Upstating time non-separable Discount factors for uncertain cash flows

Preliminary and incomplete version 21-03-06 (please quote!)

Robert Kast, CNRS, LAMETA, IDEP, kast@ensam.inra.fr

André Lapied, Université Paul Cézanne, GREQAM, IDEP, lapied@univ.u-3mrs.fr

Abstract:

The net present expected value of a cash flow is extended to non-additive integrals and discounting, in order to represent preferences of a decision maker averse to uncertainty as well as to time variability. The future is formalised as the product of the space of uncertain states and of the space of dates. Consistence of valuation with the arrivals of information is explicitly defined and assumed. In a first step the updating rule for capacities obtained by Lapied and Kast, 2006 is presented according to the usual hierarchy between uncertainty and time (discounted Choquet expectations). Then, the inversion of the hierarchy is questioned to address particular management situations. Finally, "upstating" rules for time non-separable discount factors are derived (expected non-separable discounting). Both updating and upstating rules are dynamically consistent, but are shown to violate consequentialism.

Key words: Capacities, Choquet integral, Consequentialism, Discount factors, Dynamic.
JEL classification numbers: D 81, D 83, D 92, G 31.

Upstating time non-separable Discount factors for uncertain cash flows

Preliminary and incomplete version 21-03-06 (please quote!)

Introduction

Conditioning expectations about a future cash flow makes sense in a dynamic setting: At each information arrival, the future cash flow is associated to a present certainty equivalent representing the agent's preferences given the information. The dynamic is generated by information arrivals that make preferences change. Preferences bear upon payoffs that are uncertainty contingent and also time contingent. In this paper, preferences are represented by integrals in which the agent's behaviour is grasped by a measure on uncertainty and a measure on time. In the line of Yaari, 1987, in the case of risk, Chateauneuf's, 1991, model where preferences are represented by capacities is invoked here, convex capacities grasp uncertainty aversion. Similarly, discount factors define a measure on time representing such preferences. Following Gilboa, 1989, Shalev 1997, Chateauneuf and Rebillé, 2003, they may be time non-separable and convexity of the measure on time formalises aversion to time variability of payoffs.

The usual way to deal with "expectations" about the future is to mimic the accountants' method for discounting past cash flows (therefore we'll call "discounting" the expectation about future time and reserve "expectations" for uncertainty). This consists to summarise the cash flow by a linear integral with respect to a measure of (past) time defined by discount factors. In order to apply it to future cash flows in the case they are random (in fact measurable real functions on the set representing uncertainty) the usual way consists of imposing a hierarchy between the two sets:

- First, expectations are computed at each time with respect to some measure on the uncertainty space that represent the relative importance of one state over the other: Expectations yield certainty cash equivalents.
- Then these expectations are discounted by factors that represent preferences for cash availability at one time over another one (preference for present consumption): Discounting yields a present equivalent of the expectations' cash flow.

This hierarchy is somewhat arbitrary. We argue in this paper that, alternatively, an agent may consider trajectories one by one, on which time can be discounted, and then take the expectation of the different discounted payoffs on such trajectories to obtain a certain present value. It so happen that if both expectation and discounting are computed by linear integrals, and that discount factors are independent of the trajectories, both valuations are equal.

However, if the integrals are not linear, this result doesn't hold anymore. Simple counter examples will show why. One hierarchy may be favoured over the other depending on which one out of aversion to payoff uncertainty or aversion time variability of certainty payoffs is given dominance.

Most of the literature on the subject has concentrated on uncertainty only to represent the future, in this paper instead, time is explicitly introduced and the future is the product of two measurable spaces. After having presented results using the usual hierarchy, we address the problem of favouring the hierarchy: Time and then Uncertainty.

The first section of this paper is devoted to the model and notations used in the remaining of the paper. In the second section, we summarise results of Lapied and Kast, 2006. The third section questions the inversion of the usual hierarchy referred to in the previous section and illustrates the discussion with an example of the management of solvability constraints by an insurance company. Section 4 presents the main results of this paper on the "uspstating" of discount factors in a model where discounting is computed according to a non-separable integral but expectations are additive. In the last section we explain why in non-additive models, both the updating rules and the upstating rules are dynamically consistent by construction: they violate consequentialism.

1. The model

The following notations will be kept thorough the paper:

Uncertainty is described by a finite set $S = \{1,\dots , S\}$. Time is a finite set $T = \{1, \dots , T\}$.

