A Model of Dynamic Liquidity Contracts*

ONUR ÖZGÜR†

New York University

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Abstract

The main goal of this paper is to analyze the nature of long-term liquidity contracts that arise between lenders and borrowers in the absence of perfect enforceability and when both parties are financially constrained. We study an infinite horizon dynamic contracting model between a borrower and a lender with the following features: The borrower, is credit-constrained, faces a stochastic project arrival process every period, can choose to renege each period, and can save through the lender. Projects are indivisible. The lender is resource-constrained, and can commit to the terms of the contract as long as it is ex-ante individually rational to do so. We show that: (i) Enforcement problems and endogenous resource constraints can severely curtail the possibility of financing projects, (ii) the economy exhibits investment cycles, (iii) credit is rationed if either the lender has too little capital or the borrower has too little financial collateral. This paper's technical contribution is to show the existence and characterization of financial contracts that are solutions to a non-convex dynamic programming problem.

Journal of Economic Literature Classification Numbers: C6, C7, D9, G2

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†Department of Economics, 269 Mercer St., 7th Floor, New York, NY, 10003; e-mail: onur.ozgur@nyu.edu; webpage: http://home.nyu.edu/∼oo218

1 Introduction

The level of resources of lenders have significant and positive effects on lending and economic activity. As their capital levels fall, banks become more conservative in their lending. During the credit crunch of 1990, banks started cutting back on lending immensely. Limited bank capital relative to the loan demand contributed to restrictive bank lending during the recession of 1990/91.[1] There is a large empirical literature that examines the link between bank capital and lending (see Sharpe [33] for an extensive survey).[2] There is also a theoretical literature that suggests that higher bank capital tends to increase lending (see Besanko and Kanatas [10], Thakor [37], Holmström and Tirole [23], and Diamond and Rajan [17]). The analyses in Thakor [37] and Holmström and Tirole [23] are most consistent with the findings of the empirical literature. Yet, a proper investigation of these issues necessitates a genuinely dynamic model with endogenous capital constraints on financial intermediaries.[3]

There is also empirical evidence that dynamic bank relationships help borrowers through implicit contracting (see Petersen and Rajan [28], Berger and Udell [7], Hoshi, Kashyap and Sharsftein [20, 21, 22]). On the theoretical side, Haubrich [19], Boot, Greenbaum and Thakor [12] are early examples of works to recognize the potential gains from long-term interactions between banks and borrowers. What is missing though is an explicitly dynamic model of liquidity provision generating interesting credit dynamics and rationing which could capture the stylized facts observed in the data.

This paper aims at filling these gaps. We study the nature of long-term liquidity provision between lenders and borrowers in the absence of perfect enforceability and when both parties are financially constrained. To this end, we build an infinite horizon model of long-term lending and borrowing and analyze in what ways liquidity shortages on both sides affect the evolution of the economy and investment activity in particular.

An infinitely-lived, risk neutral borrower (firm or entrepreneur) receives a project every period with some probability.[4] The project requires a certain amount of funds to be invested and has

[1]See Bernanke and Lown [9] on 'The Credit Crunch'. They give anecdotal evidence on Richard Syron, then president of the Federal Reserve Bank of Boston, calling the crunch a 'Capital Crunch'. Syron argued in a testimony before Congress that the credit crunch in New England was due to a shortage in bank capital. Banks in the region had to write down loans, forced by the real estate bubble. This in turn led to the depletion of their equity capital. In order to meet regulatory requirements, they had to sell assets and scale down their lending.

[2]Most of these studies have been conducted for US data inquiring into whether implementation of the 1998 Basel accords' capital standards caused a 'credit crunch' in the US. Sharpe [33] finds that empirical evidence suggests that loan losses have a negative and bank profitability has a positive effect on loan growth.

[3]We quote (emphasis ours) from Holmström and Tirole [23, p.690]: "*Limited intermediary capital* is a necessary ingredient in the study of credit crunches and cyclical solvency ratios." They also mention that a proper investigation of these issues requires endogenous intermediary capital and an explicitly dynamic model.

[4]This paper is not concerned with consumption smoothing. We are focusing on the asset side of the lenders' balance sheets to study liquidity provision to borrowers for productive purposes. For a classical treatment of the

a stochastic return structure. Projects have positive net present value which make them socially desirable to undertake. Moreover, the projects are indivisible and although the borrower has a given endowment every period, it is not sufficient to cover the required investment level. There is limited liability on the part of the borrower; net payments need to be nonnegative. All these in turn generate a *demand for liquidity* from the borrower. The problem is that this demand is not always matched by an associated supply of credit since the contracts are not perfectly enforceable. It is possible for the borrower to renege on the contractual clauses and run away with the return on investment. There is no commitment mechanism to prevent the borrower from defaulting. As a consequence, at the optimum, the lender offers incentive-compatible contracts and the borrower is credit constrained.

An infinitely-lived, risk neutral lender (bank or financial intermediary) provides the borrower with liquidity. If the two of them enter into a long-term liquidity provision agreement, the lender is supposed to extend credit to the borrower in states where the borrower has a demand for external financing because he wants to implement a project and the project lives for one period only. The lender can commit to the terms of the contract as long as the net payments from the contract are non-negative. The lender has a storage technology that makes it possible for the entrepreneur to accumulate wealth through the lender. We analyze an economy where there is no other way for the borrower to save. This is not restrictive in the sense that it is the simplest way to capture the idea that the return on deposits with the lender is higher than the return on the borrower's storage technology (self-insurance). Contrary to the common practice in the literature which takes it for granted that lenders have unlimited resources, we assume that lenders are resource-constrained. This is to capture in a simple way the fact that lenders also face financial frictions in raising funds (see Kashyap and Stein [25], Schneider [32]): for example, they might face liquidity shortfalls due to other financial commitments. Each borrower that a lender commits to provide liquidity to, has an opportunity cost for the lender when it comes to dealing with other borrowers. Financial collateral also plays an important role in our framework. If a borrower chooses to default, what he can confiscate is limited to the returns on the project. His deposits with the lender are then transferred to the lender.

A dynamic liquidity contract is a mechanism that specifies transfers to and payments from the lender as a function of the entire history of realizations of the random variables in the model. We first analyze the benchmark case of contracts in the absence of enforceability problems, i.e., the first-best contracts. An efficient contract needs to be efficient after any history. This insight facilitates the analysis immensely by letting us use recursive methods. In the absence of default, optimal accumulation and investment decisions are independent of the surplus sharing rule stated in the contract. All that matters then is how the surplus is shared. Optimal contracts define a

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liquidity provision of banks for consumption smoothing purposes, see Diamond and Dybvig [16]. For an analysis of intertemporal smoothing by long-lived intermediaries, see Allen and Gale [4]

Pareto frontier between the value to the borrower and the value to the lender, which is strictly decreasing. A reduction in the value to the lender is an equivalent increase in the value to the borrower; there is no loss of value because there is no possibility of default. However, it might happen that for some levels of the resources of the lender, the resource constraint for the lender binds, and projects will be passed up (although they have positive net present value); so there will be credit rationing. Nevertheless, if the agents are patient enough, the economy will accumulate resources and will start investing in those projects in finite time, with probability one.

One of our results is that first-best savings are bounded from above. Since both parties are risk neutral, saving happens only for productive purposes in this economy. One saves because one wants to be able to invest when there is a project to be invested in. This is what we call the first-order effect. There is also the second-order effect stemming from the fact that a sequence of unlucky draws can lead to drying up of resources. Agents would like to save to insure themselves against a resource depletion of this sort. However, as the resource level increases, the utility cost of saving one more dollar outweighs the gain from insurance, leading to savings being bounded. Clearly, relatively more patient agents save more, since they care about the possibility of a sequence of bad shocks more. Another feature of the first-best contracts is that the investment rule is monotonic in the level of resources. The intuition is clear: if investment is undertaken at a level of resources, it will also be undertaken at higher levels of resources.

A feasible second-best contract needs to be incentive compatible since the borrower has no other credible way to commit to the terms of the contract. The worst punishment that can be inflicted upon the borrower in case of default is exclusion from the credit markets. In that case, the borrower is compelled to consume what he confiscated plus his stream of future endowment: default leads to autarky. We borrow this assumption from the literature on dynamic contracts (see Albuquerque and Hopenhayn [2], Alvarez and Jermann [5], Atkeson [6], Kehoe and Levine [26], Marcet and Marimon [27], Thomas and Worrall [39]) and adopt it as is, since our focus is rather on liquidity provision and the effect of lender resource constraint on the nature of emerging contracts.[5]

Relative to the first-best contracts, second-best contracts feature more interesting properties. Enforcement problems and endogenous resource constraints can severely reduce the possibility of financing projects. Investment and savings are functions both of the level of resources and the surplus sharing rule, in contrast to the first-best contracts. Investment is (weakly) under-provided. Second-best savings are (weakly) less than first-best savings capturing the intuition that it does not pay off to postpone consumption/investment to self-insure against a possible 'credit crunch'.

[5]Recent work in the aforementioned literature focuses on relaxing that assumption to endogenize the value of default. Cooley, Marimon and Quadrini [14] present a model where the value of repudiation is endogenous and is affected by all the general equilibrium conditions. Phelan [29] is an earlier contribution that allows recontracting in case of default, shows the existence of a default-free equilibrium with trade and characterizes its properties.

Second-best contracts exhibit interesting dynamics. Any economy where agents are relatively patient exhibits investment cycles. The fact that lenders are resource-constrained is the reason why we obtain that result. For any initial level of resources, there is a positive probability that the economy will go into a 'credit crunch' phase. This happens because the lender's resource constraint starts binding. For economies where agents are relatively patient, this is not an absorbing state. The economy recovers in the long-run, the capital levels are restored and investment activity goes back to normal. As the economy evolves through time, the aggregate resources increase on average, which in turn increases the lag between a phase of investment and a phase of no-investment. However, in economies with relatively impatient agents, this phase is catastrophic. The aggregate system gets stuck in that phase and we observe constant stagnation.

There is another type of dynamics generated by the incentive compatible nature of the second-best contracts. Depending on the initial distribution of payoffs, different initial investment behaviors are observed. For some economies, if the initial value of the share that the borrower gets from the surplus is too small, the lender might find it too costly to make sure there is no default. Hence, initially there is inefficiency in the economy since although the projects have positive net present value, credit is rationed. This does not lead to the collapse of the relationship though. As time evolves, the share of the borrower increases (on average) and investment starts to be undertaken for the same states of resources where initially credit was declined. The intuition is that, over time there is 'collateral' being built and held by the lender in terms of promises (deposits). Since the lender has no commitment problem, these promises have to be delivered and the bargaining power of the borrower inside the relationship increases. Eventually, we see investment being implemented.

A third kind of dynamics are in the spirit of Bernanke and Gertler's [8] accelerator result. Worsening aggregate conditions lead to the worsening of the borrower's worth/financial collateral (summarized by v), leading to the fact that positive net present value projects are passed up because it is not incentive compatible to extend credit to the borrower with such little collateral. Ameliorating aggregate conditions work in the opposite direction.

Bernanke and Gertler [8] study an OLG model of business cycle dynamics where borrowers' balance sheet positions play an important role. They show that agency costs associated with the undertaking of physical investment are decreasing in the borrower's net worth. They, then, proceed to show that this general insight implies the emergence of *accelerator* effects on investment. Strengthened balance sheets of borrowers during good times in turn expand investment demand which tends to amplify the boom; weakened balance sheets during bad times work in the opposite direction. Thus, the existence of agency costs exacerbates business cycles.

There are very important distinctions between their model and the present model. Bernanke and Gertler uses an OLG model whereas the present paper is a full-fledged infinite horizon model of borrowing and lending, internalizing the gains from long-term relationships. The same kind of accelerator effect is exhibited by the dynamics of the second-best contracts in the present paper. Moreover, it happens that there is a completely different kind of cyclical behavior emerging in the current model that [8] does not have. Cycles can be efficient, i.e., we can observe investment cycles not because of agency costs but because lenders' resources are scarce and the system enters a 'capital crunch' region.

This paper also contributes a technical analysis of the existence and characterization of optimal financial contracts that are solutions to a non-convex dynamic programming problem. Indivisibility of the projects along with credit constraints make the set of feasible contracts a non-convex set. The resulting value functions are not everywhere differentiable and exhibit discontinuity points as shown in Figure 3 below. The standard methods using concave programming and/or (super)differentiability of the value functions are not of help in this context. We can still use a rather direct strategy and exploit monotonicity of the resulting operators to prove our characterization results. The problem of the value function entering the constraint set was introduced in Thomas and Worrall [39]. In our case, the problem is exacerbated by the fact that both the borrower and the lender have limited liability constraints and the continuation values should be nonnegative for both.[6] We can use an approach similar to the one in [39] to show existence of the value function. It is a bit more complicated since, in game theoretical terms, the problem in [39] is a repeated game where the current problem is a dynamic game with an unbounded state space.

There are close ties between this paper and the literature on dynamic incentives to repay. Bolton and Scharfstein [11] and Stiglitz and Weiss [35] analyze incentive effects of termination in credit relationships, in which the threat of termination by the lender provides the right incentives for the borrower to pay back the loan. The threat of termination in our paper is the indefinite exclusion from the credit market. However, what the existing literature does not take into account is the effect of financial frictions on the lender's part upon the nature of the optimal liquidity contracts. Capturing this effect is the focus of this paper, what makes it fundamentally different from the existing literature.

The literature on sovereign debt repayment uses similar ideas and techniques. Allen [3] and Eaton and Gersovitz [18] are early examples modelling the strategic considerations of a sovereign debtor. Bulow and Rogoff [13], a critique of reputation models, show that reputation alone is not enough to ensure debt repayment; default dominates repayment. Again, there are certain differences that should be noted. Monitoring is an issue in that literature to make sure that loans are used for the projects they are meant to be used for. There are intricate issues involving what kind of punishment schemes can be used, stemming from the fact that borrowers are sovereign

[6] Thomas and Worrall [39] show that the resulting dynamic program is not a standard concave programming problem and the operator is not a contraction mapping in the supremum metric, despite the presence of strict discounting. The technical reason for that is the presence of the value function in the constraints.

countries. Finally, none of these models look at the lender's side, whereas we are concerned with the case of a resource-constrained lender.

