Performance-Sensitive Debt*

Gustavo Manso
Bruno Strulovici
Alexei Tchistyi

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Abstract

This paper studies *performance-sensitive debt* (PSD), the class of debt obligations whose interest payments depend on some measure of the borrower's performance. For example, step-up bonds compensate credit rating downgrades with higher interest rates, and reward credit rating upgrades with lower interest rates. In an endogenous default setting, we develop an algorithm to value PSD obligations allowing for general payment profiles, and obtain closed-form pricing formulas in important special cases, including step-up bonds. Moreover, we provide a criterion to compare different PSD obligations in terms of their efficiency. In particular, we find that step-up bonds lead to earlier default and lower the market value of the issuing firm's equity, compared to fixed-coupon bonds of the same market value. Lastly, we analyze the implications of our results for the policy of credit-rating agencies.

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*Gustavo Manso and Alexei Tchistyi are with the Graduate School of Business, Stanford University. Bruno Strulovici is with the Management Science & Engineering Department, Stanford University. Emails: manso@stanford.edu, bstrulovici@stanford.edu and tchistyi@stanford.edu. We are extremely grateful to Darrell Duffie for insightful comments and advice throughout the development of the paper. We also thank José Blanchet, Albert Chun, Peter DeMarzo, Michael Harrison, Allan Mortensen, Ilhyock Shim, Robert Wilson and participants in the Finance 622 class project at Stanford University for helpful conversations.

1 Introduction

This paper studies *performance-sensitive debt* (PSD), the class of debt obligations whose interest payments depend on some measure of the borrower's performance. For instance, step-up bonds compensate credit rating downgrades with higher interest rates, and credit rating upgrades with lower interest rates.[1] Performance pricing loans, a large fraction of commercial loans, also tie their interest rates to some measure of the borrower's credit quality.[2]

PSD obligations, including step-up bonds and performance pricing loans, compensate debtholders for changes in the borrower's credit risk. Practitioners have not yet reached any consensus on the likely effects of these *risk-compensating* PSD schemes. While proponents laud their high-yield, low-volatility characteristics (some even finding them "too generous"[3]), critics argue that risk-compensating PSD schemes generate a vicious circle by increasing the burden of debt service during financial strains, harming the issuer even more and, eventually, harming investors.[4] Underlying this disagreement is the lack of a theoretical model to value PSD and to assess the effect of issuing PSD rather than standard debt. This latter difficulty can be formalized as follows: for a given amount of debt raised, risk-compensating PSD results in paying higher interest than standard debt in times of low performance, and lower interest in times of high performance. It is unclear, then, between the perspective of lighter debt burden in times of high performance and the increased payment strains in times of low performance, which type of debt is more desirable.

Our goal is to build a valuation model for PSD, and to investigate how different types of PSD affect the timing of default and the equity value of the issuing firm.

We develop a pricing algorithm allowing, tractably, for general payment profiles. We show that the equity value associated with PSD satisfies an ordinary differential equation with a boundary condition corresponding to zero value at default, and a "smooth-pasting" condition. We obtain closed-form pricing of PSD in important special cases, including step-up bonds.

Building on our valuation model, we find that risk-compensating PSD

[1] Step-up bonds exceed \$100bn for both US- and European-based issuers (see Lando and Mortensen (2003) and "Step lightly," CFO Magazine (January 2001).

[2] These loans represent over 70% of commercial loans (see Asquith, Beatty, and Weber (2002)).

[3] "The price of protection," Credit Magazine (September 1st, 2002)

[4] "Credit ratings can harm your wealth," Investment Adviser (December 9th, 2002).

schemes have an overall negative effect on the issuing firm. In particular, issuing risk-compensating PSD leads to earlier default and, consequently, lowers the market value of the issuing firm's equity, holding constant the amount of cash raised by the obligation.

Our results also bear implications on the behavior of credit-rating agencies. In trying to avoid the "credit-cliff dynamic", rating agencies are sometimes reluctant to downgrade distressed firms with PSD obligations in their capital structure.[5] Reluctant agencies generate distortions between actual and theoretical ratings, affecting the reliability of credit rating agencies.

Models of the valuation of risky debt can be divided into two classes. The first class treats firm's liabilities as contingent claims on its underlying assets, and bankruptcy as an endogenous decision of the firm. This set includes Black and Cox (1976), Fischer, Heinkel, and Zechner (1989), Leland (1994), Leland and Toft (1996) and Duffie and Lando (2001). In the second class of models, bankruptcy is not a decision of the firm. There is either an exogenous default boundary for the firm's assets (see Merton (1974) and Longstaff and Schwartz (1995)), or an exogenous process for the timing of bankruptcy, as in Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), and Duffie and Singleton (1999).

Das and Tufano (1996), Acharya, Das, and Sundaram (2002), Houweling, Mentink, and Vorst (2003) and Lando and Mortensen (2003) obtain pricing formulas for credit-sensitive notes using the second class of models of the valuation of risky debt. Since they consider an exogenous default process, the costs associated with performance-sensitive debt do not become apparent in their models.

In order to assess these costs, we work in the setting of Leland (1994), in which default is an endogenous decision of the firm. Instead of a fixed-coupon consol bond, we consider debt obligations in which the interest rate is linked to some performance measure of the borrower. Performance-sensitive debt is thus fully characterized in this setting by some $C : \Pi \mapsto \mathbb{R}_+$ that maps a performance measure π to the interest rate $C(\pi)$ charged on the debt. Typical performance measures are credit ratings and financial ratios such as debt-to-earnings, leverage, or interest coverage.

For PSD obligations C and D that are based on the same performance measure, we say that C is *more risk-compensating* than D if $C - D$ is non-increasing and non-constant. We prove that if C and D raise the same amount of cash, and if C is more risk-compensating than D, then C is less efficient than D, in the sense that it induces an earlier default time,

[5] See Standard & Poor's (2001).

therefore a higher present value of bankruptcy costs, and thus reduces the initial market value of the issuing equity.

Therefore, it turns out that the trade-off between the opposite effects of the more risk-compensating scheme – relatively higher coupons in times of low performance and lower coupons in times of high performance – is systematically resolved in favor of the less risk-compensating debt.[6] We propose the following interpretation for this result.

At the time of default decision, the more risk-compensating PSD requires higher interest payments, increasing the firm's losses. Although it is possible that this PSD imposes a lighter debt burden in the future, the current situation has a higher weight on equityholders' decision, and makes it less attractive for them to continue running the firm.

The remainder of the paper is organized as follows. In Section 2, we illustrate several applications. In Section 3, we present the general model and formalize the notion of PSD. Section 4 analyzes the case of asset-based PSD obligations, demonstrating their relative efficiency. In section 5, we explicitly derive the valuation of step-up and linear PSD obligations. Section 6 discusses different performance measures used in practice, and solves for the case of ratings-based PSD. Section 7 discusses the implications of our results for rating agencies policy. Section 8 provides additional discussion. Section 9 concludes.

2 Applications of PSD

This section describes PSD obligations that arise in practice. Some types of PSD obligations, such as credit-sensitive notes, performance-pricing loans and catastrophe bonds, have explicit performance-pricing provisions. Other types of PSD obligations are implicitly performance dependent because the terms of the debt are subject to renegotiation or are the result of an optimal dynamic capital strategy.

Credit-sensitive notes. A credit-sensitive note, sometimes called a step-up bond, pays an interest rate that is contractually linked with the credit rating of the borrower.

[6] This result is somewhat related to the finding by Hillion and Vermaelen (2004) that the issuance of floating-priced convertibles is followed by significant negative abnormal returns. Hillion and Vermaelen (2004) point out that the design of floating-priced convertibles encourages speculative short-selling activities by the convertible holders that can hurt the equityholders. In this paper, we are not considering convertibles or market speculation.

First issued in the late 1980s, credit-sensitive notes have recently experienced an upsurge, specially among European telecommunications companies.[7]

Performance-pricing loans. Performance-pricing loans explicitly tie their interest to some pre-specified performance measure of the borrower. Typical performance measures used for this purpose are credit ratings and such financial ratios as debt-to-earnings, leverage, or interest coverage.

In an analysis of the Loan Pricing Corporation Database, Asquith, Beatty, and Weber (2002) found that the proportion of lending agreements including performance pricing provisions covered by this database increased from 40% in 1994 to over 70% in 1998.

Put-call provisions. Suppose a debt issue has provisions allowing the lending bank to put the debt back to the issuer when some performance measure drops below a contractual threshold. When such a provision is triggered, the lending bank often renegotiates the initial terms of the loan in effect, increasing the interest rate.

The borrower may be given an option to call the loan when its credit quality improves. This permits the borrower to refinance the debt at lower interest rates after good performance. The outcome of these forms of optionality is effectively PSD.

Reset bonds. A reset bond, sometimes called a payment-in-kind (PIK) bond, has an interest rate that is adjusted periodically so that the market value of the bond is the same as its principal. In some cases the new interest rate is determined by an auction. The associate coupon rate C is thus decreasing in the credit quality of the borrower and a reset bond is, in effect, a form of PSD. Default in the junk-bond market may be induced by the rise in coupon payments of reset bonds.[8]

Short-term debt. The simplest case of PSD is short-term debt, such as commercial paper, since the coupon rate rises and falls continuously with the credit quality of the borrower. Myers (1977) argues that short-term debt may be used to mitigate agency costs. In Diamond (1991), risky firms do

[7]Houweling, Mentink, and Vorst (2003) and Lando and Mortensen (2003) study the pricing of the recent European telecommunications step-up bonds.