S is endowed with a filtration indexed by time: $F = \{F_1, \dots , F_T\}$, with $F_1 \subset \dots \subset F_T = 2^S$. T is endowed with the algebra of its parts: 2^T.

A cash flow is a positive real function:

$X: R^{SxT} \rightarrow R_+$, and $X = [X(1,1), \dots , X(s,t), \dots X(S,T)]$.

In the following, a cash flow is often considered as a stochastic process adapted to F:

$X = (X_1, \dots , X_T)$, with $X_t = X(.,t)$, F_t-measurable, for $t = 1, \dots , T$.

A cash flow can also be considered as a list of trajectories indexed by states in S: A trajectory is a measurable real function:

$\forall s \in S, X(s) = [X(s, 1), \dots , X(s, T)]$ with $X(s): (T, 2^T) \rightarrow R_+$.

Without loss of generality, let us add to any cash flow X an "initial" payoff such that: $\forall s \in S$, $\forall t \in T$ $X(0, 0) = 0 \leq X(s, t)$.

Information is yielded by F_t-measurable random variables Y_t indexed by some dates t in T.

Preferences of the agent on cash flows are decomposed into two preferences:

- Preferences on S-contingent payoffs will be represented by an expectation E with respect to a measure m if it is probability distribution, n if it is a capacity, m or $n: 2^T \rightarrow [0, 1]$ with $m(A \cup B) + m(A \cap B) = m(A) + m(B)$, but only $n(A) \leq n(B)$ if $A \subset B$. And let's note:

$\forall s \in S,\ \forall t \in T,\ Dn(s,t) = n\{s \in S / X(s,t) \geq X(s,t)\} - n\{s = 0, 1, \ldots, S / X(s,t) > X(s,t)\}$,

which collapses to $D\,m(s,t) = m(\{s\}) = m(s)$ for a probability.

With these notations, we can define the expectations of uncertain payoffs (certainty equivalents):

$$\forall t \in T,\ E(X_t) = \sum_{s=0}^{S} X_t(s)\, Dn(s,t) \qquad (1),$$

$$\text{or } E(X_t) = \sum_{s=0}^{S} X_t(s)\, m(s) \qquad (1').$$

- Preferences on T-contingent payoffs will be represented by a discounting D, i.e. an integral on trajectories with respect to discount factors p if they are time separable and r if they are not: p or $r: 2^T \rightarrow R_+$, and let's note:

$\forall s \in S,\ \forall t \in T,\ Dr(s,t) = r\{t \in T / X(s,t) \geq X(s,t)\} - r\{t = 0, 1, \ldots, T / X(s,t) > X(s,t)\}$,

which collapses to $D\,p(s,t) = p(\{t\}) = p(t)$ for a separable discounting.

We can define the discounting of payoffs trajectories (present equivalents):

$$\forall s \in S,\ D[X(s,.)] = \sum_{t=0}^{T} X(s,t)\, Dr(s,t) \qquad (2),$$

$$\text{or } D(X(s,.)] = \sum_{t=0}^{T} X(s,t)\, p(t) \qquad (2').$$

At this stage, the usual way to value a cash flow X is to impose a hierarchy between uncertainty and time: First, uncertainty is integrated at each date (Expectations of the X_t's) and then the trajectory of the certainty equivalents is integrated (present equivalent):

- First, Expectation (certainty equivalent) at each date are obtained by formula (1).
- Then, Discounting (present equivalent) on the trajectory of certainty equivalents are computed according to formula (2).
- The Discounted Expected cash flow (present certainty equivalent) is given by:

$$DE(X) = \sum_{t=0}^{T} \Big[\sum_{s=0}^{S} X(s,t)\, Dn(s,t)\Big]\ Dr(s,t) \qquad (3).$$

However, there doesn't seem to be any reason why not favour the revert hierarchy: First, time is integrated along each trajectory (present equivalents) and then the random present equivalents are integrated with respect to capacity ν, which amounts to:

- First, Discounting each trajectories (present equivalents) are computed by formula (2).
- Then, Expectation (certainty equivalent) of the present equivalents are obtained with formula (1).
- The Expected Discounted cash flow (certainty present equivalent) is given by:

$$ED(X) = \sum_{s=0}^{S} \left[\sum_{t=0}^{T} X(s,t) \; D\rho(s,t)\right] \; D\nu(s,t) \qquad (4).$$

