Subjective Expected Utility in Games

Alfredo Di Tillio

University of Pennsylvania

January 2004

Motivation

"Quality Game"

	H	L
B	3,1	0,3
N	1,0	1,2

Traditional game-theoretic analysis:

- utilities taken as given
- "reasonable" beliefs identified

But in decision theory no separation of utils and beliefs:

- utilities and beliefs generated together from prefs
- beliefs not required to be "reasonable"

Aim of this paper: reconcile the two approaches

Savage's Omelet

"Your wife has just broken five good eggs into a bowl when you come in and volunteer to finish making the omelet. A sixth egg, which for some reason must either be used for the omelet or wasted altogether, lies unbroken beside the bowl. You must decide what to do with this unbroken egg." (Savage, 1954)

	STATE	
ACT	**Good**	**Rotten**
Waste	five-egg omelet, one good egg wasted	five-egg omelet, no good eggs wasted
Use	six-egg omelet, no good eggs wasted	no omelet, five good eggs wasted

<u>Who</u> determines the true state is inessential ("Nature").

The dominant approach to the problem:

- Acts = functions mapping states into outcomes
- Attach a utility to each outcome
- Attach a probability to each move by Nature
- Choose act with highest expected utility

But: <u>what</u> are subjective probabilities and utilities?

Savage: a mathematization of preference among acts
(You bet the egg is rotten $\Leftrightarrow$ You prefer "Waste")

Interactive Omelet

"Both you and your wife hate wasting good eggs, but strongly disagree on the size of the perfect omelet: you say six eggs, she says five. Now, your wife took six of the seven eggs in your fridge. One of the seven was rotten, and only she knew which one. She has just broken five good eggs into a bowl when you come in and volunteer ..."

	"STATE"	
ACT	**Good**	**Rotten**
Waste	five-egg omelet, one good egg wasted	five-egg omelet, no good eggs wasted
Use	six-egg omelet, no good eggs wasted	no omelet, five good eggs wasted

Same physical description, but here another decision maker (wife) determines endogenously the true state. G, R are now acts for her!

The “textbook” approach to the problem:

- Attach a utility to each outcome
- Attach a probability to each move by wife
- Attach a probability to possible utilities of wife
- Attach a probability to possible probabilities of wife
- ...
- Choose act with highest expected utility

Can a Savage-like foundation be given here?

Interactive Omelet – What is Bob's Ω?

Naive approach: $\boldsymbol{\Omega = \{G, R\}}$
"I bet she put a rotten egg" *not enough!*

Bob cannot reasonably guess without precise idea of:
(i) desirability of outcomes to Alice
(ii) Alice's guess about $\boldsymbol{W, U}$

$\boldsymbol{\Omega}$ must include (i) and (ii), i.e. wife's attitude towards her acts $\boldsymbol{G, R}$ (e.g. does she obey Savage's axioms?)

"I bet she put a rotten egg, as (I bet) she is sure (would bet) I will throw it away" *better, but still not enough!*

Problem of infinite regress arises.

Problem with the "Textbook" Approach

- Infinite regress problem should not be solved assuming probabilities, because it arises before specification of Ω.
- When asking whether Bob is a Savage decision-maker, must also assess Bob's confidence in Alice being so.

  Solution: use Harsanyi-like model w/o probabilities: Alice has "action and preference-type" unknown to Bob.

  - Alice's (Bob's) possible types: Ω_B (Ω_A)
  - Each $\omega_A \in \Omega_A$ gives a pref rel $\succcurlyeq_B$ over X^{Ω_B}
  - Each $\omega_B \in \Omega_B$ gives a pref rel $\succcurlyeq_A$ over X^{Ω_A}
  - So $\succcurlyeq_B$ tells Bob's confidence in Alice's confidence in Bob's confidence in . . .

Plan of the Talk and Preview of Results

- Use hierarchies of preferences to define "canonical" uncertainty spaces for Bob ($\boldsymbol{\Omega_B}$) and Alice ($\boldsymbol{\Omega_A}$)

- Show that $\succcurlyeq_{\boldsymbol{B}}$ on $\boldsymbol{X^{\Omega_B}}$ comprehensively describes Bob's prefs and also his belief about Alice's prefs (hence about her beliefs)

- Show that a sequence of "simple" preference statements by Bob reveals:
(i) whether Bob is a SEU maximizer
(ii) whether Bob is sure Alice is, too
(iii) whether Bob is sure Alice is sure Bob is, etc.