The dynamic programming approach that we use is similar to the methods in Abreu, Pearce and Stacchetti [1], Atkeson [6] and Phelan and Stacchetti [30]. The general theme in this body of research is to use dynamic programming techniques to characterize the equilibrium value set of the repeated or dynamic game. These methods are especially powerful when it is difficult to determine what the worst *credible punishment* that can be inflicted upon the opponent is. For the purposes of the current paper, that worst punishment is already known and it is *autarky*. Hence, the boundary of the equilibrium payoff set (second-best frontier) is supported by the threat to return to autarky. This is the game theoretical explanation of why we use 'best-deviation' constraints to support the second-best strategies.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 analyzes the efficient contracts when there is no enforceability problems. In section 4, we provide a full characterization of the optimal financial contracts with limited enforceability and look at their long-run properties. Section 5 presents a summary and conclusions along with possible future research. All proofs and technical results are collected in the Appendix.

2 The Model

Time is indexed by $t = 0, 1, \ldots$. There are two agents, a borrower, indexed by E, and a lender, indexed by L, both infinitely lived. The borrower has a deterministic endowment stream of $Y > 0$ units of the only consumption good in the economy, every period. Each period, with some probability $p \in (0, 1)$, he has a project that needs to be implemented within that period. Investment requires $I > Y$ units of the consumption good whose return is $D > I$ units, in the same period, with probability q, and 0 units with probability $(1 - q)$. The net present value of a project is strictly positive, i.e., $qD - I > 0$. This is to make the problem interesting. Projects increase surplus to be shared in the economy. Let θ_t be the random variable that takes the value 1 if the agent has a project at time t, 0 otherwise. Similarly, let μ_t be the random variable that takes the value 1 if the project is a success and 0 if it is a failure. The borrower cannot store goods whereas the lender has a storage technology that brings him one unit of the good next period for every unit stored in the current one ($1 < \beta^{-1}$, hence, storing is costly). Let $H^t \equiv \{h^t = (\theta_0, \mu_0, \ldots, \theta_t, \mu_t)\}$ be the set of t-period histories of past realizations of the i.i.d. stochastic process (θ_t, μ_t), for $t = 0, 1, \ldots$ A *contract* $\sigma = (\sigma_t) = (c_t, m_t, I_t, S_{t+1})$ is a vector of sequences of functions where after any history h^{t-1}, $I_t(h^{t-1}, \theta_t)$ is the amount invested, $c_t(h^{t-1}, \theta_t, \mu_t)$ is the borrower's suggested consumption, $m_t(h^{t-1}, \theta_t, \mu_t)$ is net payments to the lender, and finally $S_{t+1}(h^{t-1}, \theta_t, \mu_t)$ is the amount transferred to the next period. Figure 1 shows the timing of actions at time t.

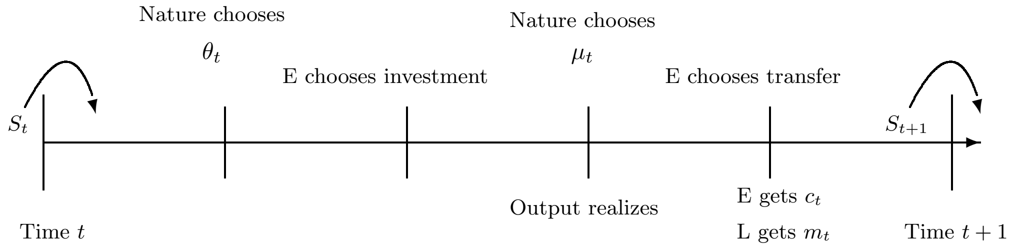

Figure 1: Timing of Decisions

The borrower is risk neutral and ranks allocations according to their consumption sequences, $c = (c_t)$

$$U^E(\sigma) = (1-\beta)\sum_{t=0}^{\infty}\beta^t c_t.$$

The lender is risk neutral and ranks allocations with respect to their sequences of net payments, $m = (m_t)$

$$U^L(\sigma) = (1-\beta)\sum_{t=0}^{\infty}\beta^t m_t.$$

and $\beta \in (0,1)$ is the common discount factor. As it is a common practice in the repeated games literature, we normalize the utility levels to be able to compare them to period utilities. The endowment stream of the borrower guarantees him Y on average. So, the autarkic level of a borrower who does not enter into a long-term contract is defined as $v^E_{aut} = Y$. The lender owns the resources amounting to $S_0 \geq 0$ units of the consumption good, initially. The lender honors all the promises that he makes whereas the borrower might choose to renege on the current contract any time she wants to do so. If the borrower chooses to default, he is excluded from the credit markets forever.

3 First-Best Problem

In this section, we begin first by solving for the efficient contracts. They constitute good benchmark cases that let us make welfare comparisons between two different institutional frameworks and use that comparison to see how the severity of incentive problems change the structure of the contracts offered in the economy. Assuming that a planner has all the information that the agents have and there are no incentive problems, an efficient contract should satisfy $\forall t$, $\forall h^t$

$$\begin{aligned} S_{t+1}(h^t) \leq \quad & S_t(h^{t-1}) - m_t(h^t) - c_t(h^t) + Y \\ & + D1_{\{\theta_t=1,\ \mu_t=1 \text{ and } I_t(h^{t-1},\theta_t)\geq I\}} - I_t(h^{t-1},\theta_t) \end{aligned} \tag{1}$$

which is an aggregate feasibility constraint. The idea that liquidity might be limited is captured by the following two constraints, $\forall t$, $\forall h^t$:

$$I_t(h^{t-1}, \theta_t) \leq S_t(h^{t-1}) + Y \tag{2}$$

which are resource constraints, and

$$c_t, m_t, I_t, S_{t+1} \geq 0 \tag{3}$$

which are the limited liability constraints (nonnegative net payments): there is a lower bound on the consumption that can be taken away from the borrower. Technically, these two types of constraints guarantee that the utility possibility set is bounded from below and above, for a given initial level of resources S_0. The fact that the lender is resource-constrained and there is limited liability are the reasons why we obtain interesting credit cycles. Figure 2 depicts the sequencing of events with a decision tree in case the investment decision is made.

A contract is said to be *feasible* if it is an element of the following set with initial stock level S,

Definition 1 *Let* $\Lambda^{FB}(S) = \{\sigma = (c_t, m_t, I_t, S_{t+1})$ *that satisfy (1)-(3), with* $S_0 = S\}$

Remark 1 *Note that* $\Lambda^{FB}(S)$ *is not convex due to the non-convexity of the production technology; technically this is due to the presence of the indicator function on the right hand side of (1). Although this complicates the analysis, the utility possibility set for each agent is nevertheless a convex interval, which makes the problem manageable on the plane of agents' utilities.*

See Figure 3 for an example of a value function P*, given* $v = 0$*, i.e., all surplus goes to the lender. The striking feature is that, there are jumps every* Y *units on the resource axis. The first major jump happens at* $(I - Y)$ *since that is the threshold level of beginning-of-period level of resources that makes investment feasible (lender's resource constraint). The other jumps are higher-order jumps. For example,* P *jumps at* $2(I - Y)$ *because if* $S = 2(I - Y) - \epsilon$ *for a very small positive* ϵ*, in case of failure, resources fall short of* $(I - Y)$ *next period and the lender's resource constraint binds. However, if* $S = 2(I - Y) + \epsilon$*, it is feasible to implement projects next period in case of a liquidity shock. The jumps to the left of* $(I - Y)$ *have a similar explanation.* P *jumps at* $(I - 2Y)$ *because it matters whether* S *is to the right or left of it when it comes to saving for the next period at the end of the period. In the former case, the resource constraint binds next period even if everything is saved; in the latter it does not.*

With this machinery at hand, We can characterize the set of efficient allocations in this economy. Such allocations are solutions to the following parameterized family of problems

$$P(v, S) = \max_{\sigma \in \Lambda^{FB}(S)} \left\{ EU^L(\sigma) \mid EU^E(\sigma) \geq v \right\} \tag{4}$$

where v is feasible in the sense that there exists a feasible contract σ that gives the borrower an ex-ante present discounted utility of at least v.

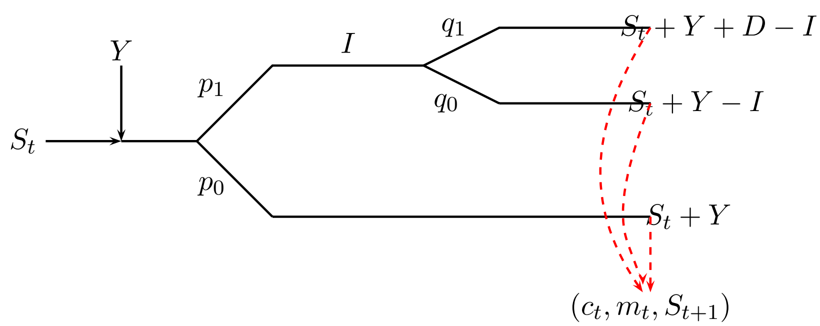

Figure 2: Decision Tree

An efficient allocation has to be efficient after any history. Otherwise, it would have been possible to replace the tail of the allocation after some history with a Pareto-improving one. This would have made the original allocation non-efficient at time $t = 0$, since all histories are reached with positive probability. This property of an efficient allocation helps us write (4) as a recursive first-best problem (RFB). It follows that

$$
\begin{aligned}
(RFB) \quad P(v,S) \quad &= \max_{(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_\theta, v_{\theta\mu}) \in \mathbb{R}_+^{18}} \sum_{\theta\mu} p_\theta q_\mu [(1-\beta) m_{\theta\mu} + \beta P(v_{\theta\mu}, S_{\theta\mu})] \\
\text{s.t.} \quad & \sum_{\theta\mu} p_\theta q_\mu [(1-\beta) c_{\theta\mu} + \beta v_{\theta\mu}] \geq v && (5) \\
\text{and } \forall \theta,\ \forall \mu: \quad & S_{\theta\mu} \leq S + Y + D 1_{\{\theta=1,\ \mu=1,\ I_\theta \geq I\}} - I_\theta - m_{\theta\mu} - c_{\theta\mu} && (6) \\
& I_\theta \leq S + Y && (7) \\
& v_{\theta\mu} \in [0, \overline{v}_{S_{\theta\mu}}]. && (8)
\end{aligned}
$$

where we use $1 - p_0 = p_1 = p$ and $1 - q_0 = q_1 = q$. The new random variables θ and μ share the same support and distribution with θ_t and μ_t, respectively, for any t. The constraint in (5) is a promise keeping constraint that guarantees that the borrower receives at least utility v, on average, from the contract offered by the lender. The one in (8) makes sure that the continuation values offered to the borrower are feasible. (6) and (7) are the recursive versions of (1) and (2), respectively. We show in the appendix the equivalence of these two programs and the existence of the value function as stated in the following proposition, along with some characterization results.

Proposition 1 *1. An optimal first-best contract exists.*

2. The optimal investment and saving policies $(v, S) \to I_\theta(v, S)$ *and* $(v, S) \to S_{\theta\mu}(v, S)$ *are*

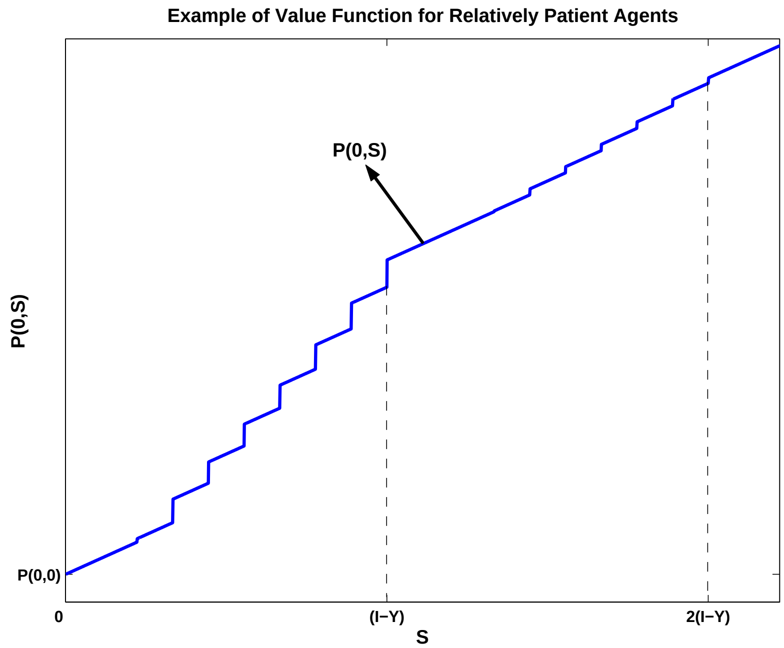

Figure 3: Jumps in the Value Function

independent of v, $\forall\theta, \forall\mu$, *i.e.,* $I_\theta(v, S) = I_\theta(v', S)$ *for any feasible* v, v' *given any* $S \geq 0$. *Similarly for* $S_{\theta\mu}$.

3. P *is strictly increasing in* S *and strictly decreasing in* v. *The Pareto Frontier is characterized by* $P(v, S) = \overline{v}_S - v$ *with* $v \in [0, \overline{v}_S]$, *where* $\overline{v}_S$ *is the highest feasible discounted utility that an agent can achieve in a feasible contract, given* $S \geq 0$.

Note that the existence result in Proposition 1 does not mention uniqueness. The reason for that is that time path of transfers is not uniquely determined as both parties are risk neutral and have the same discount factor. Second part of Proposition 1 states that in the absence of default, which party gets what portion of the surplus generated within the relationship, does not have any bearing on the choice of investment and saving plans. The intuition is clear: If no party has the power to renege on the contract, the only thing that matters is to maximize the surplus to be shared first, then to split it according to a predetermined sharing rule, v. This brings about the last part of Proposition 1: A one unit reduction in v leads to a one unit reduction in P, since the optimal investment and saving policies are not affected from this change and that it is a transferable utility environment.

Proposition 2 1. *There exists* $\overline{S}$ *such that* $S_{\theta\mu}(v, S) \leq \overline{S}$ *for all feasible* (v, S) *and all* θ, μ.

2. *The optimal investment rule is to 'invest' if and only if* $S \geq \underline{S}$ *and* $\theta = 1$*, for some* $\underline{S} \geq I - Y$*.*

First statement says that given any economy, savings are bounded from above. That is because saving is costly ($1 < \beta^{-1}$) and makes sense if it makes the continuation value jump. The intensity of those jumps decreases as S increases, since they become higher-order. After a while, it is not worthwhile to save more since the gain from saving more due to the increase in the continuation value from the jump is less than the cost of consuming less, in the current period.