Note that, in the usual net present expected value, the two hierarchies yield the same result, because of additivity:

$$DE(X) = \sum_{t=0}^{T} \left[\sum_{s=0}^{S} X(s,t)\, m(s)\right] \; \rho(t) = \sum_{s=0}^{S} \left[\sum_{t=0}^{T} X(s,t) \; \rho(t)\right] \; m(s) = ED(X) \quad (5).$$

2. **Discounting Choquet Expected cash flows**

In this section we summarise Lapied and Kast's, 2006, results where the usual hierarchy: First, Uncertainty, then Time, is favoured in the valuation of a cash flow. Furthermore, discounting is assumed to be time separable (Koopman's, 1965 axioms for preferences for present consumption, here cash value) with respect to discount factors ρ. Expectations correspond to Chateauneuf's, 1991, axioms yielding a subjective Choquet integral with respect to a capacity ν. This is the Discounted Choquet Expected process payoffs valuation, a direct extension of the traditional Discounted Expected process payoffs with classical linear expectations and time separable discounting (formula 5). Valuation (present certain payoff equivalent) of a cash flow X is then obtained as:

$$DE(X) = D[(E(X(.,1),\dots, E(.,T))] = \sum_{t=0}^{T} \left[\sum_{s=0}^{S} X_t(s)\, D\nu(s,t)\right] \; \rho(t) \qquad (3') .$$

Information is released by a process $(Y_t)_{t=2\dots T-1}$ with values is some set I that we'll assume finite. At date t, information is $[Y_t=i] \in \mathcal{F}_t$.

In order to integrate dynamics and information arrivals, two more axioms on preferences are added: Model consistency and Dynamic consistency (Sarin and Wakker, 1998, express them in terms of preferences in a static framework).

Model Consistency: for any date t and information $[Y_t = i]$, conditional preferences satisfy the same axioms (here, Koopman's + Chateauneuf's) and expectations and discounting are computed according to the same models as before.

Here, $E^{[Y_t=i]}$ is the conditional expectation with respect to a conditional capacity $n^{[Y_t=i]}$ (that will be defined implicitly). Furthermore, the conditional discounting is assumed to be independent of i: D^t. The conditional discounted expectation of X at date t when an information at date t is observed, $I_t = i$ is given by:

$$D^t\, E^{[Y_t=i]}\,(X) = \sum_{t=t+1}^{T} \Big[\sum_{s=0}^{S} X_t(s)\, Dn^{[Y_t=i]}(s,t) \Big]\ p^{t}(t).$$

In Lapied and Kast (2006), Dynamic consistency is derived from the axiom that Kreps and Porteus (1978) called Time consistency in an optimisation problem based on additive expectations. We shall keep this formalisation in the following of the paper.

Dynamic consistency:

$$\forall\, t \in T,\ DE(X) = DE[(X_1, \ldots, X_{t-1}, X_t + D^t E^{[Y_t=i]}(X), 0, \ldots, 0)].$$

Applying this formula to a particular cash payoff: $X = (0, \ldots, X_T)$, an implicit definition of a conditional Choquet integral is obtained:

$$DE(X_T)=DE[D^t E^{[Y_t=I]}(X_T)],$$

or, because the separable discounting on both sides of the equation simplifies aways:

$$E(X_T)=E[E^{[Y_t=I]}(X_T)]$$ [1].

From this implicit definition of conditional Choquet integrals, an updating rule for capacities is derived from the definition of conditional capacities: $n(A/B)= E^{[1_B=1]}(1_A)$, where 1_A is the payoff (1 if state is in A, 0 otherwise) and 1_B is the information function (B if $1_B = 1$, B^c otherwise). The rule can be explicitly computed in two cases:

- If information is comonotonic with payoffs, i.e. $A \subseteq B$ (or $B^c \subseteq A$), Bayes' rule prevails: $n\,(A/B) = \dfrac{n(A \cap B)}{n(B)}$.

[1] Notice that this is the classical implicit definition of conditional Lebesgue expectations, in the case where the capacity is a probability. To be more precise, the formula considers all integrals truncated on subsets of S.

- If information is antimonotonic with payoffs (-1_A is comonotonic with 1_B), i.e. $B^C \subseteq A$ (or $A^c \subseteq B$), Dempster-Shafer's rule prevails: $n(A/B) = \frac{n(A \cup B^C) - n(B^C)}{1 - n(B^C)}$.

