# Insider Trading with Stochastic Valuation[†]

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**Abstract**

This paper studies a model of strategic trading with asymmetric information of an asset whose value follows a Brownian motion. An insider continuously observes a signal that tracks the evolution of the asset fundamental value. At a random time a public announcement reveals the current value of the asset to all the traders. The equilibrium has two regimes separated by an endogenously determined time $T$. In $[0,T)$, the insider gradually transfers her information to the market and the market's uncertainty about the value of the asset decreases monotonically. By time $T$ all her information is transferred to the market and the price agrees with the market value of the asset. In the interval $[T,\infty)$, the insider trades large volumes and reveals her information immediately, so market prices track the market value perfectly. Despite this market efficiency, the insider is able to collect strictly positive rents after $T$.

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# 1 Introduction

This paper studies a model of strategic trading with asymmetric information of an asset whose value follows a Brownian motion. An insider receives a flow of (noisy) signals that tracks the evolution of the asset value. Other traders receive no signals and can only observe the total volume of trade. There is uncertainty about the value of the asset before the insider gets the first signal, hence the first signal generates a lumpy informational asymmetry between the insider and the rest of the market participants. The signals the insider receives later are equally informative, but they contribute only marginally to the informational asymmetry. The information advantage continues until an unpredictable time when a public announcement reveals the current value of the asset to all the traders.

Kyle (1985) introduced a dynamic model of insider trading where an insider receives only one signal and the fundamental asset value does not change over time. Through trade, the insider progressively releases her private information to the market as she exploits her informational advantage. The market is also populated by many liquidity traders that are uninformed and trade randomly. At time 0, the insider observes the value of an asset. The same information is publicly released later, at time 1, to all market participants. In each trading period in the time interval $[0, 1]$, traders submit order quantities to a risk-neutral market maker who sets prices competitively and trades in his own account to clear the market. The market maker cannot observe individual trades, but can observe the total volume of trade in each trading period. The market maker also knows (in equilibrium) the strategy of the informed trader, and sets prices efficiently conditional on past and present volumes of trade.

Kyle constructs a linear equilibrium where in each period the price adjustment is proportional to the volume of trade, and the volume the insider trades is proportional to the gap between the asset value and the current market price. The market maker's estimate of the asset value, reflected in the current market price, improves over time. As the public announcement date approaches, this estimate converges to the value of the asset and the insider trades frantically in her desire to exploit any price differential.

Our model differs from Kyle's model in three important ways. First, the fundamental value of the asset follows a Brownian motion and therefore changes continuously over time. Second, in addition to the initial observation, the insider continuously receives a signal of the current fundamental value of the asset. Third, the public announcement date is unpredictable: it has an exponential distribution.

The first difference by itself is irrelevant. In Kyle's model it makes no difference whether at time 0 the insider observes the true value of the asset or just an unbiased signal. Moreover, the model where the insider observes the true value and the value of the asset follows a Brownian motion is formally equivalent to a model where the initial observation is an unbiased signal of the final value of the asset. But this feature of our model becomes important when it is combined with the second feature. Finally, the third feature removes the force in Kyle's model behind the trade frenzy that occurs as the announcement date approaches. In our model, where the announcement date is not deterministic, the insider has no urgency to exhaust all arbitrage opportunities, and release all her private information in the process, by a particular deadline. Thus, while it is evident that in Kyle's model the price will become efficient (in the sense that it incorporates all the available information) as time reaches the announcement date, it is unclear whether in our model the insider will ever fully reveal her private information.

Our model is not the first to introduce a public announcement with random time. Back and Baruch (2004) compare the models of Kyle (1985) and Glosten and Milgrom (1985). To facilitate the comparison, they adopt a Glosten and Milgrom model with a single long-lived insider (who times her transactions strategically) and a Kyle model with a random terminal time and a risky asset that takes only the values 0 or 1.

Our model includes various special cases. The value of the asset remains constant over time if the variance of its Brownian motion is reduced to 0. Since in our model the insider observes the initial value without noise, the signals that track the value of the asset over time becomes superfluous. This version of our model is similar to Kyle's model, where the insider is endowed only with an initial piece of private information, but with a random end time. Alternatively, we can specialize our model to give the insider no initial informational advantage. This is accomplished by informing *all* traders of the initial value of the asset. In this version of the model, the insider's informational advantage arises exclusively from her ability to observe the evolution of the asset value. This is an important model in its own right. An interesting question in this model is how the insider 'manages' the information asymmetry. For example, the insider could let the information asymmetry (the variance of the uninformed traders' estimate of the current value) grow to reach asymptotically a certain limit or without bound. The larger is the information asymmetry, the more likely it is that the market price will diverge substantially from the actual value of the asset, and therefore, the larger are the profitable arbitrage opportunities. Thus, in this model as well it is not evident how much of the insider's information is incorporated in the market price and how quickly this happens. We study this special case in the process of constructing an equilibrium for our general model. It turns out that in equilibrium the insider fully reveals her information as soon as she receives it. Hence, the market price equals the asset value at all times. Yet, the insider makes strictly positive profits. In independent work, Chau and Vayanos (2006) reach the same conclusion (for this case without initial informational asymmetry) in a slightly different model. They assume that the insider receives a flow of information, the asset pays a dividend, and there is no public announcement. In addition, they assume that the market maker continuously observes a noisy signal of the value of the asset. In the absence of this noisy signal, their model would be formally equivalent to ours. Chau and Vayanos (2006) limit attention to the steady state of their model and do not study how the equilibrium approaches the steady state. One implication of our results is that in the absence of an initial information asymmetry, the steady state is reached 'immediately' (as the period length goes to 0), so although Chau and Vayanos (2006) assume that trading has been taking place indefinitely, this is not needed.

We pause now to compare this to the results of some of the seminal papers in the literature. In their celebrated paper, Grossman and Stiglitz (1980) study a trade model with asymmetric information, where consumers can acquire costly signals before they trade. They demonstrate that a rational expectations equilibrium does not exist if the cost of information is relatively low and there are no other sources of uncertainty besides the value of the risky asset (they also consider a model with supply uncertainty that does have an equilibrium). In a rational expectations equilibrium, the price is a sufficient statistic for the information of all the informed traders. Therefore, the informed traders enjoy no informational advantage and do not get compensated for the costly signals they acquire. Thus, in equilibrium, no consumer would incur the cost of acquiring information. But then, unexpectedly acquiring information would be profitable. Grossman and Stiglitz (1980) analyze a static general equilibrium model. Hellwig (1982) introduces a dynamic general equilibrium model with a risky asset whose dividends follow a Brownian motion. In order to achieve Walrasian market clearing while escaping the problematic features identified by Grossman and Stiglitz, he assumes that agents condition their demands on the current price, but that they *ignore* the informational content of that price (using only past prices to make inferences). Hellwig shows that in this model, an equilibrium exists. When the length of the period converges to 0, and therefore the price for the previous period contains almost as much information as the price for the current period, the informed traders' rents remain bounded away from zero. Moreover, as in our special case with no initial informational asymmetry, the price incorporates all the available information with (almost) no delay. Thus, in Hellwig's model, the informed traders get compensated and in equilibrium a fraction of them acquire costly information. With our simple demand protocol, with agents placing orders before learning the price, there is no need to resort to Hellwig's device of having consumers respond less than rationally to the current price. Like Hellwig, we find that information rents are bounded uniformly away from zero as the period length converges to zero, even though the difference in the information contained in this period's and last period's prices is also converging to zero. However, we do not assume perfect competition (our insider is a monopolist), and our model has a second source of uncertainty, the amount traded by liquidity traders, which is not present in Hellwig's model. So, while in Hellwig's model the total volume of trade perfectly reveals the information of the informed traders, in our model the liquidity traders' orders provide camouflage for the insider to conceal her trades. But in equilibrium, she does not.

The equilibrium of our general model has a striking feature. There is a time $T$, endogeneously determined in equilibrium, by which the insider reveals all her information (if the public announcement has not yet occurred). Thus, even though there is no deterministic deadline, the price converges to the asset value at time $T$. Moreover, time $T$ divides the equilibrium into two phases. As long as the public announcement does not occur, in the interval $[0, T)$ the insider gradually transfers her information to the market and the market's uncertainty about the value of the asset decreases to 0 monotonically. In the interval $[T, \infty)$, the insider trades large volumes and reveals her information immediately, so market prices track the asset value perfectly. Nevertheless, as we explained above, after $T$ the insider collects strictly positive rents, even when the time period converges to 0. In $[0, T)$ the insider is indifferent about her order quantities, though she trades according to a deterministic function of the current price and value of the asset. Therefore, she is indifferent about purchasing an additional share of the asset now or in the future, even though she discounts future payoffs. This is so because the market compensates her more generously in the future for any price differential. In $[T, \infty)$, her compensation, as a function of the price differential, is constant over time, and thus she is eager to cash in her rents as soon as arbitrage opportunities materialize.

We conclude the Introduction by discussing a small subset of the vast literature on insider trading.[1] Two of the most influential papers in the area of strategic trading with asymmetric information are Kyle (1985) and Glosten and Milgrom (1985). These classic papers formalize Bagehot (1971) intuitive story

[1] For a comprehensive review of this literature, and its connection to the broader market microstructure theory, we refer the reader to O'Hara (1997), Brunnermeier (2001), Biais et al. (2005), Amihud et al. (2006) and references therein.

that the market provides a mechanism to compensate informed traders for their superior information, while liquidity traders are willing to make (small) losses for the benefit of carrying out their transactions immediately. Glosten and Milgrom study a market where multiple insiders and noisy traders place orders sequentially (one at a time) to a risk-neutral and competitive specialist, who sets bid and ask prices. If the proportion of insiders is high and/or the quality of their private information is too good then the resulting bid-ask spread is too wide and the market shuts down. However, when there are few insiders with limited private information, the market does operate. Moreover, the bid-ask spread converges to zero as time goes by. Three notable extensions of the Glosten and Milgrom model are Easley and O'Hara (1987) that study the impact of block trading on the bid-ask spread, Glosten (1989) that considers a monopolist specialist that maximizes expected profits, and Dasgupta and Prat (2005) that analyse a model where some insiders receive superior signals and informed traders care about their reputations. In this last paper, in equilibrium, there is herd behavior and prices do not converge to the asset value.

More closely related to our work is the literature that builds upon Kyle (1985). In a continuous-time setting, Back (1992) considers a general distribution for the insider's private signal (Kyle assumes a normal distribution) and prove the existence and *uniqueness* of an equilibrium pricing rule. Holden and Subrahmanyam (1992) and Foster and Viswanathan (1996) consider a market with multiple competing insiders. Holden and Subrahmanyam assume a symmetric model in which the insiders are endowed with the same private information and show that insiders' competition creates a strong-form efficient market almost immediately. Foster and Viswanathan allow for heterogenous information among the insiders and show that this asymmetry reduces the degree of competition among them, and hence, the efficiency of market prices. In a one-period model with heterogeneous insiders, Spiegel and Subrahmanyam (1992) replace Kyle's uninformed liquidity traders (and their exogenous price-inelastic noisy trades) with strategic utility-maximizing agents trading for hedging purposes. They demonstrate that the welfare of uninformed traders *decreases* with the number of insiders. In a multi-period setting, Mendelson and Tunca (2004) propose an alternative endogenous liquidity trading model allowing for various type of market information; some available exclusively to the insider (tractable information) and some unavailable to all market participants (intractable information) that gets partially released over time. In contrast to Kyle's model, Mendelson and Tunca assume that the insider's private information acquisition is costly. The volume of uninformed trades decreases with market uncertainty, forcing the insider to reduce her own volume of trade. As a result, less information is acquired by the insider and information is spread out into the market more slowly.

The rest of the paper is organized as follows. Section 2 introduces the model in full generality and deals with its discrete-time version. We construct the unique linear Markovian equilibrium for this model and derive its asymptotic properties. Here we also study the special case where the fundamental value of the asset is constant over time. In Section 3 we study the limit of the discrete-time equilibrium as the length of a period goes to zero, including the special case when the value of the asset does not change. This exercise suggests an equilibrium for the continuous-time model that we pursue in Section 4. The equilibrium is composed of two distinct phases that we show paste smoothly. In Section 5 we discuss features and extensions of the continuous-time equilibrium, such as a model with multiple insiders.

## 2 Discrete Time Model

We introduce first a continuous-time model, where the fundamental value of the asset and the liquidity trader's (target) holding of the asset are described by continuous time stochastic processes. In the discrete time model that we study in this Section, trading orders are restricted to take place only at discrete times; the time between two trading dates is a period. The continuous time model we study in Section 4 removes this institutional constraint. We then construct a linear Markovian equilibrium for the discrete time model.

The market participants are the insider, the market maker and a (large) number of liquidity traders. The insider (and only her) continuously receives private information about the fundamental value of the asset. Every period $n$, the insider and the liquidity traders place buy/sell orders for a quantity of the asset. An order is a binding contract to buy/sell a quantity of the asset (the 'size of the order') at a price determined by the market maker. At the end of the period, after observing the total volume of trade, the market maker sets the price $p_n$ and trades the necessary quantity to close all orders. This trading process continues until an unpredictable random time $\tau$ when the fundamental value of the asset becomes public knowledge. At this time, the market price immediately matches the fundamental value and the insider loses her informational advantage.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with two independent standard Brownian motions $B_t^v$ and $B_t^y$, where $t \in [0, \infty)$ denotes (calendar) time. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the usual filtration generated by $(B^v, B^y)$. The value of the fundamental at time $t$ is $\bar{V}_t$, which we assume evolves over time as an arithmetic Brownian motion

$$\mathrm{d}\bar{V}_t = \bar{\sigma}_v \, \mathrm{d}B_t^v,$$

for some constant $\bar{\sigma}_v$. The initial value $\bar{V}_0$ is drawn from a normal distribution with mean $\bar{v}_0$ and variance $\bar{\Sigma}_0$. The insider alone observes the (stochastic) evolution of $\bar{V}_t$ during $t \in [0, \tau)$. The market maker and the rest of the market participants only know the distribution of $\bar{V}_0$. The random time $\tau$ when the value of the fundamental becomes public knowledge is exponentially distributed with mean $1/\theta$, and is independent of $(B^v, B^y)$.

In the discrete time model the market maker opens the floor for trading only at discrete times $\{t_n\}_{n\geq 0}$. We assume that these trading dates are evenly spaced over time (*e.g.*, once a day) so that $t_n = n\,\Delta$ for some positive constant $\Delta$. The interval of time $[t_n, t_{n+1})$ is called period $n$. For $t > 0$, let $\lfloor t \rfloor$ denote the largest integer $n$ such that $n\Delta \leq t$. The period when the fundamental value becomes public knowledge is $\nu = \lfloor \tau \rfloor$, and we assume that the announcement always occur at the end of the period. The discrete random variable $\nu$ has a geometric distribution with probability of failure $q = e^{-\theta\Delta}$.

During the trading period $[0, \tau)$, the insider and the liquidity traders simultaneously place their orders at the begining of every period. Liquidity trades are not strategic agents and they are motivated to trade for idiosyncratic reasons. They trade so as to match a moving target for their net holding of the asset. Their holding target $Y_t$ at time $t$ follows an arithmetic Brownian motion

$$\mathrm{d}Y_t = \sigma_y \, \mathrm{d}B_t^y$$

for some constant $\sigma_y$.

At trading time $t_n$, the liquidity traders place orders for a total of $y_n = Y_{t_n} - Y_{t_{n-1}}$. While the insider starts trading at time 0, the moment she starts observing her private information, the liquidity traders have been trading prior to this time and at time 0, before they place their orders, they already hold $Y_{-\Delta}$ shares of the asset. Without loss of generality, hereafter we assume that $Y_{-\Delta} = 0$. Given that $\{Y_t\}$ follows a Brownian motion, $\{y_n\}$ is a sequence of i.i.d. normal random variables with mean 0 and variance $\Sigma_y = \sigma_y^2 \Delta$. Let $x_n$ denote the order placed by the insider at trading time $t_n$, and let $X_t$ be her net holding at time $t$ (including the order she placed for the current period). That is, $X_t = 0$ for $t < 0$ and

$$X_t = \sum_{n=0}^{\lfloor t \rfloor} x_n \quad \text{for } t \geq 0.$$

Similarly, let $z_n = x_n + y_n$ denote the total volume of trade at trading time $t_n$, and let $Z_t = 0$ for $t < 0$ and

$$Z_t = \sum_{n=1}^{\lfloor t \rfloor} z_n \quad \text{for } t \geq 0.$$

Note that at each trading time $t_n$, $Z_{t_n} = X_{t_n} + Y_{t_n}$ is the total holding of the asset (including the current orders) by the insider and liquidity traders.

At the begining of each period $n$ before the fundamental value becomes public knowledge, the market maker commits to a pricing rule (that is legally binding). The rule specifies the price $p_n$ for the current period's transactions as a function of the total volume of trade $z_n$. The insider and the liquidity traders place their orders after the rule is announced. All orders are executed at the end of the period. To understand the filtration we define below, note that while the market maker commits to a rule before knowing the current period's volume of trade, the actual price is determined after learning the volume of trade. Let the price process $\{P_t\}$ be defined as follows: $P_t = p_{\lfloor t \rfloor \Delta}$ for $t \in [0, (\nu + 1)\Delta)$, and $P_t = \bar{V}_{\lfloor t \rfloor \Delta}$ for $t \in [(\nu + 1)\Delta, \infty)$.

The market maker observes the public history of prices and (total) volumes of trade. His information is represented by the filtration $\mathbb{F}^M = \{\mathcal{F}_t^M\}_{t \geq 0}$, where $\mathcal{F}_t^M = \sigma(P_s : 0 \leq s < t) \vee \sigma(Z_s : 0 \leq s \leq t)$ is the sigma algebra generated by the history of prices and holdings up to time $t$. Since information is only revealed at trading times $t_n$, in period $n$, the market maker knows the history $h_n^M = (z_0, p_0, \ldots, z_{n-1}, p_{n-1}, z_n)$.[2] Each period, the market maker learns the volume of trade before he sets the market price. The insider's information includes the public history of prices and trades, and the private history of orders she has placed and fundamental values she has observed. Her information is represented by the filtration $\mathbb{F}^I = \{\mathcal{F}_t^I\}_{t \geq 0}$, where $\mathcal{F}_t^I = \sigma((P_s, X_s, Z_s) : 0 \leq s < t) \vee \sigma(\bar{V}_s : 0 \leq s \leq t)$. That is, at trading time $t_n$, she knows the history $h_n^I = (\bar{V}_0, x_0, z_0, p_0, \bar{V}_1, \ldots, x_{n-1}, z_{n-1}, p_{n-1}, \bar{V}_n)$. The insider places her order at the begining of the period, after observing the current value of the fundamental.

The insider and the market maker are risk neutral and discount future payoffs by the discount factor $\delta > 0$. Given a trajectory $\{X_t\}$ for the insider's holding and $\{P_t\}$ for market prices, the

[2] In the tradition of game theory, we are including the trajectory of prices in the market maker's history, though he is not a strategic player. If he were a strategic player, he could deviate and set prices out of equilibrium. In that case, for a given history of trades, he would make different inferences for different trajectory of prices.

insider's payoff is

$$\Pi(P,X) = \sum_{n=0}^{\nu}[e^{-\nu\delta\Delta}\bar{V}_{t_{\nu+1}} - e^{-n\delta\Delta}p_n]x_n.$$

With uncertainty, the risk-neutral insider maximizes the expected value of $\Pi(P,X)$. Let $V_n$ denote the insider's expected discounted value of the fundamental value at time $\tau$ given that the fundamental value has not been publicly revealed yet and her information at time $t_n$. That is

$$V_n = \mathbb{E}[e^{-\delta(t_\nu - t_n)}\bar{V}_{t_{\nu+1}} \mid \nu \geq n,\, \bar{V}_{t_n}] = \mathbb{E}[e^{-(\nu-n)\delta\Delta} \mid \nu \geq n]\bar{V}_{t_n} = \left[\frac{1-q}{1-\rho}\right]\bar{V}_{t_n},$$

where $\rho = qe^{-\delta\Delta} = e^{-(\theta+\delta)\Delta}$. $V_n$ represents the current *intrinsic value* of the asset. Let $\sigma_v = \bar{\sigma}_v(1-q)/(1-\rho)$. Then

$$V_{n+1} = V_n + W_n,$$

where $\{W_n\}$ is a sequence of i.i.d. normal random variables with mean 0 and variance $\Sigma_v = \sigma_v^2\Delta$.