- Sketch applications and link to existing theory

Related Literature

♦ Kadane & Larkey (Management Science 1982)

♦ Mariotti, Battigalli (RFEB 1996)

♦ Epstein & Wang (Econometrica 1996)

♦ Mertens & Zamir (Int J Game Theory 1985)

Preliminary: Savage's Theory

- *States of the World*: a set Ω
- *Outcomes*: a set X
- *Acts*: X^{Ω}, the functions $f : \Omega \longrightarrow X$
- *Preference Relation* $\succcurlyeq$ *over Acts*

If $\succcurlyeq$ satisfies Savage's axioms, it is <u>represented</u> by

- *Utility Function*: $u : X \longrightarrow \mathbb{R}$
- *Subjective Belief*: $P \in \Delta(\Omega)$

$$f \succcurlyeq g \quad \Leftrightarrow \quad \int_{\Omega} u(f(\omega))dP(\omega) \geq \int_{\Omega} u(g(\omega))dP(\omega)$$

Primitives

- Players: $\boldsymbol{A, B}$
- Strategy Sets: $\boldsymbol{S_A, S_B}$
- Outcomes: $\boldsymbol{X}$
  (outcome function: $\phi : \boldsymbol{S_A} \times \boldsymbol{S_B} \rightarrow \boldsymbol{X}$)

Construction of Bob's Space of Uncertainty (I)

(symmetric definitions and results for Alice)

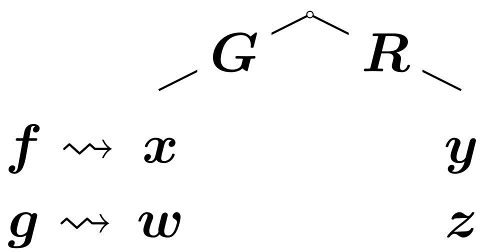
- $\Omega_B^0 = S_A$
- $\mathcal{P}_B^0 =$ all preference relations on $X^{\Omega_B^0}$
- $\Omega_B^1 = S_A \times \mathcal{P}_A^0$
- $\mathcal{P}_B^1 =$ all preference relations on $X^{\Omega_B^1}$

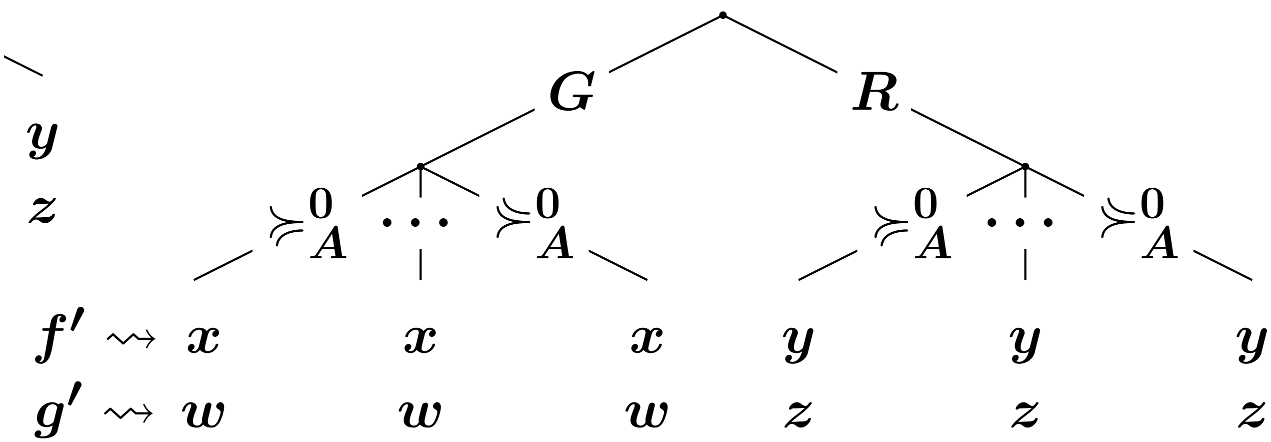
Construction of Bob's Space of Uncertainty (II)

Every act in $X^{\Omega_B^0}$ can be seen as an act in $X^{\Omega_B^1}$

In the omelet example, Bob's f and f' are the "same":