The second part mainly states that the investment rule is of the threshold nature. If the contract dictates investment for some level of resources, so does it for higher levels of resources. Remember that the first-best optimal policies are independent of v. That is why the results in the Proposition are not dependent on the promise level.

The next Proposition summarizes what we mean by investment cycles generated by the resource-constrainedness of the lender. These are efficient cycles in the sense that they are not caused by incentive problems.

Proposition 3 *Given any economy, we will observe investment cycles for economies with relatively patient agents, almost surely. For low discount rate economies, the economy gets stuck in a 'credit crunch' region with probability one, in finite time. Conditioning on a state where productive investment is undertaken in the economy, the expected number of periods it takes to go into a state of no investment is an increasing function of the discount factor and the likelihood of investments being productive.*

3.1 First-Best Contracts for Two classes of Economies: The case of Low βs

The following will be our working example in this and the next section. The economy considered is a special case of the general economy outlined above. Propositions 1 and 4 will apply. Moreover, we will be able to give a more explicit characterization of the optimal contract and use that to stress the important aspects stemming from the incentive compatibility and the resource constraint. We begin with providing the full characterization of the first-best contracts for two classes of economies.

Example 1 *Given the values for* Y, I, D, p, q*, there exist two threshold levels of the discount factor,* β_1 *and* β_2*, where* $0 < \beta_1 < \beta_2 < 1$ *such that*

1. *For any* $\beta \in (0, \beta_1)$*, a first-best contract exists and is characterized by:*
   - $I_1(v, S) = I$ *iff* $S \geq (I - Y)$*.*
   - *Do never save.*
2. *For any* $\beta \in [\beta_1, \beta_2)$*, a first-best contract exists and is characterized by:*

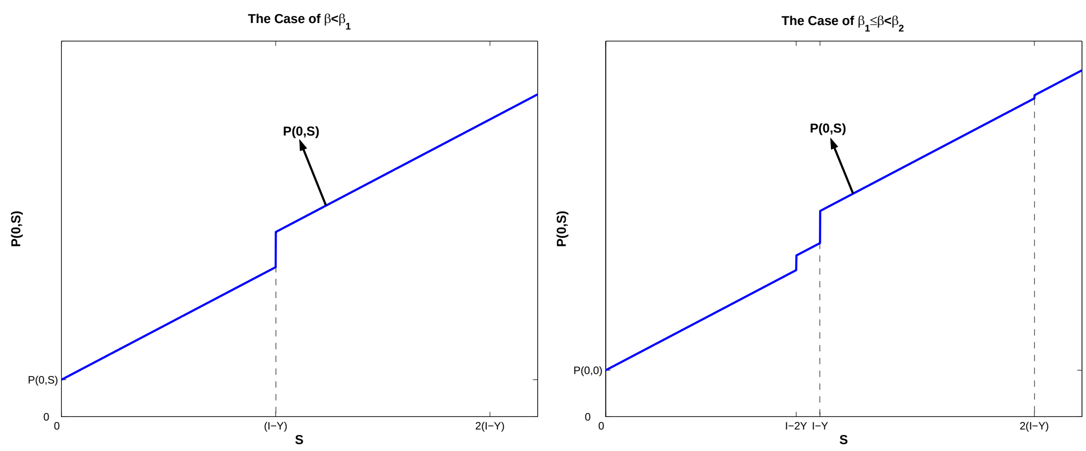

Figure 4: Maximum Surplus for Two Classes of Discount Factors

- *$I_1(v,S) = I$ iff $S \geq (I-Y)$.*
- *Save $(I-Y)$ if feasible, otherwise save 0*

Notice that v is not part of the characterization since from Proposition 1.2, the optimal investment and saving decisions are independent of the promise level. Moreover, consumption paths are not part of the characterization either. As mentioned before, that is because they are indeterminate due to the risk neutrality of the agents and that they have the same time preference.

First part of the example gives a characterization of the first-best contracts for the case of extremely impatient agents. The continuation value of the partnership is too small for these agents to save at all. The threshold discount factor $\beta_1 = \frac{(I-Y)}{(I-Y)+p_1(q_1 D - I)}$, as we show it in the proof, is the one that makes the agents indifferent between saving and not saving. The problem, de facto, becomes one of one-period project funding. Since investing is socially optimal, barring any incentive problems, investment is undertaken whenever it is feasible to do so. A close look at the threshold discount factor provides further insights. β_1 is an increasing function of I and a decreasing function of p_1, q_1, D and Y. The intuition is clear: as the probability of project arrival and the probability of success of investments increase, the marginal types start saving. That's because the continuation value of the partnership increases. Similarly for D. If Y increases, the number of periods that the agents should accumulate before they start investing decreases, which makes it worthwhile for some non-savers to start saving. A decrease in the value of I works exactly in the same direction. Hence, the set of types (discount rates) who will save becomes larger.

Second part of the example characterizes the first-best contracts for agents who are "just patient" enough to save for the next period that amount, $(I-Y)$, that will make it feasible to invest in case of a productive shock. The added value of saving more than this amount to self-insure for more than one period, is not enough to compensate for the utility cost incurred. Similarly, it is not worth building the necessary stock to be able to invest in case of a liquidity shock in the future, if the initial resources are too small. The same comparative statics exercise that we did above for β_1 can be undertaken for β_2 and shown that β_2 behaves the same way.[7]

3.1.1 Dynamics

Let the following be the set of states for the Markov aggregate system of our economy, generated by the optimal investment and saving rules.

$$\mathcal{S} \equiv \{S^* \mid S^* \text{ is the optimal saving level for some level of end-of-period resources}\}$$

Optimal savings are at the discontinuity points of the value function P. This is because saving is relatively costly and if savings are at a continuity point of P, there is always the temptation of cutting them down since the gain from saving one unit less is $(1-\beta) > \beta(1-\beta)$, the cost of continuing with one unit less (P has constant slope $(1-\beta)$ at continuity points). Hence, in the case of the first class of economies, $\mathcal{S} = \{0\}$, since saving zero is the optimal strategy for any S. This is an 'absorbing state' and the economy will be in that state forever at the period-ends, from second period on. From the second period on, no investment projects will be undertaken. Capital will be depleted and the economy will be in a constant state of stagnation.

For the second class of economies, the transitions are a bit more interesting. In this case, $\mathcal{S} = \{0, (I-Y)\}$. The Markov transition matrix, R, can be computed easily, by referring to Figure 2.

$$R \equiv \begin{bmatrix} 1 & 0 \\ p_1 q_0 & 1 - p_1 q_0 \end{bmatrix}$$

Let the first row denote state 0 and the second row represent state $(I-Y)$. For example, the probability of moving from state $(I-Y)$ to state 0 is given by $R_{21} = p_1 q_0$. The probability of ending up in the absorbing state in finite time is

$$1 - \lim_{n\to\infty} (p_1 q_0)^n = 1$$

since it is simply the complement of the event 'always in state $(I-Y)$'. Once again, the dynamics are simple. From the second period on, if the initial resources are sufficient to implement projects, the economy stays in state $(I-Y)$, for some time, with positive probability. This is a very fragile state since one bad shock (project failure) is sufficient to move the economy into the state of capital crunch where it stays forever. The lender's resource constraint binds; lender cannot provide the

[7] Although we don't have a clear explicit form for β_2, a look at Figure 7 reveals that an increase in p_1, q_1, D and Y pulls up the intersection of the function with the vertical axis, C, whereas a decrease in I takes the function down by taking its value at $\beta = 1 = -F$, down. Both these movements make the intersection of the function with the horizontal axis, β_2, move to the left.

borrower with any liquidity since he has no funds available. Positive net present value projects are passed up. The interesting feature is that these are ‘efficient dynamics’.

4 Second-Best Contracts

From now on, we are dropping the perfect commitment assumption on the part of the borrower. The lender can commit to the clauses of the current contract as long as his ex-ante participation constraint is satisfied at $t = 0$. In contrast, the borrower has the opportunity to renege on the agreement after the investment is taken and can run away with the return on investment that he confiscates. Remember that the borrower goes back to autarky, in case he defaults. A contract, then, analogously to the first-best, is said to be *feasible* if it is an element of the following set with initial stock level S,

Definition 2 *Let* $\Lambda^{SB}(S) = \{\sigma = (c_t, m_t, I_t, S_{t+1})$ *that satisfy (1)-(3), with* $S_0 = S$ *and an incentive compatibility constraint (IC), i.e.,*

$$\forall t,\ \forall h^t,\ (1-\beta)c_t(h^t) + \beta EU^E(\sigma \mid h^t) \geq (1-\beta)D1_{\{\theta_t=1,\ \mu_t=1\ \text{and}\ I_t(h^{t-1},\theta_t)\geq I\}} + \beta Y\}$$

Taking these constraints into account, second-best contracts will be the solutions to the following program

$$Q(v,S) = \max_{\sigma\in\Lambda^{SB}(S)} \left\{EU^L(\sigma) \mid EU^E(\sigma) \geq v\right\} \tag{9}$$

where v is feasible in the sense that there exists a feasible contract σ that gives the borrower at least an ex-ante utility level of v. Notice that feasible contracts for this program are the ones coming from the set of efficient contracts, $\Lambda^{FB}(S)$, that satisfy one extra condition, namely, the no-deviation constraint.

Notice that the above constraints are the “best deviation” constraints for the borrower. In general, the borrower might deviate from what the allocation prescribes in many different ways; none of these would bring him a higher payoff then the best deviation strategy would. It is in this sense that if the allocation satisfies the best-deviation constraints, then it will be incentive feasible.

The program above can be written recursively, using v and S as state variables. These two are good “summary statistics” for our program, giving all the necessary information required to be able to solve for the second-best. Hence, the recursive second-best program (RSB) is:

$$(RSB) \quad Q(v,S) = \max_{(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_\theta)\in\mathbb{R}^{14}_+} \sum_{\theta\mu} p_\theta q_\mu[(1-\beta)m_{\theta\mu} + \beta Q(v_{\theta\mu}, S_{\theta\mu})] \tag{10}$$

$$\text{s.t.} \quad \sum_{\theta\mu} p_\theta q_\mu[(1-\beta)c_{\theta\mu} + \beta v_{\theta\mu}] \geq v \tag{11}$$

$$S_{\theta\mu} \leq S + Y + D1_{\{\theta=1,\ \mu=1,\ I_\theta\geq I\}} - I_\theta - m_{\theta\mu} - c_{\theta\mu} \tag{12}$$

$$I_\theta \leq S + Y \tag{13}$$

$$(1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \geq (1-\beta)D1_{\{\theta=1,\ \mu=1,\ I_\theta \geq I\}} + \beta Y \tag{14}$$

$$v_{\theta\mu} \in [Y, \overline{v}_{S_{\theta\mu}}]. \tag{15}$$

The only addition to the set of constraints is (14), which makes sure that the borrower gets a utility level at least as high as he would get by defaulting, on the path where investment is undertaken and it is a success. On the other realization paths, the constraint is redundant since continuation values, for the borrower, need to be at least as large as Y, for the borrower to accept staying in the relationship. Otherwise, he can always go back to autarky and guarantee himself Y, on average.

The following Proposition is the heart and core of our entire analysis. It first states that a solution to the above program exists and then it proceeds with giving a full characterization of the second-best contracts that arise between the parties given the parametrization of the economy.

Proposition 4 *1. An optimal second-best contract exists.*

2. There exists an $\underline{S} \geq (I - Y)$ such that for all $S \geq \underline{S}$, there are two promise values $0 \leq v_*(S) \leq v^*(S) \leq \overline{v}_S$ such that

(a) $I_1(v, S) = 0$, for $v \in [Y, v_*(S)]$,

(b) $I_1(v, S) = I$, for $v \in [v_*(S), \overline{v}_S]$

(c) The value function in (RSB) is given by

$$Q(v,S) = \begin{cases} \overline{v}_S - v & \text{if } v \geq v^*(S) \\ \overline{v}_S - v^*(S) & \text{if } v \in [v_*(S), v^*(S)] \\ \overline{v}_S - [v^*(S) - v_*(S)] - v & \text{if } v \in [Y, v_*(S)] \end{cases}$$

(d) $v^*(S)$ is nondecreasing in S.

The statements in Proposition 4 , which are also depicted in Figure 5 have a very nice, sound economic intuition. For each level of v and S, the number of ways the incentives can affect the first-best utility levels is two. What distinguishes the figure on the left from the one on the right is the fact that, first-best optimal policies are also second-best optimal on the left. On the right, below a level of v, investment is not undertaken under the second-best rule although it is under the first-best rule. Notice that, both these cases refer to the first-best optimal 'Invest' regime. When the first-best rule is not to invest in the current period, there is no distributional conflicts arising from incentive compatibility; hence first-best and second-best utility levels coincide, which is the case in Figure 6.

Proposition 5 *1. Second-best savings are less than or equal to the first-best savings.*

2. The optimal investment rule, $(v, S) \to I_\theta(v, S)$, is a nondecreasing function of v.

Now, a couple of comments are in order. Second-best savings are less than or equal to first-best savings capturing the intuition that it does not pay off to over-save relative to the first-best level to self-insure against the possibility of ending up in a "credit crunch" regime, where there are not enough resources to undertake projects in case of a liquidity shock. The result that savings are bounded from above for a given economy is trivially implied by this statement.

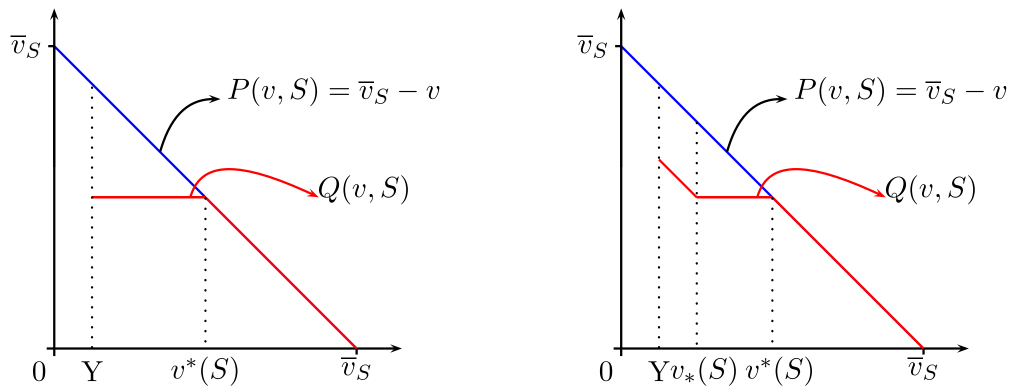

Figure 5: Second-Best Frontier for $S \geq \underline{S}$.

