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Alfredo Di Tillio

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Subjective Expected Utility in Games*

Alfredo Di Tillio

Department of Economics, University of Texas at Austin

IEP and IGIER, Università Bocconi

Address: IGIER, Università Bocconi, Via Salasco 5, 20136 Milano, Italy

E-mail: `alfredo.ditillio@unibocconi.it`

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Abstract

This paper extends Savage's subjective approach to probability and utility from decision problems under exogenous uncertainty to choice in strategic environments. Interactive uncertainty is modeled both explicitly — using hierarchies of preference relations, the analogue of beliefs hierarchies — and implicitly — using *preference structures*, the analogue of type spaces à la Harsanyi — and it is shown that the two approaches are equivalent. Preference structures can be seen as those sets of hierarchies arising when certain restrictions on preferences, along with the players' common certainty of the restrictions, are imposed. Preferences are a priori assumed to satisfy only very mild properties (reflexivity, transitivity, and monotone continuity). Thus, the results provide a framework for the analysis of behavior in games under essentially any axiomatic structure. An explicit characterization is given for Savage's axioms, and it is shown that a hierarchy of relatively simple preference relations uniquely identifies the decision maker's utilities and beliefs of all orders. Connections with the literature on beliefs hierarchies and correlated equilibria are discussed.

JEL Classification: C70; D80; D81; D82; D83

Keywords: Subjective probability; Preference hierarchies; Type spaces; Beliefs hierarchies; Common belief; Expected utility; Incomplete information; Correlated equilibria

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1. Introduction

The behavioral premises and implications of subjective expected utility (henceforth SEU) theory are well understood in one-agent situations. Taking as primitive only a preference relation on the set of acts, i.e. functions mapping the *exogenously specified set of states of the world* into the set of outcomes, Savage [23] identified the axioms one must impose on this relation in order to unambiguously identify a utility function and a subjective belief representing it. For interactive contexts, where it is further assumed informally that agents share a common assumption of the SEU hypothesis, a similar treatment has not been given.[1] This paper concerns choice in such environments, where the hypothesis that a player is a SEU maximizer, believes each other player is, believes each other player believes each other player is, and so on, is not assumed, but rather must be derived, as in Savage, from rules on subjective preference.

A game situation is described by a set of players $1, \ldots, I$, a set S^0 of states of nature, a set S^i of strategies for each player i, and a function ζ mapping $S = \times_{j=0}^{I} S^j$ into a set of outcomes Z.[2] In such an environment, it is just not clear what a state of the world should be. Indeed, in game theory the SEU hypothesis carries much more content than in decision theory. Probabilistic beliefs for each player are assumed not only on the determinants of the outcome (i.e. on the set S) but also on the players' utilities and beliefs on S, on the player's beliefs on the players' utilities and beliefs, etc.[3] To be sure, solving a player's problem amounts to choosing a strategy with highest expected utility, which can be identified based on his beliefs on S only. But, contrary to the one-agent case, beliefs are typically restricted by consistency requirements to formulate which the analyst is forced to assume beliefs about beliefs — for instance, the requirement that a player believe that another player chooses a strategy maximizing expected utility.

Thus, following Savage's approach, hence regarding the players' utilities and beliefs as representations of preference relations, one faces a conceptual and methodological problem. States of the world must be specified without explicit reference to the players' utilities and beliefs, as these are not to be treated as primitive objects. At the same time, in order to make sense out of beliefs about beliefs, its description must include the players' preferences. Furthermore, both for methodological rigour and model versatility, it would be desirable to achieve this without imposing too

[1] Kadane and Larkey [14] remarked as a "curiosity of intellectual history" that Savage's theory and game theory had "had little to do with one another despite their common heritage" from von Neumann and Morgenstern's [25] work. The only exception is Epstein and Wang's [9] paper, discussed later on in this section.

[2] Note that any uncertainty about the outcome function ζ can be modeled as uncertainty about the state of nature.

[3] This is done implicitly, following Harsanyi [12]. A set of types for every player is introduced, and for every type of every player a utility function and a subjective belief over the states of nature and the other players' strategies and types is assumed. Mertens and Zamir [21] proved that Harsanyi's idea implies no loss of generality, in the sense that any hierarchy of beliefs of a player can be generated by choosing sets of types appropriately.

many axioms a priori. One usually specifies the set of states of the world before imposing axioms; moreover, it makes little sense to ask whether a player's preferences satisfy a certain axiom, if all preferences included in the description of a state of the world automatically satisfy that axiom.

In this paper we propose a solution to this problem based on a straightforward generalization of Harsanyi's idea and similar to the one devised by Epstein and Wang [9]. The key notion introduced here is that of *preference structure*, defined by an abstract set X, an algebra $\mathcal{B}$ of subsets of X, a $\mathcal{B}$-measurable function $\sigma : X \to S$, (we assume S and Z are finite) and a $\mathcal{B}$-measurable function $\vartheta^i : X \to \Pi(X, \mathcal{B})$ for each $i = 1, \dots, I$, where $\Pi(X, \mathcal{B})$ is the set of preference relations on the set of $\mathcal{B}$-measurable maps from X to Z, endowed with a suitably specified algebra. Thus, the implicitly defined hierarchies of beliefs in a Harsanyi-like model become implicitly defined *hierarchies of preferences* here.[4] Each point in X implicitly describes every player's preference over acts of the form $f : S \to Z$, acts of the form $f : S \times \Pi^I(S) \to Z$, acts of the form $f : S \times \Pi^I(S) \times \Pi^I(S \times \Pi^I(S)) \to Z$, etc.[5] This procedure avoids assuming utilities and beliefs directly; more importantly, it makes the fact that a player's preferences satisfy certain axioms a well defined event for every player, and it allows us to formally state the hypothesis that a player's preferences satisfy the axioms and that this is *common certainty* (see below) among the players.

Preference relations are a priori assumed to satisfy only very mild axioms, namely, reflexivity, transitivity, and monotone continuity. The first two are without a doubt among the least controversial. The third axiom, also rather intuitive and automatically satisfied if the family of events is finite, is equivalent to countable additivity of the belief in the Savage representation if this exists; we prove it guarantees the analogous property even in the absence of axioms other than transitivity. The upshot of these limited restrictions is that one can use preference structures to analyze strategic situations under many possible axiomatic structures, provided our three axioms hold — we discuss this further in Subsection 6.1.

While our results demonstrate that Savage's theory is applicable to game situations,[6] it is worth

[4]More precisely, the analogy is with a model *à la* Aumann [3], where strategies appear explicitly in the description of a state. Indeed, our notion of preference structure logically corresponds to what Aumann calls "information system".

[5]We use $\Pi^I(\cdot)$ as an abbreviation for $\times_{i=1}^I \Pi(\cdot)$. In the main body of the paper we do *not* work with the full set $S \times \Pi^I(S) \times \Pi^I(S \times \Pi^I(S)) \times \cdots$. Instead, we impose *coherency* at all levels of the hierarchies. This means that we consider only those elements of $\Pi^I(S) \times \Pi^I(S \times \Pi^I(S))$ where the preferences appearing in the first and second coordinate agree (in the obvious way) on the acts of the form $f : S \to Z$, and similarly for higher orders. The alternative construction, where hierarchies are unrestricted and coherency is imposed *a posteriori*, is entirely equivalent, as we explain in Subsection 6.5. Note also that i's strategy and preferences appear in i's own uncertainty. This is done mainly for notational convenience, i.e. to avoid having to construct different spaces of states of the world for different players; we explain this further in Subsection 6.2.

[6]This is in sharp contrast with the results of Mariotti [20]. Roughly, the latter paper shows that a player's preference relation on strategies cannot satisfy Savage's axioms if various game-theoretic notions of rationality regarding the other players are imposed (see Battigalli [4] for a counterargument to these negative results). Our approach is very different from Mariotti's, mainly because his framework does not allow preference hierarchies, and these are what we use in

pointing out that they also highlight special features of the theory that arise in these contexts but are typically absent from the one-person case. First, Savage's axiom P6 requires that the space of states of the world be infinite. In one-person situations where the latter is not the case, one must appeal to objects extraneous to the model (such as an infinite sequence of coin tosses) in order to meet the requirement. Here, however, the necessary cardinality obtains automatically due to the infinite construction. Second, P6 acquires a special meaning; roughly, in our context the axiom says that a player cannot be sure about the precise hierarchies of preference relations of the other players, or even sure they belong to a given finite set. In other words, nontrivial uncertainty of the higher order beliefs is what makes beliefs themselves exist and be unique.[7]

1.1. Outline of the Analysis and Plan of the Paper

After dealing with a few inevitable preliminaries in Section 2, we introduce preference hierarchies and preference structures in Section 3, where our first main results are proved. Analogously to Mertens and Zamir's (1985) main theorem, these results show that our model carries no loss of generality. The set Ω of all sequences comprising a state of nature and, for each player, a strategy and a coherent hierarchy of preference relations, together with the algebra $\mathcal{A}$ of its cylinders, has a nice mathematical structure. In the terminology introduced in Section 2, it is a *simple space*, namely, a zero-dimensional compact Hausdorff topological space, whose family of clopen sets is precisely the algebra of events. The results show that each $\omega \in \Omega$ corresponds to a unique preference relation $\varpi^i(\omega)$ for each player i on the set of acts of the form $f : \Omega \to Z$. Furthermore, the *canonical preference structure* comprising the space $(\Omega, \mathcal{A})$, the projection of Ω on S, and the mappings ϖ^i, is isomorphic (in a natural sense) to $S \times \Pi^I(\Omega)$ and such that every other well-behaved preference structure can be mapped into it in an essentially unique way.[8] Thus, the canonical structure is *universal* in Mertens and Zamir's sense.