**Definition 1** *A strategy for the market maker is an $\mathcal{F}_t^M$-adapted process $\{P_t\}_{0\leq t\leq\tau}$, and a strategy for the insider is an $\mathcal{F}_t^I$-adapted process $\{X_t\}_{0\leq t\leq\tau}$. The profile $(P,X)$ is an equilibrium if (i) for any $n \geq 0$*

$$P_{t_n} = \mathbb{E}[e^{-\delta(t_\nu - t_n)}\bar{V}_{t_{\nu+1}} \mid \nu \geq n,\, X,\, \mathcal{F}_{t_n}^M] = \mathbb{E}[V_n \mid X,\, \mathcal{F}_{t_n}^M],$$

*and (ii) given $P$, $\mathbb{E}[\Pi(P,X)]$ is bounded above and $\{X_t\}$ maximizes $\mathbb{E}[\Pi(P,X)]$.*

We do not model explicitly competition among market makers, but we implicitly assume that our market maker competes in prices with other market makers. In equilibrium, this competition drives the market maker to set the price equal to the expected value of the asset market value at time $t_{\nu+1}$ given the history of information he has observed so far and the insider's trading strategy. The market maker only uses his history to make inferences about the past choices of the insider and therefore, indirectly, about the distribution of $\bar{V}_t$. The insider chooses her strategy so as to maximize her expected discounted profit, given that she knows how the market maker will choose prices.

In equilibrium, the market maker's expected payoff is 0 and the insider's expected payoff is positive. In expectation, the insider's profits are equal to the liquidity traders' losses. In our model the liquidity traders are very primitive and are not sensitive to losses. A more realistic assumption would require that the volume they trade decreases with the losses they make. Condition (ii) for an equilibrium makes the (minimal) requirement that the liquidity traders' losses be finite.

The model is not exactly a game and our definition of an equilibrium does not coincide with that of a Nash equilibrium. However, Kyle (1985) suggests that this definition would coincide with that of a Nash equilibrium in a game where two market makers simultaneously bid prices after observing the current volume of trade and the winner gets the right to clear the market at the wining price. To avoid collusion, we can assume that there is a large population of market makers and that each market maker participates in the bidding game only once.

We will restrict attention to Markovian equilibria with a particular state space. At the begining of period $n$, before the market maker observes the volume of trade, the state is $(n, v_{n-1}, \Sigma_{n-1})$, where

$v_{n-1}$ is the market maker's estimate of $V_n$ and $\Sigma_{n-1}$ is the variance of this estimate. Note that since $V_n = V_{n-1} + W_{n-1}$, and $W_{n-1}$ is an independent random variable with mean 0 and variance $\Sigma_v$, $v_{n-1}$ coincides with the market maker's estimate of $V_{n-1}$, but as an estimate of $V_{n-1}$, the variance is $\Sigma_{n-1} - \Sigma_v$. Since the market maker's estimate of $V_n$ depends on the strategy $X$ of the insider, the state and corresponding Markovian strategy profile need to be specified simultaneously.

**Definition 2** *A strategy profile $(P, X)$ is Markovian if for each $n$, the insider's order $x_n$ and the market maker's price $p_n$ depend only on the current state $(n, v_{n-1}, \Sigma_{n-1})$ and the signals they receive in period $n$, $V_n$ for the insider and $z_n$ for the market maker. In this case we write $x_n = X_n(v_{n-1}, \Sigma_{n-1}, V_n)$ and $p_n = P_n(v_{n-1}, \Sigma_{n-1}, z_n)$. The state evolves according to the following transition rule*

$$v_n = \mathbb{E}[V_{n+1}|v_{n-1}, \Sigma_{n-1}, z_n, X] \quad \textit{and} \quad \Sigma_n = \mathbb{E}[(V_{n+1} - v_n)^2|v_{n-1}, \Sigma_{n-1}, z_n, X], \ \textit{where}$$
$$v_{-1} = \left[\frac{1-q}{1-\rho}\right] \bar{v}_0 \quad \textit{and} \quad \Sigma_{-1} = \left[\frac{1-q}{1-\rho}\right]^2 \bar{\Sigma}_0$$

*grad If $(P, X)$ is a Markovian strategy profile, let*

$$\bar{\Pi}_n(v_{n-1}, \Sigma_{n-1}, V_n) = \mathbb{E}[\sum_{k=n}^{\nu}(V_n - e^{-(k-n)\Delta}p_k)x_k \mid v_{n-1}, \Sigma_{n-1}, V_n, (P, X)]$$

*be the insider's expected payoff for the transactions made from period $n$ until the fundamental value is publicly revealed, discounted to the end of period $n$, when the current state is $(n, v_{n-1}, \Sigma_{n-1})$ and the insider observes $V_n$. When $(P, X)$ is a Markovian equilibrium, $p_n = v_n$ for all $n$.*

Below we construct linear Markovian equilibria $(P, X)$ such that

$$P_n(v_{n-1}, \Sigma_{n-1}, z_n) = v_{n-1} + \lambda_n(\Sigma_{n-1})z_n \quad \text{and} \quad X_n(v_{n-1}, \Sigma_{n-1}, V_n) = \beta_n(\Sigma_{n-1})(V_n - v_{n-1}), \quad (1)$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ are sequences of functions $\lambda_n, \beta_n : \mathbb{R}_{++} \to \mathbb{R}_+$. In order to analyze these strategies, we need a couple of preliminary results.

Each period $n$, the market maker uses the new observation $z_n$ to update his prior distribution on $V_n$. When the insider chooses her order according to the rule $x_n = \beta_n(\Sigma_{n-1})(V_n - p_{n-1})$, $(V_n, z_n)$ has a multinormal joint distribution. The Projection Theorem (see Lemma 2 below) implies that conditional on $z_n$, $V_n$ has a normal distribution whose variance is *independent* of $z_n$. Thus, in equilibrium, the trajectory $\{\Sigma_n\}$ is *deterministic* and independent of the history of trades. Therefore the sequences $\{\lambda_n\}$ and $\{\beta_n\}$ are also deterministic and hereafter we drop the arguments $\Sigma_{n-1}$. Also, since in equilibrium $p_n = v_n$ for all $n$, hereafter we do not differentiate these two variables.

Assume that the market maker's strategy $P$ satisfies (1) for some sequence $\{\lambda_n\} \subset \mathbb{R}_{++}$. Given $P$, the insider confronts each period $n$ a non-stationary dynamic programming problem. Let $\hat{\Pi}_n(p, V)$ be the insider's total expected discounted value from period $n$ onward (discounted to the end of period $n$) when the price and intrinsic value in period $n-1$ are $(p, V)$. If $\{\lambda_n\}$ satisfies a certain transversality condition, the sequence $\{\hat{\Pi}_n\}$ satisfies a Bellman equation.

Let $\mathbb{B}$ be the space of continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbb{B}^\infty$ be the space of sequences $\Pi = \{\Pi_n\}$ such that $\Pi_n \in \mathbb{B}$ for all $n \geq 0$. Recall that $\{(y_n, W_n)\}$ is an independent sequence of i.i.d. normal random variables, with 0 mean and covariance matrix

$$\begin{bmatrix} \Sigma_y & 0 \\ 0 & \Sigma_v \end{bmatrix}$$

For any $\Pi \in \mathbb{B}^\infty$, $n \geq 0$ and $(p, V) \in \mathbb{R}^2$, let

$$b_n(\Pi_{n+1})(p, V) = \max_x \, (V - p - \lambda_n x)x + \rho\mathbb{E}\big[\Pi_{n+1}(p + \lambda_n(x + y_n), V + W_n)\big],$$

and let $B(\Pi)$ be the sequence of functions $\{B(\Pi)_n\}$, where $B(\Pi)_n = b_n(\Pi_{n+1})$ for each $n \geq 0$. When $\{\lambda_n\}$ converges to 0 'too fast' (for example, faster than $\{\omega^n\}$ for some $0 \leq \omega < \rho$), $\hat{\Pi}_n(p, V)$ is unbounded. But if, for example, $\lambda_n \geq \omega^n$ for some $\omega \geq \rho$, each $\hat{\Pi}_n(p, V)$ is bounded and $\hat{\Pi}$ satisfies the Bellman equation $\hat{\Pi} = B(\hat{\Pi})$ (that is, $\hat{\Pi}$ is a fixed point of $B$).

**Lemma 1** (Optimal Profits) *Assume that $\{P_n\}$ satisfies (1) for the sequence $\{\lambda_n\} \subset \mathbb{R}_{++}$. Let*

$$S = \sum_{n=1}^{\infty} \frac{\rho^n}{\lambda_n}.$$

*If $S = \infty$ then $\hat{\Pi}_n(p, V) = \infty$ for all $n \geq 0$ and $(p, V) \in \mathbb{R}^2$. If $S < \infty$ and there is $M > 0$ such that $\lambda_n < M$ and $\rho\lambda_n/\lambda_{n+1} \leq 1$ for all $n \geq 0$, then there exist positive sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ such that $\lambda_n\alpha_{n+1} \leq 1/2$ and $\rho\hat{\Pi}_n(p, V) = \alpha_n(p - V)^2 + \gamma_n$ for all $n \geq 0$ and $(p, V) \in \mathbb{R}^2$, and $\hat{\Pi} = B(\hat{\Pi})$.*

If for some $\omega \in (\rho, 1]$, the sequence $\{\lambda_n\}$ satisfies $\lambda_{n+1}/\lambda_n \geq \omega$ for all $n \geq 0$, then $\sum \rho^n/\lambda_n < \infty$ and $\hat{\Pi}_n$ is well defined for all $n \geq 0$. However, the condition $\lambda_{n+1}/\lambda_n > \rho$ for all $n \geq 0$ may not be sufficient. For example, if $\lambda_n = \rho^n(1 + [n + 1]^{-1})$ for all $n \geq 0$, then $\sum \rho^n/\lambda_n = \infty$ and $\hat{\Pi}_n \equiv \infty$ for all $n \geq 0$, even though $\lambda_{n+1}/\lambda_n > \rho$ for all $n \geq 0$.

**Lemma 2** (Projection Theorem for Normal Random Variables) *Consider a normally distributed two-dimensional random vector $(\xi, \eta)$. Then, $\xi$ admits the following factorization*

$$\xi = \mathbb{E}[\xi] + \frac{\mathbb{C}\mathrm{ov}[\xi, \eta]}{\mathbb{V}\mathrm{ar}[\eta]} (\eta - \mathbb{E}[\eta]) + \epsilon,$$

*where $\epsilon$ is a normally distributed random variable independent of $\eta$ with mean $\mathbb{E}[\epsilon] = 0$ and variance $\mathbb{V}\mathrm{ar}[\epsilon] = \mathbb{V}\mathrm{ar}[\xi]\,(1 - r^2)$, and $r$ is the correlation coefficient between $\xi$ and $\eta$. It follows that*

$$\mathbb{E}[\xi|\eta = z] = \mathbb{E}[\xi] + \frac{\mathbb{C}\mathrm{ov}[\xi, \eta]}{\mathbb{V}\mathrm{ar}[\eta]} (z - \mathbb{E}[\eta]) \quad \textit{and}$$
$$\mathbb{V}\mathrm{ar}[\xi|\eta = z] = \mathbb{V}\mathrm{ar}[\epsilon] = \mathbb{V}\mathrm{ar}[\xi]\,(1 - r^2).$$

An important conclusion of the Projection Theorem is that the conditional variance $\mathbb{V}\mathrm{ar}[\xi|\eta = z]$ is independent of $z$. In the context of our linear Markovian equilibrium, this fact implies that the evolution of the variance $\Sigma_n$ is independent of the volumes of trade and the insider's trading decisions.

**Theorem 1** *There exist unique sequences $\{\lambda_n\}$, $\{\beta_n\} \in \mathbb{R}_{++}$ such that the linear strategy profile $(P, X)$ defined by (1) is a Markovian equilibrium. In equilibrium, $\{\Sigma_n\}$ is a deterministic trajectory that is not affected by the (stochastic) choices of the insider and the market maker. Furthermore, there exist sequences $\{\alpha_n\} \subset \mathbb{R}_{++}$ and $\{\gamma_n\}$ such that the insider's expected payoff for $(P, X)$ satisfies*

$$\rho\,\bar{\Pi}_n(p, \Sigma, V) = \alpha_n \left(V - p\right)^2 + \gamma_n \qquad \textit{for all } n \geq 0. \tag{2}$$

Proof: The proof requires to establish three facts: (i) assuming that $X_n$ satisfies (1) for some $\beta_n$, there exists a constant $\lambda_n$ such that $\mathbb{E}[V_{n+1} \mid v_{n-1}, \Sigma_{n-1}, z_n, X_n] = v_{n-1} + \lambda_n z_n$; (ii) assuming that $\{P_n\}$ satisfies (1) for some sequence $\{\lambda_n\}$, $\{\bar{\Pi}_n\}$ satisfies (2) for some sequence $\{(\alpha_n, \gamma_n)\}$ and $\{X_n\}$ satisfies (1) for some sequence $\{\beta_n\}$; and (iii) there are unique sequences $\{\lambda_n\}$ and $\{\beta_n\}$ such that the corresponding strategy profile $(P, X)$ defined by (1) is a Markovian equilibrium.

Assume that $X_n$ is given by (1) for some constant $\beta_n$, and that $p_{n-1} = v_{n-1}$. Define the random variables $\xi = V_n - p_{n-1}$ and $\eta = \beta_n(V_n - p_{n-1}) + y_n$. Conditional on $(v_{n-1}, \Sigma_{n-1})$, the vector $(\xi, \eta)$ is normally distributed with

$$\begin{aligned} \mathbb{E}[\xi] &= 0, \quad \mathbb{V}\mathrm{ar}[\xi] = \Sigma_{n-1} \\ \mathbb{E}[\eta] &= 0, \quad \mathbb{V}\mathrm{ar}[\eta] = \beta_n^2\Sigma_{n-1} + \Sigma_y \\ \mathbb{C}\mathrm{ov}(\xi, \eta) &= \mathbb{E}[\xi(\beta_n\xi + y_n)] = \beta_n\Sigma_{n-1}, \quad \text{and} \quad r = \frac{\beta_n\sqrt{\Sigma_{n-1}}}{\sqrt{\beta_n^2\Sigma_{n-1} + \Sigma_y}} . \end{aligned}$$

By the Projection Theorem,

$$\begin{aligned} v_n &= \mathbb{E}[V_{n+1} \mid \eta = z_n] = p_{n-1} + \mathbb{E}[(V_n - p_{n-1}) + W_n \mid \eta = z_n] = p_{n-1} + \frac{\beta_n\Sigma_{n-1}}{\beta_n^2\Sigma_{n-1} + \Sigma_y} z_n \\ \mathbb{V}\mathrm{ar}[V_n \mid \eta = z_n] &= \mathbb{V}\mathrm{ar}[\xi \mid \eta = z_n] = \Sigma_{n-1}\left[1 - \frac{\beta_n^2\Sigma_{n-1}}{\beta_n^2\Sigma_{n-1} + \Sigma_y}\right] = \frac{\Sigma_{n-1}\Sigma_y}{\beta_n^2\Sigma_{n-1} + \Sigma_y} . \end{aligned}$$

Therefore,

$$\Sigma_n = \mathbb{V}\mathrm{ar}[V_{n+1} \mid \eta = z_n] = \mathbb{V}\mathrm{ar}[V_n + W_n \mid \eta = z_n] = \Sigma_v + \frac{\Sigma_{n-1}\Sigma_y}{\beta_n^2\Sigma_{n-1} + \Sigma_y} , \tag{3}$$

and $\Sigma_n$ is independent of $z_n$. Since in equilibrium, $p_n = P_n(p_{n-1}, \Sigma_{n-1}, z_n) \equiv v_n$, $P_n(p_{n-1}, \Sigma_{n-1}, z_n)$ satisfies (1) with

$$\lambda_n = \frac{\beta_n\Sigma_{n-1}}{\beta_n^2\Sigma_{n-1} + \Sigma_y} . \tag{4}$$

Now assume that $\{P_n\}$ satisfies (1) for some sequence $\{\lambda_n\}$ such that $\sum \rho^n/\lambda_n < \infty$ and $\rho\lambda_n/\lambda_{n+1} \leq 1$ for all $n \geq 1$. Then, by Lemma 1, there exist a sequence $\{(\alpha_n, \gamma_n)\}$ such that $\{\bar{\Pi}_n\}$ satisfies (2). Therefore, in period $n$, the insider's expected value $\bar{\Pi}_n(p_{n-1}, \Sigma_{n-1}, V_n)$ is

$$\begin{aligned} &\max_x \ \mathbb{E}\left[(V_n - P_n(p_{n-1}, \Sigma_{n-1}, x + y_n))x + \rho\bar{\Pi}_{n+1}(P_n(p_{n-1}, \Sigma_{n-1}, x + y_n), \Sigma_n, V_{n+1}) \mid V_n\right] \\ = &\max_x \ \mathbb{E}\left[(V_n - p_{n-1} - \lambda_n(x + y_n))x + \alpha_{n+1}(V_n + W_n - p_{n-1} - \lambda_n(x + y_n))^2 + \gamma_{n+1}\right] \\ = &\max_x \ \left[(V_n - p_{n-1} - \lambda_n x)x + \alpha_{n+1}(\lambda_n^2 x^2 - 2\lambda_n x(V_n - p_{n-1})) + C\right], \end{aligned} \tag{5}$$

where $C = \alpha_{n+1}((V_n - p_{n-1})^2 + \Sigma_v + \lambda_n^2 \Sigma_y) + \gamma_{n+1}$ is independent of $x$. This is the Bellman equation for period $n$; the right-hand side of (5) is precisely $b_n(\bar{\Pi}_{n+1}(\cdot, \Sigma_n, \cdot))(p_{n-1}, v_{n-1})$. By Lemma 1, $\lambda_n \alpha_{n+1} < 1$, so the quadratic objective function is a concave function of $x$ and the optimal solution is obtained from the first-order condition:

$$x^* = \beta_n (V_n - p_{n-1}) \quad \text{where} \quad \beta_n = \frac{1 - 2\lambda_n \alpha_{n+1}}{2\lambda_n (1 - \lambda_n \alpha_{n+1})} \,. \tag{6}$$

Thus $X_n$ defined by (1) is indeed the insider's best reply function.

