Communication and Efficiency in Auctions *

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Abstract

We study auctions under restricted communication. Bidders have valuations in a compact interval, but can only report one of a finite number of messages. We provide a characterization of welfare as well as revenue maximizing auctions in the symmetric independent private values case when bidders report simultaneously. We show that the seller choosing how to allocate the fixed communication capacity allocates it evenly over the bidders. The optimal auction is asymmetric, contrasting the symmetry of optimal auctions when communication is not restricted. The analysis is extended to the case of multiple identical objects and single unit demand.

Finally, we characterize welfare maximizing auctions under restricted communication when two bidders report sequentially and there are only two periods of reporting. These auctions can be thought of as a two step procedure. In the first step, the first bidder chooses a price from a finite menu of prices. In the second step the object is offered to the second bidder at a higher price. If the second bidder accepts it he receives the object and pays the offered price, otherwise the first bidder receives the object at the price he chose.

1 Introduction

Surprisingly little attention has been paid to how information is transmitted in auctions. Most often the issue of communication is sidestepped by applying the celebrated revelation principle (see Myerson (1979) and Myerson (1981)), thus enabling bidders to report their valuations. We consider

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optimal auction design when bidders cannot fully report their valuations due to communication restraints. Such a setup arises in many observed auctions where bidding is discretized or incremental. Perhaps the most natural application is the design of Internet auctions, where the communication is in the binary code, thus rendering the message space finite (see Blumrosen, Nisan and Segal (2007)).

Outline and Results. We consider optimal auction design, both under welfare and revenue maximization, when bidders cannot fully report their valuations due to communication restraints. We use the independent private values setup (IPV) where bidders have values distributed on a compact interval but are only allowed to choose a report from a finite set of messages. A similar model was studied by Blumrosen, Nisan and Segal (2007) (henceforth BNS). Direct comparison between their and our results is provided below.

We start with the model in which the seller is selling a single indivisible object. The seller commits to a mechanism, bidders observe their valuations and simultaneously report one of the finite number of messages they can use. Allocations and transfers are executed on the basis of reported messages. We start the analysis by showing that equilibria are in threshold strategies which enables us to show existence of a welfare maximizing equilibrium. The analysis in our paper differs from the analysis under no restrictions on communication in the sense that we need to deal with reporting strategies as well as the mechanism. Nuisance that is avoided when communication is costless by applying the revelation principle and focusing on equilibria with truthful reporting. In our case it is not a priori clear what truthful reporting means, nor is it clear what reporting strategies maximize the objective.

We proceed to show that information transmission of the bidder with the highest communication capacity[1] is restricted by the communication capacity of all the remaining bidders. In particular, if there are only two bidders and one of them, say bidder 2, can choose among k messages, then at most $k+1$ messages of bidder 1 are welfare relevant. Indeed, the seller is comparing the expected valuations of the two bidders given their reports and the strategies they are using. To each message of bidder 2 corresponds an expected value, therefore all together k expected values. Fixing bidder 2's behavior, the only welfare-relevant information provided by bidder 1 is whether his valuation is above the highest expected value of bidder 2, between the highest and the second highest,..., or

[1] Cardinality of the message space.

below the lowest. Hence at most $k+1$ distinct messages from bidder 1 are relevant.

The above stated result enables us to characterize welfare optimizing equilibria in the two bidder case. Optimal equilibria are asymmetric both in strategies as well as in the allocation rule even when both bidders are ex ante symmetric.[2] In an example borrowed from BNS both bidders have uniformly distributed valuations over the interval $[0, 1]$ and the cardinality of both bidders' message spaces is two. The highest welfare is achieved when one bidder, say bidder 1, uses threshold $\frac{2}{3}$ and the other bidder threshold $\frac{1}{3}$. The allocation rule assigns the object to bidder 1 unless he reports his lower partition cell and bidder 2 his higher partition cell; in later case the object is awarded to bidder 2. Transfers to support this equilibrium are Vickrey type transfers: the winning bidder pays the lowest value he could have, report according to his strategy, and still win. Interesting property of strategies in the welfare optimal equilibrium is what BNS call mutual centeredness. Notice that in the above outlined example bidder 1's highest threshold is equal to the expectation of bidder 2's higher partition cell and bidder 2's threshold is the expectation of bidder 1's lower partition cell. A similar property holds for the general case when bidders do not have the same distribution of valuations or the same cardinality of message spaces.

Symmetry is often required for legal, fairness or other purposes. In view of this we characterize the cost of imposing symmetry restrictions on a two bidder mechanism when bidders are ex ante symmetric. If equilibria are required to be fully symmetric, same strategies and ties broken by a flip of a fair coin, one needs almost twice as many messages for each bidder to achieve the same welfare as when no such symmetry requirements are imposed. More precisely, welfare obtained in the welfare maximizing equilibrium where each bidder has k possible messages is equal to the welfare obtained in the best symmetric equilibrium with each of the bidders having $2k-1$ messages. In the above example, we argued that the highest welfare is achieved when bidder 1 uses threshold $\frac{2}{3}$ and bidder 2 threshold $\frac{1}{3}$. Welfare in that mechanism is equal to the highest welfare achieved in the mechanism in which each bidder can use 3 messages and equilibrium is required to be symmetric. The later equilibrium has both bidders reporting according to a threshold strategy with two thresholds: $\frac{1}{3}$, $\frac{2}{3}$.

Using similar analysis, the problem of welfare maximization can be solved for the case when seller's value for the object is positive and commonly known. This in turn enables one to solve

[2] We refer to bidders as *symmetric* when they have both, the same distribution over their values, F, and the same cardinality of the message space, k.

the revenue maximization problem under the standard assumption of regular distributions. Due to Myerson (1981) the problem of revenue maximization can be transformed into the problem of maximization of virtual valuations given that they are no smaller than 0, which is equivalent to the problem of welfare maximization after virtual valuations are reinterpreted as the actual valuations.

The above results provide a solution to a more general problem in which the seller can decide how to distribute a fixed amount of communication capacity between the two bidders. In particular, he distributes the number of messages evenly.

In our main result we provide a characterization of welfare optimal auctions[3] when the seller can choose how to distribute fixed amount of communication capacity but is required to assign at least one message to each bidder and all the bidders have valuations distributed with the same distribution F. The problem can be thought of as follows. First the seller decides how to assign a fixed number of thresholds. For example, suppose there are three bidders and the seller can use at most nine messages. This corresponds to six thresholds in total since we require that each bidder has at least one message. One can fix values of arbitrary six thresholds and see what is the optimal way to assign them to bidders. We show that the optimal assignment is a *priority assignment* in which thresholds are assigned in a round robin fashion: bidder 1 obtains the highest threshold, bidder 2 the second highest, bidder 3 the third and then one starts with bidder 1 again.[4] For a graphical representation of the priority assignment see Figure 2 below. The priority assignment implies that bidder 1's top partition cell is the highest in the strong set order, bidder 2's top partition cell second highest, bidder 3's third highest and then one starts with bidder 1 again. Because bidders have the same distributions the same chain of inequalities holds for expected values of the partition cells. But this in turn yields the optimal allocation rule: the object is allocated to the bidder reporting the highest partition cell according to the strong set order. Due to the nature of the priority assignment the optimal allocation rule is the same no matter what the set of thresholds one starts with. The second step is easy: one maximizes welfare over all the possible values of thresholds while holding the assignment and the allocation rule fixed. Optimal thresholds satisfy a similar property as in the two bidder case, i.e. bidder 1's highest threshold is a convex combination of the expected value of bidder 2's top partition cell and bidder 3's top partition cell. The weights are the relative probabilities of the respective bidder having the value in the top partition cell given

[3] As in the two bidder case, we extend the analysis to profit maximization after solving the problem of welfare maximization when the seller has a commonly known positive valuation.

[4] This is, of course, up to relabling of bidders.

that at least one of the two bidders has it. The other thresholds are characterized similarly.

We also show how analysis can be extended to the problem of allocating multiple identical objects to the symmetric bidders with a single unit demand.

The above analysis assumes that the bidders report messages simultaneously. The next logical step is to explore welfare and profit maximization under sequential reporting. Here, as opposed to when there are no restrictions on communication, simultaneous reporting is not without loss of generality when cardinalities of bidders' message spaces are fixed. We study 2 bidder mechanisms with two periods of reporting. After the seller commits to the mechanism, bidder 1 reports one of k_1 messages, bidder 2 observes it and reports himself. The allocation and transfers are executed on the basis of the two reports. We show that the optimal welfare achieved in such a mechanism is equal to the optimal welfare in a simultaneous mechanism where bidder 1 has k_1 possible messages and bidder 2 has $k_1 + 1$. The optimal mechanism can be thought of as the seller offering a menu of k_1 prices to bidder 1. After bidder 1 chooses one of the prices, bidder 2 is offered the object at a higher price. If he accepts he obtains the object and pays the price he was offered, otherwise bidder 1 obtains the object at the price he chose. An analogous result is obtained under revenue maximization.

Related Literature In the paper most closely related to ours Blumrosen, Nisan and Segal (2007) study the effects of restricted communication in auctions. They show that the informationally optimal strategies are threshold strategies and that they can be supported as an equilibrium. We in addition show that all the equilibria are in threshold strategies and that the relevant communication of the bidder with the highest communication capacity is restricted by the communication capacity of all the remaining bidders, which is novel. This enables us to provide an alternative way to characterize the optimal equilibria of two bidder mechanisms where bidders have independently distributed signals and each has a message space of an arbitrary finite cardinality. BNS characterize the optimal two bidder equilibria when bidders have the same cardinality of the message space. The two bidder two messages example with uniform distribution is also due to BNS. On the other side, characterization of the cost of requiring symmetry is new.

For more than two bidders BNS have only been able to provide characterization under the assumption that all the bidders have the same distribution and each has cardinality of the message space equal to two. We solve a more general problem, still requiring symmetric distributions, by showing that the seller optimally distributes the communication capacities evenly over the bidders. Moreover we characterize the optimal equilibria under such a distribution of communication capacity. Our characterization of the profit maximizing equilibria is tighter even when restricted to their environment; i.e. bidders choose between two messages. Characterization for multiple identical objects and a single unit demand is also novel.

BNS provide a two bidder - two message example showing that sequential reporting with the same number of messages can do better than simultaneous reporting if the bidder to report second is revealed the first bidder's report. They proceed to show that at least as high a welfare as in any I bidder sequential reporting mechanism with communication requirement m (meaning that the sum of bits used by all the bidders is m) can be achieved by a simultaneous reporting mechanism with communication requirement no larger than mI. While this provides a general bound, we, on the other side, exactly characterize welfare and profit maximizing equilibria of two bidder mechanisms with two rounds of sequential reporting. BNS's result apply only to the welfare.

Blumrosen and Feldman (2006) study restricted communication in a general mechanism design problem. Among other things they show the optimality of threshold strategies under the assumption of multilinear and single crossing social-value function.

Possibility of having restricted communication was already considered in Myerson (1979) and Myerson (1981), though in different context. Myerson's work implies that neither bidders, nor the seller, can be made better off by restricting communication. Wilson (1989) and McAfee (2002) analyze the role of priority services in industries where spot markets would be expensive to organize. In Wilson's model the core of the problem is uncertainty of supply while in our case supply is commonly known but the demand is uncertain (at least from the seller's point of view). Yet in different setup, equilibria of similar form to ours are obtained by Bergemann and Pesendorfer (2007), though through a different channel. In our model bidders are unable to fully communicate their valuations, while in their model the bidders are not fully informed of the value due to the restrictions imposed on them by the seller.

Various papers study restricted communication in a fixed auction setting. Some examples are Rothkopf and Harstad (1994) in oral auctions and Athey (2001) in the first price auction. We, on the other side, are deriving the optimal auction.

Milgrom (2007) looks at a different kind of communication restrictions. He studies effects of restricted communication in combinatorial auctions where bidders instead of submitting bids for all the possible packages submit bids for a subset of those.

Dynamic auctions having a flavor of limited communication arise in numerous papers. For example, Sandholm and Gilpin (2006) analyze *take-it-or-leave-it auctions* in which seller is sequentially making offers to the bidders until a bidder accepts an offer; also see Kress & Boutilier (2004).

2 The Model

2.1 Preferences

A single indivisible object is to be sold to one of the bidders indexed by $i \in \{1, 2, ..., I\}$. Each bidder i's private value, v_i, is distributed with a commonly known atomless distribution function F_i over a compact interval $V_i = [0, 1]$. In addition we assume that the distribution function allows for everywhere positive density f_i. Let

$$V = \mathop{\times}_{i=1}^{I} V_i = [0, 1]^I .$$

As usual, we adopt the notation $v = (v_i, v_{-i})$. Prior to the auction, bidder i's valuation v_i is independently drawn from the distribution F_i and revealed only to the bidder himself. The seller can be thought of as bidder 0, his commonly known valuation being v_0. Mostly we will be dealing with the case $v_0 = 0$. Instances when this assumption is dropped will be clearly denoted, though.

Bidders are assumed to maximize their quasi-linear utilities

$$U(v_i, Q_i, T_i) = v_i Q_i - T_i,$$

where Q_i is bidder i's probability of receiving the object and T_i his monetary transfer.

2.2 Information Structure and Strategies

Here is where our model departs from the classical mechanism design literature as presented say in Krishna (2002) Chapter 5. As opposed to the case of unrestricted communication, each bidder has predetermined set of messages M_i of finite cardinality.[5,6] Let $k_i \equiv |M_i|$ denote the cardinality of the set M_i for all i, and let $M_i = \{m_1, m_2, ..., m_{k_i}\}$. Furthermore, we denote $M = \times_{i=1}^{I} M_i$ with a representative element m.

After observing their valuations bidders report to the seller simultaneously. In particular, after observing a private value v_i bidder i reports a message from M_i according to a reporting strategy μ_i, where

$$\mu_i : V_i \to M_i.$$

The vector of strategies, or strategy profile, is denoted $\mu = (\mu_1, \mu_2, ..., \mu_I)$, the set of strategies for bidder i is S_i, and the set of strategies $S = \times_{i=1}^{I} S_i$.

2.3 Mechanism

The mechanism determines the allocation, possibly stochastic, of the object in question and payments made by the bidders given the vector of reported messages. The allocation is

$$Q : M \to [0,1]^{I+1},$$

where i-th coordinate of $Q(m)$, denoted $Q_i(m)$, is interpreted as bidder i's probability of winning the object given the reported messages, and $Q_0(m)$ the probability that the seller retains the object. As usual $Q_i(m) \geq 0$ for every i and every m, and $\sum_{i=0}^{I} Q_i(m) = 1$. In the urge to suppress as much notation as possible we will define bidder i's expected probability of winning the object given his own signal v_i, the strategies μ and the allocation rule Q_i as

$$q_i\left(\mu_i(v_i), \mu_{-i}\right) = E_{v_{-i}} Q_i(\mu(v)).$$

The payment scheme can be represented as

$$T : \underset{i=1}{\overset{I}{\times}} M_i \to \mathbb{R}^I,$$

[5] One could more generally say that the mechanism designer can pick the message space for each player to be up to the cardinality k_i. This would only complicate the notation as it will be easy to see that the mechanism designer would always (weakly) prefer to use set of messages with the largest allowed cardinality.

[6] Later in the paper we will look at the more general setup in which total number of messages is fixed and the seller can choose how to allocate them over the bidders.

where i-th coordinate of $T(m)$, denoted $T_i(m)$, is bidder i's payment given the profile of reports m. Again, $t_i(\mu_i(v_i), \mu_{-i})$ denotes the expected payment of bidder i with value v_i under the profile of strategies μ :

$$t_i(\mu_i(v_i), \mu_{-i}) = E_{v_{-i}} T_i(\mu(v)).$$

Definition 1 *A mechanism (M, Q, T) consists of sets of possible messages M_i for each i, an allocation rule Q and a payment rule T as defined above. The set of all mechanisms with I bidders and $k_i = |M_i|$ messages for each bidder i is denoted by $G^I_{k_1, k_2, \ldots, k_I}$.*

Most of the time M will be fixed and clear from the context. In such a case we will use (Q, T) to denote a mechanism.

We are ready to explicitly describe bidders' behavior. The mechanism defines a Bayesian game in which each bidder is choosing a reporting strategy, μ_i, to maximize his expected payoff given other bidders' strategies:

$$u_i(\mu, v_i, q_i, t_i) = q_i(\mu_i(v_i), \mu_{-i}) v_i - t_i(\mu_i(v_i), \mu_{-i}).$$

We require that the mechanism and the strategies satisfy the interim incentive constraints, i.e. for every i:

$$q_i(\mu_i(v_i), \mu_{-i}) v_i - t_i(\mu_i(v_i), \mu_{-i}) \geq q_i(\mu_i(v_i'), \mu_{-i}) v_i - t_i(\mu_i(v_i'), \mu_{-i}),$$

for every $v_i, v_i' \in V_i$, and the interim participation constraint (IR):

$$u_i(\mu, v_i, q_i, t_i) \geq 0$$

for every $v_i \in V_i$. While the above incentive constraints are somewhat more permissible than is usual they are still necessary for a Bayesian equilibrium. To be more precise, we will be looking at the equilibria of the model, where equilibrium entails reporting strategies μ, on one side, and the mechanism given by an allocation rule Q and transfers T, on the other. Instead of inspecting the conditions of a Bayesian equilibrium one can resort to incentive compatibility (Myerson (1979)) though perhaps in our context it might be more prudent to say that a certain mechanism (Q, T) is incentive compatible with respect to the particular profile of strategies μ. This contrasts the analysis usually applied in the setting without bounds on communication. There one would apply the revelation principle and focus on equilibria in truthful reporting strategies. From that point on incentive conditions are imposed to obtain the mechanism which supports truthful reporting as an equilibrium. Something similar could be replicated in our context. Instead of explicitly dealing with reporting strategies we could use a model where bidders can fully report their values but only such allocation rules and transfers can be used, for which there exists k_1 partition of the first dimension of V, k_2 partition of second dimension of V, etc., and both the allocation rule and the transfers are measurable with respect to the algebra induced by the product partition. We did not find such an approach particularly fruitful.