Depending on the parametrization of the economy, we have either of these two figures for the second-best frontier. For each S, $v^*(S)$ is the minimum promise level below which (11) holds with inequality. The reason is first that you have to make sure that the borrower doesn't default. Hence you need to provide him with at least the default utility, on the success realization path. Second, the continuation promise levels must be accepted by the borrower. Hence, even if the initial level of v is below $v^*(S)$, the de facto payment, on average is equal to $v^*(S)$. For some parameterizations, there is another threshold level $v_*(S)$ below which "rationing the credit" is optimal although it is socially optimal to undertake the investment in the absence of incentive problems. The idea is that, if investment is undertaken, the cost of making sure that the borrower does not default might be so high that the lender prefers not extending the credit although first-best requires him to do so.

4.1 Second-Best Contracts for Two Classes of Economies: The Case of Low β's

Here, we take it from where we left in the previous section's Example 1 and analyze the behavior of the second-best contracts and compare it to the benchmark case of first-best contracts. The explicit characterization of the former for both economies makes it clear what kind of distortions the enforceability problems cause to the socially optimal allocations and utility levels.

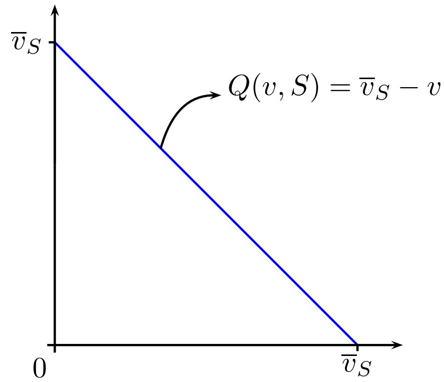

Figure 6: Second-Best Frontier for $S < \underline{S}$.

Example 2 *1. For any $\beta \in (0, \beta_1)$, a stationary second-best contract exists and is characterized by:*

(a) For $S \geq I - Y$, there exist $0 \leq v_* \leq v^* \leq \overline{v}_S$ such that

i. $I_1(v,S) = 0$, for $v \in [Y, v_*]$,

ii. $I_1(v,S) = I$, for $v \in [v_*, \overline{v}_S]$

(b) Do never save

(c) We have the figure on the right iff $p_1 > \frac{Y}{I}$.

2. For any $\beta \in [\beta_1, \beta_2)$, a stationary second-best contract exists and is characterized by:

(a) For $S \geq I - Y$, there exist $0 \leq v_*(S) \leq v^*(S) \leq \overline{v}_S$ such that

i. $I_1(v,S) = 0$, for $v \in [Y, v_*(S)]$,

ii. $I_1(v,S) = I$, for $v \in [v_*(S), \overline{v}_S]$

(b) Save $(I - Y)$ if feasible, otherwise save 0

One interesting feature of the first class of economies is the fact that we have the second-best frontier on the right hand side of Figure 5 if $p_1 > \frac{Y}{I}$. So, if the probability of a liquidity shock is high, the lender does not extend credit because it is too costly to ensure no-default in case investment is undertaken; an increase in p_1 raises the weight of that state in the expected utility computation.

5 Conclusion

In this paper, we studied the nature of long-term liquidity provision between lenders and borrowers in the absence of perfect enforceability and when both parties are financially constrained. To this end, we built an infinite horizon model of long-term lending and borrowing and analyzed in what ways liquidity shortages on both sides affect the evolution of the economy and investment activity in particular.

Stylized facts about episodes of credit crunches and financial distress in general tell us that banks become wary of lending in case their capital levels start to decrease. This in turn leads borrowers with positive net present value projects to be denied credit. The current model captures these stylized facts in a compelling way.

Relative to the first-best contracts, second-best contracts feature more interesting properties. Enforcement problems and endogenous resource constraints can severely reduce the possibility of financing projects. Investment and savings are functions both of the level of resources and the surplus sharing rule, in contrast to the first-best contracts. Investment is (weakly) underprovided. Second-best savings are (weakly) less than first-best savings capturing the intuition that it does not pay off to postpone consumption/investment to self-insure against a possible 'credit crunch'.

We also show that the economy exhibits investment cycles of two different natures. First type of cycles happen because the lenders are resource constrained; these are efficient cycles. The second type of cycles are due to incentive compatibility. The punchline is: credit is rationed if either the lender has too little capital or the borrower has too little financial collateral.

The present work's technical contribution is to show the existence and characterization of financial contracts that are solutions to a non-convex dynamic programming problem.

The next step should be to build and analyze a model in which the opportunity cost (capital constraint) of lenders is endogenized by explicitly modelling the credit markets as a dynamic game between lenders for loanable funds. The introduction of competition among banks not only enriches the story but also makes it possible to analyze in a realistic way renegotiation issues, reputation building on the part of the lender/borrower, and endogenous opportunity costs and credit rationing due to locking-up with a current set of borrowers.

There are a couple of other points that should not be left out: One should look at economies where there are both aggregate and micro level liquidity shocks and consider heterogeneity among lenders and borrowers. Heterogeneity and aggregate uncertainty introduces the possibility of intrainstitutional arrangements in a dynamic setting, that the present model cannot capture. Of course, all of this is for future work.

6 Appendix

As we mentioned before, this section contains the proofs to all propositions and technical results contained in the main body of the paper.

Proof of Proposition 1

In the first-best program, period utility functions and the space of feasible resources are unbounded. We first show that at the optimum, the set $\{EU^L(\sigma) \mid EU^E(\sigma) \geq v,\ \sigma \in \Lambda^{FB}(S)\}$ is bounded from above by the value of a relaxed program. This allows us to show that the supremum function exists for (4) and is finite-valued. This latter defines a set, F, of feasible (v, S) pairs. We define a functional space, $B(F)$, on that and an operator, T, on that functional space, which is associated with the (RFB) problem. The idea behind this construction is that 'limited liability' requires that the value function be non-negative for any feasible value of v in the first-best program. So, the value function itself enters the constraint set of the problem. This brings up the question 'what is the set of feasible values of v' for any given S. We will first prove a series of Lemmas which, put together, will deliver the result we are interested in.

Let $P^*(v, S)$ be the value of the supremum in the first-best problem in (4) parameterized by $S_0 = S \geq 0$ and a feasible v. We first show that the supremum of the sequence problem exists and is attained by a contract. Then, we proceed to show that the unique fixed point of the operator T, associated with the (RFB) problem, defined on the 'right' space of candidate value functions, is actually P^*. We then proceed to further characterize the first-best frontier and first-best contracts.

Lemma 1 *$P^*(v, S)$ exists and is attained by an optimal contract. $P^*(v, S) \leq (1-\beta)S + Y + p_1 q_1(D - I)$. In particular, $(1-\beta)S + Y \leq P^*(0, S) \leq (1-\beta)S + Y + p_1 q_1(D - I)$.*

Proof: The following program is a relaxed version of the first-best problem in (4) with $v = 0$, where the economy generates Y for sure plus $(D - I)$ with probability $p_1 q_1$, every period.

$$\begin{aligned} & \sup_{(m_t, S_{t+1})_{t=0}^{\infty}} E(1-\beta)\sum_{t=0}^{\infty}\beta^t m_t \\ \text{s.t. } \forall t,\ \forall h^t \quad & S_{t+1}(h^t) \leq S_t(h^{t-1}) - m_t(h^t) + Y + (D - I)1_{\{\theta_t = 1,\ \mu_t = 1\}} \qquad (16) \\ & m_t, S_{t+1} \geq 0 \text{ and } S_0 = S \geq 0 \text{ is given.} \end{aligned}$$

Objective is linear, constraint set is convex. So, this is a concave programming problem. Hence, the first-order conditions are necessary and sufficient for a maximum. Partially differentiating the objective with respect to $S_{t+1}(h^t)$ gives

$$-\text{Prob}(h^t)\beta^t(1-\beta) + \text{Prob}(h^t)\beta^{t+1}(1-\beta) < 0$$

which implies that $S_{t+1}(h^t) = 0$ for all h^t. This means that the agent consumes all his wealth at the end of each period. Then, the supremum is achieved and by simple algebra, is equal to

$$(1-\beta)S + Y + p_1 q_1(D - I)$$

Clearly, any contract that is feasible for the original problem is also feasible for the relaxed problem making $(1-\beta)S + Y + p_1 q_1 (D - I)$ an upper bound for the set of expected discounted utilities that the agent can get from feasible contracts. Hence, the supremum to the original problem exists and $P^*(0, S) \leq (1-\beta)S + Y + p_1 q_1 (D - I)$. A similar argument shows that $P^*(0, S)$ is an upper bound to the set of expected discounted utilities with $v \neq 0$, since the constraint set of the latter problem is smaller than that of the problem with $v = 0$. Hence, supremum is well-defined and $P^*(v, S) \leq P^*(0, S)$ for any (v, S) feasible. Supremum is attained by an optimal contract because the constraint set is a closed set defined by a sequence of weak inequalities. Hence the sup in the statement of the problem can be replaced by a max and an optimal contract can be computed for each S and v.

One feasible strategy in the original problem is to 'never invest' after any history. Conditioning on 'never investing', the optimal strategy solves

$$
\begin{aligned}
& \max_{(m_t, S_{t+1})_{t=0}^{\infty}} E(1-\beta) \sum_{t=0}^{\infty} \beta^t m_t \\
\text{s.t. } \forall t, \ \forall h^t \quad & S_{t+1}(h^t) \leq S_t(h^{t-1}) - m_t(h^t) + Y \\
& m_t, S_{t+1} \geq 0 \text{ and } S_0 = S \geq 0 \text{ is given.}
\end{aligned}
$$

By the same token as above, the maximum is achieved and is equal to

$$(1-\beta)S + Y$$

Since this is a possibly suboptimal strategy, we have

$$(1-\beta)S + Y \leq P^*(0, S)$$

Gathering both sides of the inequalities together, we obtain the stated result. □

Now, we are ready to lay down the functional equation and show that P^* is the only fixed point of the operator defining the functional equation. The important thing is to find the 'right' functional space for the recursive problem. Let F be the set of (v, S) pairs for which a feasible contract exists; namely

$$F \equiv \left\{(v, S) \in \mathbb{R}_+^2 \mid v \leq P^*(0, S)\right\}$$

It turns out that the 'right' space for our problem is the space of non-negative valued functions on F, that are non-increasing in v, non-decreasing in S and bounded above by the function $(1-\beta)S + Y + p_1 q_1 (D - I)$. Namely,

$$
\begin{aligned}
B(F) \quad = \quad & \{f : F \to \mathbb{R}_+ \mid \text{ s.t. } (i) - (iii) \text{ hold where}\} \\
& (i) \ f(v', S) \leq f(v, S) \ \text{ for } v' > v \\
& (ii) \ f(v, S) \leq f(v, S') \ \text{ for } S' > S \\
& (iii) \ 0 \leq f(v, S) \leq (1-\beta)S + Y + p_1 q_1 (D - I)
\end{aligned}
$$

Let the operator T be defined on $B(F)$ for any $(v,S)\in F$ and any $P\in B(F)$ by:

$$\begin{aligned}
(TP)(v,S) \quad &= \max_{(c_{\theta\mu},m_{\theta\mu},S_{\theta\mu},I_\theta,v_{\theta\mu})\in\mathbb{R}^{18}_+} \sum_{\theta\mu} p_\theta q_\mu[(1-\beta)m_{\theta\mu}+\beta P(v_{\theta\mu},S_{\theta\mu})] \\
\text{s.t. } \forall\theta,\ \forall\mu \quad & (v_{\theta\mu},S_{\theta\mu})\in F \qquad (17)\\
& S_{\theta\mu}\le S+Y+D1_{\{\theta=1,\ \mu=1,\ I_\theta\ge I\}}-I_\theta-m_{\theta\mu}-c_{\theta\mu}\\
& I_\theta\le S+Y\\
& \sum_{\theta\mu} p_\theta q_\mu[(1-\beta)c_{\theta\mu}+\beta v_{\theta\mu}]\ge v
\end{aligned}$$

Lemma 2 *$T:B(F)\to B(F)$ and P^* is its unique fixed point.*

Proof: $(TP)(v,S)\ge 0$ for any $(v,S)\in F$ and $P\in B(F)$ due to limited liability and the non-negativity of P. Let $\overline{P}(v,S)\equiv(1-\beta)S+Y+p_1q_1(D-I)$ be the upper bound function for elements of $B(F)$. $\overline{P}$ trivially is an element of $B(F)$. With $\overline{P}$ being the continuation function, the optimal consumption/saving decision is to consume everything at the end of the period. Conditioning on that, the optimal investment strategy is to invest whenever it is feasible. But then,

$$\begin{aligned}
(T\overline{P})(v,S)\le(T\overline{P})(0,S) \quad &= \begin{cases}(1-\beta)S+Y+p_1q_1D-(1-\beta q_0)p_1I, & \text{if } S\ge I-Y\\ (1-\beta)S+Y+\beta p_1q_1(D-I), & \text{if } S<I-Y.\end{cases}\\
&\le \overline{P}(v,S)
\end{aligned}$$

Clearly, T is monotonic in P and $(T\overline{P})$ is non-increasing in v and nondecreasing in S which makes the latter an element of $B(F)$. Hence, $0\le(TP)(v,S)\le(T\overline{P})(v,S)\le\overline{P}(v,S)$ for any $(v,S)\in F$ and $P\in B(F)$. This establishes that T is a self-map.

$P^*\in B(F)$ and by the standard arguments (see [36] Theorem 4.2), $TP^*=P^*$. The next step is to show that this is the only fixed point of the operator T. What is required is a boundedness condition, $\lim_{n\to\infty}\beta^nP(v_n,S_n)=0$, where $(v_n,S_n)_{n=0}^\infty$ is a particular realization path of a feasible contract from (S_0,v_0). So, this condition requires the value function to be bounded on any realized path from (v_0,S_0) for a feasible contract, and is sufficient to guarantee that any P that satisfies this condition is actually the supremum function. We will show that $\lim_{n\to\infty}\beta^nP(v_n,S_n)=0$ for any fixed point P of T. Since there is a finite number of states of nature each period, Theorem 4.3 in ([36]) can be modified to argue that any fixed point P of (17) that satisfies the boundedness condition is actually the supremum function P^*.