Equations (5) and (6) imply that

$$\bar{\Pi}_n(p_{n-1}, \Sigma_{n-1}, V_n) = \frac{(V_n - p_{n-1})^2}{4\lambda_n (1 - \lambda_n \alpha_{n+1})} + \alpha_{n+1}(\Sigma_v + \lambda_n^2 \Sigma_y) + \gamma_{n+1}.$$

That is

$$\frac{\alpha_n}{\rho} = [4\lambda_n (1 - \lambda_n \alpha_{n+1})]^{-1} \tag{7}$$

$$\frac{\gamma_n}{\rho} = \gamma_{n+1} + \alpha_{n+1}(\Sigma_v + \lambda_n^2 \Sigma_y). \tag{8}$$

Equilibrium conditions (3), (4) and (6) – (8) define recursively the sequence $\{(\Sigma_n, \lambda_n, \beta_n, \alpha_n, \gamma_n)\}$. As we will see below, given $\Sigma_{-1}$, each sequence is uniquely identified by the choice of $\beta_0$. However, the sequence becomes infeasible (for example, $\beta_n < 0$ for some $n$) if $\beta_0$ is not chosen properly. There is a unique choice $\beta_0^*$ that leads to a feasible sequence that also satisfies $\sum \rho^n / \lambda_n < \infty$. By Lemma 1, in this case $\bar{\Pi}$ satisfies (2) and therefore the linear Markovian strategy $(P, X)$ corresponding to $\{(\lambda_n, \beta_n)\}$ is an equilibrium. All other choices of $\beta_0$ lead to infeasible sequences or to sequences that satisfy $\sum \rho^n / \lambda_n = \infty$, and therefore, by Lemma 1, are not consistent with equilibrium. ∎

Starting from $(\Sigma_{-1}, \beta_0)$, we now recursively construct the sequence $\{(\Sigma_n, \beta_{n+1})\}$ and establish the properties invoked at the end of the previous proof. Equations (6) and (4) imply that

$$\alpha_{n+1} = \frac{1 - 2\lambda_n \beta_n}{2\lambda_n (1 - \lambda_n \beta_n)} = \frac{\Sigma_y^2 - \beta_n^4 \Sigma_{n-1}^2}{2\beta_n \Sigma_{n-1} \Sigma_y}.$$

Combining this equation with (7) and (4), we obtain

$$\alpha_n = \frac{\rho}{4\lambda_n (1 - \lambda_n \alpha_{n+1})} = \frac{\rho(1 - \lambda_n \beta_n)}{2\lambda_n} = \frac{\rho \Sigma_y}{2\beta_n \Sigma_{n-1}}. \tag{9}$$

The last two equations (with the time index shifted by 1) imply that

$$\frac{\Sigma_y^2 - \beta_n^4 \Sigma_{n-1}^2}{2\beta_n \Sigma_{n-1} \Sigma_y} = \frac{\rho \Sigma_y}{2\beta_{n+1} \Sigma_n} \quad \text{or} \quad \beta_{n+1} \Sigma_n = \rho \beta_n \Sigma_{n-1} \left[ \frac{\Sigma_y^2}{\Sigma_y^2 - \beta_n^4 \Sigma_{n-1}^2} \right] . \tag{10}$$

Equations (3) and (10) define $(\Sigma_n, \beta_{n+1})$ as a function of $(\Sigma_{n-1}, \beta_n)$. The sequence $\{(\lambda_n, \alpha_n, \gamma_n)\}$ can be derived afterwards, using equations (4), (7) and (8), once the whole sequence $\{(\Sigma_n, \beta_{n+1})\}$ has been computed first. To compute the sequence $\{(\Sigma_n, \beta_{n+1})\}$ recursively, it is convenient to introduce the following change of variables

$$A_n = \frac{\Sigma_{n-1}}{\Sigma_v} \quad \text{and} \quad B_n = \frac{\beta_n \Sigma_{n-1}}{\sqrt{\Sigma_y \Sigma_v}} \,.$$

Then, equations (3) and (10) imply that

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} F_A(A_n, B_n) \\ F_B(A_n, B_n) \end{bmatrix} \quad \text{where} \quad F_A(A_n, B_n) = 1 + \frac{A_n^2}{A_n + B_n^2} \text{ and } F_B(A_n, B_n) = \rho \left[ \frac{A_n^2 \, B_n}{A_n^2 - B_n^4} \right].$$

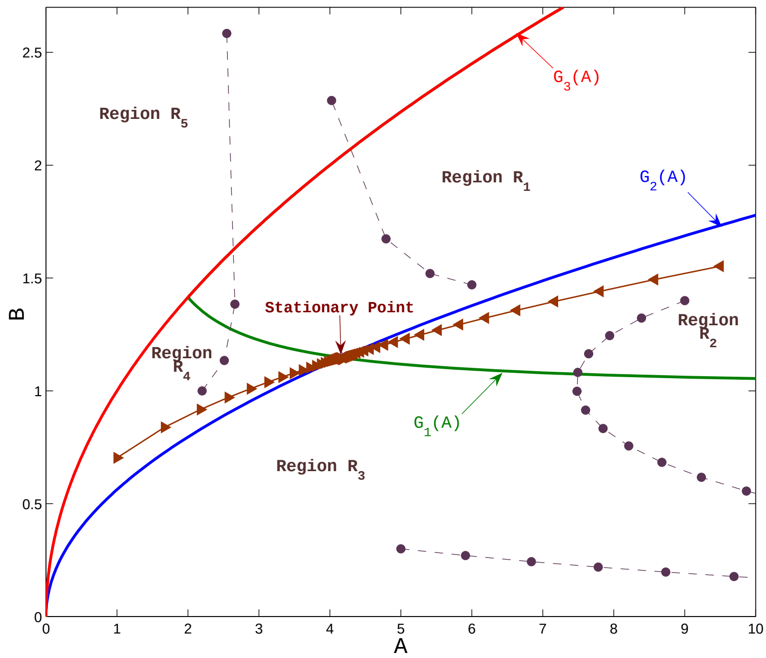


Figure 1: Partition induced by the functions $G_1$, $G_2$ and $G_3$.

Let

$$G_1(A) = \sqrt{\frac{A}{A-1}}, \quad G_2(A) = \sqrt{A}\,[1-\rho]^{1/4} \quad \text{and} \quad G_3(A) = \sqrt{A}.$$

Since in equilibrium $\beta_n > 0$ for all $n$, a point $(A, B)$ is *feasible* only if $F_B(A, B) \geq 0$, that is, only if $B \leq G_3(A)$. The function $G_1$ is defined so that $F_A(A, G_1(A)) = A$. If $B > G_1(A)$, then $F_A(A, B) < A$, and if $B < G_1(A)$, then $F_A(A, B) > A$. Similarly, the function $G_2$ is defined so that $F_B(A, G_2(A)) = B$. If $B > G_2(A)$, then $F_B(A, B) > B$, and if $B < G_1(A)$, then $F_B(A, B) < B$. As Figure 1 above shows, the graphs of these functions partition the $(A, B)$ space into 5 regions. In $R_1$, $F(A, B)$ is always to the left and higher than $(A, B)$, and any sequence $\{(A_n, B_n)\}$ with initial point $(A_0, B_0)$ in this region eventually crosses the graph of $G_3$ and becomes infeasible. In $R_2$, $F(A, B)$ is always to the left and lower than $(A, B)$. In $R_3$, $F(A, B)$ is always to the right and lower than $(A, B)$. In $R_4$, $F(A, B)$ is always to the right and higher than $(A, B)$. $R_5$ is the region of infeasible points. In Figure 1 we have also plotted four sequences, each starting in a different region. A sequence that remains feasible must start in $R_2$, $R_3$ or $R_4$, and any sequence that starts in $R_3$ always remain feasible. But not all sequences that start in $R_2$ or $R_4$ remain feasible. Sequences that start in $R_1$ always become infeasible.

By definition, the intersection of the graphs of $G_1$ and $G_2$ define a stationary point $(\hat{A}, \hat{B})$ such that $(\hat{A}, \hat{B}) = F(\hat{A}, \hat{B})$. This stationary point is

$$\hat{A} = \frac{1 + \sqrt{1-\rho}}{\sqrt{1-\rho}} \qquad \text{and} \qquad \hat{B} = \sqrt{1 + \sqrt{1-\rho}}\,.$$

The corresponding $(\hat{\Sigma}, \hat{\beta}, \hat{\lambda}, \hat{\alpha}, \hat{\gamma})$ associated with $(\hat{A}, \hat{B})$ is

$$\hat{\Sigma} = \hat{A}\Sigma_v, \quad \hat{\beta} = \frac{\hat{B}}{\hat{A}}\sqrt{\frac{\Sigma_y}{\Sigma_v}}, \quad \hat{\lambda} = \frac{1}{\hat{B}}\sqrt{\frac{\Sigma_v}{\Sigma_y}}, \quad \hat{\alpha} = \frac{\rho}{2\hat{B}}\sqrt{\frac{\Sigma_y}{\Sigma_v}} \quad \text{and} \quad \hat{\gamma} = \frac{\rho\hat{\alpha}(\Sigma_v + \hat{\lambda}^2\Sigma_y)}{1-\rho},$$

where we used the definitions of $A_n$ and $B_n$ to compute $\hat{\Sigma}$ and $\hat{\beta}$; (4) and the identities $\hat{A} = F_A(\hat{A}, \hat{B})$ and $\hat{B} = G_1(\hat{A})$ to compute $\hat{\lambda}$; (9) to compute $\hat{\alpha}$; and (8) to compute $\hat{\gamma}$. If $(A_0, B_0) = (\hat{A}, \hat{B})$, then $(A_n, B_n) = (\hat{A}, \hat{B})$ for all $n \geq 1$. Therefore, if $(\Sigma_{-1}, \beta_0) = (\hat{\Sigma}, \hat{\beta})$, then $(\Sigma_{n-1}, \beta_n, \lambda_n, \alpha_n, \gamma_n) = (\hat{\Sigma}, \hat{\beta}, \hat{\lambda}, \hat{\alpha}, \hat{\gamma})$ for all $n \geq 0$. Thus, if $\Sigma_{-1} = \hat{\Sigma}$, there is a stationary Markovian equilibrium, where

$$P_n(p_{n-1}, \Sigma_{n-1}, z_n) = p_{n-1} + \hat{\lambda} z_n \quad \text{and} \quad X_n(p_{n-1}, \Sigma_{n-1}, V_n) = \hat{\beta}(V_n - p_{n-1}) \quad \text{for all } n \geq 0.$$

In this equilibrium, the variance of the market maker's estimate remains constant: $\Sigma_n = \hat{\Sigma}$ for all $n \geq 0$. Along the stochastic equilibrium path, the fundamental value and price evolve until time $\tau$ according with the process

$$V_{n+1} = V_n + W_n \quad \text{and} \quad p_{n+1} = \left[\frac{\sqrt{1-\rho}}{1+\sqrt{1-\rho}}\right] V_n + \left[\frac{1}{1+\sqrt{1-\rho}}\right] p_n + \left[\frac{\Sigma_v}{\Sigma_y(1+\sqrt{1-\rho})}\right]^{\frac{1}{2}} y_n.$$

By continuity of the vector field $F$, there exists a curve $\mathcal{C}$, contained in $R_2 \cup R_4$ and passing through $(\hat{A}, \hat{B})$, such that $F(A, B) \in \mathcal{C}$ for all $(A, B) \in \mathcal{C}$. That is, $\mathcal{C}$ is the largest subset of $\mathbb{R}^2$ such that $F(\mathcal{C}) \subset \mathcal{C}$ and $(\hat{A}, \hat{B}) \in \mathcal{C}$. We do not have an analytical representation for $\mathcal{C}$, but we can approximate it numerically. This curve is strictly increasing, and it approaches the origin to the left (but it does not contain it). Therefore, there exists a strictly increasing function $\psi : (0, \infty) \to (0, \infty)$, such that $(A, B) \in \mathcal{C}$ if and only if $B = \psi(A)$. For any initial $A_0 > 0$, let $B_0 = \psi(A_0)$. Then the sequence $\{(A_n, B_n)\}$, where $(A_{n+1}, B_{n+1}) = F(A_n, B_n)$ for each $n$, is contained in $\mathcal{C}$ (that is, $B_n = \psi(A_n)$ for all $n \geq 0$) and therefore remains feasible forever. Moreover, $(A_n, B_n) \to (\hat{A}, \hat{B})$ as $n \to \infty$. When $A_0 < \hat{A}$ (respectively, $A_0 > \hat{A}$), $B_0 < \hat{B}$ ($B_0 > \hat{B}$) and $\{(A_n, B_n)\}$ is monotonically increasing (decreasing). Since $\psi$ is concave and

$$\lambda_n = \frac{1}{B_n}\sqrt{\frac{\Sigma_v}{\Sigma_y}} \quad \text{and} \quad \beta_n = \frac{B_n}{A_n}\sqrt{\frac{\Sigma_y}{\Sigma_v}},$$

the sequence $\{(\lambda_n, \beta_n)\}$ is also monotone and $(\lambda_n, \beta_n) \to (\hat{\lambda}, \hat{\beta})$. Therefore, $\lambda_n \geq \min\{\lambda_0, \hat{\lambda}\}$ for all $n \geq 1$, and for *any* $\omega \in (\rho, 1)$ there exists $\ell > 0$ so that $\lambda_n \geq \omega^n/\ell$. Hence, by Lemma 1, $\{\bar{\Pi}_n\}$ satisfies (2) for the sequence $\{(\alpha_n, \gamma_n)\}$ and the linear strategy $(P, X)$ associated with the sequence $\{(\lambda_n, \beta_n)\}$ is an equilibrium. In summary, for any given $\Sigma_{-1} > 0$, if we initialize

$$\beta_0 = \Psi(\Sigma_{-1}) \qquad \text{where} \qquad \Psi(\Sigma_{-1}) = \frac{\sqrt{\Sigma_y \Sigma_v}}{\Sigma_{-1}} \psi\left(\frac{\Sigma_{-1}}{\Sigma_v}\right),$$

we obtain a feasible sequence $\{(\Sigma_{n-1}, \beta_n, \lambda_n, \alpha_n, \gamma_n)\}$, and the corresponding linear strategy $(P, X)$ is a Markovian equilibrium.

For any given $\Sigma_{-1} > 0$, if $\beta_0 > \Psi(\Sigma_{-1})$, the corresponding $(A_0, B_0)$ lies above $\mathcal{C}$ (that is, $B_0 > \psi(A_0)$). In this case, we show below that the sequence $\{(A_n, B_n)\}$ will eventually become infeasible (that is, for some finite $n$, $(A_n, B_n) \in R_5$). Therefore, such choice of $\beta_0$ is not compatible with equilibrium. If $\beta_0 < \Psi(\Sigma_{-1})$ instead, the corresponding $(A_0, B_0)$ lies below $\mathcal{C}$ and the sequence

$\{(A_n, B_n)\}$ remains feasible forever. However, in this case we show below that the sequence enters region $R_3$ and remains there forever afterwards. Lemma 3 then establishes that $\sum \rho^n/\lambda_n = \infty$. Therefore, by Lemma 1, the sequence $\{\lambda_n\}$ is not consistent with equilibrium. Thus, the only feasible choice is $\beta_0 = \Psi(\Sigma_{-1})$, leading to the Markovian equilibrium described above.

We now show that if $(A_0, B_0) \in R_2$, the sequence $\{(A_n, B_n)\}$ cannot jump to $R_4$. That is, if the sequence abandons the region $R_2$, it must go to regions $R_1$ or $R_3$. Indeed, let $(A, B) \in R_2$, $(A', B') = F(A, B)$, and $c = \sqrt{1-\rho}$. Then for $(A', B')$ to be in $R_4$ we must have that $G_2(A') \le B' \le G_1(A')$, which implies that $G_2(A') \le G_1(A')$, or

$$\sqrt{c\left[1+\frac{A^2}{A+B^2}\right]} = \sqrt{\frac{c(A+A^2+B^2)}{A+B^2}} \le \sqrt{\frac{1+A^2/(A+B^2)}{A^2/(A+B^2)}} = \sqrt{\frac{A+A^2+B^2}{A^2}}$$

That is, we must have that $c/(A+B^2) \le 1/A^2$ or $\sqrt{cA}\sqrt{A-1/c} \le B$. But $(A, B) \in R_2$ implies that $B \le G_2(A) = \sqrt{cA}$ and $A > \hat{A} = (1+c)/c$. Therefore, $\sqrt{cA}\sqrt{A-1} \le \sqrt{cA}$, or $A - 1/c \le 1$, which is a contradiction.

Similarly, a sequence that starts in $R_4$ cannot jump to $R_2$. Indeed, let $(A, B) \in R_4$ and $(A', B') = F(A, B)$. For $(A', B')$ to be in $R_2$ we must have that $G_1(A') \le B' \le G_2(A')$, which implies that $G_2(A') \ge G_1(A')$, or $\sqrt{cA}\sqrt{A-1/c} \ge B$. But $(A, B) \in R_4$ implies that $B \ge G_2(A) = \sqrt{cA}$ and $A < \hat{A} = (1+c)/c$. Therefore, $\sqrt{cA}\sqrt{A-1} \ge \sqrt{cA}$, or $A - 1/c \ge 1$, which is a contradiction.

**Lemma 3** *If $\beta_0 < \Psi(\Sigma_{-1})$ then $\sum \rho^n/\lambda_n = \infty$.*

**Remarks**:

- Despite the fact that insider's trades are informative and reduce the market uncertainty, when the initial variance $\Sigma_0 < \hat{\Sigma}$, $\Sigma_n$ ends up *increasing* with $n$. In this case, the variance reduction induced by insider trading is insufficient to compensate for the additional uncertainty generated by the evolution of $\{V_n\}$.

- To carry the analysis above we had to assume a state $(n, v_{n-1}, \Sigma_{n-1})$. But, in equilibrium, $\{\Sigma_n\}$ is a monotone sequence and there is a one-to-one relationship between $n$ and $\Sigma_n$. Hence, we can reduce the state variables to $(v_{n-1}, \Sigma_{n-1})$. Indeed, the equilibrium is stationary. The continuation value for the insider in period $n$, for example, does not depend on $n$ and could be written as $\bar{\Pi}(v_{n-1}, \Sigma_{n-1}, V_n)$ instead of $\bar{\Pi}_n(v_{n-1}, \Sigma_{n-1}, V_n)$. Put a different way, if we consider another problem where the initial variance is $\Sigma_{n-1}$, its equilibrium would coincide with the continuation equilibrium from period $n$ onward of the equilibrium where the initial variance is $\Sigma_{-1}$. Similarly, we could write $\beta(\Sigma_{n-1})$ instead of $\beta_n$ (and the same is true for the other sequences that define the equilibrium).

- In our definition of an equilibrium we have ruled out profiles $(P, X)$ for which $\mathbb{E}[\Pi(P, X)] = \infty$. For each $\beta_0 \in (0, \Psi(\Sigma_1))$ there is a profile $(P, X)$ such that $\mathbb{E}[\Pi(P, X)] = \infty$ but that otherwise satisfies all the conditions for an equilibrium (see Lemma 3). For those profiles, one can show that liquidity traders always have bounded (positive or negative) payoffs, independent of whether the insider's payoff is finite or not. Hence, it is the market maker that finances the insider's infinite expected rents. When the market maker sets the price equal to the expected value of the asset (as required in equilibrium), $\mathbb{E}[\Pi(P, X)] < \infty$ implies that he makes 0 profits. But when $\mathbb{E}[\Pi(P, X)] = \infty$, he makes infinite losses. We require that in equilibrium $\mathbb{E}[\Pi(P, X)] < \infty$ because an outcome where the market maker incurs infinite losses would not be sustainable.

### 2.1 The Perfect Information Case

A special case of our model is when $\Sigma_v = 0$. In this case the insider knows from the start what the value of the fundamental will be at the time when it is revealed. This is the assumption made by Kyle (1985).

**Proposition 1** *Suppose $\Sigma_v = 0$ and let $\Sigma_{-1}$ be given. Then, there exists a linear Markovian equilibrium defined by the sequences*

$$\beta_n = \sqrt{\frac{S\Sigma_y}{\Sigma_{-1}}}\,(1+S)^{\frac{n}{2}} \qquad \textit{and} \qquad \lambda_n = \sqrt{\frac{S\Sigma_{-1}}{\Sigma_y}}\,(1+S)^{-\frac{n+2}{2}}, \qquad n \geq 0,$$

*where $S$ is the unique root in $(0,1)$ of the equation $(1+S)(1-S)^2 = \rho^2$. The resulting equilibrium satisfies*

$$\Sigma_n = \frac{\Sigma_{-1}}{(1+S)^{n+1}}, \quad \alpha_n = \frac{\rho}{2}\sqrt{\frac{\Sigma_y}{S\Sigma_{-1}}}\,(1+S)^{\frac{n}{2}}, \quad \textit{and} \quad \gamma_n = \frac{\rho^2}{2}\,\frac{\sqrt{\Sigma_{-1}\,\Sigma_y\,S}}{\sqrt{1+S}-\rho}\,(1+S)^{-\frac{n+2}{2}}.$$

Note that in equilibrium the value of $\Sigma_n$ converges to 0 as $n \to \infty$. That is, the market is asymptotically efficient as the number of periods goes to infinity. Recall that $\rho = e^{-(\theta+\delta)\Delta}$, and thus $\rho$ is decreasing function of $\theta$ and $\delta$. That is, $\rho$ decreases if the insider becomes more impatient or the public revelation of the fundamental value happens faster. When $\rho$ decreases, $S$ increases, the insider reveals her private information faster, and market efficiency increases.

We will return to this special case in Section 4, where we derive its continuous-time counterpart by letting the period length $\Delta$ go to zero.

## 3 Continuous-Time Trading as a Discrete-Time Limit

In this section, we analyze the discrete-time linear equilibrium in Theorem 1 in the limit as $\Delta$ goes 0. This limit will help us identify heuristically some distinctive features of the equilibrium which we will use in section 4 to formally derive a continuous-time solution.

First, let us explicitly rewrite the discrete-time model in terms of the calendar time $t$. Recall that for any time $t \geq 0$ the corresponding trading period is $n = \lfloor t \rfloor$. In the discrete-time model, the insider's trading strategy in period $n$ is $x_n = \beta_n (V_n - p_{n-1})$. To get a continuous-time analogue, we would like to express the insider's strategy as a trading rate per unit time. For this, we define $\beta_t = \beta_n/\Delta$ where $n = \lfloor t \rfloor$. We also define the continuous time extensions $\Sigma_t = \Sigma_n$, $\lambda_t = \lambda_n$, $\alpha_t = \alpha_n$

and $\gamma_t = \gamma_n$ where $n = \lfloor t \rfloor$. Finally, recall that $\Sigma_y = \sigma_y^2\,\Delta$, $\Sigma_v = \sigma_v^2\,\Delta$ and $\rho = e^{-\mu\,\Delta}$, where $\mu = \delta + \theta$ and $\sigma_v = \bar{\sigma}_v(1 - e^{-\theta\Delta})/(1 - e^{-\mu\Delta})$. Note that $\sigma_v \to \bar{\sigma}_v\theta/\mu$ as $\Delta \to 0$.