Furthermore, while truthful reporting is a natural candidate for a profile of strategies to prevail in the IPV model without restrictions on communication, in our model it is neither clear what truthful reporting is, nor what reporting strategies will be optimal in an equilibrium we are looking for, be it welfare or profit maximizing. Therefore equilibria with different reporting strategies will have to be considered.

We accrued enough notation to define the welfare of an equilibrium.

Definition 2 *Let $w(g,\mu)$ denote the expected welfare of an equilibrium of reporting strategies μ and a mechanism $g \equiv (M,Q,T)$, i.e.*

$$w(g,\mu) = E_{v\in[0,1]^I}\left[\sum_{i=0}^{I} Q_i(\mu(v)) \cdot v_i\right]$$

.

In addition let $w^*_{I,(k_1,k_2,\ldots,k_I)}$ denote the maximal expected welfare of equilibria in which each bidder i has k_i possible messages, i.e.

$$w^*_{I,(k_1,k_2,\ldots,k_I)} = \sup_{g\in G_{I,(k_1,k_2,\ldots,k_I)}} w(g,\mu).$$

When $k_1 = k_2 = \ldots = k_I$ we write $w^*_{I,k}$. On the other side expected profit is the sum of expected payments.

Definition 3 *Let $\pi(g,\mu)$ denote the expected profit of an equilibrium of reporting strategies μ and* *a mechanism* $g \equiv (M, Q, T)$, *i.e.*

$$\pi (g, \mu) = E_{v \in [0,1]^I} \left[\sum_{i=1}^{I} T_i (\mu (v)) \right],$$

and let $\pi^*_{I,(k_1,k_2,\ldots,k_I)}$ *denote the maximal expected profit of equilibria in which each bidder* i *has* k_i *possible messages.*

3 Welfare maximizing mechanisms

3.1 General Setup

In this section we provide basic results concerning welfare maximizing mechanisms and reporting strategies. More precise characterization under various assumptions is provided in the subsequent sections.

First part of the analysis follows a rather standard route. Normally one would proceed to show that the incentive compatibility implies monotonicity of q_i in v_i. Something similar can be done here. Nothing precludes us from using some of the technology of the direct revelation mechanism.[7] Fix an equilibrium, that is μ, Q and T, and write with a slight abuse of notation

$$q_i (v_i) = q_i \left(\left(\mu_i (v_i), \mu_{-i} \right) \right).$$

Imposing incentive compatibility yields $q_i (v_i) v_i - t_i (v_i) \geq q_i (v_i') v_i - t_i (v_i')$ and $q_i (v_i') v_i' - t_i (v_i') \geq q_i (v_i) v_i' - t_i (v_i)$ for any $v_i, v_i' \in [0, 1]$. Combining the last two equations gives $[q_i (v_i) - q_i (v_i')] [v_i - v_i'] \geq 0$. The range of q_i is restricted by the cardinality of message space of bidder i, rendering q_i weakly monotonic.

The following observation will make our life much easier. Every equilibrium has an outcome equivalent equilibrium of the same mechanism in threshold strategies. To be more precise, if profile μ is an equilibrium of the mechanism with an allocation rule Q, and transfers T, then there exists an outcome equivalent threshold reporting equilibrium μ' for the fixed Q and T.

[7] What we use is the fact that if an equilibrium is fixed one can use the composite function of the reporting rules and the allocation.

By a **threshold strategy** we mean a strategy in which for every $v_i < v_i'$ such that $\mu_i(v_i) = \mu_i(v_i')$, $v_i'' \in [v_i, v_i']$ implies $\mu_i(v_i'') = \mu_i(v_i)$.

Lemma 1 *Fix a mechanism* (M, Q, T). *Any equilibrium reporting strategies can be replicated by threshold strategies.*

Proof. Proofs of all the results are provided in the Appendices. ■

By the observation above we know that q_i is weakly increasing in v_i. Therefore if bidder i is sending the same message for both valuations v_i and v_i', such that $v_i < v_i'$, and thus inducing the same expected probability of winning the object, it has to be the case that he is also winning with the same expected probability for all the intermediate values. But then he is either reporting the same message at those values or he is reporting something else but still winning the object with the same expected probability. In the later case we can without loss of generality have the bidder report the same message on the whole interval.[8]

Lemma 1 claims that equilibrium can be replicated by a threshold strategy because technically there do exist equilibria with non-threshold strategies. As is noted above, those strategies are not very interesting since some bidder is using more than one message to convey the same meaning. Those messages can, for our purposes, be treated as one.

Threshold strategy can be described by a set of thresholds. We will adopt the habit of denoting the highest threshold of bidder i by c_1^i, the second highest by c_2^i, etc. Analogously the message reported when bidder i observes a value in the interval $\left[c_1^i, 1\right]$ is denoted by m_1, the message if he observes a value in the interval $[c_2, c_1)$ will be m_2, etc. For example, if bidder i has $|M_i| = k$ then his strategy can be described by a vector of thresholds $c^i = \left(c_1^i, c_2^i, .., c_{k-1}^i\right)$. This reduces problem of having to deal with Lebesgue measurable functions in $k^{[0,1]}$ to dealing with elements in $[0,1]^k$.

Let us pause for a moment and try to see where the analysis is leading us. In the unrestricted IPV model solving for the welfare optimal mechanism is rather easy. One invokes the revelation principle which fixes the message spaces and reduces dimensionality of the problem by mapping the reporting strategy and the allocation rule of each bidder into one object. That is, instead of looking for reporting strategies, allocations and transfers, one only needs to look for the later two Of course the mechanism has to satisfy some properties, like incentive compatibility and individual rationality. Timing of those (ex ante, interim, ex post) depends on the preferences of the author.

[8]This might not be true if one were to consider interdependent values.

Before mentioned conditions, are easily dealt with in the basic model. Even more, it is easy to achieve truthful reporting of bidders by appropriately setting the transfers. The optimal welfare is then achieved by allocating the object to the bidder with the highest value.

In our model the revelation principle does not provide much help, as pointed out earlier. The main problem is that the welfare optimizing reporting strategy is far from obvious. Clearly, truthful direct reporting is not going to help, as it is not clear what truthful reporting in our model would mean. So far we have managed to reduce the set of strategies we need to consider by establishing that it is enough to look at the set of threshold strategies.

On the other side dealing with the mechanism is much more rewarding. Most of the time we will be concerned with the allocation rules since transfers, first, do not enter the welfare directly, and second, will not cause much trouble. Perhaps it is more natural to think of the mechanism as fixed and of bidders as choosing their strategies. At this juncture we will make use of the alternative approach, though. Let μ be some fixed profile of threshold strategies. We will look at all the mechanisms for which μ is an equilibrium and select those that achieve the highest welfare. Let a welfare optimal allocation rule given μ be denoted $Q^*(\mu)$ and transfers $T^*(\mu)$. Here $Q^*(\mu)$ and $T^*(\mu)$ are functions from $\underset{i=1}{\overset{I}{\times}} M_i$ into $[0,1]^I$ and $\mathbb{R}_+^I$ respectively.

The next lemma shows that we can restrict the attention to a narrow set of allocation rules.

Lemma 2 *Fix a threshold strategy μ. Allocation rule $Q^*(\mu)$ maximizes the welfare for a given profile of strategies μ, iff it satisfies the following property. For every i and every $m \in M$*

$$Q_i^*(\mu)(m) > 0 \text{ only if for every } j,\ E\left[V_i \mid \mu_i(V_i) = m_i\right] \geq E\left[V_j \mid \mu_j(V_j) = m_j\right].$$

Moreover, there exists $T^*(\mu)$ such that μ is an equilibrium of the mechanism $(Q^*(\mu), T^*(\mu))$.

$Q^*(\mu)$ maximizes the welfare for a fixed profile of strategies μ iff for every profile of reports, m, the object is allocated to the bidder with the highest expected value. Differently, for a fixed μ ex ante welfare maximization is equivalent to interim welfare maximization.

Some clarification is needed. Let μ be some threshold strategy profile and $Q^*(\mu)$ as above. $Q^*(\mu)$ can be supported by the following Vickrey type transfers. If bidder i wins the object with a positive probability he pays the smallest value he could have, report according to his strategy μ_i and still win the object with some positive probability, p, multiplied by p. If he is not awarded the object at all he does not have to pay. Let us try to clarify the issue by the means of a simple example.

Example 1 *Two bidders have valuations independently distributed with the uniform distribution over* $[0,1]$*. In addition* $|M_1| = |M_2| = 2$*, so each bidder can use one of the two messages. Let our* μ *be a threshold strategy such that both bidders use threshold* 0.5*. If a bidder observes valuation in* $[0.5, 1]$ *he reports* m_1*, otherwise* m_2*. An allocation rule* $Q^*(\mu)$*, is as follows:* $Q_1^*(m_1, m_1) = 0.3$*,* $Q_1^*(m_1, m_2) = 1$*,* $Q_1^*(m_2, m_1) = 0$*,* $Q_1^*(m_2, m_2) = 0.7$*, where the first argument of* Q_1^* *is bidder* $1's$ *and the second bidder* $2's$ *message, and* $Q_2^* = 1 - Q_1^*$ *. It is easy to see that such an allocation rule satisfies the properties required by Lemma 2. Now, the equilibrium transfer from bidder* 1 *when* (m_1, m_1) *is reported is* $T_1^*(m_1, m_1) = 0.15$*. Indeed, the smallest valuation he could have, report according to his strategy and win with positive probability is* 0.5*, and the probability with which he wins is 0.3. Hence the transfer* 0.15*. Transfers for the other reports can be derived analogously. It is routine to verify that the profile of strategies* μ *is indeed an equilibrium of* (Q^*, T^*)*.*

Verifying that μ is an equilibrium of $(Q^*(\mu), T^*(\mu))$ follows the usual thread of reasoning. A winning bidder can not have a profitable deviation because either he would have to overbid (remember here tiebreaking might be an event of positive probability, therefore a winning bidder might want to overbid to clinch the object with probability 1) and then pay more than he values the object or he would underbid, gain nothing, but possibly win the object with a smaller probability while still paying the same price. A loosing bidder might only be able to raise his payoff by overbidding but then again he would have to pay more for a positive probability of winning an object than it is worth to him.

It might be worth pointing out that $Q^*(\mu)$ can be taken to be deterministic. After all, when two (or more) bidders have the same expected value given the strategies and reports, it is irrelevant who is awarded the object. Hence, one of them can get it with probability 1. In such a case transfers $T^*(\mu)$ are easier to specify: the winning bidder pays the smallest valuation he could have, report according to his prescribed strategy and still win the object.

Next theorem establishes existence of a welfare optimal equilibrium. That is, of a combination of strategies μ^* and an incentive compatible mechanism $(Q^*(\mu^*), T^*(\mu^*))$ that maximize the welfare.

Theorem 1 *Welfare maximizing equilibrium exists.*

Such an equilibrium is in dominant strategies; it is also ex post individually rational. This is of course due to the nature of Vickrey type transfers.

An alternative approach was taken by BNS. They show that even if one allows for any strategy profile and allocation/transfer rules (without requiring it to be an equilibrium)[9] one obtains the same result, i.e. threshold strategies coupled with the appropriate Q^* achieve the highest welfare. A strategy profile conveying the most information needed for the welfare maximization, even when bidders naively follow their strategies, is in threshold strategies. This is, of course, also true in the unrestricted IPV model. One could not dream of doing better than have bidders report truthfully and award the object to the bidder with the highest value.

The following proposition originally appeared in BNS, Theorem 3.1. We provide an alternative proof in the Appendix A.

Proposition 1 *Optimal welfare is achieved with threshold strategies.*

The optimal allocation rule is characterized by Lemma 2. Since we are not requiring strategies to form an equilibrium, transfers can remain unspecified.

Similar observation as under equilibrium analysis applies. If one is willing to think of strategies in which two or more messages are used to convey the same meaning as threshold strategies, then any welfare maximizing strategy is essentially equivalent[10] to a threshold strategy. There is a minor technical difference though, hence the qualifier 'essentially'. Under equilibrium analysis equilibrium behavior and thus welfare maximization is required for every possible profile of valuations. Whereas, when one is maximizing ex ante welfare over all the possible strategies, unruly behavior is possible on the set of measure zero.

The above results lay a foundation for the further analysis in this paper. We established that a welfare optimizing equilibrium exist and, moreover, is in threshold strategies. The next result provides an additional insight into nature of communication under welfare maximization.

Let us inspect an example first. In the two bidder case, no larger welfare can be achieved by a mechanism in which bidder 1 can use five messages and bidder 2 can use two, than by a mechanism in which bidder 1 can use three messages and bidder 2, two. Indeed, for each of bidder 2's messages expected valuation given his message and strategy can be computed. Let V_1^2 be the expected

[9] That is, one is maximizing over all the μ and assuming that bidders do not deviate.
[10] L^1 equivalent.

valuation when bidder 2 reports m_1 and V_2^2 when he reports m_2. Without loss of generality we can assume $V_1^2 \geq V_2^2$. Since the only relevant information is whose valuation, or expected valuation, is larger, bidder 1 can do no better than report whether his valuation is above V_1^2, between V_1^2 and V_2^2 , or below V_2^2. Thus at most three messages of bidder 1 are welfare relevant.

While it is intuitive that higher welfare can be achieved by increasing the number of possible messages for both bidders, as will be shown below, the same is not necessarily true when communication capacity of one bidder is increased. After all, higher welfare is achieved by a direct comparison. Increasing the richness of structure for one bidder will not help if one has nothing to match it to on the other side.

A general statement for two bidders would say that as soon as $|k_1 - k_2| > 1$, one bidder has messages that are irrelevant for welfare. More precisely, let $k_1 \geq k_2$ and define $k = \min\{k_1, k_2 + 1\}$. Then

$$w^*_{2,(k_1,k_2)} = w^*_{2,(k,k_2)}.$$

The optimal welfare in a two bidder mechanism with cardinalities of message space k_1 and k_2 is equal to the optimal welfare when k_1 is replaced with k. Similar reasoning extends to the case of I bidders.

Proposition 2 *Let $k_1 \geq k_2 \geq ... \geq k_I$ and define $k = \min\left\{k_1, \sum_{i=2}^{I} k_i + 1\right\}$. Then*

$$w^*_{I,(k_1,k_2,...,k_I)} = w^*_{I,(k,k_2,...,k_I)}.$$

If a bidder has too many messages at his disposal some of them will not contribute to the welfare. Let the bidder with the highest cardinality of the message space be bidder 1. One can devise a ranking of all the remaining bidders' expected values for each message they report. Now, the best one can do in terms of welfare is to let bidder 1 (truthfully) report whether his expected value is above the highest of those in the ranking, between the highest and the second highest, etc. To achieve this, at most $\sum_{i=2}^{I} k_i + 1$ messages are needed, meaning that what the bidder with the highest cardinality of the message space is able to communicate is bounded by the communication capability of all the remaining bidders.

3.2 Two bidders

Welfare optimizing equilibrium for the case of two bidders was first provided by BNS. For completeness we provide the result here and suggest an alternative method to obtain it. In addition, we provide a result for symmetric bidders[11] relating welfare maximizing equilibria to welfare maximizing equilibria under the requirement that the mechanism and strategies be symmetric. Characterization of communication cost of symmetry sheds light onto origins and causes of asymmetry of welfare maximizing equilibria.

A setup with two bidders is considered here. Assumptions about preferences and distributions are as stated in Section 2. We will make great use of results established in previous sections. Theorem 1 enables us to restrict attention to threshold strategies. Furthermore, by Proposition *2* we only need to solve the problem in which k_1 and k_2 are such that $|k_1 - k_2| \leq 1$. The other cases reduce to this one. Without loss of generality we can assume that $k_1 \geq k_2$ and define $k = \min\{k_1, k_2 + 1\}$, as in Proposition 2.

We call a profile of threshold strategies c^* **mutually centered**[12] if it is a solution to either of the following two systems of equation:

$$\begin{aligned} c_n^{1*} &= E\left[V_2 \mid c_{n-1}^{2*} \geq V_2 \geq c_n^{2*}\right], \text{ for } n = 1, ..., k-1 \\ c_n^{2*} &= E\left[V_1 \mid c_n^{1*} \geq V_1 \geq c_{n+1}^{1*}\right], \text{ for } n = 1, ..., k_2 - 1 \end{aligned} \tag{1}$$

or

$$\begin{aligned} c_n^{1*} &= E\left[V_2 \mid c_n^{2*} \geq V_2 \geq c_{n+1}^{2*}\right], \\ c_n^{2*} &= E\left[V_1 \mid c_{n-1}^{1*} \geq V_1 \geq c_n^{1*}\right], \end{aligned} \tag{2}$$

for $n = 1, ..., k_2 - 1$, where $c_0^i = 1$ and $c_n^i = 0$ for $n \geq \min\{k, k_i\}$ and $i \in \{1, 2\}$. In a profile of mutually centered strategies either bidder 1's highest threshold is the expectation of the bidder 2's top partition cell, bidder 2's highest threshold the expectation of the bidder 1's second highest partition cell, etc., or the roles of the two bidders' are reversed. One way or another, we get an unambiguous ranking over the expected valuations given the reported messages. It is rather easy

[11] With the same prior distribution function, that is $F_1 = F_2 = F$, and the same cardinality of the message space, $|M_1| = |M_2| = k$.

[12] The term coined by BNS.

to see that a threshold strategy satisfying this properties exists.[13] Furthermore, Lemma *2* provides a characterization of a welfare maximizing allocation rule $Q^{*}\left(c^{*}\right)$ for c^{*}. In particular, the object is allocated to the bidder reporting the higher partition cell, where higher refers to the strong set order.

Definition 4 *Inspired by BNS, we call an equilibrium a* ***priority equilibrium*** *if it uses mutually centered strategies,* c^{*}*, and an allocation rule,* $Q^{*}\left(c^{*}\right)$*, which awards the object to the bidder reporting higher partition cell.*[14]

The next proposition characterizes welfare maximizing equilibria. The result was first provided by BNS, Theorem 3.1 and Theorem 3.5, for the case $|M_1| = |M_2| = k$.

