Estimation of Objective and Risk-neutral Distributions based on Moments of Integrated Volatility*

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Abstract

In this paper, we present an estimation procedure which uses both option prices and high-frequency spot price feeds to estimate jointly the objective and risk-neutral parameters of stochastic volatility models. The procedure is based on a method of moments that uses analytical expressions for the moments of the integrated volatility and series expansions of option prices and implied volatilities. This results in an easily implementable and rapid estimation technique.

Keywords: Realized Volatility, Implied Volatility, Risk Premium, Moments of Integrated Volatility, Objective Distribution, Risk-neutral Distribution

JEL Classification: C1,C5,G1

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1 Introduction

In continuous-time modeling in finance, continuous-time processes for asset prices are combined with an absence of arbitrage argument to obtain the prices of derivative assets. Therefore, statistical inference on continuous-time models of asset prices can and should combine two sources of information, namely the price history of the underlying assets on which derivative contracts are written and the price history of the derivative securities themselves.

However, statistical modeling poses a challenge. A joint model needs to be specified, not only for the objective probability distribution which governs the random shocks observed in the economy, but also for the risk-neutral probability distribution, which allows to compute derivative asset prices as expectations of discounted payoffs. Since the two distributions have to be equivalent, there exists a link between the two through an integral martingale representation which includes the innovations associated with the primitive asset price processes and the risk premia associated with these sources of uncertainty. Moreover, state variables, observable or latent, may affect the drift and diffusion coefficients of the primitive assets and the corresponding risk premia.

The main contribution of this paper is to propose a new methodology for an integrated analysis of spot and option prices. It is based on simple generalized method-of moment (GMM) estimators of both the parameters of the asset price and state variable processes and the corresponding risk premia. To focus on the issue of the joint specification of an objective probability distribution and a risk-neutral one, we will restrict ourselves to the case of one state variable which will capture the stochastic feature of the volatility process of the underlying asset.

We will adopt a popular affine diffusion model where volatility is parameterized as follows:

$$\begin{aligned} \sigma_t^2 &= \alpha + \xi V_t \\ dV_t &= k(\theta - V_t)dt + \gamma \sigma_t dW_t^\sigma \end{aligned}$$

where V_t is a latent state variable with an innovation governed by a Brownian motion W_t^σ.. This innovation can be correlated (with a coefficient ρ) with the innovation of the primitive asset price process governed by W_t^S:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t^S \tag{1.1}$$

In a seminal paper, Hull and White (1987) have shown that, in the particular case where $\rho = 0$, the arbitrage-free option price is nothing but a conditional expectation of the Black and Scholes (1973) (BS) price, where the constant volatility parameter σ^2 is replaced by the so-called integrated volatility process: $\frac{1}{T-t}\mathcal{V}_{t,T} = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$ and where the conditional expectation is computed with respect to the risk-neutral conditional probability distribution of $\mathcal{V}_{t,T}$ given σ_t. Our method of moments will be based on analytical expressions for the moments of this integrated volatility to estimate the parameter vector of the vector $\beta = (\kappa, \theta, \gamma, \alpha, \xi, \rho)$. These expressions were derived recently by Lewis (2001) using a recursive method, and also by Bollerslev and Zhou (2002) for the first two moments. Heston (1993) has extended the analytical treatment of this option pricing formula to the case where ρ is different from zero, allowing for leverage effects and the presence of risk premia.

However, with or without correlation, the option pricing formula involves the computation of a conditional expectation of a highly nonlinear integral function of the volatility process. To simplify this computation, we propose to use an expansion of the option pricing formula in the neighborhood of $\gamma = 0$, as in Lewis (2000), which corresponds to the Black-Scholes deterministic volatility case. The coefficients of this expansion are well-defined functions of the conditional moments of the joint distribution of the underlying asset returns and integrated volatilities, which we also derive analytically. These analytical expansions will allow us to compute very quickly implied volatilities which are functions of the parameters of the processes and of the risk premia.

An integrated GMM approach using intraday returns for computing approximate integrated volatilities and option prices for computing implied volatilities allow us to estimate jointly the parameters of the processes and the volatility risk premia λ. The main attractive feature of our method is its simplicity once analytical expressions for the various conditional moments of interest are available. The great advantage of the affine diffusion model is precisely to allow an analytical treatment of the conditional moments of interest.

Bollerslev and Zhou (2002) have developed such a GMM approach based on the first two moments of integrated volatility to estimate the objective parameters of stochastic volatility and jump-diffusion models. Recently, Bollerslev, Gibson and Zhou (2004) adopted a very similar approach to ours, but considered a so-called model-free approach to recover implied volatilities.[1]

[1]See in particular Britten-Jones and Neuberger (2000), Lynch and Panigirtzoglou (2003) and Jiang and Tian (2004). The latter study shows how to implement the model-free implied volatility using observed option prices. They characterize the truncation errors when a finite range of strike prices are available in practice. To calculate the model-free implied volatility, they use a curve-fitting method and extrapolation from endpoint implied volatilities.

Only few studies have estimated jointly the risk-neutral and objective parameters, and the estimation methods used are generally much more involved. Pastorello, Renault and Touzi (2000) proposed an iterative estimation procedure that used option and returns information to provide an estimate of the objective parameters in the absence of risk premia. Poteshman (1998) extends their methodology to include correlation between returns and volatility, a non-zero price of volatility risk, and flexible nonparametric specifications for this price of risk as well as the drift and diffusion functions of the volatility process.

Chernov and Ghysels (2000)use the Efficient Method of Moments (EMM), a procedure that estimates the parameters of the structural model through a seminonparametric auxiliary density. Finally, Pan (2002) uses the Fourier transform to derive a set of moment conditions pertaining to implied states and jointly estimate jump-diffusion models using option and spot prices.[2] Pastorello, Patilea and Renault (2003) propose a general methodology of iterative and recursive estimation in structural non-adaptive models which nests all the previous implied state approaches. Compared to all these methods, the main advantage of our method is its simplicity and computational efficiency.

We show through an extensive Monte Carlo that the estimation procedure works well for both the Hull and White (1987) model and the Heston model (1993). Of course the selected moment conditions differ between the two models. Due to the presence of a correlation parameter in the Heston (1003) model, we include moment conditions involving the cross-product of returns with either integrated volatility or implied volatility. In the Hull and White (1987) case, the moment conditions were based only on moments of integrated volatility and of implied volatility.

Finally, we provide an empirical illustration of our method for the Hull and White (1987) model applied to the Deutsch Mark - U.S. Dollar exchange rate futures market. We use 5-minute returns on the exchange rate futures and daily option prices on the same futures to compute the moments and implement our methodology.

The rest of the paper is organized as follows. In Section 2, we present the general methodology, and show how we construct two blocks of moment conditions for the estimation of the models, one based on the high-frequency return measures, another on the implied volatility obtained as power series in the volatility of volatility parameter γ. Section 3 describes the moment conditions for the first block of moment conditions, while Section

[2]Duffie, Pan and Singleton (2000) have extended the moment computations to the case of affine jump-diffusion models (where jumps are captured by Poisson components), while Barndorff-Nielsen and Shephard [8] have put forward the so-called Ornstein-Uhlenbeck-like processes with a general Levy innovation. The general statistical methodology that we develop in this paper could be extended to these more general settings if a specification is chosen for the risk premia of the various jump components.

4 explains how to use option prices expansions to define model-specific implied volatilities, and how these implied volatilities can lead to the estimation of $\tilde{\beta}(\hat{\beta}, \lambda)$. Section 5 presents a Monte Carlo study for the two stochastic volatility models, with and without correlation, for several sets of parameter values. In Section 6, we provide an empirical illustration of the methodology. An extensive appendix contains all the detailed formulas for computing the various quantities of interest.

2 A general outline of the method

As stated in the introduction, two different but equivalent sets of bivariate stochastic processes are to be considered here. The objective process is taken to be the affine stochastic volatility process

$$d\begin{bmatrix} S_t \\ V_t \end{bmatrix} = \begin{bmatrix} \mu_t S_t \\ \kappa(\theta - V_t) \end{bmatrix} dt + \sqrt{\alpha + \xi V_t} \begin{bmatrix} S_t & 0 \\ \gamma\rho & \gamma\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}, \tag{2.2}$$

where S_t and V_t are the price and volatility processes. The affine qualification comes from the form of the volatility appearing in the returns process, a form which was first studied by Duffie and Kan (1996). The risk neutral process is taken to be

$$d\begin{bmatrix} S_t \\ V_t \end{bmatrix} = \begin{bmatrix} r_t S_t \\ \kappa^*(\theta^* - V_t) \end{bmatrix} dt + \sqrt{\alpha + \xi V_t} \begin{bmatrix} S_t & 0 \\ \gamma\rho & \gamma\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} d\tilde{W}_t^1 \\ d\tilde{W}_t^2 \end{bmatrix}, \tag{2.3}$$

where we assumed that only the parameters κ and θ were modified in the passage from one measure to the other. This is in accord with the risk premia structure proposed by Heston (1993), which generalizes, in the affine case, to

$$\kappa^* = \kappa - \lambda \tag{2.4}$$

$$\kappa^*(\alpha + \xi\theta^*) = \kappa(\alpha + \xi\theta), \tag{2.5}$$

the volatility risk premium being parametrized by λ. For such models, the objective parameters to be estimated are[3]

$$\beta = (\kappa, \theta, \gamma, \alpha, \xi, \rho)\,. \tag{2.6}$$

In order to define the risk-neutral set of parameters, one must have the additional parameter λ, since we shall assume the short rate r_t to be observed. By abuse of language, we will

[3]Since the drift term μ_t does not matter for option pricing purposes, we do not specify it explicitely. Moreover, the inference method we will use for the objective parameters is robust to its specification.

often refer to the risk-neutral parameters as the vector (β, λ), which is simply related to the set

$$\beta^* = (\kappa^*, \theta^*, \gamma, \alpha, \xi, \rho)\,. \tag{2.7}$$

In the next sections, we will show that high-frequency measures of returns can be used to measure the integrated volatility $\mathcal{V}_{t,T}$. Lewis (2001) proposes a method to compute conditional moments of the integrated volatility in affine stochastic volatility models. Using these, it is possible to construct a set of moment conditions $f_1(\mathcal{V}_{t,T}, \beta)$, which is such that

$$\mathrm{E}\left[f_1(\mathcal{V}_{t,T}, \beta)\right] = 0. \tag{2.8}$$

Moreover, as we will present, it is possible to define model-specific implied volatilities $V_t^{imp}(\beta, \lambda, \{c_{obs}\})$, with $\{c_{obs}\}$ being the set of observed option prices. These implied volatilities, that are not to be confounded with Black-Scholes implied volatilities, are defined to be the point-in-time volatility which gives, for given values of the risk-neutral parameters β^*, the observed option price. As we will discuss, it can be used to construct a second set of moment conditions $f_2(V_t^{imp}(\beta, \lambda, \{c_{obs}\}), \tilde{\beta})$, which depends on the values of the risk-neutral parameters β, λ taken to define the implied volatilities and on the parameters associated to the implied volatility process, which we will denote $\tilde{\beta}$. Note that if the model is correctly specified, one expects $\tilde{\beta}$ and β to be equal. This second set of moment conditions obeys the usual requirement

$$\mathrm{E}\left[f_2(V_t^{imp}(\beta, \lambda, \{c_{obs}\}), \tilde{\beta})\right] = 0, \tag{2.9}$$

Since we want $\tilde{\beta} = \beta$, it is thus possible to construct the set of moment conditions

$$E\begin{bmatrix} f_1(\mathcal{V}_{t,T}, \beta) \\ f_2(V_t^{imp}(\beta, \lambda, \{c_{obs}\}), \beta) \end{bmatrix} = 0 \tag{2.10}$$

which we would like to use to estimate the objective parameters β and the risk premium λ, both of them providing the risk-neutral parameters β^*.