(x, y, w, z denote outcomes: omelet size, eggs wasted)





- We say $\succcurlyeq_B^1 \in \mathcal{P}_B^1$ <u>agrees</u> with $\succcurlyeq_B^0 \in \mathcal{P}_B^0$ if

$$f \succcurlyeq_B^0 g \quad \Leftrightarrow \quad f' \succcurlyeq_B^1 g'$$

Construction of Bob's Space of Uncertainty (III)

Agreement is all we require. Now add higher-order prefs:

- $\Omega_B^2 = \left\{ \left(s_A, \succcurlyeq_A^0, \succcurlyeq_A^1 \right) \in S_A \times \mathcal{P}_A^0 \times \mathcal{P}_A^1 \;\middle|\; \succcurlyeq_A^1 \text{ agrees with } \succcurlyeq_A^0 \right\}$

- $\mathcal{P}_B^2 =$ all preference relations on $X^{\Omega_B^2}$

and recursively

- $$\Omega_B^{n+1} = \left\{ \left(s_A, \succcurlyeq_A^0, \ldots, \succcurlyeq_A^{n-1}, \succcurlyeq_A^n \right) \in S_A \times \mathcal{P}_A^0 \times \cdots \times \mathcal{P}_A^n \;\middle|\; \succcurlyeq_A^n \text{ agrees with } \succcurlyeq_A^{n-1} \text{ agrees with } \ldots \succcurlyeq_A^0 \right\}$$

- $\mathcal{P}_B^{n+1} =$ all preference relations on $X^{\Omega_B^{n+1}}$

Bob's Space of Uncertainty: Definition

Let

$$\Omega_B^* = \left\{ \left(s_A, \succcurlyeq_A^0, \succcurlyeq_A^1, \ldots \right) \in S_A \times \mathcal{P}_A^0 \times \mathcal{P}_A^1 \times \cdots \;\middle|\; \text{for every } n \geq 1,\ \left(s_A, \succcurlyeq_A^0, \ldots, \succcurlyeq_A^{n-1} \right) \in \Omega_B^n \right\}$$

Let

$$\pi_B^{*,n} : \Omega_B^* \longrightarrow \Omega_B^n$$

$$\left(s_A, \succcurlyeq_A^0, \succcurlyeq_A^1, \ldots \right) \mapsto \left(s_A, \succcurlyeq_A^0, \ldots, \succcurlyeq_A^{n-1} \right)$$

Then:

$$\mathcal{A}_B^* = \left\{ \left(\pi_B^{*,n} \right)^{-1}(E) \;\middle|\; n \geq 0,\ \ E \subseteq \Omega_B^n \right\}$$

is an algebra on Ω_B^* (events Bob is concerned with)

Bob's Space of Uncertainty: Properties

- $(\Omega_B^*, \mathcal{A}_B^*)$ is Bob's space of uncertainty
- Associated space of acts:

$$X^{(\Omega_B^*, \mathcal{A}_B^*)} = \left\{ f : \Omega_B^* \to X \;\middle|\; f^{-1}(x) \in \mathcal{A}_B^* \quad \forall x \right\}$$

- $\mathcal{P}_B^* =$ all preference relations on $X^{(\Omega_B^*, \mathcal{A}_B^*)}$

Proposition 1: $X^{(\Omega_B^*, \mathcal{A}_B^*)} = \left\{ \left(f \circ \pi_B^{*,n}\right) \middle| n \geq 0,\ f \in X^{\Omega_B^n} \right\}$

Consequence: Given $\succcurlyeq_A^0, \succcurlyeq_A^1, \ldots$, a unique $\succcurlyeq_A^* \in \mathcal{P}_A^*$ agrees with all of them, or:

$$\boxed{\textbf{Theorem 1}:\ \Omega_B^* \cong S_A \times \mathcal{P}_A^*}$$

Spaces of Uncertainty: Discussion

- By the theorem, "preference types" implicitly describe interactive preferences, just like Harsanyi-types
  - ▼ A "type" $\succcurlyeq_B^*$ of Bob is a pref. relation on $X^{(\Omega_B^*, \mathcal{A}_B^*)}$, where $\Omega_B^* = S_A \times \mathcal{P}_A^*$ and $\mathcal{P}_A^*$ is the "type" space of Alice.
  - ▼ Preference relations are without loss of generality, regardless of which axioms we impose.
- Ω_B^* is uncountable, and $\mathcal{A}_B^*$ is countably infinite
- Events in $\mathcal{A}_B^*$ have a double role:
  - ▼ object of thought experiments (bets) by Bob
  - ▼ describe thought experiments (preference on bets) of Alice

SEU Subspaces – Introduction

A sequence of Alice's preference relations

$$\left(s_A, \succcurlyeq_A^0, \succcurlyeq_A^1, \ldots\right) \in \Omega_B^*$$

might satisfy Savage's axioms, but her corresponding belief might put positive probability on Bob's preferences not satisfying them.