Because comonotonicity concerns payoffs that cannot be available after information is revealed, this two-sided updated rule, and hence the model, violates consequentialism[2]. It is known (Sarin and Wakker, 1998) that, otherwise, only the Gilboa and Schmeidler's, 1989, multi-priors model could be consistent with dynamic consistency, model consistency and non-additive expectations.

3. Questioning the inversion of the hierarchy between discounting time and taking uncertain payoffs expectations.

At first glance, it may seem that the sets S and T play symmetrical roles in the description of the future , however, what they represent isn't symmetrical: in S, a state s OR a state s' may obtain, while in T, a date t AND a date t' will be reached. This logical difference mixed with the mathematical similarity of the representations make the reasoning confusing and the apparent symmetry is misleading. For instance, as we shall see through the following example, the two hierarchies between the integration of time and of uncertainty considered in section 1, do not yield the same valuations of future cash flows. This is in contrast with the classical Net Present Expected Value, where formulas (3) and (4) collapse into (5) because of additivity of both measures.

In the following example, discounting is computed according to a non-negative non-decreasing measure on $(T,2^T)$, expectations are done according to a Choquet integral and the two hierarchies between Time and Uncertainty integrations are considered.

- Hierarchy 1: Uncertainty then Time

Formula (1) then formula (2) are combined to yield the present certainty equivalent of cash flow X according to formula (3):

$$DE(X) = \sum_{t=0}^{T} [\sum_{s=0}^{S} X(s,t)\, Dn(s,t)]\ \ Dr(s,t) \qquad (3).$$

- Hierarchy 2: Time then Uncertainty

[2] A counter example is given in Lapied and Kast, 2006.

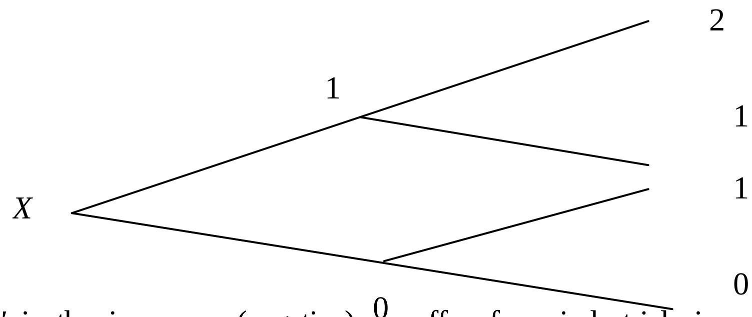
Formula (2) then formula (1) are combined to yield the certainty equivalent of present payoffs of cash flow X according to formula (4):

$$ED(X) = \sum_{s=0}^{S} \left[\sum_{t=0}^{T} X(s,t) \;\; Dr(s,t) \right] \;\; Dn(s,t) \qquad (4).$$

Consider two risks, X and X' as representing the (negative) payoffs of an insurer over two years. X corresponds to the cash flow of a credit risk, i.e the reimbursements (negative payoffs) of some insurance on default over two periods. When there are no (or few) defaults in the first period, it is a signal that the business cycle is high and is likely to increase again or begin to decrease in the second period. But, if there are many defaults in the first period, the decreasing business cycle has already began and it may rise up some or become worse in the second period. For instance, let X be the cash flow on $S = \{s_1, s_2, s_3, s_4\}$ and $T= \{1, 2\}$ with X's payoffs:

$X_1(\{s_1\}) = X_1(\{s_2\}) = 0, X_1(\{s_3\}) = X_1(\{s_4\}) = 1,$

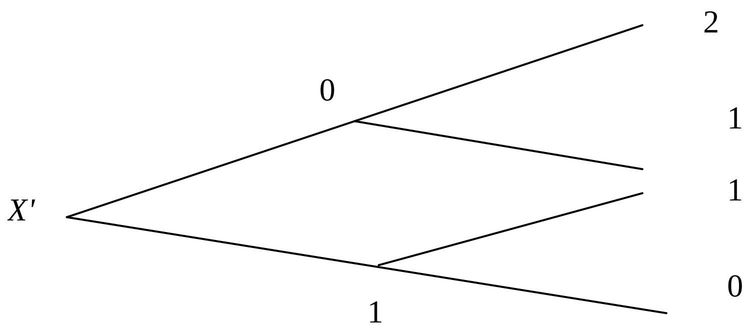
$X_2(\{s_1\}) = 0, X_2(\{s_2\}) = X_2(\{s_3\}) = 1, X_2(\{s_4\}) = 2.$