Such a set of moment conditions can be estimated efficiently using a GMM approach. This is the main approach we will follow. However, since the parameters in β appear in a very nonlinear fashion in $(V_t^{imp}(\beta, \lambda, \{c_{obs}\})$, it can be argued that estimating the system (2.10)directly by GMM is a difficult task. Therefore we also consider several other approaches. Namely, one can first restrict his attention to the first set of moment conditions, from which it is possible to obtain an estimate $\hat{\beta}$ of the objective parameters.[4]. Then, one

[4]In this case, we will be able to compare the estimators obtained with our moment conditions including skewness to the estimators of Bollerslev and Zhou (2002) based on only the first two moments of integrated volatility

considers the simplified set of moment conditions

$$E\left[\begin{array}{c} f_1(\mathcal{V}_{t,T},\beta) \\ f_2(V_t^{imp}(\hat{\beta},\lambda,\{c_{obs}\}),\beta,) \end{array}\right]=0 \tag{2.11}$$

According to Gouriéroux, Monfort and Renault (1996), there should not be any efficiency loss when comparing the system of moment conditions (2.11) to the one in (2.10).

We also consider a simpler sequential approach by using a sequential estimation procedure rather than a two-step GMM. That is, we will use the first set of moment conditions to obtain the estimate $\hat{\beta}$.

$$\mathrm{E}\left[f_1(\mathcal{V}_{t,T},\beta)\right]=0\longrightarrow\hat{\beta}. \tag{2.12}$$

Given a value of λ, we will then use the estimate $\hat{\beta}$ to obtain $\tilde{\beta}(\hat{\beta},\lambda)$;

$$\mathrm{E}\left[f_2(V_t^{imp}(\hat{\beta},\lambda,\{c_{obs}\}),\beta)\right]=0\longrightarrow\tilde{\beta}(\hat{\beta},\lambda). \tag{2.13}$$

Finally, by defining a suitable metric, a task for which the results from indirect inference can be used, we will minimize the distance between the two sets of objective parameters estimate $\hat{\beta}$ and $\tilde{\beta}(\hat{\beta},\lambda)$ in order to estimate $\hat{\lambda}$;

$$\hat{\lambda}=\min_{\lambda}||\hat{\beta}-\tilde{\beta}(\hat{\beta},\lambda)||. \tag{2.14}$$

One advantage of the sequential approach is to allow for mixed frequencies when estimating the parameters of the objective distribution, and when recovering the risk neutral parameters through option prices. Even though the sequential approach is not optimal, the loss in efficiency may not be large since the computation of the implied volatility in the f_2 set of moment conditions is based on moments of integrated volatility. Figure 1 summarizes the various estimation steps in the sequential GMM method.

3 Estimating Objective Parameters from High-Frequency Returns

From the seminal works of Andersen and Bollerslev (1998) and Barndorff-Nielsen and Shephard (2001), we know that high-frequency intraday data on returns can be used to obtain indirect information on the otherwise unobservable volatility process. The logarithmic price of an asset is assumed to obey the stochastic differential equation

$$dp_t=\mu(p_t,V_t,t)dt+\sqrt{V_t}dW_t, \tag{3.15}$$

where V_t is the squared-volatility process (which could be stochastic, particularly of the affine type we discussed above) and W_t is a standard brownian motion. Note that from

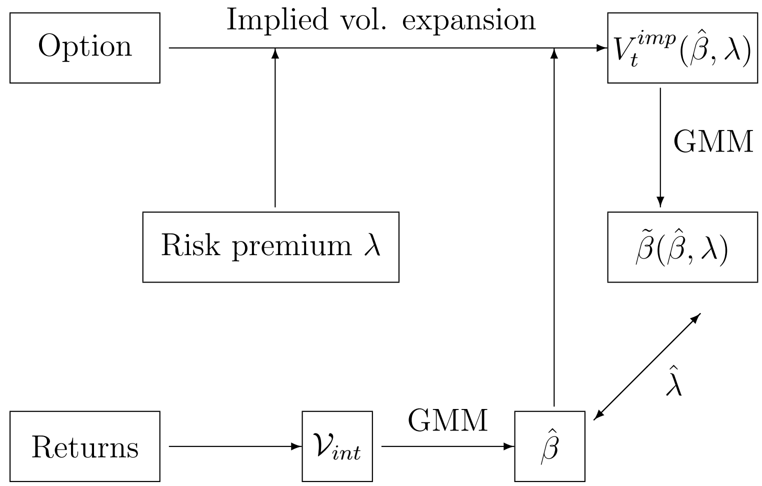

Figure 1: Overview of the sequential estimation procedure.

now on, we shall restrict our attention to the $\alpha = 0,\ \xi = 1$ case. If the drift and diffusion coefficients are sufficiently regular to guarantee the existence of a unique strong solution to the SDE, then, by the theory of quadratic variation, we have

$$\lim_{N\to\infty} \sum_{i=1}^{N} \left[p_{t+\frac{i}{N}(T-t)} - p_{t+\frac{i-1}{N}(T-t)} \right]^2 \longrightarrow \int_t^T V_s \, ds \equiv \mathcal{V}_{t,T}, \tag{3.16}$$

and $\mathcal{V}_{t,T}$ is referred to as the integrated volatility of the process V_t from time t to T. Andersen et al. (2001a, 2001b, 2003) offer a characterization of the distributional features of daily realized returns volatilities constructed from high-frequency five-minutes returns for foreign exchange and individual stocks. The finiteness of the number of measures induces a systematic error in the integrated volatility measure, and, in fact, the quadratic variation estimator will be a biased estimator of the integrated volatility if the drift term is not zero, this bias falling as the number of measures increases. Bollerslev and Zhou (2002) use such an aggregation of returns to obtain integrated volatility time series from which they estimate by GMM the parameters of Heston's (1993) stochastic volatility model. They base their estimation on a set of conditional moments of the integrated volatility, where they add to the basic conditional mean and second moment various lag-one and lag-one squared counterparts. In constructing estimates of the objective parameters of the stochastic volatility process, we follow their basic approach but introduce a new set of moment conditions involving higher moments of the integrated volatility, in particular its skewness. Lewis (2001) derives analytically all conditional moments of the integrated volatility for the class of affine stochastic volatility models (which includes the Heston

(1993) and Hull-White (1987) models). [5]

Some attention has to be devoted to information sets. Following the notation of Bollerslev and Zhou (2002), we shall define the filtration $\mathcal{F}_t = \sigma\{V_s, s \leq t\}$, that is, the sigma algebra generated by the instantaneous volatility process. Our moment conditions for the integrated volatility are originally conditional on this filtration. Since only the integrated volatility is observable, we need to introduce the discrete filtration $\mathcal{G}_t = \sigma\{\mathcal{V}_{s-1,s}, s = 0, 1, 2, \cdots, t\}$, which is the sigma algebra of observed integrated volatilities. Evidently, the filtration $\mathcal{G}_t$ is nested in the finer $\mathcal{F}_t$. This enables one to rewrite moment conditions in terms of the coarser filtration using the law of iterated expectations: $\mathrm{E}\left[\mathrm{E}\left(\cdot\,|\mathcal{F}_t\right)|\mathcal{G}_t\right] = \mathrm{E}\left(\cdot\,|\mathcal{G}_t\right)$.

3.1 The Hull and White Model

In the case where there is no correlation between returns and volatility ($\rho = 0$), we use the following set of moment conditions:

$$f_{1t}(\beta) = \begin{bmatrix} \mathcal{V}_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}_{t+1,t+2}|\mathcal{G}_t\right] \\ \mathcal{V}^2_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}^2_{t+1,t+2}|\mathcal{G}_t\right] \\ \mathcal{V}^3_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}^3_{t+1,t+2}|\mathcal{G}_t\right] \end{bmatrix} \tag{3.17}$$

The three conditional moment restrictions (3.17) can be expressed in terms of observed integrated volatilities because we have (see Appendix B) closed form formulas for $\mathrm{E}\left[\mathcal{V}^k_{t+1,t+2}|\mathcal{G}_t\right]$, $k = 1, 2, 3$, in terms of $\mathrm{E}\left[\mathcal{V}^k_{t,t+1}|\mathcal{G}_t\right]$, $k = 1, 2, 3$. We use each of the resulting three orthogonality conditions with two choices of instruments. That is, for the moment condition involving $\mathrm{E}\left[\mathcal{V}^k_{t+1,t+2}|\mathcal{G}_t\right]$, we use both a constant and $\mathcal{V}^k_{t-1,t}$ as instrumental variables, which results in six unconditional moment restrictions.[6]

[5] Duffie, Pan and Singleton (2000)provide analytical expressions for the instantaneous volatility process for such models. Bollerslev and Zhou (2002)derived analytical expressions for the mean and variance of the integrated volatility in Feller-type volatility models. To our knowledge, higher moments of the integrated volatility were not previously computed. Zhou (2003) characterized the Itô conditional moment generator for affine jump-diffusion models, and other nonlinear quadratic variance and semiparametric flexible jump models.

[6] Due to the MA(1) structure of the error terms in (3.18), the optimal weighting matrix for GMM estimation entails the estimation of the variance and only the first-order autocorrelations of the moment conditions.

$$f_{1t}(\beta) = \begin{bmatrix} \mathcal{V}_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}_{t+1,t+2}|\mathcal{G}_t\right] \\ (\mathcal{V}_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}_{t+1,t+2}|\mathcal{G}_t\right])\mathcal{V}_{t-1,t} \\ \mathcal{V}^2_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}^2_{t+1,t+2}|\mathcal{G}_t\right] \\ (\mathcal{V}^2_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}^2_{t+1,t+2}|\mathcal{G}_t\right])\mathcal{V}^2_{t-1,t} \\ \mathcal{V}^3_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}^3_{t+1,t+2}|\mathcal{G}_t\right] \\ (\mathcal{V}^3_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}^3_{t+1,t+2}|\mathcal{G}_t\right] \mathcal{V}^3_{t-1,t}) \end{bmatrix} \tag{3.18}$$