The fastest accumulation path for the problem in (16) is $S_1=S_0+Y+p_1q_1(D-I)$, $S_2=S_1+Y+p_1q_1(D-I)=S_0+2(Y+p_1q_1(D-I))$, $\ldots$, $S_n=S_0+n(Y+p_1q_1(D-I))$, $\ldots$ Clearly, it is so for the original problem, too. Hence for any realized feasible path $(v_n^*,S_n^*)_{n=0}^\infty$ of the original problem, we have

$$P(v_n^*,S_n^*)\le P(0,S_n^*)\le\overline{P}(v_n^*,S_n)$$

The discounted versions respect the same ordering

$$\beta^n P(v_n^*, S_n^*) \leq \beta^n P(0, S_n^*) \leq \beta^n \overline{P}(v_n^*, S_n)$$

Substituting for $\overline{P}(v_n^*, S_n)$, we have

$$\beta^n \overline{P}(v_n^*, S_n) = \beta^n \left[(1-\beta)(S_0 + n(Y + p_1 q_1 (D - I)) + Y + p_1 q_1 (D - I)\right]$$

which goes to zero as n goes to infinity. Therefore, so does $\beta^n P(v_n^*, S_n^*)$. By the above argument, $P = P^*$. □

We will use P for the supremum function which is also the unique fixed point of (17) by Lemma 2. $P(0, S)$ is the maximum surplus that can be generated in an economy parameterized by S. This latter follows from the fact that, the best first-best contract for the borrower would solve the same problem, modulo a relabelling of the variables and the following Lemma.

Lemma 3 *The optimal investment and saving strategies* $(I_t, S_{t+1})_{t=0}^{\infty}$ *depend only on* S*, not on* v*. Moreover,* $P(v, S) = P(0, S) - v$ *for* $v \in [0, P(0, S)]$*.*

Proof: Let $(m_t^*, I_t^*, S_{t+1}^*)_{t=0}^{\infty}$ be the optimal contract that achieves the supremum in the first-best with $v = 0$ and $S_0 = S$. Let $v \in [0, P(0, S)]$. We will first construct $(c_t^{**}, m_t^{**}, I_t^{**}, S_{t+1}^{**})_{t=0}^{\infty}$ where $c_t^{**} = \alpha m_t^*$ and $m_t^{**} = (1 - \alpha) m_t^*$, where $\alpha \in (0, 1)$ is s.t.

$$v = E(1-\beta) \sum_{t=0}^{\infty} \beta^t c_t^{**} = \alpha E(1-\beta) \sum_{t=0}^{\infty} \beta^t m_t^*$$

and $S_{t+1}^{**} = S_{t+1}^*$, $I_t^{**} = I_t^*$ as before. Clearly, this new contract is feasible and achieves the utility level v for the borrower. Now, suppose for a contradiction that the new contract is Pareto dominated by $(c_t', m_t', I_t', S_{t+1}')_{t=0}^{\infty}$, the new optimizer. Therefore,

$$v = E(1-\beta) \sum_{t=0}^{\infty} \beta^t c_t'$$

since the IR constraint for the borrower binds necessarily at the optimum and

$$P(v, S) = E(1-\beta) \sum_{t=0}^{\infty} \beta^t m_t' > E(1-\beta) \sum_{t=0}^{\infty} \beta^t m_t^{**} = E(1-\beta) \sum_{t=0}^{\infty} \beta^t m_t^* (1-\alpha)$$

But this is going to imply that the original contract for $v = 0$ could not have been optimal. Let's define the contract $(c_t'', m_t'', I_t'', S_{t+1}'')_{t=0}^{\infty}$ by $c_t''(h^t) = 0$, $m_t''(h^t) = (m_t'(h^t) + c_t'(h^t))$ with $I_t'' = I_t'$ and $S_{t+1}'' = S_{t+1}'$. This is clearly feasible for the problem with $v = 0$ and

$$
\begin{aligned}
P(0,S) \geq E(1-\beta)\sum_{t=0}^{\infty}\beta^t m_t'' &= E(1-\beta)\sum_{t=0}^{\infty}\beta^t(c_t'+m_t') \\
&> E(1-\beta)\sum_{t=0}^{\infty}\beta^t c_t' + E(1-\beta)\sum_{t=0}^{\infty}\beta^t m_t^*(1-\alpha) \\
&= \alpha P(0,S) + (1-\alpha)P(0,S) \\
&= P(0,S)
\end{aligned}
$$

a contradiction.

As to the second claim, the contract $(c_t^{**}, m_t^{**}, I_t^{**}, S_{t+1}^{**})_{t=0}^{\infty}$ is shown to be feasible and achieve the supremum for the first-best program with $v \in (0, P(0,S))$. Hence,

$$
\begin{aligned}
P(v,S) &= E(1-\beta)\sum_{t=0}^{\infty}\beta^t m_t^{**} \\
&= E(1-\beta)\sum_{t=0}^{\infty}\beta^t(1-\alpha)m_t^* \\
&= (1-\alpha)E(1-\beta)\sum_{t=0}^{\infty}\beta^t m_t^* \\
&= P(0,S) - v
\end{aligned}
$$

from the definition of α. $\square$

1. Lemma 1 with Lemma 2 proves this part.
2. Lemma 2 with Lemma 3 provides the proof.
3. That P is strictly decreasing in S is obvious from the definition of the program. The rest is by Lemma 3. $\square$

Proof of Proposition 2

1. Suppose for a contradiction that for any $\overline{S}$, there is an S' such that for some (v,S) and (θ,μ), $S' = S_{\theta\mu}(v,S) > \overline{S}$. Then, we can construct a nondecreasing sequence (S_n'):

$$
S_n' = S_{\theta_n\mu_n}(v_n, S_n) > n \quad , \text{for } n \in \mathbb{N}
$$

for some (v_n, S_n) feasible, and (θ_n, μ_n) for each n. Let W_n be the end-of-period wealth on the realized path corresponding to (v_n, S_n) and (θ_n, μ_n). Clearly, this sequence is unbounded and for each consecutive terms, we have (since first-best decisions are independent of v)

$$
(1-\beta)[W_{n+1} - S_n'] + \beta P(0, S_n') \leq (1-\beta)[W_{n+1} - S_{n+1}'] + \beta P(0, S_{n+1}')
$$

due to optimality, with at least one strict inequality. But, then this ordering is independent of the particular wealth level since we can just get rid of those from both sides of the inequality. Hence,

$$(1-\beta)[-S_1] + \beta P(0, S_1) < (1-\beta)[-S_{n+1}] + \beta P(0, S_{n+1})$$

for large n. This in turn implies that

$$\frac{P(0, S_{n+1}) - P(0, S_1)}{S_{n+1} - S_1} > \frac{(1-\beta)}{\beta} > (1-\beta)$$

Hence, P should increase, on average, with a slope larger than $(1-\beta)/\beta$, greater than $(1-\beta)$. This would imply that P should intersect the line $(1-\beta)S + Y + p_1 q_1 (D - I)$ eventually, which is a contradiction since that is an upper bound for P. Moreover, this level $\overline{S}$ is achieved by the same token above. For large levels of W, $\overline{S}$ and everything less than that will be available. Since P is bounded above and P is clearly right-continuous, there is a threshold level of end-of-period wealth after which optimal savings are $\overline{S}$. □

Proof of Proposition 3

Let the following be the set of states for the Markov aggregate system of our economy, generated by the optimal investment and saving rules.

$$\mathcal{S} \equiv \{S^* \mid S^* \text{ is the optimal saving level for some level of end-of-period resources}\}$$

Optimal savings are at the discontinuity points of the value function P as we point out in the Proof of Example 1. We know from Proposition 2.1 that this set is bounded. Clearly, $\mid \mathcal{S} \mid < \infty$, due to the discreteness of the problem. Given an economy, we obtain an optimal saving policy which defines a Markov transition matrix R, for our aggregate system. Clearly, the savings policy is monotonically nondecreasing (with more than one best response for some levels of resources for which case we use the convention of picking the smallest one). Let the states in $\mathcal{S}$ be ordered in an increasing fashion, i.e., $\mathcal{S} = \left\{S_1 < S_2 < \cdots < S_N = \overline{S}\right\}$ where $N = \mid \mathcal{S} \mid$. Hence our economy is going to be in one of these states at the end of each period with some probability. For small discount factors, $\beta \in [\,0, \beta_2)$, we show in section 3.1 that the economy gets stuck in the absorbing state $S_1 = 0$, in finite time. We can observe implementation of projects before the system gets absorbed by that state, if the initial resources are large.

For larger discount factors, there is no absorbing state. For any n, with probability $1 - p_1 q_0$, the system moves to a state $S' \geq S_n$ ($S' > S_n$, with probability at least $p_1 q_1$, for $n < N$) and to a lower state with probability $p_1 q_0$. Hence, for high discount factors, the state space is very fine. Hence, the probability of going from one state to the other is always positive which guarantees that the system will hit each state with probability one.

As the discount factor increases, the set of states becomes finer which makes it harder to fall down into the region where resource constraint of the lender binds. Second, trivially, the probability of going down to a lower state is $p_1 q_0$ from second period on. This probability decreases if q_1 goes up (q_0 goes down), which increases the expected number of periods that takes the system to go down to the 'capital crunch' region. $\square$

Proof of Example 1

We know from Proposition 1 that P exists, that there exists $\sigma = (c_{\theta\mu}, m_{\theta\mu}, I_{\theta\mu}, S_{\theta\mu})$, an optimal stationary contract and the optimal saving and investment strategies are independent of v, meaning that $P(v, S) = P(0, S) - v$. W will be used to denote end-of-period wealth.
1. We first conjecture that for any $\beta \in (0, \beta_1)$, where $\beta_1 \equiv \frac{(I-Y)}{(I-Y)+p_1(q_1 D-I)}$, $P(0, S)$ is given by

$$P(0,S) = \begin{cases} (1-\beta)\,S + Y & \text{, if } S < I - Y \\ (1-\beta)\,[S + p_1(q_1 D - I)] + Y & \text{, if } S \geq I - Y \end{cases}$$

It will be shown, next, that the conjectured function P is indeed the value function and the optimal policy functions are as specified in the proposition. The proof proceeds in two steps: First, we check if deviations from the savings strategy pay off; second, we look for an improved investment policy.

Savings: A simple arbitrage argument shows that the optimal savings need to be at the points of discontinuity: At a point of differentiability, the derivative of

$$\max_{0 \leq S' \leq S} (1-\beta)(W - S') + \beta P(0, S')$$

with respect to S' is $-(1-\beta) + \beta(1-\beta) < 0$, which implies that it pays-off to decrease S', if possible. This means that the only possible saving strategy would be to save $I - Y$ when feasible. We claim that saving 0 is optimal. To this end, we look at the difference of the maximum surpluses from saving $I - Y$ and not saving at all, in that order

$$(1-\beta)\,[W - (I - Y)] + \beta P(0, I - Y) - (1-\beta)W - \beta P(0,0)$$

which is

$$\begin{aligned} &= -(1-\beta)(I-Y) + \beta\,(P(0, I-Y) - P(0,0)) \\ &= -(1-\beta)(I-Y) + \beta(1-\beta)\,((I-Y) + p_1(q_1 D - I)) && (18) \\ &= (1-\beta)\,[-(I-Y)(1-\beta) + \beta p_1(q_1 D - I)] < 0 \\ \iff \quad & \beta < \beta_1 = \frac{(I-Y)}{(I-Y) + p_1(q_1 D - I)} && (19) \end{aligned}$$

which is true by hypothesis. All we needed was to show that there was no one-shot profitable deviation from the conjectured saving strategy, which we did.

Investment: Conditioning on the fact that $\theta = 1$ (productive shock), we need to show that $\forall\ S \geq (I - Y)$, investing gives a higher payoff than not investing does. Given the optimal saving policy, investing brings:

$$(1-\beta)\,(S+Y+q_1 D - I) + \beta P(0,0)$$

where not investing brings:

$$(1-\beta)\,(S+Y) + \beta P(0,0)$$

whose difference is

$$(1-\beta)(q_1 D - I) > 0$$

Hence, investing is optimal. It is easy to see that, this saving/investment strategy yields the conjectured maximum surplus function. Then, Proposition 1 implies that any first-best optimal contract, independent of v, will necessarily have this saving/investment strategy pair, as part of it.

2. We first conjecture that for any $\beta \in [\beta_1, \beta_2)$, where β_2 will be computed below, $P(0, S)$ is given by[8]

$$P(0,S) = \begin{cases} (1-\beta)\,S + Y & \text{if } S < I - 2Y \\ (1-\beta)\,[S-(I-2Y)] + \beta P(0, I-Y) & \text{if } I-2Y \leq S < I-Y \\ (1-\beta)\Delta\,[(I-Y)p_1 q_0 + p_1(q_1 D - I)] + Y & \text{if } I-Y \leq S < 2(I-Y) \\ (1-\beta)\,[S+Y-(I-Y)+p_1(q_1 D - I)] + \beta P(0, I-Y) & \text{if } S \geq 2(I-Y) \end{cases}$$

with $\Delta \equiv [1-\beta\,(1-p_1 q_0)]^{-1}$. Once again, we need to show that the conjectured value function is indeed the correct one. To that effect, we show that the stated saving and investment rules are the optimal ones given P.

Savings: We know that savings need to be at the discontinuity points of the value function, by the arbitrage argument that we provided before. Hence, the candidates are: $I-2Y$, $I-Y$, $2(I-Y)$ and not saving at all. The idea is to make sure that saving $I-Y$ does better than all the other possibilities:

- $\forall\ W \geq I-Y$, *saving 0 is not a better policy* which translates into

$$(1-\beta)\,[W-(I-Y)] + \beta P(0, I-Y) - (1-\beta)W - \beta P(0,0) \geq 0$$

[8] This is the proof for the case $0 < Y < I - Y$, which is more interesting since it is harder to self-finance. The proof for the case of $Y \geq I - Y$ is similar.

the left hand side of which is equivalent to

$$\begin{aligned}
&= -(1-\beta)(I-Y) + \beta\left[P(0, I-Y) - P(0,0)\right] \\
&= -(1-\beta)(I-Y) + \beta\Delta(1-\beta)\left[(I-Y)p_1q_0 + p_1(q_1D - I)\right] \qquad (20) \\
&= (1-\beta)\left[-(I-Y)(1-\beta\Delta p_1q_0) + \beta\Delta p_1(q_1D - I)\right]
\end{aligned}$$

By algebra, $(1-\beta\Delta p_1q_0) = (1-\beta)\Delta$, which implies that the last line can be written as:

$$\begin{aligned}
&= \quad (1-\beta)\Delta\left[-(I-Y)(1-\beta) + \beta p_1(q_1D - I)\right] \geq 0 \\
&\iff \quad -(I-Y)(1-\beta) + \beta p_1(q_1D - I) \geq 0 \\
&\iff \quad \beta \geq \beta_1
\end{aligned}$$

which is the case by hypothesis.

