region we know that

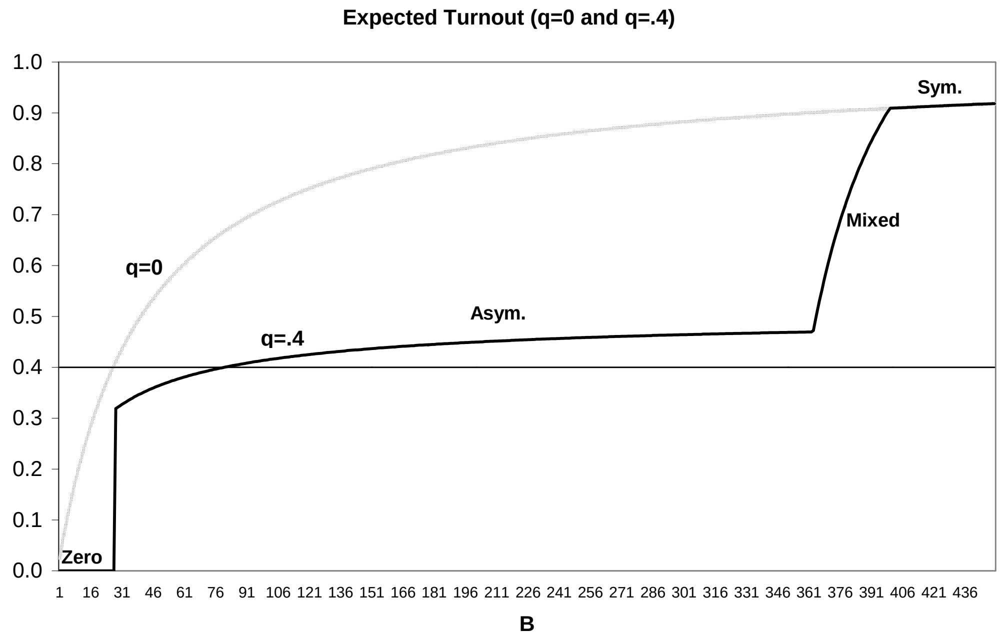
$$B \frac{q}{\rho\left(\widetilde{R}(q)\right)} = B \frac{\rho\left(S\left(\widetilde{R}(q)\right)\right)}{\rho\left(\widetilde{R}(q)\right) + \rho\left(S\left(\widetilde{R}(q)\right)\right)} - S\left(\widetilde{R}(q)\right)$$

$$R(\widehat{q}) > \widetilde{R}(q) > R^* > S\left(\widetilde{R}(q)\right)$$

it must be the case that

$$\frac{1}{2} > \frac{\rho\left(S\left(\widetilde{R}(q)\right)\right)}{\rho\left(\widetilde{R}(q)\right) + \rho\left(S\left(\widetilde{R}(q)\right)\right)} > \frac{q}{\rho\left(\widetilde{R}(q)\right)} \quad \Longrightarrow \quad \frac{\rho\left(\widetilde{R}(q)\right)}{2} > q$$

We have two versions of the turnout paradox. The weaker formulations is that, above $\underline{q}$, increasing the quorum requirement decreases the expected turnout in equilibrium. In the stronger formulation, in a certain region depicted by the boldfaced segment, the introduction of a quorum requirement has the effect of making the turnout, which without that requirement would have been above that quorum, to be instead below that quorum. The picture below shows the different turnouts that could be obtained with and without a quorum requirement for all values of the interest B.



5. Probability of Referendum Approval

The chance that the referendum passes varies depending on which region of the parameter space we are in. For any given B we have:

- For $q \in \left(0, \underline{q}\right)$
$$P(q) = 1/2$$
constant !
- For $q \in \left(\underline{q}, \widehat{q}\right)$
$$P = \alpha\left(1 - \frac{q}{\rho\left(\widetilde{R}\right)}\right) + (1-\alpha)\left(\frac{\rho\left(\widetilde{R}\right)}{\rho\left(\widetilde{R}\right) + \rho\left(\widetilde{S}\right)}\right)$$

Conjecture 11. *P is increasing for $q \in \left(\underline{q}, \widehat{q}\right)$*

$$\frac{dP}{dq} =$$

- For $q \in (\widehat{q}, \overline{q})$

$$P = 1 - \frac{q}{\rho\left(\widehat{R}\right)}$$

is decreasing as $\frac{q}{\rho\left(\widehat{R}\right)}$ is increasing in q as proved in Lemma (14). Moreover

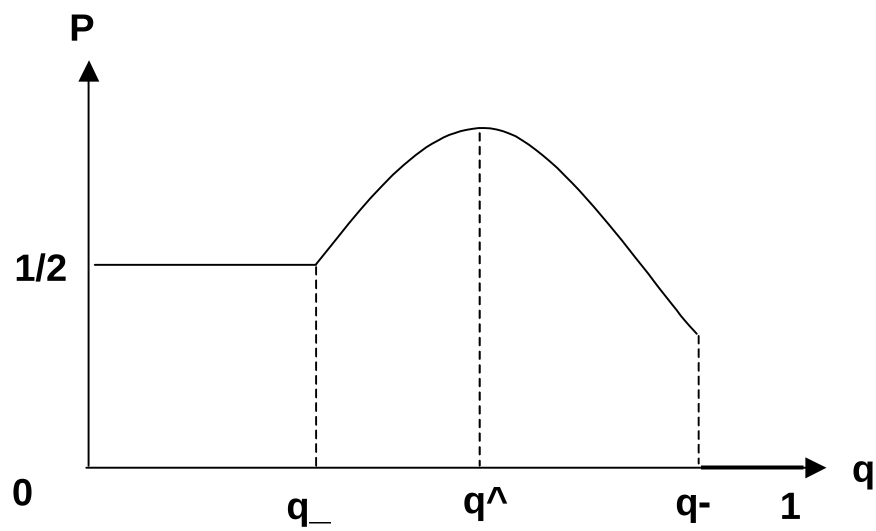
$$P(\widehat{q}) > 1/2$$

since by definition at $q = \widehat{q}$

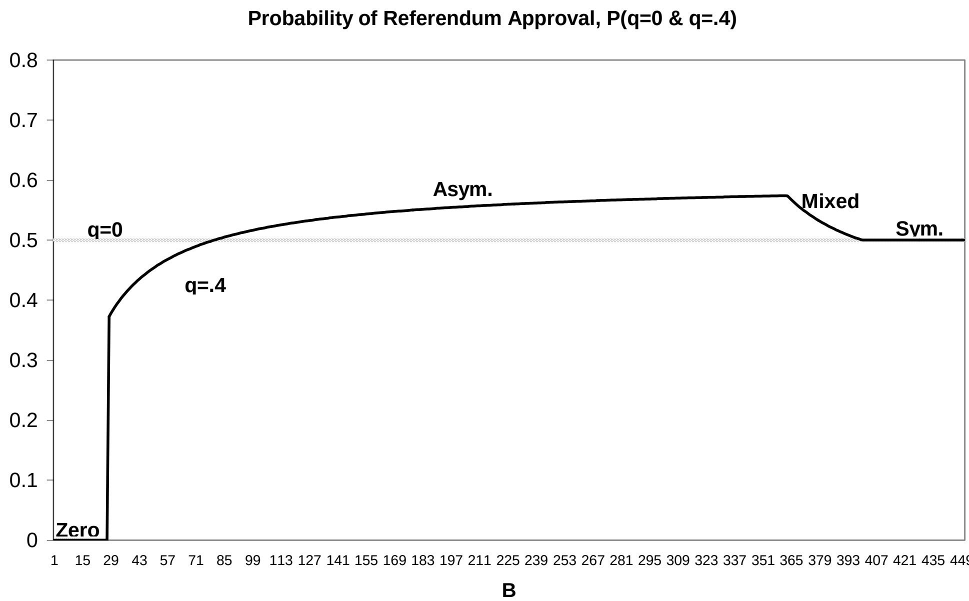
$$\pi_s\left(0, \widehat{R}\right) = \pi_s\left(\widehat{S}, \widehat{R}\right) \quad \Longrightarrow \quad 1 - P(\widehat{q}) < 1 - P\left(\widehat{S}, \widehat{R}\right) < \frac{1}{2}$$

Namely, as the profits are equal the chance of winning must be higher in the case s is spending a positive amount $\widehat{S}$.
- For $q \in (\overline{q}, 1)$

$$P = 0$$



The above picture shows how P changes as the quorum q increases for fixed interest B. The picture below shows P as a function of B with and without a quorum requirement.