3.2 The Heston Model

The Heston model has one more parameter than the Hull & White one: the correlation coefficient between the two Brownian motions, ρ. In principle, the latter could be identified even using only the marginal moments of the integrated and the spot volatility (thanks to the fact that the implied spot volatilities depend on both λ and ρ). In practice, this is not sufficient. It is mandatory to add some cross-moments between the log returns [7] and the integrated volatility to the moment conditions in (3.17). The set of twelve moment conditions used in the estimation of the objective parameters is as follows:

$$f_{1t}(\beta) = \begin{bmatrix} \mathcal{V}_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}_{t+1,t+2}|\mathcal{G}_t\right] \\ (\mathcal{V}_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}_{t+1,t+2}|\mathcal{G}_t\right])\mathcal{V}_{t-1,t} \\ \mathcal{V}^2_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}^2_{t+1,t+2}|\mathcal{G}_t\right] \\ (\mathcal{V}^2_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}^2_{t+1,t+2}|\mathcal{G}_t\right])\mathcal{V}^2_{t-1,t} \\ \mathcal{V}^3_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}^3_{t+1,t+2}|\mathcal{G}_t\right] \\ (\mathcal{V}^3_{t+1,t+2} - \mathrm{E}\left[\mathcal{V}^3_{t+1,t+2}|\mathcal{G}_t\right])\mathcal{V}^3_{t-1,t} \\ (p_{t+1} - p_t)\mathcal{V}_{t+1,t+2} - \mathrm{E}\left[(p_{t+1} - p_t)\mathcal{V}_{t+1,t+2}|\mathcal{G}_t\right] \\ ((p_{t+1} - p_t)\mathcal{V}_{t+1,t+2} - \mathrm{E}\left[(p_{t+1} - p_t)\mathcal{V}_{t+1,t+2}|\mathcal{G}_t\right])(p_{t-1} - p_{t-2})\mathcal{V}_{t-1,t} \\ (p_{t+1} - p_t)^2\mathcal{V}_{t+1,t+2} - \mathrm{E}\left[(p_{t+1} - p_t)^2\mathcal{V}_{t+1,t+2}|\mathcal{G}_t\right] \\ ((p_{t+1} - p_t)^2\mathcal{V}_{t+1,t+2} - \mathrm{E}\left[(p_{t+1} - p_t)^2\mathcal{V}_{t+1,t+2}|\mathcal{G}_t\right])(p_{t-1} - p_{t-2})^2\mathcal{V}_{t-1,t} \\ (p_{t+1} - p_t)\mathcal{V}^2_{t+1,t+2} - \mathrm{E}\left[(p_{t+1} - p_t)\mathcal{V}^2_{t+1,t+2}|\mathcal{G}_t\right] \\ ((p_{t+1} - p_t)\mathcal{V}^2_{t+1,t+2} - \mathrm{E}\left[(p_{t+1} - p_t)\mathcal{V}^2_{t+1,t+2}|\mathcal{G}_t\right])(p_{t-1} - p_{t-2})\mathcal{V}^2_{t-1,t} \end{bmatrix} \tag{3.19}$$

[7] We cannot express the moments conditions in terms of the log price of the asset, as in Bollerslev and Zhou (2002), since it is nonstationary. Attempts to run the GMM estimation with the price instead of the returns conditions resulted tended to exhibit an erratic behavior: estimates on the boundary of the admissible region, numerous error flags raised by the optimization software, and so on.

To derive the closed-form expression of the cross-moments, we used the recurrence formula provided in Appendix A of Lewis (2001). These expressions are given in Appendix B.

4 Using Implied volatilities to link objective and risk-neutral parameters

The Black-Scholes formula usually leads to incorrect option pricing, but it remains a central tool in the derivatives' industry, through the widespread use of the so-called Black-Scholes implied volatility. Since option prices are strictly increasing functions of the volatility, it is possible to invert the function globally so as to associate to an option price the volatility for which the Black-Scholes formula would provide accurate pricing. The mispricing is revealed through the fact that the implied volatility function appears to be dependent on moneyness. This dependency is often referred to as the volatility smile.

Implied volatilities can also be defined for more complex pricing models. These models will usually have extra parameters, so that the inversion of the pricing formula in the volatility parameter can be done if one sets values for the other parameters. The value of the implied volatility will therefore be dependent not only on the option price, but also on the values of the other parameters.

However, analytical option pricing formulas are generally difficult to invert and one has to use numerical procedures which are computationally intensive and whose precision has to be controlled. Moreover, implementing integral solutions such as Heston's formula can be very delicate due to divergences of the integrand in regions of the parameter space.

One way to avoid both problems is to rewrite option pricing formulas as power series around values of the parameters for which the model can be analytically solved (*i.e* it has an explicit form in terms of elementary and special functions; not an integral one). This avenue is followed by Lewis[31].

4.1 Series expansions of option pricing formulas

Since option prices are continuous in the volatility of volatility parameter γ, one can expand the pricing formula around a fixed γ, which we will set to zero, as it corresponds to a deterministic volatility model that we can solve analytically, as we will show. Generally, options will have prices $c(S, K, r, T, V_t, \beta, \lambda)$ at time t, where S is the underlying asset's price at this time, K the strike price, T the expiration date, β and λ are respectively the objective parameters and the volatility risk premium and V_t is the volatility at time t. The general expansion around $\gamma = 0$ of an option pricing formula has the form

$$c(S, K, r, T, V(t), \beta(\cdot, \gamma), \lambda) = \sum_{j=0}^{\infty} \mu_j(S, K, r, T, V(t), \beta(\cdot, 0), \lambda)\gamma^j, \tag{4.20}$$

the notation $\beta(\cdot, \gamma)$ being chosen to exhibit the explicit dependence of the parameter set β, $\beta(\cdot, 0)$ meaning that the μ_j's have no dependence in γ. If $\gamma \to 0$, the model has deterministic volatility. For affine stochastic volatility models, it reduces to

$$\frac{dV_t}{dt} = \kappa^* \left(\theta^* - V_t\right), \tag{4.21}$$

with the obvious solution

$$V_T = \theta^* + \left(V(t) - \theta^*\right) e^{-\kappa^*(T-t)}. \tag{4.22}$$

Hull and White have shown that in non-constant but deterministic volatility models, the option price is given by the Black-Scholes formula evaluated at the mean value of the integrated volatility over the option life;

$$c(S, K, r, T, V(t), \beta(\cdot, 0), \lambda) = c_{BS}(S, K, r, T, \frac{1}{T-t}\bar{\mathcal{V}}_{t,T}), \tag{4.23}$$

with

$$\begin{aligned}
\bar{\mathcal{V}}_{t,T} &= \mathrm{E}_t\left[\mathcal{V}_{t,T}\right] && (4.24)\\
&= \left[\frac{1}{T-t}\int_t^T \left[\theta^* + \left(V(t) - \theta^*\right) e^{-\kappa^*(T-s)}\right] ds\right] && (4.25)\\
&= \theta^* + \left(V(t) - \theta^*\right) \frac{1 - e^{-\kappa^*(T-t)}}{\kappa^*(T-t)}. && (4.26)
\end{aligned}$$

The first coefficient of the expansion is thus

$$\mu_0(S, K, r, T, V_t, \beta(\cdot, 0), \lambda) = c_{BS}(S, K, r, T, \frac{1}{T-t}\bar{\mathcal{V}}_{t,T}). \tag{4.27}$$

Other coefficients can be computed in an analogue way (see Appendix A for explicit computations of coefficients of the expansion up to order 6 for the Hull-and-White model).

4.2 Explicit inversion of the pricing formula: Implied volatilities

One of the main advantages of having at our disposal a series expansion of option pricing formulas, apart from the rapid and simple price computation scheme it provides, is that inverting a series is usually straightforward, so that one can define implied volatilities and compute them very efficiently. As we pointed out earlier, option prices being stricly increasing functions of the volatility, the inversion is always possible. Given the price series expansion (4.20), assume that given an observed option price c_{obs}, the volatility V_t^{imp} admits the expansion

$$V_t^{imp}(S,K,r,T,c_{obs},\beta(\cdot,\gamma),\lambda) = \sum_{j=0}^{\infty} \nu_j(S,K,r,T,c_{obs},\beta(\cdot,0),\lambda)\gamma^j. \tag{4.28}$$

If the observed option price is correctly priced by (4.20) with volatility being the implied volatility $V_t^{imp}(S,K,r,T,c_{obs},\beta(\cdot,\gamma),\lambda)$, then

$$\begin{aligned} c_{obs} &= c(S,K,r,T,V_t^{imp},\beta(\cdot,\gamma),\lambda) && (4.29)\\ &= \sum_{j=0}^{\infty} \mu_j(S,K,r,T,\sum_{k=0}^{\infty} \nu_k(S,K,r,T,c_{obs},\beta(\cdot,0),\lambda)\gamma^k,\beta(\cdot,0),\lambda)\gamma^j. && (4.30)\end{aligned}$$

Having explicit expressions for the μ_js, in which the volatility parameter will usually enter in a polynomial form, one can solve the above equation order by order in γ. From these expressions, one can define coefficients $\tilde{\nu}_j(S,K,r,T,c_{obs},\beta(\cdot,0),\lambda,\{\nu_k\})$ that are such that

$$c_{obs} = \sum_{j=0}^{\infty} \tilde{\nu}_j(S,K,r,T,c_{obs},\beta(\cdot,0),\lambda,\{\nu_k\})\gamma^j. \tag{4.31}$$

This equation is solved by imposing the conditions

$$\tilde{\nu}_0 = c_{obs} \qquad \tilde{\nu}_j = 0, \forall j \geq 1, \tag{4.32}$$

which form a system of polynomial equations for the ν_k. This system is easily solved order by order in k. Such a procedure can be understood as defining the inverse of a function $f(z)$ as the function $f^{-1}(z)$ which is such that $f(f^{-1}(z)) = z$. For instance, in mean-reverting stochastic volatility process such as the one we consider here, only the $j = 0$ and $k = 0$ terms contribute to the zeroth order term in (4.30), so that the coefficient ν_0 can be identified straightforwardly;

$$c_{obs} = \tilde{\nu}_0 = \mu_0(S,K,r,T,\nu_0(S,K,r,T,c_{obs},\beta(\cdot,0),\lambda),\beta(\cdot,0),\lambda). \tag{4.33}$$

Now, if we define V_{BS}^{imp} to be the usual Black-Scholes implied volatility, which can always be used instead of the observed option price, we can simplify this equation to

$$V_{BS}^{imp} = \theta^* + (\nu_0 - \theta^*)\frac{1-e^{-\kappa^*(T-t)}}{\kappa^*(T-t)}, \tag{4.34}$$

so that

$$\nu_0(S,K,r,T,V_{BS}^{imp},\beta(\cdot,0),\lambda) = \theta^* + (V_{BS}^{imp} - \theta^*)\frac{\kappa^*(T-t)}{1-e^{-\kappa^*(T-t)}} \tag{4.35}$$

In the following, to emphasize that the implied volatility series we defined here is parameter specific, we will refer to it as $V_t^{imp}(\beta,\lambda)$. That is, given an option data set, one can associate an implied volatility time series to every choice of the parameters β,λ. It is only when $\beta=\beta_0$ and $\lambda=\lambda_0$, the true values of the parameters, that $V_t^{imp}(\beta_0,\lambda_0)=V_t$. This will be the starting point of our GMM estimation which we will now present.

4.3 Moment Conditions for implied volatilities

Given daily option prices and some initial values for the objective parameters β_0 and the risk premium λ_0, we are able to construct an implied volatility time series. If the risk premium was suitably chosen and the objective parameters the true ones, every option on a given day would have the same implied volatility, so that the implied volatility surface would be flat, with a value equal to the point-in-time volatility V_t. Since this will not usually be the case, one could use some daily mean value of the observed implied volatilites $V_t^{imp}(\beta_0,\lambda_0)$ to generate the implied volatility time series. [8]

4.3.1 Moment conditions for the Implied Hull and White volatility

One can then consider the daily volatility series and use the moments of the instantaneous volatility in the moment conditions $f_{2t}(\beta)$ where, again, each of these conditional moment conditions are used with two instrumental variables (which are a constant and V_t^k for the moment condition involving $\mathrm{E}\left[V_{t+1}^k|\mathcal{F}_t\right]$), resulting in six unconditional moment conditions.

$$f_{2t}(V,\beta)=\begin{bmatrix}(V_{t+1}-E(V_{t+1}|\mathcal{F}_t))\\(V_{t+1}-E(V_{t+1}|\mathcal{F}_t))V_t\\(V_{t+1}^2-E(V_{t+1}^2|\mathcal{F}_t))\\(V_{t+1}^2-E(V_{t+1}^2|\mathcal{F}_t))V_t^2\\(V_{t+1}^3-E(V_{t+1}^3|\mathcal{F}_t))\\(V_{t+1}^3-E(V_{t+1}^3|\mathcal{F}_t))V_t^3\end{bmatrix} \tag{4.36}$$

These moment conditions are used in (2.10) to obtain estimates of the β and λ parameters with a joint GMM procedure. We can also use these conditions replacing β by $\hat{\beta}$ obtained with the high-frequency returns and estimate the system (2.11) by a two-step

[8]Whether strongly out or in-the-money options should be included in this mean value computation can be debated. We chose to do so when dealing with Monte Carlo generated option prices, but it is not obvious that this would be the reasonable choice when using empirical data.

