Learning and Price Volatility in Duopoly Models of Resource Depletion*

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Abstract

The aim of this paper is to provide a theoretical model that can account for price fluctuations in depletable resource markets. We do so by introducing learning into a Hotelling-style duopoly model of optimal resource depletion. Before the depletable resource becomes scarce, the self-confirming equilibrium of the model mirrors noncooperative rational expectations equilibrium in that supply is high and price is low, although learning does induce occasional upward price spikes that are followed by a long period of falling price. Once the depletable resource becomes scarce the dynamics of the market change significantly, with the self-confirming equilibrium mirroring the cooperative rather than noncooperative rational expectations equilibrium. Supply is low and price is high. Price spikes still occur but against a background of increasing prices. When

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scarcity is sufficient then the market is permanently in a self-confirming equilibrium that is equivalent to cooperative rational expectations equilibrium.

1 Introduction

Depletable or nonrenewable resources are by definition[1] characterised by a finite stock and so likely to be in the hands of a small number of suppliers. Many real world examples of depletable resources are indeed characterised by significant use of market power e.g oil, diamonds, bauxite, uranium, mercury and copper. Therefore any attempt to explain the dynamics of resource depletion should take into account *both* scarcity and market power. Indeed, the classic contribution of Hotelling (1931) describes the optimal depletion plan when a resource in fixed supply is controlled by a monopolist. With the rising influence of OPEC in the 1970s monopoly models of depletable resource markets became the subject of further analysis (see, *inter alia,* Dasgupta and Heal (1974), Stiglitz (1976) and Stiglitz and Dasgupta (1982)). However properly understanding price dynamics in these markets clearly requires understanding the interplay between a limited number of producers and so the literature also developed to consider the case of oligopoly models (see Salant (1976), Lewis and Schmalensee (1983), Loury (1986)). The focus of this research program however remained capturing the asymptotic dynamics that was the focus of Hotelling (1931) and understanding how changes in industry structure would affect these dynamics. This literature did not try and contribute to explaining higher frequency shifts in commodity prices, as in Deaton and Laroque (1992) and (1996) and Chambers and Bailey (1996).

The introduction of oligopoly into depletable resource markets introduces potentially rich market dynamics. For instance, Newbery (1980) shows that in the context of a market characterised by a strategic leader and a competitive fringe the stated policies of the market leader may be time inconsistent[2]. This result reveals the general point that in the intertemporal

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[1] See Sweeney (1993) for a definition of a depletable resource. Key characteristics are that the stock increases when the resource is used, it never decreases and no use is possible without a positive stock.

[2] Although see Salo and Tahvonen (2001) and Groot, Withagen and de Zeeuw (2003) for time consistent

setting that characterises a depletable resource market agents' *perception*s of current and future market conditions are critical in influencing the equilibrium. Further depletable resource markets are characterised by two key uncertainties - uncertainty over the relative scarcity of the commodity, e.g what level of stock is still left to be depleted? What are the prospects for increasing resource stocks through new discoveries? and also uncertainty over each supplier's market power, e.g How much resource does the supplier currently hold and how much does it think is held by its competitors? How sensitive is price to quantity supplied? These questions are subject to large uncertainties to that it is not unreasonable to expect perceptions to sometimes depart from reality leading to the possibility that the dynamics of resource depletion and commodity price fluctuations do not always reflect economic fundamentals but instead the shifting perceptions of agents.

The aim of this paper is to bring perceptions and their formation to the forefront in an otherwise standard duopoly model of resource depletion. We examine whether it is possible that changing perceptions of agents can interact with the low frequency dynamics of Hotelling to generate large shifts in commodity prices that are not connected with actual fundamentals. We do this by requiring suppliers to *learn* about the economic environment in which they operate, and show how learning introduces an additional layer of dynamics to depletable resource markets. Learning dynamics are far from trivial in our model and are easily as important as fundamentals as a driving force for price and quantities in the short and medium term. Furthermore, learning is consistent with features of real-world depletable resource markets that fundamentals find hard to explain, such as occasional rapid price increases or intermittent periods of apparent collusion between suppliers. To derive our results we allow each supplier to learn about the economic environment by estimating and updating a simple approximating

versions of these models.

model of market dynamics. The natural limit of this learning process leads suppliers into a self-confirming equilibrium of the type described by Sargent (1999). The equilibrium is self-confirming in the sense that neither supplier has an incentive to change its behaviour.

The first result we derive is that the nature of the self-confirming equilibrium of the model changes with the level of resource scarcity. At low levels of scarcity, the learning process takes suppliers to a self confirming equilibrium that is exactly the same as the non-cooperative rational expectations equilibrium of the model. Supply is high and price is low, just as it would be if suppliers did not cooperate and had rational expectations. In contrast, at high levels of scarcity the learning process leads to a self-confirming equilibrium that exactly equates to the cooperative rational expectations equilibrium of the model. Supply is low and price is high, in an outcome that replicates the case of collusion amongst suppliers colluded with rational expectations. We therefore see learning acting as a coordination device in a duopoly model of resource depletion. Intuitively, coordination begins when scarcity first gives both suppliers an incentive to reduce supply and the price of the depletable resource rises. In our learning environment, each supplier mistakenly interprets the rise in price as signalling an increase in the amount of market power it holds. The perceived increase in market power creates incentives for both suppliers to further reduce supply, a process that continues until supply converges to levels consistent with cooperative equilibrium. Our first result suggests that scarcity causes price to rise not just because it implies falling supply *per se* but also because it coordinates suppliers on a self-confirming equilibrium that corresponds to a cooperative outcome. To the casual observer it appears that scarcity causes suppliers to start colluding even though no such collusion is occurring.

The second result of our paper is that learning sometimes causes the duopoly model to deviate from self-confirming equilibrium in a significant and well-defined manner. During

these escape episodes, the price of the depletable resource initially rises rapidly to a level consistent with cooperative rational expectations equilibrium and then gradually falls back to a level consistent with non-cooperative rational expectations equilibrium. Learning therefore creates occasional upward price spikes that are followed by prolonged periods of falling price. The escape itself is triggered by a rare combination of shocks that cause both suppliers to inadvertently reduce supply at the same time. The large uncertainties in the learning environment mean that each supplier at least partially interprets the resulting price rise as a signal of increased market power, in which case they both have an incentive to reduce supply even further and the escape becomes self-fulfilling. The limit of the process is the price and supply combination that characterises cooperative rational expectations equilibrium. Looking from outside the model, the escape episode resembles an outbreak of collusion between suppliers. Supply is first restricted and then expanded in a way that mirror those we would expect during the set up and collapse of a collusive agreement.