Section 4 is devoted to the study of *common certainty components* of the universal structure, namely, sets of hierarchies obtained imposing a certain event along with the players' certainty of this event, certainty of the players' certainty, and so on. Following Savage, given a preference

order to impose axioms, players' certainty of the axioms, etc. Thus, preference hierarchies provide a workaround to the issues raised by Mariotti — thanks to Dale Stahl (private conversation) for suggesting this interpretation.

[7] It is well known that P1–P5 are not sufficient for existence of a SEU representation. This may be readily verified by a straightforward adaptation of the famous counterexample by Kraft et al [18]. With a finite set of outcomes, uniqueness of the SEU representation in general requires an infinite set of states of the world — see Gul [11] — and, more importantly, non-atomicity of the subjective belief.

[8] By well-behaved structure we mean a structure whose algebra of events involves no redundancy (points where all players choose the same strategies and have the same preferences) and no more information than the corresponding events in $\mathcal{A}$. These requirements are formalized in Section 3.

structure, we say a player is certain of an event at a point of the structure if his preferences at that point are such that he is indifferent between any two acts that coincide on that event. In order to discuss a player's certainty of a player's certainty, it becomes necessary to extend the class of events in a structure beyond the initially assumed algebra. Thus, in Section 4 we prove that, for a *simple structure* — a structure whose underlying space $(X, \mathcal{B})$ is simple — a preference relation in $\Pi(X, \mathcal{B})$ has a unique extension to a preference relation on the enlarged set of acts obtained expanding $\mathcal{B}$ to include a *closed* subset of X, i.e. an intersection of events in $\mathcal{B}$. This is enough to formally define common certainty of a closed subset of X, because if $E \subseteq X$ is closed then the subset of X where a player is certain of E is itself closed. The main result in Section 4 then states that a well-behaved preference structure is simple if and only if isomorphic to the component of the universal structure obtained imposing common certainty of a closed subset of the latter.

In Section 5 we prove several important results. First, we show that, given any space $(X, \mathcal{B})$, a preference relation in $\Pi(X, \mathcal{B})$ satisfying Savage's axioms has a unique extension to a preference relation (also satisfying Savage's axioms) on the enlarged set of acts obtained replacing $\mathcal{B}$ with the σ-algebra $\mathcal{B}^*$ generated by $\mathcal{B}$. This allows us to talk about *belief*, i.e. certainty in the presence of Savage's axioms, of any set in $\mathcal{B}^*$. Next, we show that if $\mathcal{B}^*$ is in fact *countably generated* — a pervasive assumption in the literature on type spaces, then, given any structure based on the space $(X, \mathcal{B})$, we can talk about interactive beliefs, because the subset of X where a player believes some event $E \in \mathcal{B}^*$ is shown to be itself in $\mathcal{B}^*$. The latter result is one of the conclusions of Proposition 10, a key result showing the equivalence between *Savage structures*, i.e. preference structures where Savage's axioms hold for every player at every point — and *SEU systems*, i.e. the standard object assumed in the game-theoretic literature, i.e. a measurable space with profiles of strategies, utility functions, and beliefs (over the space itself) associated to each point in the space. Finally, we prove that the *canonical Savage structure*, i.e. the structure whose underlying space is the component Ω_S obtained imposing Savage's axioms and common belief of them, features analogous universality properties. First, we prove that this structure is isomorphic to $S \times \Pi_S^I(\Omega_S)$ and also to $S \times \mathcal{U}^I \times \Delta^I(\Omega_S)$ — where Π_S denotes Savage preference relations, Δ non-atomic probability measures, and $\mathcal{U}$ non-constant utility functions (up to positive affine transformations). Second, we prove the equivalence among (i) certain well-behaved Savage structures,[9] (ii) the components of the universal structure obtained imposing Savage's axioms and common belief of a Borel set (an event in the σ-algebra generated by $\mathcal{A}$), and (iii) the components of the canonical Savage structure obtained imposing common belief of a Borel set (an event in the relative σ-algebra on Ω_S).

[9]Here by well-behaved structure we mean a little more than in Footnote 8. Namely, we also need the property that the underlying space $(X, \mathcal{B})$ is such that the pair comprising X and the σ-algebra generated by $\mathcal{B}$ is *standard Borel*.

1.2. Relationship with Earlier Literature on Hierarchies of Preferences

The idea of preference hierarchies is not new. The issues outlined above were first formally dealt with by Epstein and Wang [9]. They are motivated by the exact same problem and prove theorems analogous to ours, using preference hierarchies in a similar way. Compared to that paper, our work has the disadvantage of dealing with finite games only. Indeed, the technical structure of our model does not lend itself well to the case where, say, the set S is infinite (but see the discussion in Subsection 6.7 below). By contrast, the strategy spaces in Epstein and Wang are only assumed compact Hausdorff. However, the results in the two papers are complementary in several dimensions, and ours offers some important advantages.

First, as noted above and acknowledged by Epstein and Wang themselves, the specification of a state of the world should presume "as few preference axioms as possible". In our construction preferences are indeed rather unrestricted, whereas in their paper the restrictions are substantial. In particular, they take the outcome space to be the unit interval and assume monotone utilities on the latter, by requiring that all preferences satisfy a number of axioms; the more restrictive ones, completeness and monotonicity, are *not* automatically satisfied in the finite case. Thus, the finite case makes the model not only simpler, but also more general in this respect.

Second, our model presumes much less in terms of "computational" ability of the players, and this is not just because preferences are a priori unrestricted (in particular, possibly not complete). Indeed, even within those subspaces of the universal structure where the full force of Savage's axioms is imposed, players are only assumed able to rank relatively uncomplicated acts; at each level of the hierarchies there are a finite number of acts, a finite number of events these may depend on, and a finite number of outcomes. Thus, in principle, one could elicit a player's entire hierarchy of beliefs by means of a *sequence* of questions, each involving a finite number of alternatives.[10] By contrast, in Epstein and Wang's paper, at each level a player is assumed to make preference comparisons between acts depending on a very complex family of events.[11]

Third, and perhaps more importantly, our paper is more oriented towards applications, though it does not concretely discuss any. Epstein and Wang's paper adopts an approach analogous to Brandenburger and Dekel [7], focusing on the extension results (analogous to our result establishing existence and uniqueness of the mappings ϖ^i) and the consequent isomorphism results (analogous to our isomorphism results regarding Ω and $S \times \Pi^I(\Omega)$ first and Ω_S and $S \times \Pi^I_\mathrm{S}(\Omega_\mathrm{S})$ next). However, it does not address the question of whether, how exactly, and why in modeling a

[10] It is worth pointing out that, when working with conventional beliefs hierarchies based on a set S, one must deal with spaces of very high cardinality, even to describe second or third order beliefs, and this is true even if S is finite.

[11] The problem of classes of events too rich for an individual to conceive is the main motivation for Kopylov's [17] generalizations of Savage's theorem. One of Kopylov's results, crucial for our proofs, is recorded in Section 5.

concrete strategic scenario one can without loss of generality assume a Harsanyi's type space-like construct — though objects similar to our preference structures are indeed briefly discussed as an illustration (see page 1348 of [9]). By contrast, in this paper the precise correspondence between the abstract notion of preference structure and the corresponding common certainty subspaces of the universal structure is investigated in detail. The exercise is far from vacuous, since, as explained above and proved in the paper, not all preference structures can be thought of as common certainty components; only (and all) the sufficiently well-behaved ones can. Needless to say, establishing the precise sense in which abstractly and implicitly defined hierarchies correspond to concretely and explicitly specified ones is the very motivation behind all Mertens and Zamir-like papers. For this reason, we think that the full-fledged analysis of preference structures given here is desirable, and we also hope it will prove useful — after all, just like in traditional applications one works with Harsanyi's types rather than directly with hierarchies, preference structures are what one would work with in the possible concrete applications of our model.[12]

2. Preliminaries

An *uncertainty space* (or, more simply, a *space*) is a pair $(X, \mathcal{B})$ where X is a set and $\mathcal{B}$ is an algebra of subsets of X called *events*. If the specific algebra $\mathcal{B}$ is irrelevant or clear from the context, we may refer to the set X alone as a space; for instance, if X is finite, then it is understood endowed with the algebra of all its subsets. Any subset X' of X equipped with its relative algebra, i.e. the family of all sets of the form $X' \cap E$ where $E \in \mathcal{B}$, is a *subspace* of X.