6. Expected Party Profits

The expected profits of the party depend on which region of the parameter space we are in. For any given B we have:

- For $q \in \left(0, \underline{q},\right)$

$$\pi_s = \pi_r = \left(\frac{B}{2} - R^*\right)$$

  are constant in q.
- For $q \in \left(\underline{q}, \widehat{q}\right)$

$$\begin{aligned}
\mathrm{E}\left(\pi_s\right) &= B\frac{q}{\rho\left(\widetilde{R}\right)} = B\frac{\rho\left(\widetilde{S}\right)}{\rho\left(\widetilde{R}\right)+\rho\left(\widetilde{S}\right)} - \widetilde{S} \\
\mathrm{E}\left(\pi_s\right) &< \left(\frac{B}{2} - R^*\right) = \pi_s \\
\frac{d\mathrm{E}\left(\pi_s\right)}{dq} &< 0
\end{aligned}$$

since as $\widetilde{R} > R^*$, both inequalities are a consequence of revealed profitability for s.

$$\begin{aligned}
\mathrm{E}(\pi_r) &= B\left(\alpha\left(1-\frac{q}{\rho\left(\widetilde{R}\right)}\right)+(1-\alpha)\left(\frac{\rho\left(\widetilde{R}\right)}{\rho\left(\widetilde{R}\right)+\rho\left(\widetilde{S}\right)}\right)\right)-\widetilde{R} \\
\mathrm{E}(\pi_r) &> \left(\frac{B}{2}-R^*\right)=\pi_r
\end{aligned}$$

as long as $\widetilde{R} \in (R^*, 2R^*)$ since $\mathrm{E}(\pi_s) < \pi_s$ and

$$\mathrm{E}(\pi_s)+\mathrm{E}(\pi_r)=\left(B-\widetilde{R}\right)>(B-2R^*)=\pi_s+\pi_r$$

It remains to be seen whether

$$0<\frac{d\mathrm{E}(\pi_r)}{dq}=-\frac{d\widetilde{R}}{dq}-\frac{d\mathrm{E}(\pi_s)}{dq}$$

- For $q \in (\widehat{q}, \overline{q})$

$$\mathrm{E}(\pi_s)=B\frac{q}{\rho\left(\widehat{R}\right)}, \quad \mathrm{E}(\pi_r)=B\left(1-\frac{q}{\rho\left(\widehat{R}\right)}\right)-\widehat{R}$$

For $q = \widehat{q}$ the profits of s are lower than in the symmetric equilibrium as

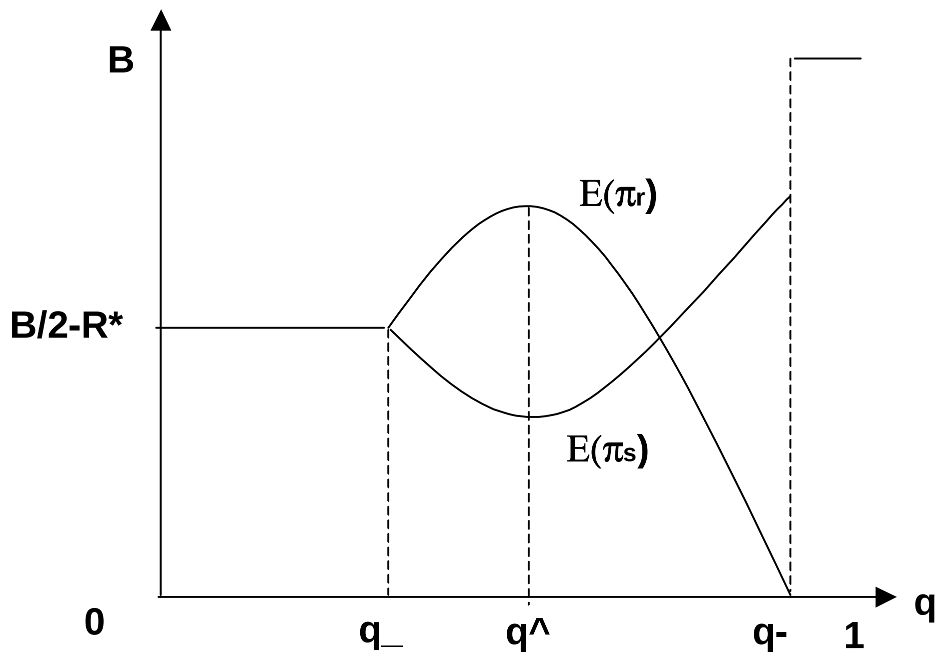
$$\mathrm{E}(\pi_s)=B\frac{q}{\rho\left(\widehat{R}\right)}=B\frac{\rho\left(\widehat{S}\right)}{\rho\left(\widehat{R}\right)+\rho\left(\widehat{S}\right)}-\widehat{S}<\left(\frac{B}{2}-R^*\right)$$

where, since as $\widehat{R} > R^*$, the last inequality is a consequence of revealed profitability for s. As proved in Lemma (14) for $q \in (\widehat{q}, \overline{q})$, increasing the quorum benefits party s and hurts party r

$$\frac{d\mathrm{E}(\pi_s)}{dq}>0, \quad \frac{d\mathrm{E}(\pi_r)}{dq}<0$$

- For $q \in (\overline{q}, 1)$

$$\pi_s = B, \quad \pi_r = 0$$



7. Normative Aspects

7.1. **Effective Quorum Target.** A voting quorum requirement is apparently there to make sure that, for a referendum to be valid, there is enough "interest" for the issue at stake. Since the "interest" is measured by the voter turnout, the voting quorum requirement should take into account that the turnout is endogenous and try to correct for that. Suppose q is the quorum target that a given sovereign country wants to enforce for its referenda. Ideally, a spending profile that is an equilibrium without the quorum requirement and that yields an expected turnout above q should remain an equilibrium even with the quorum requirement. This happens if and only if, for such spending profiles, the zero spending strategy is not a profitable deviation profitable for status quo party. Namely, the quorum busting strategy, which is always available to the status quo party, should be used only to bust the expected turnouts below q. Any referendum is characterized by an exogenous interest B for the issue at stake in the referendum. In the symmetric positive spending equilibrium this exogenous interest B determines the equilibrium spending $(S^* = R^*)$ and the expected turnout

$\mathrm{E}(T)$ as follows

$$\begin{aligned}(S^* = R^*) \quad &: \quad \frac{\rho'(R^*)}{\rho(R^*)} = 4B^{-1} \\ \mathrm{E}(T) \quad &= \quad \rho(R^*)\end{aligned}$$

For any voting quorum q, there exists a threshold value B_q below which, in the positive spending equilibrium, the expected turnout is below q. Namely, if $B < B_q$ then $\mathrm{E}(T) < q$ in the positive spending equilibrium. This threshold is defined by:

$$B_q : \mathrm{E}(T) = \rho\left(R_q^*\right) = q$$

Ideally, only in the referenda with $B < B_q$ by the status quo party should want to use its quorum busting zero spending strategy. For a given q, the zero spending strategy is the best response of the status quo party for the values of B such that

$$\left(\frac{1}{2} - R^* B^{-1}\right) \rho(R^*) \leq q$$

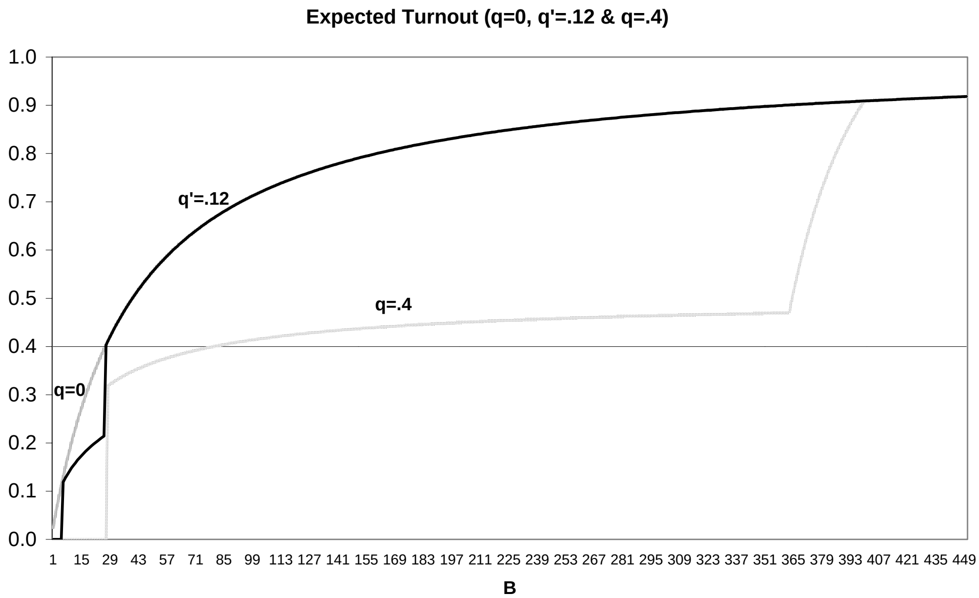
We can map any voting quorum q into an *effective* voting quorum q' as

$$q' = \left(\frac{1}{2} - R_q^* B_q^{-1}\right) q$$

A sovereign country that wants to enforce a voting quorum target of q should set as voting quorum q' instead. This policy guarantees, firstly, that the status quo party uses the quorum busting strategy in all the referenda with $B < B_q$ (which would imply $\mathrm{E}(T) < q$ in the positive spending equilibrium) and, secondly, that the positive spending equilibrium survives in all referenda with $B > B_q$ (which implies $\mathrm{E}(T) > q$). The effective quorum target q' corrects for the endogeneity of the party spending and is *less than half* of the original voting quorum q. For instance

$$\begin{aligned}\rho(R) \quad &= \quad 1 - e^{-\alpha R} \implies R^* = \frac{1}{\alpha}\ln\left(1 + \frac{\alpha B}{4}\right) \implies \mathrm{E}(T) = \frac{\alpha B}{4 + \alpha B} = q \\ &\implies B_q = \frac{4q}{\alpha(1-q)} \implies q' = \left(\frac{q}{2} + \frac{1-q}{4}\ln(1-q)\right) < \frac{q}{2}\end{aligned}$$

The following picture shows how in this example to target a quorum of $q = 0.4$, an effective quorum of $q'(0.4) = 0.12$ is needed. The effective quorum can corrects the turnout paradox in the sense that it makes the expected turnout be below a quorum of $q = 0.4$ only when the expected turnout would have been below quorum anyway.