[8]"The Junk-Bond Time Bombs Could Go Off," Business Week (April 9th, 1990).

not issue short-term debt in order to avoid early liquidation. Guedes and Opler (1996) provide empirical evidence supporting both claims.

Catastrophe bonds. Catastrophe (CAT) bonds, usually issued by insurance companies, promises coupons that are reduced in case total losses in the insurance industry are above a pre-specified threshold.[9]

Dynamic capital structure. In a setting with taxes and bankruptcy costs, the optimal amount of debt outstanding varies with asset level. When the asset level increases, for example, issuers are better off by issuing more debt, since this gives them higher tax benefits. On the other hand, when the asset level decreases, debt reductions are optimal, ignoring transaction costs, as they reduce the present value of bankruptcy costs. The net effect, under some conditions, is PSD. This setting is studied in Goldstein, Ju, and Leland (1998).

3 The General Model

We begin by specifying a general model. Further assumptions will be added in later sections. We consider a generalization of the optimal liquidation models of Fischer, Heinkel, and Zechner (1989) and Leland (1994)[10].

A firm generates cash flows at the rate δ_t, at each time t. We assume that δ is a diffusion defined by

$$d\delta_t = \mu_\delta(\delta_t)dt + \sigma_\delta(\delta_t)dB_t, \tag{1}$$

where μ_δ and σ_δ satisfy the classic assumptions for the existence of a unique strong solution[11] to (1) on a fixed probability space $(\Omega, \mathcal{F}, P)$ with the information filtration $(\mathcal{F}_t)$ generated by the standard Brownian motion B. For simplicity, we assume that all agents are risk-neutral. There is a constant risk-free interest rate r, with $\mu_\delta < r - \varepsilon$ for some positive constant ε. The market value A_t at time t of the future cash flows of the firm is then

$$A_t = E_t\left[\int_t^\infty e^{-r(s-t)}\,\delta_s\,ds\right] < \infty \tag{2}$$

[9] See Fitch IBCA (2001) for a survey of the market for CAT bonds.

[10] While previous model specifications are limited to geometric Brownian motion, we consider here a general diffusion model

[11] See for example Karatzas and Shreve (1991)

where E_t denotes the $\mathcal{F}_t$-conditional expectation. By the Markov property, A_t only depends on $\{\delta\}_{s\leq t}$ through δ_t. Specifically, there exists a smooth function $A:\mathbb{R}\rightarrow\mathbb{R}$ such that $A_t = A(\delta_t)$, which implies that $\{A_t\}_{t\geq 0}$ is a diffusion:

$$dA_t = \mu(A_t)dt + \sigma(A_t)dB_t. \tag{3}$$

For the sake of ulterior computations, we assume

Condition 1 *μ and σ are smooth and bounded and σ is coercive.*[12]

Since $E_t[\delta_s]$ is increasing in δ_t, $A(\cdot)$ is increasing, which implies the existence of a continuous inverse function $\delta:\mathbb{R}\rightarrow\mathbb{R}$ such that $\delta_t = \delta(A_t)$.

We consider a performance measure represented by an $\mathcal{F}_t$-adapted stochastic process $(\pi_t)_{0\leq t<\infty}$ taking values in a totally ordered, topological space Π. In general, π_t can be any statistic that measures the firm's ability and willingness to serve its debt obligations in the future. Financial ratios and credit ratings are among commonly used performance measures. A *performance-sensitive debt* (PSD) obligation is a claim on the firm that promises a non-negative payment rate that may vary with the performance measure of the firm. Formally, a PSD obligation $C(\cdot)$ is a measurable function $C:\Pi\rightarrow\mathbb{R}$, such that the firm pays $C(\pi_t)$ to the debtholders at time t.[13] For example, the consol bond of Leland (1994) is a degenerate case of PSD. The reader should note that, while our earlier sections dealt mostly with "risk-compensating" PSD (that pay higher coupons when performance worsens), our definition of PSD encompasses more general kinds of PSD.

Given a PSD obligation C, the firm's optimal liquidation problem[14] is to choose a default time $\widehat{\tau}$ to maximize its initial equity value W_0^C, given the debt structure C. That is,

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[12] In the one-dimensional case, coerciveness means that there exists a real number $\underline{\sigma}$ such that $0<\underline{\sigma}\leq\sigma$. Throughout, smoothness means continuous differentiability.

[13] We are considering perpetual debt, which is a standard simplifying assumption for the endogenous default framework. See, for example, Leland (1994). However, our model can be extended to the case of finite average debt maturity, if we assume that debt is continuously retired at par at a constant fractional rate. See Leland (1998) for more on this approach. Leland and Toft (1996) use more general finite-maturity debt framework. However, due to the complexity of their model, most of their results are obtained using simulations.

[14] Firms usually have standard fixed-coupon bonds together with different types of PSD obligations in their capital structure. In this case, the total outstanding debt of the firm is still PSD, but one has to sum the payment rates of all debt obligations issued by the firm when determining the total payment rate of the firm, which is the relevant payment rate for liquidation purposes.

$$W_0^C \equiv \sup_{\widehat{\tau}\in\mathcal{T}} E\left[\int_0^{\widehat{\tau}} e^{-rt}[\delta_t - (1-\theta)C(\pi_t)]\,dt\right], \tag{4}$$

where $\mathcal{T}$ is the set of $\mathcal{F}_t$ stopping times and θ is the corporate tax rate on earnings. If τ^* is the optimal liquidation time, then the market value of the equity at time $t < \tau^*$ is

$$W_t^C = E_t\left[\int_t^{\tau^*} e^{-r(s-t)}[\delta_s - (1-\theta)C(\pi_s)]\,ds\right]. \tag{5}$$

Analogously, the market value U_t^C of the PSD obligation C at time t is

$$U_t^C \equiv E_t\left[\int_t^{\tau^*} e^{-r(s-t)}C(\pi_s)\,ds\right] + E_t\left[e^{-r(\tau^*-t)}\left(A_{\tau^*} - \rho(A_{\tau^*})\right)\right], \tag{6}$$

where $\rho(.)$ defines the portion of the asset value lost at bankruptcy. We assume that ρ is an increasing function such that $0 \le \rho(x) \le x$ for all $x \ge 0$. If δ_t is lower than $(1-\theta)C(\pi_t)$, equity holders have a net negative dividend rate.[15] Equity holders will continue to operate a firm with negative dividend rate if the firm's prospects are good enough to compensate for the temporary losses.

4 Asset-Based PSD

In all the applications of PSD listed in the Section 2, the interest rate charged to the borrower depends on the borrower's credit quality. Since the market value A of assets is a time-homogeneous Markov process, the current asset level A_t is the only state variable in our model, and any measure of the borrower's earnings prospect at time t is solely determined by A_t.

Therefore, it is natural to consider the asset level A_t as a performance measure. An *asset-based PSD* is a PSD whose coupon rate only depends on the current asset level. Specifically, an asset-based PSD is a measurable function $C : \mathbb{R}_+ \to \mathbb{R}$, under which the firm pays coupons at rate $C(A_t)$ at time t. Using this definition, we derive valuation and efficiency results for asset-based PSD.

[15] Limited liability is satisfied if the negative dividend rate is funded by dilution, for example through share purchase rights issued to current shareholders at the current valuation.

4.1 Valuation

Given an asset-based PSD, the initial value of the equity is:[16]

$$W(A_0) \equiv \sup_{\widehat{\tau} \in \mathcal{T}} E\left[\int_0^{\widehat{\tau}} e^{-rt}\left[\delta(A_t) - (1-\theta)C\left(A_t\right)\right]\ dt\right].$$

The Markov property and time homogeneity imply that there exist asset levels A_B and A_H with $A_B < A_0 < A_H$, such that an optimal default time of the firm is of the form $\tau^* = \min(\tau(A_B), \tau(A_H))$, where $\tau(x) \equiv \inf\{t : A_t = x\}$. Even though the existence of an upper asset boundary A_H above which the firm would default is mathematically possible, we exclude this unnatural possibility with the following condition.

Condition 2 *There exist levels $\underline{x} < \bar{x}$ and a positive constant $\underline{c}$ such that*

1. $(1-\theta)C(x) \geq \delta(x)$ *if and only if* $x \leq \bar{x}$.
2. $(1-\theta)C(x) \geq \delta(x) + \underline{c}$ *for* $x \leq \underline{x}$.

The first part of Condition 2 states that for asset levels higher than $\bar{x}$, the cash flow rate is higher than the coupon payment rate. It can be easily verified that, under this condition, $A_H = +\infty$, so that the optimal default time simplifies to $\tau^* = \tau(A_B)$. Therefore, equity holders' optimal problem can be expressed without loss of generality as:

$$W(x) = \sup_{y<x} \tilde{W}(x,y), \tag{7}$$

where

$$\tilde{W}(x,y) \equiv E_x\left[\int_0^{\tau(y)} e^{-rt}\left[\delta(A_t) - (1-\theta)C(A_t)\right] dt\right].$$

In order to derive an ordinary differential equation (ODE) for W, we impose the following condition on C:

Condition 3 *The PSD obligation C is such that:*

1. *There exist nonnegative constants k_1 and k_2 that satisfy*

$$0 \leq (1-\theta)C(y) \leq k_1 + k_2 y.$$

[16] Throughout this section, we omit the superscript C and the subscript 0 whenever there is no ambiguity.