Our final contribution is to show that scarcity interacts with escape dynamics in economically meaningful ways. Escapes by definition happen from the self-confirming equilibrium to a point equivalent to cooperative rational expectations equilibrium, so by our first result escapes become less dramatic as scarcity forces the self-confirming equilibrium closer to the cooperative rational expectations equilibrium. Scarcity also increases the probability and duration of an escape because suppliers are already partially coordinated by the learning mechanism. The rare combination of shocks needed to trigger an escape is more likely when both suppliers are reducing supply due to scarcity, but return to self-confirming equilibrium is slower so the escape episode lasts longer. The combination of increased probability of an escape and slower return to self-confirming equilibrium after an escape means that the market spends progressively more time away from self-confirming equilibrium as scarcity becomes more important.

As a consequence, our model predicts that scarcity brings permanently higher prices and an end to upward price spikes caused by escape dynamics.

The remainder of this paper is organised as follows: In Section 2 we describe the mechanism determining price in our duopoly model of resource depletion and show how suppliers form and base their decisions on a perceived model of the market. Section 3 derives the self-confirming equilibrium of our model, and shows how scarcity causes it to change from being the noncooperative to being the cooperative rational expectations equilibrium. The escape dynamic properties of the model are analysed in Section 4, which uses stochastic approximation techniques to calculate how the model's dominant escape path is affected by resource scarcity. A final Section 5 concludes.

2 Model specification

In this section we outline our core model of a depletable resource market characterised by duopoly. We intend this as a general model to capture our key insights rather than a specific model matched to any particular commodity market[3]. Naturally the specific implications of our analysis will vary depending on the precise market setting but a general duopoly model is all that is required to show the potential for learning and shifting perceptions to provide rich additional dynamics over and above the role of fundamentals.

2.1 Determination of market price

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[3] Although see Ellison and Scott (2007) for a specific application of this approach to the crude oil market.

The price p_t of a homogenous depletable resource is assumed to depend on total market supply through a linear demand curve:

$$p_t = a - b\sum_{i=1}^{2} q_{it}, \qquad (2.1)$$

where $a > 0$ and $b > 0$ are structural parameters. The two suppliers $i = 1, 2$ are assumed to have imperfect control over the quantity of depletable resource they supply to the market. Each supplier sets intended supply $\hat{q}_{it}$ but makes a control error W_{it} so market supply q_{it} is given by:

$$q_{it} = \hat{q}_{it} + W_{it}, \qquad (2.2)$$

where W_{it} is i.i.d. with mean zero and standard deviation σ. Control errors are uncorrelated across suppliers and have sufficiently bounded support to ensure that supply is always positive.

2.2   Supplier perceptions

The two suppliers in our model know that market price is determined by the linear demand curve (2.1) but have to learn the values of the structural parameters a and b. Furthermore, each supplier is assumed to only observe its own supply to the market. The latter assumption means a supplier has to impute the supply of its competitor before it can learn about the structural parameters. In the market, supply arises as the solution to a dynamic optimisation problem constrained by perceptions of market power and scarcity. This makes it difficult to calculate the supply of the competitor. We therefore permit a minor departure from full rationality and allow each supplier to use an approximating model to impute the supply of its competitor. The approximating model we assume is one where each supplier believes the supply of its competitor has had no systematic variation in the recent past. In self-confirming equilibrium without resource scarcity this will indeed be the case, so our approximating model is only initially misspecified off the equilibrium path. The perception that competitor supply

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has no systematic variation means that each supplier sees recent market data as generated by the following process:

$$p_t = (a - bq_j + \eta_t) - bq_{it}, \tag{2.3}$$

in which case recent data can be used to estimate a regression of the form:

$$p_t = \gamma_{it}^0 + \gamma_{it}^1 q_{it} + \eta_{it} \tag{2.4}$$

and obtain estimates of $(a - bq_j)$ and $-b$. The presence of the residual η_{it} in the regression equation represents an acknowledgement by the supplier that it only has an approximating model of price determination. The perception of no systematic variation in competitor supply means η_{it} is seen as orthogonal to q_j and standard econometric techniques produce estimates $(\hat{\gamma}_{it}^0 \ \hat{\gamma}_{it}^1)$ that are unbiased predictors of $(a - bq_j)$ and $-b$. In response to the perception that (2.3) only holds in recent data, we assume that each supplier estimates their regression equation using discounted least squares:

$$\hat{\gamma}_{it+1} = \hat{\gamma}_{it} + \varepsilon R_{it}^{-1} X_{it}' \left(p_t - X_{it} \hat{\gamma}_{it}' \right), \tag{2.5}$$

$$R_{it+1} = R_{it} + \varepsilon \left(X_{it}' X_{it} - R_{it} \right), \tag{2.6}$$

where $X_{it} = (1 \ q_{it})$, $\hat{\gamma}_{it} = (\hat{\gamma}_{it}^0 \ \hat{\gamma}_{it}^1)$ and ε is the rate at which data is discounted. These standard recursive updating equations can also be interpreted as describing the constant gain learning algorithm suppliers use to form perceptions of the economic environment they operate in.

2.3 Supplier decisions

The decision faced by a supplier is how much depletable resource to supply to the market now, given their current knowledge of market conditions and expectations of how market power and scarcity will develop in the future. This is a difficult problem to solve because the optimal

depletion plan depends not only on the evolution of the seller's own perceptions but also on the evolution of perceptions held by its competitor. Some simplifying assumptions are needed to make the problem tractable for the supplier. The first assumption we make is to rule out exploration and the discovery of new supplies of the depletable resource. We also abstract from extraction and storage costs by assuming the resource can be brought to market at zero marginal cost at any time. Under these assumptions, the problem of a supplier reduces to calculating the optimal depletion plan for a given stock of resource Q_{it}. The most difficult part is then for the supplier to form perceptions of how market power will develop in the future. This ultimately depends on what a supplier expects their competitor to do and how much confidence they have in their estimates of the structural parameters of the model. We operationalise this aspect by assuming that the supplier maximises anticipated rather than expected profits, in the sense suggested by Kreps (1998). Under this decision criterion, the supplier projects forward by assuming that the degree of market power it holds is known with certainty and will remain unchanged in the future. Unchanging market power means the supplier anticipates facing the same linear demand curve in the future as it does now. The current perception of the demand curve is summarised by equation (2.4) and parameter estimates $(\hat{\gamma}_{it}^0 \ \hat{\gamma}_{it}^1)$ so the (approximately) optimal depletion plan solves the following problem:

$$\max_{\hat{q}_{it}} E_t \sum_{k=0}^{\infty} \beta^k p_{t+k} q_{it+k}$$

$$s.t.$$

$$p_{t+k} = \hat{\gamma}_{it}^0 + \hat{\gamma}_{it}^1 q_{it+k} + \eta_{it+k},$$

$$q_{it+k} = \hat{q}_{it+k} + W_{it+k},$$

$$\sum_{k=0}^{\infty} q_{it+k} = Q_{it}. \tag{2.7}$$

We solve the supplier's problem by first forming the Lagrangian:

$$\mathcal{L} = \sum_{k=0}^{\infty} \beta^j p_{t+k} q_{it+k} - \lambda_{it} \left[\sum_{k=0}^{\infty} q_{it+k} - Q_{it} \right]. \tag{2.8}$$

The first order condition with respect to $\hat{q}_{it+k}$ is:

$$\beta^k (\hat{\gamma}_{it}^0 + 2\hat{\gamma}_{it}^1 q_{it+k}) = \lambda_{it}, \tag{2.9}$$

from which it follows that supply in the first period is given by:

$$\hat{q}_{it} = \frac{\lambda_{it} - \hat{\gamma}_{it}^0}{2\hat{\gamma}_{it}^1}. \tag{2.10}$$

The value of the Lagrange multiplier λ_{it} can be calculated by defining T_{it} as the time to exhaustion of the depletable resource. At the time of exhaustion, optimality requires the supply of the resource to be zero so first order condition (2.9) reduces to:

$$\beta^{T_i} \hat{\gamma}_{it}^0 = \lambda_{it}, \tag{2.11}$$

which defines a relationship between T_i and λ_{it}. In similar fashion, optimality requires the stock of the depletable resource to be zero at the time of exhaustion. Summing the first order condition (2.9) between t and T_i then gives a second relationship between T_i and λ_{it}:

$$T_i \hat{\gamma}_{it}^0 + 2\hat{\gamma}_{it}^1 Q_{it} = \lambda_{it} \frac{1 - \beta^{-T_i}}{1 - \beta^{-1}} \tag{2.12}$$

Equations (2.11) and (2.12) simultaneously determine the time to exhaustion and the Lagrange multiplier, the latter determining market supply through equation (2.10).

An important feature of anticipated profit maximising is that the supplier only implements the first period supply of the optimal depletion plan. By the time the second period comes round the supplier has updated the estimates of the parameters in its demand curve, and so needs to formulate a new optimal depletion plan. The supplier therefore learns and reoptimises

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each period. The inconsistency between learning and assuming unchanging demand when optimising implies a departure from rationality, but Kreps (1998) argues that the departure is likely to lead to only small approximation errors. Maximising anticipated instead of expected profits may also be a robust strategy when the supplier lacks confidence in its understanding of the environment it operates in.

3   Self-confirming equilibrium

The first question we ask is whether the learning processes in our model naturally guide suppliers into a self-confirming equilibrium of the type discussed by Sargent (1999). Our answer is that they do but the nature of the self-confirming equilibrium depends on the degree of resource scarcity. We derive these results using stochastic approximation techniques to analyse the *mean dynamics* of the continuous time analogue of our model, thereby tracing out the expected evolution and limit point of supplier beliefs.

3.1   Mean dynamics

The mean dynamics of beliefs can be derived by re-writing the supplier's recursive updating scheme (2.5) and (2.6) as:

$$\frac{\hat{\gamma}_{it+1} - \hat{\gamma}_{it}}{\varepsilon} = R_{it}^{-1} X_{it}' \left(p_t - X_{it} \gamma_{it}' \right), \tag{3.1}$$

$$\frac{R_{it+1} - R_{it}}{\varepsilon} = \left(X_{it}' X_{it} - R_{it} \right). \tag{3.2}$$

Equations (3.1) and (3.2) describe a discrete-time approximation of a continuous time process peturbed by shocks. Taking the limit as $\varepsilon \to 0$, the approximation error tends to zero and a weak law of large numbers ensures that the stochastic element becomes negligible.

In the limit, the mean dynamics of beliefs can therefore be described by a pair of ordinary differential equations:

$$\dot{\gamma}_i = R_i^{-1} X_i' \left(p - X_i \gamma_i' \right), \tag{3.3}$$

$$\dot{R}_i = X_i' X_i - R_i. \tag{3.4}$$

Expressions for p and $X_i = (1 \ q_i)$ can be obtained from equations (2.1) and (2.2) determining market price, and equation (2.10) for market supply. After taking expectations, the mean dynamics of beliefs are then given by:

$$\dot{\gamma}_i = R_i^{-1} \left(\begin{array}{c} a - b \left(\frac{\lambda_i - \gamma_i^0}{2\gamma_i^1} + \frac{\lambda_j - \gamma_j^0}{2\gamma_j^1} \right) - \frac{\gamma_i^0}{2} - \frac{\lambda_i}{2} \\ \left(\frac{\lambda_i - \gamma_i^0}{2\gamma_i^1} \right) \left[a - b \left(\frac{\lambda_i - \gamma_i^0}{2\gamma_i^1} + \frac{\lambda_j - \gamma_j^0}{2\gamma_j^1} \right) - \frac{\gamma_i^0}{2} - \frac{\lambda_i}{2} \right] - (b + \gamma_i^1)\sigma \end{array} \right), \tag{3.5}$$

$$\dot{R}_i = \left(\begin{array}{cc} 1 & \frac{\lambda_i - \gamma_i^0}{2\gamma_i^1} \\ \frac{\lambda_i - \gamma_i^0}{2\gamma_i^1} & \left(\frac{\lambda_i - \gamma_i^0}{2\gamma_i^1} \right)^2 + \sigma \end{array} \right) - R_i. \tag{3.6}$$