Imposing that all preferences in the spaces Ω_A^* and Ω_B^* satisfy the axioms is <u>not</u> enough.

Modelling "Sureness"

Given B's acts $f, g \in X^{(\Omega_B^*, \mathcal{A}_B^*)}$ and an event $E \in \mathcal{A}_B^*$, write fEg for the act that is f on E and g on $\Omega_B^* \setminus E$.

Following Savage, we say Bob is <u>sure</u> at $(s_B, \succcurlyeq_B^*) \in \Omega_A^*$ that some event $E \in \mathcal{A}_B^*$ obtains, provided that

$$fAg \sim_B^* fAh \qquad \forall f, g, h \in X^{(\Omega_B^*, \mathcal{A}_B^*)}$$

Can Bob be sure about some $\Omega_B \subseteq \Omega_B^*$ not in $\mathcal{A}_B^*$?

Yes. If Ω_B is <u>projective</u> i.e. completely described by its projections $E^0 = \pi_B^{*,0}(\Omega_B)$, $E^1 = \pi_B^{*,1}(\Omega_B)$, ..., or

$$\omega_B \in \Omega_B \quad \Leftrightarrow \quad \pi_B^{*,n}(\omega_B) \in E^n \quad \forall n$$

Evident Subspaces

Say $\Omega_A \subseteq \Omega^*_A$ and $\Omega_B \subseteq \Omega^*_B$ are <u>evident</u> if

$$\forall\, n \qquad \forall\, \left(s_B, \succcurlyeq^*_B\right) \in \Omega_A \qquad \forall\, \left(s_A, \succcurlyeq^*_A\right) \in \Omega_B$$

Alice is $\succcurlyeq^*_A$-sure about $\left(\pi^{*,n}_A\right)^{-1}\left(\pi^{*,n}_A(\Omega_A)\right)$

Bob is $\succcurlyeq^*_B$-sure about $\left(\pi^{*,n}_B\right)^{-1}\left(\pi^{*,n}_B(\Omega_B)\right)$

Projective, evident subspaces are "autonomous" uncertainty spaces: at every point inside, both players are (meaningfully) "commonly sure" they are inside.

Are there projective, evident subspaces where <u>all</u> Savage's axioms hold?

NO (but please hold on)

Savage's Axioms

P1 (Ordering): $\succcurlyeq$ complete, transitive

P2 (Sure-Thing I): $fEh \succcurlyeq gEh \;\Leftrightarrow\; fEh' \succcurlyeq gEh'$

P3 (Sure-Thing II): E not sure: $x \succcurlyeq y \Leftrightarrow xEf \succcurlyeq yEf$

P4 (Qualitative Probability): $x \succ y$ and $x' \succ y'$:
$xEy \succcurlyeq xFy \Leftrightarrow x'Ey' \succcurlyeq x'Fy'$

P5 (Nontriviality): $x \succ y$ for some x, y

P6 (Continuity): Let f, g be acts such that $f \succ g$, and $x \in X$. There exists a (measurable) partition $E_1, \ldots, E_N$ of Ω such that $xE_nf \succ g$ and $f \succ xE_ng$ for all $1 \leq n \leq N$.