2. C is right continuous on $[0, \infty)$ *and has left limits on* $(0, \infty)$.

Using the strong Markov property of $\{A_t\}_{t \geq 0}$,

$$\tilde{W}(x, y) = f(x) - \xi(x, y) f(y) \tag{8}$$

where[17] for $x > y$,

$$\xi(x, y) = E_x[e^{-\tau(y)}],$$

and

$$f(x) = E_x \left[\int_0^\infty e^{-rt} \left[\delta(A_t) - (1 - \theta) C(A_t) \right] dt \right].$$

The next lemma shows that, under Condition 2, the default triggering level A_B is strictly positive.

Lemma 1 *Under Condition 2, there exists a level* $\tilde{x}$ *such that any optimal default time* τ *satisfies* $\tau \leq \tau(\tilde{x})$ *almost surely.*

An important consequence of Lemma 1 is that default occurs with positive probability. Our next theorem characterizes the solution of the optimal stopping problem (7).

Theorem 1 *If a PSD C satisfies Conditions 1–3, the following statements are equivalent:*

1. A_B *is an optimal default triggering level:*

$$W(x) = E_x \left[\int_0^{\tau(A_B)} e^{-r(s-t)} [\delta(A_s) - (1 - \theta) C(A_s)] \, ds \right].$$

2. $W(x)$ *and* A_B *satisfy:*

- *(i)* $A_B \in (0, \bar{x})$.
- *(ii) W is continuously differentiable and* W' *is bounded and left and right differentiable.*
- *(iii) W vanishes on* $[0, A_B]$ *and satisfies the following ODE at any point of continuity of C:*

$$\frac{1}{2}\sigma^2(x) W''(x) + \mu(x) W'(x) - rW(x) + \delta(x) - (1 - \theta) C(x) = 0. \tag{9}$$

[17] Previous assumptions on μ and σ imply that ξ is well-defined, continuous, increasing in y and less that 1 (see Karatzas and Shreve (1991)).

A proof is given in the Appendix.[18]

The continuous differentiability of W and the fact that W is 0 on $[0, A_B]$ imply that $W'(A_B) = 0$, which is known as the *smooth-pasting condition*. Theorem 1 provides a method for solving the firm's optimal liquidation problem. The proposed algorithm is the following

1. Determine the set of continuously differentiable functions that solve ODE (9) at every continuity point of C. It can be shown that any element of this set can be represented with two parameters,[19] say L_1 and L_2.

2. Determine A_B, L_1, and L_2 using the following conditions:

a. $W(A_B) = 0$.
b. W' is bounded.
c. $W'(A_B) = 0$.
d. $A_B \in (0, \bar{x})$.

We interpret (a) as the boundary condition on the solution at the point A_B of the ODE. Condition (b) says that $W'(x)$ remains bounded as $x \to +\infty$ and constitutes the second boundary condition on the solution of the ODE. The smooth-pasting condition (c) is interpreted as the first order optimization condition that defines the optimal bankruptcy boundary. Condition (d) verifies that condition 2.(i) of Theorem 1 is satisfied.

Now that we know how to price the equity associated with PSD, we can also price the PSD itself. Using the fact that the sum of the equity value, the PSD value, and the expected losses resulting from the bankruptcy is the sum of the asset level and the present value of the tax benefits, we obtain the PSD pricing formula:

$$U\left(A_t\right) = \frac{1}{1-\theta}\left[A_t - W\left(A_t\right) - \left[\rho(A_B) + \theta(A_B - \rho(A_B))\right]\xi(A_t, A_B)\right].$$

[18] The Appendix also gives two separate equations involving the right and left derivatives of W' at discontinuity points of C (cf. equations (28) and (29)).

[19] In fact, we really consider here solutions of coupled equations (28) and (29), which boil down to the ODE (9) at any continuity point of C. One can easily check that the set of solutions of the coupled equations still is a two-dimensional vector space.

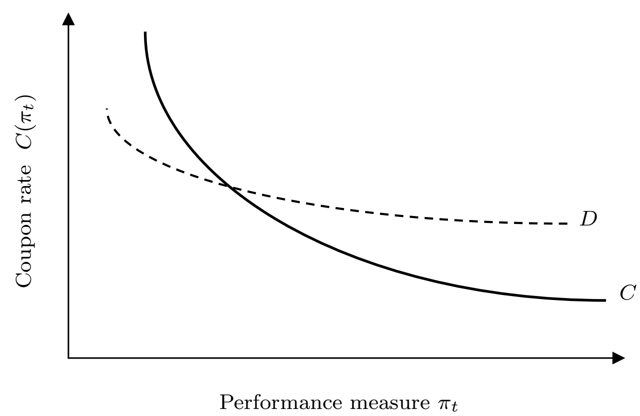

Figure 1: C is more risk-compensating than D

4.2 The Relative Efficiency of Asset-Based PSD

In this subsection, we study the relative efficiency of alternative asset-based PSD. Specifically, we derive a partial order, by "efficiency," among alternative PSD issues that raise the same amount of cash. We need the following definitions and condition, that we state in terms of a general performance measure π. These will also be used in Section 6.2, for the case of credit ratings.

Definition 1 (Relative Efficiency). *Let C and D be PSD that raise the same funds, $U_0^C = U_0^D$. We say that C is* less efficient *than D if it determines a lower equity price, that is, if $W_0^C < W_0^D$.*

Definition 2 (Risk Compensating). *Let C and D be PSD issues based on the same performance measure. We say that C is* more risk-compensating *than D if $C - D$ is a non-increasing, not constant function.*

Figure 1 illustrates the "risk-compensating" concept.

Condition 4 (Efficiency Domain). *A PSD issue C is said to be in its efficiency domain if, for any constant $\alpha > 0$, we have $U_0^{C-\alpha} < U_0^C$, where $C - \alpha$ denotes a PSD issue that pays $C(A_t) - \alpha$ at time t.*

Condition 4 means that it is not possible to raise the same amount of cash as C by a constant downward shift in its coupon rate. For example, a bond paying a fixed coupon rate c raises an increasing amount of cash as c increases, until c reaches a point at which the loss due to precipitated default dominates the gain due to the increase of coupon payment (as in Figure 2). The forms of PSD that we consider are in their efficiency domain, for otherwise efficiency in the sense of Definition 1 can be trivially improved by uniformly reducing the interest rate paid.

Theorem 2 *Suppose C and D both are asset-based PSD, satisfying $U_0^C = U_0^D$ and Conditions 1–4. If C is more risk-compensating than D, then C is less efficient than D.*

A proof of the theorem is given in the appendix.

The above result is supported by the following intuition. Equityholders decide to declare bankruptcy when coupon payments become too high compared to dividends. At this time, the firm pays higher interest rates with C than with D and, while there is a possibility that the situation be reversed in the future, the urgency of the current situation increases the firm's incentive to declare bankruptcy.

This intuition can be further illustrated by the opposite, extreme example of a bond paying a coupon rate equal to the dividend rate $C(A_t) = \delta(A_t)$. This coupon rate decreases to zero as the asset level goes to zero. The coupon payments never exceed the dividends, so the firm never goes bankrupt. Such a bond transfers all the value of the firm to debtholders, and, if it could qualify as "debt" for tax purposes, would reduce tax payments to zero since the tax benefit resulting from coupon payments is equal to the tax on the dividends. Equityholders could decide to buy all of the debt, in which case this bond allows them to receive all of their dividends in form of coupon payments.

Corollary 1 *Let C be a PSD issue satisfying Conditions 1–4. If C is non-increasing and not constant, it is less efficient than the fixed-interest PSD issue raising the same amount of cash and verifying Condition 4. If C is non-decreasing and not constant, it is more efficient than any fixed-interest PSD issue raising the same amount of cash.*

The result suggests that, in many settings, the issuer would choose the least risk-compensating form of debt that qualifies as "debt" for tax treatment.

The following numerical example compares "one-step" PSD issues C that raise the same amount M, in the class $\mathcal{C}_M$ of PSD defined by

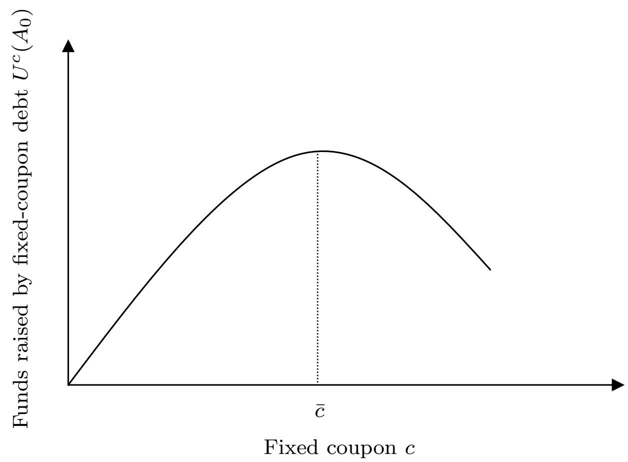

Figure 2: A fixed-coupon bond is in its efficiency domain if $c \in [0, \bar{c}]$.

$$C(A_t) = \begin{cases} C_1, & A_t \geq G_2 \\ C_2, & A_t < G_2 \end{cases},$$

such that $C_2 \geq C_1$ and $U^C(A_0) = M$.

We assume that the asset is a geometric Brownian motion with parameters $\mu = 0.01$, $\sigma = 0.1$, and that $\rho(x) = 0.25x$, $\theta = 0$, $r = 0.03$, $A_0 = 100$, $G_2 = 80$, and $M = 50$, M can be raised by issuing a bond that promises to pay a fixed coupon rate of 2. To see the inefficiency of step-up bonds, we compute for one-step PSD issues in $\mathcal{C}_M$ the present value of bankruptcy losses, which is by definition

$$Q(C) \equiv 0.25E\left[e^{-r\tau(A_B^C)}A_B^C\right] = 0.25A_B^C\left(\frac{A_0}{A_B^C}\right)^{-\gamma_1},$$

where[20] $\gamma_1 = \dfrac{m + \sqrt{m^2 + 2r\sigma^2}}{\sigma^2}$ and $m = \mu - \frac{\sigma^2}{2}$. According to Definition 2, (C_1, C_2) is more risk-compensating than (C_1', C_2') if $C_2 - C_1 > C_2' - C_1'$.