X' is the insurance (negative) payoffs of an industrial risk insurance. In this case, a difficulty occurring during the first period, when the production is not very large yet, yields an information and is assumed to develop prevention measures able to reduce the risk in the second period, when production becomes important and the risk would induce higher casualties. For instance, let X's payoffs be such that:

$X'_1(\{s_1\}) = X'_1(\{s_2\}) = 1, X'_1(\{s_3\}) = X'_1(\{s_4\}) = 0,$

$X'_2(\{s_1\}) = 0, X'_2(\{s_2\}) = X'_2(\{s_3\}) = 1, X'_2(\{s_4\}) = 2.$



Suppose that non-separable discount factors r and capacity n are such that:

$r(1, 2) = 1.8$, $r(1) = 0.9$, $r(2) = 0.8$,[3]

$n(\{s_1, s_2, s_3, s_4\}) = 1$, $n(\{s_1, s_2, s_4\}) = 0.9$, $n(\{s_2, s_3, s_4\}) = 0.6$,

$n(\{s_1, s_2\}) = n(\{s_2, s_4\}) = n(\{s_3, s_4\}) = 0.5$, $n(\{s_2\}) = 0.1$, $n(\{s_4\}) = 0.3$.[4]

Then, following the second hierarchy:

$D[X(s_1)] = 0$, $D[X(s_2)] = r(2)$, $D[X(s_3)] = r(1, 2)$, $D[X(s_4)] = r(1, 2) + r(2)$,

$ED(X) = r(2)\, n(\{s_2, s_3, s_4\}) + [r(1, 2) - r(2)]\, n(\{s_3, s_4\}) + r(2)\, n(\{s_4\}) = 1.22$,

$D[X'(s_1)] = r(1)$, $D[X'(s_2)] = r(1, 2)$, $D[X'(s_3)] = r(2)$, $D[X'(s_4)] = 2\, r(2)$,

$ED(X') = r(2) + [r(1) - r(2)]\, n(\{s_1, s_2, s_4\}) + [2\, r(2) - r(1)]\, n(\{s_2, s_4\})$

$\quad + [r(1, 2) - 2r(2)]\, n(\{s_2\}) = 1.26$.

According to the first hierarchy:

$E(X_1) = n(\{s_3, s_4\})$, $E(X_2) = n(\{s_2, s_3, s_4\}) + n(\{s_4\})$,

$DE(X) = r(1, 2)\, n(\{s_3, s_4\}) + r(2)\, [n(\{s_2, s_3, s_4\}) + n(\{s_4\}) - n(\{s_3, s_4\})] = 1.22$,

$E(X'_1) = n(\{s_1, s_2\})$, $E(X'_2) = n(\{s_2, s_3, s_4\}) + n(\{s_4\})$,

$DE(X') = r(1, 2)\, n(\{s_1, s_2\}) + r(2)\, [n(\{s_2, s_3, s_4\}) + n(\{s_4\}) - n(\{s_1, s_2\})] = 1.22$.

The inequality $ED(X') > ED(X)$ shows that X' hedges better than X against payoffs variations in time, given the decision maker is adverse to these variations. The insurance premium will be less for X' than for X. The criterion ED cannot capture this effect because $ED(X) = ED(X')$.
This example induces us to consider the second hierarchy in next section.

4. Non-separable discounting and additive expectation of a cash flow

---

[3] This corresponds to preference for present because $r(1) > r(2)$ and variation-aversion because $r(1, 2) > r(1) + r(2)$ (convexity, or super-modularity).

[4] This corresponds to risk-aversion (convexity).

In this section, we consider the special case where the measure on uncertainty is a (subjective) probability m and the hierarchy is the second one: First time is integrated with respect to non-separable discount factors on each trajectory, then uncertainty is integrated with respect to m.
The present value of cash flow X is then given by its Expected Discounted process payoffs:

$$ED(X) = \sum_{s=0}^{S} \left[\sum_{t=0}^{T} X(s,t) \;\; D\rho(s,t)\right] \; m(s) \qquad (4').$$

Let information be given by Y_t, an F_t-measurable function into $(I, 2^I)$, where $t = 2, \dots, T-1$. The two supplementary axioms of section 2 have to be modified in their expressions, they become:

Model Consistency: for any date t and information $[Y_t = i]$, valuations are done with the same formulas as before.