Savage's Axioms: Comment

- In terms of *coherent betting*,
  - ▼ P1–P5 express *rationality*
  - ▼ P6 does not (better seen as a property)

- Fundamental technical/conceptual difference:
  - ▼ P1–P5 are satisfied by $\succcurlyeq_{\boldsymbol{B}}^{*}$ on $\boldsymbol{X}^{(\Omega_B^*, \mathcal{A}_B^*)}$ iff

    they hold for $\succcurlyeq_{\boldsymbol{B}}^{\mathbf{0}}, \succcurlyeq_{\boldsymbol{B}}^{\mathbf{1}}, \ldots$ on $\boldsymbol{X}^{\Omega_B^{\mathbf{0}}}, \boldsymbol{X}^{\Omega_B^{\mathbf{1}}}, \ldots$
  - ▼ This makes no sense for P6

- Justifying P6 usually requires an extraneous "fair coin" (it is a pure "richness" property)

The Meaning of P6

P6 has two "parts":

(i) The algebra or $\boldsymbol{\sigma}$-algebra of events must be infinite (the "coin" must be tossed infinitely many times)

(ii) Every event is made up of arbitrarily "small" events (the "coin" cannot be <u>totally</u> biased)

Here (i) comes for <u>free</u>, and (ii) means:

▶ Bob cannot be sure about the thought experiments of Alice when these become "very hypothetical" . . .

Expected Utility Subspaces (I)

Can players be "commonly sure" about P1–P6? Almost . . . ★

Proposition 4: $\exists$ projective subsets $\overline{\boldsymbol{\Omega}}_i \subseteq \boldsymbol{\Omega}_i^*$ such that

(i) $\overline{\boldsymbol{\Omega}}_A, \overline{\boldsymbol{\Omega}}_B$ are evident.

(ii) $\forall (s_i, \succcurlyeq_i^0, \succcurlyeq_i^1, \ldots) \in \overline{\boldsymbol{\Omega}}_j \quad \forall n \quad \exists \succcurlyeq_i^* \in \boldsymbol{\mathcal{P}}_i^*$ that satisfies P1–P6 and extends $\succcurlyeq_i^n$.

(iii) For any (not nec. proj.) $\boldsymbol{\Omega}_i' \subseteq \boldsymbol{\Omega}_i^*$ satisfying (i), (ii), we have $\boldsymbol{\Omega}_i' \subseteq \overline{\boldsymbol{\Omega}}_i$.

(iv) $\overline{\boldsymbol{\Omega}}_A, \overline{\boldsymbol{\Omega}}_B$ are, like $\boldsymbol{\Omega}_A^*, \boldsymbol{\Omega}_B^*$, uncountable (they are also Cantor sets).

★ order of quantifiers in (ii) crucial: $\exists \succcurlyeq_i^*$ s.t. $\forall n$ is <u>false</u>

Expected Utility Subspaces (II)

Proposition 5: Let

$$\overline{\Omega}_B^{\mathbf{P6}} = \big\{(s_A, \succcurlyeq_A^*) \in \overline{\Omega}_B \;\big|\; \succcurlyeq_A^* \text{ satisfies P6}\big\}.$$

Then $\overline{\Omega}_B^{\mathbf{P6}}$ and $\overline{\Omega}_B \setminus \overline{\Omega}_B^{\mathbf{P6}}$ have the same projections.

Thus, P1–P5 and extendability to P6 exploit definition of projective evident subsets to the fullest extent . . .

$\Rightarrow$ from now on work inside $\overline{\Omega}_A, \overline{\Omega}_B$ with induced algebras $\overline{\mathcal{A}}_i = \big\{\overline{\Omega}_i \cap E \;\big|\; E \in \mathcal{A}_i^*\big\}$

. . . but the scope of the definition does not reach beyond that, so . . .

$\Rightarrow$ need σ-algebras $\overline{\mathcal{F}}_i = \sigma\big(\overline{\mathcal{A}}_i\big)$

Expected Utility Subspaces (III)

Problem: need to extend preferences $\succcurlyeq_i$ on $X^{(\overline{\Omega}_i, \overline{\mathcal{A}}_i)}$ to the much bigger family $X^{(\overline{\Omega}_i, \overline{\mathcal{F}}_i)}$. Not possible (in a unique way) in general, but:

Proposition 6: If $(s_i, \succcurlyeq_i) \in \overline{\Omega}_j^{\mathbf{P6}}$, then there is a unique $\succcurlyeq_i^{\mathbf{P6}}$ on $X^{(\overline{\Omega}_i, \overline{\mathcal{F}}_i)}$ that satisfies P1–P6 and extends $\succcurlyeq_i^*$.

Thus, it makes sense to define: i is <u>sure</u> about event $E \in \overline{\mathcal{F}}_i$ at $(s_i, \succcurlyeq_i) \in \overline{\Omega}_j^{\mathbf{P6}}$, provided that E is $\succcurlyeq_i^{\mathbf{P6}}$-null.