GMM procedure. Finally, one can use the sequential approach described in (2.12), (2.13), and (2.14).

4.3.2 Moment conditions for the Implied Heston Volatility

We proceed in the same way for the Heston model and use a similar of moment conditions as (3.19) with V replacing $\mathcal{V}$ and $\mathcal{F}$ replacing $\mathcal{G}$:

$$f_{2t}(\beta) = \begin{bmatrix} V_{t+1,t+2} - \mathrm{E}\left[V_{t+1,t+2}|\mathcal{F}_t\right] \\ (V_{t+1,t+2} - \mathrm{E}\left[V_{t+1,t+2}|\mathcal{F}_t\right])\mathcal{V}_{t-1,t} \\ V^2_{t+1,t+2} - \mathrm{E}\left[V^2_{t+1,t+2}|\mathcal{F}_t\right] \\ (V^2_{t+1,t+2} - \mathrm{E}\left[V^2_{t+1,t+2}|\mathcal{F}_t\right])V^2_{t-1,t} \\ V^3_{t+1,t+2} - \mathrm{E}\left[V^3_{t+1,t+2}|\mathcal{F}_t\right] \\ (V^3_{t+1,t+2} - \mathrm{E}\left[V^3_{t+1,t+2}|\mathcal{F}_t\right])V^3_{t-1,t} \\ (p_{t+1}-p_t)V_{t+1,t+2} - \mathrm{E}\left[(p_{t+1}-p_t)V_{t+1,t+2}|\mathcal{F}_t\right] \\ ((p_{t+1}-p_t)V_{t+1,t+2} - \mathrm{E}\left[(p_{t+1}-p_t)V_{t+1,t+2}|\mathcal{F}_t\right])(p_{t-1}-p_{t-2})V_{t-1,t} \\ (p_{t+1}-p_t)^2V_{t+1,t+2} - \mathrm{E}\left[(p_{t+1}-p_t)^2V_{t+1,t+2}|\mathcal{F}_t\right] \\ ((p_{t+1}-p_t)^2V_{t+1,t+2} - \mathrm{E}\left[(p_{t+1}-p_t)^2V_{t+1,t+2}|\mathcal{F}_t\right])(p_{t-1}-p_{t-2})^2V_{t-1,t} \\ (p_{t+1}-p_t)V^2_{t+1,t+2} - \mathrm{E}\left[(p_{t+1}-p_t)V^2_{t+1,t+2}|\mathcal{F}_t\right] \\ ((p_{t+1}-p_t)V^2_{t+1,t+2} - \mathrm{E}\left[(p_{t+1}-p_t)V^2_{t+1,t+2}|\mathcal{F}_t\right])(p_{t-1}-p_{t-2})V^2_{t-1,t} \end{bmatrix} \tag{4.37}$$

The expressions for the moments are provided in Appendix B.

5 A Monte Carlo Study

In order to assess how the estimation methods proposed in the previous sections perform, we conducted a Monte Carlo study for both the Hull-White and the Heston models, that is two stochastic volatility models with and without leverage effect (that is, $\rho = 0$) respectively. It should be mentioned that any model that admits a formulation of the option pricing formula in terms of a power series could be considered with the same methodology.

To test the efficiency of the methods, we generated, using usual Monte Carlo techniques, several sets of high-frequency price-volatility time series and the associated sets of daily option prices.

5.1 Hull and White Model

We chose to study four sets of parameters. The volatility parameters are chosen in the same way as in Bollerslev and Zhou (2002), in order to compare with their results. The four sets of parameters are as follows:

- $(\kappa, \theta, \gamma, \lambda) = (0.1, 0.25, 0.1, 0.05)$: A stationary process with high risk premium.
- $(\kappa, \theta, \gamma, \lambda) = (0.1, 0.25, 0.1, 0.02)$: The same process with a lower volatility risk premium
- $(\kappa, \theta, \gamma, \lambda) = (0.03, 0.25, 0.1, 0.01)$: A highly persistent volatility process (nearly unit-root)
- $(\kappa, \theta, \gamma, \lambda) = (0.1, 0.25, 0.2, 0.05)$: A quasi non-stationary process with high volatility of volatility parameter

Note that these are daily parameters. As in Bollerslev and Zhou, we normalized θ so that the yearly volatility is $\frac{\sqrt{240 \times \theta}}{100}$, with 240 being the number of days per year we chose. This means that to $\theta = 0.25$ is associated a yearly volatility of 7.74%.

As we just said, each year was defined as 240 days, which were further subdivised in 80 five-minute periods. The quadratic variations $\mathcal{V}_{t,t+1}$ were aggregated over those 80 periods, giving daily integrated volatilies, whereas the option prices were computed from the mid-day price and spot volatility of the underlying asset (i.e. the 40-th observation in each day). In turn, each 5-minute interval is actually subdivided in ten 30 seconds subintervals, and the SDE is simulated using the finest grid.

Option prices were computed by using the Hull and White's model specificity that option prices can be expressed as mean values of the Black-Scholes price over the integrated volatility distribution. Explicit expressions for the option prices and implied volatility series up to the sixth order in γ can be found in Appendix C. However, option prices were obtained by simulating volatility trajectories to approximate the integrated volatility distribution. We wanted to avoid using our option pricing formula in order to validate its use. However, if enough trajectories are simulated, the simulated and series expansion price are quasi indistinguishable (at least in the region where the expansion is valid and has enough precision). One could reduce the number of drawn trajectories to add noise to option prices.

The risk premium structure is chosen as in the Heston's (1993) paper, that is, risk-neutral (denoted by stars) and objective parameters are related by

$$\kappa^* = \kappa - \lambda \tag{5.38}$$

$$\theta^* \kappa^* = \theta \kappa \tag{5.39}$$

$$\gamma^* = \gamma. \tag{5.40}$$

Usually, one would expect λ to be positive, so that the asymptotic volatility is higher, meaning that option prices will also be. This is however not important in what follows, although all the examples we chose have positive risk premium.

We applied the four estimation methods described in section 3. The GMM estimation procedure was conducted with a Newey-West kernel with a lag length of two.[9]

Results of the estimations are provided in Table 1, Panels A to D. Statistics were obtained by estimating parameters over 5000 independent sets of 4-year data (960 observations). A first general remark could be that all estimators show little bias and that the method denoted Ψ_1 where all the moment conditions are taken jointly appears to be the most efficient overall. In any case it is better than the two other methods that deliver an estimation of the risk premium. With respect to the β vector, the RMSE of gamma is smallest for the $\hat{\beta}$ estimator. Since the volatility of volatility coefficient estimation is intimately linked with the data frequency, this is not a surprise. The estimate could be improved if one had access to mid and end-of-day option prices. Finally, the volatility risk premium is also nicely recovered. The RMSE for lambda in Ψ_1 remains quite small, except maybe for the last configuration of parameters (Panel D). This may be due to the fact that the process is in this case quasi non-stationary, with a high volatility of volatility parameter. The error is also relatively large for the other parameters in this case.

A comparison of the $\hat{\beta}$ estimates with the GMM estimates of Bollerslev and Zhou (2002) reveals that the RMSE are in general smaller with the moment conditions that we specified. An exception is θ, which seems to be slightly better estimated by their set of moment conditions. The difference between the two sets is the introduction in our estimators of the third-moment conditions, while in their selection of instruments they included the lagged squared integrated volatility.

Finally, one element that was not put forward before is the large excess kurtosis present in the distributions of the parameter estimates for almost all the estimation methods. It does not always affect the same parameter but it is very pervasive. This point should be investigated further in future research.

5.2 Heston Model

For this model, the sampling scheme used for the Hull and White model is not coherent with the timing used to evaluate the cross moments included in the moment conditions. It

[9]We checked that results were quite insensitive to this choice.

requires prices and spot volatilities observed at the beginning of the period over which the quadratic variation is computed. For this reason, we modified the sampling scheme to use opening prices instead of mid-day prices.

Moreover, the previous Monte Carlo strategy does not work in the Heston model. Basically, the problem is that the in-the-money option prices simulated in the way we proceeded violate often the lower bound $C \geq S - Ke^{-r\tau}$, which makes it impossible to compute the associated Black and Scholes volatility. In the Hull and White model this problem was absent because C could be computed as the expectation of $C_{BS}(S, \mathcal{V}_{t,t+\tau})$ with respect to the distribution of the integrated volatility. A similar expression also exists for the Heston model (see formula (C.3), but in this case C_{BS} is not evaluated in S, but in S rescaled by a function of the volatility trajectory.

Of course increasing the number of trajectories may attenuate the issue, but it does not provide a complete solution, and it significantly increases the computational burden. For this reason, in the following experiments the option prices were evaluated using the series expansion formula in implied volatility and adding a random error term. We studied the characteristics of the error terms implicitly introduced in the option prices by the MC strategy used to evaluate them in the Hull and White model, and found out that they looked fairly similar to $\mathcal{N}(0, \omega^2)$ added to the implied B&S volatilities, with $\omega = 4 \times 10^{-7}$. This is the distribution from which the random noises in option prices (via the implied B&S volatilities) are drawn in the experiments concerning the Heston model.

The optimal weighting matrices have been computed in a second step using Newey - West with two lags for estimators based on f_1 only and on both f_1 and f_2, and with one lag for those based on f_2 only.

We considered the same 4 sets of values for β as in the Hull and White experiments, and two possible values for the additional parameter ρ: -0.5 and -0.2. The results are provided in Table 2. The parameter sets referred to in the table are defined as follows:

Label	κ	θ	γ	ρ	λ
1a	0.1	0.25	0.1	-0.5	0.05
1b	0.1	0.25	0.1	-0.2	0.05
2a	0.1	0.25	0.1	-0.5	0.02
2b	0.1	0.25	0.1	-0.2	0.02
3a	0.03	0.25	0.1	-0.5	0.01
3b	0.03	0.25	0.1	-0.2	0.01
4a	0.1	0.25	0.2	-0.5	0.05
4b	0.1	0.25	0.2	-0.2	0.05

For each parameters configuration, the MC experiment consists of 5,000 replications of 960 observations each.

In order to identify the correlation parameter ρ, these experiments considered six additional cross-moment conditions in addition to those used in the experiments conducted in the Hull and White model. Nevertheless, ρ is clearly the most difficult parameter to estimate. In particular, this is witnessed by the uniformly highest – sometimes by a very significant margin – RMSE among all the parameters, and some mean and median bias in many of the estimators considered.

A closer look also suggests that the bias in $\widehat{\rho}$ characterizes $\hat{\beta}_{QV}$, and also those based upon it ($\widetilde{\beta}$ and $\widehat{\psi}_2$). Hence, it seems that, in general, estimators more closely based on quadratic variations tend to underestimate the leverage, roughly by 20% of the true value.

One might think that this behavior is due to the fact that the quadratic variations are only estimates of the true integrated volatilities, and that the measurement errors they embed may lead to the underestimation highlighted above. To check this hypothesis, we estimated β using also the true integrated volatilities. It turns out that the underestimation of ρ is roughly the same as before. Hence, it is not due to the measurement error in quadratic variations.