Proposition 3 *Optimal welfare in a 2 bidder auction with* $|M_1| = k_1, |M_2| = k_2$ *is achieved in a priority equilibrium.*

Proposition 3 provides a necessary condition for the equilibrium. In the case of $k_1 = k_2$, the proposition, and the proof, is silent as to whether one should use system of equations (1) or (2). For the case when $k_1 > k_2$ we can be somewhat more precise, though. In that case system of equations (1) applies at the optimum, and therefore the allocation rule Q^{*} is pinned down.

Although some intuition can be grasped from Figure 1 below, a short explanation will be helpful. The optimal mechanism works as follows: bidder i observes his private valuation and reports the message m_n when his valuation belongs to the interval $\left[c_n^i, c_{n-1}^i\right)$; $n \in \{1, ..., k_i\}$.[15] The seller observes the reports and awards the object to the bidder with the highest expected value given strategies c^{*}. Mechanism in a welfare maximizing equilibrium is asymmetric, as are the reporting strategies. For example, one of the two bidders, denote him by j, has priority over the other. That is, if the reports are (m_1, m_1) the object is awarded to bidder j. Indeed $E\left[V_j \mid \mu_j\left(V_j\right) = m_1\right] > E\left[V_{-j} \mid \mu_{-j}\left(V_j\right) = m_1\right]$, since $c_0^j = c_0^{-j}$ and $c_1^j = E\left[V_{-j} \mid c_0^{-j} \geq V \geq c_1^{-j}\right]$. The same is true more generally. Whenever the two bidders report the same message bidder j wins the object. When the reports are different the bidder with the higher message wins, where by higher we mean the message with the lower index.

[13] BNS claim uniqueness. Their proof is unfortunately erroneous.

[14] The term 'priority' stems from the fact that when both bidders report the same message, assuming that message m_1 corresponds to the highest partition cell etc., ties are consistently broken in the favor of one bidder. The definition was introduced by BNS in the setup where both bidders have the same number of messages.

[15] When his valuation is 1 he reports message m_1.

Clearer intuition can be given as to why welfare optimal thresholds are mutually centered. Focusing on the prioritized bidder j, it is easy to see that the only thing that changes between him reporting message m_1 and m_2 is whether he wins or loses against bidder $-j$ who reported m_1. Therefore, when bidder j's value v_j is equal to c_1^{j*} conditional expected welfare should not depend on whether he reports m_1 or m_2. Suppose it did and suppose it would be strictly better if he reported m_1. Then, by continuity, reporting m_1 would also be welfare optimal for values just below c_1^{1*}, which would contradict the optimality of the threshold. But if at $v_j = c_1^{j*}$ it should be irrelevant for welfare whether bidder j is loosing or winning against bidder $-j$ who is reporting m_1, it has to be the case that c_1^{j*} equals the expected valuation of bidder $-j$ when the later is reporting the top partition cell. That is, $c_1^{j*} = E\left[V_{-j} \mid 1 \geq V_{-j} \geq c_1^{-j*}\right]$. The same intuition applies for other thresholds.

To make the analysis more vivid we provide a simple example originally appearing in BNS.

Example 2 *We consider the case of two bidders, each of which has two possible messages and the prior distribution over their messages is uniform on the interval* $[0, 1]$*. The welfare maximizing mechanism, with corresponding equilibrium, is the one in which bidder* 1*'s threshold is* $2/3$*. That is, if his valuation is in the interval* $[2/3, 1]$ *he reports* $\mu_1(v_1) = m_1$*, otherwise he reports* m_2*. Bidder* $2'$*s threshold is* $1/3$*. The allocation rule is* $Q_1(m_1, m_1) = Q_1(m_1, m_2) = Q_1(m_2, m_2) = 1, Q_1(m_2, m_1) = 0$*, where the first argument of* $Q_1(\cdot, \cdot)$ *is bidder* $1'$*s report, and* $Q_2(m_n, m_l) = 1 - Q_1(m_n, m_l)$*, where* $n, l \in \{1, 2\}$*. Transfers are given by* $T_1(m_1, m_1) = \frac{2}{3}, T_1(m_1, m_2) = T_2(m_2, m_1) = T_2(m_2, m_2) = 0$ *and* $T_2(m_2, m_1) = \frac{1}{3}, T_2(m_1, m_1) = T_2(m_1, m_2) = T_2(m_2, m_2) = 0$*.*

Welfare in this case is $\frac{35}{54}$*. For comparison, if no restrictions are imposed on communication one can achieve welfare of* $\frac{2}{3} = \frac{36}{54}$*; as in Vickrey (1961). Although we severely reduced means of communication the decrease in welfare is only about* 2.8%*.*

Even in the case where both bidders have the same prior distribution over valuations, $F_1 = F_2 = F$, the welfare optimizing equilibria are asymmetric. Asymmetry is twofold, both in strategies as well as in the mechanism. This brings us to another point. Symmetric treatment of bidders is often required in an auction, either for legal purposes or for fairness. In the fallowing paragraphs we characterize the relation between welfare optimal equilibria and welfare optimal symmetric equilibria.

Symmetric setup allows for the clearest analysis; i.e. $F_1 = F_2 = F$, $|M_1| = |M_2| = k$. Starting with an asymmetric setup would cause ambiguities with definition of symmetric strategy as well as mechanism. We call strategies symmetric if $\mu_1(\cdot) = \mu_2(\cdot) = \mu(\cdot)$, and mechanism symmetric if $Q_1(m_l, m_n) = Q_2(m_n, m_l)$ and $T_1(m_n, m_l) = T_2(m_l, m_n)$ for every $n, l \in \{1, ..., k\}$, where the first argument of all allocation and transfer rules is bidder 1's message and the second bidder 2's.

Requiring symmetric treatment of the two bidders can be interpreted in two ways. On one side, one could require that the equilibrium be symmetric, that way the allocation is symmetric not only given the messages, but also given the values of the two bidders. On the other side, one could require a weaker condition, that the allocation rule be symmetric. We will not express preference over those two treatments and, luckily enough, analysis will not require it.

Let us elaborate on the issue. If one is to require symmetric equilibrium the optimal allocation rule awards the object to the bidder reporting higher partition cell, where in the case of equivalent reports each bidder obtains the object with probability one half. Characterization of welfare optimal strategies (requiring that both bidders use the same strategy) is now easily obtained from the first order condition to be

$$c_n^* = E\left[X \mid c_{n-1}^* \geq X \geq c_{n+1}^*\right]$$

for every $n \in \{1, 2, ..., k-1\}$, where each bidder's strategy is given by the vector of thresholds $\left(c_1^*, c_2^*, ..., c_{k-1}^*\right)$.

As pointed out earlier one could alternatively just require that the allocation rule be symmetric. Fortunately enough we show that the optimal welfare can still be achieved in symmetric strategies.

Lemma 3 *Optimal welfare in a symmetric mechanism can be achieved with symmetric strategies.*

From this point on we talk about optimal symmetric equilibria when requiring symmetry. As the proof of the preceding lemma already hints, there is a rather interesting link between optimal equilibria and optimal symmetric equilibria. Exploring it further yields a sharper insight into and intuition for the cause of asymmetry in the welfare optimal equilibria.

Let $\widetilde{c} = \left(\widetilde{c}_1^1, \widetilde{c}_1^2, \widetilde{c}_2^1, ..., \widetilde{c}_{k-1}^1, \widetilde{c}_{k-1}^2\right)$ be a vector of threshold strategies satisfying $\widetilde{c}_1^1 \geq \widetilde{c}_1^2 \geq \widetilde{c}_2^1 \geq ... \geq \widetilde{c}_{k-1}^1 \geq \widetilde{c}_{k-1}^2$. Notice that we do not require mutual centeredness here. What this structure does give us, though, is an order over expected values conditional on reports, no matter what the underlying distribution F is. Remember, such a threshold strategy can be supported as an equilibrium by an allocation rule (and appropriate transfers) allocating the object to bidder 1 unless the message of bidder 2 has a lower index, in which case the object is awarded to bidder 2. Finally, define $\widehat{c}_1 = \widetilde{c}_1^1, \widehat{c}_2 = \widetilde{c}_1^2, ..., \widehat{c}_{2k-3} = \widetilde{c}_{k-1}^1, \widehat{c}_{2k-2} = \widetilde{c}_{k-1}^2$.

Proposition 4 *Let $F_1 = F_2 = F$, $|M_1| = |M_2| = k$ and let $\widetilde{c}$ be a vector of threshold strategies as above. The welfare obtained by the equilibrium with $\widetilde{c}$ and the allocation rule awarding the object to the bidder with the highest expected value, $\widetilde{Q}(\widetilde{c})$, is equal to the welfare obtained by a symmetric equilibrium in which both bidders use a strategy described by thresholds $\widehat{c}_1, \widehat{c}_2, ..., \widehat{c}_{2k-2}$, combined with the allocation rule $\widehat{Q}(\widehat{c})$, which always awards the object to the bidder reporting the higher partition cell.*[16]

Furthermore, the maximal welfare achieved in an equilibrium of a mechanism with $|M_1| = |M_2| = k$ is equal to the maximal welfare achieved in a symmetric equilibrium of a mechanism with $|M_1| = |M_2| = 2k-1$.

The above proposition provides a strong relation between welfare optimizing equilibria and the best one can do when requiring symmetry of play. In particular, if one requires symmetric equilibria, one needs almost twice as many messages for each bidder to achieve the highest welfare obtained under no symmetry requirements.

The result can be explained as follows. Define $T_j = [\widehat{c}_j, \widehat{c}_{j-1})$ for $j = 2, ..., 2k-2$ and $T_1 = [\widehat{c}_1, 1]$. We say that a bidder with a valuation v is of type T_j if $v \in T_j$. The lower the j the higher the type the bidder is. For the fixed strategy, the seller cannot distinguish between two valuations in an interval $[\widehat{c}_j, \widehat{c}_{j-1})$, therefore we might as well bunch them into a type. Optimally the bidder with the higher type should win the object,[17] and if the two bidders are of the same type the object can be awarded to either bidder. Strategy $\widehat{c}$ does not distinguish between all the types for each bidder,[18] i.e. when bidder 1 is of type T_1 he can indeed report so, but he can not report whether he is of type T_2 or type T_3; he can at most report that he is of $T_2 \cup T_3$. This turns out not to be an issue in the two bidder case.

When bidder 1 is of type T_1 he wins no matter what type bidder 2 is. When bidder 2 is of type T_1 he wins against all the types of bidder 1, except T_1. But if both bidders are of type T_1 the welfare does not depend on who is awarded the object so it might as well be bidder 1. When bidder

[16] The tie breaking rule here is irrelevant. For example, if both report the same partition element the winner could be decided by a flip of a fair coin.

[17] The allocation rule used is the one that awards the object to the bidder with a higher expected value given the messages and the strategies used.

[18] It is not measurable with respect to the algebra induced by types.

1 is of type T_2, therefore reporting message m_2, he can only lose if bidder 2 reports m_1. But this can only happen in an equilibrium if bidder 2 is either of type T_1 or T_2. In either of the two cases it is welfare optimal to award him the object. Following the same reasoning we see that a bidder of a certain type never looses against a bidder of a lower type. Hence the equilibrium described by thresholds $\widetilde{c} = \left(\widetilde{c}_1^1, \widetilde{c}_1^2, \widetilde{c}_2^1, ..., \widetilde{c}_{k-1}^1, \widetilde{c}_{k-1}^2\right)$ is welfare equivalent to the symmetric equilibrium where each of the bidders has a strategy described by thresholds $\widehat{c}_1, \widehat{c}_2, ..., \widehat{c}_{2k-2}$.

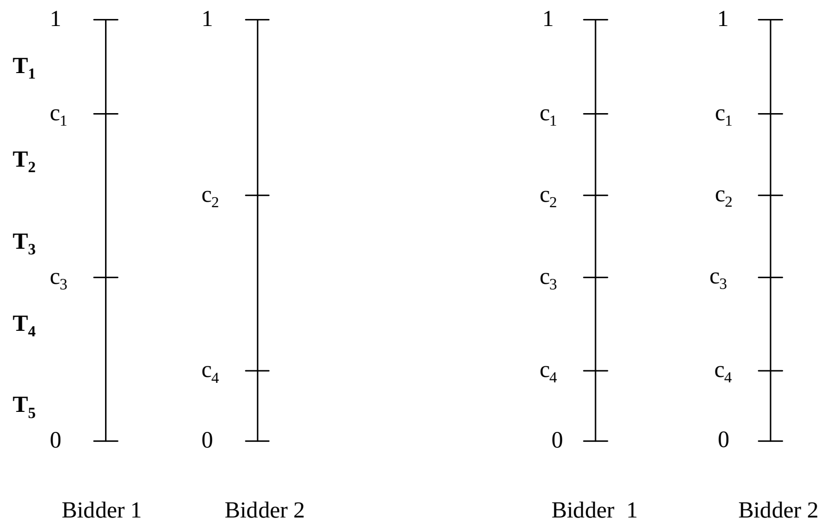

Figure 1: A symmetric two bidder - three message case.

Until now we assumed that the object to be auctioned has no value to the seller, i.e. $v_0 = 0$. While this is rather convenient, it might be of interest to explore relaxing this assumption. This will be of great benefit to us when considering the revenue maximization.

Here we shall assume $|M_1| = |M_2| = k$ and $F_1 = F_2 = F$. An informed reader will observe that the analysis to follow easily extends to the asymmetric case, both in distributions and cardinality of the message space. Before we proceed some additional notation is required. Let

$c^{\#} = \left(c_1^{1\#}, c_1^{2\#}, c_2^{1\#}, ..., c_{k-1}^{1\#}, c_{k-1}^{2\#}\right)$ be defined by:

$$
\begin{aligned}
c_{k-1}^{2\#} &= v_{0}, \\
c_{k-1}^{1\#} &= \frac{F\left(c_{k-2}^{2\#}\right) - F\left(c_{k-1}^{2\#}\right)}{F\left(c_{k-2}^{2\#}\right)} E\left[V \mid c_{k-2}^{2\#} \geq V \geq c_{k-1}^{2\#}\right] + \frac{F\left(c_{k-1}^{2\#}\right)}{F\left(c_{k-2}^{2\#}\right)} v_{0}, \\
c_{k-2}^{2\#} &= E\left[V \mid c_{k-2}^{1\#} \geq V \geq c_{k-1}^{1\#}\right], \\
c_{k-2}^{1\#} &= E\left[V \mid c_{k-3}^{2\#} \geq V \geq c_{k-2}^{2\#}\right], \\
&\ldots \\
c_{1}^{2\#} &= E\left[V \mid c_{1}^{1\#} \geq V \geq c_{2}^{1\#}\right], \\
c_{1}^{1\#} &= E\left[V \mid 1 \geq V \geq c_{1}^{2\#}\right].
\end{aligned}
$$

Strategy profile $c^{\#}$ closely resembles a mutually centered profile of strategies. The slight difference is that bidder 2's lowest threshold equals the seller's value and all the other thresholds build on it.

Definition 5 *An equilibrium is called a* ***modified priority equilibrium*** *if it uses the strategies given by* $c^{\#}$ *and an allocation rule that awards the object to the bidder with the highest expected value unless both bidders report their lowest partition cell in which case the object is allocated to the seller.*

The following proposition originally appeared in BNS.

Proposition 5 *For any* v_0*, optimal welfare is achieved in a priority equilibrium or a modified priority equilibrium.*

Having the seller with a commonly known value v_0 guarantees expected welfare of at least v_0. There is no need to distinguish between values below v_0 as in that case the seller will be awarded the object anyway. More precisely, any strategy that has some bidder reporting more than one message with conditional valuation not exceeding v_0 can be improved upon.[19] Those messages can be merged into one, while the expected value given the new message is still maintained below v_0. The additional message can now be used to separate the values above v_0 and increase efficiency.

[19] Remember that we are dealing with the case where both bidders have k messages and thus messages are a scarce resource. In the case where one bidder has significantly more messages to his disposal than the other some messages will have to be wasted.

Therefore at most one message is used to convey the meaning 'my expected value is bellow v_0'. If v_0 is small it might not be a good idea to even bother using a message for such a purpose. In that case the problem is best solved as if neglecting the seller's value altogether.

Furthermore, if it is indeed optimal for bidder 2 to announce for some values that his expected value is below v_0, it should be so for all $v < v_0$. It will never be optimal to set $c_{k-1}^2 < v_0$. If bidder 2 tries to signal that his value is below v_0, he might as well do that for all such values. That is, instead of reporting message m_k for values in $[0, a)$, for some $a < v_0$, he should do it for $[0, v_0)$.

Intuition for the thresholds given by $c^{\#}$ is similar to the case of mutually centered strategies. Slight modification is needed for $c_{k-1}^{1\#}$. When bidder $1's$ valuation is $c_{k-1}^{1\#}$ it should not matter for welfare whether he reports m_{k-1} or m_k. The only two cases where this matter is when bidder 2's valuation is either in $\left[c_{k-1}^{2\#}, c_{k-2}^{2\#}\right)$ or in $\left[0, c_{k-1}^{2\#}\right)$. But $c_{k-1}^{1\#}$ is set to just offset this effect.

Sharper characterization can be given in certain cases.

Corollary 1 *Let $v_0 \geq E\left[V\right]$. Then the optimal welfare is achieved in a modified priority equilibrium.*

By Proposition 5 we know the optimal welfare is achieved in a priority or in a modified priority equilibrium. If it were to be in a priority equilibrium then the lowest partition cell of either bidder would have expected value below $E\left[V\right]$ and therefore below v_0. But then optimally the seller would retain the object, contradicting the fact that equilibrium is a priority equilibrium.

Finally, we present an example which should enlighten this notationally intense analysis.