Furthermore, the conditional discounting D^t is assumed to be independent of i and $ED^{[Y_t=i]}$ is the conditional discounted expectation. The value of X at date t when $Y_t = i$ (an information at date t) is observed is:

$$ED^{[Y_t=i]} = \sum_{s=0}^{S} D^t[X(s)]\, m^{[Y_t=i]}(s) = \sum_{s=0}^{S} \left[\sum_{t=0}^{T} X(s,t) \;\; D\rho^t(s,t)\right] m^{[Y_t=i]}(s).$$

.

Then, **Dynamic consistency** becomes**:**

$\forall\, t = 1, \dots, T\text{-}1,\ ED(X) = ED[(X_1(s), \dots, X_{t-1}(s), X_t(s) + E^{[Y_t=i]}D^t[X(s)], 0, \dots, 0)_{s\in [Y_t=i]}]$.

Or, in this model:

$$ED(X) = \sum_{s=0}^{S} m(s)\{\sum_{t<t} X(s,t)D\rho(s,t)] + \rho(t)[X_t(s) + \sum_{s\in[Y_t=i]} m^{[Y_t=i]}(s)\sum_{t>t} X(s,t)\,D\rho^t(s,t)]\}.$$

We first concentrate on riskless cash flows $X = (x_1, \dots, x_T)$ where all x_t' are payoffs.

Let us consider a set of date: $E \subset T$, and, in order to simplify notations, let: $t^- = \{1, \dots, t-1\}$ and $t^+ = \{t+1, \dots, T\}$.

$ED(1_E) = D(1_E) = \rho(E)$, $\qquad ED^t(1_E) = \rho^t(E \cap t^+)$, $\qquad$ so that if we note:

$1^t_E = (\,(1_{E\cap\{t\}})_{t\in t^-}, 1_{E\cap t^*} + \rho^t(E\cap t^+), 0, \dots, 0)$,

Dynamic consistency can be written as: $\qquad ED(1_E) = \rho(E) = ED[(1^t_E)]$.

Proposition 4 : Under the Model and Dynamic consistency axioms, we have the "upstating" formula for discount factors applied to a subset E of T:

1. If $t \in E$: $$r^t(E \cap t^+) = \frac{r(E) - r[(E \cap t^-) \cup \{t\}]}{r(\{t\})} \qquad (4.1).$$

2. If $t \notin E$, and $r(E) \geq r[(E \cap t^-) \cup \{t\}]$:

$$r^t(E \cap t^+) = \frac{r(E) - r[(E \cap t^-) \cup \{t\}] + r(\{t\})}{r(\{t\})} \qquad (4.2).$$

3. If $t \notin E$, and $r(E) \leq r[(E \cap t^-) \cup \{t\}]$:

$$r^t(E \cap t^+) = \frac{r(E) - r(E \cap t^-)}{r[(E \cap t^-) \cup \{t\}] - r(E \cap t^-)} \qquad (4.3).$$

Proof:

We have to consider two cases:

1. If $t \in E$:

$$ED(1_E^t) = r[(E \cap t^-) \cup \{t\}] + r(\{t\}) r^t(E \cap t^+).$$

2. $t \notin E$, again we have two separate two sub-cases:

2.1. $r^t(E \cap t^+) \geq 1$, then:

$$ED(1_E^t) = r[(E \cap t^-) \cup \{t\}] + r(\{t\})[r^t(E \cap t^+) - 1].$$

2.2. $r^t(E \cap t^+) \leq 1$, then:

$$ED(1_E^t) = r[(E \cap t^-) \cup \{t\}] r^t(E \cap t^+) + r(E \cap t^-)[1 - r^t(E \cap t^+)].$$

Dynamic consistency is $ED(1_E) = ED(1_E^t)$ and yields the "upstating" formulas under the equivalent conditions given in the proposition.