Figure 3 shows the relationship between the present value of bankruptcy losses and the degree of risk-compensation $(C_2 - C_1)$ associated with the PSD. One can see that the fixed-coupon PSD results in a bankruptcy cost of 2.8, while being worth of 50. On the other hand, as the difference $(C_2 - C_1)$ rises, the bankruptcy cost climbs quickly.

[20] See Section 5

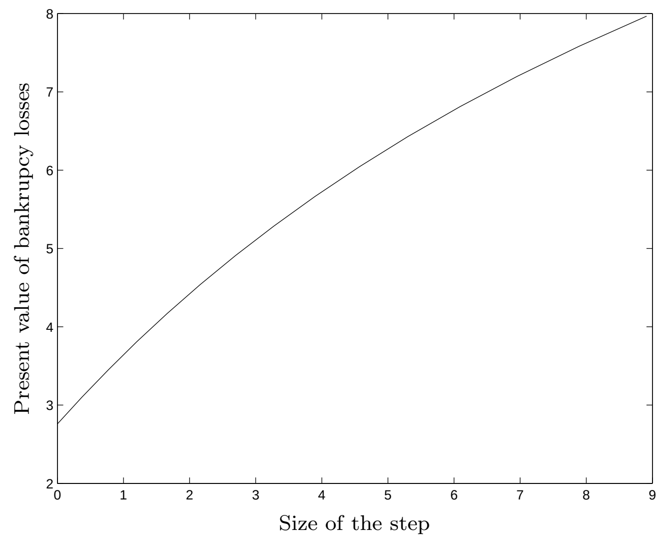

Figure 3: Present value of bankruptcy costs when we increase the size of the step.

5 Examples of Asset-Based PSD

In this section, we solve our model explicitly for two important cases: step-up and linear PSD issues. Step-up PSD is more likely to be seen in practice, while linear PSD has a convenient pricing formula. Throughout this section, we assume that the asset process is a geometric Brownian motion with drift μ and volatility σ^2. This implies that $\delta(x) = (r - \mu)x$, and that $\xi(x, y) = \left(\frac{x}{y}\right)^{-\gamma_1}$, where $\gamma_1 = \dfrac{m + \sqrt{m^2 + 2r\sigma^2}}{\sigma^2}$ and $m = \mu - \frac{\sigma^2}{2}$.

5.1 Step-Up PSD

Step-up performance-sensitive debt is defined to be a PSD obligation whose coupon payment is a non-increasing step function of the asset level. For a decreasing sequence $\{G_i\}_{i=1}^{I+1}$ of asset levels such that $G_1 = +\infty$ and $G_{I+1} = A_B$, the coupon rate of a step-up PSD obligation can be represented as

$$C\left(A_t\right) = \bar{C}_i \text{ whenever } A_t \in [G_{i+1}, G_i), \tag{10}$$

where $\{\bar{C}_i\}_{i=1}^{I}$ is an increasing sequence of constant coupon rates. With this coupon structure, the general solution of the ODE (9) is

$$W(x) = \begin{cases} 0, & x \leq A_B \ , \\ L_i^{(1)} x^{-\gamma_1} + L_i^{(2)} x^{-\gamma_2} + x - \frac{(1-\theta)\bar{C}_i}{r}, & G_{i+1} \leq x \leq G_i \ , \end{cases} \tag{11}$$

for $i = 2, \ldots, I+1$, where $\gamma_1 = \dfrac{m + \sqrt{m^2 + 2r\sigma^2}}{\sigma^2}$, $\gamma_2 = \dfrac{m - \sqrt{m^2 + 2r\sigma^2}}{\sigma^2}$, $m = \mu - \dfrac{\sigma^2}{2}$, and where $L_i^{(1)}$ and $L_i^{(2)}$ are constants to be determined shortly. According to Theorem 1,

$$W(A_B) = 0 \tag{12}$$

and

$$W'(A_B) = 0 \ , \tag{13}$$

and $W(\cdot)$ is continuously differentiable. In particular, for $i = 2, \ldots, I$,

$$W(G_i-) = W(G_i+) \, , \, W'(G_i-) = W'(G_i+) \ . \tag{14}$$

Because the market value of equity is non-negative and cannot exceed the asset value[21],

$$L_1^{(2)} = 0. \tag{15}$$

The system (12)-(15) has $2I+1$ equations with $2I+1$ unknowns ($L_i^{(j)}$, $j \in \{1,2\}$, $i \in \{1, \ldots, I\}$, and A_B). Substituting (11) into (12)-(15) and solving gives

$$L_I^{(1)} = \frac{(\gamma_2 + 1) A_B - \gamma_2 \frac{c_2}{r}}{(\gamma_1 - \gamma_2) A_B^{-\gamma_1}} \ , \tag{16}$$

$$L_I^{(2)} = \frac{-(\gamma_1 + 1) A_B + \gamma_1 \frac{c_2}{r}}{(\gamma_1 - \gamma_2) A_B^{-\gamma_2}} \ , \tag{17}$$

$$L_j^{(1)} = L_I^{(1)} + \frac{\gamma_2}{(\gamma_1 - \gamma_2) r} \sum_{i=j}^{I-1} \frac{c_{i+1} - c_i}{G_{i+1}^{-\gamma_1}}, \ j = 2, \ldots, I \ , \tag{18}$$

$$L_j^{(2)} = L_I^{(2)} - \frac{\gamma_1}{(\gamma_1 - \gamma_2) r} \sum_{i=j}^{I-1} \frac{c_{i+1} - c_i}{G_{i+1}^{-\gamma_2}}, \ j = 2, \ldots, I \ , \tag{19}$$

$$0 = -(\gamma_1 + 1) A_B + \frac{\gamma_1}{r} \left(c_I - \sum_{i=1}^{I-1} (c_{i+1} - c_i) \left(\frac{A_B}{G_{i+1}} \right)^{-\gamma_2} \right) \ , \tag{20}$$

[21] Since $\gamma_1 > 0$ and $\gamma_2 < 0$, the term $L_K^2 x^{-\gamma_2}$ would necessarily dominate the other terms in the equation (11) violating the inequality $0 \leq W(x) \leq x$, unless $L_1^2 = 0$.

where, for convenience, we let $c_i \equiv (1-\theta)\bar{C}_i$.

Although we do not have an explicit solution for these parameters, equations (16)-(19) express $L_j^{(i)}$ as a function of A_B, which, in turn, solves (20). One can verify that (20) has a unique solution on the interval $\left(0, \widehat{A}_B\right)$,[22] where $\widehat{A}_B \equiv \gamma_1 c_I/(r(\gamma_1+1))$ is the default-triggering level of assets for a consol bond with the fixed-coupon rate c_I.

5.2 Linear PSD

Consider the coupon scheme given by

$$C(x) = \beta_0 - \beta_1 x \ ,$$

with $\beta_0 > 0$.

Applying Theorem 1, the corresponding equity value is

$$W(x) = \lambda \left(x - A_B \left(\frac{x}{A_b} \right)^{-\gamma_1} \right) - \frac{\beta_0}{r} \left(1 - \left(\frac{x}{A_B} \right)^{-\gamma_1} \right) \ , \tag{21}$$

and the optimal bankruptcy boundary is

$$A_B = \frac{\gamma_1 \beta_0}{\lambda (1+\gamma_1) r} \ ,$$

where $\lambda = \frac{r-\mu+\beta_1}{r-\mu}$.

When $\beta_1 = 0$, $\lambda = 1$, and the formula (21) for W corresponds to the fixed coupon case with $C = \beta_0$. As expected, W is increasing in β_1 due to the reduction in the coupon rate.

[22] Since $\gamma_1 \in (0,\infty)$ and $\gamma_2 \in (-\infty, 0)$, the left-hand-side of (20) converges to $\frac{\gamma_1}{r} c_I > 0$ as A_B goes to 0, and equals $-\frac{\gamma_1}{r} \sum_{i=1}^{I-1} (c_{i+1} - c_i) \left(\frac{A_B}{G_{i+1}} \right)^{-\gamma_2} < 0$ for $A_B = \widehat{A}_B$, where

$$\widehat{A}_B \equiv \frac{\gamma_1 c_I}{r(\gamma_1 + 1)} \ .$$

One can verify that the left-hand-side is a strictly decreasing function of A_B. Here, $\widehat{A}_B$ is the default-triggering level of assets for a consol bond with fixed-coupon c_I. Our step-up PSD pays several different coupon rates, and all of them are greater or equal than c_I. Therefore, A_B should be no greater than $\widehat{A}_B$, and (20) has a unique solution for A_B on the interval $\left(0, \widehat{A}_B\right)$.

6 Performance Measures

Earlier, we derived valuation formulas and an inefficiency theorem for PSD obligations whose coupon payments are determined by the asset level of the firm. Since, in our model, A_t incorporates all information about future earnings of the firm, the asset level is the natural choice for a performance measure.

In practice, however, PSD contracts are usually written in terms of performance measures such as credit ratings and financial ratios. In this section, we explicitly consider the valuation and relative efficiency of PSD obligations based on these other performance measures.