Example 3 *Two bidders have valuations uniformly distributed over the interval $[0, 1]$. As we already know, the optimal thresholds in the case when the object has no value for the seller, i.e. $v_0 = 0$, are $c_1^* = \frac{2}{3}$, where c_1^* is optimal threshold of bidder 1, and $c_2^* = \frac{1}{3}$, where c_2^*, is the optimal threshold of bidder 2. The optimal welfare is $w^* = \frac{35}{54}$.*

When $v_0 > 0$ Proposition 5 tells us that optimal welfare can be achieved by a priority or a modified priority equilibrium. We computed ex ante welfare achieved in the priority equilibrium. As of the modified priority equilibrium. It is easily seen to be the case that $c^{1\#} = \frac{1+v_0^2}{2}$ and $c^{2\#} = v_0^2$, where object is allocated to bidder 1 if he reports the top cell, to bidder 2 if he reports the top cell and bidder 1 the bottom cell and the seller retains the object if both bidders report the bottom cell. The welfare is then $w^{\#}(v_0) = \frac{5+2v_0^2+v_0^4}{8}$. On the other side, $E\left[X \mid c^{1\#} \geq X \geq 0\right] = \frac{1+v_0^2}{4}$, which *is no larger than* v_0 *as long as* $v_0 > 2 - \sqrt{3}$*. Remember, we need* $E\left[X \mid c^{1\#} \geq X \geq 0\right] \leq v_0$*, for the seller should optimally retain the object when bidders report their lower partition cell. Now,* $w(v_0)$ *is strictly increasing in* v_0*, so let* $\overline{v}_0$ *be such that* $w(\overline{v}_0) = \frac{35}{54}$*. An easy computation shows* $\overline{v}_0 = \sqrt{\frac{4}{3}\sqrt{\frac{2}{3}} - 1} \approx 0.298$*. Thus as long as*

$$v_0 \geq \sqrt{\tfrac{4}{3}\sqrt{\tfrac{2}{3}} - 1},$$

which is indeed larger than $2 - \sqrt{3}$*, welfare in the modified priority equilibrium is at least as high as in the priority equilibrium* $w(\widehat{c}) \geq w(c^*)$*. The analysis above shows that for* $v_0 < \sqrt{\frac{4}{3}\sqrt{\frac{2}{3}} - 1}$ *the highest welfare is achieved in the priority equilibrium with tresholds* $(c_1^*, c_2^*) = \left(\frac{2}{3}, \frac{1}{3}\right)$ *and for* $v_0 \geq \sqrt{\frac{4}{3}\sqrt{\frac{2}{3}} - 1}$ *the highest welfare can be obtained by the modified priority equilibrium with tresholds* $\left(c^{1\#}, c^{2\#}\right) = \left(\frac{1+v_0^2}{2}, v_0^2\right)$.

3.3 More than two bidders

In the two bidders case the characterization is readily obtained by observing that expected values corresponding to two bidders' messages need to satisfy a particular order structure. Either it has to be the case that $E[V_1 \mid \mu_1(V_1) = m_1] \geq E[V_2 \mid \mu_2(V_2) = m_1] \geq E[X_1 \mid \mu_1(V_1) = m_2] \geq ...$ or a similar chain of inequalities should hold with the roles of the two bidders interchanged. In any case the idea is that for no two messages should some bidder's conditional expectations come consecutive in such a ranking, for this would be a waste of welfare relevant information.[20] The order of conditional expected values given the messages allows one to recover the optimal allocation rule. After all, Lemma 2 implies that the object should always be allocated to the bidder with the highest conditional expected value. Now the optimal strategies are easily shown to be mutually centered from the first order conditions.

The case of multiple bidders turns out to be somewhat harder to solve. A similar line of reasoning as outlined in the preceding paragraph shows that if one was to devise a ranking of expected valuations corresponding to bidders' messages, for no bidder two expected valuations

[20] We are refering to the case $|k_1 - k_2| \leq 1$, which is the only interesting.

should appear consecutive. While this gave the full structure in the two bidder case this is not so in general. For a more precise characterization subtler analysis is needed.

In what follows we provide a characterization for the case of three ex ante symmetric bidders, each of them having message space of equal cardinality, $|M_1| = |M_2| = |M_3| = k$ and $F_1 = F_2 = F_3 = F$, where F, as earlier, is an atomless distribution function on some compact interval, say $[0, 1]$, with everywhere positive density f. The analysis for the case of I bidders is a straightforward extension, though rather notationally intractable.

By Lemma 1 we can restrict the analysis to threshold strategies, in particular, to threshold strategies with $k-1$ thresholds. We denote thresholds of bidder i by $c_1^i, c_2^i, ..., c_{k-1}^i$. One can entertain many different scenarios as of what a welfare maximizing equilibrium looks like: it might be the case that $c_1^1 \geq c_1^2 \geq c_2^1 \geq c_1^3 \geq c_2^2 \geq c_2^3 \geq ...$, or maybe $c_1^1 \geq c_1^2 \geq c_1^3 \geq c_2^1 \geq c_2^2 \geq c_2^3 \geq ...$, or maybe again something completely different. Not knowing the order of thresholds and thus the order of conditional expected values given the reports seems to be an obstacle to obtaining a closed form solution for the welfare. Luckily enough there is a way around that.

The following tools will help our analysis. Any profile of threshold strategies[21] can be described by a set of thresholds $C = \{c_1, c_2, ..., c_{3k-3}\} \subset [0, 1]$, where $c_1 \geq c_2 \geq ... \geq c_{3k-3}$, and a function that assigns $k-1$ thresholds to each of the bidders. More precisely:

Definition 6 *Function*

$$\iota : \{c_1, c_2, ..., c_{3k-3}\} \to \{1, 2, 3\},$$

with the restriction $\left|\iota^{-1}\{i\}\right| = k-1$ *for each* $i \in \{1, 2, 3\}$, *is called an* ***assignment function.***

Assignment function assigns to each bidder $k-1$ thresholds from the set $\{c_1, c_2, ..., c_{3k-3}\}$. Any profile of threshold strategies can now be described by a pair C and ι. In addition we define the **priority assignment** ι^* by

$$\begin{aligned} \iota^*(c_{3j+1}) &= 1, \\ \iota^*(c_{3j+2}) &= 2, \\ \iota^*(c_{3j+3}) &= 3, \end{aligned}$$

where $j \in \{0, 1, 2, ..., k-2\}$. The priority assignment assigns the thresholds in a round robin

[21]Remember we showed earlier that any profile of threshold strategies can be supported as an equilibrium.

fashion: the highest threshold is assigned to bidder 1, second highest to bidder 2, third to bidder 3, fourth again to bidder 1, etc. For further intuition see Figure 2.

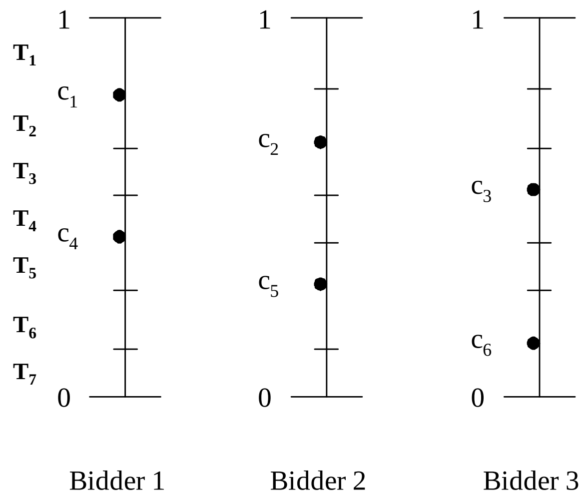

Figure 2: Threshold strategies generated by the priority assignment. The actual thresholds are denoted by dots.

The profile of strategies represented by some set of thresholds C and the priority assignment ι^* has a very convenient property. The ordering of conditional expectations given the bidders' reports is

$$E[X \mid 1 \geq X \geq c_1] > E[X \mid 1 \geq X \geq c_2] > E[X \mid 1 \geq X \geq c_3] > \tag{3}$$

$$> E[X \mid c_1 \geq X \geq c_4] > ... > E[X \mid c_{3k-3} \geq X \geq 0]. \tag{4}$$

For every profile of strategies that can be represented by a priority assignment, welfare is maximized by the following allocation rule Q^*: Q^* awards the object to bidder 1 if none of the other two bidders reports a higher partition cell,[22] to bidder 2 if bidder 1 reports a lower cell and bidder 3 does not report a higher, and to bidder 3 in the remaining cases. Now we are ready to state the result. Again, we should like to point out that the analysis extends to any number of bidders I.

Theorem 2 *The highest welfare in an equilibrium of a mechanism with* 3 *symmetric bidders is*

[22] Suppose bidders $1, 2$ and 3 report messages m_s, m_p, m_q respectively. Then $Q_1(m_s, m_p, m_q) = 1$ iff $s \leq \min\{p, q\}$ and 0 otherwise.

achieved with an allocation rule Q^* and a profile of threshold strategies given by the priority assignment ι^* and a set of thresholds $\left\{c_1^*, c_2^*, ..., c_{3k-3}^*\right\}$ characterized by:

$$\begin{aligned} c_j^* &= \frac{F\left(c_{j-1}^*\right)\left[F\left(c_{j-2}^*\right)-F\left(c_{j+1}^*\right)\right]}{F\left(c_{j-2}^*\right)F\left(c_{j-1}^*\right)-F\left(c_{j+1}^*\right)F\left(c_{j+2}^*\right)} E\left[X \mid c_{j-2}^* \geq X \geq c_{j+1}^*\right] \\ &\quad + \frac{F\left(c_{j+1}^*\right)\left[F\left(c_{j-1}^*\right)-F\left(c_{j+2}^*\right)\right]}{F\left(c_{i-2}^*\right)F\left(c_{j-1}^*\right)-F\left(c_{j+1}^*\right)F\left(c_{j+2}^*\right)} E\left[X \mid c_{j-1}^* \geq X \geq c_{j+2}^*\right], \end{aligned} \tag{5}$$

for $j \in \{1, 2, ..., 3k-3\}$.

As usual, $c_j = 1$ for $j \leq 0$ and $c_j = 0$ for $j \geq 3k-2$. The above threshold strategies and the allocation rule can be supported as an equilibrium by the Vickrey type transfers $T^*\left(c^*, \iota^*\right)$.

We will now outline the idea of the proof. As we already pointed out, for different threshold strategies different allocation rules are optimal. All of them have the same characteristic: the object is awarded to a bidder with the highest expected value given the reported messages and threshold strategies. See Lemma 2. As we also pointed out earlier, one can not establish the order of expected values without explicitly writing down the equations for the particular distribution F. This fact can be remedied.

We fix the set of thresholds C as described above and look for the profile of strategies with the highest welfare among the strategies that can be described by C. Maximization is thus over the assignment functions and allocation rules. We establish that the optimal welfare is achieved by the priority assignment ι^* and the allocation rule Q^*. The details of the proof can be found in the Appendix C; some intuition is provided below.

The outlined analysis brakes down the set of all possible strategies into smaller sets after which the local optima for each of these subsets of strategies are computed. Local optima have two things in common. First, the reporting strategies can be described by a set of thresholds and the priority assignment. Second, all of the local optima use the same allocation rule Q^*. Now one can optimize over the set of local optima holding the assignment function and the allocation rule fixed. This is a much simpler problem.

The next couple of paragraphs serve the purpose of explaining why the priority assignment maximizes the welfare for the fixed set of thresholds. Rather than providing the proof here we outline a certain characteristic of the priority assignment or the profile of strategies given by it. Fix $c_1 > c_2 > ... > c_{3k-3}$ and define types as follows: $T_1 = [c_1, 1], T_j = [c_j, c_{j-1})$ for $j = 2, ..., 3k-2$. Optimally one would like to communicate as much information as possible. The best one could hope for given the restrictions would be to fully communicate types. The highest welfare one could achieve in such a case would be:

$$\overline{w}(c) = \sum_{j=1}^{3k-2} \left\{ F(c_{j-1})^2 + F(c_{j-1}) F(c_j) + F(c_j)^2 \right\} \left[F(c_{j-1}) - F(c_j) \right] E \left[X \mid X \in T_j \right].$$

Unfortunately this is not the case in our model. Here we focus on the profile of strategies generated by the priority assignment and the allocation rule Q^*. We compare the welfare obtained with full reporting of types to the welfare obtained under the priority assignment. First we consider what happens with welfare when each of the bidders is of type T_1. When bidder 1 is of type T_1, he wins no matter what partition cells the other bidders report. When bidder 2 is of type T_1, he wins the object unless bidder 1 is of type T_1. If both of them are of the same type there is no loss of welfare by awarding the object to bidder 1. Things take a bad turn when bidder 3 is of type T_1. He wins unless bidder 1 is of type T_1, or bidder 2 is of type T_1 or T_2. Under the priority assignment bidder 2 can not report whether he is type T_1 or T_2, but rather that he is $T_1 \cup T_2$. But now the problem is that when bidder 3 is of type T_1, bidder 2 of type T_2, and bidder 1 of a type lower than T_1, the object will be awarded to bidder 2 and not to 3 as would be welfare optimal. See the figure below. In such a case we say that bidder 3 of type T_1 loses against bidder 2 of type T_2.[23] It should be quite intuitive that given the values of thresholds there is no conceivable way in which one could do better. No other assignment function can improve the welfare. For details see the proof in the Appendix C.

Considering further cases, with bidder 1 of type T_2 and so forth, one can see that under the priority assignment for every type T_j there exists exactly one bidder of that type who is losing against exactly one other bidder of type T_{j+1}.

Instead of welfare $\overline{w}$ we now get:

$$\begin{aligned} w(q^*, \iota^*, c) \;\; = \;\; & \sum_{j=1}^{3k-2} \left\{ F(c_{j-1})^2 + F(c_{j-1}) F(c_j) + F(c_j)^2 \right\} \left[F(c_{j-1}) - F(c_j) \right] E \left[X \mid X \in T_j \right] \\ & - \sum_{j=1}^{3k-2} F(c_{j-1}) \left[F(c_{j-1}) - F(c_j) \right] \left[F(c_j) - F(c_{j+1}) \right] \left\{ E \left[X \mid X \in T_j \right] - E \left[X \mid X \in T_{j+1} \right] \right\}. \end{aligned}$$

The first part of the right hand-side represents the welfare one would achieve if all bidders could

[23] To be more specific: bidder 3 of type T_1 loses agains bidder 2 of type T_2 when bidder 1 is of at most type T_2. We will often omit the last part in hope that this does not couse too much confusion.

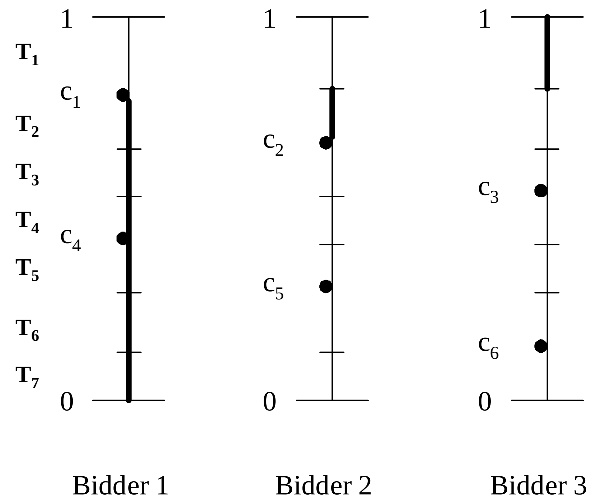

Figure 3: Bidder 3 of type T_1 is losing against bidder 2 of type T_2, when bidder 1 is not type T_1.

fully report their type, whereas the second is the loss of welfare incurred because they cannot do so.

Finally we are ready to say something about the optimal thresholds provided by the theorem. Similar intuition as in the 2 bidder case can be provided. At the optimum it should not matter, from the welfare point of view, whether the bidder observing value c_j reports the message m_j, which corresponds to interval $[c_j, c_{j-3})$,[24] or m_{j+1}, which corresponds to the interval $[c_{j+3}, c_j)$. If, for example, it would be strictly welfare improving for him to report m_j, then by continuity that would also be the case for the values just bellow the threshold c_j, which would contradict the optimality. What changes between reporting m_j and m_{j+1} is whether the bidder wins against the other two bidders when they report the message corresponding to $[c_{j-2}, c_{j+1})$ or $[c_{j-1}, c_{j+2})$, but the expected value when that occurs is given by the equation (5).

For clarity we started with the requirement $|M_1| = |M_2| = |M_3| = k$. The same results can be obtained with a weaker assumption. Let $|M_1| = k_1$, $|M_2| = k_2$ and $|M_3| = k_3$. We can label the bidders so that $k_1 \geq k_2 \geq k_3$. As long as $k_1 - k_3 \leq 1$, the same analysis as above applies. Some caution is required. In the completely symmetric case we could relabel the bidders as we wanted, whereas in the more general case the bidder with the highest threshold needs to be a bidder with the highest cardinality of message space.

[24] Bidder with treshold c_j also has tresholds c_{j-3} and c_{j+3} under the priority assignment.

Before proceeding to revenue maximization let us point out another interesting result.

Corollary 2 *Let $K = 3k$ for some $k \in \mathbb{N}$, $\kappa = \{(k_1, k_2, k_3) : k_i \in \mathbb{N}$ for $i \in \{1, 2, 3\}$ and $k_1 + k_2 + k_3 = K\}$. Then*

$$w^*_{3,k} = \max_{(k_1,k_2,k_3)\in\kappa} w^*_{3,(k_1,k_2,k_3)}.$$

The result can be easily modified for K that is not divisible by 3. The Corollary states that the highest welfare among mechanisms with the same total number of messages is achieved by the one in which all the bidders have the same cardinality of the message space.[25] If the social planner (the seller) could chose how to distribute communication capacity he would allocate it evenly over the bidders. Or differently, one could think of the seller as assigning a fixed set of thresholds. He assigns the highest threshold to bidder 1, the second highest to bidder 2, etc.

The result given by the corollary follows from the proof since we never use the restriction $\left|\iota^{-1}\{i\}\right| = k - 1$. Therefore, ι^* does the best for any set of thresholds even when assignments can distribute thresholds arbitrarily.

As in the 2 bidder case we can now characterize the welfare optimizing equilibria for the case $v_0 > 0$. We assume $k_1 = k_2 = k_3 = k$, although as before the analysis is easily modified for the case $\max\{k_1, k_2, k_3\} - \min\{k_1, k_2, k_3\} \leq 1$. As in the two bidders case we call an equilibrium a priority equilibrium if it is as in the Theorem 2. To define the modified priority equilibrium we define the appropriate thresholds.