- $\forall\, W \geq I - Y$, *saving* $I - 2Y$ *is not a better policy.* Same line of argument yields

$$\begin{aligned}
&= (1-\beta)\left(W - (I-Y)\right) + \beta P(0, I-Y) - (1-\beta)\left(W - (I-2Y)\right) - \beta P(0, (I-2Y)) \\
&= -(1-\beta)Y + \beta[P(0, I-Y) - P(0, (I-2Y))] \\
&= -(1-\beta)Y + \beta(1-\beta)P(0, I-Y) \\
&= (1-\beta)\left[-(1-\beta)Y + \beta\Delta(1-\beta)\left[(I-Y)p_1q_0 + p_1(q_1D-I)\right]\right] \\
&> (1-\beta)\left[-(1-\beta)(I-Y) + \beta\Delta(1-\beta)\left[(I-Y)p_1q_0 + p_1(q_1D-I)\right]\right] \\
&\geq 0
\end{aligned}$$

where the last inequality follows from (20).

- $\forall\, W \geq 2(I-Y)$, *saving* $2(I-Y)$ *is not a better policy.*

$$\begin{aligned}
&= (1-\beta)\left(W - (I-Y)\right) + \beta P(0, I-Y) - (1-\beta)\left(W - 2(I-Y)\right) - \beta P(0, 2(I-Y)) \\
&= (1-\beta)(I-Y) + \beta[P(0, I-Y) - P(0, 2(I-Y))] \\
&= (1-\beta)(I-Y) + \beta[P(0, I-Y) - (1-\beta)[(I-Y) + p_1(q_1D-I) + Y] - \beta P(0, I-Y)] \\
&= (1-\beta)(I-Y) + \beta\left[P(0, I-Y)(1-\beta) - (1-\beta)\left[(I-Y) + p_1(q_1D-I) + Y\right]\right] \\
&= (1-\beta)^2(I-Y) + \beta(1-\beta)\left[P(0, I-Y) - Y - p_1(q_1D-I)\right] \\
&= (1-\beta)^2(I-Y) + \beta(1-\beta)\left[[(I-Y)p_1q_0 + p_1(q_1D-I)](1-\beta)\Delta - p_1(q_1D-I)\right] \\
&= (1-\beta)^2(I-Y) + \beta(1-\beta)^2\Delta p_1q_0(I-Y) + \beta(1-\beta)^2\Delta p_1(q_1D-I) \\
&\quad -\beta(1-\beta)p_1(q_1D-I) \\
&= (I-Y)(1-\beta)^2[1+\beta p_1q_0\Delta] - \beta(1-\beta)[1-\Delta(1-\beta)]p_1(q_1D-I) \\
&= (I-Y)(1-\beta)^2[1+\beta p_1q_0\Delta] - \beta(1-\beta)\beta p_1q_0\Delta p_1(q_1D-I) \\
&= (1-\beta)\left[(I-Y)(1-\beta)(1+\beta p_1q_0\Delta) - \beta^2 p_1q_0\Delta p_1(q_1D-I)\right]
\end{aligned}$$

Now, let

$$\begin{aligned} r &:= p_1 q_0 \\ C &:= I - Y > 0 \\ F &:= r p_1 (q_1 D - I) > 0 \end{aligned}$$

which makes the last line into

$$\begin{aligned} &= (1-\beta)\left[C(1-\beta)(1+\frac{\beta r}{1-\beta(1-r)}) - \frac{\beta^2 F}{1-\beta(1-r)}\right] \\ &= \frac{(1-\beta)}{1-\beta(1-r)}\left[C(1-\beta)(1-\beta+2\beta r) - \beta^2 F\right] \\ &= \frac{(1-\beta)}{1-\beta(1-r)}\left[\beta^2\left(C(1-2r)-F\right) - 2\beta C(1-r) + C\right] \end{aligned}$$

Set

$$\begin{aligned} A &:= \frac{(1-\beta)}{1-\beta(1-r)}\left[\beta^2\left(C(1-2r)-F\right) - 2\beta C(1-r) + C\right] \\ B &:= \beta^2\left(C(1-2r)-F\right) - 2\beta C(1-r) + C \end{aligned}$$

We have B, a quadratic function of β, whose determinant is

$$\begin{aligned} & [2C(1-r)]^2 - 4C\left[C(1-2r) - F\right] \\ =\ & 4C^2 r^2 + 4CF > 0 \end{aligned}$$

Hence the equation has two real roots. Call them x_1 and x_2 and assume wlog that $x_1 < x_2$. Here are some facts that we use:

1. B evaluated at $\beta = 1$ is

$$B \mid_{\beta=1} = C(1-2r) - F - 2C(1-r) + C = -F < 0$$

2. A evaluated at the first threshold β_1 is

$$\begin{aligned} A \mid_{\beta=\beta_1} &= (1-\beta_1)\left[(I-Y)(1-\beta_1)(1+\beta_1 r\Delta) - \beta_1^2 r\Delta p_1(q_1 D - I)\right] \\ &> (1-\beta_1)\left[(I-Y)(1-\beta_1) - \beta_1 p_1(q_1 D - I)\right] \\ &= 0 \end{aligned}$$

the inequality due to the fact that $(1+\beta_1 r\Delta) > \beta_1 r\Delta$; the last equality from the definition of the threshold β_1.

3. A and B evaluated at $\beta = 0$ give

$$A \mid_{\beta=0} = C > 0 = B \mid_{\beta=0}$$

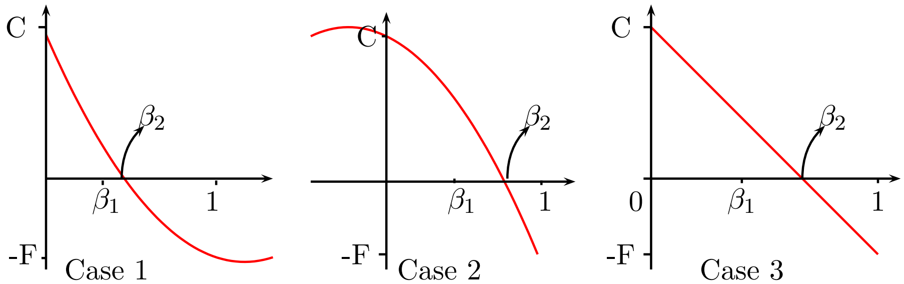

Figure 7: Three Different Cases for β_2

4. The sign of A is determined by B since

$$\frac{\partial}{\partial \beta}\left(\frac{1-\beta}{1-\beta(1-r)}\right) < 0$$

and

$$\frac{1-\beta}{1-\beta(1-r)} \mid_{\beta=0} = 1 \quad \text{and} \quad \frac{1-\beta}{1-\beta(1-r)} \mid_{\beta=1} = 0$$

We have three cases to consider, depending on the coefficient of the highest exponent. The task in each of them is to show that $0 < \beta_1 < \beta_2 < 1$. The listed facts combined with the restriction on the coefficient of the highest order term in each case delivers the result.

- Case 1: $C(1-2r) - F > 0$. This is the case where the product of the roots $x_1 x_2 = \frac{C}{C(1-2r)-F} > 0$, the sum of the roots $x_1 + x_2 = \frac{2C(1-r)}{C(1-2r)-F} > 0$ and the quadratic troughs at $\beta = \frac{2C(1-r)}{2[C(1-2r)-F]} > 1$. Hence, we have two positive roots, $0 < x_1 < 1 < x_2$.
- Case 2: $C(1-2r) - F < 0$. The product of the roots $x_1 x_2 < 0$, the sum of the roots $x_1 + x_2 < 0$ and the function peaks at $\beta = \frac{2C(1-r)}{2[C(1-2r)-F]} < 0$. So, we have 1 positive and 1 negative root, $x_1 < 0 < x_2 < 1$.
- Case 3: $C(1-2r) - F = 0$. This is the linear case. B becomes $-\beta 2C(1-r) + C$ which assumes the value $-C(1-2r) = -F < 0$ at $\beta = 1$. Hence the intersection of the line with the horizontal axis happens in the interval $(0, 1)$.

The pictures corresponding to each case are as follows.

As depicted in the figures associated with each case, B is positive for $\beta \in (0, \beta_2)$ where $0 < \beta_1 < \beta_2$. This implies that the original difference between maximum surpluses from saving $I - Y$ and saving $2(I - Y)$ is positive. Hence, saving $I - Y$ pays more than the latter.

We showed that saving $I - Y$ dominates all of the other saving policies, hence it is optimal.

Investment: There are two different regimes to consider:

- $\forall\ S \geq 2(I-Y)$, Investing brings

  $$q_1\left[(1-\beta)\left(S+Y+D-I-(I-Y)\right)+\beta P(0,I-Y)\right]+$$
  $$q_0\left[(1-\beta)\left(S+Y-I-(I-Y)\right)+\beta P(0,I-Y)\right]$$

  where not investing brings:

  $$(1-\beta)\left(S+Y-(I-Y)\right)+\beta P(0,I-Y)$$

  whose difference is

  $$(1-\beta)(q_1 D-I)>0$$

  Hence, investing is the optimal strategy in that range, given the optimal saving strategy.

- $\forall\ S\in[I-Y,\ 2(I-Y)]$, investing gives

  $$q_1\left[(1-\beta)\left(S+Y+D-I-(I-Y)\right)+\beta P(0,I-Y)\right]+$$
  $$q_0\left[(1-\beta)\left(S+Y-I\right)\right)+\beta P(0,0)]$$

  where not investing brings:

  $$(1-\beta)\left(S+Y-(I-Y)\right)+\beta P(0,I-Y)$$

  whose difference is

  $$(1-\beta)[(q_1 D-I)-\beta q_0\Delta p_1(q_1 D-I)+$$
  $$(I-Y)q_0-\beta\Delta p_1 q_0^2(I-Y)>0$$

  Hence, investing is optimal in this range, too.

Therefore, investment policy is the threshold rule stated in the proposition. □

Proof of Proposition 4

The existence of the second-best recursive contract is a little involved. The reason is that the value function has to be non-negative for any feasible value of v in the second-best program. So, the value function itself enters the constraint set of the problem which makes it into a nonstandard dynamic programming problem.

1. Let F and $B(F)$ be defined as they are in the proof of Proposition 1. Let the operator T be defined on $B(F)$ for any $(v,S)\in F$ and any $Q\in B(F)$ by:

$$(TQ)(v,S) \quad = \quad \max_{(c_{\theta\mu},m_{\theta\mu},S_{\theta\mu},I_\theta,v_{\theta\mu})\in\mathbb{R}_+^{18}} \sum_{\theta\mu} p_\theta q_\mu[(1-\beta)m_{\theta\mu}+\beta Q(v_{\theta\mu},S_{\theta\mu})]$$

$$
\begin{aligned}
\text{s.t. } \forall\theta,\ \forall\mu \qquad & Q(v_{\theta\mu}, S_{\theta\mu}) \geq 0 \qquad (21)\\
& S_{\theta\mu} \leq S + Y + D1_{\{\theta=1,\ \mu=1,\ I_\theta \geq I\}} - I_\theta - m_{\theta\mu} - c_{\theta\mu}\\
& I_\theta \leq S + Y\\
& \sum_{\theta\mu} p_\theta q_\mu [(1-\beta)c_{\theta\mu} + \beta v_{\theta\mu}] \geq v\\
& (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \geq (1-\beta)D1_{\{\theta=1,\ \mu=1,\ I_\theta \geq I\}} + \beta Y
\end{aligned}
$$

We know from before that the first-best value function, $P \in B(F)$. With one extra constraint to consider, the feasible set for the above problem is smaller than that of the first-best problem; hence $TP \leq P$. Clearly, T is a monotone operator which implies that $T^n P \leq T^{n-1} P$, all n, by a simple induction argument. Hence, for each (v, S), $((T^n P)(v, S))$ is a decreasing sequence which is bounded from below by 0. So, it should converge pointwise to a limit value, say, to $P_\infty(v, S)$.

- We need to show that P_∞ is a fixed point of the operator T. For any given (v, S), consider the sequence of optimal actions taken at each iteration of the operator T, $(c^n_{\theta\mu}, m^n_{\theta\mu}, S^n_{\theta\mu}, I^n_\theta, v^n_{\theta\mu})$. Since $(T^n(P)(v, S))$ is a decreasing sequence, the constraint in (21) is not going to relax as n increases, which makes the feasible set of values that the sequence of optimal actions live in, a compact set. Hence, $(c^n_{\theta\mu}, m^n_{\theta\mu}, S^n_{\theta\mu}, I^n_\theta, v^n_{\theta\mu})$ has a convergent subsequence, converging to $(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_\theta, v_{\theta\mu})$. The sequence satisfies all the constraints of the problem, for each n; so, the limit contract should do so too since these are all inequality constraints. In particular, $T^n(P)(v, S) \geq 0$, for all n and $T^n(P)(v, S) \to P_\infty(v, S)$ hence $P_\infty(v, S) \geq 0$, too. So, this limit contract is feasible and provides the borrower with an ex-ante discounted utility of at least v. The actual optimal contract should do at least as good which implies that $(TP_\infty)(v, S) \geq P_\infty(v, S)$. We also know, from the monotonicity of the operator T, that, for all n, $(T^n P)(v, S) \leq (T^{n-1} P)(v, S)$ hence $(T^{n-1} P)(v, S) \geq P_\infty(v, S)$. Therefore, $(T^n P)(v, S) \geq (TP_\infty)(v, S)$ for all n which implies that the limit of that sequence admits the same ordering: $(T^n P)(v, S) \to P_\infty(v, S) \geq (TP_\infty)(v, S)$. We showed that $P_\infty(v, S) \geq (TP_\infty)(v, S)$ and $P_\infty(v, S) \leq (TP_\infty)(v, S)$ which implies that $P_\infty(v, S) = (TP_\infty)(v, S)$ hence P_∞ is a fixed point of the operator T.