Theorem 1, establishes that there exists a unique equilibrium where the processes $\lambda_t$ and $\beta_t$ are deterministic functions of $\Sigma_{t-\Delta}$. This equilibrium is defined by equations (3), (4), (8), (9) and (10) and satisfies

$$\Sigma_t = \sigma_v^2\Delta + \frac{\sigma_y^2\,\Sigma_{t-\Delta}}{\beta_t^2\Sigma_{t-\Delta}\,\Delta + \sigma_y^2}, \quad \lambda_t = \frac{\beta_t\Sigma_{t-\Delta}}{\beta_t^2\,\Sigma_{t-\Delta}\,\Delta + \sigma_y^2} \quad \text{and} \quad \beta_{t+\Delta}\Sigma_t = e^{-\mu\,\Delta}\left[\frac{\beta_t\,\sigma_y^2\,\Sigma_{t-\Delta}}{\sigma_y^2 - \beta_t^4\Sigma_{t-\Delta}^2\,\Delta^2}\right].$$

Furthermore, the insider's expected profit-to-go function at the beginning of period $\lfloor t \rfloor$ satisfies

$$\Pi_{\lfloor t \rfloor}(V, p) = \alpha_t\,(V - p)^2 + \gamma_t,$$

where

$$\alpha_t = \frac{e^{-\mu\,\Delta}\,\sigma_y^2}{2\beta_t\,\Sigma_{t-\Delta}} \quad \text{and} \quad \gamma_t\,e^{\mu\,\Delta} = \gamma_{t+\Delta} + \alpha_{t+\Delta}(\sigma_v^2 + \lambda_t^2\,\sigma_y^2)\,\Delta.$$

Figure 3 depicts the values of $\beta_t$ (left panel) and $\Sigma_t$ (right panel) for different values of $\Delta$ as function of the calendar time $t$. For any $\Delta' < \Delta$ and for any $t \geq 0$, $\beta_t(\Delta') > \beta_t(\Delta)$ and $\Sigma_t(\Delta') < \Sigma_t(\Delta)$. The

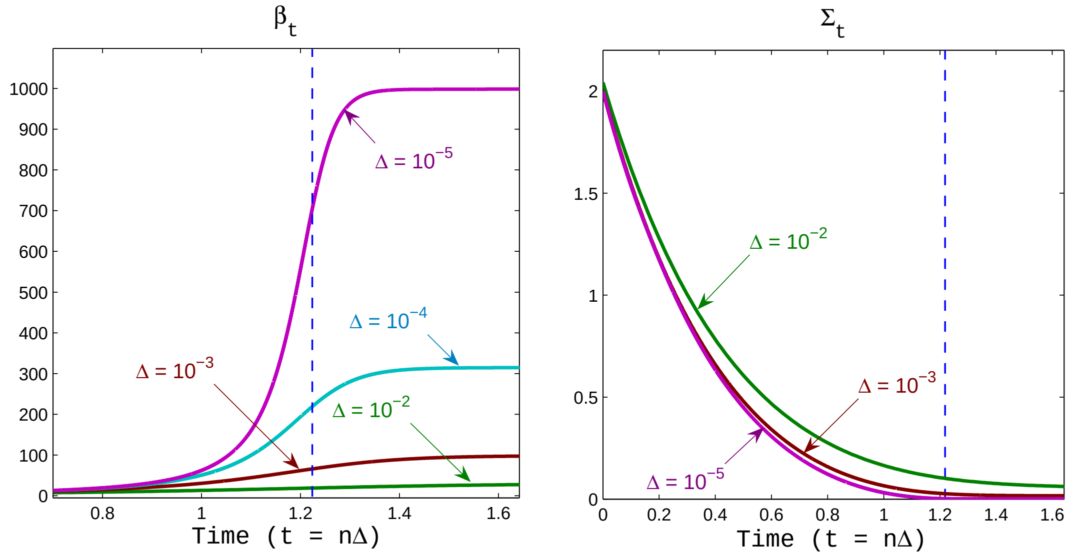


Figure 2: Evolution of $\beta_t$ (left panel) and $\Sigma_t$ (right panel) as a function of the calendar time $t = n\,\Delta$ for different values of $\Delta$. (Data: $\sigma_y^2 = 5$, $\sigma_v^2 = 0.5$, $\mu = 1$ and $\Sigma_0 = 2$.)

intuition for the first inequality is clear. As $\Delta \to 0$, the liquidity traders' volume of trade per period becomes increasingly noisier relative to the insider's volume of trade per period. More precisely, the insider's volume of trade is $\beta_t(V_t - p_t)\Delta$, while the standard deviation of the liquidity traders' volume of trade is $\sigma_y\sqrt{\Delta}$. Since $\sqrt{\Delta}/\Delta \to \infty$ as $\Delta \to 0$, when $\Delta$ decreases the insider can afford to trade faster without revealing more information. The behavior of $\beta_t$ (as a function of $t$) appears to have two distinctive phases as $\Delta \downarrow 0$. First, as $t \to 0$, $\beta_t$ converges to a fixed value $\beta_0$ independent of $\Delta$.

Second, as $t \to \infty$, $\beta_t$ converges to $\beta_\infty$ which, as a function of $\Delta$, diverges to $+\infty$ when $\Delta$ goes to 0. Furthermore, Figure 3 suggests the following stronger result

$$\lim_{\Delta \downarrow 0} \beta_t = \infty, \quad \text{for all } t \geq T,$$

where $T$ is a finite time represented by the vertical dashed line in Figure 3. Recall that in the time interval $[t_n, t_{n+1})$ the insider trades the amount $\beta_{t_n}(V_{t_n} - p_{t_n})\,\Delta$. Hence, for $t \geq T$ the insider's trading rate grows arbitrarily large as $\Delta$ goes to zero. When $\beta_t$ is large, the market maker is able to differentiate insider trading from liquidity trading. Hence, when $\Delta$ is small and $t \geq T$, the insider is revealing her private information very fast. This effect is captured in the right panel of Figure 3 that shows $\Sigma_t$ decreasing monotonically to zero and staying arbitrarily closed to zero for $t \geq T$ as $\Delta \downarrow 0$. In other words, for $t \geq T$, the market is asymptotically efficient as the period length goes to zero.

Recall that for any $\Delta > 0$, the equilibrium $(\Sigma_t, \beta_t, \lambda_t, \alpha_t, \gamma_t)$ converges to a stationary point $(\hat{\Sigma}, \hat{\beta}, \hat{\lambda}, \hat{\alpha}, \hat{\gamma})$ as $t$ goes to infinity. In terms of $\Delta$, this limit is given by

$$\hat{\Sigma} = \left(\frac{1 + \sqrt{1 - e^{-\mu\,\Delta}}}{\sqrt{1 - e^{-\mu\,\Delta}}}\right) \sigma_v^2\,\Delta, \quad \hat{\beta} = \frac{1}{\Delta}\sqrt{\frac{1 - e^{-\mu\,\Delta}}{1 + \sqrt{1 - e^{-\mu\,\Delta}}}}\,\frac{\sigma_y}{\sigma_v}, \quad \hat{\lambda} = \sqrt{\frac{1}{1 + \sqrt{1 - e^{-\mu\,\Delta}}}}\,\frac{\sigma_v}{\sigma_y},$$

$$\hat{\alpha} = \frac{e^{-\mu\,\Delta}\,\hat{\lambda}}{2}\,\frac{\sigma_y^2}{\sigma_v^2} \qquad \text{and} \qquad \hat{\gamma} = \frac{e^{-\mu\,\Delta}\,\hat{\alpha}}{1 - e^{-\mu\,\Delta}}\,(\sigma_v^2 + \hat{\lambda}^2\,\sigma_y^2)\,\Delta.$$

If we let $\Delta \downarrow 0$ the stationary equilibrium converges to

$$\lim_{\Delta \downarrow 0} (\hat{\Sigma}, \hat{\beta}, \hat{\lambda}, \hat{\alpha}, \hat{\gamma}) = \left(0, \infty, \frac{\sigma_v}{\sigma_y}, \frac{\sigma_y}{2\,\sigma_v}, \frac{\sigma_y\,\sigma_v}{\mu}\right). \tag{11}$$

The first two limits are consistent with our previous discussion: When $\Delta \downarrow 0$, the market becomes asymptotically efficient ($\hat{\Sigma} = 0$ and the insider trading rate grows arbitrarily large ($\hat{\beta} = \infty$) as $t \to \infty$. The limit above also shows that the insider makes positive profits in the limiting regime since $\hat{\alpha}$ and specially $\hat{\gamma}$ are both positive.

# 4 Continuous Time Model

In this section, we derive an "equilibrium" for the model in which trading occurs continuously over time. More precisely, the strategy profile we construct is not an equilibrium of a continuous-time game; instead, it is the limit of equilibria for a family of continuous time models where the insider's trading rate is uniformly bounded. This construction is required to introduce technical constraints in the strategy space of the insider that capture the natural limits of what is possible in a discrete time model, while at the same time preserving existence of equilibrium.

Similar to the discrete-time model, we define the intrinsic value $V_t$ to be the expected discounted value of the fundamental at time $\tau$ given the insider's information at time $t$. That is,

$$V_t = \mathbb{E}[e^{-\delta(\tau - t)}\,\bar{V}_\tau \mid \mathcal{F}_t^I, t > \tau] = \frac{\theta}{\theta + \delta}\,\bar{V}_t.$$

We also define $\sigma_v = \bar{\sigma}_v\,\theta/(\theta + \delta)$ so that $V_t$ is a driftless Brownian motion with dynamics

$$\mathrm{d}V_t = \sigma_v\,\mathrm{d}B_t^v.$$

A strategy profile is a pair of processes $(X, P)$, where $X_t \in \mathcal{F}^I_t$ is the insider's cumulative trading volume up to time $t$, and $P_t \in \mathcal{F}^M_t$ is the price set by the market maker at time $t$. For a given profile $(X, P)$, the insider's expected discounted payoff, $\Pi(P, X)$, is defined as follows

$$\Pi(P,X) = \mathbb{E}\left[e^{-\delta\,\tau}\,\bar{V}_\tau\,X_\tau - \int_0^\tau e^{-\delta\,t} P_t\,\mathrm{d}X_t - \int_0^\tau e^{-\delta\,t}\mathrm{d}[X,P]_t\right],$$

where $[X, P]_t$ is the quadratic covariation between $X_t$ and $P_t$.[3]

Given the space $\mathcal{C}$ of continuous processes adapted to the insider's information $\mathcal{F}^I_t$ , a continuous-time equilibrium is a profile $(X, P)$ with the following properties: $(i)$ given $P$, $\Pi(X, P)$ is bounded and $X \in \mathcal{C}$ maximizes[4] $\Pi(X, P)$, and $(ii)$ the price process $P$ satisfies the equilibrium condition

$$P_t = \mathbb{E}\left[V_t \,\middle|\, \mathcal{F}^M_t, X\right] \qquad 0 \le t \le \tau,$$

given the insider's trading strategy $X$.

For the analysis that follows, we find convenient to rewrite the insider's payoff using the following identity

$$e^{-\delta\,\tau}\,\bar{V}_\tau\,X_\tau = \int_0^\tau e^{-\delta\,\tau}\,\bar{V}_t\,\mathrm{d}X_t + \int_0^\tau e^{-\delta\,\tau}\,X_t\,\mathrm{d}\bar{V}_t + \int_0^\tau e^{-\delta\,\tau}\,\mathrm{d}[X,\bar{V}]_t,$$

where $[X, \bar{V}]_t$ is the quadratic covariation between $X_t$ and $\bar{V}_t$. Plugging back this identity in $\Pi$, taking expectation and canceling the stochastic integral with respect to the martingale $\bar{V}_t$, we get

$$\begin{aligned}\Pi(P,X) &= \mathbb{E}\left[\int_0^\tau \left(e^{-\delta\,\tau}\,\bar{V}_t - e^{-\delta\,t}\,P_t\right)\mathrm{d}X_t + \int_0^\tau e^{-\delta\,t}\mathrm{d}[X,V]_t - \int_0^\tau e^{-\delta\,t}\mathrm{d}[X,P]_t\right]\\ &= \mathbb{E}\left[\int_0^\infty e^{-\mu\,t}(V_t - P_t)\,\mathrm{d}X_t + \int_0^\infty e^{-\mu\,t}\mathrm{d}[X,V]_t - \int_0^\infty e^{-\mu\,t}\mathrm{d}[X,P]_t\right],\end{aligned}$$

where the second equality is based on the fact that $\tau$ is exponentially distributed with rate $\theta$ and is independent of $\mathcal{F}^I_t$. Recall also the definition $\mu = \delta + \theta$.

Our construction of a continuous-time limit equilibrium $(P, X)$ builds on the features that we heuristically derived in the previous section. That is, the limit equilibrium has two phases: an absolutely continuous phase in the interval $[0, T)$ in which $X$ has bounded variation and a singular phase in the interval $[T, \infty)$ in which $X$ has unbounded variation, for some switching time $T > 0$. In the absolutely continuous phase, the insider trades at a rate $\beta_t\,(V_t - P_t)$ and the market maker adjusts prices at a rate $\lambda_t$, where $\beta, \lambda : [0, T) \to \mathbb{R}_+$, so that

$$\mathrm{d}X_t = \beta_t\,(V_t - P_t)\,\mathrm{d}t \qquad \text{and} \qquad \mathrm{d}P_t = \lambda_t\,\mathrm{d}Z_t, \qquad t < T.$$

In the interval $[0, T)$ the variance $\Sigma_t$ decreases from $\Sigma_0$ to 0. In the singular phase $[T, \infty)$, $\Sigma_t \equiv 0$, the market maker adjusts the price at a constant rate $\lambda_T$ and the insider buys/sells at an infinite rate driving the gap between the price and the valuation instantaneously to 0. That is,

$$\mathrm{d}X_t = \frac{\mathrm{d}V_t}{\lambda_T} - \mathrm{d}Y_t \qquad \text{and} \qquad \mathrm{d}P_t = \lambda_T\,\mathrm{d}Z_t = \mathrm{d}V_t, \qquad t \ge T.$$

[3]Intuitively, this term arises because the price paid by the insider is computed 'at the end of the period', and therefore it includes the effect of the insider's 'last trade' $\mathrm{d}X_t$. For a formal derivation, see equation (11) in Back (1992).

[4]We rule out discontinuities in $X$ because they would immediately inform the market marker that he is mispricing the asset.

### 4.1 Continuous-Time Equilibrium with Restricted Trading

Since the continuous-time extension of a discrete-time trading strategy (as we defined it in Section 3) is always of bounded variation, it would be natural to let $\mathcal{C}$ be the set of continuous processes of *bounded variation* adapted to $\mathcal{F}_t^I$. However, exactly because the insider would like to use (in equilibrium) a strategy of unbounded variation, a continuous-time model with such a space $\mathcal{C}$ would not have an equilibrium. Our purpose is to approximate the discrete-time equilibrium for $\Delta$ small by a continuous-time strategy profile. We construct such a strategy profile as the limit of a sequence of continuous-time equilibria in which the insider trading rate is uniformly bounded. More specifically, for each $\bar{\beta} > 0$, we consider the restricted strategy space $\mathcal{C}(\bar{\beta})$ for the insider of all processes $X \in \mathcal{C}$ such that

$$\mathrm{d}X_t = \beta_t \, (V_t - P_t) \, \mathrm{d}t$$

for some process $\beta_t$ adapted to $\mathcal{F}_t^I$ with $|\beta_t| \leq \bar{\beta}$ for all $t \geq 0$. The continuous-time model with this strategy space for the insider does have equilibria. We construct our continuous-time approximation of the discrete-time equilibrium (for $\Delta$ small) as the limit of a continuous-time sequence of equilibria as $\bar{\beta} \to \infty$.

With a bounded trading rate, the two phases of the equilibrium are characterized by $(i)$ $\beta_t < \bar{\beta}$ in the 'absolutely continuous' phase $[0, T)$, and $(ii)$ $\beta_t = \bar{\beta}$ in the 'singular phase' $[T, \infty)$.

When $X$ is of bounded variation, the quadratic covariations $[X, V]_t$ and $[X, P]_t$ are both zero, and accordingly the last two terms of $\Pi(P, X)$ drop out and

$$\Pi(X, P) = \mathbb{E}\left[\int_0^\infty e^{-\mu\, t} \, \beta_t \, (V_t - P_t)^2 \, \mathrm{d}t\right].$$

Let us suppose that the market maker uses the pricing rule

$$\mathrm{d}P_t = \lambda_t \, \mathrm{d}Z_t, \qquad t \geq 0, \tag{12}$$

for some nonnegative process $\lambda_t$. Hence, under the restriction $X \in \mathcal{C}(\bar{\beta})$, the evolution of $P_t$ is governed by the following SDE

$$\mathrm{d}P_t = \lambda_t \, [\beta_t (V_t - P_t) \mathrm{d}t + \sigma_y \, \mathrm{d}B_t^y].$$

Note that both the insider's payoff function and the dynamics of $P_t$ depend on $P_t$ and $V_t$ only through their difference. Thus, we find it convenient to define the price differential process $M_t = V_t - P_t$ with dynamics

$$\mathrm{d}M_t = -\lambda_t \, \beta_t \, M_t \, \mathrm{d}t + \sigma_v \, \mathrm{d}B_t^v - \lambda_t \, \sigma_y \, \mathrm{d}B_t^y, \qquad t \geq 0.$$

The process $\sigma_v \, B_t^v - \lambda_t \, \sigma_y \, B_t^y$ is a driftless Gaussian process with variance $\sigma_t^2 = \sigma_v^2 + \lambda_t^2 \, \sigma_y^2$. Therefore,

$$\mathrm{d}M_t = -\lambda_t \, \beta_t \, M_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t \qquad t \geq 0,$$

where $B_t$ is a standard Brownian motion. We define the value function

$$\begin{aligned} \widehat{\Pi}(t, M) \quad &= \quad \sup_{|\beta_t| \leq \bar{\beta}} \; \mathbb{E}\left[\int_t^\infty e^{-\mu\,(s-t)} \, \beta_s \, M_s^2 \, \mathrm{d}s\right] \\ &\quad\; \text{s.t.} \quad \mathrm{d}M_s = -\lambda_s \, \beta_s \, M_s \, \mathrm{d}t + \sigma_s \, \mathrm{d}B_s, \quad s \geq t, \quad \text{and} \quad M_t = M. \end{aligned} \tag{13}$$

The process $\widehat{\Pi}(t,M)$ is the insider's optimal expected profit-to-go starting at time $t$ with an initial price differential $M_t = M$. We note that $M_t$ and $\widehat{\Pi}$ depend on both the pricing policy $\lambda$ and $\bar{\beta}$. When we wish to emphasize this dependence we will include $\lambda$ and/or $\bar{\beta}$ as part of their arguments, for example, we will write $M_t(\bar{\beta})$ or $\widehat{\Pi}(t,M,\lambda,\bar{\beta})$.

Later, we solve the optimization problem (13) and show that it admits a deterministic solution $\beta_t$. But first, let us characterize the market maker's equilibrium condition.

**Proposition 2** *Suppose the insider selects a deterministic trading rate $\beta_t$, and the market maker chooses the pricing rule (12). Then, the market maker's equilibrium condition $P_t = \mathbb{E}[V_t|\mathcal{F}_t^M, X]$ is satisfied if $\lambda_t$ is a deterministic function such that*

$$\Sigma_t\,\beta_t = \sigma_y^2\,\lambda_t \qquad \textit{and} \qquad \dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2\,\lambda_t^2, \tag{14}$$

*where $\Sigma_t = \mathbb{E}[(V_t - P_t)^2|\mathcal{F}_t^M, X]$ and $\dot{\Sigma}_t$ represents its first derivative with respect to $t$.*

The equilibrium that we construct below is defined by a pair of deterministic nonnegative processes $(\lambda_t, \beta_t)$ that solve (13) and satisfy (14).[5]

### 4.1.1 Equilibrium in $[T,\infty)$

First, we consider the singular phase of the equilibrium for $t \geq T$. It turns out that in this region, $e^{-\mu t}/\lambda_t$ is strictly decreasing.