The mean dynamics show how beliefs evolve as a function of current beliefs and the value of the Lagrange multiplier on the supplier's resource constraint. To complete the description of the mean dynamics of the model we therefore need to specify the dynamic evolution of the Lagrange multiplier, which itself is the solution to the continuous time analogues of equations (2.11) and (2.12). Solving the equations simultaneously gives an implicit expression for the Lagrange multiplier:

$$\left(1 + \log \lambda_i - \log \gamma_i^0 \right) \gamma_i^0 + 2\gamma_i^1 Q_i \log \beta = \lambda_i, \tag{3.7}$$

from which it follows by total differentiation that:

$$\left(\log \lambda_i - \log \gamma_i^0 \right) \dot{\gamma}_i^0 + \gamma_i^0 \frac{\dot{\lambda}_i}{\lambda_i} + 2\gamma_i^1 \dot{Q}_i \log \beta + 2\dot{\gamma}_i^1 Q_i \log \beta = \dot{\lambda}_i. \tag{3.8}$$

We eliminate the rate of resource depletion $\dot{Q}_i$ by recognising that it equals the amount of resource supplied to the market, i.e. $\dot{Q}_i = -q_i$. Equation (3.8) can further be simplified by

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using equation (3.7) to substitute out for the stock of depletable resource Q_i, leaving our final expression for the dynamics of the Lagrange multiplier as:

$$\frac{\dot{\lambda}_i}{\lambda_i} = \left(\frac{\log \lambda_i - \log \gamma_i^0}{\lambda_i - \gamma_i^0}\right) \dot{\gamma}_i^0 + \left(1 - \gamma_i^0 \frac{\log \lambda_i - \log \gamma_i^0}{\lambda_i - \gamma_i^0}\right) \frac{\dot{\gamma}_i^1}{\gamma_i^1} - \log \beta. \tag{3.9}$$

Equations (3.5), (3.6) and (3.9) fully characterise the mean dynamics of our model. Their limit point is the self-confirming equilibrium.

3.2 Self-confirming equilibrium without scarcity

Proposition 1 *When there is no resource scarcity, the self-confirming equilibrium is equivalent to the **noncooperative** rational expectations equilibrium of the model.*

Proof. We look for a self-confirming equilibrium in which beliefs are stable so $\dot{\gamma}_i = \dot{\gamma}_j = 0$. When there is no resource scarcity, the Lagrange multipliers λ_i and λ_j are zero and simple algebraic manipulation of equations (3.5) and (3.6) shows that the self-confirming equilibrium has the form:

$$\gamma_i = \gamma_j = \begin{pmatrix} \frac{2}{3}a \\ -b \end{pmatrix}, \tag{3.10}$$

$$R_i = R_j = \begin{pmatrix} 1 & \frac{a}{3b} \\ \frac{a}{3b} & \left(\frac{a}{3b}\right)^2 + \sigma \end{pmatrix}. \tag{3.11}$$

From the supply equation (2.10) and the market demand curve (2.1), it follows that supply and price in self-confirming equilibrium are:

$$q_i = q_j = \frac{a}{3b}, \tag{3.12}$$

$$p = \frac{a}{3}. \tag{3.13}$$

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To show that the self-confirming equilibrium is equivalent to noncooperative rational expectations equilibrium we solve for rational expectations equilibrium when suppliers do not cooperate. Under rational expectations, the market demand curve (2.1) is known and firm i takes the supply of firm j as given in its profit maximisation problem:

$$\max_{\hat{q}_{it}} E_t \sum_{k=0}^{\infty} \beta^j p_{t+k} q_{it+k}$$

$$s.t.$$

$$p_{t+k} = a - b(q_{it+k} + q_{jt+k}),$$

$$q_{it+k} = \hat{q}_{it+k} + W_{it+k}. \tag{3.14}$$

The first order condition is a reaction function for firm i supply conditional on firm j supply:

$$\hat{q}_{it} = \frac{a - b\hat{q}_{jt}}{2b}. \tag{3.15}$$

By analogy, the reaction function for firm j supply conditional on firm i supply is:

$$\hat{q}_{jt} = \frac{a - b\hat{q}_{it}}{2b}. \tag{3.16}$$

Equations (3.15) and (3.16) simultaneously determine market supply and combine with the linear demand curve (2.1) to determine market price. The noncooperative rational expectations equilibrium is therefore:

$$q_i = q_j = \frac{a}{3b}, \tag{3.17}$$

$$p = \frac{a}{3}, \tag{3.18}$$

which coincides with the self-confirming equilibrium when there is no resource scarcity. Appendix A shows that the self-confirming equilibrium is e-stable. ∎

3.3 Self-confirming equilibrium with scarcity

Proposition 2 *When there is sufficient resource scarcity, the self-confirming equilibrium is equivalent to the **cooperative** rational expectations equilibrium of the model.*

Proof. We begin the proof by asserting that a self-confirming equilibrium exists in which beliefs are stable so $\dot{\gamma}_i = \dot{\gamma}_j = 0$ in the limit as the depletable resource becomes scarce. From equation (3.9) it follows that in self-confirming equilibrium:

$$\frac{\dot{\lambda}_i}{\lambda_i} = -\log \beta \tag{3.19}$$

and the standard Hotelling result applies that the shadow price of the depletable resource rises at the rate of time preference. From equation (18) it also follows that as $\lambda_i \to \infty$ we obtain well-defined limits for the (appropriately scaled) second moment matrix R_i:

$$\lim_{\lambda_i \to \infty} \left(\frac{R_i^{12}}{\lambda_i} \right) = -\frac{1}{2\gamma_i^1 (\log \beta - 1)}, \tag{3.20}$$

$$\lim_{\lambda_i \to \infty} \left(\frac{R_i^{22}}{\lambda_i^2} \right) = -\frac{1}{4(\gamma_i^1)^2 (2\log \beta - 1)}. \tag{3.21}$$