Proposition 12. *In the asymmetric equilibrium*

$$E(T) > q \quad \Longleftrightarrow \quad P > 1/2$$

The proof is trivial as, in this equilibrium by definition

$$P = 1 - \frac{q}{\rho(R)} = 1 - \frac{1}{2}\frac{q}{\mathrm{E}(T)}$$

obviously when the status quo is not mobilizing, if the turnout is expected to reach the quorum then the referendum is more likely to pass than not, and conversely. The following picture (with: $\rho(R) = 1 - e^{-R}$ and $q = 0.4$) illustrates this fact.

7.2. **Welfare Effects of Quorum.** To develop this section the asymmetric case of $B_r \neq B_q$ should be solved first. This is also necessary to generate an equilibrium with positive spending by both parties with $q = 50\%$ or higher.

Any positive voting quorum requirement gives an advantage to the status quo. This status quo bias is a necessary feature of any quorum. However, the voting quorum avoids unnecessary rent dissipation in some cases. When the interest on the issue for the reform party B_r is low enough then the status quo party will use the zero spending strategy to bust the quorum hence avoiding unnecessary spending. When the

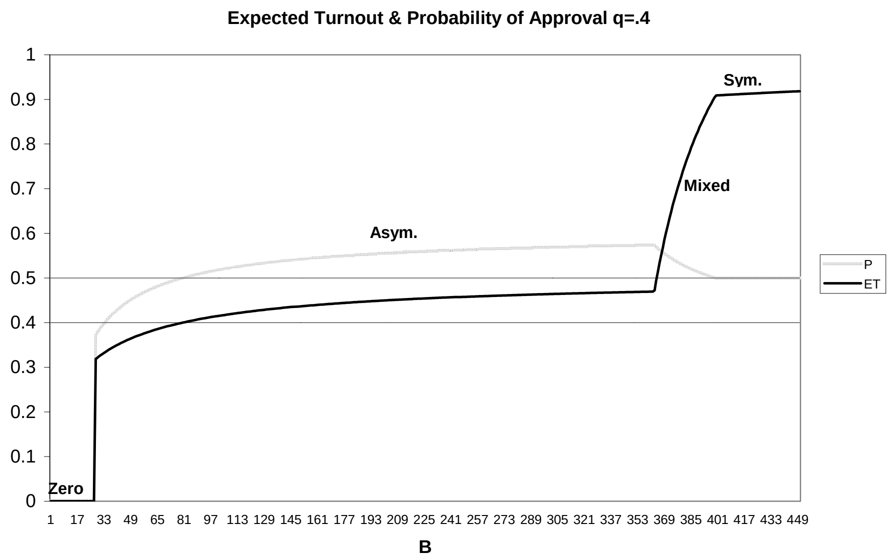


FIGURE 1

interest of the reform party B_r is high enough then the quorum busting strategy is not a profitable deviation for the status quo party and the positive spending equilibrium remains.

8. Majority Quorum

In this section we analyze the majority quorum scenario and then compare it to the voting quorum scenario.

8.1. **Probability P Regions.** Suppose the requirement is that the reform party needs a majority above some threshold $m \in \left[0, \frac{1}{2}\right]$ in order to win the referendum. Then

$$\begin{aligned} P &= \Pr\left(\left(v_R > m\right) \cap \left(v_R > v_S\right)\right) \\ &= \Pr\left(\left(\widetilde{r} > \frac{m}{\rho\left(R\right)}\right) \cap \left(\widetilde{r} > \frac{\rho\left(S\right)}{\rho\left(R\right) + \rho\left(S\right)}\right)\right) \\ &= 1 - \max\left(\min\left(1, \frac{m}{\rho\left(R\right)}\right), \frac{\rho\left(S\right)}{\rho\left(R\right) + \rho\left(S\right)}\right) \\ &= \min\left(\left(1 - W\right)^{+}, 1 - K\right) \end{aligned}$$

where we define

$$\frac{1}{M} = \frac{1}{\rho(R)} + \frac{1}{\rho(S)}, \qquad W = \frac{m}{\rho(R)}, \qquad K = \frac{\rho(S)}{\rho(R)+\rho(S)}$$

$$(1-W)^{+} = \begin{cases} \left(1-\frac{m}{\rho(R)}\right) & \text{if } \rho(R) > m \\ 0 & \text{if } \rho(R) \leq m \end{cases}$$

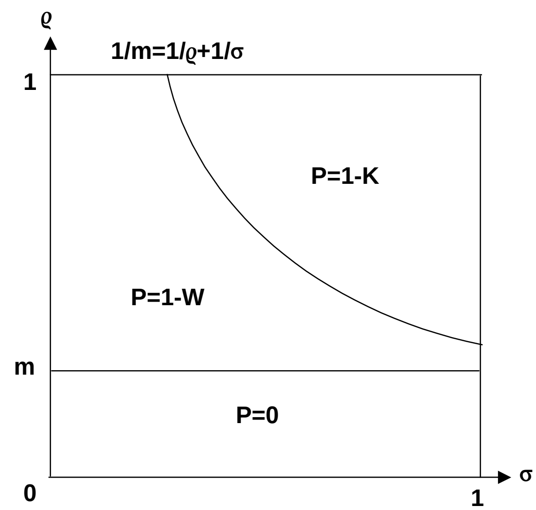
Given the implication

$$(1-W) < (1-K) \iff \frac{1}{M} > \frac{1}{m}$$

we have

$$P = \begin{cases} (1-K) & \text{if } \frac{1}{M} < \frac{1}{m} \\ (1-W)^{+} & \text{if } \frac{1}{M} \geq \frac{1}{m} \end{cases}$$

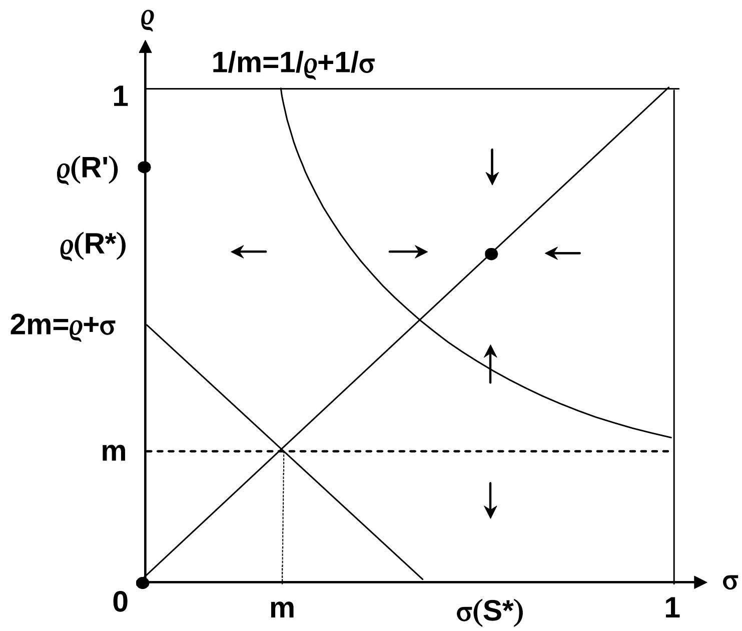
So we have three probability regions



In the region where $P = 1-K$ party r's biggest concern is having the majority of the votes. In the region where $P = 1-W$ party r's biggest concern is reaching the majority quorum: reaching it is less likely than reaching the majority of votes. In the region where $P = 0$ party r never reaches the majority quorum. Party s affects P only when $P = 1-K$. In the rest of the space P depends only on R and not on S, so s will spend not to try to mobilize voters there. Note that

$$W = Q(S=0)$$

which means that the majority quorum in the region $P = 1-W$ is identical to the voting quorum when there is zero spending on the status-quo side ($S = 0$). So, all the analysis for the region $P = 1-W$ has already been analyzed in the voting quorum case. Indeed, the probability regions are very similar and the equilibria are identical as we now show.



Proposition 13. *For any (B, m) the equilibrium in the majority quorum regime is the same as the equilibrium in the voting quorum regime with $(B, q = m)$.*

9. Summary

If the rationale for the presence of a voting quorum in a referendum is the interest of the electorate for the issue at stake, then it is flawed because the interest (measured as turnout) is endogenous. The quorum is giving an advantage to the status-quo, that can win also by losing the popular vote provided that the turnout is low enough. Indeed the status-quo supporting party is strategic and exploits often this advantage using a quorum busting strategy.

Ideally you would like to avoid the cases in which the status-quo party busts the symmetric equilibrium but the turnout in the symmetric equilibrium is above quorum. If you want to make referendums valid only if there is enough interest q, then considering that parties act strategically a voting quorum of $q/2$ or less does better than a voting quorum of q: if you put a voting quorum of $q/2$ to target an interest of q then you avoid this situation more often. The symmetric equilibrium is the same in fact. Assume $q = 50\%$. If in the symmetric equilibrium the expected turnout for each party is expected to be $(30\%, 30\%)$ say, then in the voting quorum the status-quo party has probably an incentive to break it and mobilize zero, whereas in the in the $q/2$ quorum regime not, because he is more likely to lose then. If the turnout for each party is expected to be rather say $(20\%, 20\%)$, then the status-quo party might break the equilibrium, but in that case the total turnout or real interest for the issue would have been below q anyway, so the quorum busting is what we want.