6.1 General performance measures

We assume that performance measures reflect the borrower's capacity and willingness to repay the debt. Throughout this section, we assume that $\{A_t\}_{t\geq 0}$ follows a geometric Brownian motion with drift μ and volatility σ^2 (see Section 5 for the implications of this assumption). With μ and σ given, the borrower's asset level A_t and chosen default triggering boundary A_B fully determine its default characteristics at any time t. Since A_B is not directly observed by outsiders, the performance measure π_t is a function $\bar{\pi}(A_t, \tilde{A}_B)$, where $\tilde{A}_B$ is the perceived default triggering level of assets. Although we do not make explicit use of this condition, it is natural to assume that $\bar{\pi}(\cdot,\cdot)$ is nondecreasing in A_t and nonincreasing in $\tilde{A}_B$.

A PSD obligation C therefore pays the coupon $C(\pi_t) = C(\bar{\pi}(A_t, \tilde{A}_B))$. The Markov structure and the time homogeneity of the setting imply that any optimal default time of the firm can be simplified to a default triggering boundary hitting time $\tau(A_B)$ (still imposing Condition 2). In this setting, a consistency problem arises, as the default triggering level chosen by the firm may depend on the perceived default triggering level. With y denoting the actual default triggering level of the firm, the value of the equity is

$$\tilde{W}(x, y, \tilde{A}_B) = E_x \left[\int_0^{\tau(y)} e^{-rt} \left[(r-\mu)\, A_t - (1-\theta) C(\bar{\pi}(A_t, \tilde{A}_B)) \right] dt \right].$$

Knowing that the firm seeks to maximize the value of the equity, the ratings agency therefore chooses an $\tilde{A}_B$ that solves the fixed point equation:

$$A_B \in \arg\max_{y \leq x} \tilde{W}(x, y, A_B). \tag{22}$$

This equation may have one or several solutions, or no solution at all. To avoid ambiguity, we impose the following condition.

Condition 5 *There exists a unique positive solution of equation (22).*

Given Condition 5, the coupon rate paid by the PSD obligation at time t is $C(\bar{\pi}(A_t, A_B))$. Since A_B does not change over time, this PSD, which is defined under performance measure π, is equivalent to an asset-based PSD $\tilde{C}$, defined by $\tilde{C}(A_t) \equiv C(\bar{\pi}(A_t, A_B))$. Equation (22) implies that C and $\tilde{C}$ have the same optimal default boundary A_B. Hence, provided that $\tilde{C}$ satisfies Conditions 2,3, and 4, we can compare C in terms of efficiency with asset-based PSD obligations that satisfy the same Conditions by applying Theorem 2. In particular, if $\tilde{C}(A_t)$ is a nonincreasing nonnegative function, then a fixed-coupon bond with the same market value is more efficient than C.

If π can only take finitely many values, then $\tilde{C}(A_t)$ satisfies Conditions 2 and 3. Thus, we have proven the following theorem.

Theorem 3 *Suppose that a performance measure π can only take a finite number of values, and that a PSD C is nonincreasing and nonnegative. Suppose Conditions 4 and 5 are satisfied. Then, a fixed-coupon PSD D that satisfies Condition 4, and has the same market value as C ($U_0^C = U_0^D$), is more efficient than C.*

6.2 Ratings-based PSD

We consider I different credit ratings, $1, \ldots, I$, with 1 the highest ("Aaa" in Moody's ranking) and I the lowest ("C" in Moody's ranking). We let R_t denote the issuer's credit rating at time t. We say that $C \in \mathbb{R}^I$ is a ratings-based PSD obligation if it pays interest at the rate C_i whenever $R_t = i$, with $C_{i+1} \geq C_i > 0$, for i in $\{1, \ldots, I-1\}$. Thus, a *ratings-based PSD is more risk-compensating than a fixed coupon PSD.*

We say that an *accurate* rating agency is one whose credit ratings are a function of the probability of default over a given time horizon T. Naturally, higher ratings correspond to lower default probabilities.

The default time for a ratings-based PSD is a stopping time of the form $\tau(A_B) = \inf\{s : A_s \leq A_B\}$, for some A_B. Therefore, the current asset level A_t is a sufficient statistic for $P(\tau(A_B) \leq T \,|\, \mathcal{F}_t)$, for any $T \geq t$. A rating policy is thus given by some $G : \mathbb{R} \mapsto \mathbb{R}^{I+1}$ that maps a default boundary A_B into rating transition thresholds, any such that $R_t = i$ whenever $A_t \in [G_{i+1}(A_B), G_i(A_B))$. In our setting, this policy has the form[23]

$$G(A_B) = A_B\, g, \tag{23}$$

[23]Since A_t is a geometric Brownian motion, its first passage time distribution is an

where $g \in \mathbb{R}^I$ is such that $g_1 = +\infty$, $g_{I+1} = 1$, and $g_i \geq g_{i+1}$.

The results developed for step-up PSD can be applied to ratings-based PSD. In particular, the maximum-equity-valuation problem (4) is solved by $\tau(A_B) = \inf\{s : A_s \leq A_B\}$, where A_B solves equation (20).

Plugging (23) into (20), we obtain

$$A_B = \frac{\gamma_1}{(\gamma_1 + 1)r}\widehat{C}, \tag{24}$$

where

$$\widehat{C} = \sum_{i=1}^{I} \left[\left(\frac{1}{g_{i+1}} \right)^{-\gamma_2} - \left(\frac{1}{g_i} \right)^{-\gamma_2} \right] c_i,$$

and $c_i = (1-\theta)C_i$. We note that the ratings-based PSD issue C has the same default boundary A_B as that of a fixed-coupon bond paying coupons at the rate $\widehat{C}$.

Plugging (24) into (16)-(19), (11), and (6), one obtains closed-form expressions for the market value W of equity and the market value U of debt for any ratings-based PSD obligation.

We now derive the inefficiency theorem for the case of ratings-based PSD. We keep the same definitions as in Section 4, except that the performance measure now corresponds to credit ratings, and not asset levels.

Theorem 4 *Suppose C and D are ratings-based PSD, satisfying $U_0^C = U_0^D$ and Condition 4. If C is more risk-compensating than D, then C is less efficient than D.*

The proof of the theorem is given in the appendix.

Corollary 2 *Let C be a ratings-based PSD issue satisfying Conditions 2,3, and 4. If C is not constant, it is less efficient than any fixed-interest PSD issue raising the same amount of cash and verifying Condition 3.*

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inverse Gaussian:

$$P(\tau(A_B) \leq T \,|\, \mathcal{F}_t) = 1 - \Phi\left(\frac{m(T-t) - x}{\sigma\sqrt{T-t}}\right) + e^{\frac{2mx}{\sigma^2}}\Phi\left(\frac{x + m(T-t)}{\sigma\sqrt{T-t}}\right),$$

where, $x = ln\left(\frac{A_B}{A_t}\right)$, $m = \mu - \frac{1}{2}\sigma^2$, A_t is the current level of assets and Φ is the normal cumulative distribution function. Since $P(\tau(A_B) \leq T \,|\, \mathcal{F}_t)$ depends on A_t only through $\frac{A_B}{A_t}$, we have the linearity of $G(\cdot)$.

7 Rating Agency Policy

Credit ratings differ from other measures because of the circularity issues that are imposed. In a ratings-based PSD obligation, the rating determines the coupon rate, which affects the optimal default decision of the issuer. This, in turn, influences the rating. We have so far assumed that rating agencies are accurate, in the sense that they assign credit ratings according to the probability of default over a time horizon T. In this section, we discuss what can happen when credit-rating agencies fail to account for the effect of credit-rating changes on the firm's financial standing.

Only after recent deteriorations in credit qualities of several major companies did rating agencies begin to worry about the unintended adverse effects of rating triggers.[24] Even after several incidences of default and cascading downgrades related to ratings-based PSD, it is not difficult to find examples of reluctance by rating agencies to incorporate the negative consequences of ratings-based PSD into credit ratings.[25] The following passage is from Standard & Poor's (2001):

> (...) How is the vulnerability of rating triggers reflected all along in a company's ratings? Ironically, it typically is not a rating determinant, given the circularity issues that would be posed. To lower a rating because we might lower it makes little sense – specially if that action would trip the trigger!

Another reason that rating triggers may not be incorporated into credit ratings is that often, due to confidentiality constraints, they are not publicly disclosed by the issuer. Some steps have already been taken to punish issuers who refuse to provide information about their rating triggers, although there is still no legal procedure to enforce disclosure.[26]

We say that an agency is *unresponsive* if it ignores, when assigning credit ratings, the adverse effects of rating triggers on the liquidation of the firm.

We suppose, for purpose of illustration, that a firm having a fixed-coupon note C refinances its outstanding debt by issuing a ratings-based PSD obligation D. Figure 4 plots the accurate agency policy $G(\,\cdot\,)$, which is obtained from (23), and equityholders' optimal default strategies $A_B^C(\,\cdot\,)$ and $A_B^D(\,\cdot\,)$, which are obtained from (20). Points 1 and 5 in the figure yield the solution to (22) before and after the refinancing of the debt takes place. With

[24] See Moody's (2001) and Standard & Poor's (2001).

[25] Moody's adopted a more critical view of ratings trigger after recent default events. See Moody's (2001).

[26] See Moody's (2002).