Re-writing the first line of equation (3.5) in terms of R_i^{22}/λ_i^2 and R_i^{12}/λ_i gives:

$$\dot{\gamma}_i^0 = \frac{1}{(R_i^{22}/\lambda_i^2) - (R_i^{12}/\lambda_i)^2} \left[\begin{array}{c} \left(\frac{R_i^{22}}{\lambda_i^2} \right) \left(a - b \left(\frac{\lambda_i - \gamma_i^0}{2\gamma_i^1} + \frac{\lambda_j - \gamma_j^0}{2\gamma_j^1} \right) - \frac{\gamma_i^0}{2} - \frac{\lambda_i}{2} \right) \\ - \left(\frac{R_i^{12}}{\lambda_i} \right) \left(\left(\frac{\lambda_i - \gamma_i^0}{2\gamma_i^1 \lambda_i} \right) \left[a - b \left(\frac{\lambda_i - \gamma_i^0}{2\gamma_i^1} + \frac{\lambda_j - \gamma_j^0}{2\gamma_j^1} \right) - \frac{\gamma_i^0}{2} - \frac{\lambda_i}{2} \right] - \frac{(b + \gamma_i^1)\sigma}{\lambda_i} \right) \end{array} \right], \tag{3.22}$$

which has a unique solution for $\dot{\gamma}_i = \dot{\gamma}_j = 0$ when $\lambda_i, \lambda_j \to \infty$:

$$\gamma_i = \gamma_j = \left(\begin{array}{c} a \\ -2b \end{array} \right). \tag{3.23}$$

The supply and price in self-confirming equilibrium are defined by:

$$q_i = \frac{a - \lambda_i}{4b}, \tag{3.24}$$

$$q_j = \frac{a - \lambda_j}{4b}, \tag{3.25}$$

$$p = \frac{a}{2} + \frac{\lambda_i + \lambda_j}{4}. \tag{3.26}$$

To show that the self-confirming equilibrium is equivalent to cooperative rational expectations equilibrium we solve for supply and price when the two suppliers know the demand curve and maximise joint profits. In this case the profit maximisation problem is:

$$\max_{\hat{q}_{it} + \hat{q}_{jt}} E_t \sum_{k=0}^{\infty} \beta^j p_{t+k}(q_{it+k} + q_{jt+k})$$

$$s.t.$$

$$p_{t+k} = a - b(q_{it+k} + q_{jt+k}),$$

$$q_{it+k} = \hat{q}_{it+k} + W_{it+k},$$

$$q_{jt+k} = \hat{q}_{jt+k} + W_{jt+k},$$

$$\sum_{k=0}^{\infty} q_{it+k} + q_{jt+k} = Q_{it} + Q_{jt}. \tag{3.27}$$

The first order condition determines total supply to the market:

$$\hat{q}_i + \hat{q}_j = \frac{a - \lambda}{2b}, \tag{3.28}$$

so in the simple case where suppliers supply half the market each we obtain:

$$\hat{q}_i = \hat{q}_j = \frac{a - \lambda}{4b}, \tag{3.29}$$

$$p = \frac{a}{2} + \frac{\lambda}{2}, \tag{3.30}$$

which is exactly the self-confirming equilibrium with resource scarcity. Appendix B shows that the self-confirming equilibrium is e-stable. ∎

3.4 Numerical example

Our mathematical results show that the nature of the self-confirming equilibrium changes from noncooperative to cooperative rational expectations equilibrium as the depletable resource becomes more scarce. To look at the transition between these two cases we consider a numerical example. We set $a = 2$, $b = 0.1$ and the standard deviation of control errors $\sigma = 0.1$. The initial resource stock is set to $Q_0 = 5.3746$ for each supplier to normalise the initial time to exhaustion T_0 to unity. For the discount factor we consider an annual model with 100 years to exhaustion, so $\beta = 0.95^{100}$. The first figure we present is the evolution of γ.
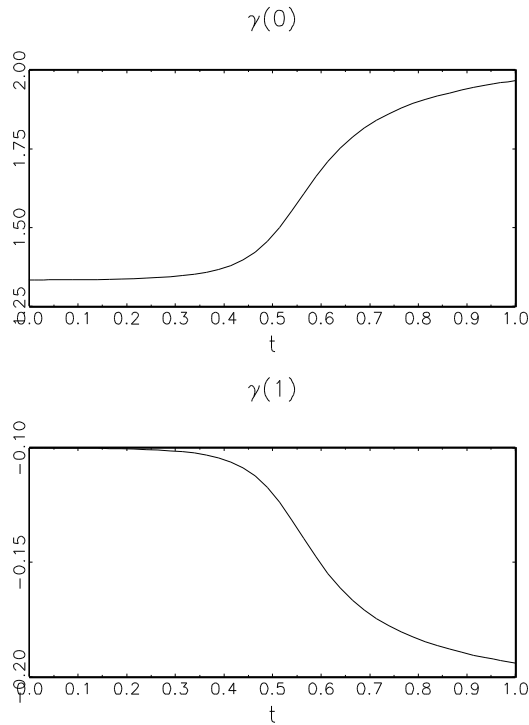

Figure 1: Evolution of beliefs as depletable resource becomes scarce

Figure 1 confirms the first result of our paper. In the first panel, belief γ^0 is $(2a/3) = 4/3$ when the resource is relatively abundant and then changes to $a = 2$ as the resource

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becomes scarce. In the second panel, belief γ^1 changes from $-b = -0.1$ to $-2b = -0.2$. The intuition for this is that scarcity acts as a coordination device in our learning model. When the resource is abundant there is no coordination between suppliers and changes in supply are dominated by random control errors that are independent across suppliers. Random control errors feed through to price with multiplier $-b$ in the true linear demand curve (1), so on average demand appears to be relatively price elastic. When the resource becomes scarce, changes in supply become increasingly dominated by scarcity considerations. A supplier then sees its own contraction in supply having a large effect on price - because the other supplier is also contracting supply - and concludes that demand is relatively price inelastic.
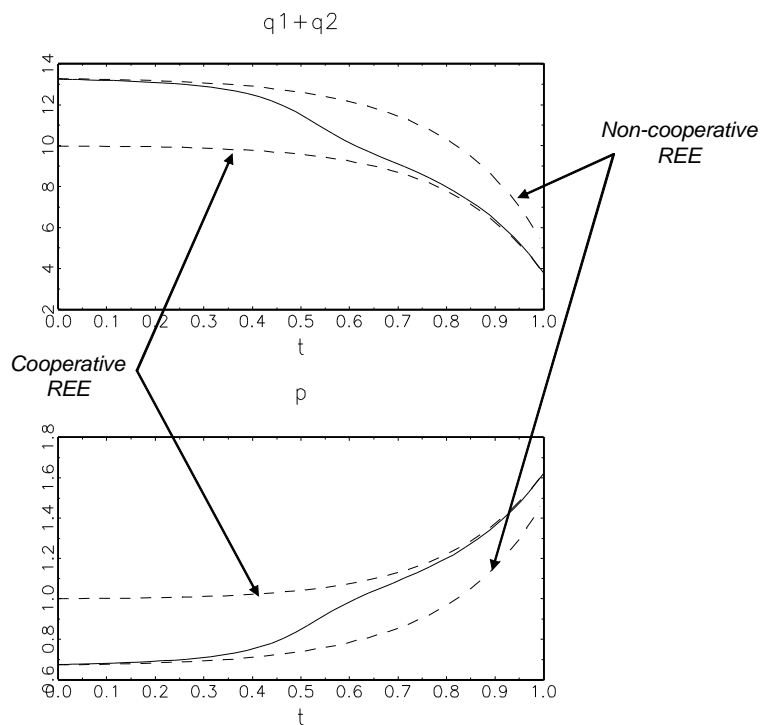