In the more familiar case where $E = \{1, \ldots, t\}$ we have the following:

Corollary :

Under the Model and the Dynamic consistency axioms, the discount factor on future time after information is released, "upstated" by the information state is:

$$r^t(\{t+1, \ldots, T\}) = \frac{r(\{1, \ldots, T\}) - r(\{1, \ldots, t\})}{r(\{t\})} \qquad (4.1.1).$$

Proof:

Even though proposition 2 is a special case of formula 4.1, we can give here a simpler and more direct proof. The expected discounted value of the unique trajectory is:

$$ED(X) = D(X) = \sum_{t=0}^{T} x_t\, Dr(t).$$

Similarly, according to Model Consistency, the conditional discounted value of X at date t is:

$$D^t(X) = \sum_{t=0}^{T} x_t\, Dr^t(t).$$

Consider the special constant cash flow $1_T = (1, \dots , 1)$. From Dynamic consistency:

$\forall\ t = 2, \dots , T\text{-}1,\ D(X) = D[(x_1, \dots , x_{t-1}, x_t + D^t(x_{t+1}, \dots , x_T), 0, \dots , 0)]$.we have:

$D^t(1_T) = r^t(\{t+1, \dots , T\})$ and $D(1_T) = r(\{1, \dots , t\}) + r(\{t\})r^t(\{t+1, \dots, T\})$.

So that:

$$r^t(\{t+1,\dots,T\}) = \frac{r(\{1,\dots,T\}) - r(\{1,\dots,t\})}{r(\{t\})}.$$

Using these upstating formulas to derive the conditional value of uncertain cash flow is left as an exercise for the time being (the time the authors finish it, of course!)

5. The role of consequentialism

We can see, through the inversion of the hierarchy, the two meanings of consequentialism. From the philosophical definition of consequentialism (e.g. Mac Clennen, 1990): decisions only depend on their *future possible consequences*. The definition excludes consequences that cannot be reached given the available information, but also excludes past consequences, i.e. the dependence on past results (behaviour regarding regrets, necessity to take more risk in order to catch up past losses, etc.).

We see these two contradictions to consequentialism appear respectively in the updating and the upstating rules presented in this paper: In the first approach (Discounted Choquet Expectations), *future* payoffs that are not *possible* after information is revealed still matter in the valuation. In the second approach (Expectated non-separable Discounting) the *past* payoffs that should not matter after information makes them obsolete, still interfere with the valuation of the future payoffs.

Let us state consequentialism more formally to better understand the reason why it may be violated when time is explicitly introduced in the representation of the future. In the case of the more familiar first hierarchy (uncertainty then time) and the updating rule, consequentialism can be expressed as in Sarin and Wakker, 1998, where only S represents the future.

We have to compare four risks (measurable functions on S) X, X' and Y, Y' such that:

$\forall s \in B^c$, $X(s) = X'(s)$ and $Y(s) = Y'(s)$ and $\forall s \in B$ $X(s) = Y(s)$ and $X'(s) = Y'(s)$.

Within the model of section 2, **consequentialism for measurable functions** can be sated as:

$$D^t E^{[Y_t=i]} (X) \geq D^t E^{[Y_t=i]} (X') \Leftrightarrow D^t E^{[Y_t=i]} (Y) \geq D^t E^{[Y_t=i]} (Y')$$

This expresses that, after information "i" is revealed at time t, X, X' and Y, Y' have the same future consequences.

Lapied and Kast's, 2006, show that their updating rules violate consequentialism because payoffs that are not following information "i" may reverse the preference order.

In order to express consequentialism for certain trajectories, where only time represents the future, let us consider four such trajectories.

Consequentialism for trajectories:

$\forall$ X, X', Y, Y' such that:

$$X = (x_1,\ldots,x_t, x_{t+1},\ldots,x_T),\ X' = (x_1,\ldots,x_t, x'_{t+1},\ldots,x'_T),$$

$$Y = (y_1,\ldots,y_t, x_{t+1},\ldots,x_T),\ Y' = (y_1,\ldots,y_t, x'_{t+1},\ldots,x'_T), \text{ then}$$

X is preferred to X' after information at date t, if and only if Y is preferred to Y' after information at date t. Or, within the model of section 4, consequentialism can be formulated by:

$$D^t (X) \geq D^t (X') \Leftrightarrow D^t (Y) \geq D^t (Y').$$

Proposition 5: The upstating rules derived from Model and Dynamic consistency for non-separable discounting, violate consequentialism for trajectories.

Proof: The proof is based on a counter-example.