**Proposition 3** *Suppose the market maker uses the pricing strategy (12) for some deterministic function $\lambda_t$ such that $e^{-\mu t}/\lambda_t$ is strictly decreasing in $t \geq T$. Then, the insider optimal strategy is to set $\beta_t = \bar{\beta}$ for all $t \geq T$. The value function $\widehat{\Pi}(t,M)$ is given by*

$$\widehat{\Pi}(t,M) = \int_t^\infty e^{-\mu\,(s-t)}\,\bar{\beta}\,\left[M^2\,e^{-2\bar{\beta}\int_t^s \lambda_u\,\mathrm{d}u} + \int_t^s \sigma_u^2\,e^{-2\bar{\beta}\int_u^s \lambda_r\,\mathrm{d}r}\,\mathrm{d}u\right]\mathrm{d}s, \qquad t \geq T$$

*and the price differential $M_t$ has mean reverting dynamics and satisfies*

$$M_t = M_T\,e^{-\bar{\beta}\int_T^t \lambda_s\,\mathrm{d}s} + \int_T^t \sigma_s\,e^{-\bar{\beta}\int_s^t \lambda_u\,\mathrm{d}u}\,\mathrm{d}B_s, \qquad t \geq T.$$

Since $\beta_t = \bar{\beta}$, the insider buys or sells as fast as possible depending on whether $M_t > 0$ or $M_t < 0$, respectively. This strategy is not surprising under the assumption that the market maker adjusts prices at a rate $\lambda_t$ such that $e^{-\mu t}/\lambda_t$ is decreasing. In fact, for example, a constant $\lambda_T$ implies that the market price at a time $t$ is only affected by the insider's cumulative trade up to time $t$ and not by its distribution over time prior to $t$. This, together with the fact that information leakage can occur at any time, gives the insider no incentive to delay any profitable trade.

Proposition 3 defines half of the equilibrium condition. The other half is based on the market maker's equilibrium condition (14). It turns out that for this condition to hold the rate $\lambda_t$ cannot be arbitrary.

[5]Note that if $\lambda_t < 0$ then the insider would like to set $\beta_t = \bar{\beta} > 0$. However, by Proposition 2 and the fact that $\Sigma_t \geq 0$, $\lambda_t$ and $\beta_t$ must have the same sign.

**Proposition 4** *Suppose the insider uses the trading strategy $\beta_t = \bar{\beta}$ for all $t \geq T$ and the market maker uses the pricing strategy (12). Let $L = 2\sigma_v \bar{\beta}/\sigma_y$. Then, the equilibrium condition $P_t = \mathbb{E}[V_t | \mathcal{F}_t^M, X]$ is satisfied if only if*

$$\lambda_t = \frac{\sigma_v}{\sigma_y} \left[ \frac{e^{Lt} + K}{e^{Lt} - K} \right] \quad \textit{for all} \quad t \geq T,$$

*for some constant $K$. Moreover, $\lambda_t \geq 0$ and $e^{-\mu t}/\lambda_t$ is decreasing for all $t \geq T$ only if*

$$0 \leq \frac{e^{LT} + K}{e^{LT} - K} \leq \frac{\mu}{L} + \sqrt{\frac{\mu^2}{L^2} + 1}.$$

Depending on whether $K > 0$ or $K < 0$ the resulting function $\lambda_t$ is decreasing or increasing, respectively, in $t$. Since in the equilibrium of the discrete-time model $\lambda_t$ is decreasing in $t$, we will later require that $K \geq 0$.

Let us conclude this section combining the results in Propositions 3 and 4 to compute the insider's expected payoff-to-go $\widehat{\Pi}(t, M)$ and the dynamics of the price differential $M_t$ for $t \geq T$. For this, note first that

$$\int \lambda_t \, \mathrm{d}t = \frac{1}{\bar{\beta}} \ln \left( e^{\frac{L\,t}{2}} - K\, e^{\frac{-L\,t}{2}} \right) = \frac{1}{\bar{\beta}} \left[ \ln \left( e^{L\,t} - K \right) - \frac{L\,t}{2} \right].$$

It follows then that the insider's payoff satisfies $\widehat{\Pi}(t, M) = \alpha_t \, M^2 + \gamma_t$ for $t \geq T$, where

$$\alpha_t = \bar{\beta}\, e^{-(L-\mu)\,t} \left( e^{L\,t} - K \right)^2 \int_t^\infty e^{(L-\mu)\,s} \left( e^{L\,s} - K \right)^{-2} \mathrm{d}s \tag{15}$$

$$\gamma_t = \bar{\beta}\, e^{\mu\,t} \int_t^\infty e^{(L-\mu)\,s} \left( e^{L\,s} - K \right)^{-2} \int_t^s \sigma_u^2 \, e^{-L\,u} \left( e^{L\,u} - K \right)^2 \mathrm{d}u \, \mathrm{d}s. \tag{16}$$

Similarly, in equilibrium, the price differential $M_t$ satisfies

$$M_t \left( e^{\frac{L\,t}{2}} - K\, e^{-\frac{L\,t}{2}} \right) = M_T \left( e^{\frac{L\,T}{2}} - K\, e^{-\frac{L\,T}{2}} \right) + \int_T^t \sigma_s \left( e^{\frac{L\,s}{2}} - K\, e^{-\frac{L\,s}{2}} \right) \mathrm{d}B_s.$$

### 4.1.2 Equilibrium in $[0, T)$

To find the equilibrium in $[0, T)$, we first consider the insider's problem for the pricing rule (12) with $\lambda_t = \lambda_0 \, e^{-\mu\,t}$. Based on the results in the previous section, the insider's expected profit-to-go at time $T$ is given by $\widehat{\Pi}(T, M) = \alpha_T \, M_T^2 + \gamma_T$, with $\alpha_T$ and $\gamma_T$ defined by (15) and (16). Hence, the insider's value function $\widehat{\Pi}(t, M)$ for $t \in [0, T)$ is given by

$$\begin{aligned} \widehat{\Pi}(t, M) \;&=\; \sup_{|\beta_t| \leq \bar{\beta}} \; \mathbb{E} \left[ \int_t^T e^{-\mu\,(s-t)} \, \beta_s \, M_s^2 \, \mathrm{d}s + e^{-\mu\,(T-t)} \, \widehat{\Pi}(T, M_T) \right] \\ & \quad \text{s.t.} \quad \mathrm{d}M_s = -\lambda_s \, \beta_s \, M_s \mathrm{d}s + \sigma_s \, \mathrm{d}B_s \quad \text{and} \quad M_t = M. \end{aligned}$$

**Proposition 5** *Suppose the market maker uses the pricing rule (12) for the deterministic function $\lambda_t = \lambda_0 \, e^{-\mu\,t}$ for $t \in [0, T)$. Then, $\widehat{\Pi}(t, M)$ admits a quadratic solution*

$$\widehat{\Pi}(t, M_t) = \alpha_t \, M_t^2 + \gamma_t, \quad t \in [0, T),$$

*with*

$$\alpha_t = \frac{1}{2\,\lambda_0}\, e^{\mu\, t} \qquad \textit{and} \qquad \gamma_t = \left[\gamma_0 - \frac{\sigma_y^2}{8\alpha_0\mu} - \alpha_0\sigma_v^2 t\right] e^{\mu t} + \frac{\sigma_y^2}{8\alpha_0\mu} e^{-\mu t}, \quad t \in [0, T),$$

*where* $\lambda_0 = e^{\mu T}/[2\alpha_T]$ *and* $\gamma_0$ *is chosen so that* $\lim_{t\uparrow T}\gamma_t = \gamma_T$.

Note that Proposition 5 does not specify the value of $\beta_t$. The reason for this is that the (HJB) optimality conditions for the insider's control problem leave $\beta_t$ undetermined. The pricing rule (12) with $\lambda_t = \lambda_0\, e^{-\mu\, t}$ for $t \in [0, T)$ makes the insider indifferent about her trading rates in $[0, T)$. Similar to a mixed strategy equilibrium, her trading rate is determined by the market maker's equilibrium conditions. The following result follows directly from Proposition 2.

**Proposition 6** *Suppose the insider uses a deterministic trading strategy* $\beta_t$ *and the market maker uses a deterministic strategy* $\lambda_t = \lambda_0\, e^{-\mu\, t}$. *Then, the pair* $(\lambda_t, \beta_t)$ *satisfies the market maker's equilibrium condition* $P_t = \mathbb{E}[V_t|\mathcal{F}_t^M, X]$ *if and only if*

$$\beta_t = \frac{\sigma_y^2\, \lambda_0\, e^{-\mu\, t}}{\Sigma_t} \qquad \textit{where} \qquad \Sigma_t = \Sigma_0 + \sigma_v^2\, t - \frac{\lambda_0^2\, \sigma_y^2}{2\, \mu}\left(1 - e^{-2\, \mu\, t}\right).$$

Note that the constraint on the insider's trading rate, $|\beta_t| \le \bar\beta$, reduces to

$$\lambda_0\, \sigma_y^2\, e^{-\mu\, t} \le \bar\beta\, \left[\Sigma_0 + \sigma_v^2\, t - \frac{\lambda_0^2\, \sigma_y^2}{2\, \mu}\left(1 - e^{-2\, \mu\, t}\right)\right], \qquad t \in [0, T).$$

### 4.1.3 Equilibrium in $[0, \infty)$

We now combine the results of the previous two sections to derive an equilibrium in the entire interval $[0, \infty)$ under the restriction $|\beta_t| \le \bar\beta$. Based on our previous analysis, an equilibrium is fully characterized by four parameters: $T$, $K$, $\lambda_0$, $\gamma_0$. As stated in Proposition 5, $\gamma_0$ must be adjusted to ensure that $\gamma_t$ is continuous at $T$. However, the choice of $\gamma_0$ does not interfere with the choice of the other parameters in any way. We now focus in the choice of $(k, \lambda_0, T)$.

Let $k$ be such that $K = ke^{\mu T}$, and let $h(L) = [\mu + \sqrt{\mu^2 + L^2}]/L$. The bounds on $K$ in Proposition 4 are equivalent to

$$-1 < k \le \frac{h(L) - 1}{h(L) + 1}\,,$$

and the upper limit is approximately equal to $\frac{\mu}{2L}$ for $L$ (or $\bar\beta$) large. For any $k$ in that interval, the remaining two parameters $\lambda_0$ and $T$ must be chosen so that $\alpha_t$ and $\Sigma_t$ are continuous at $T$. That is

$$\frac{1}{2\lambda_0}\, e^{\mu T} = \alpha_T \qquad \text{and} \qquad \Sigma_0 + \sigma_v^2\, T - \frac{\lambda_0^2\, \sigma_y^2}{2\mu}\left[1 - e^{-2\mu T}\right] = \frac{\sigma_v\, \sigma_y}{\bar\beta}\left[\frac{1 + k}{1 - k}\right],$$

where $\alpha_T$ is given by (15). One can show that for each $-1 < k \le [h(L) - 1]/[h(L) + 1]$, there is a unique $(\lambda_0(k), T(k))$ that solves these two equations. Therefore, for each $\bar\beta > 0$, there is a continuum of equilibria. All the equilibria have a *discontinuous* function $\lambda_t$; indeed, $\lambda_{T-} > \lambda_{T+}$. The size of the discountinuity $\lambda_{T-} - \lambda_{T+}$ is smallest when $k = [h(L) - 1]/[h(L) + 1]$.

In the unique equilibrium of the discrete-time model, the sequence $\{\lambda_n\}$ is decreasing, for any period length $\Delta$. Therefore, to consruct a continuous-time approximation of the discrete-time equilibrium, we need to restrict attention to continuous-time equilibria for which $\lambda_t$ is decreasing. That is, we must choose

$$1 \le \frac{e^{LT}+K}{e^{LT}-K} \le \frac{\mu}{L} + \sqrt{\frac{\mu^2}{L^2}+1} \qquad \text{or} \qquad 0 \le k \le \frac{h(L)-1}{h(L)+1}\,.$$

### 4.2 Limiting Solution as $\bar\beta \to \infty$

To obtain the continuous-time limit equilibrium, we let $\bar\beta$ go to infinity. We have to be careful, however, with the interpretation of this limit because of the nature of the insider trading strategy. According to Proposition 3, as $\bar\beta$ grows large so does the insider trading rate in $[T,\infty)$. In the limit as $\bar\beta \to \infty$, the insider wants to trade at an infinite rate. In the language of optimal control, the insider is exerting *singular* control, that is, she is using a trading strategy that is not absolutely continuous with respect to time. Nevertheless, the following Theorem guarantees that both $\widehat\Pi(t,M,\bar\beta)$ and $M_t(\bar\beta)$ admit a well-defined limit as $\bar\beta \to \infty$.

**Theorem 2** *For each $\bar\beta > 0$, let $(\hat k(\bar\beta), \hat\lambda_0(\bar\beta), \hat\gamma_0(\bar\beta), \hat T(\bar\beta))$ be an equilibrium associated with $\bar\beta$, such that $\hat k(\bar\beta) \ge 0$ (so that the corresponding $\lambda_t$ is decreasing). Assume that $(\hat k(\bar\beta), \hat\lambda_0(\bar\beta), \hat\gamma_0(\bar\beta), \hat T(\bar\beta)) \to (k,\lambda_0,\gamma_0,T)$. Then, the limit $(k,\lambda_0,\gamma_0,T)$ is independent of the sequence, $k=0$, and $\lambda_0 = \frac{\sigma_v}{\sigma_y}\, e^{\mu T}$, where $T$ is the unique nonnegative root of the equation*

$$\Sigma_0 + \sigma_v^2\, T = \sigma_v^2 \left[ \frac{e^{2\,\mu\, T}-1}{2\,\mu} \right].$$

*The continuous-time limit equilibrium has two distinct phases separated by the switching time $T$.*

- Absolutely Continuous Phase in $[0,T)$: *In this phase, the insider's trading strategy and market maker's pricing rule are given by*

$$\mathrm{d}X_t = \beta_t\,(V_t - P_t)\,\mathrm{d}t \qquad \textit{and} \qquad \mathrm{d}P_t = \lambda_t\,\mathrm{d}Z_t \qquad t<T,$$

*where $\beta_t$ and $\lambda_t$ are the two deterministic functions*

$$\beta_t = \frac{\sigma_v\,\sigma_y\,e^{\mu(T-t)}}{\Sigma_t} \qquad \textit{and} \qquad \lambda_t = \frac{\sigma_v}{\sigma_y}\,e^{\mu\,(T-t)}, \quad t<T.$$

*The variance of the market maker's estimate of $V_t$ is given by*

$$\Sigma_t = \Sigma_0 + \sigma_v^2\,t - \sigma_v^2 e^{2\mu T} \left[ \frac{1-e^{-2\,\mu\,t}}{2\mu} \right] \qquad t<T,$$

*which decreases monotonically to 0 in $[0,T)$.*

- Singular Phase in $[T,\infty)$: *In this phase, the (inverse) market depth is constant: $\lambda_t = \sigma_v/\sigma_y$, and the limiting market maker's pricing rule satisfies*

$$\mathrm{d}P_t = \frac{\sigma_v}{\sigma_y}\,\mathrm{d}Z_t, \qquad t \ge T.$$

*In addition, the price differential $M_t(\bar{\beta})$ converges weakly to 0 over compacts in $[T, \infty)$. As a result, the insider's trading strategy converges weakly to $X_t = X_T + \sigma_y \left[(B^v_t - B^v_T) - (B^y_t - B^y_T)\right]$ and $\Sigma_t = 0$ for all $t \geq T$.*

- Insider's Payoff: *Let $(T-t)^+ = \max\{0, T-t\}$. As $\bar{\beta} \to \infty$, the insider's value function $\widehat{\Pi}(t, M_t, \bar{\beta})$ converges to the quadratic function*

$$\widehat{\Pi}(t, M_t) = \alpha_t\, M_t^2 + \gamma_t \qquad \text{where} \qquad \alpha_t = \frac{\sigma_y}{2\,\sigma_v}\, e^{-\mu\,(T-t)^+} \qquad \text{and}$$

$$\gamma_t = \frac{\sigma_y\,\sigma_v}{4\,\mu}\left[3\,e^{-\mu\,(T-t)^+} + e^{\mu\,(T-t)^+}\right] + \frac{\sigma_y\,\sigma_v}{2}\,(T-t)^+\,e^{-\mu\,(T-t)^+}.$$

The previous result summarizes a number of important features of the limit equilibrium. One of the remarkable property of this equilibrium is the existence a finite time $T$, endogenously determined, at which market efficiency is reached and preserved thereafter. Indeed, the fact that $M_t \Rightarrow 0$ for $t \geq T$ means that after $T$, the insider is cashing out her private information instantaneously. Despite this market efficiency, the insider is able to collect positive rents in $[T, \infty)$ since $\widehat{\Pi}(t, 0) > 0$. The source of these rents is the continuous inflow of new information that the insider gets from privately observing the evolution of $V_t$. From the market maker's perspective, the strategy profile in $[T, \infty)$ validates his work in a rather strong sense. In fact, the market maker is concerned with setting prices so that $P_t = \mathbb{E}[V_t|\mathcal{F}^M_t]$. Theorem 2 implies that $P_t$ converges uniformly on compact sets to $V_t$ in $[T, \infty)$.[6] Hence, in the limit as $\bar{\beta} \to \infty$, the market maker knows exactly the intrinsic value of the asset and the price reflects this value at all times $t \geq T$. As a result, the insider trading volume, $X_t$ behaves as a martingale after $T$. It is also interesting to note that $X_t - X_T$ is independent of $\sigma_v$.

Intuitively, the existence of a finite threshold time $T$ is the result of two forces that influence the insider trading strategy in opposite directions. On one hand, the fact that $\widehat{\Pi}(T, 0) > 0$ implies that the insider is able to collect positive rents after $T$, despite the fact that market prices are efficient after this time. Hence, given the unpredictability of the information leakage, the insider is anxious to collect these rents as quickly as possible pushing the value of $T$ towards 0. On the other hand, the market maker's choice of a decreasing (inverse) market depth, $\lambda_t$, gives the insider incentives to slow down her trading activity pushing $T$ away from 0. In equilibrium, the choice of $\lambda_t$ is such that these two forces compensate each other and the insider gradually reveals her private information resulting in a finite time $T$ bounded away from 0.

The informational rents after time $T$ are due to the continuous inflow of new information that the insider gets by privately tracking the evolution of $V_t$. In the absence of these rents, either because $V_t$ is constant or because the insider looses her capacity to track $V_t$, the insider would have no incentive to speed up her trading and market efficiency would only be reached asymptotically ($T = \infty$).

[6]This follows from the Skorohod Representation Theorem and the fact that $M_t = V_t - P_t$ converges weakly to (the continuous process) 0.

**Theorem 3** *When $\sigma_v = 0$, there exists a continuous-time linear Markovian equilibrium where*

$$\Sigma_t = \Sigma_0\, e^{-2\,\mu\, t}, \qquad \beta_t = \sqrt{\frac{2\,\mu\,\sigma_y^2}{\Sigma_0}}\, e^{\mu\, t}, \qquad \lambda_t = \sqrt{\frac{2\,\mu\,\Sigma_0}{\sigma_y^2}}\, e^{-\mu\, t}, \tag{17}$$

$$\alpha_t = \frac{e^{\mu\, t}}{2}\sqrt{\frac{\sigma_y^2}{2\,\mu\,\Sigma_0}} \qquad \textit{and} \qquad \gamma_t = \frac{e^{-\mu\, t}}{2}\sqrt{\frac{\Sigma_0\,\sigma_y^2}{2\,\mu}}. \tag{18}$$

PROOF: using the results in section 2.1, and replacing $n = t/\Delta$, we get the following equilibrium as a function of $t$ and $\Delta$.

$$\Sigma_t = \frac{\Sigma_0}{(1+S)^{\frac{t}{\Delta}}}, \qquad \beta_t = \frac{1}{\Delta}\sqrt{\frac{\sigma_y^2\,\Delta\, S}{\Sigma_0}}\,(1+S)^{\frac{t}{2\Delta}}, \qquad \lambda_t = \sqrt{\frac{\Sigma_0\, S}{\sigma_y^2\,\Delta}}\,(1+S)^{-(1+\frac{t}{2\Delta})},$$

$$\alpha_t = \frac{e^{-\mu\,\Delta}}{2}\sqrt{\frac{\sigma_y^2\,\Delta}{\Sigma_0\, S}}\,(1+S)^{\frac{t}{2\Delta}} \qquad \text{and} \qquad \gamma_t = \frac{e^{-2\,\mu\,\Delta}}{2}\,\frac{\sqrt{\Sigma_0\,\sigma_y^2\,\Delta\, S}}{\sqrt{1+S} - e^{-\mu\,\Delta}}\,(1+S)^{-(2+\frac{t}{2\Delta})},$$

where $S$ is the unique root in $[0,1]$ of the equation $(1+S)\,(1-S)^2 = e^{-2\,\mu\,\Delta}$ and $\Sigma_0 = \Sigma_{-\Delta}\,(1+S)^{-1}$. For $\Delta$ small, it follows that $S = 2\,\mu\,\Delta + O(\Delta)$. Hence, in the limit as $\Delta \downarrow 0$, we get (17) and (18). ∎

Note that when $\sigma_v = 0$, in equilibrium $\Sigma_t \to 0$ as $t \to \infty$, but $\Sigma_t > 0$ for all $t \geq 0$. Also, the trading rate $\beta_t$ remains bounded for all $t \geq 0$. Let us conclude this section discussing other remarks about the limiting solution in Theorem 2.