Let

$$\begin{aligned}
c^{\#}_{3k-3} &= v_0 \qquad (6)\\
c^{\#}_{3k-4} &= \frac{F\left(c^{\#}_{3k-6}\right)-F\left(c^{\#}_{3k-3}\right)}{F\left(c^{\#}_{3k-6}\right)} E\left[X \mid c^{\#}_{3k-6} \geq X \geq c^{\#}_{3k-3}\right] + \frac{F\left(c^{\#}_{3k-3}\right)}{F\left(c^{\#}_{3k-6}\right)} v_0\\
c^{\#}_{3k-5} &= \frac{F\left(c^{\#}_{3k-7}\right)-F\left(c^{\#}_{3k-4}\right)}{F\left(c^{\#}_{3k-7}\right)+F\left(c^{\#}_{3k-6}\right)-F\left(c^{\#}_{3k-7}\right)F\left(c^{\#}_{3k-6}\right)} E\left[X \mid c^{\#}_{3k-7} \geq X \geq c^{\#}_{3k-4}\right] +\\
&\quad + \frac{F\left(c^{\#}_{3k-6}\right)-F\left(c^{\#}_{3k-3}\right)}{F\left(c^{\#}_{3k-7}\right)+F\left(c^{\#}_{3k-6}\right)-F\left(c^{\#}_{3k-7}\right)F\left(c^{\#}_{3k-6}\right)} E\left[X \mid c^{\#}_{3k-6} \geq X \geq c^{\#}_{3k-3}\right] +\\
&\quad + \frac{F\left(c^{\#}_{3k-4}\right)F\left(c^{\#}_{3k-3}\right)}{F\left(c^{\#}_{3k-7}\right)+F\left(c^{\#}_{3k-6}\right)-F\left(c^{\#}_{3k-7}\right)F\left(c^{\#}_{3k-6}\right)} v_0.
\end{aligned}$$

Thresholds $c^{\#}_1$ through $c^{\#}_{3k-6}$ are as in Theorem 2 after replacing indices * with $^{\#}$. For the

[25] The requirement is that each bidder gets at least one message.

strategies defined by $c^{\#}$, the optimal allocation rule $Q^{\#}$ allocates the object as Q^{*}, except in the case where all 3 bidders report their lowest partition cell, in which case the object is retained by the seller. Now we can say that an equilibrium is a modified priority equilibrium if the reporting strategies can be described by thresholds satisfying conditions (6), a priority assignment, ι^{*} and the allocation rule $Q^{\#}$. Clearly this can be supported as an equilibrium by the Vickrey type transfers as before.

Next theorem is analogous to Proposition 5.

Theorem 3 *Let $v_0 > 0$. Optimal welfare is achieved in a priority or in a modified priority equilibrium.*

We do not provide a formal proof, as it is easily obtained by combining considerations given in proof of Theorem 2 and Proposition 5.

The theorem states that when the seller's value is strictly positive it either does not matter and the optimal equilibrium is as in the case $v_0 = 0$, or it does matter and the thresholds are given by the system of equations (6). In the later case the object is allocated as in the optimal priority equilibrium except when all the bidders report their lowest partition cell in which case the seller retains the object.

4 Revenue Maximization

So far the focus was on welfare rather than revenue maximization. This is purely for expositional value, as the problem of revenue maximization can be transformed into welfare maximization (see Myerson (1981)).

Revenue maximization hardly needs an introduction. Seminal work by Myerson (1981) and Riley and Samuelson (1981) has laid a cornerstone for many a work since. The two afore mentioned papers impose no restriction on communication, neither does most of the subsequent literature. The prominent counterexample is the paper by BNS. They provide a characterization of profit maximizing equilibria for the case of 2 bidders where each of them has a regular but not necessarily identical distribution and for the case of I bidders where each has 2 possible messages and all have the same regular distribution over valuations.

We provide a solution to a more general problem by showing that the profit maximizing seller distributes communication capacity evenly over the bidders and characterizing profit maximizing equilibria under such a uniform distribution. Even for the case where each bidder has only two possible messages our characterization is sharper than the one of BNS. While allowing for any number of bidders, we require they all have the same regular distribution function, F, over $[0,1]$.[26]

Let F be the distribution function and f its density. f is regular if the virtual valuation, $\psi(v) \equiv v - \frac{1-F(v)}{f(v)}$, is a monotone, strictly increasing function of v. The key observation we use is the one of Myerson (1981) showing equivalence between maximal expected revenue and expected value of the maximum of virtual valuation (provided it is nonnegative). Myerson was, however, operating under no restrictions on communication. Nothing prevents us from replicating Myerson's exercise even under the assumption of restricted message spaces. The main point from Myerson's exercise that is relevant to us is that for every profile of valuation v the object is allocated to the bidder with the highest virtual valuation given that it surpasses 0; otherwise the auctioneer retains the object.

The problem of revenue maximization can be solved the following way. Using Myerson's techniques it can be transformed into the maximization over virtual valuations. Now one can solve the optimal welfare problem, as above, where virtual valuations are treated as actual valuations. Notice that the original distribution over the valuation of the bidder is not the distribution of the virtual valuation. The issue is easily dealt with when the virtual valuation is strictly increasing, which is why we made the assumption of regular distributions in the first place. In particular, if F is the distribution of the valuation, $G(t) = F\left(\psi^{-1}(t)\right)$ is the distribution of the virtual valuation.

Now is also clearer why attention was devoted to the welfare maximization under the assumption that the auctioneer has a positive value of the object. Even if we are solving the revenue maximization problem under the assumption that the seller has no value for the object, the transformed problem might be one of welfare maximization under the assumption that the seller's value is above the lowest value bidders could entertain. For example if all the bidders have uniformly distributed values over $[0,1]$, the transformed problem has the values uniformly distributed over the interval $[-1,1]$ and the seller has value $v_0 = 0$. After one solves for optimal thresholds in the transformed problem, one can map those strategies into strategies in the original problem.

Theorem 4 *Let* $|M_i| = k_i$ *and* $F_i = F$ *for every* $i \in \{1,2,...,I\}$ *and some regular distribution function* F*, and let* $\max\{k_1,k_2,...,k_i\} - \min\{k_1,k_2,...,k_i\} \leq 1$*. Then the revenue maximizing individually rational equilibrium is achieved by a modified priority equilibrium.*

[26] As pointed out earlier BNS require the same assumption for the case of more than 2 bidders.

As a clarification, a modified priority equilibrium in a literal sense corresponds to the modified priority equilibrium of the auxiliary welfare maximization problem of maximization over virtual valuations. The result obtained in such a problem has to be mapped back into our primary setup. Notice that the characterization here is stronger than the one in BNS. Their result states that the welfare maximizing equilibrium is achieved in a priority or in a modified priority equilibrium (BNS Theorem 4.2).

Now, an analogous result to Corollary 1 can be stated for revenue maximization.

Corollary 3 *Let* $K = 3k$ *for* $k \in \mathbb{N}$, $k \geq 2$, *and* $\kappa = \{(k_1, k_2, k_3) : k_i \in \mathbb{N} \text{ for } i \in \{1, 2, 3\} \text{ and } k_1 + k_2 + k_3 = K\}$.[27] *Then*

$$\pi_k^* = \max_{(k_1,k_2,k_3)\in\kappa} \pi_{k_1,k_2,k_3}^*.$$

Results from welfare maximization can be naturally imbedded into the revenue maximization environment. Among the mechanisms using K messages altogether, mechanisms with uniform distribution of messages over the bidders do the best. To put it differently, if the seller can choose how to allocate communication capacity he should allocate it evenly. After noticing symmetry between the problem of welfare and profit maximization, we can provide an analogous result to Proposition 2. Let k be defined as in Proposition 2.

Proposition 6

$$\pi_{k_1,k_2,\ldots,k_I}^* = \pi_{k,k_2,\ldots,k_I}^*$$

Since the problem of profit maximization is solved by converting it into the problem of welfare maximization the result is immediate.

5 Multiple Identical Units and Single Unit Demand

In an IPV setting with no restrictions on communication, the problem of allocating multiple identical units to bidders with a single unit demand is treated much like the problem of allocating a single object. Under no restrictions on communication, one fixes the reporting strategies of bidders so that they report truthfully and modifies the allocation rule and transfers in the obvious way. Our setup, on the other side, requires that we also deal with the reporting strategies.

[27] Assumption that K is divisible by 3 is made only for clarity of exposition.

In this section we provide a characterization of welfare optimal equilibria when each bidder has single unit demand and multiple identical objects are to be sold. We provide some general results as well as a more precise characterization in the symmetric case. The case of three bidders is considered, as more general case would bring about notational chaos while contributing very little to the exposition.

The only interesting case, of course, is the one of selling two objects. The case of three objects is rather trivial since the welfare is optimized by awarding an object to each of the bidders without any questions asked.

Let F_1, F_2 and F_3 be distributions of bidders' valuations. As earlier we require that distributions be atomless with everywhere positive density. For convenience we assume they are on $[0, 1]$. As before, let $|M_1| = k_1, |M_2| = k_2$ and $|M_3| = k_3$. We assume that $v_0 = 0$ for ease of exposition. Analysis is easily extended to the case $v_0 > 0$ by use of techniques developed above.

With a slight modification a mechanism can be defined much like in Section 2. Although we still require $Q_i(m) \in [0, 1]$ for every profile of messages m, since we have single unit demand, we now have

$$\sum_{i=1}^{3} Q_i(m) \leq 2.$$

The results from the beginning of Section 2 still go through. That is, an equilibrium strategy can be replicated by a threshold strategy and optimal welfare can be achieved by an equilibrium in threshold strategies. Furthermore, any threshold strategy is an equilibrium of the mechanism in which, given the reports and the strategies, the two bidders with the highest expected values obtain the object. The transfers to support such equilibria are the Vickrey type transfers adapted to the environment: a bidder's payment equals the lowest value he could have, report according to the equilibrium strategy and win the object.

Natural modification of Proposition 2 can be stated for the environment at hand. Expected values given the strategy and reported message can be calculated for bidders 2 and 3. This yields at most $k_2 + k_3$ values. On the other side, as opposed to the single unit supply, we do not really care whether bidder 1's valuation is the highest or the second highest. Either way he is awarded a unit. Therefore at most $k_2 + k_3$ messages of bidder 1 are welfare relevant.

Proposition 7 *Let $k = \min\{k_1, k_2 + k_3\}$. Then*

$$w^*_{3,(k_1,k_2)} = w^*_{3,(k,k_2)}.$$

In the remainder of the section we provide a characterization of welfare optimizing equilibria, as in the single object case, for the case of symmetric bidders, i.e. $F_1 = F_2 = F_3 = F$ and $|M_1| = |M_2| = |M_3| = k$.

Let Q^* be defined as follows. Suppose bidders 1, 2 and 3 report messages m_s, m_p and m_q respectively. Then $Q_1^*(m_s, m_p, m_q) = 1$ as long as $\max\{p, q\} \geq s$ and $Q_1^*(m_s, m_p, m_q) = 0$ otherwise. $Q_2^*(m_s, m_p, m_q) = 1$ if $p \leq q$ or $s > p$ and $Q_2^*(m_s, m_p, m_q) = 0$ otherwise. Finally $Q_3^*(m_s, m_p, m_q) = 1$ if $\max\{s, p\} > q$ and $Q_3^*(m_s, m_p, m_q) = 0$ otherwise.

Q^* awards an object to bidder 1 if neither of the remaining bidders reports a higher ranked message, to bidder 2 if bidder 3 does not report a higher ranked message or bidder 1 reports a lower ranked message and to bidder 3 if at least one of the other two bidders reports a lower ranked message. It is easy to verify that such an allocation rule satisfies the above requisite properties.

Theorem 5 *The optimal welfare is achieved by the allocation rule Q^* and the profile of threshold strategies characterized by the assignment rule ι^* and the set of thresholds $\{c_1^*, c_2^*, ..., c_{3k-3}^*\}$ given by:*

$$\begin{aligned} c_j^* &= \frac{\left[1-F\left(c_{j-1}^*\right)\right]\left[F\left(c_{j-2}^*\right)-F\left(c_{j+1}^*\right)\right]}{F\left(c_{j-2}^*\right)+F\left(c_{j-1}^*\right)-F\left(c_{j-1}^*\right)F\left(c_{j-2}^*\right)-F\left(c_{j+1}^*\right)-F\left(c_{j+2}^*\right)+F\left(c_{j+1}^*\right)F\left(c_{j+2}^*\right)} E\left[X \mid c_{j-2}^* \geq X \geq c_{j+1}^*\right] \\ &\quad + \frac{\left[1-F\left(c_{j+1}^*\right)\right]\left[F\left(c_{j-1}^*\right)-F\left(c_{j+2}^*\right)\right]}{F\left(c_{j-2}^*\right)+F\left(c_{j-1}^*\right)-F\left(c_{j-1}^*\right)F\left(c_{j-2}^*\right)-F\left(c_{j+1}^*\right)-F\left(c_{j+2}^*\right)+F\left(c_{j+1}^*\right)F\left(c_{j+2}^*\right)} E\left[X \mid c_{j-1}^* \geq X \geq c_{j+2}^*\right], \end{aligned}$$

for $j \in \{1, 2, ..., 3k - 3\}$.

Again, $c_j^* = 0$ for $j > 3k - 3$ and $c_j^* = 1$ for $j < 1$. The intuition can be grasped through the following example. Suppose bidder 1 observes value c_4^*. Given his value it should not matter whether he reports message m_2 corresponding to $[c_4^*, c_1^*]$, or message m_3 corresponding to $[c_7^*, c_4^*)$. If the conditional welfare were different in those two cases, suppose it were strictly higher when bidder 1 reports m_2. Then by continuity the same would be true for values just above c_4^*. But then the original threshold could possibly not be optimal.

Whether bidder 1 is reporting m_2 or m_3 has an effect on welfare in two cases: (1) when bidder 3 reports the partition cell $[c_3^*, 1]$ and bidder 2 reports $[c_5^*, c_2^*]$ and (2) when bidder 3 reports the partition cell $[c_6^*, c_3^*]$ and bidder 2 reports either $[c_5^*, c_2^*]$ or $[c_2^*, 1]$. But then at the optimum c_4^* should be set to offset those two possibilities.

Finally we can write the expected welfare of an optimal equilibrium as

$$\begin{aligned} w_k^* &= \sum_{j=1}^{3k-2} \left[3F\left(c_{j-1}^*\right) - F\left(c_{j-1}^*\right)^2 + 3F\left(c_j^*\right) - F\left(c_j^*\right)^2 - F\left(c_{j-1}^*\right) F\left(c_j^*\right)\right] \left[F\left(c_{j-1}^*\right) - F\left(c_j^*\right)\right] * \\ &\quad *E\left[X \mid c_{j-1}^* \geq X \geq c_j^*\right] - \sum_{j=1}^{3k-2} \left[1 - F\left(c_j^*\right)\right] \left[F\left(c_j^*\right) - F\left(c_{j+1}^*\right)\right] \left[F\left(c_{j-1}^*\right) - F\left(c_j^*\right)\right] * \\ &\quad * \left[E\left[X \mid c_{j-1}^* \geq X \geq c_j^*\right] - E\left[X \mid c_j^* \geq X \geq c_{j+1}^*\right]\right]. \end{aligned}$$

The first line can be interpreted as the welfare which one would achieve if all the bidders had thresholds $\left\{c_1^*, c_2^*, ..., c_{3k-3}^*\right\}$. The second line is the loss incurred because one cannot do so.

6 Sequential Reporting

6.1 Short Introduction

So far we were explicitly assuming simultaneous reporting.[28] BNS provide an example with two bidders, the uniform distribution over $[0, 1]$ and two messages for each bidder to show that a sequential mechanism can achieve strictly higher welfare with the same amount of communication. More precisely, the seller commits to a mechanism, one of the bidders reports first (bidder 1), the report is revealed to the second bidder after which the second bidder reports himself. Letting the bidder who reports second condition on the first bidder's report enables one to achieve strictly higher welfare than in the case when no such conditioning is allowed. It is true, though, that communication by which the second bidder is informed of first bidder's report is not accounted for.

The observation above contrasts the revelation principle (see Myerson (1979) and Myerson (1981)), which implies that any welfare obtained in an equilibrium of an auction mechanism can be obtained by an equilibrium of a direct revelation mechanism. Welfare of the optimal mechanism with sequential reporting in which bidders use a certain number of messages can not be replicated by the simultaneous reporting of bidders with the same total number of messages for each bidder.

Clearly, welfare achieved in a Bayesian equilibrium of a mechanism with restricted communica-

[28] The bidders could report sequentially but no information should be transmitted between them.

tion can be replicated by a mechanism in which there are no communication requirements. BNS show that at least as high a welfare as in a sequential I-bidder mechanism with m bits[29] can be achieved in an equilibrium of a simultaneous mechanism with mI bits (Theorem 6.1). Again, the communication by which bidders are informed about other bidders' reports is not accounted for in the calculation. While BNS provide a bound we show an exact characterization for a two bidder two period case.

In what follows we solve for welfare and profit maximizing equilibria of the following types of two bidders mechanisms. The seller commits to a mechanism and bidders observe their private values. First to report is bidder 1; he chooses one of a finite number of messages. Bidder 2 observes it and chooses one of the finite number of messages himself. Finally the allocation and transfers are executed on the basis of the two reports. Two assumptions should be emphasized: First, there are only two periods of reporting. While this is more general than simultaneous reporting it is still somewhat restrictive. Second, requiring that in each period only one bidder sends a message is restrictive given that we are using only two periods of reporting. Finally one can observe that our equilibria achieve the highest welfare among all two bidders 1 bit mechanisms, be it simultaneous or sequential.[30]

6.2 Model

The assumptions on preferences and on distributions of the two bidders are the same as in the previous sections. The differences are in timing, information, and consequently, in strategies. We assume that bidder 1 reports first: he observes his private valuation v_1 and reports a message, i.e.

$$\mu_1 : V_1 \to M_1.$$

Bidder 2 reports after observing his own valuation and bidder 1's message, i.e.

$$\mu_2 : V_2 \times M_1 \to M_2.$$

[29] m bits $= 2^m$ messages.

[30] If one is willing to sidestep dealing with bits, one can state, more generally, that this kind of equilibrium is optimal if each bidder can use up to at most 3 messages. In that case no more than two periods make sense.