Putting a voting quorum of q has exactly the same effect in all circumstances as putting a *majority quorum* of q, which perhaps out of equilibrium looks like a much tougher condition. Yet all the equilibria are always the same and so is the advantage given to the status quo. This tells you that the voting quorum imposes a disadvantage to the reform stronger than naively suggested by out of equilibrium reasoning. Note the two regimes are stated quite differently: majority quorum states that the reform party must win the election and also have at least m percent of the potential voters in order to win; voting quorum states something that looks much fairer but much sneakier in fact: the referendum is *not valid* unless there is a turnout of at least q percent (without explicitly mentioning the fact that the status quo prevails otherwise). If parties use spending strategically (as they do), then the two quorums are exactly the same, which is quite surprising at first given the premises and the phrasing.

The voting quorum may also be there for a sheer conservative reason: to give explicit advantage to the status-quo in order to preserve it more often. In that case, the law should perhaps be less deceptive and give the status-quo an advantage directly and explicitly using the majority quorum, since after all as we show majority quorum is the same as voting quorum.

10. Appendix 1: P Regions

Proof of Proposition. 1 Define the proportion of supporters mobilized by each party as

$$\rho = \rho\left(R\right), \qquad \sigma = \rho\left(S\right)$$

Define M as

$$\frac{1}{M} = \frac{1}{\rho} + \frac{1}{\sigma}$$

then M is increasing with the mobilization effort of each party. We distinguish two broad cases depending on whether M is high enough relative to the quorum or not. If

$$M < \frac{2}{q}$$

then the mobilization efforts of the party are high enough so that M is above the quorum requirement, and conversely. This quorum threshold is represented by the continuous line in figure 1. Note that if $q = 0$, then M is never below quorum (and the curved line coincides with both axes) then as q increases the line moves up and right continuously up to $q = 1$ when mobilization is never above quorum (and the curved line reduces to point $(1, 1)$). We have four cases depending on whether $\frac{1}{M} \gtrless \frac{2}{q}$ and whether $\rho \gtrless \sigma$. As you can see in the picture, the probability is continuous also across boundaries so we omit the boundary cases, which are self-explanatory.

(1) If $\frac{1}{M} > \frac{2}{q}$ and s mobilizes more, then

$$\left(\frac{1}{M} > \frac{2}{q} \cap \rho < \sigma\right) \iff K > Q$$

$$P = \Pr\left(r < Q \cap r > K\right) = 0$$

(2) If $\frac{1}{M} > \frac{2}{q}$ and r mobilizes more, then

$$\left(\frac{1}{M} > \frac{2}{q} \cap \rho > \sigma\right) \iff K < Q$$

$$P = \Pr\left(r > Q \cap r > K\right) = \begin{cases} 0 & \text{if } Q > 1 \iff \rho < q \\ 1 - Q & \text{if } Q < 1 \iff \rho > q \end{cases}$$

(3) If $\frac{1}{M} < \frac{2}{q}$ and s mobilizes more, then

$$\left(\frac{1}{M} < \frac{2}{q}\right) \cap (\rho < \sigma) \iff K < Q$$

$$P = \Pr\left(r < Q \cap r > K\right) = \begin{cases} 1 - K & \text{if } Q > 1 \iff \rho > q \\ Q - K & \text{if } Q < 1 \iff \rho < q \end{cases}$$

(4) If $\frac{1}{M} < \frac{2}{q}$ and r mobilizes more, then

$$\left(\frac{1}{M} < \frac{2}{q}\right) \cap (\rho > \sigma) \iff K > Q$$

$$P = \Pr\left(r > Q \cap r > K\right) = 1 - K$$

Summarizing we have 4 possible values of P which identify the 4 probability regions in figure 1.

$$\begin{aligned}
P &= 0 && \Longleftrightarrow && \frac{1}{M} > \frac{2}{q} \cap ((\rho < \sigma) \cup (\rho > \sigma \cap \rho < q)) = \frac{1}{M} > \frac{2}{q} \cap \rho < q \\
P &= 1 - Q && \Longleftrightarrow && \frac{1}{M} > \frac{2}{q} \cap (\rho > \sigma \cap \rho > q) = \frac{1}{M} > \frac{2}{q} \cap \rho > q \\
P &= 1 - K && \Longleftrightarrow && \frac{1}{M} < \frac{2}{q} \cap ((\rho < \sigma \cap \rho > q) \cup (\rho > \sigma)) = \frac{1}{M} < \frac{2}{q} \cap \rho > q \\
P &= Q - K && \Longleftrightarrow && \frac{1}{M} < \frac{2}{q} \cap \rho < \sigma \cap \rho < q = \frac{1}{M} < \frac{2}{q} \cap \rho < q
\end{aligned}$$

□

11. Appendix 2: Pure Strategy Equilibria

Proof of Proposition. 2 Assume $q = 0$. For all given values of S, the profit function $\pi_r(S,R)$ is continuous for all $R \geq 0$, twice differentiable for all $R > 0$ and single peaked in R as $\frac{\partial^2 \pi_r(S,R)}{\partial R^2} < 0$, and likewise for $\pi_s(S,R)$. Hence, for any pair of values (S^*, R^*) which jointly solve the two first order conditions we have $S^* = R^*$ as

$$\begin{aligned}
\frac{\rho'(R^*)\,\rho(S^*)}{(\rho(R^*) + \rho(S^*))^2} &= B^{-1} = \frac{\rho'(S^*)\,\rho(R^*)}{(\rho(R^*) + \rho(S^*))^2} \\
&\Longrightarrow \frac{\rho'(R^*)}{\rho(R^*)} = B^{-1}\frac{(\rho(R^*) + \rho(S^*))^2}{\rho(S^*)\,\rho(R^*)} = \frac{\rho'(S^*)}{\rho(S^*)} \\
&\Longrightarrow S^* = R^*
\end{aligned}$$

Since the function

$$\gamma(R^*) \equiv \frac{\rho'(R^*)}{4\rho(R^*)} = B^{-1}$$

is decreasing and its codomain set is the positive real numbers, then the equilibrium

$$S^* = R^* = \gamma^{-1}\left(B^{-1}\right)$$

exists and is unique for any $B > 0$.

Assume $q > 0$. $\pi_r(S^*, R)$ is single peaked in the $P = 1 - K$ region, it is increasing in the $P = Q - K$ region, and non positive in the $P = 0$ region. Hence, $\pi_r(S^*, R)$ is globally single peaked at $R = R^*$. The symmetric profile $S^* = R^* = \gamma^{-1}(B^{-1})$ for $q = 0$ is an equilibrium for $q > 0$ if and only if both $S^* = R^*$ lies in the $P = 1 - K$ region and s does not deviate to zero: $\pi_s(S^*, R^*) \geq \pi_s(0, R^*)$, namely

$$(\rho(R^*) > q) \cap \left(\frac{1}{\rho(R^*)} + \frac{1}{\rho(S^*)} < \frac{2}{q}\right) \cap \left(B\frac{\rho(S^*)}{\rho(R^*) + \rho(S^*)} - S^* \geq B\frac{q}{\rho(R^*)}\right)$$

that is for

$$q \in \left[0, \underline{q}\right]$$

where

$$\begin{aligned}
\underline{q} &= \min\left(\rho\left(R^*\right), 2\frac{\rho\left(S^*\right)\rho\left(R^*\right)}{\rho\left(R^*\right)+\rho\left(S^*\right)}, \frac{\rho\left(S^*\right)\rho\left(R^*\right)}{\rho\left(R^*\right)+\rho\left(S^*\right)} - S^* B^{-1}\rho\left(R^*\right)\right) \\
&= \left(\frac{1}{2} - B^{-1}S^*\right)\rho\left(R^*\right) \\
&= \left(\frac{1}{2} - B^{-1}\gamma^{-1}\left(B^{-1}\right)\right)\rho\left(\gamma^{-1}\left(B^{-1}\right)\right) \in \left(0, \frac{1}{2}\right)
\end{aligned}$$

So the necessary and sufficient condition is that s does not deviate to zero. □

In order to prove Proposition (5), we first prove some useful Lemmata.

Lemma 14.

$$\frac{d\widehat{R}}{dq} > 0, \quad \frac{d\pi_s\left(0, \widehat{R}\right)}{dq} > 0, \quad \frac{d\pi_r\left(0, \widehat{R}\right)}{dq} < 0, \quad \frac{d\pi_s\left(\widehat{S}, \widehat{R}\right)}{dq} < 0$$

Proof of Lemma. 14 $\widehat{R}$ uniquely solves by definition

$$\frac{q}{\rho\left(\widehat{R}\right)}\frac{\rho'\left(\widehat{R}\right)}{\rho\left(\widehat{R}\right)} = B^{-1}$$

Since the RHS is constant in q while the LHS is increasing in q and decreasing in $\widehat{R}$, then $\widehat{R}$ is increasing in q, i.e. $\frac{d\widehat{R}}{dq} > 0$.

As for the profits

$$\pi_s\left(0, \widehat{R}\right) = B\frac{q}{\rho\left(\widehat{R}\right)}, \quad \pi_r\left(0, \widehat{R}\right) = B\left(1 - \frac{q}{\rho\left(\widehat{R}\right)}\right) - \widehat{R},$$

it suffices to study the behavior $q/\rho\left(\widehat{R}\right)$. Since $\widehat{R}$ is increasing in q, then $\rho'\left(\widehat{R}\right)/\rho\left(\widehat{R}\right)$ is decreasing in q since it is decreasing in $\widehat{R}$. Hence, $q/\rho\left(\widehat{R}\right)$ must be increasing in q.