**Remarks**:

- The switching time $T$ is independent of $\sigma_y$. On the other hand, $T$ decreases with both $\sigma_v$ and $\mu$ and increases with $\Sigma_0$. Furthermore, as $\sigma_v \downarrow 0$, this switching time diverges to $+\infty$ and the resulting profile coincides with the equilibrium derived in Theorem 3.

- The evolution of $M_t = V_t - P_t$ in $[0, T)$ is given by

$$\mathrm{d}M_t = -\frac{\sigma_v^2\, M_t\, e^{2\,\mu\,(T-t)}}{\Sigma_t}\,\mathrm{d}t + \sigma_v\,\mathrm{d}B_t^v - \sigma_v\, e^{\mu\,(T-t)}\,\mathrm{d}B_t^y.$$

  Since $\Sigma_t$ converges to 0 as $t$ approaches $T$, it follows that $M_t$ has mean-reverting paths converging to 0 (a.s.) as $t \uparrow T$.

- By construction the insider's value function, $\widehat{\Pi}(t, M; \bar\beta)$ is continuous for every $\bar\beta < \infty$. However, $\widehat{\Pi}(t, M; \bar\beta)$ is not differentiable at $t = T$. For limit equilibrium of Theorem 2, $\gamma_t$ is differentiable everywhere but $\alpha_t$ is only differentiable almost everywhere, except at $t = T$. However, according to the previous remark, in equilibrium $M_T = 0$ a.s. Hence, $\widehat{\Pi}(t, M_t)$ is essentially smooth at $T$ in the following probabilistic sense

$$\lim_{t\uparrow T}\left[\dot\alpha_t\, M_t^2 + \dot\gamma_t\right] = \lim_{t\downarrow T}\left[\dot\alpha_t\, M_t^2 + \dot\gamma_t\right] \quad \text{(a.s.) for all initial condition } M_0 = M.$$

- One can show that the equilibrium in Theorem 2 satisfies the smooth-pasting condition

$$\lim_{t \uparrow T} \frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} = 0.$$

This is in contrast to the equilibria obtained in models that assume a fixed announcement date (*e.g.*, Kyle (1985)), where $\Sigma_t$ has a strictly negative slope before $T$ and 0 slope after $T$.

- Finally, one can show that the insider's ex-ante (before acquiring her private information) expected payoff satisfies

$$\mathbb{E}[\widehat{\Pi}(t, M_t)] = \alpha_t\, \mathbb{E}[(V_t - P_t)^2] + \gamma_t = \alpha_t\, \Sigma_t + \gamma_t = \frac{\sigma_y\, \sigma_v}{\mu} \cosh(\mu\,(T-t)^+) \qquad t \geq 0.$$

We can also interpret $\mathbb{E}[\widehat{\Pi}(t, M_t)]$ as the market best estimate of the insider's expected payoff-to-go from time $t$ on. Hence, from the market point of view the insider's expected payoff decreases monotonically with time in $[0, T)$ and stays constant after $T$. Note that because $T$ is independent of $\sigma_y$, the insider's ex-ante expected payoff grows linearly with $\sigma_y$.

# 5 Discussion

## 5.1 The Effect of Noisy Information

A key difference between our model and those in the existing literature on strategic trading is that our insider continuously updates her knowledge of the fundamental value of the asset. The volatility coefficient $\sigma_v$ determines the amount of information asymmetry between the insider and the rest of the market. The higher is $\sigma_v$ the larger is the advantage of the insider.

As noted above, the switching time $T$ decreases with $\sigma_v$, that is, the insider is willing to reveal her private information faster as the fundamental value becomes more volatile. The following result shows that this effect holds in a strong sense.

**Proposition 7** *The value of $\Sigma_t$ weakly decreases with $\sigma_v$ for all $t \geq 0$. On the other hand, the insider's ex-ante expected payoff $\mathbb{E}[\widehat{\Pi}(t, M_t)] = \alpha_t\, \Sigma_t + \gamma_t$ is weakly increasing in $\sigma_v$ for all $t \geq 0$.*

In other words, the more volatile is the fundamental value, the faster the price adjusts to the current intrinsic value. However, this efficiency come at a cost. Indeed, the insider is willing to trade away her private information faster because the market maker is willing to compensate her for doing so. Hence, we expect market prices to be more informative when the volatility of the fundamental value is higher. For example, in the special case in which there is no volatility ($\sigma_v = 0$), market efficiency ($\Sigma_t = 0$) is only reached asymptotically as $t \to \infty$ and the insider's ex-ante payoff is minimized.

## 5.2 Market Efficiency and Insider's Expected Payoff

The solution in Theorem 2 reveals that despite the fact that the market reaches full informational efficiency –that is, market price perfectly tracks the value of the asset– after time $T$, the insider still makes positive rents. In what follows, we show that this limiting solution cannot be an equilibrium of the game in continuous time.

Recall that the insider's expected payoff-to-go after time $T$ can be written as

$$\Pi(T,P,X)=\mathbb{E}\left[\int_T^\infty e^{-\mu\,(t-T)}(V_t-P_t)\,\mathrm{d}X_t+\int_T^\infty e^{-\mu\,(t-T)}\,\mathrm{d}[X,V]_t-\int_T^\infty e^{-\mu\,(t-T)}\,\mathrm{d}[X,P]_t\right].$$

Let $(P,X)$ be the profile in Theorem 2. After time $T$, the market maker's pricing strategy $P$ is given by $\mathrm{d}P_t=\lambda_T\mathrm{d}Z_t$, where $\lambda_T=\sigma_v/\sigma_y$, and the insider's cumulative volume traded is a martingale process such that $\mathrm{d}X_t=\sigma_y\,[\mathrm{d}\,B_t^v-\mathrm{d}B_t^y]$. Thus, the first stochastic integral with respect to $X_t$ has 0 expectation and the quadratic covariations between $X_t$ and $V_t$ and between $X_t$ and $P_t$ satisfy $\mathrm{d}[X,V]_t=\sigma_y\,\sigma_v\,\mathrm{d}t$ and $\mathrm{d}[X,P]_t=\lambda_t\,\sigma_y^2\mathrm{d}t=\sigma_y\,\sigma_v\,\mathrm{d}t$ respectively. It follows that

$$\Pi(T,P,X)=0\neq\lim_{\bar\beta\to\infty}\widehat{\Pi}(T,M;\bar\beta)=\frac{\sigma_v\sigma_y}{\mu}.$$

That is, $\Pi$ has a discontinuity at $(P,X)$ as $X$ is approached by strategies of bounded variation.

# Appendix

**Proof of Lemma 1.**

For each $n \geq 0$ and each $k \geq 0$, consider the finite horizon problem for the insider where the fundamental value is made public for sure at the end of period $n+k$ if it has not been publicly revealed before. Let $\Pi_{k,n}(p,V)$ be the insider's optimal discounted value from period $n$ onward in this problem, when the price and fundamental value in period $n-1$ are $(p,V)$. Obviously, $\Pi_{k,n}(p,V) \leq \hat{\Pi}_n(p,V)$ (because the insider can always choose $x_s = 0$ for all $s > n+k$) and $\lim_{k\to\infty} \Pi_{k,n}(p,V) = \hat{\Pi}_n(p,V)$ for all $n \geq 0$ and all $(p,V) \in \mathbb{R}^2$.

We first show inductively in $k$ that for each $n$, either

$$\Pi_{k,n}(p,V) = \frac{a_{k,n}}{\lambda_n}(V-p)^2 + \frac{b_{k,n}}{\lambda_n}\Sigma_v + c_{k,n}\lambda_n\Sigma_y \tag{19}$$

for some constants $(a_{k,n}, b_{k,n}, c_{k,n})$, or $\Pi_{k,n} \equiv \infty$. When $k = 0$,

$$\Pi_{0,n}(p,V) = \max\ (V - p - \lambda_n x)x = \frac{(V-p)^2}{4\lambda_n},$$

so $\Pi_{0,n}$ satisfies (19) with $a_{0,n} = 1/4$ and $b_{0,n} = c_{0,n} = 0$ for all $n \geq 1$. By induction, assume first that $\Pi_{k,n+1}$ satisfies (19) for a given $(k,n)$. We then show that either $\Pi_{k+1,n}$ also satisfies (19) or $\Pi_{k+1,n} \equiv \infty$. We have that

$$\begin{aligned}\Pi_{k+1,n}(p,V) &= \max_{x\in\mathbb{R}}\ (V-p-\lambda_n x)x + \rho\mathbb{E}[\Pi_{k,n+1}(V+W_n, p+\lambda_n(x+Y_n))] \\ &= \max_{x\in\mathbb{R}}\ (V-p-\lambda_n x)x + \rho\left[\frac{a_{k,n+1}}{\lambda_{n+1}}[(V-p-\lambda_n x)^2 + \Sigma_v + \lambda_n^2\Sigma_y] + \frac{b_{k,n+1}}{\lambda_{n+1}}\Sigma_v + c_{k,n+1}\lambda_{n+1}\Sigma_y\right].\end{aligned}$$

When $\rho\lambda_n a_{k,n+1}/\lambda_{n+1} < 1$, the quadratic objective function is concave and

$$\Pi_{k+1,n}(p,V) = \frac{\lambda_{n+1}(V-p)^2}{4\lambda_n[\lambda_{n+1} - a_{k,n+1}\rho\lambda_n]} + \rho\left[\frac{\Sigma_v}{\lambda_{n+1}}[a_{k,n+1} + b_{k,n+1}] + \Sigma_y\left[a_{k,n+1}\frac{\lambda_n^2}{\lambda_{n+1}} + c_{k,n+1}\lambda_{n+1}\right]\right].$$

Hence $\Pi_{k+1,n}$ satisfies (19) with

$$a_{k+1,n} = \frac{1}{4}\left[1 - a_{k,n+1}\frac{\rho\lambda_n}{\lambda_{n+1}}\right]^{-1} \tag{20}$$

$$b_{k+1,n} = \frac{\rho\lambda_n}{\lambda_{n+1}}[a_{k,n+1} + b_{k,n+1}] \quad \text{and} \quad c_{k+1,n} = \rho\left[a_{k,n+1}\frac{\lambda_n}{\lambda_{n+1}} + c_{k,n+1}\frac{\lambda_{n+1}}{\lambda_n}\right]. \tag{21}$$

When $a_{k,n+1}\rho\lambda_n/\lambda_{n+1} \geq 1$, the quadratic objective function is convex, and $\Pi_{k+1,n} \equiv \infty$. By induction, now assume instead that $\Pi_{k,n+1} \equiv \infty$. Then $\Pi_{k+n+1-s,s} \equiv \infty$ for all $s = 0, \ldots, n$. This concludes our proof by induction.

Let us now assume that $\sum \rho^n/\lambda_n = \infty$. In this case we will show that $\Pi_{k,n}(p,v) \to \infty$ as $k \to \infty$, for all $n$ and $(p,v)$.

*Special Case*: When $\lambda_{n+1}/\lambda_n = \rho$ for all $n \geq 0$, the sequences $\{a_{k,n}\}$, $\{b_{k,n}\}$ and $\{c_{k,n}\}$ are independent of $n$ and

$$a_{k+1} = \frac{1}{4(1-a_k)}, \quad b_{k+1} = b_k + a_k \quad \text{and} \quad c_{k+1} = \rho^2 c_k + a_k.$$

These difference equations have the solutions

$$a_k = \frac{k+1}{2(k+2)}\ , \quad b_k = a_0 + \cdots + a_{k-1} \quad \text{and} \quad c_k = a_{k-1} + a_{k-2}\rho^2 + \cdots + a_0\rho^{2(k-1)}.$$

Since $1/4 \leq a_k < a_{k+1} < 1/2$ for all $k$ and $a_k \to 1/2$, we have that $b_k \to \infty$ and $c_k \to [2(1-\rho^2)]^{-1}$. Therefore, as $\Pi_{k,n}(p,v) \geq b_k/\lambda_n$, $\Pi_{k,n}(p,v) \to \infty$ for all $n$ and $(p,v)$.

The situation $\lambda_{n+1}/\lambda_n = \rho$ for all $n \geq 1$ represents a limit case. If the sequence $\{\lambda_n\}$ goes to 0 faster, then for any $n$ there exists $k$ such $\Pi_{k,n} \equiv \infty$. Suppose for example that $\lambda_{n+1}/\lambda_n \leq \rho 3/4$ for all $n$. The function $f(a,d) = [4(1-ad)]^{-1}$ is increasing in $a$ and $d$ (when $ad < 1$). Therefore, $a_{k,n} \geq \hat{a}_k$ for all $(k,n)$, where $\hat{a}_0 = 1/4$ and $\hat{a}_{k+1} = f(\hat{a}_k, 4/3)$ for all $k \geq 0$. Since $\hat{a}_1 = 3/8$, $\hat{a}_2 = 1/2$, and $\hat{a}_3 = 3/4$, we have that $1 \leq a_{3,n+1}\rho\lambda_n/\lambda_{n+1}$, and $\Pi_{4,n} \equiv \infty$ for all $n$.

In the general case, since $a_{0,n} = 1/4$ and $\rho\lambda_n/\lambda_{n+1} > 0$ for all $n \geq 1$, it is easy to see (by induction) that (20) implies $a_{k,n} > 1/4$ for all $k \geq 1$ and $n \geq 0$. Fix $n \geq 0$. For any $k \geq 1$, if there exists $j \in \{1,\ldots,k\}$ such that $1 \leq a_{k-j.t+j}\rho\lambda_{n+j}/\lambda_{n+j+1}$, then $\Pi_{k-j+1,n+j-1} \equiv \infty$, which implies that $\Pi_{k,n} \equiv \infty$. If $1 > a_{k-j.t+j}\rho\lambda_{n+j}/\lambda_{n+j+1}$ for all $j \in \{1,\ldots,k\}$, then (21) implies that

$$b_{k,n} \geq \frac{\rho\lambda_n}{\lambda_{n+1}}\left[\frac{1}{4} + b_{k-1,n+1}\right] \geq \frac{\rho\lambda_n}{\lambda_{n+1}}\left[\frac{1}{4} + \frac{\rho\lambda_{n+1}}{\lambda_{n+2}}\left[\frac{1}{4} + b_{k-2,n+2}\right]\right] \cdots \geq \frac{\lambda_n}{4}\left[\frac{\rho}{\lambda_{n+1}} + \cdots + \frac{\rho^k}{\lambda_{n+k}}\right].$$

Note that

$$\sum_{j=1}^{\infty} \frac{\rho^j}{\lambda_{n+j}} = \frac{1}{\rho^n}\sum_{j=t+1}^{\infty}\frac{\rho^j}{\lambda_j} = \frac{1}{\rho^n}\left[\sum_{j=1}^{\infty}\frac{\rho^j}{\lambda_j} - \sum_{j=1}^{n}\frac{\rho^j}{\lambda_j}\right] = \infty.$$

Therefore, either $\Pi_{k,n} \equiv \infty$ for some $k \geq 1$, which implies that $\hat{\Pi}_n \equiv \infty$, or for all $k \geq 1$,

$$\hat{\Pi}_n(p,v) \geq \Pi_{k,n}(p,v) \geq \frac{\Sigma_v}{\lambda_n} b_{k,n} \geq \frac{\Sigma_v}{4}\sum_{j=1}^{k}\frac{\rho^j}{\lambda_{n+j}}.$$

Since the right-hand side of the last inequality converges to $\infty$ as $k \to \infty$, $\hat{\Pi}_n \equiv \infty$ in this case as well.

Finally, assume that $\sum \rho^n/\lambda_n < \infty$ and $\rho\lambda_n/\lambda_{n+1} \leq 1$ for all $n \geq 1$ In this case we will show that each $\hat{\Pi}_n(p,V)$ is a quadratic function of $(V-p)$ and that $\hat{\Pi} = B(\hat{\Pi})$.

Since $a_{0,n} = 1/4$, it is easy to show by induction that $1/4 < a_{k,n} < 1/2$ for all $k \geq 1$ and $n \geq 0$. Recall that $f(a,d) = [4(1-ad)]^{-1}$ is increasing in $a$ and $d$. Since $a_{1,n+1} > 1/4 = a_{0,n+1}$ for all $n \geq 0$, $a_{2,n} = f(a_{1,n+1}, d_n) > f(a_{0,n+1}, d_n) = a_{1,n}$ for all $n \geq 0$. Repeating this argument forward, we conclude that $\{a_{k,n}\}_{k=1}^{\infty}$ is an increasing sequence and it must converge. Let $\alpha_n = \lim_{k\to\infty}\rho a_{k,n}/\lambda_n$. Since $\rho\lambda_n/\lambda_{n+1} \leq 1$ and $a_{k,n+1} < 1/2$ for all $k$, $\lambda_n\alpha_{n+1} \leq 1/2$.

Since $a_{k,n} < 1/2$ for all $k \geq 0$ and $n \geq 0$,

$$b_{k,n} \leq \frac{\rho\lambda_n}{\lambda_{n+1}}\left[\frac{1}{2} + b_{k-1,n+1}\right] \leq \cdots \leq \frac{\lambda_n}{2}\left[\frac{\rho}{\lambda_{n+1}} + \cdots + \frac{\rho^k}{\lambda_{n+k}}\right] < \frac{\lambda_n}{2\rho^n}\sum_{j=t+1}^{\infty}\frac{\rho^j}{\lambda_j} < \infty.$$

By induction in $k$, we now show that $b_{k,n} < b_{k+1,n}$ for all $k \geq 0$ and $n \geq 1$. Clearly $b_{0,n} = 0 < b_{1,n}$ for all $n \geq 0$. Since $a_{k,n+1} < a_{k+1,n+1}$, if the inequality holds for $(k,n)$, then

$$b_{k+1,n} = d_n[a_{k,n+1} + b_{k,n+1}] < d_n[a_{k+1,n+1} + b_{k+1,n+1}] = b_{k+2,n}.$$

That is, for each $n \geq 0$, the sequence $\{b_{k,n}\}_{k=0}^{\infty}$ is increasing and hence it must converge. Solving (21) we obtain

$$c_{k,n} = \frac{1}{\lambda_n} \sum_{j=1}^{k} \rho^j \frac{\lambda_{n+j-1}^2}{\lambda_{n+j}} a_{k-j,n+j}.$$

One can show that for each $n \geq 0$, the sequence $\{c_{k,n}\}_{k=1}^{\infty}$ is increasing, and since $\lambda_s \leq M$ and $a_{j,s} < 1/2$ for all $j$ and $s$,

$$c_{k,n} \leq \frac{M^2}{2\lambda_n} \sum_{j=1}^{k} \frac{\rho^j}{\lambda_{n+j}} < \frac{M^2}{2\lambda_n \rho^n} \sum_{j=t+1}^{\infty} \frac{\rho^j}{\lambda_j} < \infty,$$

the sequence must converge. Let $\gamma_n = \lim_{k\to\infty} \rho[b_{k,n}\Sigma_v/\lambda_n + c_{k,n}\lambda_n\Sigma_y]$ and define $\hat{\Pi}_n$ by $\rho\hat{\Pi}_n(p,v) = \alpha_n(v-p)^2 + \gamma_n$.