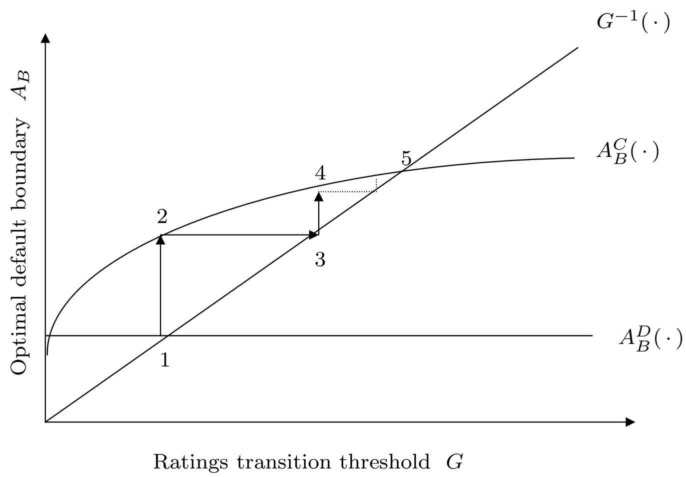

Figure 4: Rationalizing the credit cliff dynamic.

an accurate rating agency, issuance of ratings-based PSD obligations thus triggers a chain reaction that ceases only when it reaches point 5. This chain reaction, which we call *credit-cliff* dynamic, might induce a drastic downgrade or even immediate default if $A_B^C > A_0$.

By ignoring the effects of ratings triggers, an unresponsive rating agency may avoid the perverse effects associated with the credit-cliff dynamic. In the context of figure 4, an unresponsive rating agency would interrupt the chain reaction at point 2, leading to a lower optimal default boundary than in the case of an accurate rating agency.

One would then be tempted to say that the outcome of a ratings-based PSD with an unresponsive rating agency is superior to the one with an accurate one. We claim that this is not necessarily true. With unresponsive rating agencies, credit ratings do not reflect true probabilities of default and are thus less informative. Moreover, firms may be tempted to issue more risk-compensating ratings-based PSD, compensating for the unresponsiveness of rating agencies.

8 Additional Discussion

Even though our main result is that more risk-compensating PSD obligations lead to higher inefficiency, companies do issue these obligations in practice. In order to understand why this is the case, one could introduce market frictions such as adverse selection, moral hazard, contracting costs, or incomplete markets. Since these would complicate the model, we confine ourselves to an intuitive discussion of these issues.

Performance-sensitive debt may be used to solve the adverse selection problem, which arises because of information asymmetries at the time of debt issuance. In order to see this, we assume there are two firms that are identical except for their initial asset levels. That is, both firms' future cash flows are given by (1), but the "high" type has a higher initial level of assets than the "low" type. Assuming that their initial levels of assets are not observable by the market, the firm with the high assets may issue risk-compensating PSD that pays a lower initial coupon but has a higher associated bankruptcy boundary than that of the low-type firm that issued the fixed coupon debt. A lower asset level means that the firm is closer to bankruptcy. A further increase in the bankruptcy boundary would be costlier for the low-type firm. As a result, "low" type would not be willing to pool with the "high" type. On the other hand, despite the inefficiencies related to the risk-compensating PSD, the "high" type firm benefits overall from revealing its type by reducing its interest payments. Thus, the inefficiency cost associated with the risk-compensating PSD could be viewed as a signaling cost paid by the "high" type firm. Numerical examples support this intuition.

Moral hazard could also justify the use of performance-sensitive debt. By punishing shareholders with higher interest rates after a bad performance, PSD obligations may reduce a manager's ability to shift wealth in favor of shareholders. We have solved a simple numerical example in which a firm that has access to high-risk and low-risk technologies issues step-up bonds in order to avoid losses from the asset-substitution effect.

Contracting costs may be another reason for some types of PSD. When the credit quality of the borrower changes, the issuer and the investors in its debt often get involved in costly negotiation over the terms of the debt. An increase in credit quality may prompt the borrower to seek refinancing of its debt on better terms. On the contrary, the lender may demand higher interest payments in compensation for the deterioration in credit quality. Some types of PSD may resolve the renegotiation problem by automatically adjusting the interest rates.

Asquith, Beatty, and Weber (2002) and Beatty, Dichev, and Weber (2002) indeed found empirical evidence that private debt contracts are more likely to include performance pricing schemes when asymmetric information, moral hazard or recontracting costs are significant. Our paper, however, establishes that solving these problems with PSD comes with a cost.

We have so far assumed that all the agents in the economy are risk-neutral. It is straightforward, however, to extend our results to the case of risk-averse agents, in the absence of arbitrage (specifically, assuming the existence of an equivalent martingale measure).

If markets are incomplete, performance sensitive debt might be issued to meet the demands of risk-averse investors, providing them with hedge against credit deterioration of the firm. Our results suggest, however that financial guarantors, rather than the debt issuing firms, should be providing this kind of hedge.

Our inefficiency results hold for alternative definitions of financial distress. If we assume, for example, that default happens when assets do not generate enough cash flow to meet current obligations[27], then it is easy to see that a more risk-compensating PSD will lead to more inefficiency. In this *flow-based* insolvency definition, however, shareholders declare bankruptcy even though it may be still possible to issue additional equity to cover the shortage.

9 Conclusion

In this paper, we analyze the properties of performance sensitive debt using an endogenous default model. Although many types of debt contracts are performance-sensitive, they have received little attention in the literature. Endogeneity of the firm's default decision allows us to analyze the efficiency of different types of PSD.

Our main finding is that, given the same initial funds raised by sale of debt, more risk-compensating PSD leads to earlier default and, consequently, lowers the market value of the issuing firm's equity. An intuitive explanation of this result is that higher interest payments from financially distressed companies lead to higher losses, thus precipitating the default decision.

Catastrophe bonds, whose coupon rate is reduced automatically when the insurance company experiences hardship due to a high volume of insurance claims, are an example of "more" efficient debt. The majority of PSD issues, however, have an inefficient step-up feature. This leads us to believe that inefficient PSD is used to solve agency problems arising from existing market imperfections, such as adverse selection, moral hazard and contracting costs.

[27]This setting is studied in Kim, Ramaswamy, and Sundaresan (1993).

In addition, we develop a convenient method of valuing PSD. We obtain closed-form expressions for the equity prices associated with step-up, linear and rating-based PSD.

We also discuss the policy of credit-rating agencies. Inconsistent rating of PSD can generate a credit-cliff dynamic, as well as hurt market participants by providing misleading information about default risks.

10 Appendix

Proof of Lemma 1. The proof is based on the following claim:

Claim: There exists a level $\tilde{x}$ such that $\forall x \leq \tilde{x}$, $W(x) = \sup_{\tau} W(x, \tau) = 0$.

Proof. From Condition 2, there exist positive constants $\underline{x}$ and $\underline{c}$ such that $(1-\theta)C(x) > \delta(x) + \underline{c}$ for all $x \leq \underline{x}$. Let $\Xi = \sup_{\tau} W(\underline{x}, \tau) < \infty$. For any stopping time τ and $x < \underline{x}$,

$$\begin{aligned} W(x,\tau) &= E_x\left[1_{\tau<\tau(\underline{x})}\int_0^{\tau} e^{-rt}\left(\delta(A_t)-(1-\theta)C(A_t)\right)dt\right] + \\ &\quad E_x\left[1_{\tau>\tau(\underline{x})}\int_0^{\tau} e^{-rt}\left(\delta(A_t)-(1-\theta)C(A_t)\right)dt\right] \\ &\leq -\frac{\underline{c}}{r}E_x[\left(1-e^{-r\tau}\right)1_{\tau<\tau(\underline{x})}] + \\ &\quad E_x\left\{\left[-\frac{\underline{c}}{r}\left(1-e^{-r\tau(\underline{x})}\right)+\xi(x,\underline{x})\Xi\right]1_{\tau>\tau(\underline{x})}\right\}. \end{aligned}$$

Let $x^* > 0$ be the unique solution (in x) of $-\frac{\underline{c}}{r}\left(1-e^{-r\tau(\underline{x})}\right)+\xi(x,\underline{x})\Xi = 0$. Since ξ is nondecreasing in x, we have for all $x \leq \tilde{x} = \underline{x} \wedge x^*$, $W(x,\tau) \leq -\frac{\underline{c}}{r}E[(1-e^{-r\tau})\,1_{\tau<\tau(\tilde{x})}] \leq 0$, the optimum $W(x,\tau) = 0$ being reached for $\tau \equiv 0$. This claim proves that, starting from any level x and for any stopping time τ, the stopping time $\tau^- = \tau \wedge \tau(\tilde{x})$ is at least as good as τ. In other words, we can restrict ourselves, in our search for optimality, to the set of stopping times $\tilde{T} = \{\tau \text{ s.t. } \tau \leq \tau(\tilde{x})\}$.

Proof of Theorem 1. First, we prove the necessary conditions, then the sufficient conditions.

1. The proof of the necessary conditions is based a series of lemmas:

Lemma 2 *Under Conditions 1–3, f is continuously differentiable and f' is bounded and left and right differentiable. Moreover, f satisfies the following equations:*

$$\begin{array}{l} \frac{1}{2}\sigma^2(x)f_l''(x) + \mu(x)f'(x) - rf(x) + \delta(x) - (1-\theta)C_l(x) = 0 \\ \frac{1}{2}\sigma^2(x)f_r''(x) + \mu(x)f'(x) - rf(x) + \delta(x) - (1-\theta)C(x) = 0 \end{array}, \qquad (25)$$

where $f_l''(x)$ (resp. $f_r''(x)$) is the left (resp. right) derivative of f' at x, and $C_l(x)$ is the left limit of C at x.