The profit

$$\pi_s\left(\widehat{S}, \widehat{R}\right) = B\frac{\rho\left(\widehat{S}\right)}{\rho\left(\widehat{R}\right) + \rho\left(\widehat{S}\right)} - \widehat{S}$$

is decreasing in q as $\widehat{R}$ is increasing in q and a smaller $\widehat{R}$ results in higher profits of s by revealed profitability, namely for any $R\left(<\widehat{R}\right)$ and S, the best response to R, we have

$$\pi_s\left(\widehat{S},\widehat{R}\right)<\pi_s\left(\widehat{S},R\right)<\pi_s\left(S,R\right)$$

$\square$

Lemma 15.

$$\widetilde{q}\equiv\frac{\rho\left(R^*\right)}{4}$$

Has the properties

$$\begin{array}{rcl}\widehat{R}\left(\widetilde{q}\right) & = & R^*=S^* \\ \widetilde{q} & < & \underline{q}\end{array}$$

$$q=\widetilde{q}\implies\widehat{S}=S^*,\quad q\neq\widetilde{q}\implies\widehat{S}<S^*$$

Proof of Lemma. 15 $\widehat{R}$ and R^* uniquely solve by definition respectively

$$\begin{array}{rcl}\widehat{R} & : & q\dfrac{1}{\rho\left(\widehat{R}\right)}\dfrac{\rho'\left(\widehat{R}\right)}{\rho\left(\widehat{R}\right)}=B^{-1} \\ R^* & : & \dfrac{\rho'\left(R^*\right)}{4\rho\left(R^*\right)}=B^{-1}\end{array}$$

For $q=\widetilde{q}$, $\widehat{R}$ solves the same equation as R^*

$$\widehat{R}\left(\widetilde{q}\right)=R^*\iff\frac{\rho\left(R^*\right)}{4\rho\left(\widehat{R}\right)}\frac{\rho'\left(\widehat{R}\right)}{\rho\left(\widehat{R}\right)}=\frac{\rho'\left(R^*\right)}{4\rho\left(R^*\right)}=B^{-1}$$

Next, by definition

$$\begin{array}{rcl}\widetilde{q}<\underline{q} & \iff & \dfrac{1}{4}\rho\left(R^*\right)<\rho\left(R^*\right)\left(\dfrac{1}{2}-B^{-1}R^*\right) \\ \iff R^*<\dfrac{B}{4} & \iff & B^{-1}>\gamma\left(\dfrac{B}{4}\right)\end{array}$$

as $\gamma\left(\cdot\right)$ is monotonically decreasing the inequality sign is inverted on the last step. Since $\gamma\left(\cdot\right)=\rho'\left(\cdot\right)/4\rho\left(\cdot\right)$, we have

$$\widetilde{q}<\overline{q}\iff\Gamma\left(x\right)\equiv\frac{x\rho'\left(x\right)}{\rho\left(x\right)}<1\quad\text{where } x\equiv\frac{B}{4}>0$$

To prove that $\Gamma(x) < 1$, first note that $\Gamma(x)$ is differentiable hence continuous for $x > 0$. Second,

$$\Gamma(x) \geq 1 \quad \Longrightarrow \quad \Gamma'(x) = \left(\frac{\rho'(x)}{\rho(x)}(1 - \Gamma(x)) + \frac{x\rho''(x)}{\rho(x)}\right) < 0$$

Hence if $\Gamma(x)$ starts below 1 it can never increase to 1. So

$$\lim_{x \to 0} \Gamma(x) \leq 1 \quad \Longrightarrow \quad \Gamma(x) < 1$$

and the premise is true given that we assumed

$$\lim_{x \to 0}(x\rho'(x)) = 0, \qquad \lim_{x \to 0}(x\rho''(x)) = 0$$

as by l' Hopital rule

$$\begin{aligned} \rho(0) &= 0 \quad \Longrightarrow \quad \lim_{x \to 0} \Gamma(x) = \lim_{x \to 0} \frac{\rho' + x\rho}{\rho'} = 1 \\ \rho(0) &> 0 \quad \Longrightarrow \quad \lim_{x \to 0} \Gamma(x) = 0 \end{aligned}$$

Next, $\widehat{S}$ and S^* uniquely solve

$$\begin{aligned} \rho'(S^*) \frac{\rho(R^*)}{(\rho(R^*) + \rho(S^*))^2} &= B^{-1} \\ \rho'\left(\widehat{S}\right) \frac{\rho\left(\widehat{R}\right)}{\left(\rho\left(\widehat{R}\right) + \rho\left(\widehat{S}\right)\right)^2} &= B^{-1} \end{aligned}$$

Then

$$q < \widetilde{q} \quad \Longrightarrow \quad \widehat{R} < R^* = S^* \quad \Longrightarrow \quad \rho\left(\widehat{R}\right) < \rho(R^*)$$

As

$$R \lesseqgtr S^* \quad \Longleftrightarrow \quad \frac{\partial}{\partial R}\left(\frac{\rho(R)}{(\rho(R) + \rho(S))^2}\right) \gtreqless 0 \quad \Longleftrightarrow \quad \frac{\partial R}{\partial S} \gtreqless 0$$

of course, the above is true also for R best response to S of party r. For any $S \neq S^*$

$$S \lessgtr S^* \quad \Longleftrightarrow \quad \frac{\partial R}{\partial S} \gtrless 0 \quad \Longrightarrow \quad R < S^*$$

So, for any $\widehat{R} \neq R^*$

$$\rho'(S^*) \frac{\rho\left(\widehat{R}\right)}{\left(\rho\left(\widehat{R}\right) + \rho(S^*)\right)^2} < \rho'(S^*) \frac{\rho(R^*)}{(\rho(R^*) + \rho(S^*))^2} = B^{-1}$$

Hence

$$q \neq \widetilde{q} \quad \Longrightarrow \quad \widehat{S} < S^*$$

□

Proof of Proposition. 5 First, we show that

$$\underline{q} < \widehat{q} < \frac{1}{2}$$

and that these thresholds are properly defined. Define

$$\begin{aligned}
C(q) &= \pi_s\left(\widehat{S},\widehat{R}\right) - \pi_s\left(0,\widehat{R}\right) \\
&= \left(B\frac{\rho\left(\widehat{S}\right)}{\rho\left(\widehat{R}\right)+\rho\left(\widehat{S}\right)} - \widehat{S}\right) - \left(B\frac{q}{\rho\left(\widehat{R}\right)}\right) \\
D(q) &= \pi_s(S^*,R^*) - \pi_s(0,R^*) \\
&= \left(B\frac{\rho(S^*)}{\rho(R^*)+\rho(S^*)} - S^*\right) - \left(B\frac{q}{\rho(R^*)}\right)
\end{aligned}$$

Hence

$$\widehat{q} : C(\widehat{q}) = 0, \qquad \underline{q} : D\left(\underline{q}\right) = 0$$

Obviously $D'(q) < 0$, from Lemma (14) $C'(q) < 0$. So the thresholds $\underline{q}$ and $\widehat{q}$ are uniquely defined. From Lemma (15) $\widetilde{q} < \underline{q}$. All this implies

$$D(\widetilde{q}) = C(\widetilde{q}) > 0 \quad \text{and} \quad \widetilde{q} < \widehat{q}$$

Hence, for $q \geq \widetilde{q}$ the following implication is true

$$D'(q) < C'(q) \quad \Longrightarrow \quad \underline{q} < \widehat{q}$$

The conclusion is true if the premise is true, namely if

$$\begin{aligned}
D'(q) &< C'(q) \\
-B\frac{1}{\rho(R^*)} &< -B\frac{\rho\left(\widehat{S}\right)\rho'\left(\widehat{R}\right)}{\left(\rho\left(\widehat{R}\right)+\rho\left(\widehat{S}\right)\right)^2}\frac{d\widehat{R}}{dq} - B\frac{1}{\rho\left(\widehat{R}\right)} + \frac{d\widehat{R}}{dq} \\
\frac{1}{\rho(R^*)} &> \frac{1}{\rho\left(\widehat{R}\right)} + \frac{d\widehat{R}}{dq}\left(\frac{\rho\left(\widehat{S}\right)\rho'\left(\widehat{R}\right)}{\left(\rho\left(\widehat{R}\right)+\rho\left(\widehat{S}\right)\right)^2} - B^{-1}\right)
\end{aligned}$$

Since, for $q \geq \widetilde{q}$ we have $\widehat{R} \geq R^*$, and since $\frac{d\widehat{R}}{dq} > 0$ by Lemma (14), the premise is true if the above term in brackets is negative. By the last result of Lemma (15) we have that for all S

$$\left(\frac{\rho'(R)\rho(S)}{(\rho(R)+\rho(S))^2} = B^{-1}\right) \quad \Longrightarrow \quad R \leq R^*$$

Hence, since the LHS is decreasing in R we have and since $\widehat{R} > R^*$, we have

$$\frac{\rho\left(\widehat{S}\right)\rho'\left(\widehat{R}\right)}{\left(\rho\left(\widehat{R}\right)+\rho\left(\widehat{S}\right)\right)^2} < B^{-1}$$

Next, we show that

$$\underline{q} < \overline{q}$$

and that $\overline{q}$ is properly defined. Recall that

$$\pi_r\left(0,\widehat{R}\right) = B\left(1-\frac{q}{\rho\left(\widehat{R}\right)}\right)-\widehat{R}$$