The estimates of β based on the true spot volatilities, $\hat{\beta}_V$, also exhibit a downward bias w.r.t. ρ, albeit less pronounced.

In general, the best estimator – in term of bias and RMSE, irrespectively from the parameter – is $\widehat{\psi}_1$. Its only critical point seems to be that its performance worsens when γ increases (cases 4a and 4b), but this also happens with the alternative estimators.

6 An Application to Exchange Rate Data

7 Concluding Remarks

In this paper, we proposed a two-step procedure that permits the joint estimation of objective and risk-neutral parameters for stochastic volatility models. This approach uses fully the information available in the sense that it can adapt to different measure frequency for both the option and returns data it uses. This enables one to use high-frequency intraday return measures and daily option data.

The analytical expressions for the moments of integrated volatility in affine stochastic volatility models that one of the authors of this paper derived recently [32] enable us to obtain explicit expansions of the implied volatility, a crucial point in our procedure. Using these, the method is computationally simple since no simulations or numerical function inversions are involved. Moreover, the first results indicate that the method is comparable in precision to others, even though its building blocks have not been optimized.

Many points are still to be adressed. We first intend to generalize the procedure to all affine stochastic volatility models. This is a tedious computational exercice, but involves no conceptual evolution. Many statistical questions have to be looked at concerning the efficiency of the method and how it could be improved. Finally, we wish to be able to generalize the approach to Levy processes, as they were introduced in a series of recent papers by Barndorff-Nielsen and Shephard [8, **?**].

Appendix

A Computation Method and Expressions for Moments of Integrated Volatility

We are interested in the integrated volatility of the following bivariate stochastic system

$$d\begin{bmatrix} p_t \\ V_t \end{bmatrix} = \begin{bmatrix} \mu + AV_t \\ \kappa(\theta - V_t) \end{bmatrix} dt + \sqrt{V_t}\begin{bmatrix} 1 & 0 \\ \rho\gamma & \sqrt{1-\rho^2}\gamma \end{bmatrix}\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}. \tag{A.41}$$

Observing that the volatility process does not have any explicit price dependence, one can study it separately. It obeys the SDE

$$dV_t = \kappa(\theta - V_t)dt + \gamma\sqrt{V_t}\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right). \tag{A.42}$$

If $-1 \leq \rho \leq 1$, the diffusive part of this stochastic differential equation $(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t))$ is a standard brownian motion W (it has mean zero, variance one and null higher cumulants). This means that even in the presence of leverage, if one restricts his attention to volatility, the process becomes

$$dV_t = \kappa(\theta - V_t)dt + \gamma\sqrt{V_t}dW_t, \tag{A.43}$$

which will be our starting point to compute integrated volatility moments. It is known that, under such processes, the conditional expectation of the volatility is

$$\begin{aligned} \mathrm{E}\left[V_u|V(t)\right] &= \theta + e^{-\kappa(u-t)}\left(V(t)-\theta\right) \\ &\equiv \alpha_{u-t}V(t) + \beta_{u-t}, \end{aligned}$$

where we defined $\alpha_v \equiv e^{-\kappa v}$ and $\beta_v \equiv \theta(1 - e^{-\kappa v})$ [14].

We can use this result to compute the conditional expectation of the integrated volatility;

$$\begin{aligned} \mathrm{E}\left[\mathcal{V}_{t,T}|V(t)\right] &= \mathrm{E}\left[\int_t^T V_u du \middle| V(t)\right] \\ &= \int_t^T \mathrm{E}\left[V_u|V(t)\right] du \\ &= \int_t^T \left(\theta + e^{-\kappa(u-t)}\left(V(t)-\theta\right)\right) du \\ &\equiv a_{T-t}V(t) + b_{T-t}, \end{aligned}$$

with $a_{T-t} \equiv \int_t^T \alpha_{u-t}du$ and $b_{T-t} \equiv \int_t^T \beta_{u-t}du$.

A.1 Formulas for Higher Moments

In order to compute higher moments, let us consider the V_t-dependent random variable $\mathrm{E}\left[\mathcal{V}_{t,T}|V(t)\right]$:

$$\mathrm{E}\left[\mathcal{V}_{t,T}|V(t)\right] = \int_t^T \mathrm{E}_t\left[V_u\right] du. \tag{A.44}$$

If one defines $G(u,t) = \mathrm{E}_t\left[V_u\right]$, it is clear from the law of iterated expectations that $G(u,t)$ is a martingale in t. Thus, Itô's lemma implies that

$$d\mathrm{E}\left[\mathcal{V}_{t,T}|V(t)\right] = -V_t dt + a_{T-t}\gamma\sqrt{V_t}dW_t. \tag{A.45}$$

Taking integer powers and expectations on each side we obtain

$$\mathrm{E}_t\left[\left(\mathcal{V}_{t,T} - \mathrm{E}[\mathcal{V}_{t,T}]\right)^n\right] = \mathrm{E}_t\left[\left(\int_t^T a_{T-s}\sqrt{V_s}dW_s\right)^n\right]; \tag{A.46}$$

where by $\mathrm{E}_t\left[\cdot\right]$, we mean $\mathrm{E}\left[\cdot|V_t\right]$. This formula gives us a way to construct all central moments. The computation of this integral is however far from trivial. The interested reader will find in [32] details of the computation for the third and fourth central moments. We will content ourselves of giving the explicit form of the three first central moments for Feller-like stochastic volatility processes (both models considered here are in that class). The variance has the form [10]:

$$E[(\mathcal{V}_T - E(\mathcal{V}_T))^2] = A_T V + B_T \tag{A.47}$$

where

$$\begin{aligned} A_T &= \frac{2\gamma^2\left(\sinh(T\kappa) - T\kappa\right)}{e^{T\kappa}\kappa^3} \\ B_T &= \frac{\gamma^2\left(\theta + 4e^{T\kappa}\theta\left(1+T\kappa\right) + e^{2T\kappa}\left(-5\theta + 2T\theta\kappa\right)\right)}{2e^{2T\kappa}\kappa^3} \end{aligned}$$

The third central moment has the form

$$E[(\mathcal{V}_T - E(\mathcal{V}_T))^3] = M_T V + N_T \tag{A.48}$$

with:

$$\begin{aligned} M_T &= \frac{3\gamma^4\left(-1 + 2e^{3T\kappa} - 2e^{T\kappa}\left(1+2T\kappa\right) + e^{2T\kappa}\left(1 - 2T\kappa\left(1+T\kappa\right)\right)\right)}{2e^{3T\kappa}\kappa^5} \\ N_T &= \frac{\gamma^4\left(\theta + 6e^{T\kappa}\left(\theta + T\theta\kappa\right) + 2e^{3T\kappa}\left(-11\theta + 3T\theta\kappa\right) + 3e^{2T\kappa}\theta\left(5 + 2T\kappa\left(3+T\kappa\right)\right)\right)}{2e^{3T\kappa}\kappa^5} \end{aligned}$$

[10] For simplicity of notation we write the moments at time $t = 0$.

Finally, the fourth moment can be shown to be:

$$E[(\mathcal{V}_T - E(\mathcal{V}_T))^4] = Q_T V^2 + R_t V + S_T \tag{A.49}$$

with:

$$\begin{aligned}
Q_T &= \frac{12\gamma^4(-(T\kappa) + \sinh(T\kappa))^2}{e^{2T\kappa}\kappa^6} \\
R_T &= \frac{\gamma^4}{e^{4T\kappa}\kappa^7}\left(3\theta\kappa\left(-1+e^{2T\kappa}-2e^{T\kappa}T\kappa\right)\left(1+4e^{T\kappa}(1+T\kappa)+e^{2T\kappa}(-5+2T\kappa)\right)\right. \\
&\quad +\gamma^2\left(-3+15e^{4T\kappa}-12e^{2T\kappa}(1+T\kappa)(1+2T\kappa)-6e^{T\kappa}(2+3T\kappa)\right. \\
&\quad \left.\left.-2e^{3T\kappa}(-6+T\kappa(3+2T\kappa(3+T\kappa)))\right)\right) \\
S_T &= \frac{\gamma^4\theta}{4e^{4T\kappa}\kappa^7}\left(3\theta\kappa\left(1+4e^{T\kappa}(1+T\kappa)+e^{2T\kappa}(-5+2T\kappa)\right)^2\right. \\
&\quad +\gamma^2\left(3+24e^{T\kappa}(1+T\kappa)+3e^{4T\kappa}(-93+20T\kappa)+12e^{2T\kappa}(7+2T\kappa(5+2T\kappa))\right. \\
&\quad \left.\left.+8e^{3T\kappa}(21+T\kappa(27+2T\kappa(6+T\kappa)))\right)\right)
\end{aligned}$$

The explicit expressions for these moments are the ones used in the GMM estimation and option pricing expansions.

A.2 Filtration coarsening and moment conditions

We will show here how one can go from moment conditions of the type $\mathrm{E}\left[\mathcal{V}_{t,t+1}^k|\mathcal{F}_t\right]$, whose expressions are explicitely dependent on the unobservable instantaneous volatility V_t, to the conditions $\mathrm{E}\left[\mathcal{V}_{t+1,t+2}^k|\mathcal{G}_t\right]$, who depend solely on the integrated volatility process. Let us recall that the filtration $\mathcal{G}_t$ is the filtration linked to the integrated volatility process, and is thus coarser than the filtration $\mathcal{F}_t$, linked to the volatility process.

Starting from $\mathrm{E}\left[\mathcal{V}_{t+1,t+2}|\mathcal{F}_{t+1}\right] = aV_{t+1} + b$ and knowing that $\mathrm{E}\left[V_{t+1}|V_t\right] = \alpha V_t + \beta$, we have

$$\begin{aligned}
\mathrm{E}\left[\mathrm{E}\left[\mathcal{V}_{t+1,t+2}|\mathcal{F}_{t+1}\right]|\mathcal{F}_t\right] &= a\mathrm{E}\left[V_{t+1}|\mathcal{F}_t\right] + b &\text{(A.50)}\\
&= a\alpha V_t + a\beta + b. &\text{(A.51)}
\end{aligned}$$

This can be rewritten as

$$\mathrm{E}\left[\mathcal{V}_{t+1,t+2}|\mathcal{F}_t\right] = \alpha\mathrm{E}\left[\mathcal{V}_{t,t+1}|\mathcal{F}_t\right] + \beta \tag{A.52}$$

so that, by the law of iterated expectations, we have

$$\mathrm{E}\left[\mathcal{V}_{t+1,t+2}|\mathcal{G}_t\right] = \alpha\mathrm{E}\left[\mathcal{V}_{t,t+1}|\mathcal{G}_t\right] + \beta. \tag{A.53}$$

The last equation being written solely in terms of observable quantities, it can be used in a GMM estimation procedure. The same method can be applied to higher moment conditions. Starting from

$$\mathrm{Var}\left[\mathcal{V}_{t+1,t+2}|\mathcal{F}_{t+1}\right] = AV_{t+1} + B, \tag{A.54}$$

we can write

$$\begin{aligned}
\mathrm{E}\left[\mathcal{V}_{t+1,t+2}^2|\mathcal{F}_{t+1}\right] &= \mathrm{Var}\left[\mathcal{V}_{t+1,t+2}|\mathcal{F}_{t+1}\right] + \left(\mathrm{E}\left[\mathcal{V}_{t+1,t+2}|\mathcal{F}_{t+1}\right]\right)^2 \\
&= a^2V_{t+1}^2 + (2ab + A)V_{t+1} + (b^2 + B).
\end{aligned}$$