Figure 2: Evolution of self-confirming equilibrium as depletable resource becomes scarce

Figure 2 plots the evolution of supply and price in self-confirming equilibrium as the resource becomes scarce. For comparison, we include the supply and prices that would prevail

if the suppliers were in noncooperative and cooperative rational expectations equilibrium. As expected, in the top panel supply matches that in noncooperative rational expectations equilibrium when the resource is abundant, but as the resource becomes scarce there is a change towards consistency with cooperative rational expectations equilibrium. The bottom panel of the figure shows the corresponding evolution of market price.

4   Escape episodes

The second question we ask is whether the model has the ability to deviate from the mean dynamics in a significant and predictable way. As in the previous section, we obtain our results using the techniques of stochastic approximation to analyse the continuous time analogue of our model. We characterise the *escape dynamics* of the system and derive the way in which the model is most likely to deviate from its mean dynamics.

4.1   Escape dynamics

The question posed in escape dynamic analysis is what is the most likely path for beliefs if they deviate significantly (escape) from their mean dynamics. To answer this, we need a way of selecting the most likely path amongst all candidate escape paths. A natural metric is the likelihood function of the shocks needed to drive beliefs along each escape path. The path that minimises this function is the dominant escape path, representing the path of least resistance for beliefs to escape. The formal analysis of escape dynamics in economic models is laid out in the pioneering work of Williams (2001), where the dominant escape path is characterised by solving an optimal control problem. The method involves choosing a series of perturbations to mean dynamics that is most likely to cause beliefs to escape from a neighbourhood around

the self-confirming equilibrium. Mathematically, the dominant escape path is given by the solution to the following optimal control problem:

$$\bar{S} = \inf_{\dot{v}} \frac{1}{2} \int_0^t \dot{v}(s)' Q(\gamma(s), R(s), \lambda(s))^{-1} \dot{v}(s) ds$$

$$s.t.$$

$$\dot{\gamma} = R^{-1} \bar{g}(\gamma, \lambda) + \dot{v}$$

$$\dot{R} = \bar{M}(\gamma, \lambda) - R$$

$$\dot{\lambda} = \bar{f}(\gamma, R, \lambda)$$

$$\gamma(0) = \bar{\gamma}, \ M(0) = \bar{M}, \ \lambda(0) \ given, \ \gamma(t) \notin G \ for \ some \ 0 < t < T \tag{4.1}$$

The optimal control problem works by perturbing the mean dynamics of the model (3.5) - (3.6) by a factor $\dot{v}$ and asking which series of perturbations is most likely to cause beliefs to escape. The function $\dot{\lambda} = f(\gamma, R, \lambda)$ is a direct summary of equation (3.9). In the objective, $Q(\gamma, R, \lambda)$ is a weighting function that measures the likelihood of the shocks needed to perturb beliefs by $\dot{v}$.[4] We initialise beliefs at their self-confirming values and define a neighbourhood G around the self-confirming equilibrium that beliefs must escape from. The outcome of the optimal control problem is the series of belief perturbations that occur along the dominant escape path.

4.2 Dominant escape path

The dominant escape path solves optimal control problem (4.1). To find the solution we define the Hamiltonian (4.2), where a, b and c are co-state vectors for the evolution of γ, R and λ.

$$\mathcal{H} = aR^{-1} \bar{g}(\gamma, \lambda) - \frac{1}{2} a' Q(\gamma, R, \lambda) a + b \cdot (\bar{M}(\gamma, \lambda) - R) + c \bar{f}(\gamma, R, \lambda) \tag{4.2}$$

---

[4]An analytic expression for $Q(\gamma, R, \lambda)$ is given in Appendix C.

The Hamiltonian is convex so first order conditions (4.3) - (4.8) necessarily hold along the the dominant escape path.

$$\dot{\gamma} \;\; = \;\; R^{-1}\bar{g}(\gamma,\lambda) - Q(\gamma,R,\lambda)a \qquad (4.3)$$

$$\dot{R} \;\; = \;\; \bar{M}(\gamma,\lambda) - R \qquad (4.4)$$

$$\dot{\lambda} \;\; = \;\; f(\gamma,R,\lambda) \qquad (4.5)$$

$$\dot{a} \;\; = \;\; -\mathcal{H}_\gamma \qquad (4.6)$$

$$\dot{b} \;\; = \;\; -\mathcal{H}_R \qquad (4.7)$$

$$\dot{c} \;\; = \;\; -\mathcal{H}_\lambda \qquad (4.8)$$

The first order conditions form a system of ordinary differential equations. They characterise a family of escape paths, with each path being indexed by different initial values of the co-state vectors. The dominant escape path is the member of this family that achieves the escape with the most likely series of belief perturbations. A solution to the optimal control problem can therefore be obtained by searching over all possible initial values of a, b and c, applying equations (4.3) - (4.8), and choosing initial values that imply beliefs perturbations that are most likely in terms of the $Q(\gamma,R,\lambda)$ metric.