Define

$$\begin{aligned} D(q) &= \pi_s(S^*,R^*)-\pi_s(0,R^*) \\ &= B\left(\frac{1}{2}-\frac{q}{\rho(R^*)}\right)-R^* \end{aligned}$$

Hence

$$\begin{aligned} \underline{q} &: D\left(\underline{q}\right)=0 \quad \text{with } \underline{q}>\widetilde{q} \\ \overline{q} &: \pi_r\left(0,\widehat{R}\right)=0 \end{aligned}$$

then by the envelop theorem

$$\frac{d\pi_r\left(0,\widehat{R}\right)}{dq} = \frac{\partial\pi_r\left(0,\widehat{R}\right)}{\partial q} = -\frac{B}{\rho\left(\widehat{R}\right)} < 0$$

so $\overline{q}$ is uniquely defined. Since

$$\begin{aligned} q &= \widetilde{q} \implies \widehat{R}=R^* \\ q &> \widetilde{q} \implies \widehat{R}>R^* \end{aligned}$$

then

$$0 < D(\widetilde{q}) = B\left(\frac{1}{2}-\frac{\widetilde{q}}{\rho(R^*)}\right)-R^* < B\left(1-\frac{\widetilde{q}}{\rho(R^*)}\right)-R^* = \pi_r(\widetilde{q})$$

$$\frac{dD(q)}{dq} = -\frac{B}{\rho(R^*)} < -\frac{B}{\rho\left(\widehat{R}\right)} = \frac{d\pi_r\left(0,\widehat{R}\right)}{dq} < 0 \qquad \text{for } q>\widetilde{q}$$

then $\underline{q} < \overline{q}$, as $D(q)$ is smaller and decreases faster than π_r.

Lastly, we show that

$$B \gtrless \overline{B} \iff \widehat{q} \lesseqgtr \overline{q}$$

Recall that for any given $B > 0$

$$\begin{aligned} C(q,B) &= B\left(\frac{\rho\left(\widehat{S}\right)}{\rho\left(\widehat{R}\right)+\rho\left(\widehat{S}\right)}-\frac{q}{\rho\left(\widehat{R}\right)}\right)-\widehat{S} \\ \pi_r(q,B) &= B\left(1-\frac{q}{\rho\left(\widehat{R}\right)}\right)-\widehat{R} \\ \widehat{q} &: \quad C(\widehat{q},B)=0, \qquad \overline{q}:\pi_r(\overline{q},B)=0 \end{aligned}$$

Hence

$$\pi_r(\widetilde{q},B) = B\left(1-\frac{\widetilde{q}}{\rho(R^*)}\right)-R^* > B\left(\frac{1}{2}-\frac{\widetilde{q}}{\rho(R^*)}\right)-R^* = C(\widetilde{q},B) > 0$$

and we know that

$$\frac{dC}{dq} = -B\frac{1}{\rho\left(\widehat{R}\right)}+B\frac{d\widehat{R}}{dq}\left(B^{-1}-\frac{\rho\left(\widehat{S}\right)\rho'\left(\widehat{R}\right)}{\left(\rho\left(\widehat{R}\right)+\rho\left(\widehat{S}\right)\right)^2}\right) > -\frac{B}{\rho\left(\widehat{R}\right)} = \frac{d\pi_r}{dq}$$

In sum, π_r is larger than C at $q = \widetilde{q}$, but decreases faster, hence it may be that for some $q(B)$ they C and π_r cross at some value $q(B)$, namely

$$q(B) : \pi_r(q(B), B) = C(q(B), B) \equiv \pi_r(B)$$

Then

$$\pi_r(B) \lesseqgtr 0 \quad \Longleftrightarrow \quad \widehat{q} \lesseqgtr \overline{q}$$

We are left to show that $\pi_r(B)$ is decreasing, so we want

$$\begin{aligned} \frac{d\pi_r(B)}{dB} &= 1-\frac{q(B)}{\rho\left(\widehat{R}\right)}-B\frac{q'(B)\rho\left(\widehat{R}\right)-q(B)\rho'\left(\widehat{R}\right)\frac{d\widehat{R}}{dB}}{\left(\rho\left(\widehat{R}\right)\right)^2}-\frac{d\widehat{R}}{dB} \\ &= -\frac{B}{\rho\left(\widehat{R}\right)}\frac{dq(B)}{dB}+\left(1-\frac{q(B)}{\rho\left(\widehat{R}\right)}\right) < 0 \end{aligned}$$

Hence, we want

$$\frac{dq(B)}{dB} > \frac{1-\frac{q(B)}{\rho\left(\widehat{R}\right)}}{\frac{B}{\rho\left(\widehat{R}\right)}} = B^{-1}\left(\rho\left(\widehat{R}\right)-q(B)\right)$$

where

$$\begin{aligned} q(B) &: \quad (\pi_r - C) = 0 \\ \frac{dq(B)}{dB} &= -\frac{\frac{d(\pi_r - C)}{dB}}{\frac{d(\pi_r - C)}{dq}} \end{aligned}$$

.....TO BE CONTINUED. □

Proof of Proposition. 6

As for $\underline{q}(B)$, note that

$$\frac{d\underline{q}}{dB} = \rho'(R^*)\left(\frac{\partial R^*}{\partial B}\left(\frac{1}{4} - \frac{R^*}{B}\right) + \frac{1}{4}\frac{R^*}{B}\right).$$

Since

$$\frac{\partial R^*}{\partial B} = \frac{1}{4 - B\frac{\rho''(R^*)}{\rho'(R^*)}} > 0,$$

if $\frac{R^*}{B} < \frac{1}{4}$, it follows that $\frac{d\underline{q}}{dB} > 0$. Finally, $\frac{R^*}{B} < \frac{1}{4}$ if and only if $\Gamma(x) \equiv \frac{x\rho'(x)}{\rho(x)} < 1$ for $x > 0$, which is true by the proof of Lemma (15). Since

$$\lim_{B\to 0} R^* = 0, \quad \lim_{B\to\infty} R^* = \infty, \quad \lim_{B\to\infty} \frac{R^*}{B} = \lim_{B\to\infty} \frac{\partial R^*}{\partial B} \leq \frac{1}{4},$$

it follows that

$$\lim_{B\to 0} \underline{q} = 0, \quad \lim_{B\to\infty} \frac{d\underline{q}}{dB} = 0, \quad \lim_{B\to\infty} \underline{q} \in \left[\frac{1}{4}, \frac{1}{2}\right].$$

In particular, a sufficient condition for $\lim_{B\to\infty} \underline{q} = \frac{1}{2}$ is $\lim_{x\to\infty} \frac{\rho''(x)}{\rho'(x)} = c < 0$ (this is true for example in the case of $\rho(x) = 1 - e^{-\alpha x}$, and $\alpha > 0$).

As for $\widehat{q}(B)$, first let

$$\begin{aligned} \widehat{R} &= \widehat{R}(q, B) \\ \widehat{S} &= \widehat{S}(q, B), \end{aligned}$$

and note that

$$\begin{aligned} \frac{\partial \widehat{R}}{\partial q} &= \frac{1}{q}\frac{1}{\frac{2\rho(\widehat{R})}{qB} - \frac{\rho''(\widehat{R})}{\rho'(\widehat{R})}} \in \left(0, \frac{B}{2\rho\left(\widehat{R}\right)}\right) \\ \frac{\partial \widehat{R}}{\partial B} &= \frac{1}{B}\frac{1}{\frac{2\rho(\widehat{R})}{qB} - \frac{\rho''(\widehat{R})}{\rho'(\widehat{R})}} = \frac{q}{B}\frac{\partial \widehat{R}}{\partial q} \in \left(0, \frac{q}{2\rho\left(\widehat{R}\right)}\right) \end{aligned}$$

Therefore,

$$\frac{d\widehat{q}}{dB}=\frac{\rho\left(\widehat{R}\left(\widehat{q}\right)\right)\left(\frac{\widehat{S}(\widehat{q})}{B^2}+\left(B^{-1}-\frac{\rho\left(\widehat{S}(\widehat{q})\right)\rho'\left(\widehat{R}(\widehat{q})\right)}{\left(\rho\left(\widehat{R}(\widehat{q})\right)+\rho\left(\widehat{S}(\widehat{q})\right)\right)^2}\right)\frac{\partial\widehat{R}(\widehat{q})}{\partial B}\right)}{1-\rho\left(\widehat{R}\left(\widehat{q}\right)\right)\left(B^{-1}-\frac{\rho\left(\widehat{S}(\widehat{q})\right)\rho'\left(\widehat{R}(\widehat{q})\right)}{\left(\rho\left(\widehat{R}(\widehat{q})\right)+\rho\left(\widehat{S}(\widehat{q})\right)\right)^2}\right)\frac{\partial\widehat{R}(\widehat{q})}{\partial\widehat{q}}}>0$$

$$\lim_{B\to 0}\widehat{q}=0,\qquad \lim_{B\to\infty}\widehat{q}\in\left[\lim_{B\to\infty}\underline{q},\frac{1}{2}\right],$$

where we used

$$\frac{d\widehat{R}\left(\widehat{q}\left(B\right),B\right)}{dB}=\frac{\partial\widehat{R}\left(\widehat{q}\left(B\right),B\right)}{\partial\widehat{q}}\frac{d\widehat{q}}{dB}+\frac{\partial\widehat{R}\left(\widehat{q}\left(B\right),B\right)}{\partial B}>0$$

$$\frac{\widehat{S}\left(\widehat{q}\right)}{B}\in\left(0,\frac{1}{2}\right),\qquad \frac{\rho\left(\widehat{S}\left(\widehat{q}\right)\right)}{\rho\left(\widehat{R}\left(\widehat{q}\right)\right)+\rho\left(\widehat{S}\left(\widehat{q}\right)\right)}\in\left(0,\frac{1}{2}\right)$$

$$\frac{1}{2}\geq\lim_{B\to\infty}\widehat{q}\geq\lim_{B\to\infty}\underline{q}\in\left[\frac{1}{4},\frac{1}{2}\right].$$