2.1. Basic Notations and Definitions

We take as given a finite set Z of *outcomes*, a finite set S^0 of *states of nature*, a finite set $\{1, \ldots, I\}$ of *players*, and for each player i a finite set S^i of *strategies*, and let $S = S^0 \times S^1 \times \cdots \times S^I$. To avoid trivialities, we assume that Z contains at least two distinct elements.

Let $(X, \mathcal{B})$ be a space. An *act* is a function $f : X \to Z$ such that $f^{-1}(z) \in \mathcal{B}$ for every $z \in Z$. The constant act mapping every $x \in X$ to the same $z \in Z$ is denoted by z. Given two acts f, g and an event E, we write fEg for the act that coincides with f on E and with g on $X \setminus E$. The set of all acts is denoted $F(X, \mathcal{B})$ or, if the specific algebra $\mathcal{B}$ needs no emphasis, just $F(X)$. Note that, if $\mathcal{B}$ is countable, then $F(X)$ is countable; indeed, since Z is finite, the set of constant

[12]Finally, we hope the relevance of these considerations justifies this paper's otherwise unfortunate length, indeed mainly due to the extensiveness of the analysis regarding well-behaved structures, common certainty components, etc.

acts is finite, and every act whose range has at most $n \geq 2$ distinct outcomes can be written as zEf, where $z \in Z$, $E \in \mathcal{B}$, and f is an act whose range has at most $n-1$ outcomes.

If π is a binary relation on $F(X)$, i.e. a subset of $F(X) \times F(X)$, then for all acts $f, g \in F(X)$ we write $(f, g) \dot{\in} \pi$ as an abbreviation for $(f, g) \in \pi \not\ni (g, f)$. A *preference relation* on $F(X)$ is a reflexive and transitive binary relation π on $F(X)$ satisfying the following property:

Monotone Continuity. Let $f, g \in F(X)$, let $z \in Z$, and take a sequence of events E_n such that $E_n \downarrow \varnothing$. If $(f, g) \dot{\in} \pi$, then, for all n sufficiently large, $(zE_n f, g) \dot{\in} \pi$ and $(f, zE_n g) \dot{\in} \pi$.

The set of all preference relations on $F(X, \mathcal{B})$ is denoted $\Pi(X, \mathcal{B})$ and always assumed endowed with the algebra generated by the sets of the form

$$\left\{\pi \in \Pi(X, \mathcal{B}) \ : \ (f, g) \in \pi\right\}$$

where $f, g \in F(X, \mathcal{B})$. The product space $\times_{i=1}^{I} \Pi(X, \mathcal{B})$ is abbreviated as $\Pi^I(X, \mathcal{B})$. If reference to the specific algebra $\mathcal{B}$ is superfluous, we may write $\Pi(X)$ instead of $\Pi(X, \mathcal{B})$ and $\Pi^I(X)$ instead of $\Pi^I(X, \mathcal{B})$. Note that $\Pi(X)$ is finite whenever X is finite, and in this case the algebra on $\Pi(X)$ specified above is precisely the algebra of all subsets of $\Pi(X)$.

Let $(Y, \mathcal{C})$ be another space. A function $\varphi : X \to Y$ is *measurable* $\mathcal{B}/\mathcal{C}$ (or just *measurable*) if $\varphi^{-1}(E) \in \mathcal{B}$ for all $E \in \mathcal{C}$. This induces a measurable function $\widehat{\varphi} : \Pi(X) \to \Pi(Y)$, namely,

$$\pi \mapsto \left\{(f, g) \in F(Y) \times F(Y) : (f \circ \varphi, g \circ \varphi) \in \pi\right\}.$$

Note that $\widehat{\varphi}$ is injective if every $f \in F(X)$ satisfies $f = g \circ \varphi$ for some $g \in F(Y)$. This is clearly true if X is a subspace of Y and φ is the inclusion mapping. Moreover, we have the following

Lemma 1. *Let X and Y be uncertainty spaces. If $\varphi : X \to Y$ is a measurable bijection and φ^{-1} is measurable, then the induced mapping $\widehat{\varphi} : \Pi(X) \to \Pi(Y)$ is also a measurable bijection with measurable inverse; the latter is the mapping from $\Pi(Y)$ to $\Pi(X)$ induced by φ^{-1}.*

Proof. Assume that φ is a bijection and φ^{-1} is a measurable function. By the latter, every $f \in F(X)$ can be written as $g \circ \varphi$ for some $g \in F(Y)$, so $\widehat{\varphi}$ is injective. Moreover, for every $\pi \in \Pi(Y)$,

$$\widehat{\varphi}\left(\left\{(f, g) \in F(X) \times F(X) : (f \circ \varphi^{-1}, g \circ \varphi^{-1}) \in \pi\right\}\right) = \pi,$$

which proves that $\widehat{\varphi}$ is onto and also proves the last statement. □

Finally, given two spaces X and Y and a (not necessarily measurable) map $\phi : Y \to \Pi^I(X)$, we write ϕ^i for the function from Y to $\Pi(X)$ that maps $y \in Y$ into the ith coordinate of $\phi(y)$.

2.2. Simple Spaces

An uncertainty space $(X, \mathscr{B})$ is *simple* if $\mathscr{B}$ is a countable base for a compact Hausdorff topology on X. A particular case is when X is finite. The product of a finite family of spaces — which, unless otherwise noted, we assume endowed with the product algebra — is clearly a simple space if each space in the family is simple. When dealing with simple spaces, we will speak of open sets, continuous functions, and so on, without explicit reference to their topological structures; in each case, the understanding is that the relevant topology on the simple space at hand is the topology generated by the events in that space.[13]

Lemma 2. *A subset of a simple space is an event if and only if it is both closed and open. A subspace of a simple space is simple if and only if it is closed.*[14]

Proof. An open set in a simple space is a union of events; thus, a closed (hence compact) and open set in a simple space is the union of a finite family of events, hence an event. The other direction is obvious; each event and its complement are both open, hence both closed. Clearly, a subset of a simple space is compact Hausdorff if and only if it is closed in its relative topology. Since the latter is the same as the topology generated by its relative algebra, the second claim follows. □

Note that if X is a simple space, then every reflexive and transitive binary relation on $F(X)$ is automatically a preference relation on $F(X)$, i.e. monotone continuity is trivially satisfied. Indeed, every sequence of events E_n such that $E_n \downarrow \varnothing$ must satisfy $E_n = \varnothing$ for every n sufficiently large, because X is compact, each E_n is closed, and the sequence has the finite intersection property.

Proposition 1. *If X is a simple space, then $\Pi(X)$ is a simple space.*

We conclude this subsection recording a few more properties of simple spaces. If X and Y are simple spaces, then $\varphi : X \to Y$ is measurable if and only if continuous (first part of Lemma 2); in this case the induced $\widehat{\varphi} : \Pi(X) \to \Pi(Y)$ is also continuous. Moreover, we have the following.

Lemma 3. *Let X and Y be uncertainty spaces. Assume X is simple, and let $\varphi : X \to Y$ be a measurable function. If φ is onto, then the induced mapping $\widehat{\varphi} : \Pi(X) \to \Pi(Y)$ is onto.*

[13]Thus, the product of a finite family of simple spaces (resp. a subspace of a simple space) is understood endowed with the topology generated by the product algebra (resp. its relative algebra). No confusion can arise, as this is the same as the product topology (resp. the relative topology induced by the topology on the larger space).

[14]A topological space whose topology has a countable base of clopen (closed and open) sets is called *zero-dimensional*, and zero-dimensional compact Hausdorff topological spaces are called *Boolean*. Thus, what we call *simple space* is a Boolean topological space stripped down to its algebra of clopen sets. For an excellent treatment of Boolean spaces, see Koppelberg [16] — our proof of Lemma 2 is taken from there.