- Clearly, each fixed point Q of the operator T corresponds to a second-best contract. It is the standard unravelling idea. Start with an initial (v, S); the optimal actions in period 1 are given by $(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_\theta, v_{\theta\mu})$. Let σ_1 be defined as $c_1(\theta, \mu) = c_{\theta\mu}$, $m_1(\theta, \mu) = m_{\theta\mu}$, $S_2(\theta, \mu) = S_{\theta\mu}$ and $I_1(\theta) = I_\theta$. The second period contract conditional on the realization of θ, μ in the first period is the optimal action vector starting with an initial $(v_{\theta\mu}, S_2(\theta, \mu))$, and so on by repeatedly applying the operator T. The contract σ constructed this way satisfies all the constraints of the original second-best problem and delivers the borrower and the lender the utility levels v and $Q(v, S)$, respectively. Moreover, it is optimal from each history on; hence it is a second-best contract.

- We know that $P \geq Q$ where Q is a fixed point of T, from above, which leads to $T^n P \geq T^n Q = Q$. But then, the latter also holds in the limit: $P_\infty \geq T^n Q = Q$. Since every fixed point of T corresponds to a second-best contract and by the optimality of Q, we also have $P_\infty \leq Q$. Hence $P_\infty = Q$.

In summary, we showed that a fixed point, Q, of the operator T exists and can be computed by an iterative application of T, starting initially with the first-best value function P. In addition, a second-best contract exists that is associated with that value function which delivers the borrower and the lender the utility levels v and $Q(v, S)$, respectively.

2. The existence of $\underline{S}$ follows from Proposition 2.2. The idea is that if $v = \overline{v}_S$, i.e., the whole surplus goes to the borrower, then the first-best rules are implemented. So, for $v = \overline{v}_S$ and $S \geq \underline{S}$, 'invest' is the optimal strategy. Now, the sketch of the proof is as follows: as we know from the existence proof, starting from the first-best value function P, repeated application of the operator T on P leads to the second-best value function Q. At each iteration, we will show that the optimal policy rules are of the monotonic nature and that in the limit, they converge to the stated form. So, initially, we assume the continuations are given by P and solve

$$\begin{aligned}
(TP)(v,S) \quad &= \max_{(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_\theta, v_{\theta\mu}) \in \mathbb{R}_+^{18}} \sum_{\theta\mu} p_\theta q_\mu [(1-\beta) m_{\theta\mu} + \beta P(v_{\theta\mu}, S_{\theta\mu})] \\
\text{s.t. } \forall\theta,\ \forall\mu \quad & P(v_{\theta\mu}, S_{\theta\mu}) \geq 0 \\
& S_{\theta\mu} \leq S + Y + D1_{\{\theta=1,\ \mu=1,\ I_\theta \geq I\}} - I_\theta - m_{\theta\mu} - c_{\theta\mu} \\
& I_\theta \leq S + Y \\
& \sum_{\theta\mu} p_\theta q_\mu [(1-\beta) c_{\theta\mu} + \beta v_{\theta\mu}] \geq v \\
& (1-\beta) c_{\theta\mu} + \beta v_{\theta\mu} \geq (1-\beta) D1_{\{\theta=1,\ \mu=1,\ I_\theta \geq I\}} + \beta Y
\end{aligned} \tag{22}$$

where $P(v, S) = \overline{v}_S - v$. So, the above program can be written as

$$\begin{aligned}
(TP)(v,S) \quad &= \max_{(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_\theta, v_{\theta\mu}) \in \mathbb{R}_+^{18}} \sum_{\theta\mu} p_\theta q_\mu [(1-\beta) \left(S + Y + D1_{\{\theta=1,\ \mu=1,\ I_\theta \geq I\}} - I_\theta - S_{\theta\mu}\right) \\
& +\beta \overline{v}_S - (1-\beta) c_{\theta\mu} - \beta v_{\theta\mu}] \\
\text{s.t. } \forall\theta,\ \forall\mu \quad & P(v_{\theta\mu}, S_{\theta\mu}) \geq 0 \\
& I_\theta \leq S + Y
\end{aligned}$$

$$\sum_{\theta\mu} p_\theta q_\mu [(1-\beta) c_{\theta\mu} + \beta v_{\theta\mu}] \geq v \tag{23}$$

$$(1-\beta) c_{\theta\mu} + \beta v_{\theta\mu} \geq (1-\beta) D1_{\{\theta=1,\ \mu=1,\ I_\theta \geq I\}} + \beta Y \tag{24}$$

Savings: Now, the first-best saving rule maximizes the first part of the lender's objective and is independent of the optimal choice of $(c_{\theta\mu}, v_{\theta\mu})$.
Investment: There are two different regimes. Let $v^{*1}(S) := (1-\beta) p_1 q_1 D + \beta Y$.

1. If $v \geq v^{*1}(S)$: The (IR) constraint (23) binds. The reason is that, if the investment is undertaken, this is the minimum amount that the borrower needs to be provided with, ex-ante, in order to prevent default. If it does not bind, you can always lower some continuation values without violating any of the incentive compatibility constraints. So, $EU^E = v$. The alternative is not to invest. But the comparison is exactly that of the first-best investment decision whose answer is to 'invest'.

2. If $v < v^{*1}(S)$: The constraint (23) holds with strict inequality. $EU^E = v^{*1}(S)$. If there is no investment, $EU^E = v$. So, the comparison is between

$$P(v, S) - v^{*1}(S)$$

which is the first-best level of surplus minus the average ex-ante payment to E, in case of investment and

$$(1 - \beta)(S + Y - S'_{FB}) + \beta \overline{v}_{S'_{FB}} - v$$

in case of 'not investing', where S'_{FB} is the first-best level of savings. Hence, there is a level ${v_*}^1(S) > Y$ (${v_*}^1(S) = Y$ if it is optimal to invest always) such that it is optimal to invest for $v \geq {v_*}^1(S)$ and not to invest for $v \leq {v_*}^1(S)$. [9]

So, we have a full characterization of TP, i.e.,

1. Do not invest for $v \in [Y, {v_*}^1(S)]$; Invest for $v \in [{v_*}^1(S), \overline{v}_S]$

2. Save according to the first-best rule.

3. TP is given by

$$TP(v, S) = \begin{cases} \overline{v}_S - v & \text{if } v \geq v^{*1}(S) \\ \overline{v}_S - v^{*1}(S) & \text{if } v \in [{v_*}^1(S), v^{*1}(S)] \\ \overline{v}_S - [v^{*1}(S) - {v_*}^1(S)] - v & \text{if } v \in [Y, {v_*}^1(S)] \end{cases}$$

We know that if the whole surplus goes to the borrower , i.e., $v = \overline{v}_S$, the first-best rule is implemented. In the second iteration of the above problem (T^2P), for the same S, there is a nonempty interval of values of v (including $\overline{v}_S$) for which the optimal rule is to invest. That is because of the continuity of the problem w.r.t. v and the fact that P and TP coincide for high values of v. Similar reasoning guarantees that if there is an interval of values for which the optimal rule is 'not to invest' for the first iteration, there is such an interval for the second iteration, since the continuation value is lower than the original continuation value ($TP \leq P$). Then, we need to show two things: (i) T^nP has the same shape as TP and (ii) $v^*(S)$ is nondecreasing. These two combined will deliver the result. For $n = 1$, it is trivially true. For $n > 1$, assume that it is

[9] It is not always the case that investing is the optimal decision no matter what v is, as Example 2 shows.

true for $n-1$. We know that the optimal rule is to invest for $v \geq v^{*n-1}(S)$ since the first-best and second-best values coincide for that interval. $T^nP \leq T^{n-1}P$ from the existence part, hence $v^{*n}(S) \geq v^{*n-1}(S)$ since otherwise $T^nP(v^{*n}(S), S) > T^{n-1}P(v^{*n}(S), S)$, a contradiction. For $v < v^{*n-1}(S)$, $EU^E = v^{*n}(S)$ from the same token as above. We know from above that $v_*^{n-1}(S)$ is such that

$$(1-\beta)(S+Y-S'_{FB}) + \beta\overline{v}_{S'_{FB}} - v_*^{n-1}(S) = P(v,S) - v^{*n-1}(S)$$

Since $v^{*n}(S) \geq v^{*n-1}(S)$, there is a $v_*^n(S) \geq v_*^{n-1}(S)$ (where $v_*^n(S) - v_*^{n-1}(S) = v^{*n}(S) - v^{*n-1}(S)$) such that the equality still holds and it is optimal to invest for $v \geq {v_*}^n(S)$ and not to invest for $v \leq {v_*}^n(S)$. Hence, T^nP has the same shape as TP. Now, $(v^{*n}(S))$ forms a monotonically non-decreasing sequence of thresholds, bounded from above by $\overline{v}_S$. So, it should converge to $v^*(S)$ as $n \to \infty$. But, trivially then, $({v_*}^n(S)) \to v_*(S)$.
Let $S' > S$. So, $\overline{v}_{S'} > \overline{v}_S$ since P is strictly increasing from Proposition 1.3. The fact that the individual rationality constraint is tight means that, conditioning on investing today, a constraint will be violated if you do not deliver at least $v^*(S)$ to the borrower. For $S' > S$, the same contract will violate that same constraint, since the constraints are stationary. Hence, $v^*(S') \geq v^*(S)$. That concludes the proof. □

Proof of Proposition 5

1. Suppose for a contradiction that $S_{\theta\mu} > S_{FB,\theta\mu}$. Let $(c_{\theta\mu}, v_{\theta\mu}, S_{\theta\mu})$ be the optimal second-best contract given (v, S) is the state variable. On a realized path $(\theta\mu)$, utility to L is

$$(1-\beta)\left[W - c_{\theta\mu} - S_{\theta\mu}\right] + \beta Q(v_{\theta\mu}, S_{\theta\mu})$$

which can be written as

$$(1-\beta)\left[W - c_{\theta\mu} - S_{\theta\mu}\right] + \beta \begin{cases} \overline{v}_{S_{\theta\mu}} - v_{\theta\mu} & \text{if } v_{\theta\mu} \geq v^*(S_{\theta\mu}) \\ \overline{v}_{S_{\theta\mu}} - v^*(S_{\theta\mu}) & \text{if } v_{\theta\mu} \in [v_*(S_{\theta\mu}), v^*(S_{\theta\mu})] \\ \overline{v}_{S_{\theta\mu}} - [v^*(S_{\theta\mu}) - v_*(S_{\theta\mu})] - v_{\theta\mu} & \text{if } v_{\theta\mu} \in [Y, v_*(S_{\theta\mu})] \end{cases}$$

which is equivalent to

$$(1-\beta)\left[W - S_{\theta\mu}\right] + \beta \begin{cases} \overline{v}_{S_{\theta\mu}} \\ \overline{v}_{S_{\theta\mu}} \\ \overline{v}_{S_{\theta\mu}} - [v^*(S_{\theta\mu}) - v_*(S_{\theta\mu})] \end{cases} - \begin{cases} (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \\ (1-\beta)c_{\theta\mu} + \beta v^*(S_{\theta\mu}) \\ (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \end{cases} \tag{25}$$

Necessary condition for optimality requires that either

$$v_{\theta\mu} \geq v^*(S_{\theta\mu}) \geq v^*(S_{FB,\theta\mu})$$

or

$$v_{\theta\mu} \leq v_*(S_{FB,\theta\mu}) \leq v^*(S_{FB,\theta\mu}) \leq v^*(S_{\theta\mu})$$

which implies that $(c_{\theta\mu}, v_{\theta\mu})$ is feasible also under the first-best saving rule $S_{FB,\theta\mu}$ since $S_{\theta\mu} > S_{FB,\theta\mu}$. So, the payoff to the lender on the realized path from saving according to the first-best rule and paying the borrower $(c_{\theta\mu}, v_{\theta\mu})$, is better than the original scheme since the LHS of (25) is maximized at $S_{\theta\mu} = S_{FB,\theta\mu}$, i.e.,

$$(1-\beta)\left[W - S_{\theta\mu}\right] + \beta \begin{cases} \overline{v}_{S_{\theta\mu}} \\ \overline{v}_{S_{\theta\mu}} \\ \overline{v}_{S_{\theta\mu}} - \left[v^*(S_{\theta\mu}) - v_*(S_{\theta\mu})\right] \end{cases} - \begin{cases} (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \\ (1-\beta)c_{\theta\mu} + \beta v^*(S_{\theta\mu}) \\ (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \end{cases}$$

$$< (1-\beta)\left[W - S_{FB,\theta\mu}\right] + \beta \begin{cases} \overline{v}_{S_{FB,\theta\mu}} \\ \overline{v}_{S_{FB,\theta\mu}} \\ \overline{v}_{S_{FB,\theta\mu}} - \left[v^*(S_{FB,\theta\mu}) - v_*(S_{FB,\theta\mu})\right] \end{cases} - \begin{cases} (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \\ (1-\beta)c_{\theta\mu} + \beta v^*(S_{FB,\theta\mu}) \\ (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \end{cases}$$

if $v_{\theta\mu} \leq v_*(S_{\theta\mu})$, or

$$(1-\beta)\left[W - S_{\theta\mu}\right] + \beta \begin{cases} \overline{v}_{S_{\theta\mu}} \\ \overline{v}_{S_{\theta\mu}} \\ \overline{v}_{S_{\theta\mu}} - \left[v^*(S_{\theta\mu}) - v_*(S_{\theta\mu})\right] \end{cases} - \begin{cases} (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \\ (1-\beta)c_{\theta\mu} + \beta v^*(S_{\theta\mu}) \\ (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \end{cases}$$

$$< (1-\beta)\left[W - S_{FB,\theta\mu}\right] + \beta \begin{cases} \overline{v}_{S_{FB,\theta\mu}} \\ \overline{v}_{S_{FB,\theta\mu}} \\ \overline{v}_{S_{FB,\theta\mu}} \end{cases} - \begin{cases} (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \\ (1-\beta)c_{\theta\mu} + \beta v^*(S_{FB,\theta\mu}) \\ (1-\beta)c_{\theta\mu} + \beta v^*(S_{FB,\theta\mu}) \end{cases}$$

if $v_{\theta\mu} > v_*(S_{\theta\mu})$ since Q is strictly increasing in S. Therefore, saving more than the first-best rule makes L strictly worse off, which concludes the proof.

2. This is implied by Proposition 4.2. Assuming the latter is true, pick a feasible (v, S) pair. $I_0(v, S) = 0$ trivially for any feasible (v, S) since it does not pay off to allocate resources to investment when there is no project. For a given S, I_1 is trivially monotonically increasing in v, since it is a step function from Proposition 4.4.