As for $\overline{q}\left(B\right)$,

$$\begin{aligned}\frac{d\overline{q}}{dB} &= B^{-2}\begin{pmatrix}-\left(\left(\frac{\partial\widehat{R}(\overline{q})}{\partial\overline{q}}\frac{d\overline{q}}{dB}+\frac{\partial\widehat{R}(\overline{q})}{\partial B}\right)B-\widehat{R}\left(\overline{q}\right)\right)\rho\left(\widehat{R}\left(\overline{q}\right)\right)+\\ +B^2\left(1-\frac{\widehat{R}(\overline{q})}{B}\right)\rho'\left(\widehat{R}\left(\overline{q}\right)\right)\left(\frac{\partial\widehat{R}(\overline{q})}{\partial\overline{q}}\frac{d\overline{q}}{dB}+\frac{\partial\widehat{R}(\overline{q})}{\partial B}\right)\end{pmatrix}\\ &= \frac{-\frac{\frac{\partial\widehat{R}(\overline{q})}{\partial B}}{B}\rho\left(\widehat{R}\left(\overline{q}\right)\right)+\frac{\widehat{R}(\overline{q})}{B^2}\rho\left(\widehat{R}\left(\overline{q}\right)\right)+\left(1-\frac{\widehat{R}(\overline{q})}{B}\right)\rho'\left(\widehat{R}\left(\overline{q}\right)\right)\frac{\partial\widehat{R}(\overline{q})}{\partial B}}{1+\frac{\frac{\partial\widehat{R}(\overline{q})}{\partial\overline{q}}}{B}\rho\left(\widehat{R}\left(\overline{q}\right)\right)-\left(1-\frac{\widehat{R}(\overline{q})}{B}\right)\rho'\left(\widehat{R}\left(\overline{q}\right)\right)\frac{\partial\widehat{R}(\overline{q})}{\partial\overline{q}}}\\ &= \frac{\widehat{R}\left(\overline{q}\right)}{B^2}\rho\left(\widehat{R}\left(\overline{q}\right)\right)>0\end{aligned}$$

where the last equality is obtained by substituting back the equation for $\overline{q}\left(B\right)$. Moreover,

$$\lim_{B\to 0}\overline{q}=0,\qquad \lim_{B\to\infty}\frac{d\overline{q}}{dB}=0,\qquad \lim_{B\to\infty}\overline{q}\geq\frac{1}{2},$$

where we used

$$\frac{\widehat{R}\left(\overline{q},B\right)}{B}\in\left(0,1\right)$$

and

$$\begin{aligned}\frac{d\widehat{R}(\overline{q},B)}{dB} &= \frac{\partial\widehat{R}(\overline{q},B)}{\partial\overline{q}}\frac{d\overline{q}}{dB}+\frac{\partial\widehat{R}(\overline{q},B)}{\partial B}=\frac{\partial\widehat{R}(\overline{q},B)}{\partial\overline{q}}\left(\frac{d\overline{q}}{dB}+\frac{\overline{q}}{B}\right)\\ &= \frac{1}{2-\rho\left(\widehat{R}(\overline{q},B)\right)\frac{\rho''\left(\widehat{R}(\overline{q},B)\right)}{\left(\rho'\left(\widehat{R}(\overline{q},B)\right)\right)^2}}>0\end{aligned}$$

$$\lim_{B\to 0}\widehat{R}(\overline{q}) = 0,\quad \lim_{B\to\infty}\widehat{R}(\overline{q})=\infty,\quad \lim_{B\to\infty}\frac{\widehat{R}(\overline{q})}{B}=\lim_{B\to\infty}\frac{d\widehat{R}(\overline{q})}{dB}\leq\frac{1}{2}$$

In particular, if $\lim_{x\to\infty}\frac{\rho''(x)}{(\rho'(x))^2}=-\infty$, then $\lim_{B\to\infty}\overline{q}=1$ (this is true for $\rho(x)=1-e^{-\alpha x}$, and $\alpha>0$). □

12. Appendix 3: Mixed Strategy Equilibrium

Proof of Lemma. 7 First, defining $S=S(R)$ as the best response of party s to R and recalling that

$$\begin{aligned}C(R,q) &= \pi_s(S,R)-\pi_s(0,R)\\ &= B\left(\frac{\rho(S)}{\rho(R)+\rho(S)}-\frac{q}{\rho(R)}\right)-S\end{aligned}$$

The indifference condition that defines $\widetilde{R}(q)$ is

$$\widetilde{R}(q):C(R,q)=0$$

Then $S^*=S(R^*)$ and $\widehat{S}=S\left(\widehat{R}\right)$, so by definition

$$\widetilde{R}\left(\underline{q}\right)=R^*,\quad \widetilde{R}(\widehat{q})=\widehat{R}$$

Next, trivially we have $\frac{\partial C}{\partial q}<0$, hence for $q\in\left(\underline{q},\widehat{q}\right)$ we have

$$C(R^*,q)<C\left(R^*,\underline{q}\right)=0,\qquad C\left(\widehat{R},q\right)>C\left(\widehat{R},\widehat{q}\right)=0$$

If for $q\in\left(\underline{q},\widehat{q}\right)$ it is true that

$$\frac{\partial C}{\partial R}>0\quad\text{for all }\ R\in\left[R^*,\widehat{R}\right]$$

then, for any $q\in\left(\underline{q},\widehat{q}\right)$ there exists a unique $\widetilde{R}\in\left(R^*,\widehat{R}\right)$ such that $C\left(\widetilde{R},q\right)=0$.

It is left to show that $\frac{\partial C}{\partial R}>0$ when $R\in\left[R^*,\widehat{R}\right]$. Using the fact

$$B\frac{\rho'(S(R))\,\rho(R)}{(\rho(R)+\rho(S(R)))^2}=1.$$

we have

$$\begin{aligned}\frac{\partial C}{\partial R} &= B\frac{q\rho'(R)}{(\rho(R))^2} - B\frac{\rho'(R)\rho(S(R))}{(\rho(R)+\rho(S(R)))^2} + \left(B\frac{\rho'(S(R))\rho(R)}{(\rho(R)+\rho(S(R)))^2} - 1\right)\frac{\partial S(R)}{\partial R} \\ &= B\left(\frac{q\rho'(R)}{(\rho(R))^2} - \frac{\rho'(R)\rho(S(R))}{(\rho(R)+\rho(S(R)))^2}\right)\end{aligned}$$

It is left to show that the term in brackets above is positive. Since we know that $\widehat{R}$ and R^* are such that

$$\frac{q\rho'\left(\widehat{R}\right)}{\left(\rho\left(\widehat{R}\right)\right)^2} = B^{-1}, \qquad \frac{\rho'(R^*)\rho(S(R^*))}{(\rho(R^*)+\rho(S(R^*)))^2} = B^{-1}$$

then for $R \in \left[R^*, \widehat{R}\right]$

$$\frac{q\rho'(R)}{(\rho(R))^2} > B^{-1}, \qquad \frac{\rho'(R)\rho(S(R))}{(\rho(R)+\rho(S(R)))^2} < B^{-1}$$

as the term $\left(\frac{\rho'(R)\rho(S(R))}{(\rho(R)+\rho(S(R)))^2}\right)$ is decreasing in R for $R \geq R^*$. Indeed, its derivative is negative

$$\frac{\rho''(R)\rho(S)(\rho(R)+\rho(S)) + \rho'(R)((\rho(R)-\rho(S))\rho'(S)S' - 2\rho(S)\rho'(R))}{(\rho(R)+\rho(S))^3} < 0$$

as for $R \geq R^*$ we have

$$\rho(R) > \rho(S), \qquad S' < 0$$

Finally, since $C(R,q)$ is differentiable in both arguments, then by the implicit function theorem $\widetilde{R}(q)$ is differentiable and

$$\frac{\partial \widetilde{R}}{\partial q} = -\frac{\frac{\partial C}{\partial \widetilde{R}}}{\frac{\partial C}{\partial q}} > 0$$

Since $\widetilde{S} > 0$ is the best response to $\widetilde{R} > R^*$, then by the proof of Lemma (15) $\frac{\partial \widetilde{S}}{\partial \widetilde{R}} < 0$ and therefore

$$\frac{\partial \widetilde{S}}{\partial q} = \frac{\partial \widetilde{S}}{\partial \widetilde{R}}\frac{\partial \widetilde{R}}{\partial q} < 0$$

□

Proof of Proposition. 8 By construction, $\widetilde{R}$ makes party s indifferent between the two best responses 0 and $S\left(\widetilde{R}\right)$. So s is indifferent between either strategy. We have an equilibrium if s chooses the mix $(\alpha, 1-\alpha)$