Proof From Condition 1, there exists a fundamental solution[28] $\zeta(x, s, y, t)$ with the same generator as $\{A_t\}_{t\geq 0}$, such that for $s < t$,

$$\mathbb{P}_{x,s}[A_t \in \mathcal{B}] = \int_{\mathcal{B}} \zeta(x, s, y, t) dy$$

for any Borel subset $\mathcal{B}$ of $\mathbb{R}$ and

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2 \zeta}{\partial x^2}(x, s, y, t) + \mu(x)\frac{\partial \zeta}{\partial x}(x, s, y, t) + \frac{\partial \zeta}{\partial s}(x, s, y, t) = 0. \qquad (26)$$

If C is continuous, letting $\phi(x) = \delta(x) - (1-\theta)C(x)$, Friedman (1975) and an application of Fubini theorem imply that

$$f(x) = \int_{\mathbb{R}} \phi(y) \left[\int_0^\infty e^{-rt} \zeta(x, 0, y, t) dt \right] dy,$$

which, by time homogeneity of $\{A_t\}_{t\geq 0}$, implies that

$$f(x) = \int_{\mathbb{R}} \phi(y) \left[\int_0^\infty e^{-rt} \zeta(x, -t, y, 0) dt \right] dy. \qquad (27)$$

When C is discontinuous, the second part of Condition 3 implies that there is a countably finite number of discontinuities. A limit argument using approximating continuous functions then shows that (27) also holds in this case. To derive an ODE when C is continuous, a straightforward differentiation of (27) using (26) shows (25), which boils down to a single equation at any continuity point. When C is discontinuous, differentiation applied to all continuity points of C shows that (25) holds at such points, while right and left limit arguments at discontinuity points show that (25) holds at these points as well. The boundedness of f' comes from the boundedness of $\frac{\partial \zeta}{\partial x}(x, v)$, proved in Friedman (1975), and the fact that μ_δ is uniformly bounded away from r. ■

[28] See Friedman (1975).

Corollary 3 *W satisfies the following equations on $[A_B, \infty)$:*

$$\frac{1}{2}\sigma^2(x)W_l''(x) + \mu(x)W'(x) - rW(x) + \delta(x) - (1-\theta)C_l(x) = 0 \quad (28)$$

$$\frac{1}{2}\sigma^2(x)W_r''(x) + \mu(x)W'(x) - rW(x) + \delta(x) - (1-\theta)C(x) = 0, \quad (29)$$

where $W_l''(x)$ (resp. $W_r''(x)$) is the left (resp. right) derivative of W' at x, and $C_l(x)$ is the left limit of C at x. In particular, W solves ODE (9) at any continuity point of C.

Proof From Lemma 2 and (8), $\hat{W}(x, y)$ is continuous with respect to y. From Lemma 1, and compactness of $[0, x]$ there exists a level $A_B > 0$ such that $W(x) = \hat{W}(x, A_B)$. The proof is then straightforward from Lemma 2 and (8). ■

Corollary 4 *W' is bounded on* $[0, \infty)$

Proof Straightforward, from (8) and the fact that f' is bounded on $[0, \infty)$. ■

Corollary 5 *If a PSD obligation C satisfies Conditions 1–3, then $\tilde{W}(x, y)$ is continuously differentiable in both components, and $\frac{\partial \tilde{W}}{\partial x}$ is left and right differentiable in x.*

Proof This comes directly from the Lemma 2 and equation (8). ■

Lemma 3 *If a PSD obligation C satisfies Conditions 1–3, then the optimal default boundary A_B verifies $\frac{\partial \tilde{W}}{\partial x}(A_B, A_B) = 0$.*

Proof From (7) and Lemma 1, it follows that A_B satisfies $\frac{\partial \tilde{W}}{\partial A}(x, A_B) = 0$. Moreover, we have for any y, $\tilde{W}(y, y) = 0$ (since the firm defaults immediately). Differentiating this last equation and using the fact that $\frac{\partial \tilde{W}}{\partial x}(x, A_B) = 0$ yields $\frac{\partial \tilde{W}}{\partial x}(A_B, A_B) = 0$. ■

Combining equation (8), the above lemmas, and using the fact that $W(x) = \tilde{W}(x, A_B)$ concludes the proof of all necessary conditions but one. It remains to show that $A_B \leq \bar{x}$, which is immediate since, for $A_t > \underline{x}$, the cash flow rate exceeds the coupon rate implying that it is never optimal to default at this level.

2. The verification of the sufficient conditions is similar to the proof of Proposition 2.1 in Duffie and Lando (2001). Define a stochastic process χ_t as

$$\chi_t = e^{-rt} W(A_t) + \int_0^t e^{-rs} \phi_s \, ds \ ,$$

where for $x > A_B$, $W(x)$ is the solution of the ODE that satisfies all the conditions listed in the theorem, and $W(x) = 0$ for $x \leq A_{\dot{B}}$.

Since W is C^1, an application of Itô's formula leads to

$$d\chi_t = e^{-rt} d(A_t) \ dt + e^{-rt} W'(A_t) \sigma(A_t) dB_t, \tag{30}$$

where

$$d(x) \equiv \frac{1}{2} W''(x) \sigma^2(x) + W'(x) \mu(x) - rW(x) + \phi(x).$$

Since by assumption W' is bounded, the second term is a martingale, and since $E_x \left[\int_0^\infty \left(e^{-rt} W'(A_t) \sigma(A_t) \right)^2 dt \right] < \infty$, $\int_0^t e^{-rs} W'(A_s) \sigma A_s \, dB_s$ is a uniformly integrable martingale, which implies that $E_x \left[\int_0^\tau e^{-rs} W'(A_s) \sigma A_s \, dB_s \right] = 0$ for any stopping time τ. By the assumptions of the theorem

$$\phi(A_B) \leq 0. \tag{31}$$

This inequality means that when the firm declares bankruptcy, its cash flow $\delta = (r - x) A_B$ is less than the coupon payment. It is easy to verify that the drift of χ_t is never positive: $d(x)$ vanishes for $x > A_B$ since W solves the ODE, and negative for $x < A_B$, because of the inequality (31) and $W(x) = 0$ for $x < A_B$. Because of the non-positive drift, for any stopping time $T \in \mathcal{T}$, $q_0 \geq E(\chi_T)$, meaning

$$W(A_0) \geq E \left[\int_0^T e^{-rs} \phi_s \, ds + e^{-rT} W(A_T) \right] \ .$$

For the stopping time τ, we have

$$W(A_0) = E \left[\int_0^\tau e^{-rs} \phi_s \, ds \right] \geq E \left[\int_0^T e^{-rs} \phi_s \, ds \right],$$

where the inequality follows from non-negativity of W. Therefore, the stopping time τ maximizes the value of the equity. ■

Proof of Theorem 2. The proof is based on the following lemma:

Lemma 4 *Let C and D be asset-based PSD satisfying Conditions 1–3, and $A_B^C \leq A_B^D$. If $h \equiv C - D$ is not constant on $[A_B^D, \infty)$ and changes sign at most once from positive to negative on $[A_B^D, \infty)$, then, $W_0^C(x) > W_0^D(x)$ for any starting asset level $x \in (A_B^C, \infty)$.*

Proof Without loss of generality, we assume that the tax rate θ is zero. First, assume that $A_B^C = A_B^D = A_B$. Since h changes sign at most once from positive to negative on $[A_B, \infty)$, there exist constants A_1, A_2 verifying $A_B \leq A_1 \leq A_2$ and such that $h > 0$ for $A \in [A_B, A_1)$, $h = 0$ for $A \in (A_1, A_2)$, and $h < 0$ for $A \in (A_2, \infty)$.[29]
We first consider the case where $A_1 = A_B$. Then necessarily $A_2 < \infty$, otherwise h would be constant on $[A_B, \infty)$. Thus, h is zero on $[A_B, A_2)$ and negative on (A_2, ∞). It is easy to verify that for any PSD C with initial asset level x and defaulting boundary A_B, we have

$$U_0^C(x) = E_x\left[\int_0^{\tau(A_B)} e^{-rs} C(A_s)\, ds\right] + (A_B - \rho(A_B))\xi(A_0, A_B). \tag{32}$$

Since (A_2, ∞) has a positive measure, (32) implies that $U_0^D(x) > U_0^C(x)$ for all $x \in (A_B, \infty)$. Equation (6) then allows to conclude that $W_0^C(x) > W_0^D(x)$ for all $x \in (A_B, \infty)$.
Now we consider the case in which $A_1 > A_B$. Thus, $h(A_B) > 0$. We will first show that $W_0^C(x) > W_0^D(x)$ for all $x \in (A_B, A_1)$. From equations (28) and (29), we have for $H(x) \equiv W_0^C(x) - W_0^D(x)$:

$$\frac{1}{2}H_l''(x)\sigma^2(x) + H'(x)\mu(x) - rH(x) - h_l(x) = 0 \tag{33}$$

$$\frac{1}{2}H_r''(x)\sigma^2(x) + H'(x)\mu(x) - rH(x) - h(x) = 0, \tag{34}$$

where $H_l''(x)$ (resp. $H_r''(x)$) is the left (resp. right) derivative of H' at x, and $h_l(x)$ is the left limit of h at x, which exists according to Condition 3 and Theorem 1. Also from Theorem 1, $W^i(A_B) = 0$ and $(W^i)'(A_B) = 0$ for $i = C, D$. Therefore, $H(A_B) = H'(A_B) = 0$. Since $h(A_B) > 0$, it follows from equation (34) that $H_r''(A_B) > 0$. This implies that $H'(x) > 0$ and $H(x) > 0$ in a right neighborhood of A_B. Precisely, there exists $\eta > 0$, such that $H'(x) > 0$ and $H(x) > 0$ for $x \in (A_B, A_B + \eta)$. We will now prove by contradiction that $H'(x) > 0$ for all $x \leq A_1$. Letting y denote the first time