Proof of Example 2

We know that both P and Q exist and that $0 \leq Q(v, S) \leq P(v, S)$ for any feasible (v, S).

1. It will be shown first that the saving strategy is as in the first-best case. Then, the optimal threshold-investment behavior will be characterized fully.

Savings: The 'trick' here is to solve the alternate program where the continuation is replaced by P, then go back and show that one achieves the same utility level with that solution in the original problem, as well. On a realized path, optimal contract, on this path, should solve

$$\begin{array}{rll} & \max\limits_{(c_{\theta\mu}, v_{\theta\mu}, S_{\theta\mu})} (1-\beta)\left[W - c_{\theta\mu} - S_{\theta\mu}\right] + \beta Q(v_{\theta\mu}, S_{\theta\mu}) & \\ \text{s.t.} & (1-\beta)c_{\theta\mu} + \beta v_{\theta\mu} \geq (1-\beta)D1_{\{\theta=1,\mu=1,I_\theta \geq Y\}} + \beta Y & (*) \\ & \sum\limits_{\theta\mu} p_\theta q_\mu[(1-\beta)c_{\theta\mu} + \beta v_{\theta\mu}] \geq v \in [Y, \overline{v}_S] \quad (**) & \end{array}$$

given the optimal choices for the other paths. Let's first consider the alternate program where we continue with the first-best value function, P, given that both $(*)$ and $(**)$ are satisfied

$$\begin{aligned} & \max(1-\beta)\left[W - c_{\theta\mu} - S_{\theta\mu}\right] + \beta P(v_{\theta\mu}, S_{\theta\mu}) \\ = \; & \max(1-\beta)\left[W - c_{\theta\mu} - S_{\theta\mu}\right] + \beta\left[P(0, S_{\theta\mu}) - v_{\theta\mu}\right] \\ = \; & \max(1-\beta)\left[W - S_{\theta\mu}\right] + \beta P(0, S_{\theta\mu}) - (1-\beta)c_{\theta\mu} - \beta v_{\theta\mu} \end{aligned}$$

LHS of this problem is maximized at $S_{\theta\mu} = 0$, independently of the RHS. That's because any utility level arising from an optimal $(c_{\theta\mu}, v_{\theta\mu})$ pair that is feasible under an alternative saving strategy can be replicated by a corresponding contract (c', v') under the surplus-maximizing saving strategy. Since there is no saving, the only incentive compatible continuation value is $v_{\theta\mu} = Y$ ($P(0,0) = Y$ from Proposition 1). This scheme is feasible under the original program too. Since

$$0 \leq Q(Y, 0) \leq P(Y, 0) = 0$$

the first inequality from the definition of second-best and the second from Proposition 1, we achieve the same utility under the original second-best program that we do under the alternate program. Hence, the necessary condition for optimality for a second-best contract is not saving at all as in the first-best case.

Investment: Given the optimal saving rule, there are two possible investment strategies:

1. If there is no investment ($I_1(v, S) = 0$), second-best problem solves

$$\begin{array}{rl} & \max\limits_{v,c} (1-\beta)\left[S + Y - c\right] \\ \text{s.t.} & (1-\beta)c + \beta Y \geq v \in [Y, \overline{v}_S] \end{array}$$

   In the second-best optimum, the constraint should bind and

$$c = \left(\frac{v - \beta Y}{1-\beta}\right) \leq \frac{\overline{v}_S - \beta Y}{1-\beta} \leq \frac{\overline{v}_S}{1-\beta} = \frac{(1-\beta)[S+Y]}{1-\beta} = S + Y$$

Moreover, $c \geq 0$ clearly, which makes it feasible. Notice that we are assuming that $c_{\theta\mu} = c$ here. This is just one of the solutions because of the linearity of the problem. However, all solutions leave L and E with the same utility levels. Hence,

$$EU^L = (1-\beta)\left[S + Y - \frac{v - \beta Y}{1-\beta}\right] = (1-\beta)S + (Y - v)$$

2. If there is investment ($I_1(v,S) = I \leq S + Y$), since the optimal saving rule implies $S_{\theta\mu} = 0$ and $v_{\theta\mu} = Y$, L's program becomes

$$\begin{aligned} &\max \sum_{\theta\mu} p_\theta q_\mu (1-\beta)\left(S + Y + D1_{\{\theta=1,\, \mu=1,\, I_\theta \geq I\}} - I_\theta - c_{\theta\mu}\right) \\ \text{s.t.} \quad & \sum_{\theta\mu} p_\theta q_\mu [(1-\beta)c_{\theta\mu} + \beta Y] \geq v \qquad (26) \\ & c_{\theta\mu} \geq D1_{\{\theta=1,\, \mu=1,\, I_\theta \geq I\}} \qquad (27) \end{aligned}$$

Let λ and γ be the Lagrange multipliers for the IR constraint, in (26), and the IC constraint $c_{11} \geq D$, in (27), respectively. First order conditions for c_{11}, $c_{\theta\mu}$ for $\theta\mu \neq 1$, are, in that order

$$-1 + \lambda + \frac{\gamma}{(1-\beta)p_1 q_1} \leq 0, \quad c_{11} \geq 0, \quad \left[-1 + \lambda + \frac{\gamma}{(1-\beta)p_1 q_1}\right] c_{11} = 0 \qquad (28)$$

$$-1 + \lambda \leq 0, \quad c_{\theta\mu} \geq 0, \quad [-1 + \lambda]\, c_{\theta\mu} = 0 \qquad (29)$$

Let $v^* \equiv \max\{\ Y, (1-\beta)p_1 q_1 D + \beta Y\ \}$. This quantity is crucial in determining whether (26) binds or not.

(a) If $v \in [Y, v^*]$, (26) is an inequality, which implies that $\lambda = 0$. Then, (29) implies that $c_{\theta\mu} = 0$ for $\theta\mu \neq 1$. Finally, $c_{11} = D$ since otherwise, decreasing c_{11} would increase the objective without violating any constraints, which would be a contradiction to optimality. Therefore, the expected utility to L from investing is

$$EU_I^L = (1-\beta)\left[S + p_1(q_1 D - I)\right] + Y - v^*$$

where the corresponding level from non-investing is

$$EU_{NI}^L = (1-\beta)S + Y - v.$$

Now, $EU_I^L - EU_{NI}^L \to (1-\beta)p_1(q_1 D - I) > 0$ as $v \to v^*$. As $v \to Y$, $EU_I^L - EU_{NI}^L \to (1-\beta)(Y - p_1 I)$. If the latter is negative ($p_1 > \frac{Y}{I}$), by the Intermediate Value Theorem, there exists a $v_* = (1-\beta)p_1 I + \beta Y \in [Y, v^*)$ such that $EU_I^L - EU_{NI}^L < 0$ for $v \in [Y, v_*)$. If it is nonnegative ($p_1 \leq \frac{Y}{I}$), it means that investing is optimal for all $v \in [Y, v^*)$, i.e., $v_* = Y$.

(b) If $v \in (v^*, \overline{v}_S]$, (26) binds. $\lambda = 1$, hence the slope of the second-best frontier is -1. If $\lambda < 1$, (28) and (29) imply that $c_{\theta\mu} = 0$ for $\theta\mu \neq 1$ and $c_{11} = D$, which would imply, in turn, that $v = v^*$. But, this latter is a contradiction. Therefore, the expected utility to L from investing is

$$EU_I^L = (1-\beta)\left[S + p_1(q_1 D - I)\right] + Y - v$$

where the corresponding level from non-investing is

$$EU_{NI}^L = (1-\beta)S + Y - v.$$

So, $EU_I^L - EU_{NI}^L = (1-\beta)p_1(q_1 D - I) > 0$ which makes investing the optimal decision for all $v \in (v^*, \overline{v}_S]$

2. The proof of the second part follows the same reasoning and machinery that the first one does. For that reason, it will be omitted. □

References

[1] Abreu, D., Pearce, D. and Stacchetti, E. (1990), "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring", *Econometrica*, 58, 1041-1063.

[2] Albuquerque, R. and Hopenhayn, H. A. (2002), "Optimal Lending Contracts and Firm Dynamics", forthcoming *Review of Economic Studies*.

[3] Allen, F. (1983), "Credit Rationing and Payment Incentives", *Review of Economic Studies*, 50, 639-646.

[4] Allen, F. and Gale, D. (1997), "Financial Markets, Intermediaries, and Intertemporal Smoothing", *Journal of Political Economy*, 105, 523-546.

[5] Alvarez, F. and Jermann, U. J. (2000), "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica*, 59, 1069-1089.

[6] Atkeson, A. (1991), "International Lending with Moral Hazard and Repudiation", *Econometrica*, 59, 1069-1089.

[7] Berger A. N. and Udell, G. (1994), "Did Risk-Based Capital Allocate Bank Credit and Cause a Credit Crunch in the U.S.?", *Journal of Money, Credit and Banking*, 26, 585-628.

[8] Bernanke, B. S. and Gertler, M. (1989), "Agency Cost, Net Worth and Business Fluctuations," *American Economic Review*, 79, 14-31.

[9] Bernanke, Ben S. and Lown, C. S. (1991), "The Credit Crunch," *Brookings Papers on Economic Activity* , 2, 205-247.

[10] Besanko, D. and Kanatas, G. (1996), "The Regulation of Bank Capital: Do Capital Standards Promote Bank Safety?" *Journal of Financial Intermediation*, 5:2, 160-183.

[11] Bolton, P. and Scharfstein, D. (1990), "A THeory of Predation Based on Agency Problems in Financial Contracting", *American Economic Review*, 80, 93-106.

[12] Boot, A., Greenbaum, S. and Thakor, A. (1993), "Reputation and Discretion in Financial Contracting,“ *American Economic Review*, 83(5), 1165-1183.

[13] Bulow, J. and Rogoff, K. (1989), "Sovereign Debt: Is to Forgive to Forget?", *American Economic Review*, 79, 43-50.

[14] Cooley, T., Marimon R. and Quadrini V. (2004), "Aggregate Consequences of Limited Contract Enforceability", *Journal of Political Economy*, 112, 817-847.

[15] DeMarzo, P. M. and Fishman, M. J. (2000), "Optimal Long-Term Financial Contracting with Privately Observed Cash Flows", Stanford University and Northwestern University working paper.

[16] Diamond, D. and Dybvig, P. (1983), "Bank Runs, Liquidity, and Deposit Insurance", *Journal of Political Economy*, 91, 401-419.

[17] Diamond, D. and Rajan, R. (2000), "A Theory of Bank Capital," *Journal of Finance*, 55, 2431-2465.

[18] Eaton, J. and Gersovitz, M. (1981), "Debt with Potential Repudiation: Theoretical and Empirical Analysis", *Review of Economic Studies*, 48, 289-309.

[19] Haubrich, J. (1989), "Financial Intermediation, Delegated Monitoring, and Long-Term Relationships," *Journal of Banking and Finance*, 13, 9-20.

[20] Hoshi, T., Kashyap, A. and Scharfstein, D. (1990a), "Bank Monitoring and Investment: Evidence from the Changing Structure of Japanese Corporate Banking Relationships,“ in R. Glenn Hubbard, editor, Asymmetric Information, Corporate Finance and Investment (University of Chicago Press, Chicago, Illinois).

[21] Hoshi, T., Kashyap, A. and Scharfstein, D. (1990b), "The Role of Banks in Reducing the Costs of Financial Distress in Japan,“ *Journal of Financial Economics*, 27, 67-88.

[22] Hoshi, T., Kashyap, A. and Scharfstein, D. (1991), "Corporate Structure, Liquidity and Investment: Evidence from Japanese Industrial Groups," *Quarterly Journal of Economics*, 106, 33-60.

[23] Holmström, B. and Tirole, J. (1997), "Financial Intermediation, Loanable Funds, and The Real Sector," *Quarterly Journal of Economics*, 112, 663-692.

[24] Holmström, B. and Tirole, J. (1998), "Private and Public Supply of Liquidity," *Journal of Political Economy*, 106, 1-40.

[25] Kashyap, A. and Stein, J. (2000), "What do a Million Observations on Banks Say about the Transmission of Monetary Policy?" A*merican Economic Review*, 90:3, 407-428.

[26] Kehoe, T. J. and Levine, D. K. (1993), "Debt-Constrained Asset Markets," *Review of Economic Studies*, 60, 865-888.

[27] Marcet, A. and Marimon, R. (1992), "Communication, Commitment and Growth", *Journal of Economic Theory*, 58, 219-249.

[28] Petersen, M. and Rajan, R. (1995), "The Effect of Credit Market Competition on Lending Relationships," *Quarterly Journal of Economics*, 110, 407-443.

[29] Phelan, C. (1995), "Repeated Moral Hazard and One-Sided Commitment", *Journal of Economic Theory*, 66, 488-506.

[30] Phelan, C. and Stacchetti, E. (2001), "Sequential Equilibria in a Ramsey Tax Model", *Econometrica*, 69, 1491-1518.

[31] Ray, D. (2002), "The Time Structure of Self-Enforcing Agreements", *Econometrica*, 70, 547-582.

[32] Schneider, M. (2001), "Borrowing Constraints in a Dynamic Model of Bank Asset and Liability Management," mimeo, NYU.

[33] Sharpe, S. (1995), "Bank Capitalization, Regulation, and the Credit Crunch: A Critical Review of the Research Findings." Finance and Economics Discussion Series paper No. 95/20, Board of Governors of the Federal Reserve System.

[34] Stiglitz, J. E. and Weiss, A. (1981), Credit Rationing in Markets with Imperfect Information, *American Economic Review*, 71, 393-410.

[35] Stiglitz, J. E. and Weiss, A. (1983), Incentive Effects of Termination: Applications to the Credit and Labor Markets, *American Economic Review*, 71, 393-410.

[36] Stokey, N. L. and Lucas, R. E. (with Prescott, E. C.) (1989), *Recursive Methods in Economic Dynamics*, Cambridge, MA: Harvard University Press

[37] Thakor, A. (1996), "Capital Requirements, Monetary Policy, and Aggregate Bank Lending: Theory and Empirical Evidence," *Journal of Finance*, 51:1, 279-324.

[38] Thomas, J. and Worrall, T. (1988), "Self-Enforcing Wage Contracts", *Review of Economic Studies*, 55, 541-554.

[39] Thomas, J. and Worrall, T. (1994), "Foreign Direct Investment and the Risk of Expropriation", *Review of Economic Studies*, 61, 81-108.