Let $\mathbb{Q}$ be the set of $f \in \mathbb{B}$ such that for some $a, b \in \mathbb{R}$, $f(p,v) = a(v-p)^2 + b$, and define the norm $||f|| = \max\{|a|, |b|\}$. Let $\mathbb{Q}_n = \{a(v-p)^2 + b \mid a\rho\lambda_n < 1\}$. Then, $b_n : \mathbb{Q}_n \to \mathbb{Q}_n$ is continuous. Therefore, for each $n \geq 0$,

$$\hat{\Pi}_n = \lim_{k\to\infty} \Pi_{k,n} = \lim_{k\to\infty} b_n(\Pi_{k,n+1}) = b_n(\hat{\Pi}_{n+1}).$$

That is, $\hat{\Pi} = B(\hat{\Pi})$. ∎

**Proof of Lemma 3.**

When $\beta_0 < \Psi(\Sigma_{-1})$, the sequence $\{(A_n, B_n)\}$ remains feasible forever. Moreover, for some finite $N$, $(A_n, B_n) \in R_3$ for all $n \geq N$. Therefore $A_n < A_{n+1}$ for all $n \geq N$ and $A_n \to \infty$. Recall that the graphs of $G_1$ and $G_2$ intersect at $(\hat{A}, \hat{B})$, and that $(A, B) \in R_3$ and $A \geq \hat{A}$ imply that $B \leq G_1(A)$. The function $h(A) = (A-1)^2/[A(A-2)]$ is decreasing for all $A > 2$, and $h(A) \to 1$ as $A \to \infty$. Let $\omega \in (\rho, 1)$. Without loss of generality, assume that $N$ is such that $A_N \geq \hat{A}$ and $h(A_N) \leq \omega/\rho$. Then, $B_n \leq G_1(A_n)$ for all $n \geq N$, and therefore for all $n \geq N$,

$$B_{n+1} = F_B(A_n, B_n) = \rho \left[\frac{A_n^2 B_n}{A_n^2 - B_n^4}\right] \leq \rho \left[\frac{A_n^2 B_n}{A_n^2 - [G_1(A_n)]^4}\right] = \rho h(A_n) B_n \leq \omega B_n. \tag{22}$$

Since $B_N \leq \hat{B}$, this implies that $B_n \leq \hat{B}\omega^{n-N}$ for all $n \geq N$. From (4),

$$\lambda_n = \frac{\beta_n \Sigma_{n-1}}{\beta^2 \Sigma_{n-1} + \Sigma_y} = \frac{A_n B_n}{A_n + B_n^2} \sqrt{\frac{\Sigma_v}{\Sigma_y}} < B_n \sqrt{\frac{\Sigma_v}{\Sigma_y}}.$$

Since we would like to show that $\sum \rho^n/\lambda_n = \infty$, we need a tighter upper bound on $B_n$. Note, however, that

$$B_{n+1} = F_B(A_n, B_n) = \rho \left[\frac{A_n^2 B_n}{A_n^2 - B_n^4}\right] \geq \rho B_n \quad \text{for all } n \geq 0,$$

so there is not a lot of slack in the previous upper bound (22) for $B_{n+1}$.

For any $\epsilon > 0$, let $N^* > N$ be such that $\hat{B}\omega^{N^*-N} < \epsilon$. Then, for all $n \geq N^*$,

$$A_{n+1} = F_A(A_n, B_n) = 1 + \frac{A_n^2}{A_n + B_n^2} \geq 1 + \frac{A_n}{1 + \epsilon^2/A_n} \geq 1 + A_n \left[1 - \frac{\epsilon^2}{A_n}\right] = A_n + (1 - \epsilon^2).$$

Let $e = 1 - \epsilon^2$. Then $A_{N^*+n} \geq A_{N^*} + ne > ne$ for all $n \geq N^*$. Feeding this bound back into (22), we obtain that

$$B_{N^*+n+3} \leq \rho h((n+2)e) B_{N^*+2+n} \leq \cdots \leq \rho^n h((n+2)e) h((n+1)e) \cdots h(3e) B_{N^*+3}.$$

Choose $\epsilon < 1/4$ so that $\epsilon^2 < 1/16$. Then, for all $k \geq 3$,

$$\begin{aligned} h(ke) &= \frac{[k-1-k\epsilon^2]^2}{[k-k\epsilon^2][k-2-k\epsilon^2]} = 1 + \frac{1}{k(k-2) - 2k(k-1)\epsilon^2 + k^2\epsilon^4} \\ &< 1 + \frac{1}{k[k-2-2(k-1)\epsilon^2]} < 1 + \frac{8}{k[7k-15]} \leq 1 + \frac{4}{k^2}\,. \end{aligned}$$

Let

$$H_n = \left[1 + \frac{4}{1^2}\right]\left[1 + \frac{4}{2^2}\right] \cdots \left[1 + \frac{4}{n^2}\right] \quad \text{and} \quad a_n = \frac{1}{H_n} = \left[\frac{1^2}{1^2+4}\right] \cdots \left[\frac{n^2}{n^2+4}\right].$$

Note that $[1 + 4/1^2][1 + 4/2^2] = 10$. Hence, $B_{N^*+n+3} < \rho^n B_{N^*+3} H_n / 10$. Therefore,

$$\sqrt{\frac{\Sigma_v}{\Sigma_y}} \sum_{n \geq 1} \frac{\rho^n}{\lambda_n} > \sum_{n \geq 1} \frac{\rho^n}{B_n} > \sum_{n \geq 1} \frac{10 \rho^{N^*+3+n}}{\rho^n H_n B_{N^*+3}} = \frac{10}{B_{N^*+3}} \rho^{N^*+3} \sum_{n \geq 1} a_n.$$

Gauss's test (see, for example, Knopp 1990) states that if

$$\frac{a_{n+1}}{a_n} = 1 - \frac{c}{n} - \frac{g_n}{n^\epsilon}$$

where $\epsilon > 1$ and $\{g_n\}$ is bounded, then $\sum a_n$ converges when $c > 1$ and diverges when $c \leq 1$. In our case

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(n+1)^2 + 4} = 1 - \left[\frac{4n^2}{(n+1)^2+4}\right] \frac{1}{n^2}\,,$$

so $c = 0$ and $\epsilon = 2$. Therefore $\sum a_n = \infty$ and $\sum \rho^n / \lambda_n = \infty$. ∎

**Proof of Proposition 1.**

Consider equations (3) and (10) with $\Sigma_v = 0$. Let us introduce the change of variable $S_n = \beta_{n+1}^2 \Sigma_n / \Sigma_y$. Replacing $\beta_n$ by $S_n$, equations (3) and (10) become

$$\Sigma_n = \frac{\Sigma_{n-1}}{S_{n-1} + 1} \qquad \text{and} \qquad S_n = \frac{\rho^2 S_{n-1}}{(1 + S_{n-1})(1 - S_{n-1})^2}.$$

With a partial substitution, equation (10) can also be written as $\beta_{n+1}\Sigma_n = \rho \beta_n \Sigma_{n-1}/(1 - S_{n-1}^2)$. Therefore, for $\{S_n\}$ to be compatible with equilibrium it must be that $0 < S_n \leq 1$ for all $n$. Since the dynamics of $S_n$ on the right equation are independent of $\Sigma_n$ we can solve this equation independently of the first equation. A solution for this equation is $S_n = S$ for all $n \geq 0$, where $S$ is a root of the equation

$$(1 + S)(1 - S)^2 = \rho^2.$$

The function $f(x) = (1+x)(1-x)^2$ is monotonically decreasing in $(0, 1)$ with $f(1) = 0 < \rho < 1 = f(0)$ and we conclude that there is a unique root $S \in (0, 1)$. If we set $S_{-1} = S$ then $S_n = S$ for all $n \geq 0$. In this case, the evolution of $\Sigma_n$ is given by

$$\Sigma_n = \frac{\Sigma_{-1}}{(1 + S)^{n+1}} \qquad \text{for all } n \geq 0. \tag{23}$$

Thus, the variance decreases geometrically over time. The evolution of $\beta_n$ follows directly from the definition of $S_n$. Similarly, the value of $\alpha_{n+1}$ can be computed from equation (9). Finally, to get the value of $\gamma_n$ we iterate equation (8) to get

$$\begin{aligned}\gamma_n &= \rho\gamma_{n+1} + \rho\alpha_{n+1}\,\lambda_n^2\,\Sigma_y \\ &= \rho^2\gamma_{n+2} + \rho^2\alpha_{n+2}\,\lambda_{n+1}^2\,\Sigma_y + \rho\alpha_{n+1}\,\lambda_n^2\,\Sigma_y \\ &= \sum_{k=1}^{\infty}\rho^k\alpha_{n+k}\,\lambda_{n+k-1}^2\,\Sigma_y + \lim_{k\to\infty}\rho^k\,\gamma_{n+k}.\end{aligned}$$

Replacing $\alpha_n$ and $\lambda_n$ one can show that the summation converges to the value of $\gamma_n$ stated in the proposition, which also shows that the limit converges to 0 since $0 < \rho < 1$.

Let $g(x) = \rho^2 x/f(x)$, so that $S_n = g(S_{n-1})$. The function $g : [0,1) \to \mathbb{R}$ is convex, $g(0) = 0$, $\lim_{x\to 1} g(x) = \infty$, and $g(S) = S$. If we set $S_{-1} > S$, then the sequence generated by $S_n = g(S_{n-1})$ increases monotonically until $S_n > 1$ for some $n$. That is, the sequence becomes infeasible. If we set $S_{-1} < S$, then the sequence generated by $S_n = g(S_{n-1})$ decreases monotonically to 0, and the corresponding sequence $\{\lambda_n\}$ converges to 0 'too fast', making $\sum \rho^n/\lambda_n = \infty$. Therefore, only the choice $S_{-1} = S$ is consistent with equilibrium. ∎

**Proof of Proposition 2.**

The condition $P_t = \mathbb{E}[V_t|\mathcal{F}_t^M]$ implies that $P_t$ is the orthogonal projection $V_t$ on $\mathcal{F}_t^M$ in $L^2$. Hence, we can interpret the market maker's equilibrium condition as the solution to a classical Kalman-Bucy filtering problem. Let the signal process be the value of the fundamental $V_t$, with dynamics

$$\mathrm{d}V_t = \sigma_v\,\mathrm{d}B_t^v,$$

and the observation process be the price process $P_t$, with dynamics

$$\mathrm{d}P_t = \lambda_t\,\mathrm{d}Z_t = \beta_t\,\lambda_t\,(V_t - P_t)\,\mathrm{d}t + \sigma_y\,\lambda_t\,\mathrm{d}B_t^y.$$

Let $v_t$ be the corresponding optimal (in mean square sense) filtering estimate of $V_t$ and $\Sigma_t$ be the filtering error. Then, the equilibrium condition is $P_t = v_t$.

The generalized Kalman filter conditions for the pair $(V_t, P_t)$ are given by

$$\mathrm{d}v_t = \frac{\Sigma_t\,\beta_t}{\lambda_t\,\sigma_y^2}\,[\mathrm{d}P_t - \lambda_t\,\beta_t\,(v_t - P_t)\,\mathrm{d}t] \qquad \text{and} \qquad \dot{\Sigma}_t = \sigma_v^2 - \frac{(\Sigma_t\,\beta_t)^2}{\sigma_y^2}.$$

To recover the identity $P_t = v_t$ we need to impose that

$$\frac{\Sigma_t\,\beta_t}{\lambda_t\,\sigma_y^2} = 1 \qquad \text{or equivalently} \qquad \Sigma_t\,\beta_t = \lambda_t\,\sigma_y^2.$$

This equality together with the border condition $v_0 = P_0$ imply that $v_t = P_t$ for all $t > 0$. This equality also implies that $(\Sigma_t\,\beta_t)^2 = \lambda_t^2\,\sigma_y^4$. Therefore, the second filtering condition leads to the differential equation

$$\dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2\,\lambda_t^2,$$

which completes the proof of the Lemma. ∎

**Proof of Proposition 3.**

Recall the definition of $\widehat{\Pi}(t,M)$ given by (13). From standard dynamic programming theory, we know that under certain regularity conditions $\widehat{\Pi}(t,M)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \max_{|\beta|\leq\bar{\beta}} \left\{ -\lambda_t\, \beta\, M\, \widehat{\Pi}_M(t,M) + \frac{\sigma_t^2}{2}\, \widehat{\Pi}_{MM}(t,M) + \widehat{\Pi}_t(t,M) - \mu\, \widehat{\Pi}(t,M) + \beta\, M^2 \right\}, \tag{24}$$

where $\widehat{\Pi}_M$ and $\widehat{\Pi}_{MM}$ are the first and second partial derivatives of $\widehat{\Pi}(t,M)$ with respect to $M$, and $\widehat{\Pi}_t$ is the partial derivative with respect to $t$. Our characterization of $\widehat{\Pi}(t,M)$ works in two steps. First, we will restrict the control $\beta_t$ to be a deterministic function of $t$. Under this restriction, we will derive the (open-loop) solution

$$\widehat{\Pi}(t,M) = \int_t^\infty e^{-\mu\,(s-t)}\, \bar{\beta}\, \mathbb{E}[M_s^2 | M_t = M]\, \mathrm{d}s. \tag{25}$$

In the second step, we will invoke a so called *verification theorem* (*e.g.*, Fleming and Soner 1993, Theorem 5.1) to show that this solution solves the HJB equation in (24) with $\beta_t = \bar{\beta}$.

<u>Step 1</u>: Open-Loop Solution

Suppose the insider uses a *deterministic* strategy $\beta_t$. Define $N_s = \mathbb{E}[M_s^2 | M_t = M]$. Then, Ito's lemma implies that $\dot{N}_s = \sigma_s^2 - 2\,\lambda_s\,\beta_s\,N_s$ and the insider's objective function can be written as

$$\int_t^\infty \frac{e^{-\mu\,(s-t)}}{2\,\lambda_s}\, [\sigma_s^2 - \dot{N}_s]\, \mathrm{d}t.$$

Integrating by parts, we get

$$\begin{aligned} \widehat{\Pi}(t,M) \;=\; & \max_{|\beta_t|\leq\bar{\beta}} \left\{ \int_t^\infty e^{\mu\,t}\, N_s\, \mathrm{d}\left(\frac{e^{-\mu\,s}}{2\,\lambda_s}\right) - \lim_{s\to\infty} \frac{e^{-\mu\,(s-t)}}{2\,\lambda_s}\, N_s \right\} + \frac{M^2}{2\,\lambda_t} + \int_t^\infty e^{-\mu\,(s-t)}\, \frac{\sigma_s^2}{2\,\lambda_s}\, \mathrm{d}s \\ & \text{s.t.} \quad \dot{N}_s = \sigma_s^2 - 2\,\lambda_s\,\beta_s\,N_s \quad \text{and} \quad N_t = M. \end{aligned} \tag{26}$$

Because $N_s$ is nonnegative and $e^{-\mu\,s}/\lambda_s$ is strictly decreasing, it follows that the insider wants to make $N_s$ as small as possible for all $s \geq t \geq T$. Since $\dot{N}_s = \sigma_s^2 - 2\,\lambda_s\,\beta_s\,N_s$, $N_s$ is minimized by choosing $\beta_s = \bar{\beta}$ for all $s \geq T$. Replacing $\beta_s = \bar{\beta}$, we can integrate the resulting linear ODE between $t$ and $s$ to get

$$N_s = M^2\, e^{-2\,\bar{\beta}(\Lambda_s - \Lambda_t)} + \int_t^s \sigma_u^2\, e^{-2\,\bar{\beta}\,(\Lambda_s - \Lambda_u)}\, \mathrm{d}u, \tag{27}$$

for the auxiliary function $\Lambda_t = \int_0^t \lambda_s\, \mathrm{d}s$. Finally, replacing this expression in the insider's payoff (25) we get an explicit expression for $\widehat{\Pi}(t,M)$.

<u>Step 2</u>: Verification

We now show that the open-loop solution given by (25) solves the HJB optimality condition (24) using the control $\beta_t = \bar{\beta}$ for all $t$. Straightforward calculations show that for the value function $\widehat{\Pi}(t,M)$ defined by (25), the HJB reduces to

$$0 = \max_{|\beta|\leq\bar{\beta}} \left\{ M^2\, (\beta - \bar{\beta}) \left[ 1 - 2\,\lambda_t\,\bar{\beta} \int_t^\infty e^{-\mu\,(s-t)}\, e^{-2\,\bar{\beta}\,(\Lambda_s - \Lambda_t)}\, \mathrm{d}s \right] \right\}.$$

Hence, it suffices to show that the term inside the square brackets is positive for all $t \geq T$ to conclude that $\beta_t = \bar{\beta}$ is an optimal control and that $\widehat{\Pi}(t, M)$ defined by (25) is the insider value function (without restricting $\beta_t$ to be deterministic).

$$\begin{aligned} 2\,\lambda_t\,\bar{\beta} \int_t^{\infty} e^{-\mu\,(s-t)}\, e^{-2\,\bar{\beta}\,(\Lambda_s-\Lambda_t)}\,\mathrm{d}s &= \lambda_t\, e^{\mu\, t+2\,\bar{\beta}\,\Lambda_t} \int_t^{\infty} \frac{e^{-\mu\, s}}{\lambda_s}\, e^{-2\,\bar{\beta}\,\Lambda_s}\,\mathrm{d}(2\,\bar{\beta}\,\Lambda_s) \\ &< \lambda_t\, e^{\mu\, t+2\,\bar{\beta}\,\Lambda_t}\, \frac{e^{-\mu\, t}}{\lambda_t} \int_t^{\infty} e^{-2\,\bar{\beta}\,\Lambda_s}\,\mathrm{d}(2\,\bar{\beta}\,\Lambda_s) \\ &= e^{2\,\bar{\beta}\,\Lambda_t}\left[-e^{-2\,\bar{\beta}\,\Lambda_s}\right]_t^{\infty} = 1 - e^{-2\,\bar{\beta}\,(\Lambda_\infty-\Lambda_t)} \leq 1. \end{aligned}$$

The first inequality is based on the assumption that $e^{-\mu\, s}/\lambda_s$ is strictly decreasing. The second inequality is based on the fact that $\Lambda_t = \int_0^t \lambda_s\,\mathrm{d}s$ is increasing in $t$. This shows that $\widehat{\Pi}(t, M)$ defined by (25) is the insider's value function and the optimal control is $\beta_t = \bar{\beta}_t$ for all $t \geq T$.