On the other side a mechanism is defined as earlier. The allocation rule is

$$Q : M_1 \times M_2 \to [0,1]^3 ,$$

and the transfer scheme

$$T : M_1 \times M_2 \to \mathbb{R}^2_+ .$$

We impose the following incentive constraints. After bidder 2 observes $\mu_1(v_1)$:

$$\begin{aligned} & Q_2(\mu_1(v_1), \mu_2(v_2, \mu_1(v_1)))\, v_2 - T_2(\mu_1(v_1), \mu_2(v_2, \mu_1(v_1))) \\ \geq \;& Q_2\left(\mu_1(v_1), \mu_2\left(v_2', \mu_1(v_1)\right)\right) v_2 - T_2\left(\mu_1(v_1), \mu_2\left(v_2', \mu_1(v_1)\right)\right), \end{aligned} \tag{7}$$

for every v_2, v_2'. Bidder 1 has no prior information but his own signal, thus for every v_1 :

$$\begin{aligned} & \int \left[Q_1(\mu_1(v_1), \mu_2(v_2, \mu_1(v_1)))\, v_1 - T_1(\mu_1(v_1), \mu_2(v_2, \mu_1(v_1)))\right] f_2(v_2)\, dv_2 \\ \geq \;& \int \left[Q_1\left(\mu_1(v_1'), \mu_2\left(v_2, \mu_1(v_1')\right)\right) v_1 - T_1\left(\mu_1(v_1'), \mu_2\left(v_2, \mu_1(v_1')\right)\right)\right] f_2(v_2)\, dv_2 \end{aligned} \tag{8}$$

for every $v_1' \in [0,1]$. The above IC constraints are somewhat generous: we only require that under the prescribed strategies none of the bidders wants to deviate to an action that he was supposed to be played for some valuation. Deviations to messages that were not supposed to be used according to his prescribed strategy are allowed. We solve the welfare and the profit maximization problem under this *relaxed* problem and argue that the solution is indeed a Bayesian equilibrium.

Equivalently loose participation constraints are:

$$Q_2(\mu_1(v_1), \mu_2(v_2, \mu_1(v_1)))\, v_2 - T_2(\mu_1(v_1), \mu_2(v_2, \mu_1(v_1))) \geq 0,$$

for every v_1 and v_2, for bidder 2, and

$$\int \left[Q_1(\mu_1(v_1), \mu_2(v_2, \mu_1(v_1)))\, v_1 - T_1(\mu_1(v_1), \mu_2(v_2, \mu_1(v_1)))\right] f_2(v_2)\, dv_2 \geq 0$$

for every v_1 for bidder 1. By using methodology of Lemma 1 one can easily see that strategies satisfying conditions (7) and (8) have to be threshold strategies.[31] Perhaps a clarification is in

[31] Again, up to equating messages for which the bidder wins with the same expected probability.

order: μ_2 is a threshold strategy with respect to μ_1 if $\mu_2(\cdot, \mu_1(v_1))$ is a threshold strategy for every v_1. Then $\mu = (\mu_1, \mu_2)$ is in threshold strategies if μ_1 is a threshold strategy and μ_2 is a threshold strategy with respect to μ_1. The proof of the following Lemma is omitted because of its simplicity.

Lemma 4 *For any profile of strategies μ and a mechanism (Q, T) satisfying IC conditions μ is in threshold strategies.*

We denote bidder 1's thresholds by $c = (c_1, ..., c_{k_1-1})$ where $c_1 \geq c_2 \geq ... \geq c_{k_1-1}$. As above, the message corresponding to bidder 1's top partition cell is m_1, to second highest m_2, etc. Bidder 2's thresholds after observing message m_j, $j \in \{1, 2, ..., k_1\}$, are $d(m_j) = \left(d_1^j, d_2^j, ..., d_{k_2-1}^j\right)$. A strategy of bidder 2 is $d = (d(m_1), d(m_2), ..., d(m_{k_1}))$. Remainder of the analysis draws heavily on the results obtained in the setup with simultaneous reporting; therefore we are somewhat less formal. We will neglect the incentive constraints at first. That is, we will try to find the largest welfare one can obtain in threshold strategies assuming that bidders blindly follow the prescribed strategies. In the end we will show that such strategies and the optimal allocation rule can indeed be supported as a Bayesian equilibrium.

For any fixed profile of threshold strategies given by (c, d), the optimal allocation rule Q^* adheres to the characterization of Lemma 2. Given the threshold strategies and the reports, the object is allocated to the bidder with the highest expected value. In any equilibrium in the second round of reporting, bidder 1's message and the corresponding expected value are known; therefore the only relevant information from bidder 2 is whether his value is above or below the expected value of bidder 1. But then at most two message of bidder 2 are relevant, i.e. there is one threshold. We label the two messages by U and D, the first corresponding to the upper partition cell and the later to the lower.[32] With a slight abuse of notation we denote bidder 2's threshold after observing message m_j by d_j; now, $d = (d_1, d_2, ..., d_{k_1})$. Making use of Proposition 2 and the fact that bidder 1 has k_1 messages, one observes that there is no need to partition bidder 2's valuation interval into more than $k_1 + 1$ partition cells. This, in turn, means that the highest welfare achieved in our mechanism with two periods of sequential reporting is no larger than the welfare one can achieve by simultaneous reporting where bidder 1 has k_1 and bidder 2, $k_1 + 1$ messages, i.e. $w^*_{2,(k_1,k_1+1)}$. In what follows we show that this upper bound on welfare can be achieved.

[32] To be more precise. If $k_2 > 2$ one should specify what happens if bidder 2 announces one of the messages that are not supposed to be reported in the equilibrium. There are several ways of remedying this problem, one of them being: if bidder 2 announces a message which was not supposed to occur he obtains the object with probability zero and does not pay.

Let c^* and d^* be mutually centered strategies, as given by the System of Equations (2), i.e. $d_1^* = E[V_1 \mid 1 \geq V_1 \geq c_1^*]$, $c_1^* = E[V_2 \mid d_1^* \geq V_1 \geq d_2^*]$, etc. Here d_j^* is bidder 2's threshold after bidder 1 reported message m_j, for $j \in \{1, ..., k_1\}$. The optimal allocation rule, Q^*, awards the object to the bidder with the highest expected value:

$$\begin{aligned} Q_1^*(m_j, U) &= 1 - Q_2^*(m_j, U) = 0, \\ Q_1^*(m_j, D) &= 1 - Q_2^*(m_j, D) = 1, \end{aligned}$$

for all $j \in \{1, 2, ..., k_1\}$. Now we are only left to specify the transfers to support mutually centered strategies (c^*, d^*) and the allocation rule Q^* as an equilibrium. Bidder 2's transfers are[33]

$$\begin{aligned} T_2^*(m_j, U) &= d_j^*, \\ T_2^*(m_j, D) &= 0. \end{aligned} \tag{9}$$

for $j \in \{1, ..., k_1\}$. These are the usual Vickrey type transfers. If bidder 2 wins, the payment he has to make is equal to the smallest valuation he could have, report according to his threshold and still win. Bidder 1's transfers are somewhat more interesting:

$$\begin{aligned} T_1^*(m_j, U) &= 0, \\ T_1^*(m_j, D) &= \sum_{i=j}^{k_1} \frac{F_2(d_i^*) - F_2(d_{i+1}^*)}{F_2(d_j^*)} c_i^*, \end{aligned}$$

for all $j \in \{1, ..., k_1\}$. Bidder 1's transfers can be interpreted as expected Vickrey transfers. That is, if bidder 1 wins after reporting m_j, bidder 2's message only reveals that bidder 2's valuation is in $\left[0, d_j^*\right]$. If bidder 1 would have reported m_{j+1} the information obtained from bidder 2 would have been different. The transfer of bidder 1 accounts for that. When he reports m_j he wins the object with probability $F_2\left(d_j^*\right)$. Given that he wins, bidder 2's valuation is in the interval $\left[d_{j+1}^*, d_j^*\right]$ with probability $\frac{F_2(d_j^*) - F_2(d_{j+1}^*)}{F_2(d_j^*)}$, in which case bidder 1 could have value c_j^*, report according to his strategy and still win. Given that bidder 1 wins, bidder 2's valuation is in the interval $\left[d_{j+2}^*, d_{j+1}^*\right]$ with probability $\frac{F_2(d_{j+1}^*) - F_2(d_{j+2}^*)}{F_2(d_j^*)}$, whence bidder 1 would need to value the object at least c_{j+1}^*,

[33] Notice that the optimal transfers depend, not only on the reports, but on the treshold strategies (c^*, d^*). Presently we supress this in notation hopeing that no confusion will arise.

report according to his prescribed threshold strategy and still win. The rest of the explanation follows the same pattern.

Proposition 8 *Optimal welfare of a two period mechanism with sequential reporting can be achieved in a Bayesian equilibrium with mutually centered strategies,* (c^*, d^*)*, and the mechanism* (Q^*, T^*)*.*

All the usual disclaimers apply. Mutual centeredness is a necessary condition for an equilibrium; thus there might exist mutually centered strategies that are not welfare optimizing. Furthermore, we lay no claims on uniqueness of a welfare optimal equilibrium. Since the optimal transfers depend directly on the strategies, the same observations apply to them. On the other side, the allocation rule is pinned down. Though there might be multiple mutually centered profiles of strategies that achieve the optimal equilibrium, in all of the optimal equilibria the allocation rule is the same, i.e. Q^*.

The above analysis deals with the case $v_0 = 0$. Analysis can be easily extended to the case $v_0 > 0$. Fix v_0. Let $c^{\#}$ and $d^{\#}$ be vectors of thresholds as given in the system of equations (6),[34] i.e.

$$\begin{aligned} d^{\#}_{k_1} &= v_0, \\ c^{\#}_{k_1-1} &= \tfrac{F_2\left(d^{\#}_{k_1-1}\right) - F_2\left(d^{\#}_{k_1}\right)}{F_2\left(d^{\#}_{k_1-1}\right)} E\left[V_2 \mid d^{\#}_{k_1-1} \geq V_2 \geq d^{\#}_{k_1}\right] + \tfrac{F_2\left(d^{\#}_{k_1}\right)}{F_2\left(d^{\#}_{k_1-1}\right)} v_0, \end{aligned}$$

etc. The accompanying allocation rule, $Q^{\#}$, is as Q^* except for the reports (m_{k_1}, D) when the object is retained by the seller. The supporting transfers are similar as above. Bidder 2, his transfers denoted $T_2^{\#}$, pays the lowest value he could have, report according to his threshold strategy, given bidder 1's report, and still win. Bidder 1's transfers are expected Vickrey transfers modified to account for the fact that the seller might retain the object:

$$\begin{aligned} T_1^{\#}(m_j, U) &= T_1^{\#}(m_{k_1}, D) = 0, \text{ for all } j \in \{1, 2, ..., k_1\} \\ T_1^{\#}(m_j, D) &= \sum_{i=j}^{k_1-1} \tfrac{F_2\left(d_i^{\#}\right) - F_2\left(d_{i+1}^{\#}\right)}{F_2\left(d_j^{\#}\right)} c_i^{\#}, \text{ for all } j \in \{1, 2, ..., k_1 - 1\}. \end{aligned}$$

The result can now be stated to say that the welfare optimal equilibrium is achieved in an equilibrium with mutually centered strategies (c^*, d^*) and (Q^*, T^*) or in an equilibrium with modified

[34] $\left(c^{\#}, d^{\#}\right)$ depends on v_0, although we suppress this in notation.

mutually centered strategies $\left(c^{\#}, d^{\#}\right)$ and the mechanism $\left(Q^{\#}, T^{\#}\right)$.

Finally, one can tackle the problem of revenue maximization by transforming it into the auxiliary problem of welfare maximization in which virtual valuations are treated as actual valuations. After solving the auxiliary problem one can map the solution back into the original setup.

7 Concluding Remarks

We provide a novel analysis of auctions with restricted communication. Several lessons are to be taken from it. First, when communication is restricted the revelation principle does not apply in the usual sense: simultaneous reporting of bidders is not without loss of generality. Second, communication transmitted by the bidder with the highest cardinality of the message space is bounded by the sum of cardinalities of the message spaces of all the remaining bidders (plus one). Relevant information stems from comparison of bidders' valuations. Third, when bidders have symmetric distributions the seller optimally distributes communication capacity evenly over the bidders, although the auction they bid in afterwards is asymmetric. Finally, we provide an equivalence result between the maximal welfare of a certain mechanism with simultaneous reporting and the maximal welfare achieved in a two bidder two period sequential mechanism.

We explore the effects of bounded communication in the best understood auction environment, that is, in the IPV model with a single indivisible object. Although we provide quite a comprehensive analysis much work is still to be done. In future work we plan to explore restricted communication in various other settings, such as auctions with interdependent values (some preliminary work was done by Blumrosen and Feldman (2006)) and bilateral bargaining.

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Appendix A. Proofs of Subsection 3.1

Proof of Lemma 1. Let μ be an equilibrium of an incentive compatible mechanism (Q,T). Let $v_i < v_i'$ be such that $\mu_i(v_i) = \mu_i(v_i')$ and $v_i'' \in [v_i, v_i']$. By incentive compatibility we have

$$q\left(\mu_i(v_i), \mu_{-i}\right) v_i - t_i\left(\mu_i(v_i), \mu_{-i}\right) \geq q\left(\mu_i\left(v_i''\right), \mu_{-i}\right) v_i - t_i\left(\mu_i\left(v_i''\right), \mu_{-i}\right).$$

By interchanging v_i and v_i'' and consolidating the two inequalities one obtains

$$\left(q_i\left(\mu_i(v_i), \mu_{-i}\right) - q_i\left(\mu_i\left(v_i''\right), \mu_{-i}\right)\right)\left[v_i - v_i''\right] \geq 0,$$

which in turn implies $q_i\left(\mu_i(v_i), \mu_{-i}\right) \leq q_i\left(\mu_i\left(v_i''\right), \mu_{-i}\right)$. Similarly $q_i\left(\mu_i\left(v_i''\right), \mu_{-i}\right) \leq q_i\left(\mu_i\left(v_i'\right), \mu_{-i}\right)$. Now, since $\mu_i(v_i) = \mu_i(v_i')$ we have $q_i\left(\mu_i(v_i), \mu_{-i}\right) = q_i\left(\mu_i\left(v_i''\right), \mu_{-i}\right) = q_i\left(\mu_i\left(v_i'\right), \mu_{-i}\right)$. Therefore either $\mu_i(v_i'') = \mu_i(v_i)$ or $\mu_i(v_i'') \neq \mu_i(v_i)$, but then one can identify $\mu_i(v_i'')$ with $\mu_i(v_i)$ since $q_i\left(\mu_i(v_i), \mu_{-i}\right) = q_i\left(\mu_i\left(v_i''\right), \mu_{-i}\right)$ in any case. ■

Proof of Lemma 2. Fix the profile of reporting strategies μ. Let (Q,T) be some incentive compatible mechanism for μ. If for every realization of a message profile a bidder (or possibly bidders) with the highest expected value wins, we are done. Otherwise there exists some combination of reports $\mu(v)$ and two bidders i and j such that $E[V_i \mid \mu_i(V_i) = \mu_i(v_i)] < E\left[V_j \mid \mu_j(V_j) = \mu_j(v_j)\right]$ and $q_i\left(\mu_i(v_i), \mu_{-i}\right) > 0$. It is easy to see that retaining the same thresholds, given by the strategy μ, and always awarding the object to the bidder with the highest expected value conditional on the reports yields no smaller expected welfare. Let $Q^*(\mu)$ be some such allocation rule. We only need to check that this allocation can actually be supported by transfers. Indeed, Vickrey transfers tailored to our setup will do: if a bidder wins the object with positive probability, he pays the lowest possible valuation he could have, report according to his strategy and still win with positive probability, say p, multiplied by p. To make it even simpler, one can take $Q^*(\mu)$ to be deterministic and the winning bidder pays the smallest valuation he could have, report according to his strategy and still win the object.

The other direction is immediate. ■

Proof of Theorem 1. By Lemma 1 and Lemma 2 we can confine ourselves to threshold strategies, c, and an allocation rules of type $Q^*(c)$. Now we only need to establish the existence of an optimal threshold strategy.

Let C^i be the set of all threshold strategies for bidder i, i.e. $C^i = \left\{c^i \in [0,1]^{k_i - 1} : c^i_1 \geq c^i_2 \geq ... \geq c^i_{k_1-1}\right\}$. Using the properties of $Q^*(c^*)$, welfare as a function of a profile of threshold strategies, c, can now be written as

$$\begin{aligned} w(c) &= \sum_{t_1=1}^{k_1-1}\sum_{t_2=1}^{k_2-1}...\sum_{t_I=1}^{k_I-1} \Pr\left(c^1_{t_1} \geq V_1 \geq c^1_{t_1+1}\right)\Pr\left(c^2_{t_2} \geq V_2 \geq c^2_{t_2+1}\right)\cdot ... \cdot \Pr\left(c^I_{t_I} \geq V_n \geq c^I_{t_I+1}\right) \qquad (10) \\ &\cdot \max\left\{E\left[V_1 \mid c^1_{t_1} \geq V_1 \geq c^1_{t_1+1}\right], E\left[V_2 \mid c^2_{t_2} \geq V_2 \geq c^2_{t_2+1}\right], ..., E\left[V_I \mid c^I_{t_I} \geq V_3 \geq c^I_{t_I+1}\right]\right\}, \end{aligned}$$

where $\Pr\left(c^i_{t_i} \geq V_i \geq c^i_{t_i+1}\right) = F_i\left(c^i_{t_i}\right) - F_i\left(c^i_{t_i+1}\right)$ for each i. Clearly the set $C = \mathop{\times}_{i=1}^{n} C^i$ is compact in $\mathbb{R}^{k_1+k_2+...+k_I-I}$. Furthermore, w is continuous in c, which establishes the existence of an optimum, i.e. c^* and $Q^*(c^*)$. We already argued in the text that c^* and $Q^*(c^*)$ can be supported by the Vickrey type transfers $T^*(c^*)$. ■

Proof of Proposition 1. Fix a strategy μ and an allocation rule Q. Transfers here are of no importance.