Proof. Assume that $\varphi : X \to Y$ is onto, and let $\pi \in \Pi(Y)$. We have to find a preference relation on $F(X)$ that is mapped into π by the function $\widehat{\varphi}$. Consider the set

$$\big\{(f \circ \varphi, g \circ \varphi) : (f, g) \in \pi\big\} \cup \big\{(f, f) : f \in F(X)\big\}.$$

This is a well defined binary relation on $F(X)$, as ontoness of φ ensures that, if $f, f' \in F(Y)$ and $f \circ \varphi = f' \circ \varphi$, then $f = f'$. Since it contains (f, f) for every $f \in F(X)$, it satisfies reflexivity; transitivity is directly inherited from π; monotone continuity is automatic, since X is simple. $\square$

3. Coherent Preference Hierarchies

The space S embodies the *basic uncertainty* faced by the players. In the standard game-theoretic framework, *strategic uncertainty* is formalized by means of *coherent beliefs hierarchies* based on this space. In our framework, following Epstein and Wang [9], it is modeled by means of *coherent hierarchies of preference relations*. In this section we construct the space of all such hierarchies, establish its properties, and show that one can describe preference hierarchies implicitly, much like one describes coherent beliefs hierarchies by means of Harsanyi's types.

Define $\Omega_0 = S$ and $\Omega_1 = \Omega_0 \times \Pi^I(\Omega_0)$. Let $\rho_0 : \Omega_1 \to \Omega_0$ and $\delta_1 : \Omega_1 \to \Pi^I(\Omega_0)$ be the natural projections, and let $\widehat{\rho}_0 : \Pi(\Omega_1) \to \Pi(\Omega_0)$ be the mapping induced by ρ_0. Proceeding recursively, define Ω_{n+1} for $n \geq 1$ as the largest subspace of $\Omega_n \times \Pi^I(\Omega_n)$ such that the following diagram — where $\rho_n : \Omega_{n+1} \to \Omega_n$ and $\delta_{n+1} : \Omega_{n+1} \to \Pi^I(\Omega_n)$ are the natural projections, as before — commutes for every player i:

$$\begin{array}{ccc}
\Omega_{n+1} & \xrightarrow{\;\delta^i_{n+1}\;} & \Pi(\Omega_n) \\
\rho_n \big\downarrow & & \big\downarrow \widehat{\rho}_{n-1} \\
\Omega_n & \xrightarrow{\;\delta^i_n\;} & \Pi(\Omega_{n-1})
\end{array}$$

As before, let $\widehat{\rho}_n$ be the mapping from $\Pi(\Omega_{n+1})$ to $\Pi(\Omega_n)$ induced by ρ_n.[15]

Lemma 4. *For all $n \geq 0$, the projection $\rho_n : \Omega_{n+1} \to \Omega_n$ is onto. For all $\omega_n \in \Omega_n$, in fact, $\rho_n^{-1}(\omega_n)$ has at least two distinct elements. Thus, the induced $\widehat{\rho}_n : \Pi(\Omega_{n+1}) \to \Pi(\Omega_n)$ is onto.*

[15] Commutativity of the diagram is what we refer to as coherency. We follow Mertens and Zamir's [21] approach to coherency, whereby the property is built-in rather instead of being imposed *a posteriori* (as Epstein and Wang [9] instead do, following Brandenburger and Dekel [7]). We could have followed the latter route, obtaining essentially the same results obtained here; a more formal discussion on this, however, must wait till Subsection 6.5.

The space of all *coherent preference hierarchies* is the set

$$\Omega = \left\{(\omega_0, \omega_1, \ldots) \in \Omega_0 \times \Omega_1 \times \cdots \;:\; \rho_n(\omega_{n+1}) = \omega_n \quad \forall n \geq 0\right\}$$

with the algebra $\mathcal{A}$ of all sets of the form $\varrho_n^{-1}(A)$, where $n \geq 0$, $A \subseteq \Omega_n$, and ϱ_n is the natural projection of Ω on Ω_n. Note that ϱ_n is onto for all n, hence $\mathcal{A}$ is indeed a well defined algebra. The induced $\widehat{\varrho}_n : \Pi(\Omega) \to \Pi(\Omega_n)$ is, in fact, also onto, as the following result (by Lemma 3) implies.

Proposition 2. $(\Omega, \mathcal{A})$ *is a simple uncertainty space.*

Proof. The sequence (Ω_n, ρ_n) is a projective system of finite (hence compact Hausdorff) topological spaces. By construction, the set Ω endowed with the relative product topology (which has $\mathcal{A}$ as a base) is the projective limit of the system, hence a compact Hausdorff topological space. □

The space Ω closes our construction in the sense of leaving no further uncertainty undescribed. This is established in the following proposition. Paralleling the analogous results in the literature on beliefs hierarchies,[16] the proposition provides the first step towards our main theorems.

Proposition 3. *There exists a unique mapping* $\varpi : \Omega \to \Pi^I(\Omega)$ *such that*

$$\widehat{\varrho}_{n-1} \circ \varpi^i \;=\; \delta_n^i \circ \varrho_n \tag{1}$$

for every $i = 1, \ldots, I$ *and every* $n \geq 1$. *The mapping* ϖ *is measurable and onto. The mapping* $\omega \mapsto (\varrho_0(\omega), \varpi(\omega))$ *from* Ω *into* $S \times \Pi^I(\Omega)$ *is a measurable bijection with measurable inverse.*

3.1. Preference Structures

The following is a straightforward generalization of Harsanyi's [12] notion of a type space, whereby coherent preference hierarchies are implicitly described, just as coherent beliefs hierarchies are implicitly described by Harsanyi's types.

Definition 1. A *preference structure* (or, more simply, a *structure*) is a tuple $(X, \mathcal{B}, \sigma, \vartheta)$ where $(X, \mathcal{B})$ is a space and $\sigma : X \to S$ and $\vartheta : X \to \Pi^I(X)$ are measurable functions. A structure is *simple* (resp. a σ*-structure*) if the underlying uncertainty space is simple (resp. a measurable space, i.e. a space such that the algebra of events is in fact a σ-algebra).

[16] See Proposition 2 in Brandenburger and Dekel [7] and Theorem 2.9 in Mertens and Zamir [21].

A structure $(X, \mathcal{B}, \sigma, \vartheta)$ generates a set of hierarchies in a natural way, as follows. First, let $\gamma_0 = \sigma$ and write $\widehat{\gamma}_0$ for the induced mapping from $\Pi(X, \mathcal{B})$ to $\Pi(\Omega_0)$. Then, recursively for all $n \geq 0$, define $\gamma_{n+1} : X \to \Omega_{n+1}$ as

$$x \mapsto \Big(\gamma_n(x), \big(\widehat{\gamma}_n\big(\vartheta^1(x)\big), \ldots, \widehat{\gamma}_n\big(\vartheta^I(x)\big)\big)\Big)$$

and $\widehat{\gamma}_{n+1} : \Pi(X) \to \Pi(\Omega_{n+1})$ as the mapping induced by γ_{n+1}. Finally, define $\gamma : X \to \Omega$ as

$$x \mapsto \big(\gamma_0(x), \gamma_1(x), \ldots\big).$$

Since σ and ϑ are measurable, and since $\gamma_n = \rho_n \circ \gamma_{n+1}$ for all $n \geq 0$ by construction, the functions $\gamma_0, \gamma_1, \ldots$ are all well defined and measurable, hence so is γ. The mapping γ is the *generator* of the structure, and its range is the set of hierarchies *generated* by the structure.

Borrowing terminology from the literature on coherent beliefs hierarchies, we say $(X, \mathcal{B}, \sigma, \vartheta)$ is a *complete* structure if the mapping from X into $S \times \Pi^I(X)$ such that $x \mapsto (\sigma(x), \vartheta(x))$ is onto.[17] If, in addition, the inverse of this mapping exists and is measurable, then we write

$$X \cong S \times \Pi^I(X).$$

The notion of complete structure should not be confused with the notion of complete preference relation. Instead, our nomenclature is suggested and justified by the following.

Proposition 4. *The generator of a simple and complete structure is onto.*

3.2. Preference Morphisms

In order to formalize equivalence between structures we need a notion of isomorphism, whereby isomorphic structures generate the same set of hierarchies, and structures generating the same set of hierarchies are isomorphic. The following definition provides the necessary starting point.

Definition 2. A *morphism* from a structure $(X, \mathcal{B}, \sigma, \vartheta)$ to a structure $(Y, \mathcal{C}, \varsigma, \theta)$ is a pair of

[17] The terms *complete* and *belief complete* have been used, with analogous meaning, by various authors (for instance, Battigalli and Siniscalchi [5] and Brandenburger et al. [8]). We use the term *belief complete* later on in the paper, when dealing with structures where Savage's axioms hold — we avoid doing so here, and just say *complete*, because at this point our preference hierarchies are allowed to be very far from having a probabilistic structure.

measurable functions $\alpha : S \to S$ and $\varphi : X \to Y$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & S \\ \sigma \big\uparrow & & \big\uparrow \varsigma \\ X & \xrightarrow{\varphi} & Y \\ \vartheta \big\downarrow & & \big\downarrow \theta \\ \Pi(X, \mathcal{B}) & \xrightarrow{\widehat{\varphi}} & \Pi(Y, \mathcal{C}) \end{array}$$

where $\widehat{\varphi} : \Pi(X) \to \Pi(Y)$ is the map induced by φ, commutes for every $i = 1, \dots, I$. The two structures are *isomorphic* if, in addition, α and φ are bijections and $(\alpha^{-1}, \varphi^{-1})$ is also a morphism.