4.3   Numerical example

To illustrate the nature of the dominant escape path we return to our numerical example with $a = 2$, $b = 0.1$, $\sigma = 0.1$ and $\beta = 0.95^{100}$. The first dominant escape path we report is for the case of no resource scarcity. It is shown in the top-left panel of Figure 3. The dominant escape path in our model is similar to the escape paths in Williams (2001), with beliefs spending a long time near the self-confirming equilibrium before escaping rapidly at $t \approx 2$ to new values close to $\gamma_i = (a, -2b)$. The mechanism causing beliefs to escape is the

same as in Williams (2001). Intuitively, an escape happens when a sequence of control errors causes both suppliers to simultaneously contract supply. The resulting increase in price causes each supplier to start to believe that demand is price inelastic, which gives further incentives for suppliers to contract supply. Price rises again and there is even more reason to contract supply in the belief that demand is inelastic. This reinforces beliefs in inelastic demand and price rises rapidly as supply contracts. The sequence of control errors that triggers the escape is a series of negative W_i and W_j shocks. With both control errors negative, the initial rise in price induces suppliers to contract supply enough to trigger an escape episode.
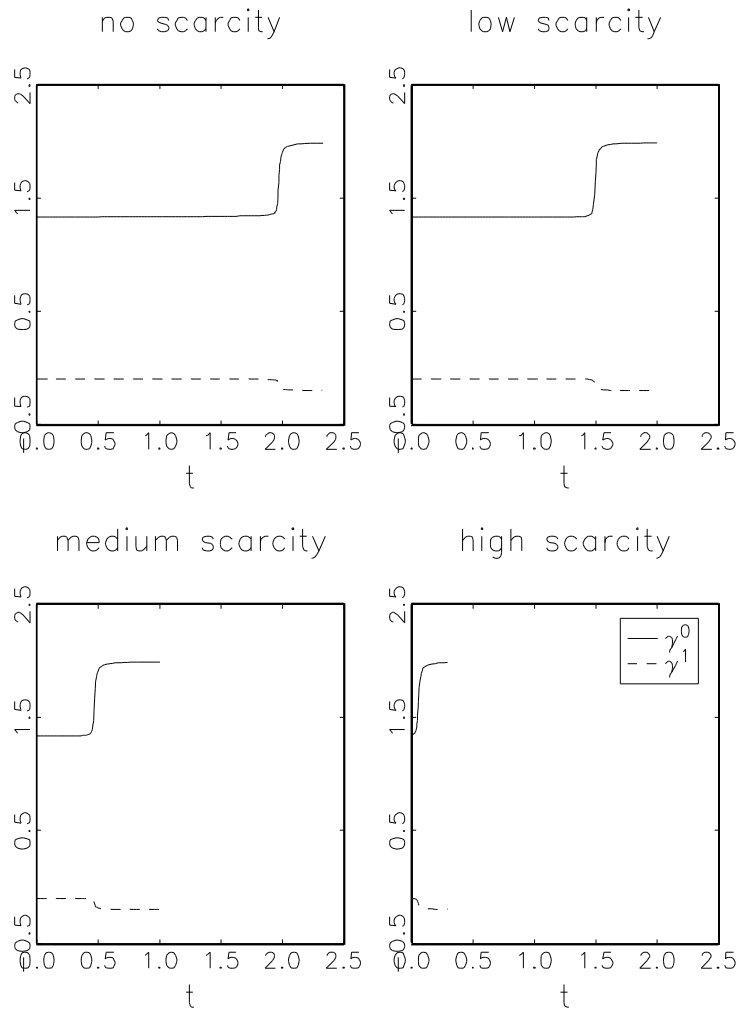
Figure 3: Dominant escape paths with different levels of resource scarcity

The remaining panels of Figure 3 show how the presence of scarcity affects escape dynamics. Starting from the top two panels we see that the introduction of scarcity brings forward the timing of the escape along the dominant escape path. In other words, the escape episode happens sooner the more scarce is the depletable resource. The pattern continues in the bottom two panels, the latter of which showing that the escape becomes almost instantaneous

when scarcity reaches high levels. Table 1 shows the same behaviour in an alternative format by reporting the relationship between the time to escape and the level of resource scarcity (as measured by the time to ultimate exhaustion of the depletable resource).

Time to exhaustion	Time to escape
4	2.02
3.5	2.05
3	2.03
2.5	1.53
2	1.04
1.5	0.51
1	0.09

Table 1: Relationship between time to exhaustion and time to escape

The fall in time to escape can be explained by scarcity causing escape episodes to occur against a background of falling supply and increasing price. In such circumstances, it is more likely that the special combination of control errors needed to trigger an escape will occur. Recall that simultaneous negative control errors were needed to trigger an escape in the no scarcity case. This is also true here, but the natural contraction of supply induced by scarcity means that smaller control errors are needed to trigger an escape. Escape episodes are therefore more likely to happen sooner.

5 Conclusions

Depletable resource markets are characterised by a finite stock and frequently significant market power. Since Hotelling (1931) the implications of this for the optimal rate of resource exhaustion and the behaviour of prices have been analysed. However, the focus of this analysis has been on the low frequency behaviour of prices that distinguishes nonrenewable resources from renewable commodities. By contrast our focus is on how uncertainty and shifting perceptions can influence high frequency shifts in commodity prices. In an environment characterised by uncertainty over the extent of scarcity and the degree of market power possessed by each agent we show how learning and changes in agents perceptions can introduce substantial market volatility over and above that suggested by fundamentals.

We show that in a world characterised by no concerns over scarcity our duopoly model reaches a self confirming equilibrium which is the non-cooperative Cournot-Nash equilibrium in which each producer supplies a large quantity and the market price of the commodity is low. In the case where scarcity is apparent to producers then the self confirming equilibrium is instead characterised by the cooperative Cournot-Nash equilibrium whereby each producer restricts output and the market price is high. In effect scarcity acts as a coordinating device and even though producers do not collude they restrict output and arrive at the cooperative outcome. Therefore as scarcity increases for a commodity the price experiences a shift to the higher cooperative solution.