Let: $X = 1_E = (1, 0, 1, 0)$ and $X' = 1_{E'} = (1, 0, 0, 1)$

$Y = 1_F = (0, 0, 1, 0)$ and $Y' = 1_{F'} = (0, 0, 0, 1)$

And information fall at date $\tau = 2$. Then, from proposition 4.2:

$r(E) = r(1, 3)$, $r(E') = r(1, 4)$, $r(F) = r(3)$, $r(F') = r(4)$,

$$r[(E \cap t^-) \cup \{t\}] = r[(E' \cap t^-) \cup \{t\}] = r(1,2),$$

$$r[(F \cap t^-) \cup \{t\}] = r[(F' \cap t^-) \cup \{t\}] = r(2).$$

Suppose furthermore that[5]:

$r(4) < r(3) < r(2)$ and $r(1, 2) < r(1, 3) < r(1, 4)$.

For E and E' we are in the case 2 and for F and F' we are in the case 3 of proposition 4.2, then:

$$D^t(1_E) = \frac{r(1,3) - r(1,2) + r(2)}{r(2)}, \; D^t(1_{E'}) = \frac{r(1,4) - r(1,2) + r(2)}{r(2)},$$

$$D^t(1_F) = \frac{r(3)}{r(2)}, \; D^t(1_{F'}) = \frac{r(4)}{r(2)},$$

and: $D^t(1_E) < D^t(1_{E'})$, while $D^t(1_F) > D^t(1_{F'})$, in violation of consequentialism.

Concluding remarks

The results of this paper are founded on the explicit expression of time that imposes a dynamic setting. Then, Dynamic consistency is the founding expression of an implicit definition of conditional non-additive integrals under the axiom of Model consistency. We know from Sarin and Wakker, 1998, that if furthermore Consequentialism was imposed as an axiom, only Giloboa and Schmeidler's, 1989, multi-priors models would be consistent with non-additive expectations. A similar result would apply to non-separable discounting. This is in contradiction with the preference representation models referred to in this paper, so it is no wonder that the results violate consequentialism.

The same is to be expected for the more general model where both expectations and discounting are non-additive (formula 3). This is left to be addressed in further research that are needed to explore the valuation methods relevant for the management of uncertain cash flows such that reinsurance portfolios and the valuation of flexible investments in precautionary devices. Regarding the type of (scientific) uncertainty for the future that necessitates precaution (instead of prevention), the further assumption that discount factors are not dependant on the uncertain states need be questioned: In such situations, a measure on

[5] This corresponds to preference for present and fear for temporal concentration of payoffs.

$S \times T$ should represent the agent's preferences on future payoffs, and discount factors as well as capacities should be derived from it.

References:

- Chateauneuf, A. (1991). "On the use of capacities in modelling uncertainty aversion and risk aversion", *Journal of Mathematical Economics* 20, 349-369.
- Chateauneuf, A., R. Kast, and A. Lapied. (2001). "Conditioning capacacities and Choquet integrals: the role of comonotony", *Theory and Decision* 51, 367-386.
- Chateauneuf, A., and Y. Rébillé. (2004). "Some characterizations of non-additive multi-period models", *Mathematical Social Sciences* 48, 235-250.
- Gilboa, I. (1989). "Expectations and variations in multi-period decisions", *Econometrica* 57, 1153-59.
- Gilboa, I., and D. Schmeidler. (1989), "Maximin Expected Utility With a Non-Unique Prior", *Journal of Mathematical Economics*, 18, 141-153.
- Koopmans, T. (1972). "Representation of preference orderings over time", in C. Mac Guire, and R. Radner (eds.), *Decision and organization*, North Holland, 79-100.
- Kreps, D., and E. Porteus. (1978). "Temporal resolution of uncertainty and dynamic choice theory", *Econometrica* 46, 1, 185-200.
- Lapied A., and Kast R. (2006), "Updating Choquet valuation and discounting informatation arrivals", WP Lameta, University of Montpellier.
- Mac Clennen, E. (1990). *Rationality and Dynamic Choice*, Fundamental explorations, Cambridge University Press, Penn.
- Sarin, R., and P. Wakker. (1998). "Dynamic choice and NonExpected Utility", *Journal of Risk and Uncertainty* 17, 87-119.

- Shalev, J. (1997). "Loss aversion in a multiperiod model", *Mathematical Social Sciences* 33, 203-226.
- Yaari, M. (1987). "The dual theory of choice under risk", *Econometrica* 55, 95-115.