The following handy result provides an easy way to check whether a morphism is, in fact, an isomorphism. For this it suffices to verify that inverses exist and are measurable.

Lemma 5. *Let (α, φ) be a morphism from a structure $(X, \mathcal{B}, \sigma, \vartheta)$ to a structure $(Y, \mathcal{C}, \varsigma, \theta)$. If α and φ are bijections and φ^{-1} is measurable, then $(\alpha^{-1}, \varphi^{-1})$ is also a morphism.*

Proof. Let $\widehat{\varphi} : \Pi(X) \to \Pi(Y)$ be the mapping induced by φ. Since (α, φ) is a morphism, we have $\alpha \circ \sigma = \varsigma \circ \varphi$ and $\widehat{\varphi} \circ \vartheta^i = \theta^i \circ \varphi$ for every player i. Thus, if α and φ are bijections, then $\sigma \circ \varphi^{-1} = \alpha^{-1} \circ \varsigma$ and, by Lemma 1, also $\vartheta^i \circ \varphi^{-1} = (\widehat{\varphi})^{-1} \circ \theta^i$. Again by Lemma 1, the mapping from $\Pi(Y)$ to $\Pi(X)$ induced by φ^{-1} is precisely $(\widehat{\varphi})^{-1}$, so the proof is complete. □

An important particular case of Definition 2 is the following. Take two preference structures $(X, \mathcal{B}, \sigma, \vartheta)$ and $(Y, \mathcal{C}, \varsigma, \theta)$ such that $(X, \mathcal{B})$ is a subspace of $(Y, \mathcal{C})$ and σ is the restriction of ς to X. If the identity function on S and the inclusion mapping $\iota : X \to Y$ constitute a morphism, then we say $(X, \mathcal{B}, \sigma, \vartheta)$ is a *substructure* of $(Y, \mathcal{C}, \varsigma, \theta)$. Note that, in this case,

$$\big\{(f, g) \in F(Y) \times F(Y) \ : \ f \circ \iota = g \circ \iota\big\} \subseteq \theta^i(x) \quad \forall i = 1, \dots, I, \ \forall x \in X, \tag{2}$$

by reflexivity of $\vartheta^i(x)$. In other words, player i is indifferent, according to $\theta^i(x)$, between any two acts that coincide on X. Conversely, given a structure $(Y, \mathcal{C}, \varsigma, \theta)$, a subspace X of Y which satisfies (2) can be made into a substructure. For this it suffices to endow X with the restriction of ς to X and, for each $i = 1, \dots, I$, the mapping from X to $\Pi(X)$ such that

$$x \mapsto \big\{(f \circ \iota, g \circ \iota) \ : \ (f, g) \in \theta^i(x)\big\}. \tag{3}$$

This mapping is, by (2), indeed well defined, and in this case we say X *induces* (or, with slight abuse of terminology, *is*) a substructure of the structure $(Y, \mathcal{C}, \varsigma, \theta)$.

Lemma 6. *Let* (α, φ) *be a morphism from a structure* $(W, \mathcal{D}, \tau, \eta)$ *to a structure* $(Y, \mathcal{C}, \varsigma, \theta)$*, and let* $X = \varphi(W)$*. Let* $\phi : W \to X$ *be the mapping such that* $w \mapsto \varphi(w)$*. Then* X *is a substructure of the structure* $(Y, \mathcal{C}, \varsigma, \theta)$ *and, furthermore,* (α, ϕ) *is a morphism.*

3.3. The Universal Preference Structure

Recall that $\varpi : \Omega \to \Pi^I(\Omega)$ denotes the mapping whose existence and uniqueness are established in Proposition 3. We refer to $(\Omega, \mathcal{A}, \varrho_0, \varpi)$ as the *canonical* preference structure. Its generated set of hierarchies is, of course, Ω itself; its generator is the identity. More generally, if a subset $\Omega' \subseteq \Omega$ induces a substructure, the set of hierarchies generated by the latter is Ω', and the generator is the inclusion of Ω' into Ω. In fact, the identity on S and the generator γ of *any* structure $(X, \mathcal{B}, \sigma, \vartheta)$ constitute a morphism from the latter into the canonical structure, since the generator γ and the induced $\widehat{\gamma} : \Pi(X) \to \Pi(\Omega)$ clearly satisfy

$$\widehat{\gamma} \circ \vartheta^i = \varpi^i \circ \gamma \qquad \forall i = 1, \ldots, I. \tag{4}$$

Indeed, if α is the identity on S and (α, φ) is a morphism from $(X, \mathcal{B}, \sigma, \vartheta)$ to $(\Omega, \mathcal{A}, \varrho_0, \varpi)$, then it is immediate to see that $\varphi = \gamma$.[18] Moreover, we have the following.

Theorem 1. $\Omega \cong S \times \Pi^I(\Omega)$.

Proof. This is a restatement of part (ii) of Proposition 3 in the language of structures. □

Clearly, isomorphic structures generate the same hierarchies. Conversely, structures that generate the same set of hierarchies and satisfy both requirements in the following definition are isomorphic, indeed isomorphic to a substructure of the canonical structure, as Theorem 2 below shows.

Definition 3. A structure $(X, \mathcal{B}, \sigma, \vartheta)$ is *minimal* (resp. *non-redundant*) if $\mathcal{B}$ is (resp. if every two distinct elements of X are separated by) the smallest algebra $\mathcal{B}'$ such that

$$\sigma^{-1}(s) \in \mathcal{B}' \qquad \forall s \in S, \tag{5}$$

$$\left\{x \in X : (f, g) \in \vartheta^i(x)\right\} \in \mathcal{B}' \qquad \forall i = 1, \ldots, I, \ \forall f, g \in F(X, \mathcal{B}'). \tag{6}$$

The *associated minimal structure* is the structure $(X, \underline{\mathcal{B}}, \sigma, \underline{\vartheta})$ where $\underline{\mathcal{B}}$ is the intersection of all algebras of subsets of X satisfying both (5) and (6) above, and $\underline{\vartheta} : X \to \Pi^I(X, \underline{\mathcal{B}})$ is defined by

$$\underline{\vartheta}^i(x) = \vartheta^i(x) \cap \left[F\big(X, \underline{\mathcal{B}}\big) \times F\big(X, \underline{\mathcal{B}}\big)\right] \qquad \forall i = 1, \ldots, I, \ \forall x \in X.$$

[18] In other words, the structure $(\Omega, \mathcal{A}, \varrho_0, \varpi)$ is *universal* in the sense of Heifetz and Samet [13].

Note that the intersection of all algebras satisfying (5) and (6) is again an algebra with these properties, so the latter definitions are meaningful. Analogously to the probabilistic case, non-redundancy requires distinct elements of X to differ in terms of states of nature, strategies, or preferences.[19] Minimality says that those subsets of X, which measurability of σ and ϑ requires to belong to $\mathcal{B}$, are the only (basic) events the players can reason about. These properties are characterized as follows.

Proposition 5. *A structure $(X, \mathcal{B}, \sigma, \vartheta)$ is minimal if and only if $\mathcal{B}$ is the smallest algebra $\mathcal{B}'$ such that the generator of the structure is measurable $\mathcal{B}'/\mathcal{A}$. Moreover, whether it is minimal or not, the structure is non-redundant if and only if its generator is injective.*

As an obvious consequence of Proposition 5, every substructure of the canonical structure, and indeed every substructure of a minimal and non-redundant structure, is both minimal and non-redundant. Now using Lemma 6 we conclude that if Ω' is the set of hierarchies generated by some structure, then Ω' is a substructure of the canonical structure, hence it satisfies (2), which here is

$$\big\{(f, g) \in F(\Omega) \times F(\Omega) \,:\, f \circ \iota = g \circ \iota\big\} \subseteq \varpi^i(\omega) \quad \forall i = 1, \ldots, I,\ \forall \omega \in \Omega', \tag{7}$$

where ι denotes the inclusion from Ω' to Ω. Based on our observation following (2), we then have the following characterization: a set of hierarchies Ω' is generated by some structure if and only if it satisfies property (7), that is, if and only if it induces a substructure of the canonical structure.