We show that the existence of these two different self confirming equilibria, each dependent on the perception of scarcity, opens up the possibility of escape dynamics. Given a certain configuration of shocks each producer is led to believe that scarcity is binding and their degree of monopoly power has increased. They each respond by cutting back production and so produce sharp increases in price. This reinforces their original perceptions leading to further

cut backs in production and sharp jumps in price as the market shifts to the cooperative solution. However, eventually producers realise their mistaken perceptions and prices return once more to their non-cooperative and lower level. During the period of escape dynamics prices of depletable resources are characterised by sharp upward increases and then more gradual declines. In this way learning dynamics provide substantial volatility over and above that driven by fundamentals. However fundamentals do exert an influence on the nature of these dynamics. As scarcity becomes more apparent in reality then the jumps in prices towards the cooperative solution become smaller and these higher prices are more robust and persistent. Our analytical findings confirm the existence of these escape dynamics but our simulations confirm not just that these exist but they occur with a frequency and magnitude that adds considerable volatility to the price dynamics of finite resources.
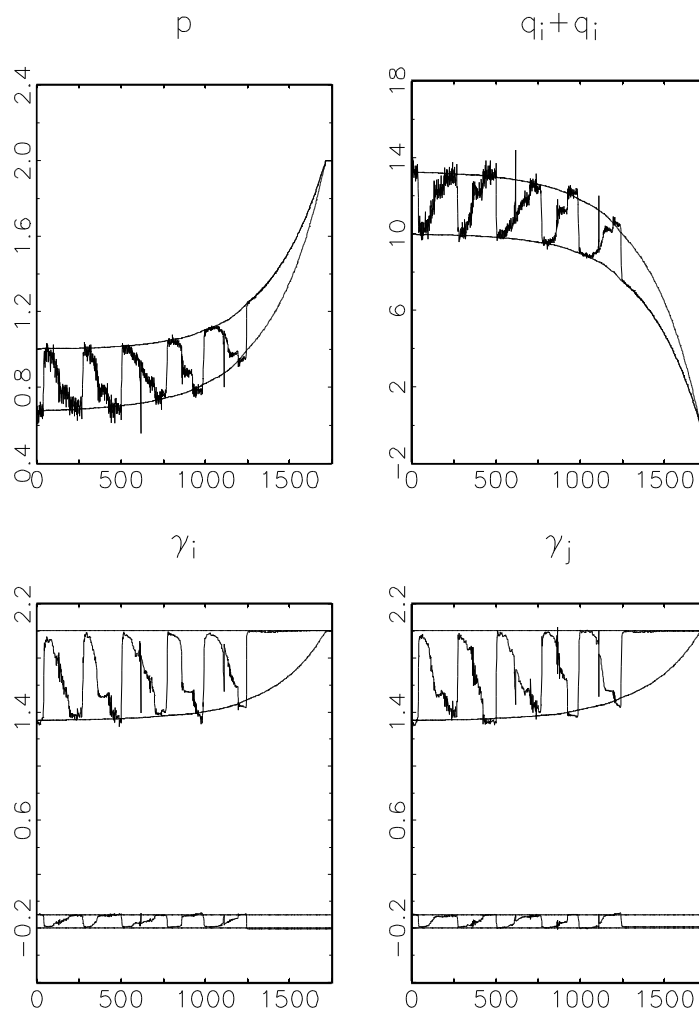
Figure 4: Simulated path of price, supply and beliefs

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A E-stability of self-confirming equilibrium without resource scarcity

To show that the self-confirming equilibrium is e-stable it is sufficient to show that the eigenvalues of the Jacobian have negative realy parts when evaluated at the self-confirming equilibrium. The Jacobian of the system is defined by:

$$
J = \begin{pmatrix}
\frac{\partial \dot{\gamma}_i}{\partial \gamma_i} & \frac{\partial \dot{\gamma}_i}{\partial \gamma_j} & \frac{\partial \dot{\gamma}_i}{\partial R_i} & \frac{\partial \dot{\gamma}_i}{\partial R_j} \\[4pt]
\frac{\partial \dot{\gamma}_j}{\partial \gamma_i} & \frac{\partial \dot{\gamma}_j}{\partial \gamma_j} & \frac{\partial \dot{\gamma}_j}{\partial R_i} & \frac{\partial \dot{\gamma}_j}{\partial R_j} \\[4pt]
\frac{\partial \dot{R}_i}{\partial \gamma_i} & \frac{\partial \dot{R}_i}{\partial \gamma_j} & \frac{\partial \dot{R}_i}{\partial R_i} & \frac{\partial \dot{R}_i}{\partial R_j} \\[4pt]
\frac{\partial \dot{R}_j}{\partial \gamma_i} & \frac{\partial \dot{R}_j}{\partial \gamma_j} & \frac{\partial \dot{R}_j}{\partial R_i} & \frac{\partial \dot{R}_j}{\partial R_j}
\end{pmatrix},
\tag{A.1}
$$

which when evaluated at self-confirming equilibrium reduces to:

$$
J|_{SCE} = \begin{pmatrix}
\left.\frac{\partial \dot{\gamma}_i}{\partial \gamma_i}\right|_{SCE} & \left.\frac{\partial \dot{\gamma}_i}{\partial \gamma_j}\right|_{SCE} & 0 & 0 \\[4pt]
\left.\frac{\partial \dot{\gamma}_j}{\partial \gamma_i}\right|_{SCE} & \left.\frac{\partial \dot{\gamma}_j}{\partial \gamma_j}\right|_{SCE} & 0 & 0 \\[4pt]
\left.\frac{\partial \dot{R}_i}{\partial \gamma_i}\right|_{SCE} & 0 & -I & 0 \\[4pt]
0 & \left.\frac{\partial \dot{R}_j}{\partial \gamma_j}\right|_{SCE} & 0 & -I
\end{pmatrix}.
\tag{A.2}
$$

The eigenvalues of the $-I$ identity matrices trivially have negative real parts so a sufficient condition for e-stability is that the following matrix has eigenvalues with negative real parts:

$$
\begin{pmatrix}
\left.\frac{\partial \dot{\gamma}_i}{\partial \gamma_i}\right|_{SCE} & \left.\frac{\partial \dot{\gamma}_i}{\partial \gamma_j}\right|_{SCE} \\[4pt]
\left.\frac{\partial \dot{\gamma}_j}{\partial \gamma_i}\right|_{SCE} & \left.\frac{\partial \dot{\gamma}_j}{\partial \gamma_j}\right|_{SCE}
\end{pmatrix}.
\tag{A.3}
$$

Simple but tedious calculations show that the matrix is of the form:

$$
\begin{pmatrix}
-1 & 0 & -\frac{1}{2} & -\frac{a}{3b} \\[4pt]
0 & -1 & 0 & 0 \\[4pt]
-\frac{1}{2} & -\frac{a}{3b} & -1 & 0 \\[4pt]
0 & 0 & 0 & -1
\end{pmatrix},
\tag{A.4}
$$

and the eigenvalues $-0.5, -1, -1, -1.5$ all have negative real parts. The self-confirming equilibrium is e-stable.

B E-stability of self-confirming equilibrium with resource scarcity

To follow.

C Analytical expression for $Q(\gamma, R, \lambda)$

To follow.