Welfare can be written as

$$\begin{aligned} w &= \sum_j \left\{\int \left[\sum_{m_{-j}} \Pr(m_{-j}) Q_j\left(\mu_j(v_j), m_{-j}\right)\right] v_j dF_j\right\} \qquad (11) \\ &= \int \left[\sum_{m_{-i}} \Pr(m_{-i}) Q_i(\mu_i(v_i), m_{-i})\right] v_i dF_i + \sum_{j\neq i}\left\{\int \left[\sum_{m_{-j}} \Pr(m_{-j}) Q_j\left(\mu_j(v_j), m_{-j}\right)\right] v_j dF_j\right\}, \end{aligned}$$

where

$$\Pr(m_{-j}) = \int \mathbf{1}_{\left[\mu_{-j}(v_{-j}) = m_{-j}\right]} dF_{-j}.$$

In addition, define

$$q_j\left(\mu_j(v_j)\right) = \left[\sum_{m_{-i}} \Pr(m_{-j}) Q_j\left(\mu_j(v_j), m_{-j}\right)\right]$$

for any $j \in \{1, 2, ..., I\}$.

Notice that the second term on the right hand side of the equation (11) depends only on the measure (induced by F_i) on which bidder i reports a certain message and not on particular values for which he does it. In particular, any strategy, μ_i', that uses each message with the same probability as strategy μ_i, leaves that term unaltered.

Without loss of generality (one can always relabel the messages)

$$q_i(m_{i1}) \geq q_i(m_{i2}) \geq ... \geq q_i(m_{ik}).$$

Now, we define a new threshold strategy, μ_i', for bidder i by

$$\begin{aligned}
\mu_i'(v_i) &= m_{i1} \text{ for } v_i \in [c_1, 1], \text{ where } c_1 \text{ is defined by } \int_{c_1}^{1} dF_i = \int \mathbf{1}_{[\mu_i(v_i)=m_{i1}]} dF_i, \\
\mu_i'(v_i) &= m_{i2} \text{ for } v_i \in [c_2, c_1], \text{ where } c_2 \text{ is defined by } \int_{c_2}^{c_1} dF_i = \int \mathbf{1}_{[\mu_i(v_i)=m_{i2}]} dF_i,
\end{aligned}$$

etc. Such a strategy is well defined. The claim is that by replacing strategy μ_i with μ_i' and holding strategies of other bidders and the allocation rule fixed the welfare does not decrease.

Indeed, clearly

$$\int_0^t q_i\left(\mu_i'(v_i)\right) dF_i \leq \int_0^t q_i\left(\mu_i(v_i)\right) dF_i$$

for every $t \in [0,1]$ and

$$\int_0^1 q_i\left(\mu_i'(v_i)\right) dF_i = \int_0^1 q_i\left(\mu_i(v_i)\right) dF_i.$$

Then by the first order stochastic dominance type of an argument

$$\int v_i q_i\left(\mu_i(v_i)\right) dF_i \leq \int v_i q_i\left(\mu_i'(v_i)\right) dF_i.$$

Going from μ_i to μ_i', we shift the mass at which bidder i is reporting message m_{i1} and therefore winning the object with the highest expected probability. This forms an interval for the highest valuations while keeping the probability with which message m_{i1} is reported constant. Next we shift the mass at which bidder i is reporting message m_{i2} just below the interval at which the bidder reports m_{1i} and so on. Clearly μ_i' is a threshold strategy. It is also rather easy to verify that it increases the first term in the above equation for welfare (we already claimed that the second term is unaltered) and thus it does not decrease welfare. Finally, we can repeat the procedure over all the bidders to obtain a threshold strategy μ' that achieves at least as high a welfare as μ did. Now we can show that the optimal threshold strategy exists the same way as in the proof of the Theorem 1. ■

Proof of Proposition 2. The proposition is trivial in the case $k_1 \leq \sum_{i=2}^{I} k_i + 1$. Let $k_1 > \sum_{i=2}^{I} k_i + 1$ and let μ^* be some welfare maximizing profile of strategies whose existence was proved in Theorem 1. For all the bidder 2 through I one can compute expected values corresponding to each of their partition cells and form a ranking of those expected values. The only information of importance under welfare maximization is whose value is the highest. All the welfare-relevant information is whether bidder 1's expected value is higher than the highest in the ranking of the remaining bidders, below the highest but above the second highest, etc., thus using at most $\sum_{i=2}^{I} k_i + 1$ messages. ■

Appendix B. Proofs of Subsection 3.2

Proof of Proposition 3. As already noted in the text, by Theorem 1 we can restrict ourselves to threshold strategies, and by Proposition 2 we only need to consider the case $|k_1 - k_2| \leq 1$.

The proof proceeds by induction over the cardinality of message space, where it is assumed that $k_1 \geq k_2$. Characterization is readily obtained when $k_1 = k_2 = 2$. We then show that a strictly higher welfare can be achieved with $k_1 = 3$, $k_2 = 2$ and all the messages are used and characterize the optimal strategies. One then proceeds the same way to the case of $k_1 = k_2 = 3$.

More formally. Characterization is easily obtained for the case $|M_1| = |M_2| = 2$. Each bidders' strategy is described by a single threshold. We denote bidder 1's threshold by c^1 and bidder 2's by c^2. Now, given the threshold strategies $c = \left(c^1, c^2\right)$ only two things can happen, either

$$E\left[V_1 \mid 1 \geq V_1 \geq c^1\right] \geq E\left[V_2 \mid 1 \geq V_2 \geq c^2\right] \tag{12}$$

and the optimal welfare (given the strategy described by the thresholds c) is

$$\begin{aligned} w(c) &= [1 - F_1(c_1)]\, E\left[X_1 \mid 1 \geq V_1 \geq c^1\right] + F_1\left(c^1\right)\left[1 - F_2\left(c^2\right)\right] E\left[V_2 \mid 1 \geq V_2 \geq c^2\right] \\ &\quad + F_1(c_1)\, F_2(c_2)\, E\left[X_1 \mid c^1 \geq V_1 \geq 0\right], \end{aligned}$$

or the roles of bidder 1 and 2 are reversed. Remember, the allocation we are using allocates, conditioning on the strategies, the object to the bidder with the highest expected value given the observed reports.

Either way, taking the first order conditions will give us mutual centeredness. As we already proved, the optimum exists, and it is clearly not on the boundary. Thus it is either the case that

$$\begin{aligned} c^{1*} &= E\left[V_2 \mid 1 \geq V_2 \geq c^{2*}\right], \\ c^{2*} &= E\left[V_1 \mid c^{1*} \geq V_1 \geq 0\right], \end{aligned} \tag{13}$$

or

$$\begin{aligned} c^{2*} &= E\left[V_1 \mid 1 \geq V_1 \geq c^{1*}\right], \\ c^{1*} &= E\left[V_2 \mid c^{2*} \geq V_2 \geq 0\right]. \end{aligned}$$

The remainder of the proof follows an inductive step. We show that the strategies have to be mutually centered for the case of one bidder having cardinality of message space three and the other two. From there we proceed to the case when both bidders have three possible messages.

Suppose $|M_1| = |M_2|+1 = 3$, the other case is handled the same way. In the optimal solution for the case of two messages for each bidder, $c^* = \left(c^{1*}, c^{2*}\right)$, it either has to be the case that inequality (12) holds with the strict inequality (mutual centeredness) or the reversed inequality holds.

In the **first case**, we have the following order (see the system of equations (13)):

$$\begin{aligned} E\left[V_1 \mid 1 \geq V_1 \geq c^{1*}\right] &> E\left[V_2 \mid 1 \geq V_2 \geq c^{2*}\right] > E\left[V_1 \mid c^{1*} \geq V_1 \geq 0\right] \\ &> E\left[V_2 \mid c^{2*} \geq V_2 \geq 0\right], \end{aligned}$$

and the welfare

$$\begin{aligned} w^*_{2,2} &= \left[1 - F_1\left(c^{1*}\right)\right] E\left[V_1 \mid 1 \geq V_1 \geq c^{1*}\right] + F_1\left(c^{1*}\right)\left[1 - F_2\left(c^{2*}\right)\right] E\left[V_2 \mid 1 \geq V_2 \geq c^{2*}\right] \\ &\quad + F_1\left(c^{1*}\right) F_2\left(c^{2*}\right) E\left[V_1 \mid c^{1*} \geq V_1 \geq 0\right]. \end{aligned}$$

Holding the first threshold of bidder 1 fixed, and relabeling it to $c_1^1 = c^{1*}$, we can set his second threshold, c_2^1, so that

$$E\left[V_1 \mid c_1^1 \geq V_1 \geq c_2^1\right] > E\left[V_2 \mid c^{2*} \geq V_2 \geq 0\right] > E\left[V_1 \mid c_2^1 \geq V_1 \geq 0\right],$$

thus making use of all three messages. We denote the welfare under these new strategies, and the corresponding new allocation rule, $w_{3,2}$. Now

$$\begin{aligned} w_{3,2} - w_{2,2}^* &= \left[F_1\left(c_1^1\right) - F_1\left(c_2^1\right)\right] F_2\left(c^{2*}\right) E\left[V_1 \mid c_1^1 \geq V_1 \geq c_2^1\right] + F_1\left(c_2^1\right) F_2\left(c^{2*}\right) E\left[V_2 \mid c^{2*} \geq V_2 \geq 0\right] \\ &\quad - F_1\left(c_1^1\right) F_2\left(c^{2*}\right) E\left[V_1 \mid c_1^1 \geq V_1 \geq 0\right] \\ &= F_1\left(c_2^1\right) F_2\left(c_2^*\right) \left\{E\left[V_2 \mid c^{2*} \geq V_2 \geq 0\right] - E\left[V_1 \mid c_2^1 \geq V_1 \geq 0\right]\right\} > 0. \end{aligned}$$

This shows that by utilizing all three messages of bidder 1 and two messages of bidder 2 we can achieve strictly higher welfare than by using 2 messages of each bidder. We denote the optimal threshold strategy for the case $|M_1| = |M_2| + 1 = 3$ by $c^* = \left(c_1^{1*}, c_2^{1*}, c_1^{2*}\right)$. The above statement implies that it has to be the case that

$$\begin{aligned} E\left[V_1 \mid 1 \geq V_1 \geq c_1^{1*}\right] &\geq E\left[V_2 \mid 1 \geq V_2 \geq c_1^{2*}\right] \geq E\left[V_1 \mid c_1^{1*} \geq V_1 \geq c_2^{1*}\right] \\ &\geq E\left[V_2 \mid c_1^{2*} \geq V_2 \geq 0\right] \geq E\left[V_1 \mid c_2^{1^*} \geq V_1 \geq 0\right]. \end{aligned}$$

Suppose not. Then in a ranking, as above, it would have to be the case that two bidder 1's expected values would come in consecutive, thus conveying the same message. But we already know that by utilizing all three messages a higher welfare can be achieved. The thresholds can now be characterized from the first order conditions:

$$\begin{aligned} c_1^{1*} &= E\left[V_2 \mid 1 \geq V_2 \geq c_1^{2*}\right], \\ c_2^{1*} &= E\left[V_2 \mid c_1^{2*} \geq V_2 \geq 0\right], \\ c_1^{2*} &= E\left[V_1 \mid c_1^{1*} \geq V_1 \geq c_2^{1*}\right]. \end{aligned}$$

In the **second case**, instead of adding an additional message of bidder 1 at the bottom, one adds it at the top.

Next we prove the result for $|M_1| = |M_2| = 3$. Since in the case $|M_1| = |M_2| + 1 = 3$, $E\left[V_1 \mid 1 \geq V_1 \geq c_1^{1*}\right] > E\left[V_2 \mid 1 \geq V_2 \geq c_1^{2*}\right]$, one can add a threshold at the top for bidder 2 and increase the welfare. Hence $w^*_{3,3} > w^*_{3,2}$. As in the previous case, one argues that the expected values, given the partition cells of both bidders, have to alternate. This yields two possible cases. Either $E\left[V_1 \mid 1 \geq V_1 \geq c_1^1\right] \geq E\left[V_2 \mid 1 \geq V_2 \geq c_1^2\right]$, or the other way around. In either case one can write out the welfare and obtain mutually centered strategies from the first order conditions.

The process can be iterated to obtain a solution for the general case $|M_1| = k_1$, $|M_2| = k_2$, for any k_1, k_2 such that $\max\{k_1, k_2\} \geq 2$. The case $k_1 = k_2 = 1$ is not particularly interesting as there are no thresholds, and the case of $|k_1 - k_2| > 1$ can be reduced to the case $|k_1 - k_2| = 1$. ■

Proof of Lemma 3. For ease of exposition and clarity we provide a proof for $k = 3$. The proof is easily, albeit with notational inconvenience, extended to the general case.

Let $c = (c_1, c_2)$ and $d = (d_1, d_2)$ be the threshold strategies of bidder 1 and 2 respectively. Given that the allocation rule is symmetric, the welfare is

$$\begin{aligned}
w &= 0.5 \sum_{i=1}^{3} \left[F(c_{i-1}) - F(c_i)\right]\left[F(d_{i-1}) - F(d_i)\right]\left\{E\left[V \mid c_{i-1} \geq V \geq c_i\right] + E\left[V \mid d_{i-1} \geq V \geq d_i\right]\right\} + \\
&\quad + \sum_{i=1}^{2} \left(\left[F(c_{i-1}) - F(c_i)\right] F(d_i)\, E\left[V \mid c_{i-1} \geq V \geq c_i\right] + F(c_i)\left[F(d_{i-1}) - F(d_i)\right] E\left[V \mid d_{i-1} \geq V \geq d_i\right]\right) \\
&= 0.5 \left\{ \begin{array}{c} \left[1 - F(c_1)\right] E\left[X \mid 1 \geq X \geq c_1\right] + F(c_1)\left[1 - F(d_2)\right] E\left[X \mid 1 \geq X \geq d_2\right] + \\ F(c_1)\, F(d_2)\, E\left[X \mid c_1 \geq X\right] \end{array} \right\} + \\
&\quad 0.5 \left\{ \begin{array}{c} \left[1 - F(d_1)\right] E\left[X \mid 1 \geq X \geq d_1\right] + F(d_1)\left[1 - F(c_2)\right] E\left[X \mid 1 \geq X \geq c_2\right] + \\ F(d_1)\, F(c_2)\, E\left[X \mid d_1 \geq X\right] \end{array} \right\},
\end{aligned}$$

where the second equality follows after some rearranging. The crucial point to see is that the maximization over c_1 and d_2 (the first summand) is separated from maximization over c_2 and d_1. The first summand is equivalent to a half of the welfare where each bidder has 2 messages and bidder 1 has the priority. We already know this has a solution. Let such optimal thresholds be $c_1 = a^*$ and $d_2 = b^*$.

Since bidders have the same distribution, the second summand represent the same problem: it represents welfare maximization when both bidders have two messages and bidder 2 has the priority. But then the same solution can be applied: $d_1 = a^*, c_2 = b^*$.

Thus we have a solution $c_1 = d_1 = a^*$ and $c_2 = d_2 = b^*$, which proves our claim. ■

Proof of Proposition 4. The first part of the proposition was proved in the text. For the second part, showing the equivalence between two welfare maximizing equilibria, let w be the welfare of the optimal equilibrium when $|M_1| = |M_2| = k$ and w' be the optimal welfare achieved in a symmetric equilibrium of a mechanism with $|M_1| = |M_2| = 2k-1$. If $w' > w$ then one can obtain welfare w' in a mechanism with $|M_1| = |M_2| = k$ by setting $c_1^1 = c_1, c_1^2 = c_2, ..., c_{k-1}^2 = c_{2k-2}$. But then clearly w is not the optimal welfare. The case $w' < w$ is handled similarly. ■

Proof of Proposition 5. In the text we explained why in a welfare maximizing optimum no threshold is below v_0. The remainder of the analysis is split into two cases, depending on whether given the optimal vector of thresholds, $\widetilde{c}$, we have $E\left[X \mid c_{k-1}^1 \geq X \geq 0\right] \leq v_0$ or $E\left[X \mid c_{k-1}^1 \geq X \geq 0\right] > v_0$. That is, the cases depend on whether bidder 1's lowest message is conveying that his expected value is above or below v_0. In the first case c_{k-1}^2 has to be equal to v_0; in the other case $v_0 > 0$ is nonbinding and the solution is equal to the one in the case $v_0 = 0$.

If $E\left[X \mid c_{k-1}^1 \geq X \geq 0\right] < v_0$, then when both bidders report message m_k the object is awarded to the seller. This case corresponds to maximizing

$$w = \sum_{j=0}^{k-2} F\left(c_j^2\right) \int\limits_{c_{j+1}^1}^{c_j^1} vdF(v) + \sum_{j=0}^{k-2} F\left(c_{j+1}^1\right) \int\limits_{c_{j+1}^2}^{c_j^2} vdF(v) + F\left(c_{k-1}^1\right) F\left(c_{k-1}^2\right) v_0 \tag{14}$$

$$\text{s.t.} \quad E\left[X \mid c_{k-1}^1 \geq X \geq 0\right] \leq v_0 \quad . \tag{15}$$

If, on the other side, $E\left[X \mid c_{k-1}^1 \geq X \geq 0\right] \geq v$, then bidder 1 obtains the object when both bidders bid m_k. The relevant maximization problem is

$$w = \sum_{j=0}^{k-2} F\left(c_j^2\right) \int\limits_{c_{j+1}^1}^{c_j^1} vdF(v) + \sum_{j=0}^{k-2} F\left(c_{j+1}^1\right) \int\limits_{c_{j+1}^2}^{c_j^2} vdF(v) + F\left(c_{k-1}^2\right) \int\limits_{0}^{c_{k-1}^1} vdF(v) \tag{16}$$

$$\text{s.t.} \quad E\left[X \mid c_{k-1}^1 \geq X \geq 0\right] \geq v_0 \tag{17}$$

One obtains the highest welfare by computing the optimum of both problems and comparing which is higher.

Some clarifications may be needed. When the condition of the problem defined in (14) is not binding, one obtains $\widehat{c}_{k-1}^2 = v_0$. If the welfare is to be optimized, it should not matter whether the bidder with the value $\widehat{c}_{k-1}^2$ is reporting message m_k or m_{k-1}. The only thing that changes between those two messages is whether bidder 2 is winning or losing against the seller with the value v_0.