Theorem 2. *A structure $(X, \mathcal{B}, \sigma, \vartheta)$ is minimal and non-redundant if and only if it is isomorphic to the substructure of $(\Omega, \mathcal{A}, \varrho_0, \varpi)$ induced by the set of hierarchies generated by $(X, \mathcal{B}, \sigma, \vartheta)$.*

Proof. Sufficiency follows from the fact that the set of hierarchies generated by a structure induces a substructure of $(\Omega, \mathcal{A}, \varrho_0, \varpi)$. To prove necessity, suppose $(X, \mathcal{B}, \sigma, \vartheta)$ is minimal and non-redundant, and let γ be its generator. The latter is injective by non-redundancy and Proposition 5, thus the mapping $\varphi : X \to \gamma(X)$ such that $x \mapsto \gamma(x)$ is a bijection. By minimality, and again by Proposition 5, the inverse of φ is measurable. Let α be the identity on S. By Lemma 5, it suffices to show (α, φ) is a morphism. This follows at once from Lemma 6. □

In view of Theorems 1 and 2, following the analogous terminology in the literature on coherent hierarchies of beliefs, the structure $(\Omega, \mathcal{A}, \varrho_0, \varpi)$ will be also called the *universal preference structure*. The next two sections are devoted to the study of certain substructures of it, namely,

[19] Since these aspects are supposed to describe all relevant uncertainty of the model, it is not clear how to interpret a redundant structure. Non-redundancy of a structure is in fact equivalent to injectivity of its generator, as Proposition 5 below shows.

those arising as the result of imposing various restrictions on preferences, along with the players' common certainty — to be defined presently — of the restrictions.

4. Common Certainty and Simple Structures

The set of hierarchies generated by a structure that is not complete is, in general,[20] a strict subset of Ω. First, the projection of this subset on the space Ω_n may fail to be onto for some n (hence fail to be onto for all n sufficiently large), so that some states of nature or strategies, or certain preferences on $F(\Omega_{n-1})$, never occur. Second, even if its projection on Ω_n is onto for all n, the generated set of hierarchies may still be a strict subset of Ω, so that certain preferences on $F(\Omega, \mathcal{A})$ are ruled out.[21] In any case, since only hierarchies satisfying restrictions of one form or the other (or both) appear in the generated set of hierarchies, one would like to interpret the structure as a model where not only are the restrictions true, but each player is certain of this, is certain of the fact that all players are certain, and so on.

The latter additional restrictions, however, have not been formally modeled yet. Thus, verifying that such interpretation is formally sound requires establishing a link between the explicit restrictions — formulated within the universal structure via a formal notion of certainty — and the preference structures where those restrictions are only implicitly assumed. This is indeed our task in this section and the next. In this section we define the notion of certainty of events in a structure, prove that for simple structures this notion extends to *closed* subsets, and finally show that the non-redundant simple structures are precisely (isomorphic to) the sets of hierarchies obtained imposing iterated certainty of a closed subset of the universal structure.

Definition 4. Let $(X, \mathcal{B})$ be a space and let π be a preference relation on $F(X, \mathcal{B})$. An event $E \in \mathcal{B}$ is *null according to* π, or just π*-null*, if $(fEg, gEh) \in \pi$ for all $f, g, h \in F(X, \mathcal{B})$. If $(X, \mathcal{B}, \sigma, \vartheta)$ is a structure, then we say player i is *certain of* $E \in \mathcal{B}$ *at* $x \in X$, or that E is $\vartheta^i(x)$*-certain*, provided that $X \setminus E$ is null according to $\vartheta^i(x)$.

Note that an event that is a subset of a null event is necessarily also null, because two acts that coincide on the latter must also coincide on the former. Furthermore, as an obvious consequence of transitivity, the union of a finite family of null events is null. Monotone continuity actually guarantees the following, much stronger result.

[20] We do not know whether the converse of Proposition 4 is true (i.e. we do not know whether every structure whose generator is onto must be complete), though we conjecture it is not.

[21] For example, take the set of all hierarchies $\omega \in \Omega$ such that $(f, g) \notin \varpi^i(\omega)$ for some player i and some $f, g \in F(\Omega, \mathcal{A})$. This is the set of hierarchies where not all players are indifferent between all acts. It is a strict subset of Ω, but its projection on each Ω_n is clearly Ω_n, so it satisfies property (7) and is thus generated by some structure.

Proposition 6. *Let X be a space, let $\pi \in \Pi(X)$, and take a sequence of events A_n such that $A = \cup_n A_n$ is an event. Then A is π-null if and only if each A_n is π-null.*

The latter proposition establishes the analogue of the well known *knowledge continuity* property of probability measures (the intersection of a countable family of events having probability one also has probability one). Whereas the latter stems from countable additivity of probability measures, the claim in the proposition above heavily depends on monotone continuity. As we formally state below, monotone continuity is, in the presence of Savage's axioms, equivalent to countable additivity of the belief in the Savage representation, so the claim in the proposition would be no surprise if π were assumed to satisfy Savage's axioms. However, the proposition does not assume anything beyond reflexivity, transitivity, and monotone continuity. Indeed, reflexivity is not used at all in the proof. Thus, while the proposition will also prove useful in the analysis below, it also establishes a result of some independent interest, namely, that the key properties behind knowledge continuity are just transitivity and monotone continuity.

4.1. Certainty and Common Certainty in Simple Structures

For simple structures, the definition of certainty given above can be extended to intersections of (a fortiori countable) families of events, i.e. to closed subsets.

Proposition 7. *Let $(X, \mathcal{B})$ be a simple space, let π be a preference relation on $F(X, \mathcal{B})$, let $A_1, A_2, \ldots$ be a sequence of π-null events in $\mathcal{B}$, and let $A = \cup_n A_n$. Let $\mathcal{B}^+$ be the algebra generated by $\mathcal{B} \cup \{A\}$. There is a unique preference relation π^+ on $F(X, \mathcal{B}^+)$ such that*

$$\pi^+ \cap \left[F(X, \mathcal{B}) \times F(X, \mathcal{B})\right] = \pi. \tag{8}$$

Moreover, if the union of a sequence of events in $\mathcal{B}$ belongs to $\mathcal{B}^+$, then it is null according to π^+ if and only if each event in the sequence is π-null. In particular, A is null according to π^+.

Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a simple structure, take a sequence of events $E_n \in \mathcal{B}$, and let $E = \cap_n E_n$. Based on the latter result, we say player i is *certain of E at $x \in X$*, or that E is $\vartheta^i(x)$*-certain*, provided that E_n is $\vartheta^i(x)$-certain for all n. By the second part of the proposition, this is indeed well-defined; in other words, certainty of E does not depend on the particular sequence of events E_n chosen; if E_n is $\vartheta^i(x)$-certain for all n, then, for any other sequence of events $E'_n \in \mathcal{B}$ such that $E = \cap_n E'_n$, we have that E'_n is $\vartheta^i(x)$-certain for all n. Thus, for every player i we define

$$C^i(E) = \left\{x \in X \,:\, \text{player } i \text{ is certain of } E \text{ at } x\right\}.$$

The fundamental and obvious property of the latter set is that it is itself closed. Indeed,

$$C^i(E) = \bigcap_{n\geq 1} \bigcap_{f,g,h\in F(X)} \left\{x \in X \,:\, (hE_n f, hE_n g) \in \vartheta^i(x)\right\}.$$

Thus, we can define recursively: E is 1-*mutually certain at* $x \in X$ if E is $\vartheta^i(x)$-certain for every player i; E is $(m+1)$-*mutually certain at* $x \in X$ if E is m-mutually certain at x and, moreover,

$$MC_m(E) = \left\{x' \in X \,:\, E \text{ is } m\text{-mutually certain at } x'\right\}$$

is $\vartheta^i(x)$-certain for every player i; finally, E is *commonly certain at* $x \in X$ if E is m-mutually certain at x for all m. Observe that, since $MC_m(E)$ is closed for every m, so is

$$CC(E) = \left\{x \in X \,:\, E \text{ is commonly certain at } x\right\}.$$

Given a simple structure $(X, \mathcal{B}, \sigma, \vartheta)$, a subspace of $(X, \mathcal{B})$ having the form $E \cap CC(E)$, where $E \subseteq X$ is closed, will be called a *closed common certainty component* (or just *closed component*) of the structure. We stress that m-mutual certainty, common certainty, and closed components are defined only for simple structures and closed subsets of them. The following proposition collects some important facts about mutual and common certainty. These are in fact straightforward generalizations of analogous, well known results concerning probabilistic beliefs.[22]

Proposition 8. *Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a simple structure. For every sequence of closed subsets A_n of X, letting $A = \cap_n A_n$, the following hold:*

(i) $MC_m(A) = \cap_n MC_m(A_n)$ *for every* $m \geq 1$.

(ii) If B is a closed subset of X and $A \subseteq B$, then $MC_1(A) \subseteq MC_1(B)$.

(iii) If $A \subseteq MC_1(A)$, then $A \subseteq CC(A)$.

A subspace of $(X, \mathcal{B})$ is a closed component of $(X, \mathcal{B}, \sigma, \vartheta)$ if and only if it induces a simple substructure of $(X, \mathcal{B}, \sigma, \vartheta)$.