Taking expectations, we have

$$\mathrm{E}\left[\mathrm{E}\left[\mathcal{V}_{t+1,t+2}^2|\mathcal{F}_{t+1}\right]|\mathcal{V}_t\right] = a^2\mathrm{E}\left[V_{t+1}^2|\mathcal{F}_t\right] + (2ab + A)\mathrm{E}\left[V_{t+1}|\mathcal{F}_t\right] + (b^2 + B) \tag{A.55}$$

Replacing the moments of the instantaneous volatility by their explicit expressions and using the formulas we have for the first two moments of integrated volatility , we can rewrite this expression as

$$\begin{aligned}
\mathrm{E}\left[\mathcal{V}_{t+1,t+2}^2|\mathcal{F}_t\right] = {} & \alpha^2\mathrm{E}\left[\mathcal{V}_{t,t+1}|\mathcal{F}_t\right] \\
&+\frac{1}{a}\left(a^2(C + 2\alpha\beta) + (\alpha - \alpha^2)(2ab + A)\right)\mathrm{E}\left[\mathcal{V}_{t,t+1}|\mathcal{F}_t\right] \\
&-\frac{b}{a}\left(a^2(C + 2\alpha\beta) + (\alpha - \alpha^2)(2ab + A)\right) \\
&+\left(a^2(D + \beta^2) + \beta(2ab + A) + (1 - \alpha^2)(b^2 + B)\right),
\end{aligned}$$

where

$$C = \frac{\gamma^2}{\kappa}\left(e^{-\kappa} - e^{-2\kappa}\right) \tag{A.56}$$

and

$$D = \frac{\gamma^2\theta}{2\kappa} \tag{A.57}$$

Coarsening the filtration leads to the desired moment conditions. We shall stop computations here, but it clear that the other moment conditions can be computed in the same way.

B Expressions for the Cross-Moments in the Heston Model

We use the following expressions of the cross moments:

$$E[(p_{t+1} - p_t)V_{t+1}|\mathcal{F}_t] = \frac{\gamma(V_t\kappa + \theta(-1 + e^{\kappa} - \kappa))\rho}{e^{\kappa}\kappa}$$

$$E[(p_{t+1}-p_t)^2V_{t+1}|\mathcal{F}_t] = \frac{1}{2e^{2\kappa}\kappa^2}\ \big[2(-1+e^\kappa)V_t^2\kappa + 2V_t(\theta\kappa(2+e^{2\kappa}+e^\kappa(-3+\kappa))$$
$$+\gamma^2(1+e^\kappa(-1+\kappa+\kappa^2\rho^2))) + \theta(2(-1+e^\kappa)\theta\kappa(1+e^\kappa(-1+\kappa))$$
$$+\gamma^2(-1+e^{2\kappa}(1+4\rho^2) - 2e^\kappa(\kappa+2\rho^2+2\kappa\rho^2+\kappa^2\rho^2)))\big]$$

$$E[(p_{t+1}-p_t)V_{t+1}^2|\mathcal{F}_t] =$$
$$\frac{\gamma(2V_t^2\kappa^2 + (-1+e^\kappa)\theta(-1+e^\kappa-\kappa)(\gamma^2+2\theta\kappa) + V_t(\gamma^2+2\theta\kappa)(-1-2\kappa+e^\kappa(1+\kappa)))\rho}{e^{2\kappa}\kappa^2}$$

These moments – along with the first three marginal moments of V_t – are used twice: once with the spot volatilities filtered from the option prices, and once with the observed integrated volatilities. In the latter case, we need the expressions of V_t and V_t^2 in terms of $E(\mathcal{V}_{t,t+1}|\mathcal{F}_t)$ and $E(\mathcal{V}_{t,t+1}^2|\mathcal{F}_t)$. The solutions are given by:

$$V_t = \frac{E(\mathcal{V}_{t,t+1}|\mathcal{F}_t) - b}{a}$$
$$V_t^2 = \frac{Ab + ab^2 - aB - (A-2ab)E(\mathcal{V}_{t,t+1}|\mathcal{F}_t) + aE(\mathcal{V}_{t,t+1}^2|\mathcal{F}_t)}{a^3}$$

where a, b, A and B are the coefficients appearing in the conditional expectation and variance of the integrated volatility. Substituting the expression of V_t and V_{t+1} in the first cross moment above, we get:

$$E\left[(p_{t+1}-p_t)\frac{E(\mathcal{V}_{t+1,t+2}|\mathcal{F}_{t+1}) - b}{a}\middle|\mathcal{F}_t\right] = \frac{1}{e^\kappa\kappa}\left[\gamma\left(\frac{E(\mathcal{V}_{t,t+1}|\mathcal{F}_t)-b}{a}\kappa + \theta(-1+e^\kappa-\kappa)\right)\rho\right]$$

By the Law of Iterated Expectations,

$$E\left[(p_{t+1}-p_t)\frac{\mathcal{V}_{t+1,t+2} - b}{a}\middle|\mathcal{F}_t\right] = \frac{1}{e^\kappa\kappa}\left[\gamma\left(\frac{E(\mathcal{V}_{t,t+1}|\mathcal{F}_t)-b}{a}\kappa + \theta(-1+e^\kappa-\kappa)\right)\rho\right]$$

Finally, the moment conditions have to be computed by conditioning to the "discrete time" filtration $\mathcal{G}_t = \{(p_s, \mathcal{V}_{s-1,s}),\ s = t, t-1, \ldots\} \subset \mathcal{F}_t$. Iterating the expectations we get:

$$E\left[(p_{t+1}-p_t)\frac{\mathcal{V}_{t+1,t+2} - b}{a}\middle|\mathcal{G}_t\right] = \frac{1}{e^\kappa\kappa}\left[\gamma\left(\frac{E(\mathcal{V}_{t,t+1}|\mathcal{G}_t)-b}{a}\kappa + \theta(-1+e^\kappa-\kappa)\right)\rho\right]$$

An analogous procedure leads to implementable moment conditions derived from the expressions of $E[(p_{t+1}-p_t)^2V_{t+1}|\mathcal{F}_t]$ and $E[(p_{t+1}-p_t)V_{t+1}^2|\mathcal{F}_t]$ above.

C Option prices and implied volatility series expansions

In this first appendix, we present the explicit series expansions for two well-known option models: the Hull-White and Heston models. While Hull and White's model could be seen as the zero-leverage limit of Heston's model, its formulation in terms of an integral over the integrated volatility conditional distribution simplifies greatly computations, especially since we have analytical expressions for the integrated volatility conditional moments.

C.1 Hull and White's model

It is widely known, since the pioneering work of Hull and White, that for the class of stochastic volatility models we consider, if there is **no leverage**, the price of an o ption is the expectation value, over the integrated volatility risk-neutral distribution, of the Black and Scholes price with volatility equal to the integrated volatility. Coarsely, an European call of maturity $T-t$ and strike price K on an asset valued S at time t has a price

$$c_{HW}(S,K,t,T,V(t)) = \int dP^{\mathcal{V}(t,T)} c_{BS}(S,K,t,T,\frac{1}{T-t}\mathcal{V}(t,T)), \tag{C.58}$$

where $P^{\mathcal{V}(t,T)}$ is the conditional density of the integrated volatility between t and T knowing the instantaneous volatility at time t.

To be more concise, we shall denote the Black-Scholes price on an asset of volatility V by $c_{BS}(V)$. The integration over the distribution of volatilities can be approximated by expanding the integrated volatility around its mean. To do so, one needs analytical expressions for the integrated volatility's central moments, which are given in appendix B, and the sensibilities of the Black and Scholes prices to a change in volatility. These sensibilities can be computed without much effort; computations are shown at the end of this section. Using these, we can extend Hull and White's result and show that $c_{HW}(V(t))$ (we omit the extra parameters) can be written as (in order to simplify notations, we will write $\overline{\mathcal{V}} \equiv \frac{1}{T-t}\mathrm{E}_t\mathcal{V}_{t,T}$, and $\mathcal{V}$ for $\frac{1}{T-t}\mathcal{V}_{t,T}$)).

$$\begin{aligned}
c_{HW}(V(t)) &= c_{BS}(\overline{\mathcal{V}}) + \frac{1}{2}\frac{\partial^2 c_{BS}(x)}{\partial x^2}\bigg|_{\overline{\mathcal{V}}} \mathrm{Var}(\mathcal{V}) + \frac{1}{6}\frac{\partial^3 c_{BS}(x)}{\partial x^3}\bigg|_{\overline{\mathcal{V}}} \mathrm{E}_t\left[(\mathcal{V}-\bar{\mathcal{V}})^3\right] + \cdots \\
&= c_{BS}(\overline{\mathcal{V}}) + \frac{1}{2}\frac{S\sqrt{T-t}\Phi'(d_1)(d_1d_2-1)}{4(\overline{\mathcal{V}})^{3/2}}\mathrm{Var}(\mathcal{V}) + \\
&\quad + \frac{1}{6}\frac{S\sqrt{T-t}\Phi'(d_1)\left((d_1d_2-3)(d_1d_2-1)-(d_1^2+d_2^2)\right)}{8(\overline{\mathcal{V}})^{5/2}}\mathrm{E}_t\left[(\mathcal{V}-\bar{\mathcal{V}})^3\right] \\
&\quad + \frac{1}{24}\frac{S\sqrt{T-t}\Phi'(d_1)}{16(\overline{\mathcal{V}})^{\frac{7}{2}}} \times \\
&\quad\quad \big((d_1d_2-5)(d_1d_2-3)(d_1d_2-1)+4d_1d_2- \\
&\quad\quad\quad (d_1^2+d_2^2)(3d_1d_2-9)\big)\, \mathrm{E}_t\left[(\mathcal{V}-\bar{\mathcal{V}})^4\right],
\end{aligned}$$

where we recall that

$$d_1 = \frac{\log(S/K) + \left(r+\frac{\overline{\mathcal{V}}}{2}\right)(T-t)}{\sqrt{\overline{\mathcal{V}}(T-t)}}, \tag{C.59}$$

$$d_2 = d_1 - \sqrt{\overline{\mathcal{V}}(T-t)}, \tag{C.60}$$

$$c_{BS}(\overline{\mathcal{V}}) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \tag{C.61}$$

and $\Phi(\cdot)$ denotes the cumulative function of a standard normal distribution. This gives us a closed form series expansion for option prices. The explicit expressions for the moments of integrated volatility are given in Appendix B. As can be seen there, all moments can be organized as polynomials in γ. If one collects powers of γ, the option price expansion can be reorganized as

$$c_{HW}(V(t)) = c_{BS}(\overline{\mathcal{V}}) + c_{HW}^{(2)}(V(t))\,\gamma^2 + c_{HW}^{(4)}(V(t))\,\gamma^4 + c_{HW}^{(6)}(V(t))\,\gamma^6 + \mathcal{O}(\gamma^8), \tag{C.62}$$

where

$$c_{HW}^{(2)}(V(t)) = \frac{1}{2}\frac{S\sqrt{T-t}\Phi'(d_1)(d_1d_2-1)}{4(\overline{\mathcal{V}})^{3/2}}(A_{T-t}V(t)+B_{T-t}) \tag{C.63}$$

$$c_{HW}^{(4)}(V(t)) = \frac{1}{6}\frac{S\sqrt{T-t}\Phi'(d_1)\left((d_1d_2-3)(d_1d_2-1)-(d_1^2+d_2^2)\right)}{8(\overline{\mathcal{V}})^{5/2}}(M_{T-t}V(t)+N_{T-t}) \tag{C.64}$$

$$+\frac{1}{24}\frac{S\sqrt{T-t}\Phi'(d_1)}{16(\overline{\mathcal{V}})^{\frac{7}{2}}}\left((d_1d_2-5)(d_1d_2-3)(d_1d_2-1)+4d_1d_2-\right. \tag{C.65}$$

$$\left.(d_1^2+d_2^2)(3d_1d_2-9)\right)Q_{T-t}V(t)^2 \tag{C.66}$$

$$c_{HW}^{(6)}(V(t)) = \frac{1}{24}\frac{S\sqrt{T-t}\Phi'(d_1)}{16(\overline{\mathcal{V}})^{\frac{7}{2}}}\left((d_1d_2-5)(d_1d_2-3)(d_1d_2-1)+4d_1d_2-\right. \tag{C.67}$$

$$\left.(d_1^2+d_2^2)(3d_1d_2-9)\right)(R_{T-t}V(t)+S_{T-t}) \tag{C.68}$$

In the notations of the second appendix, $\overline{\mathcal{V}} = a_{T-t}V(t) + b_{T-t}$. Since the coefficients $A_{T-t}, B_{T-t}, M_{T-t}, N_{T-t}, Q_{T-t}, R_{T-t}, S_{T-t}, a_{T-t}$ and b_{T-t} do not depend either on γ or $V(t)$, it will be simple to perform the series inversion. Note however that d_1 and d_2 do depend on $V(t)$.