Even more, if one was to solve this problem, not paying attention to the condition would lead to a system of equations much like the one in (1) with the only difference being

$$\begin{aligned} c_{k-1}^1 &= \frac{\left[F\left(c_{k-2}^2\right)-F\left(c_{k-1}^2\right)\right]}{F\left(c_{k-2}^2\right)} E\left[X \mid c_{k-2}^2 \geq X \geq c_{k-1}^2\right] + \frac{F\left(c_{k-1}^2\right)}{F\left(c_{k-2}^2\right)} v_0 \\ c_{k-1}^2 &= v_{0}. \end{aligned}$$

Furthermore, if it happens to be the case that after solving the unrestricted problem one gets $E\left[X \mid \widehat{c}_{k-1}^1 \geq X \geq 0\right] \geq v_0$, the actual equation for the welfare that one should apply is (16). But, raising the threshold $\widehat{c}_{k-1}^2$ up to the level $E\left[X \mid \widehat{c}_{k-1}^1 \geq X \geq 0\right]$, one does not decrease the welfare. Therefore the solution is the solution to the system of equations under the problem where the seller's value is 0. ■

Appendix C. Proofs of Subsection 3.3

Proof of Theorem 2. Let us briefly describe the mechanics of the proof before we plunge into formalities. As hinted in the paragraphs following the theorem, the proof proceeds in two steps. In the first step we maximize welfare over a subset of strategies. In particular, we fix values of $3k-3$ thresholds, $c_1 > c_2 > c_3 > ... > c_{3k-3}$, and look at the optima over all the possible threshold strategies generated by these thresholds (each bidder having $k-1$ thresholds). We sidestep dealing with incentives until the very end of the proof, where we show that the welfare optimal equilibrium of strategies and allocation rule can be supported as an equilibrium with appropriate transfers.

In the second step we observe that these local optima have a rather convenient property: the strategies are induced by the priority assignment ι^* and the optimal allocation rule is Q^*. Now we can fix those two and optimize the system over the sets of thresholds.

Step 1. As noted in the text, every threshold strategy profile in which each bidder uses $k-1$ thresholds can be described by a set of thresholds and an assignment rule ι. We fix values of $3k-3$ thresholds in $[0,1]$, $c_1 > c_2 > c_3 > ... > c_{3k-3}$. The case where inequalities are weak will be commented upon later. Type T_j is defined to be $[c_j, c_{j-1})$, where $c_0 = 1$ and $c_{3k-2} = 0$. We call the type T_j higher than T_k when $j < k$.

For any allocation rule, Q, total welfare can be obtained from computing the welfare for every combination of types and summing it over all the possibilities. We use notation $\mu_i\left(T_j\right)$ to denote $\mu_i\left(x\right)$ when $x \in T_j$. All the combinations of types can be covered by the following procedure: We start with bidder 1 being type T_1 and consider all the combinations of bidder 2 and 3, then we consider bidder 2 being type T_1, bidder 1 being any type but T_1 and bidder 3 being any type. Next, we have bidder 3 being type T_1 and bidders 2 and 3 any type but type T_1. Going through with types T_2 and on, one analyses all the possibilities. The important property of the procedure is that when one considers a certain bidder of a certain type one need not consider other bidders being of a higher type since this was taken care of in previous steps.

The next couple paragraphs use the point made in the text. It would be welfare optimal, given the thresholds, if each bidder could reveal his type. Since this is not the case, there is some loss in welfare compared to the case where all the bidders could fully report their types T_j. Notice the resemblance to the analysis in the symmetric two bidder case. We bound the loss from below and show that the priority assignment, ι^*, and the allocation rule Q^* achieve that bound. The bound is given type by type: for each type, T_j, there exists at least one bidder of this type, i_1, who loses against exactly one bidder, i_2, of type T_{j+1} when the third bidder, i_3, is of at most type T_{j+1}. That is, when the bidder i_1 is of type T_j, bidder i_2 of type T_{j+1} and bidder i_3 of at most type T_{j+1}, bidder i_2 is awarded the object. Figure 3 provides a graphical representation of the case when bidder 3 of type T_1 loses against bidder 2 of type T_2 when bidder 1 is of at most type T_1.

Optimal welfare, given the fixed thresholds, would be achieved if all the types would win against all the lower types. Again, it is irrelevant who wins when multiple bidders are of the same type and this type is the highest. If full communication of types was possible, then the welfare achieved from at least one bidder being of type T_j and nobody else being a higher type would be:

$$\overline{w}_j = \left[F\left(c_{j-1}\right) - F\left(c_j\right)\right] \left\{ \sum_{i=1}^{3} F\left(c_{j-1}\right)^{3-i} F\left(c_j\right)^{i-1} \right\} E\left[X \mid c_{j-1} \geq X \geq c_j\right],$$

and the total welfare

$$\overline{w} = \sum_{j=1}^{3k-2} \overline{w}_j.$$

The equation for $\overline{w}_j$ can be thought of as follows. When bidder 1 is of type T_j, and thus has expected value $E\left[X \mid c_{j-1} \geq X \geq c_j\right]$, which happens with probability $\left[F\left(c_{j-1}\right) - F\left(c_j\right)\right]$, he wins against both other bidders of at most type T_j, which happens with probability $F\left(c_{j-1}\right)^2$. When bidder 2 is of type T_j he obtains the object when bidder 1 is of at most type T_{j+1} and bidder 3 of at most T_j, which happens with probability $F\left(c_{j-1}\right) F\left(c_j\right)$. And when bidder 3 is of type T_j, he is awarded the object when both other bidders are of a lower type which happens with probability $F(c_j)^2$. Summing over all $\overline{w}_j$ yields total welfare $\overline{w}$.

Unfortunately, achieving this welfare is unattainable under our communication restrictions. As already hinted we derive what is the smallest loss in comparison to $\overline{w}$ that can be achieved. We fix an assignment rule ι and a type T_j, where $j < 3k-2$. Let $\iota(T_j) = 1$. This is just for convenience; we could always relabel the bidders. At this point we just want to obtain a lower bound on the welfare when at least one of the bidders is type T_j and nobody else is of a higher type.

Let a be the smallest $l \in \mathbb{N}$ such that $\iota(c_{j+l}) = 2$.[35] That is, c_{j+a} is the first threshold after c_j that is assigned to bidder 2 under the assignment ι. Then we have $\iota(c_{j+a}) = 2$. Similarly, let b be the smallest $l \in \mathbb{N}$ such that $\iota(c_{j+l}) = 3$. Hence, $\iota(c_{j+b}) = 3$. Again without loss of generality we can assume $a < b$ (otherwise relabel the bidders). We denote the message the bidders report when they observe T_j by m_j. Maximization of welfare for types T_j implies that when one bidder is of type T_j, therefore reporting m_j, and the other bidders are reporting lower partition cells, then he should be allocated the object.

There are a couple of cases to consider. When bidders 1 and 2 report message m_j and bidder 3 reports a lower message, the object is optimally allocated to bidder 1. Why, if it was allocated to bidder 2 then it would also be allocated to him when he was either of the types $T_{j+1}, T_{j+2}, ..., T_{j+a}$;[36] therefore bidder 1 of type T_j would be losing against all this lower types of bidder 2. The same kind of comparisons show that whenever bidder 1 reports m_j and no higher messages are reported by the other two bidders, bidder 1 should win the object.

Comparison between bidders 2 and 3 shows that the object should be awarded to bidder 2 when the two of them report message m_j and bidder 1 reports a lower message. Now the welfare from types T_j is

$$\begin{aligned} w_j \quad = \quad & [F(c_{j-1}) - F(c_j)] F(c_{j-1})^2 E[X \mid c_{j-1} \geq X \geq c_j] + \\ & [F(c_{j-1}) - F(c_j)] F(c_{j-1}) F(c_j) E[X \mid c_{j-1} \geq X \geq c_j] + \\ & [F(c_{j-1}) - F(c_j)] F(c_j) F(c_{j+a}) E[X \mid c_{j-1} \geq X \geq c_j] + \\ & [F(c_{j-1}) - F(c_j)] F(c_j) [F(c_j) - F(c_{j+a})] E[X \mid c_j \geq X \geq c_{j+a}], \end{aligned}$$

[35] One can always think of threshold $c_{3k-1} = 0$ as being allocated to all of the bidders; therefore a is well defined.

[36] The first threshold below c_j that is assinged to bidder 2 is c_{j+a}. Therefore he is reporting the same message for all those types.

where the first summand is the welfare achieved when bidder 1 is of type T_j and the other two bidders are at most of type T_j. The second summand is due to bidder 2 being type T_j, bidder 1 being at most type T_{j+1} and bidder 3 being at most type T_j. Last two summands are due to bidder 3 being type T_j and bidder 1 being at most type T_{j+1}. The first part comes from the fact that he is winning against types of bidder 2 up to T_{j+a+1}, and the second part that he is losing against all the type T_{j+1} through T_{j+a} of bidder 2.

w_j can now be rewritten:

$$w_j = \overline{w}_j - [F(c_{j-1}) - F(c_j)] F(c_j) [F(c_j) - F(c_{j+a})] \{E[X \mid c_{j-1} \geq X \geq c_j] - E[X \mid c_j \geq X \geq c_{j+a}]\}.$$

The welfare from bidder being type T_j is the optimal welfare one would achieve with full communication of types minus the loss incurred due to the fact that bidder 3 of type T_j is losing against lower types of bidder 2. Loss is clearly minimized by setting a to 1. That is, by an assignment rule ι' such that $\iota'(c_j) = 1, \iota'(c_{j+1}) = 2$. For $a = 1$ the only case where a bidder of type T_j loses against a bidder of a lower type is when bidder 3 is of type T_j, bidder 2 is of type T_{j-1} and bidder 1 is of a type lower than T_j, thus reporting message bellow m_j.

The analysis above shows that even if we could adapt the allocation rule as we wanted to for every type T_j, the best one could achieve with the given thresholds is that for each type there is one bidder that is losing against exactly one bidder of exactly one type lower. Although the analysis so far has not pinned down the value of b, we show it to be optimally set at 2. Also notice that $a = 1$ says that it is never optimal to allocate two consecutive thresholds to the same bidder.

First, one can easily see that the priority assignment, ι^*, together with an allocation rule that gives the highest priority to bidder 1 and the second highest to bidder 2, attains this optimum. That is, of the types T_1, only bidder 3 is loosing against type T_2 of bidder 2 when bidder 1 is of at most T_2. When of type T_2 bidder 1 is losing against bidder 3 of type T_3, when bidder 2 is of at most type T_3, etc. For every type T_j, there is exactly one bidder who is losing against exactly one other bidder of type T_{j+1} when the third bidder is of at most T_{j+1}.

Indeed one can show that the priority assignment ι^* is the only assignment that achieves the optimum, up to the relabeling of bidders. We start the analysis with type T_1. Without loss of generality let $\iota^*(c_1) = 1$. By the above characterization ($a = 1$) one of the other two bidders should obtain c_2. Again without loss of generality let $\iota^*(c_2) = 2$. Now suppose $\iota^*(c_3) = 1$. Two things could happen. When bidder 1 reports m_2, bidder 2 reports m_2 and bidder 3 reports m_1, either bidder 3 is awarded the object and then types T_3 and T_4 of bidder 3 are winning against type T_2 of bidder 1, or bidder 1 is awarded the object in which case bidder 3 of type T_1 is losing against bidder 1 of types 2 and 3. Iterating the process, one can see that ι^* does strictly better than any other assignment rule.[37]

The following should be observed: the optimal allocation rule awards the object to bidder 1 unless one of the other two bidders report a higher message; it is awarded to bidder 2 if bidder 1 reports a lower message and bidder 3 reports at most as high a message; and it is awarded to bidder 3 if both other bidder are reporting strictly lower messages.

That is, Q^* is defined as follows. Let m_k be bidder 1's message, m_l bidder 2's and m_n bidder 3's, then $Q_1^*(m_k, m_l, m_n) = 1$ if $k \geq \max\{l, n\}$ and 0 otherwise. $Q_2^*(m_k, m_l, m_n) = 1$ if $l \geq \max\{k+1, n\}$ and 0 otherwise and finally $Q_3^*(m_k, m_l, m_n) = 1$ if $n \geq \max\{k+1, l+1\}$ and 0 otherwise.

Step 2. What we proved above is that for every set of thresholds as above one does the best by using priority assignment, ι^*, and the allocation rule Q^*.

Welfare can now be written as

$$w\left(q^*, \iota^*, c\right) = \sum_{j=1}^{3k-2} F\left(c_{j-2}\right) F\left(c_{j-1}\right) \int_{c_j}^{c_{j-3}} x dF.$$

Maximizing over c yields

$$\begin{aligned} c_j^* &= \frac{F\left(c_{j-1}^*\right)\left[F\left(c_{j-2}^*\right)-F\left(c_{j+1}^*\right)\right]}{F\left(c_{j-2}^*\right)F\left(c_{j-1}^*\right)-F\left(c_{j+1}^*\right)F\left(c_{j+2}^*\right)} E\left[X \mid c_{j-2}^* \geq X \geq c_{j+1}^*\right] \\ &\quad + \frac{F\left(c_{j+1}^*\right)\left[F\left(c_{j-1}^*\right)-F\left(c_{j+2}^*\right)\right]}{F\left(c_{i-2}^*\right)F\left(c_{j-1}^*\right)-F\left(c_{j+1}^*\right)F\left(c_{j+2}^*\right)} E\left[X \mid c_{j-1}^* \geq X \geq c_{j+2}^*\right], \end{aligned} \tag{18}$$

for every $j \in \{1, 2, ..., 3k-3\}$ .It is easy to see that welfare is maximized in the interior. That is it cannot happen that for some j $c_j^* = c_{j+1}^*$. Therefore our consideration of the case $c_1 > c_2 > ... > c_{3k-3}$ is justified. ■

Proof of Theorem 3. The existence theorem can be easily modified to account for $v_0 > 0$. Knowing that the welfare optimal equilibrium exists, only two things can happen. Either the seller

[37] Up to the relabling of bidders.

never retains the object and the optimal equilibrium is the priority equilibrium, or he retains the object for some reports in which case it is easy to show that an optimal equilibrium is a modified priority equilibrium by slightly modifying the proofs of Theorem 2 and Proposition 5. ■

Appendix D. Proofs of Sections 4, 5 and 6

Most of the results of Section 4 build on previous results in the paper. Here we only provide a sketch of the proof of Theorem 4.

Proof of Theorem 4. How one can transform the problem of profit maximization into an auxiliary problem of welfare maximization was described in the text; for further details one should see BNS. We denote the random variable corresponding to the virtual valuation by W and its support by $[\underline{w}, \overline{w}]$. From Theorem 3 we know that the solution to the auxiliary problem is either a priority or a modified priority equilibrium. Now we argue that it cannot be a priority equilibrium. The crucial observation that we use is that the expected value of the virtual valuation is 0 (see Krishna (2002), p. 69). Suppose the optimal welfare in the auxiliary problem could be achieved in a priority equilibrium. Without loss of generality we can assume that bidder 1 is the one who is awarded the object when all the bidders report their lowest partition cell. Let $\underline{c}$ be bidder 1's lowest threshold in an optimal priority equilibrium; clearly $\underline{c} < \overline{w}$. Then $E[W \mid \underline{c} \geq W \geq \underline{w}] < E[W] = 0$, which means that when all the bidders are reporting their lowest partition cell the welfare can be increased if the seller retains the object, contradicting the assumed optimality of the priority game.
■

Proof of Proposition 8. In the text we establish that at most two messages of bidder 2 are relevant when he is called upon to report. That is, given the strategy and the observed message of bidder 1, the only relevant information is whether bidder 2's valuation is above or below bidder 1's expected valuation. On the other side, k_1 messages of bidder 1 partition his valuation space into k_1 partition cells. Proposition 2 implies that at most $k_1 + 1$ messages of bidder 2 are informationally relevant. Alternatively, optimal welfare of our two bidder mechanism with two periods of reporting is bounded above by the optimal welfare of the simultaneous reporting mechanism where bidder 1 has k_1 messages and bidder 2 has $k_1 + 1$, i.e. $w^*_{2,(k_1,k_1+1)}$. We need to show that this upper bound can be attained in a Bayesian equilibrium of the mechanism with sequential reporting.

Let c^{1**} and c^{2**} be mutually centered threshold strategies of bidder 1 and 2 respectively in some welfare optimal equilibrium under simultaneous reporting and $|M_1| = k_1 = |M_2| - 1$. Defining

$c_j^* = c_j^{1**}$ for $j \in \{1, 2, ..., k_1 - 1\}$ and $d_j^* = c_j^{2**}$ for $j \in \{1, 2, ..., k_1\}$ one can readily see that the strategy given by (c^*, d^*) combined with the allocation rule Q^* achieves welfare $w^*_{2,(k_1,k_1+1)}$ if the bidders can be induced to report according to it. Bidder 2's incentives are easily aligned by the transfers, T_2^*, which require him to pay (when he wins) the lowest valuation he could have, report according to his threshold and still win given the bidder 1's report. If he is not awarded the object he does not need to pay. Bidder 1's transfers require a little more work. Let's start at the bottom. Suppose that bidder 1 announces m_{k_1} and bidder 2 reports D. In such a case bidder 1 is awarded the object and we can define $T_1^* (m_{k_1}, D) = 0$. Clearly $T_1^* (m_j, U) = 0$ for all $j \in \{1, 2, ..., k_1\}$. Now consider that bidder 1 reports m_{k_1-1} and bidder 2 reports D. If the threshold strategy is to be an equilibrium then when bidder 1's valuation is $c^*_{k_1-1}$ he should just be indifferent between reporting message m_{k_1} and m_{k_1-1}, i.e.:

$$F_2 \left(d^*_{k_1-1} \right) \left[c^*_{k_1-1} - T_1^* \left(m_{k_1-1}, D \right) \right] = F_2 \left(d^*_{k_1} \right) c^*_{k_1-1}.$$

But this yields the transfer given by the equation (9). It is easy to see now that for all valuations above $c^*_{k_1-1}$ bidder 1 prefers to report m_{k_1-1} over m_{k_1}. By recursively applying the logic above one can obtain all the remaining transfers. ■