Combining the last claim in the proposition with the results obtained in the previous section, we finally arrive at the main result about common certainty and simple structures.

Theorem 3. *A preference structure is simple and non-redundant if and only if it is isomorphic to the substructure of the canonical structure induced by a closed component of the latter.*

[22] See, for instance, the classic Monderer and Samet [22].

Proof. Sufficiency is an immediate consequence of Proposition 8, whence necessity will also follow, using Theorem 2, once we show that every simple and non-redundant structure is minimal. Thus, suppose $(X, \mathcal{B}, \sigma, \vartheta)$ is simple and non-redundant, let γ be its generator, and let Ω' be the generated set of hierarchies. Since $(X, \mathcal{B})$ and $(\Omega, \mathcal{A})$ are simple spaces and γ is continuous, and since Ω' is compact and $(X, \mathcal{B}, \sigma, \vartheta)$ is non-redundant, the mapping $\varphi : X \to \Omega'$ such that $x \mapsto \gamma(x)$ is injective, onto, and continuous, hence a homeomorphism (as X is compact and Ω' is Hausdorff). Since Ω' is a compact and hence closed subspace of Ω, by Lemma 2 every subset of Ω' that is both closed and open in Ω' has the form $\Omega' \cap A$, where $A \in \mathcal{A}$. Thus, every subset of X that is both closed and open in X (hence every event in X, by Lemma 2) has the form $\varphi^{-1}(\Omega' \cap A) = \gamma^{-1}(A)$, where $A \in \mathcal{A}$. Thus, by Proposition 5, $(X, \mathcal{B}, \sigma, \vartheta)$ is minimal. □

Thus, the simple and non-redundant structures are (up to isomorphism) those sets of hierarchies obtained imposing common certainty of a closed subset of hierarchies. Such structures are the natural choice in a number of important cases. Section 6 below briefly discusses some of them.

5. Common Belief and Standard Savage Structures

For a preference structure $(X, \mathcal{B}, \sigma, \vartheta)$ where the axioms of Savage hold everywhere, the notion of certainty further extends to all sets in the σ-algebra generated by $\mathcal{B}$. This extension result provides the key to all results in this section. We begin recording Savage's theory and some extensions.

5.1. Savage Preferences and Savage Structures

Definition 5. Let $(X, \mathcal{B})$ be a space. A preference relation $\pi \in \Pi(X, \mathcal{B})$ is *Savage* if it satisfies the following (for all $f, g, h, h' \in F(X, \mathcal{B})$, all $z, z', z'', z''' \in Z$, all $A, B \in \mathcal{B}$):

P1. If $(f, g) \notin \pi$, then $(g, f) \in \pi$. If $(f, g) \in \pi$ and $(g, h) \in \pi$, then $(f, h) \in \pi$.

P2. If $(fAh, gAh) \in \pi$, then $(fAh', gAh') \in \pi$.

P3. If A is not π-null, then $(zAf, z'Af) \in \pi$ if and only if $(z, z') \in \pi$.

P4. If $(z, z') \dot{\in} \pi$ and $(z'', z''') \dot{\in} \pi$, then $(zAz', zBz') \in \pi$ if and only if $(z''Az''', z''Bz''') \in \pi$.

P5. There exist $\overline{z}, \underline{z} \in Z$ such that $(\overline{z}, \underline{z}) \dot{\in} \pi$.

P6. If $(f, g) \dot{\in} \pi$, then there is a finite partition $\{A_1, \ldots, A_N\} \subseteq \mathcal{B}$ of X such that $(zA_n f, g) \in \pi$ and $(f, zA_n g) \in \pi$ for all $1 \leq n \leq N$.

The subspace of $\Pi(X, \mathcal{B})$ consisting of all Savage preference relations on $F(X, \mathcal{B})$ is denoted $\Pi_{\mathsf{S}}(X, \mathcal{B})$. The product space $\times_{i=1}^{I} \Pi_{\mathsf{S}}(X, \mathcal{B})$ is denoted $\Pi_{\mathsf{S}}^{I}(X, \mathcal{B})$. A *Savage structure* is a preference structure where P1–P6 hold at every state for every player — in other words, a preference structure $(X, \mathcal{B}, \sigma, \vartheta)$ such that $\vartheta^i(x) \in \Pi_{\mathsf{S}}(X, \mathcal{B})$ for every $x \in X$ and every $i = 1, \ldots, I$.

The next result, henceforth referred to as *Savage's Theorem*, is in fact an extension of Savage's original result. Before stating it, we need the following.

Definition 6. Let $(X, \mathcal{B})$ be a space. A *belief* on $\mathcal{B}$ is a countably additive function $\mu : \mathcal{B} \to [0, 1]$ with $\mu(X) = 1$. A belief μ is *finely ranged* if for all $A \in \mathcal{B}$, all $\epsilon > 0$, and all $0 \leq p \leq \mu(A)$ there exists $A \supseteq B \in \mathcal{B}$ such that $-\epsilon < \mu(B) - p < \epsilon$. A belief μ is *convex ranged* if for all $A \in \mathcal{B}$ and all $0 \leq p \leq \mu(A)$ there exists $A \supseteq B \in \mathcal{B}$ such that $\mu(B) = p$. The set of all finely ranged beliefs on $\mathcal{B}$ is denoted by $\Delta_{\mathsf{FR}}(X, \mathcal{B})$, the set $\times_{i=1}^{I} \Delta_{\mathsf{FR}}(X, \mathcal{B})$ by $\Delta_{\mathsf{FR}}^{I}(X, \mathcal{B})$. The set of all convex ranged beliefs on $\mathcal{B}$ is denoted by $\Delta_{\mathsf{CR}}(X, \mathcal{B})$, the set $\times_{i=1}^{I} \Delta_{\mathsf{CR}}(X, \mathcal{B})$ by $\Delta_{\mathsf{CR}}^{I}(X, \mathcal{B})$. A *utility function* is a mapping $u : Z \to [0, 1]$ such that $\max_{z \in Z} u(z) = 1$ and $\min_{z \in Z} u(z) = 0$. The set of all utility functions is denoted $\mathcal{U}$, and the set $\times_{i=1}^{I} \mathcal{U}$ is denoted $\mathcal{U}^I$. Given another space Y and a function $\phi : Y \to \Delta_{\mathsf{FR}}^{I}(X, \mathcal{B})$, the function mapping $y \in Y$ into the coordinate of $\phi(y)$ corresponding to player i is denoted ϕ^i. The analogous convention is adopted for functions mapping Y into $\Delta_{\mathsf{CR}}^{I}(X, \mathcal{B})$ or into $\mathcal{U}^I$.

Savage's Theorem. *Let $(X, \mathcal{B})$ be a space. A preference relation π on $F(X, \mathcal{B})$ is Savage if and only if there exist $(u, \mu) \in \mathcal{U} \times \Delta_{\mathsf{F}}(X, \mathcal{B})$ such that, for all $f, g \in F(X, \mathcal{B})$,*

$$(f, g) \in \pi \quad \textit{if and only if} \quad \sum_{z \in Z} u(z)\mu\big(f^{-1}(z)\big) \geq \sum_{z \in Z} u(z)\mu\big(g^{-1}(z)\big). \tag{9}$$

In this case, μ and u are unique.

Proof. Kopylov [17] proves that a binary relation π on $F(X, \mathcal{B})$ satisfies P1–P6 if and only if there exist a nonconstant function $u : Z \to \mathbb{R}$ and a finitely additive $\mu : \mathcal{B} \to [0, 1]$ with $\mu(X) = 1$ such that (9) holds, and also proves uniqueness. The equivalence (in the presence of P1–P6) between countable additivity of μ and Monotone Continuity of π has been proved in Villegas [24] and Arrow [1] for the case where $\mathcal{B}$ is a σ-algebra. Their proofs taken verbatim are valid even if $\mathcal{B}$ is assumed to be an algebra. □

The pair (u, μ) satisfying (9) is the *Savage representation* of (or *represents*, or is *induced* by) the preference relation π. Note that an event $E \in \mathcal{B}$ is π-null if and only if $\mu(E) = 0$.

Lemma 7. *Let* $(X, \mathcal{B})$ *be a space, let* $\mathcal{B}^*$ *be the* σ*-algebra generated by* $\mathcal{B}$*, and let* μ *be a belief on* $\mathcal{B}$*. There exists a unique belief* μ^* *on* $\mathcal{B}^*$ *whose restriction to* $\mathcal{B}$ *is* μ*. Moreover,* μ *is finely ranged if and only if* μ^* *is convex ranged (equivalently, if and only if* μ^* *is non-atomic).*

This extension result about *beliefs* carries over to *preferences*, as the following proposition establishes. The latter allows us to extend the notion of certainty, at those points in a structure where the axioms of Savage hold, to the σ-algebra generated by the algebra of events in the structure.