(with α on $S=0$) such that the best response of party r is $\widetilde{R}$, namely let

$$R(\alpha) \equiv \arg\max_R \left(\alpha\pi_r(0,R) + (1-\alpha)\pi_r\left(S\left(\widetilde{R}\right),R\right)\right) = \widetilde{R} \tag{2}$$

and note that it must be the case that $R(\alpha) \in (R(0), R(1))$, where

$$\begin{aligned}
R(1) &\equiv \arg\max_R \left(B\left(1-\frac{q}{\rho(R)}\right) - R\right) = \widehat{R} > \widetilde{R} \\
R(0) &\equiv \arg\max_R \left(B\left(\frac{\rho(R)}{\rho(R)+\rho\left(S\left(\widetilde{R}\right)\right)}\right) - R\right) = R'' < R^* < \widetilde{R}
\end{aligned}$$

The objective

$$\left(\alpha\pi_r(0,R) + (1-\alpha)\pi_r\left(S\left(\widetilde{R}\right),R\right)\right) =$$

$$= \left(\alpha\left(B\left(1-\frac{q}{\rho(R)}\right)-R\right) + (1-\alpha)\left(B\left(\frac{\rho(R)}{\rho(R)+\rho\left(S\left(\widetilde{R}\right)\right)}\right)-R\right)\right)$$

is concave for all α hence the FOC delivers uniquely the correct α, namely

$$\begin{aligned}
B^{-1} &= \alpha\frac{q\rho'(R(\alpha))}{(\rho(R(\alpha)))^2} + (1-\alpha)\frac{\rho'(R(\alpha))\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho(R(\alpha))+\rho\left(S\left(\widetilde{R}\right)\right)\right)^2} \\
\alpha &= \frac{B^{-1} - \frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^2}}{\frac{q\rho'\left(\widetilde{R}\right)}{\left(\rho\left(\widetilde{R}\right)\right)^2} - \frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^2}}
\end{aligned}$$

Next, we note that.

$$\alpha\left(\underline{q}\right) = \alpha\left(\widetilde{R}=R^*\right) = 0, \quad \alpha(\widehat{q}) = \alpha\left(\widetilde{R}=\widehat{R}(\widehat{q})\right) = 1$$

Finally,

$$\frac{d\alpha}{dq} = \frac{\partial\alpha}{\partial\widetilde{R}}\frac{\partial\widetilde{R}}{\partial q} + \frac{\partial\alpha}{\partial q} > 0$$

We know by the proof of Lemma (7) that $\frac{\partial\widetilde{R}}{\partial q} > 0$. We can show that

$$\frac{\partial\alpha}{\partial\widetilde{R}} > 0$$

Since α is defined by the equation

$$T\left(\alpha,\widetilde{R}\right)\equiv\alpha\frac{q\rho'\left(\widetilde{R}\right)}{\left(\rho\left(\widetilde{R}\right)\right)^{2}}+(1-\alpha)\frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^{2}}-B^{-1}=0$$

by the implicit function theorem

$$\begin{aligned}\frac{\partial\alpha}{\partial\widetilde{R}} &= -\frac{\partial T\left(\alpha,\widetilde{R}\right)}{\partial\widetilde{R}}/\frac{\partial T\left(\alpha,\widetilde{R}\right)}{\partial\alpha}\\ &= \frac{-\frac{\partial}{\partial\widetilde{R}}\left(\alpha\frac{q\rho'\left(\widetilde{R}\right)}{\left(\rho\left(\widetilde{R}\right)\right)^{2}}+(1-\alpha)\frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^{2}}\right)}{\frac{q\rho'\left(\widetilde{R}\right)}{\left(\rho\left(\widetilde{R}\right)\right)^{2}}-\frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^{2}}}\end{aligned}$$

Since, at the numerator $\alpha\frac{q\rho'\left(\widetilde{R}\right)}{\left(\rho\left(\widetilde{R}\right)\right)^{2}}$ and $(1-\alpha)\frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^{2}}$ are both decreasing in $\widetilde{R}$, the sign of $\frac{\partial\alpha}{\partial\widetilde{R}}$ is equal to the sign of the denominator

$$\frac{q\rho'\left(\widetilde{R}\right)}{\left(\rho\left(\widetilde{R}\right)\right)^{2}}-\frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^{2}}=\frac{1}{\alpha}\left(B^{-1}-\frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^{2}}\right).$$

Since $\frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^{2}}$ is decreasing in $\widetilde{R}$, and $\widetilde{R}>R''$ (i.e. the best response of r to $S\left(\widetilde{R}\right)$) then

$$\frac{1}{\alpha}\left(B^{-1}-\frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^{2}}\right)>0$$

However,

$$\frac{\partial\alpha}{\partial q}=-\frac{\frac{\rho'\left(\widetilde{R}\right)}{\left(\rho\left(\widetilde{R}\right)\right)^{2}}\left(B^{-1}-\frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^{2}}\right)}{\left(\frac{q\rho'\left(\widetilde{R}\right)}{\left(\rho\left(\widetilde{R}\right)\right)^{2}}-\frac{\rho'\left(\widetilde{R}\right)\rho\left(S\left(\widetilde{R}\right)\right)}{\left(\rho\left(\widetilde{R}\right)+\rho\left(S\left(\widetilde{R}\right)\right)\right)^{2}}\right)^{2}}<0$$

TO BE CONTINUED..... □

13. Appendix 4: Expected Turnout

We know that

$$\lim_{q\to\underline{q}}\frac{\partial E\left(T\right)}{\partial q}<0$$

As

$$\lim_{q\to\underline{q}} \frac{\partial E(T)}{\partial q} = \frac{1}{2}\left(\rho'(R^*) - \lim_{q\to\underline{q}} \frac{\partial\alpha}{\partial\widetilde{R}}\rho(R^*)\right)\lim_{q\to\underline{q}}\frac{\partial\widetilde{R}}{\partial q} =$$
$$\frac{1}{2}\left(\rho'(R^*) - \frac{4\rho'(R^*) - B\rho''(R^*)}{16\underline{q} - B\rho'(R^*)}\rho(R^*)\right)\lim_{q\to\underline{q}}\frac{\partial\widetilde{R}}{\partial q} <$$
$$\frac{1}{2}\left(\rho'(R^*) - \frac{4\rho'(R^*)}{16\underline{q} - B\rho'(R^*)}\rho(R^*)\right)\lim_{q\to\underline{q}}\frac{\partial\widetilde{R}}{\partial q} =$$
$$\frac{\rho'(R^*)}{2}\left(1 - \frac{4\rho(R^*)}{16\rho(R^*)\left(\frac{1}{2} - \frac{R^*}{B}\right) - B\rho'(R^*)}\right)\lim_{q\to\underline{q}}\frac{\partial\widetilde{R}}{\partial q} < 0$$

and

$$\begin{aligned} 16\rho(R^*)\left(\frac{1}{2} - \frac{R^*}{B}\right) - B\rho'(R^*) - 4\rho(R^*) &< 4\rho(R^*) - B\rho'(R^*) = 0 \\ \lim_{q\to\underline{q}}\frac{\partial\alpha}{\partial\widetilde{R}} &= \frac{4\rho'(R^*) - B\rho''(R^*)}{16\underline{q} - B\rho'(R^*)} > 0 \end{aligned}$$

We know that

$$\lim_{q\to\widehat{q}} E(T(q)) = \frac{\rho(R(\widehat{q}))}{2} < \rho(R^*) = E\left(T\left(\underline{q}\right)\right)$$

as

$$\frac{\rho(R(\widehat{q}))}{2} = \frac{\rho'(R(\widehat{q}))\widehat{q}B}{2\rho(R(\widehat{q}))} < \frac{\rho'(R^*)B}{4} = \rho(R^*)$$

14. Appendix 5: Majority Quorum

Proof of Proposition . 13 First, the no quorum symmetric profile (S^*, R^*) is an equilibrium if and only if m is such that

$$(\rho(R^*) > m) \cap \left(\frac{1}{\rho(R^*)} + \frac{1}{\rho(S^*)} < \frac{1}{m}\right) \cap \left(B\frac{\rho(S^*)}{\rho(R^*) + \rho(S^*)} - S^* \geq B\frac{m}{\rho(R^*)}\right)$$

that is for

$$m \in [0, \underline{m}]$$

where

$$\begin{aligned} \underline{m} &= \min\left(\rho(R^*), \frac{\rho(S^*)\rho(R^*)}{\rho(R^*) + \rho(S^*)}, \frac{\rho(S^*)\rho(R^*)}{\rho(R^*) + \rho(S^*)} - S^*B^{-1}\rho(R^*)\right) \\ &= \left(\frac{1}{2} - S^*B^{-1}\right)\rho(R^*) \\ &= \underline{q} \end{aligned}$$

So the necessary and sufficient condition is again that s does not deviate to zero and if $m = q$ the symmetric equilibrium existence conditions are the same in the majority of voting quorum regimes.

Second, the value that satisfies the FOC for r in there region $P = 1 - W$

$$\widehat{R}(m) : m\frac{\rho'\left(\widehat{R}\right)}{\rho^2\left(\widehat{R}\right)} = B^{-1}$$

So $\widehat{R}(m)$ is the best response to $S = 0$ as long as it gives a non-negative payoff

$$B\left(1 - \frac{m}{\rho\left(\widehat{R}(m)\right)}\right) - \widehat{R}(m) \geq 0$$

in which case the best response is $R = 0$, by which for any $m > 0$ party r loses for sure at no cost. It is easy to see by just substituting q with m, that for any B all the boundaries for existence of the equilibria in the voting quorum case coincide with the boundaries for the majority quorum case, moreover the equilibria are the same. □

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