[29] By convention [a,a) and (a,a) equal the empty set. The precise values at A_1 and A_2 are unimportant.

when $H'(y) = 0$, we have necessarily $H(y) > 0$. From equation (33) and the fact that $h(y) \geq 0$ for $y \leq A_1$, it follows that $H_l''(y) > 0$, contradicting the fact that y was the first time where $H'(y) = 0$. Therefore, $H'(x) > 0$ and $H(x) > 0$ on $(A_B, A_1]$. Last, we prove that $H(x) > 0$ on (A_1, ∞). By definition of W^C, W^D, and A_B, we have:

$$\begin{aligned} W_0^C(x) &= E_x^Q \left[\int_0^{\tau^*} q_t \left(\delta_t - C\left(A_t\right)\right) dt \right] \text{ and} \\ W_0^D(x) &= E_x^Q \left[\int_0^{\tau^*} q_t \left(\delta_t - D\left(A_t\right)\right) dt \right], \end{aligned}$$

where $q_t = e^{-rt}$, $\tau^* = \tau(A_B)$. Therefore,

$$H(x) = -E_x^Q \left[\int_0^{\tau^*} q_t h(A_t) dt \right].$$

It follows that for any $x > A_1$, we have, since $\tau(A_1) < \tau(A_B) = \tau^*$ and $\int_0^{\tau^*} = \int_0^{\tau(A_1)} + \int_{\tau(A_1)}^{\tau^*}$,

$$H(x) = -E_x^Q \left[\int_0^{\tau(A_1)} q_t h(A_t) \right] + E_x^Q (e^{-r\tau(A_h)}) H(A_1).$$

Since $h(.)$ is non-positive on (A_1, ∞) and we have seen that $H(A_1) > 0$, it follows that $H(x) > 0 \ \forall x \in (A_B, \infty)$, which concludes the proof of the lemma in the case $A_B^C = A_B^D = A_B$. Now we consider the case where $A_B^C < A_B^D$. Then, $W_0^C(x) > 0$ and $W_0^D(x) = 0$ for $x \in (A_B^C,\ A_B^D]$, whence the claim holds trivially on this interval. The rest of the proof is identical to the first part for $x > A_B^D$. ■

From this lemma, we will first conclude the proof of the theorem in the case of asset-based PSD. We proceed by contradiction. We assume first that $A_B^C = A_B^D = A_B$. Then, the pair (C, D) satisfies the conditions of the lemma, which allows to conclude that $W_0^C(x) > W_0^D(x) \ \forall x > A_B$. By formula (6), we conclude in particular that for $x = A_0$, $U_0^C < U_0^D$, which contradicts the hypothesis of Theorem 1. We now assume that $A_B^C < A_B^D$. Then, we can lower the value of the interests paid by D uniformly, proceeding by translation: we consider the PSD D_ε that pays the interest function $D_\varepsilon = D - \varepsilon$. Then, with the assumption that D is in the efficiency domain of its translation class (Condition 4), we have $U_0^{D_\varepsilon} < U_0^D = U_0^C$. On the other hand, since the interest payments are getting lower as ε increases, there exists an $\varepsilon_0 > 0$ such that $A_B^{D_{\varepsilon_0+}} \le A_B^C \le A_B^{D_{\varepsilon_0-}}$. Moreover, since $h = C - D$ is non-increasing and not constant, so is $h_\varepsilon \equiv C - D_\varepsilon = C - D + \varepsilon$. In particular, h_ε is not constant and changes sign at most once. Since D satisfies Conditions 2 and 3, it is easy to verify that so does D_ε, $\forall \varepsilon > 0$. Therefore, the pairs (C, D_ε) with ε in a left neighborhood of ε_0 satisfy the hypothesis of the lemma, which implies that $W_0^C(x) > W_0^{D_{\varepsilon_0}}(x)$[30] for any starting asset level $x \in (A_B^C, \infty)$. By (6), we conclude that $U_0^C < U_0^{D_\varepsilon}$ for any ε in a right neighborhood of ε_0, which contradicts the fact that $U_0^{D_\varepsilon} \le U_0^D = U_0^C$ for all $\varepsilon > 0$. ■

Proof of Theorem 4. The proof is based on the proof of Theorem 2. In the case of ratings-based PSD obligations it is easy to see that Conditions 1–3 are automatically satisfied. We suppose first that $A_B^C = A_B^D$. This implies that $G(A_B^C) = G(A_B^D)$. From Lemma 4, $U_0^C > U_0^D$. This contradicts the fact that $U_0^C = U_0^D$. Now suppose that $A_B^C < A_B^D$. Take $\varepsilon > 0$ such that $A_B^C = A_B^{D_\varepsilon}$. Then $G(A_B^C) = G(A_B^{D_\varepsilon})$ and Lemma 4 implies that $U_0^C < U_0^{D_\varepsilon}$. Condition 2, on the other hand implies that $U_0^{D\varepsilon} < U_0^D = U_0^C$ and we have a contradiction. Therefore $A_B^C > A_B^D$. Since $U_0^C = U_0^D$, the result follows from (6). ■

References

Acharya, V., S. Das, and R. Sundaram, 2002, A Discrete-Time Approach to No-arbitrage Pricing of Credit Derivatives with Rating Transitions, *Financial Analysts Journal* May-June, 28–44.

Asquith, P., A. Beatty, and J. Weber, 2002, Performance Pricing in Debt Contracts, Working paper, Penn State University and MIT.

Beatty, P., I. Dichev, and J. Weber, 2002, The Role and Characteristics of Accounting-based Performance Pricing in Private Debt Contracts, Working paper, Michigan, Penn State and MIT.

Black, F., and J. Cox, 1976, Valuing Corporate Securities: Some Effects of Bond Identures Provisions, *Journal of Finance* 31, 351–367.

Das, S., and P. Tufano, 1996, Pricing Credit Sensitive Debt when Interest Rates, Credit Rates and Credit Spreads are Stochastic, *Journal of Financial Engineering* 5, 161–198.

[30] Here, we use the fact that $W_0^{D\varepsilon}(x)$ is continuous in ε, which is an easy consequence of Corollary 5

Diamond, D., 1991, Debt Maturity Choice and Liquidity Risk, *Quarterly Journal of Economics* 109, 709–737.

Duffie, D., and D. Lando, 2001, The Term Structure of Capital Spreads with Incomplete Accounting Information, *Econometrica* 69, 633–664.

Duffie, D., and K. Singleton, 1999, Modelling the term structure of defaultable bonds, *Review of Financial Studies* 12, 687–720.

Fischer, E., R. Heinkel, and J. Zechner, 1989, Dynamic Capital Structure Choice: Theory and Tests, *Journal of Finance* 44, 19–40.

Fitch IBCA, Duff & Phelps, 2001, The Operation and Evolution of Catastrophe-Linked Bonds, *Special Report*.

Friedman, A., 1975, *Stochastic Differential Equations and their Applications* vol. 1. (Academic Press New York).

Goldstein, R., N. Ju, and H. Leland, 1998, A Generalized Modigliani-Miller Model of Dynamic Capital Structure, Working paper, Fisher College of Business and Haas School of Business.

Guedes, J., and T. Opler, 1996, The Determinants of the Maturity of Corporate Debt Issues, *Journal of Finance* 51, 1809–1833.

Hillion, P., and T. Vermaelen, 2004, Death Spiral Convertibles, *Journal of Financial Economics* 71, 381–415.

Houweling, P., A. Mentink, and T. Vorst, 2003, Valuing Euro Rating-Triggered Step-Up Telecom Bonds, Working paper, Erasmus University Rotterdam.

Jarrow, R., D. Lando, and S. Turnbull, 1997, A Markov Model for the Term Structure of Credit Risk Spreads, *Review of Financial Studies* 10, 481–523.

Jarrow, R., and S. Turnbull, 1995, Pricing Derivatives on Financial Securities Subject to Credit Risk, *Journal of Finance* 50, 53–86.

Karatzas, I, and S. E. Shreve, 1991, *Brownian Motion and Stochastic Calculus*. (Springer-Verlag New York).

Kim, I., K. Ramaswamy, and S. Sundaresan, 1993, Valuation of Corporate Fixed-Income Securities, *Financial Management* Special Issue on Financial Distress, 117 – 131.

Lando, D., and A. Mortensen, 2003, Mispricing of Step-Up bonds in the European Telecom Sector, Working paper, Copenhagen Business School.

Leland, H., 1994, Corporate Debt Value, Bond Covenants, and Optimal Capital Structure, *Journal of Finance* 49, 1213–1252.

Leland, H., 1998, Agency Costs, Risk Management, and Capital Structure, *Journal of Finance* 53, 1213–1243.

Leland, H., and K. Toft, 1996, Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads, *Journal of Finance* 51, 987–1019.

Longstaff, F., and E. Schwartz, 1995, Valuing Risky Debt: A new approach, *Journal of Finance* 50, 789–820.

Merton, R., 1974, On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, *Journal of Finance* 2, 449–470.

Moody's, 2001, The Unintended Consequences of Rating Triggers, *Special Comment*.

Moody's, 2002, Moody's Analysis of US Corporate Rating Triggers Heightens Need for Increased Disclosure, *Special Comment*.

Myers, S., 1977, Determinants of Corporate Borrowings, *Journal of Financial Economics* 5, 147–175.

Standard & Poor's, 2001, Playing Out the Credit Cliff Dynamic, *Standard & Poor's Ratings Criteria*.