Since $\beta_t \equiv \bar{\beta}$, it follows that $M_t$ has mean reverting dynamics

$$\mathrm{d}M_t = -\lambda_t\,\bar{\beta}\,M_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t.$$

We can integrate this equation using the integrating factor $\exp(\bar{\beta}\,\Lambda_t)$. Indeed, from a straightforward application of Itô's lemma we get

$$\mathrm{d}\big(e^{\bar{\beta}\,\Lambda_t}\,M_t\big) = \sigma_t\, e^{\bar{\beta}\,\Lambda_t}\,\mathrm{d}B_t,$$

and integrating this equation between $T$ and $t$ we get

$$M_t = M_T\, e^{-\bar{\beta}\,(\Lambda_t-\Lambda_T)} + \int_T^t \sigma_s\, e^{-\bar{\beta}\,(\Lambda_t-\Lambda_s)}\,\mathrm{d}B_s, \qquad t \geq T$$

which completes the proof. ∎

**Proof of Proposition 4.**

According to Proposition 2, the market maker equilibrium condition reduces to

$$\Sigma_t\,\bar{\beta} = \lambda_t\,\sigma_y^2 \qquad \text{and} \qquad \dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2\,\lambda_t^2.$$

Therefore, the second filtering equation leads to the ODE

$$\frac{\sigma_y^2}{\bar{\beta}}\,\frac{\mathrm{d}\lambda_t}{\mathrm{d}t} = \sigma_v^2 - \sigma_y^2\,\lambda_t^2.$$

Solving for $\lambda_t$ we get

$$\lambda_t = \frac{\sigma_v}{\sigma_y}\left[\frac{e^{Lt}+K}{e^{Lt}-K}\right],$$

for some constant of integration $K$, where $L = 2\sigma_v\bar{\beta}/\sigma_y$. To ensure that $e^{-\mu t}/\lambda_t$ is decreasing for $t \geq T$ we need to impose

$$\frac{\mu\, e^{-\mu\, t}\,\lambda_t + e^{-\mu\, t}\,\dot{\lambda}_t}{\lambda_t^2} \geq 0 \qquad \text{or equivalently} \qquad \mu\,\lambda_t + \frac{\bar{\beta}}{\sigma_y^2}[\sigma_v^2 - \sigma_y^2\lambda_t^2] \geq 0.$$

For $\lambda_t \geq 0$, this last inequality is satisfied only if

$$\lambda_t \leq \frac{1}{2\bar{\beta}} \left[ \mu + \sqrt{\mu^2 + \left( \frac{2\bar{\beta}\sigma_v}{\sigma_y} \right)^2} \right].$$

This condition together with $\lambda_t \geq 0$ are satisfied if and only if

$$0 \leq \frac{e^{LT} + K}{e^{LT} - K} \leq \frac{\mu}{L} + \sqrt{\frac{\mu^2}{L^2} + 1}. \qquad \blacksquare$$

**Proof of Proposition 5.**

The dynamic programming HJB equation for $\widehat{\Pi}(t, M)$ is

$$0 = \max_{|\beta| \leq \bar{\beta}} \left\{ -\lambda_t \, \beta \, M \, \Pi_M + \frac{1}{2} \, (\sigma_v^2 + \lambda_t^2 \, \sigma_y^2) \, \Pi_{MM} + \Pi_t - \mu \, \Pi + M^2 \, \beta \right\}, \qquad t \in [0, T)$$

where $\widehat{\Pi}_M$ ($\widehat{\Pi}_{MM}$) and $\widehat{\Pi}_t$ are the first (second order) partial derivative of $\widehat{\Pi}$ with respect to $M$ and $t$, respectively. The border condition is $\lim_{t \to T^-} \widehat{\Pi}(t, M) = \widehat{\Pi}(T, M)$. Let us guess a quadratic solution $\widehat{\Pi}(t, M_t) = \alpha_t \, M_t^2 + \gamma_t$ for this equation, where $\alpha_t$ and $\gamma_t$ are two deterministic functions of $t$. Then, the HJB reduces to

$$0 = \max_{|\beta| \leq \bar{\beta}} \left\{ \left[ \beta \, (1 - 2\lambda_t \, \alpha_t) + \dot{\alpha}_t - \mu \, \alpha_t \right] M^2 + \alpha_t (\sigma_v^2 + \lambda_t^2 \sigma_y^2) + \dot{\gamma}_t - \mu \gamma_t \right\}.$$

In addition, assume that $1 - 2\lambda_t \alpha_t = 0$. Then, $\alpha_t = \alpha_0 e^{\mu t}$ with $\alpha_0 \lambda_0 = 1/2$ and $\dot{\alpha}_t - \mu \alpha_t = 0$. Moreover,the HJB becomes singular (*i.e.*, independent of the control $\beta$). Finally, $\gamma_t$ must satisfy

$$\alpha_t (\sigma_v^2 + \lambda_t^2 \sigma_y^2) + \dot{\gamma}_t - \mu \gamma_t = 0.$$

This equation is solved by

$$\gamma_t = \left[ \gamma_0 - \frac{\sigma_y^2}{8\alpha_0 \mu} - \alpha_0 \sigma_v^2 t \right] e^{\mu t} + \frac{\sigma_y^2}{8\alpha_0 \mu} e^{-\mu t}.$$

The values of $\alpha_0$ (or equivalently $\lambda_0$) and $\gamma_0$ are obtained by imposing the *value-matching* condition $\lim_{t \uparrow T} \widehat{\Pi}(t, M) = \widehat{\Pi}(T, M)$ for all $M$. That is, $\alpha_0 = \alpha_T \, e^{-\mu \, T}$ and $\gamma_0$ is chosen so that $\lim_{t \uparrow T} \gamma_t = \gamma_T$. $\blacksquare$

**Proof of Theorem 2.**

Let us first prove that $\lambda_t$ converges to the constant $\sigma_v / \sigma_y$ for $t \geq T$ as $\bar{\beta} \to \infty$. Recall that

$$1 \leq \frac{e^{LT} + K}{e^{LT} - K} \leq \frac{\mu}{L} + \sqrt{\frac{\mu^2}{L^2} + 1}.$$

Since $L = 2\sigma_v \bar{\beta} / \sigma_y$ it follows that

$$\lim_{\bar{\beta} \to \infty} \frac{\mu}{L} + \sqrt{\frac{\mu^2}{L^2} + 1} = 1 \qquad \text{and so} \qquad K = 0 \quad \text{and} \quad \lim_{\bar{\beta} \to \infty} \lambda_t = \frac{\sigma_v}{\sigma_y}, \quad t \geq T.$$

This result and Proposition 2 imply that

$$\Sigma_T = \frac{\sigma_y^2\,\lambda_T}{\bar\beta} \to 0 \quad \text{as } \bar\beta \to \infty.$$

As a result, in the limit as $\bar\beta \to \infty$ the threshold time $T$ solves

$$\Sigma_T = \Sigma_0 + \sigma_v^2\,T - \sigma_v^2\left[\frac{e^{2\,\mu\,T}-1}{2\,\mu}\right] = 0.$$

Since $K = 0$ (and $k = 0$), equation (15) implies that

$$\alpha_T = \lim_{\bar\beta\to\infty}\ \bar\beta\,e^{(L+\mu)\,T}\int_T^\infty e^{-(L+\mu)\,s}\,\mathrm{d}s = \lim_{\bar\beta\to\infty}\ \frac{\bar\beta}{L+\mu} = \frac{\sigma_y}{2\sigma_v}\,.$$

Therefore $\lambda_0 = e^{\mu\,T}/[2\alpha_T] = \sigma_v/\sigma_y$.

It only remains to prove the weak convergence of $M_t(\bar\beta)$ to 0 as $\bar\beta \to \infty$. For this we will invoke Theorem 2.1 in Prokhorov (1956) and prove the convergence of the finite-dimensional distributions of $\{M_t(\bar\beta)\}$ to 0 together with the compactness of the sequence $\{M_t(\bar\beta)\}$ (see also Billingsley 1999, Chapter 2). Let $\mathcal{T} = [T_1, T_2]$ with $T < T_1 < T_2$. In what follows, we define $\Lambda(t,s) = \int_t^s \lambda_u\,\mathrm{d}u$, $\underline{\lambda}_{\mathcal{T}} = \min\{\lambda_t\,:\,t\in\mathcal{T}\}$, $\bar\lambda_{\mathcal{T}} = \max\{\lambda_t\,:\,t\in\mathcal{T}\}$ and $\bar\sigma_{\mathcal{T}} = \max\{\sigma_t\,:\,t\in\mathcal{T}\}$.

Let $\{t_1, t_2, \ldots, t_n\} \in \mathcal{T}$. For each $t\in\mathcal{T}$, $M_t(\bar\beta)$ satisfies

$$M_t(\bar\beta) = M_T\,e^{-\bar\beta\,\Lambda(T,t)} + \int_T^t \sigma_s\,e^{-\bar\beta\,\Lambda(s,t)}\,\mathrm{d}B_s.$$

Therefore, the random vector $(M_{t_1}(\bar\beta), M_{t_2}(\bar\beta), \ldots, M_{t_n}(\bar\beta))$ has a Gaussian distribution. We now show that this distribution converges to the distribution of the constant vector $(0,\ldots,0)$. Let us denote by $\mu^M(\bar\beta)$ and $\Sigma^M(\bar\beta)$ its mean vector and variance-covariance matrix, respectively. It follows that the $i^{\text{th}}$ component of $\mu^M(\bar\beta)$ is given by

$$\mu_i^M(\bar\beta) = \mathbb{E}[M_{t_i}(\bar\beta)] = \mathbb{E}[M_T]\,e^{-\bar\beta\,\Lambda(T,t_i)}, \qquad i = 1,\ldots,n.$$

Similarly, the covariance between the $i^{\text{th}}$ and $j^{\text{th}}$ components in $\Sigma^M(\bar\beta)$ is given by (assume $t_i \le t_j$)

$$\begin{aligned}
\Sigma_{ij}^M(\bar\beta) &= \mathbb{E}[\big(M_{t_i}(\bar\beta) - \mu_i^M(\bar\beta)\big)\,\big(M_{t_j}(\bar\beta) - \mu_j^M(\bar\beta)\big)] \\
&= \mathbb{E}\left[\left(\int_T^{t_i}\sigma_s\,e^{-\bar\beta\,\Lambda(s,t_i)}\,\mathrm{d}B_s\right)\left(\int_T^{t_j}\sigma_s\,e^{-\bar\beta\,\Lambda(s,t_j)}\,\mathrm{d}B_s\right)\right] \\
&= \mathbb{E}\left[\left(\int_T^{t_i}\sigma_s\,e^{-\bar\beta\,\Lambda(s,t_i)}\,\mathrm{d}B_s\right)\left(e^{-\bar\beta\,\Lambda(t_i,t_j)}\int_T^{t_i}\sigma_s\,e^{-\bar\beta\,\Lambda(s,t_i)}\,\mathrm{d}B_s + \int_{t_i}^{t_j}\sigma_s\,e^{-\bar\beta\,\Lambda(s,t_j)}\,\mathrm{d}B_s\right)\right] \\
&= e^{-\bar\beta\,\Lambda(t_i,t_j)}\,\mathbb{E}\left[\left(\int_T^{t_i}\sigma_s\,e^{-\bar\beta\,\Lambda(s,t_i)}\,\mathrm{d}B_s\right)^2\right] = e^{-\bar\beta\,\Lambda(t_i,t_j)}\left(\int_T^{t_i}\sigma_s\,e^{-2\,\bar\beta\,\Lambda(s,t_i)}\,\mathrm{d}s\right).
\end{aligned}$$

The fourth equality uses the fact that $B_t$ has independent increment so that two stochastic integrals with non-overlapping ranges are uncorrelated. The fifth equality uses Itô's isometry. Therefore, as $\bar\beta$ goes to infinity we get

$$\lim_{\bar\beta\to\infty}\mu_i^M(\bar\beta) = 0 \qquad \text{and} \qquad \lim_{\bar\beta\to\infty}\Sigma_{ij}^M(\bar\beta) = 0, \qquad \text{for all } i,j = 1,\ldots,n.$$

We conclude that the distribution of $(M_{t_1}(\bar\beta), M_{t_2}(\bar\beta), \ldots, M_{t_n}(\bar\beta))$ converges to the distribution of the constant $(0, \ldots, 0)$.

We now prove that $\{M_t(\bar\beta) : \bar\beta > 0\}$ is tight. For this we show that there exists a constant $K$ independent of $\bar\beta$ such that

$$\mathbb{E}\left[\left(M_{t_2}(\bar\beta) - M_{t_1}(\bar\beta)\right)^2\right] \le K\,|t_2 - t_1|, \qquad \text{for all } t_1, t_2 \in \mathcal{T}.$$

Indeed, by the definition of $M_t(\bar\beta)$ and Itô's isometry it follows for $t_1 \le t_2$ that

$$\mathbb{E}\left[\left(M_{t_2}(\bar\beta) - M_{t_1}(\bar\beta)\right)^2\right] = \left(M_T^2\, e^{-2\bar\beta\Lambda(T,t_1)} + \int_T^{t_1} \sigma_s^2 e^{-2\bar\beta\Lambda(s,t_1)}\mathrm{d}s\right)\left(1 - e^{-\bar\beta\Lambda(t_1,t_2)}\right)^2 + \int_{t_1}^{t_2} \sigma_s^2 e^{-2\bar\beta\Lambda(s,t_2)}\mathrm{d}s.$$

Since $(1-\exp(-x))^2 \le 2\,x$ for all $x \ge 0$, it follows that

$$\left(1 - e^{-\bar\beta\Lambda(t_1,t_2)}\right)^2 \le 2\,\bar\beta\Lambda(t_1,t_2) \le 2\,\bar\beta\,\bar\lambda_{\mathcal{T}}\,(t_2 - t_1).$$

As a result, we have that

$$\mathbb{E}\left[\left(M_{t_2}(\bar\beta) - M_{t_1}(\bar\beta)\right)^2\right] \le \left[2\,\bar\lambda_{\mathcal{T}}\left(M_T^2\,\bar\beta\, e^{-2\bar\beta\Lambda(T,t_1)} + \int_T^{t_1} \sigma_s^2\,\bar\beta\, e^{-2\bar\beta\Lambda(s,t_1)}\mathrm{d}s\right) + \bar\sigma_{\mathcal{T}}^2\right](t_2 - t_1).$$

Since $t_1 \ge T_1 > T$, it is not hard to show that

$$\bar\beta\, e^{-2\bar\beta\Lambda(T,t_1)} \le \frac{e^{-1}}{2\,\Lambda(T,t_1)} \le \frac{e^{-1}}{2\,\Lambda(T,T_1)}.$$

In addition,

$$\int_T^{t_1} \sigma_s^2\,\bar\beta\, e^{-2\bar\beta\Lambda(s,t_1)}\mathrm{d}s \le \bar\sigma_{\mathcal{T}}^2 \int_T^{t_1} \bar\beta\, e^{-2\bar\beta\,\underline{\lambda}_{\mathcal{T}}\,(t_1-s)}\mathrm{d}s = \frac{\bar\sigma_{\mathcal{T}}^2}{2\,\underline{\lambda}_{\mathcal{T}}}.$$

Hence, we can choose the constant $K$ to be equal to

$$K = \bar\lambda_{\mathcal{T}}\left(\frac{M_T^2\, e^{-1}}{\Lambda(T,T_1)} + \frac{\bar\sigma_{\mathcal{T}}^2}{\underline{\lambda}_{\mathcal{T}}}\right) + \bar\sigma_{\mathcal{T}}^2,$$

which is independent of $\bar\beta$. Hence, $\{M_t(\bar\beta)\}$ is tight. If $M_T = 0$ a.s. then we can repeat the previous steps with $T_1 = T$.

Finally, we prove the weak convergence of the insider's trading strategy. Let us denote by $X_t(\bar\beta)$ the insider's trading strategy when $\beta_t = \bar\beta$. It follows from Proposition 3 that $X_t(\bar\beta)$ has the following dynamics

$$\mathrm{d}X_t(\bar\beta) = \bar\beta\, M_t(\bar\beta)\,\mathrm{d}t = \frac{1}{\lambda_t}\left[\sigma_v\,\mathrm{d}B_t^v - \lambda_t\,\sigma_y\,\mathrm{d}B_t^y - \mathrm{d}M_t(\bar\beta)\right], \qquad t \ge T.$$

Integrating from $T$ to $t$ we get

$$X_t(\bar\beta) = X_T + \sigma_y\left[(B_t^v - B_T^v) - (B_t^y - B_T^y)\right] - \frac{\sigma_y}{\sigma_v}\,(M_t(\bar\beta) - M_T).$$

Since $M_T = 0$, $\lambda_t \to \sigma_v/\sigma_y$ and $M_t(\bar\beta)$ converges weakly to 0 as $\bar\beta \to \infty$ for all $t > T$, it follows that

$$X_t(\bar\beta) \overset{\bar\beta\to\infty}{\Longrightarrow} X_t = X_T + \sigma_y\left[(B_t^v - B_T^v) - (B_t^y - B_T^y)\right], \qquad t \ge T. \qquad \blacksquare$$

**Proof of Proposition 7.** Recall from Theorem 2 that $\Sigma_t$ satisfies

$$\Sigma_t = \Sigma_0 + \sigma_v^2\, t - \sigma_v^2\, e^{2\mu T} \left[\frac{1 - e^{-2\,\mu\, t}}{2\mu}\right] \qquad t < T,$$

and $\Sigma_T = 0$ for all $t \geq T$, where $T \geq 0$ is the unique solution to

$$\Sigma_0 + \sigma_v^2\, T = \sigma_v^2 \left[\frac{e^{2\,\mu\, T} - 1}{2\,\mu}\right].$$

Since $T$ decreases with $\sigma_v$, it suffices to prove that $\Sigma_t$ decreases with $\sigma_v$ for $t < T$.

In what follows, and without lost of generality, we will normalize the value of $\mu$ such that $2\,\mu = 1$ (this is equivalent to re-scaling time). With this normalization, the derivative of $\Sigma_t$ ($t < T$) with respect to $\sigma_v^2$ is equal to

$$\frac{\partial \Sigma_t}{\partial \sigma_v^2} = t - e^T\,(1 - e^{-t}) - \sigma_v^2\, e^T\,(1 - e^{-t})\,\frac{\partial T}{\partial \sigma_v^2}, \qquad t < T.$$

In addition, from the definition of $T$ it follows that

$$\frac{\partial T}{\partial \sigma_v^2} = \frac{1}{\sigma_v^2}\left[\frac{1 + T - e^T}{e^T - 1}\right].$$

Plugging back this value on $\frac{\partial \Sigma_t}{\partial \sigma_v^2}$ we get that for $t < T$

$$\frac{\partial \Sigma_t}{\partial \sigma_v^2} = t - (1 - e^{-t})\left[\frac{T}{1 - e^{-T}}\right] \leq 0.$$

The inequality follows from the fact that $t/(1 - e^{-t})$ is an increasing function of $t$.

Let us now prove the monotonicity of the insider's ex-ante expected payoff. Given the normalization $2\,\mu = 1$, this payoff is given by

$$\mathbb{E}[\widehat{\Pi}(t, M_t)] = 2\,\sigma_y\,\sigma_v\,\cosh\left(\frac{1}{2}\,(T - t)^+\right) \qquad t \geq 0.$$

Note that to prove the monotonicity of $\mathbb{E}[\widehat{\Pi}(t, M_t)]$ with respect to $\sigma_v$ it is enough to focus on the case $t \leq T$. The derivative of $\Pi_t$ with respect to $\sigma_v$ is given by

$$\begin{aligned}
\frac{\partial \mathbb{E}[\widehat{\Pi}(t, M_t)]}{\partial \sigma_v} &= 2\,\sigma_y\,\cosh\left(\frac{1}{2}\,(T - t)\right) + \sigma_y\,\sigma_v\,\sinh\left(\frac{1}{2}\,(T - t)\right)\frac{\partial T}{\partial \sigma_v} \\
&= 2\,\sigma_y\,\cosh\left(\frac{1}{2}\,(T - t)\right) + 2\,\sigma_y\,\sinh\left(\frac{1}{2}\,(T - t)\right)\left[\frac{1 + T - e^T}{e^T - 1}\right] \\
&= 2\,\sigma_y\,\sinh\left(\frac{1}{2}\,(T - t)\right)\left[\frac{T}{e^T - 1}\right] + 2\,\sigma_y\,\exp\left(\frac{T - t}{2}\right) \geq 0. \qquad \blacksquare
\end{aligned}$$

# References


Amihud, Y., H. Mendelson, L.H. Pedersen. 2006. Liquidity and asset prices. *Foundations and Trends in Finance* **1** 269–364.

Back, K. 1992. Insider trading in continuous time. *Review of Financial Studies* **5** 387–409.

Back, K., S. Baruch. 2004. Information in securities markets: Kyle meets glosten and milgrom. *Econometrica* **72**(2) 433–465.

Bagehot, W. 1971. The only game in town. *Financial Analyst Journal* **22** 12–14.

Biais, B., L. Glosten, C. Spatt. 2005. Market microstructure: a survey of microfundations, empirical results and policy implications. *Journal of Financial Markets* **8** 217–264.

Billingsley, P. 1999. *Convergence of Probability Measures*. 2nd ed. John Wiley & Sons, Inc., New York.

Brunnermeier, M.K. 2001. *Asset Pricing under Asymmetric Information*. Oxford University Press, New York.

Chau, M., D. Vayanos. 2006. Stron-form efficiency with monopolistic insiders. Tech. rep., London School of Economics.

Dasgupta, A., A. Prat. 2005. Reputation and asset prices: A theory of information cascades and systematic mispricing. Tech. rep., London Business School.

Easley, D., M. O'Hara. 1987. Price, trade size, and information in securities markets. *Journal of Financial Economics* **19** 69–90.

Fleming, W.H., H.M. Soner. 1993. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York.

Foster, F.D., S. Viswanathan. 1996. Strategic trading when agents forecast the forecasts of others. *The Journal of Finance* **LI**(4) 1437–1478.

Glosten, L. 1989. Insider trading, liquidity and the role of monopolist specialist. *Journal of Business* **62** 211–235.

Glosten, L., P. Milgrom. 1985. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics* **14** 71–100.

Grossman, S., J. Stiglitz. 1980. On the impossibility of informationally efficient markets. *America Economic Review* **70** 393–408.

Hellwig, M.F. 1982. Rational expectations equilibrium with conditioning on past prices: A mean-variance example. *J. of Economic Theory* **26** 279–312.

Holden, G.W., A. Subrahmanyam. 1992. Long-lived private information and imperfect competition. *The Journal of Finance* **XLVII**(1) 247–270.

Knopp, K. 1990. *Theory and Applications of Infinite Series*. Dover, New York.

Kyle, A. 1985. Continuous auctions and insider trading. *Econometrica* **53** 1315–1335.

Mendelson, H., T. Tunca. 2004. Strategic trading, liquidity, and information acquisition. *Review of Financial Studies* **17** 295–337.

O'Hara, M. 1997. *Market Microstucture Theory*. Blackwell Publishing, Massachusetts.

Prokhorov, Y. 1956. Convergence of random processes and limit theorems in probability theory. *Theor. Prob. and Appl.* **1**(2) 157–214.

Spiegel, M., A. Subrahmanyam. 1992. Informed speculation and hedging in a noncompetitive securities market. *Review of Financial Studies* **5** 307–329.