We want to identify the coefficients ν_j of the implied volatility up to order γ^6. Recall that the expansion is defined as

$$V_t^{imp}(S,K,r,T,c_{obs},\beta,\lambda) = \sum_{j=0}^{\infty}\nu_j(S,K,r,T,c_{obs},\beta(\cdot,0),\lambda)\gamma^j. \tag{C.69}$$

As we pointed out before, if one inserts the expansion (C.69) in (C.62) and further expands in powers of γ, the coefficients ν_j can be identified by restricting the equation to be satisfied order by order in γ. The zeroth order condition is thus

$$c_{obs} = c_{BS}(V_{BS}^{imp}) = c_{BS}(a_{T-t}\nu_0 + b_{T-t}). \tag{C.70}$$

This means that, as we announced earlier,

$$\nu_0 = \frac{V_{BS}^{imp} - b_{T-t}}{a_{T-t}}. \tag{C.71}$$

The expressions being heavy, we will only show how one can extract the second order in gamma contribution. This can be done by truncating the series (C.69) to

$$V_t^{imp}(S,K,r,T,c_{obs},\beta,\lambda) = \frac{V_{BS}^{imp} - b_{T-t}}{a_{T-t}} + \nu_1\gamma + \nu_2\gamma^2 + \mathcal{O}(\gamma^3) \tag{C.72}$$

If one expands the coefficient $c_{HW}^{(2)}(V_t^{imp})\gamma^2$ in powers of γ and retains terms up to the second order, he obtains

$$c_{HW}^{(2)}(V_t^{imp})\gamma^2 = \frac{\exp\left(\frac{\left(\frac{V_{BS}^{imp}(T-t)}{2}+2Log[\frac{S}{K}]\right)^2}{8V_{BS}^{imp}(T-t)}\right)}{32a_{T-t}\sqrt{2\pi(T-t)}(V_{BS}^{imp})^{5/2}} \times \tag{C.73}$$

$$\Bigg(S(-a_{T-t}B_{T-t} + A_{T-t}(b_{T-t} - V_{BS}^{imp})) \times \tag{C.74}$$

$$\left(V_{BS}^{imp}(T-t)(4 + V_{BS}^{imp}(T-t)) - 4\log\left[\frac{S}{K}\right]^2\right)\Bigg)\gamma^2 \tag{C.75}$$

$$+\mathcal{O}(\gamma^3). \tag{C.76}$$

Expanding in the same way $c_{BS}(\overline{\mathcal{V}})$ gives

$$c_{BS}(\overline{\mathcal{V}}) = c_{BS}(V_{BS}^{imp}) + \frac{a_{T-t}e^{-\frac{V_{BS}^{imp}(T-t)}{8}-\frac{\log\left[\frac{S}{K}\right]^2}{2V_{BS}^{imp}(T-t)}}KT\sqrt{\frac{S}{K}}}{2\sqrt{2\pi V_{BS}^{imp}(T-t)}}\nu_2\gamma^2 + \mathcal{O}(\gamma^3). \tag{C.77}$$

The condition

$$c_{obs} = c_{BS}(V_{BS}^{imp}) = c_{BS}(\overline{\mathcal{V}}) + c_{HW}^{(2)}(V_t^{imp})\gamma^2 \tag{C.78}$$

will then be satisfied only if $\nu_1 = 0$ and

$$\nu_2 = \frac{\left(a_{T-t}B_{T-t} + A_{T-t}(V_{BS}^{imp} - b_{T-t})\right)\left(V_{BS}^{imp}(T-t)(4 + V_{BS}^{imp}(T-t)) - 4\log\left[\frac{S}{K}\right]^2\right)}{16a_{T-t}^2(V_{BS}^{imp})^2(T-t)}. \tag{C.79}$$

This gives us the inversion up to the second order in γ^4, since one can show that $\nu_j = 0$ for all odd j's. The inversion could be performed for higher orders, but expressions quickly become heavy so that we will not report results here; when considering explicit examples, the coefficient ν_4 was also used.

C.2 Sensibilities of the Black-Scholes call price to volatility

In this subsection, we show how one can compute the successive derivatives of the Black-Scholes formula with respect to the volatility. Giving us the terms appearing in the Hull and White formula, it enables us to go one step further and thus to take into account the higher moments of the integrated volatility.

Recall that

$$c_{BS}(V) = S\Phi(d_1) - Ke^{-rt}\Phi(d_2), \tag{C.80}$$

where

$$d_1 = \frac{\log(S/K) + \left(r + \frac{V}{2}\right)t}{\sqrt{Vt}}, \tag{C.81}$$

$$d_2 = d_1 - \sqrt{Vt}. \tag{C.82}$$

We will first derive some useful identities. Deriving explicitely d_1 and d_2 with respect to V we obtain

$$\frac{\partial d_1}{\partial V} = -\frac{1}{2V}d_2 \tag{C.83}$$

$$\frac{\partial d_2}{\partial V} = -\frac{1}{2V}d_1. \tag{C.84}$$

Also, since $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy$, we have

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{C.85}$$

$$\Phi''(x) = -x\Phi'(x). \tag{C.86}$$

The surprisingly simple form of the derivatives will enable us to find simple expressions for the derivatives of $c_{BS}(V,t)$. The first one is readily computable:

$$\begin{aligned}\frac{\partial c_{BS}(V)}{\partial V} &= S\Phi'(d_1)\frac{\partial d_1}{\partial V} - Ke^{-rt}\Phi'(d_2)\frac{\partial d_2}{\partial V} \\ &= -\frac{1}{2V}\left(S\Phi'(d_1)d_2 - Ke^{-rt}\Phi'(d_2)d_1\right)\end{aligned}$$

To simplify this further, let's observe that, using the definition of d_2 in terms of d_1, we have

$$\begin{aligned}\Phi'(d_2) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} = \Phi'(d_1)\exp(\log\frac{S}{K} + rt) \\ &= \frac{S}{Ke^{-rt}}\Phi'(d_1)\end{aligned}$$

So that

$$\begin{aligned}\frac{\partial c_{BS}(V)}{\partial V} &= -\frac{1}{2V}S\Phi'(d_1)\left(d_2 - d_1\right) \\ &= \frac{1}{2V}S\Phi'(d_1)\sqrt{Vt} \\ &= \frac{S\Phi'(d_1)\sqrt{t}}{2\sqrt{V}}.\end{aligned}$$

Getting a simple form for the first derivatives enables us to derive the following ones easily. For the second derivative:

$$\begin{aligned}\frac{\partial^2 c_{BS}(V)}{\partial V^2} &= \frac{S\sqrt{t}}{2}\left(\frac{-1}{2V^{\frac{3}{2}}}\Phi'(d_1) + \Phi''(d_1)\frac{-1}{2V^{\frac{3}{2}}}d_2\right) \\ &= \frac{S\sqrt{t}\Phi'(d_1)}{4V^{\frac{3}{2}}}\left(d_1 d_2 - 1\right),\end{aligned}$$

which is the expression appearing in Hull and White's formula. The third one is computed similarly;

$$\frac{\partial^3 c_{BS}(V)}{\partial V^3} = \frac{S\sqrt{t}\Phi'(d_1)}{8V^{\frac{5}{2}}}\left\{((d_1 d_2 - 3)(d_1 d_2 - 1) - (d_1^2 + d_2^2)\right\},$$

One can also compute the fourth derivative

$$\begin{aligned}\frac{\partial^4 c_{BS}(V)}{\partial V^4} &= \frac{S\sqrt{t}\Phi'(d_1)}{16V^{\frac{7}{2}}}\{(d_1 d_2 - 5)(d_1 d_2 - 3)(d_1 d_2 - 1) + 4d_1 d_2 \\ &\qquad -(d_1^2 + d_2^2)(3d_1 d_2 - 9)\}.\end{aligned}$$

This permits us to go steps further in the series development and take into account the third and fourth central moments of the integrated volatility.

C.3 Heston's Model

The price of an European call option in Heston's model can also be written as an expectation value of the Black-Scholes option pricing under the stochastic volatility distribution. Romano and Touzi (1997) have shown that, under the stochastic volatility process:

$$d\begin{bmatrix} S_t \\ V_t \end{bmatrix} = \begin{bmatrix} \mu_t S_t \\ \kappa(\theta - V_t) \end{bmatrix} dt + \sqrt{V_t}\begin{bmatrix} S_t & 0 \\ \gamma\rho & \gamma\sqrt{1-\rho^2} \end{bmatrix}\begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}, \tag{C.87}$$

the call option price $c(S_0, V_0, T)$ can be written as

$$c(S_0, V_0, T) = \mathrm{E}\left[c_{BS}(S_0 \Omega_T(V_0), \bar{V}_T(V_0)\right], \tag{C.88}$$

where

$$\Omega_T(V_0) = \exp\left(\rho \int_0^T \sqrt{V}_T dW_t^2 - \frac{1}{2}\rho^2 \int_0^T V_t dt\right) \tag{C.89}$$

$$\bar{V}_T \equiv \frac{1}{T}(1-\rho^2)\int_0^T V_t dt. \tag{C.90}$$

Using simulations, this formula provides a quite rapid and precise method to compute option prices. Moreover, one can show how the inverse Fourier transform method and this formulation of Heston's model are linked. Performing a numerical inversion of the transform is indeed nothing more than integrating over the distribution of some suitably constructed variable, which can be shown to be linked to integrated volatility and the Girsanov-like initial price.