Expected Utility Subspaces (IV)

Fundamental measurability property of common beliefs:

- $\overline{\Omega}_j^{\mathsf{P6}} \in \overline{\mathcal{F}}_j$

- $\overline{\Omega}_j^{\sigma,1} := \left\{ (s_i, \succcurlyeq_i) \in \overline{\Omega}_j^{\mathsf{P6}} \;\middle|\; \overline{\Omega}_i^{\mathsf{P6}} \text{ is } \succcurlyeq_i\text{-sure} \right\} \in \overline{\mathcal{F}}_j$

- $\overline{\Omega}_j^{\sigma,2} := \left\{ (s_i, \succcurlyeq_i) \in \overline{\Omega}_j^{\sigma,1} \;\middle|\; \overline{\Omega}_i^{\sigma,1} \text{ is } \succcurlyeq_i\text{-sure} \right\} \in \overline{\mathcal{F}}_j$

. . .

- $\overline{\Omega}_j^{\sigma,n+1} := \left\{ (s_i, \succcurlyeq_i) \in \overline{\Omega}_j^{\sigma,n} \;\middle|\; \overline{\Omega}_i^{\sigma,n} \text{ is } \succcurlyeq_i\text{-sure} \right\} \in \overline{\mathcal{F}}_j$

Expected Utility Subspaces (V)

Conclude that:

$$\Omega_j^{\sigma} := \bigcap_{n \geq 1} \overline{\Omega}_j^{\sigma,n} \in \overline{\mathcal{F}}_j$$

and finally, putting

$$\mathcal{F}_j^{\sigma} := \left\{ \Omega_j^{\sigma} \cap E \;\middle|\; E \in \overline{\mathcal{F}}_j \right\}$$

$$\mathcal{P}_j^{\sigma} := \text{all P1–P6 preference relations on } X^{(\Omega_j^{\sigma}, \mathcal{F}_j^{\sigma})}$$

Main Theorem of SEU in Games

$$\boxed{\textbf{Theorem 2}\text{: } \Omega^{\sigma}_{B} \;\cong\; S_A \times \mathcal{P}^{\sigma}_{A} \;\cong\; S_A \times \mathcal{U}_A \times \Delta(\Omega^{\sigma}_{A})}$$

where:

- $\triangleright$ $\mathcal{U}_A = \left\{ u : X \to \mathbb{R} \;\middle|\; u \text{ is non-constant} \right\} \Big/ \text{pos.aff.trans.}$
- $\triangleright$ $\Delta(\Omega^{\sigma}_{A})$ = all non-atomic, countably additive probability measures on $(\Omega^{\sigma}_{A}, \mathcal{F}^{\sigma}_{A})$
- $\triangleright$ Second $\cong$ is $f \succcurlyeq_A g \;\Leftrightarrow\; \int_{\Omega^{\sigma}_{A}} (u \circ f) dP \geq \int_{\Omega^{\sigma}_{A}} (u \circ g) dP$

Conclusions

- Harsanyi's (1967/68) idea generalizes to preference relations.

- The universal space Ω^*_B answers the question of what is Bob's state of the world in the game with Alice. The construction is prior to all axioms.

- Savage's P1–P6 give universal Harsanyi-type spaces $\Omega^\sigma_A, \Omega^\sigma_B$ (as in Mertens and Zamir (1985)). Foundations for SEU <u>and</u> Harsanyi's model (nonatomic case) are thus provided.

- Going back to Harsanyi requires P6, a reasonable property that is also vital for existence/uniqueness of SEU representations. Finite-type models are problematic.

Topics for Further Research

- Measuring Utility in Games: A simple procedure
- The space $\Omega^{\sigma} = \Omega^{\sigma}_{A} \times \Omega^{\sigma}_{B}$ is as in Aumann (1974)
  - ▼ $\mathcal{F}^{\sigma}_{A}$ contains the Bob-secret events
  - ▼ ex-post subjective correlated equilibria
- Complete and incomplete information unified
  - ▼ finite type-spaces seem inconsistent with SEU
  - ▼ players cannot be commonly sure about <u>both</u> utilities and beliefs
  - ▼ link to purification and higher-order beliefs lit.
- Other axiomatics (e.g. Machina and Schmeidler (1992))