It is straightforward to show (using Girsanov theorem and simple stochastic calculus) that

$$\mathrm{E}\left[\Omega_T(V_0)\right] = 1 \tag{C.91}$$

and

$$\mathrm{E}\left[\bar{V}_T(V_0)\right] = (1-\rho^2)\left(\theta + (V_0-\theta)\frac{1-e^{-\kappa T}}{\kappa T}\right) \equiv \bar{V}_T^*. \tag{C.92}$$

The latter is nothing but $(1-\rho^2)$ times the mean integrated volatility. The identification of these mean values enables us to expand the option pricing formula around these:

$$c(S_0,V_0,T) = \sum_{p,q=0}^{\infty} \frac{1}{p!q!} \frac{\partial^{p+q} c_{BS}}{\partial V^p}\bigg|_{S_0,\bar{V}_T^*} S_0^q \mathrm{E}\left[(\Omega_T-1)^q(\bar{V}_T-\bar{V}_T^*)^p\right] \tag{C.93}$$

Defining

$$R^{(p,q)}(S,V,T) \equiv \frac{\left(\frac{\partial}{\partial V}\right)^p \left(S\frac{\partial}{\partial S}\right)^q c_{BS}}{\frac{\partial c_{BS}}{\partial V}}, \tag{C.94}$$

this can be rewritten as

$$c(S_0,V_0,T) = \sum_{p,q=0}^{\infty} \frac{R^{(p,q)}(S_0,\bar{V}_T^*,T)}{p!q!} \mathrm{E}\left[(\Omega_T-1)^q(\bar{V}_T-\bar{V}_T^*)^p\right] \partial c_{BS}\partial V\bigg|_{S_0,\bar{V}_T^*}. \tag{C.95}$$

In terms of the moneyness

$$X \equiv \log[\frac{S}{Ke^{-rT}}], \tag{C.96}$$

and the mean integrated volatility

$$Z \equiv \frac{1}{(1-\rho^2)}\bar{V}_T^*, \tag{C.97}$$

the first values of $R^{(p,q)}$ are the following:

$$\begin{aligned}
R^{(1,0)} &= 1 \\
R^{(2,0)} &= T\left(\frac{X^2}{2Z^2} - \frac{1}{2Z} - \frac{1}{8}\right) \\
R^{(1,1)} &= \left(-\frac{X}{Z} + \frac{1}{2}\right) \\
R^{(1,2)} &= \left(\frac{X^2}{Z^2} - \frac{X}{Z} - \frac{1}{4Z}(4-Z)\right) \\
R^{(2,2)} &= T\left(\frac{1}{2}\frac{X^4}{Z^4} - \frac{1}{2}\frac{X^3}{Z^3} - 3\frac{X^2}{Z^3} + \frac{1}{8}\frac{X}{Z^2}(12+Z) + \frac{1}{32}\frac{1}{Z^2}(48-Z^2)\right).
\end{aligned}$$

Using these expressions, one can solve for implied volatility with a second-order expansion:

$$\begin{aligned}V_{BS}^{imp}(S,V,T) &= v(V,T)+\frac{\gamma}{\tau}J_1(\kappa,\theta,\rho,V,T)R^{(1,1)}\\&+\gamma^2\left(\frac{1}{\tau^2}J_3(\kappa,\theta,\rho,V,T)R^{(2,0)}\right.\\&+\frac{1}{\tau}J_4(\kappa,\theta,\rho,V,T)R^{(1,2)}\\&\left.+\frac{1}{2\tau^2}\left(J_1(\kappa,\theta,\rho,V,T)\right)^2\left[R^{(2,2)}-(R^{(1,1)})^2R^{(2,0)}\right]\right)+\mathcal{O}(\gamma^3)\end{aligned}$$

where the expressions for the J are given by:

$$J_1(\kappa,\theta,\rho,V,T)=\begin{cases}\frac{1}{2}\rho VT^2, & \text{if } \kappa=0\\ \frac{\rho}{\kappa^2}\left(\left(\theta+(1+\kappa T)(\theta-V(0))\right)e^{-\kappa T}+(\kappa T-2)\,\theta+V(0)\right), & \text{if not}\end{cases} \tag{C.98}$$

$$J_2(\kappa,\theta,\rho,V,T)=0 \tag{C.99}$$

$$J_3(\kappa,\theta,\rho,V,T)=\begin{cases}\frac{1}{6}VT^3, & \text{if } \kappa=0\\ \frac{\left(\theta-2V+e^{2\kappa T}((-5+2\kappa T)\theta+2V)+4e^{\kappa T}(\theta+\kappa T\theta-\kappa TV)\right)}{4e^{2\kappa T}\kappa^3}, & \text{if not}\end{cases} \tag{C.100}$$

$$J_4(\kappa,\theta,\rho,V,T)=\begin{cases}\frac{1}{6}\rho^2VT^3 & \text{if } \kappa=0\\ \frac{\rho^2((6+2e^{\kappa T}(-3+\kappa T)+\kappa T(4+\kappa T))\theta+(-2+2e^{\kappa T}-\kappa T(2+\kappa T))V)}{2e^{\kappa T}\kappa^3} & \text{if not}\end{cases} \tag{C.101}$$

In order to be able to extract an Heston-implied volatility from the observable Black-Scholes implied volatility. Namely, we will identify the coefficient λ_i such that

$$V=\lambda_0(S,V_{BS}^{imp},T)+\lambda_1(S,V_{BS}^{imp},T)\gamma+\lambda_2(S,V_{BS}^{imp},T)\gamma^2+\mathcal{O}(\gamma^3). \tag{C.102}$$

Such a development makes sense, because if γ goes to zero, the Heston model becomes a deterministic volatility model (in fact, if κ goes to zero, we recover the Black-Scholes model). Computing the coefficients λ_0, λ_1 and λ_2 is a straightforward but tedious exercice. One inserts the series expansion for V in the series expansion for V_{BS}^{imp} and equates the powers of γ.

The coefficients $\lambda_0, \lambda_1, \lambda_2$ of the inverse series for the spot volatility $V_t=\lambda_0+\lambda_1\gamma+\lambda_2\gamma^2$ are finally given by:

$$\lambda_0=\frac{-\theta+e^{\kappa\tau}\,\theta+e^{\kappa\tau}\,V_{BS}^{imp}\,\kappa\,\tau-e^{\kappa\tau}\,\theta\,\kappa\,\tau}{-1+e^{\kappa\tau}}$$

$$\lambda_1 = \frac{-(\rho\,(2\,X - V_{BS}^{imp}\,\tau)\,(-(e^{\kappa\tau}\,V_{BS}^{imp}\,\kappa\,\tau\,(-1+e^{\kappa\tau}-\kappa\,\tau)) + \theta\,(1+e^{2\,\kappa\,\tau} - e^{\kappa\tau}\,(2+\kappa^2\,\tau^2))))}{2\,(-1+e^{\kappa\tau})^2\,V_{BS}^{imp}\,\kappa\,\tau}$$

$$\begin{aligned}
\lambda_2 \;=\; & \frac{1}{32\,e^{\kappa\tau}\,(-1+e^{\kappa\tau})^3\,V_{BS}^{imp^3}\,\kappa^3\,\tau^3}\;\Big[-16\,e^{\kappa\tau}\,V_{BS}^{imp^2}\,X\,\kappa\,\rho^2\,\tau^2\,(e^{\kappa\tau}\,V_{BS}^{imp}\,\kappa^2\,\tau^2\,(2+\kappa\,\tau \\
& +e^{\kappa\tau}\,(-2+\kappa\,\tau)) + \theta\,(-2+2\,e^{3\,\kappa\,\tau}+e^{\kappa\tau}\,(6-\kappa^3\,\tau^3) - e^{2\,\kappa\,\tau}\,(6+\kappa^3\,\tau^3))) \\
& +4\,X^2\,(2\,\theta^2\,\rho^2\,(1+e^{2\,\kappa\,\tau}-e^{\kappa\tau}\,(2+\kappa^2\,\tau^2))^2 + 2\,e^{\kappa\tau}\,V_{BS}^{imp^2}\,\kappa^2\,\tau^2\,(-1+e^{3\,\kappa\,\tau}\,(-1+\rho^2) \\
& +e^{2\,\kappa\,\tau}\,(1+2\,\kappa\,\tau+2\,\rho^2\,(-1-3\,\kappa\,\tau+\kappa^2\,\tau^2)) + e^{\kappa\tau}\,(1-2\,\kappa\,\tau+\rho^2\,(1+6\,\kappa\,\tau+3\,\kappa^2\,\tau^2))) \\
& +V_{BS}^{imp}\,\theta\,\kappa\,\tau\,(1+e^{4\,\kappa\,\tau}\,(3+4\,\rho^2) + 2\,e^{\kappa\tau}\,(-3+\kappa\,\tau+2\,\rho^2\,(-1+\kappa\,\tau)) \\
& -2\,e^{3\,\kappa\,\tau}\,(5-\kappa\,\tau+2\,\kappa^2\,\tau^2+2\,\rho^2\,(3-\kappa\,\tau-\kappa^2\,\tau^2+\kappa^3\,\tau^3)) \\
& -4\,e^{2\,\kappa\,\tau}\,(-3+\kappa\,\tau-\kappa^2\,\tau^2+\rho^2\,(-3+2\,\kappa\,\tau+\kappa^2\,\tau^2+2\,\kappa^3\,\tau^3)))) \\
& +V_{BS}^{imp}\,\tau\,(-2\,\theta^2\,\rho^2\,(12+V_{BS}^{imp}\,\tau)\,(1+e^{2\,\kappa\,\tau}-e^{\kappa\tau}\,(2+\kappa^2\,\tau^2))^2 \\
& -2\,e^{\kappa\tau}\,V_{BS}^{imp^2}\,\kappa^2\,\tau^2\,(-4-V_{BS}^{imp}\,\tau+e^{3\,\kappa\,\tau}\,(-4-V_{BS}^{imp}\,\tau+\rho^2\,(-4+V_{BS}^{imp}\,\tau)) \\
& +e^{2\,\kappa\,\tau}\,((4+V_{BS}^{imp}\,\tau)\,(1+2\,\kappa\,\tau) - 2\,\rho^2\,(-4+V_{BS}^{imp}\,\tau)\,(1-\kappa\,\tau+\kappa^2\,\tau^2)) \\
& +e^{\kappa\tau}\,((-4-V_{BS}^{imp}\,\tau)\,(-1+2\,\kappa\,\tau) - \rho^2\,(-4+V_{BS}^{imp}\,\tau)\,(-1+2\,\kappa\,\tau+\kappa^2\,\tau^2))) \\
& -V_{BS}^{imp}\,\theta\,\kappa\,\tau\,(4+V_{BS}^{imp}\,\tau+e^{4\,\kappa\,\tau}\,(3\,(4+V_{BS}^{imp}\,\tau) - 4\,\rho^2\,(-4+3\,V_{BS}^{imp}\,\tau)) \\
& +2\,e^{\kappa\tau}\,((4+V_{BS}^{imp}\,\tau)\,(-3+\kappa\,\tau) + 2\,\rho^2\,(-4+4\,\kappa\,\tau+V_{BS}^{imp}\,\tau\,(3+\kappa\,\tau))) \\
& +2\,e^{3\,\kappa\,\tau}\,((-4-V_{BS}^{imp}\,\tau)\,(5-\kappa\,\tau+2\,\kappa^2\,\tau^2) + 2\,\rho^2\,(4\,(-3+\kappa\,\tau+\kappa^2\,\tau^2-\kappa^3\,\tau^3) \\
& +V_{BS}^{imp}\,\tau\,(9+\kappa\,\tau+\kappa^2\,\tau^2+\kappa^3\,\tau^3))) - 4\,e^{2\,\kappa\,\tau}\,((-4-V_{BS}^{imp}\,\tau)\,(3-\kappa\,\tau+\kappa^2\,\tau^2) \\
& +\rho^2\,(V_{BS}^{imp}\,\tau\,(9+2\,\kappa\,\tau+\kappa^2\,\tau^2) + 4\,(-3+2\,\kappa\,\tau+\kappa^2\,\tau^2+2\,\kappa^3\,\tau^3))))))\Big]
\end{aligned}$$

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