Proposition 9. *Let* $(X, \mathcal{B})$ *be a space and let* $\mathcal{B}^*$ *denote the* σ*-algebra generated by* $\mathcal{B}$*. A preference relation* π *on* $F(X, \mathcal{B})$ *is Savage if and only if there exists a Savage preference relation* π^* *on* $F(X, \mathcal{B}^*)$ *such that*

$$\pi^* \cap \left[F(X, \mathcal{B}) \times F(X, \mathcal{B})\right] = \pi. \tag{10}$$

In this case, π^* *is unique and furthermore, if* μ *denotes the belief on* $\mathcal{B}$ *induced by* π *and* μ^* *denotes the belief on* $\mathcal{B}^*$ *induced by* π^**, then* μ *is the restriction of* μ^* *to* $\mathcal{B}$*. In particular, an event* $E \in \mathcal{B}$ *is* π*-null if and only if it is* π^**-null.*

The preference relation π^* in the latter result will be called the *extension* of π.

Definition 7. Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a structure, let $\mathcal{B}^*$ be the σ-algebra generated by $\mathcal{B}$, and let $E \in \mathcal{B}^*$. Player i *believes* E *at* $x \in X$ if the preference relation $\vartheta^i(x)$ is Savage and, furthermore, $X \setminus E$ is null according to the extension of $\vartheta^i(x)$ to $F(X, \mathcal{B}^*)$.

Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a structure and let $\mathcal{B}^*$ be the σ-algebra generated by $\mathcal{B}$. For each $E \in \mathcal{B}^*$ we define

$$C^i_{\mathrm{S}}(E) = \left\{x \in X \ : \ \text{player } i \text{ is certain of } E \text{ at } x\right\} \quad \text{and} \quad MC_{\mathrm{S},1}(E) = \cap_{i=1}^{I} C^i_{\mathrm{S}}(E).$$

Observe that $C^i_{\mathrm{S}}(X)$ is the set all $x \in X$ such that $\vartheta^i(x)$ is Savage — in particular, the structure is Savage if and only if $X = MC_{\mathrm{S},1}(X)$. Note also that, by Proposition 6 and by the last claim in Proposition 9, the definition above indeed agrees with the definition of certainty given earlier for closed subsets of simple structures — i.e., if $(X, \mathcal{B}, \sigma, \vartheta)$ is simple and $E \subseteq X$ is closed, then

$$C^i_{\mathrm{S}}(E) = C^i_{\mathrm{S}}(X) \cap C^i(E).$$

In order to speak of mutual and common belief we must show that $C^i_{\mathrm{S}}(E) \in \mathcal{B}^*$ for each $E \in \mathcal{B}^*$. For sufficiently well-behaved structures, this is indeed one of the consequences of Proposition 10 below. Meanwhile, we prove the following.

Lemma 8. *Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a structure and let $\mathcal{B}^*$ be the σ-algebra generated by $\mathcal{B}$. If $\mathcal{B}$ is countable, then, for every $i = 1, \dots, I$ and every P = P1,...,P6,*

$$\left\{x \in X : \vartheta^i(x) \textit{ satisfies } P\right\} \in \mathcal{B}^*.$$

In particular, $C^i_{\mathsf{S}}(X) \in \mathcal{B}^*$.

Now, using Proposition 9 and the uniqueness in Savage's Theorem, taking a (not necessarily Savage) structure $(X, \mathcal{B}, \sigma, \vartheta)$ and again letting $\mathcal{B}^*$ be the σ-algebra generated by $\mathcal{B}$, for every player i one can define induced mappings

$$\lambda^i : C_{\mathsf{S}}(X) \to \Pi_{\mathsf{S}}(X, \mathcal{B}^*), \quad \upsilon^i : C_{\mathsf{S}}(X) \to \mathcal{U}, \quad \text{and} \quad \beta^i : C_{\mathsf{S}}(X) \to \Delta_{\mathsf{CR}}(X, \mathcal{B}^*)$$

via the function ϑ^i. Namely, λ^i maps $x \in MC_{\mathsf{S},1}(X)$ into the extension of $\vartheta^i(x)$ to $F(X, \mathcal{B}^*)$, whereas υ^i and β^i map x into the utility function and belief on $\mathcal{B}^*$ induced by $\lambda^i(x)$, respectively. The following important result builds on this observation.

Proposition 10. *Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a structure, let $\mathcal{B}^*$ be the σ-algebra generated by $\mathcal{B}$, assume $\mathcal{B}$ is either a countable algebra or a countably generated σ-algebra,[23] fix a player i, and let*

$$\lambda^i : C_{\mathsf{S}}(X) \to \Pi_{\mathsf{S}}(X, \mathcal{B}^*), \quad \upsilon^i : C_{\mathsf{S}}(X) \to \mathcal{U}, \quad \textit{and} \quad \beta^i : C_{\mathsf{S}}(X) \to \Delta_{\mathsf{CR}}(X, \mathcal{B}^*)$$

be induced by ϑ^i. Then, for all $E \in \mathcal{B}^*$, all $z \in Z$, all $u \in \mathbb{R}$, all $p \in \mathbb{R}$, and all $f, g \in F(X, \mathcal{B}^*)$,

$$\left\{x \in X : x \in C^i_{\mathsf{S}}(X) \textit{ and } \beta^i(x)(E) > p\right\} \in \mathcal{B}^*, \tag{11}$$

$$\left\{x \in X : x \in C^i_{\mathsf{S}}(X) \textit{ and } \upsilon^i(x)(z) > u\right\} \in \mathcal{B}^*, \tag{12}$$

$$\left\{x \in X : x \in C^i_{\mathsf{S}}(X) \textit{ and } (f, g) \in \lambda^i(x)\right\} \in \mathcal{B}^*. \tag{13}$$

Thus, $(X, \mathcal{B}, \sigma, \vartheta)$ is a Savage structure if and only if $\left(X, \mathcal{B}^*, \sigma, (\lambda^i)_{i=1}^I\right)$ is a Savage σ-structure.

An immediate corollary of this proposition is that we can define mutual and common belief. Under the notations and assumptions of the theorem, for every $E \in \mathcal{B}^*$ we have

$$C^i_{\mathsf{S}}(E) = \bigcap_{k \geq 1} \left\{x \in X : x \in C^i_{\mathsf{S}}(X) \text{ and } \beta^i(x)(E) > 1 - 1/k\right\}, \tag{14}$$

[23]The latter means that $\mathcal{B} = \mathcal{B}^*$ and there exists a countable algebra $\mathcal{B}'$ that generates $\mathcal{B}$.

so $C^i_{\mathsf{S}}(E) \in \mathcal{B}^*$ by (11) and hence also $MC_{\mathsf{S},1}(E) \in \mathcal{B}^*$. Thus, we can define

$$MC_{\mathsf{S},m+1}(E) = MC_{\mathsf{S},m}(E) \ \cap \ MC_{\mathsf{S},1}(MC_{\mathsf{S},m}(E))$$

recursively for all $m \geq 1$, and finally

$$CC_{\mathsf{S}}(E) = \cap_{m\geq 1} MC_{\mathsf{S},m}(E).$$

A subspace of $(X, \mathcal{B})$ of the form $E \cap CC_{\mathsf{S}}(E)$, where $E \in \mathcal{B}^*$, will be called a *Savage common belief component* (or just *Savage component*) of the structure $(X, \mathcal{B}, \sigma, \vartheta)$. Note that the latter is not assumed to be simple, nor is it assumed to be Savage — but if it is both simple and Savage, then a closed subspace is a Savage component if and only if it is a closed component. However, we should stress that we do require that $\mathcal{B}$ be either a countable algebra or a countably generated σ-algebra, and that $E \in \mathcal{B}^*$ — otherwise, mutual belief, common belief, and Savage components are not defined (just like mutual certainty, common certainty, and closed components are defined only for simple structures and closed subsets). Similarly to the first part of Proposition 8, by (14) and by countable additivity of the belief $\beta^i(x)$ for all $i = 1, \ldots, I$ and all $x \in C^i_{\mathsf{S}}(X)$, we have

$$MC_{\mathsf{S},m}(\cap_n E_n) = \cap_n MC_{\mathsf{S},m}(E_n) \tag{15}$$

for every $m \geq 1$ and every sequence of events E_n in $\mathcal{B}^*$. Moreover, $MC_{\mathsf{S},m}(D) \subseteq MC_{\mathsf{S},m}(E)$ for all $m \geq 1$ and all $D, E \in \mathcal{B}^*$ such that $D \subseteq E$. Finally, $E \subseteq CC_{\mathsf{S}}(E)$ for all $E \in \mathcal{B}^*$ satisfying $E \subseteq MC_{\mathsf{S},1}(E)$. Similarly to the last claim in Proposition 8, here we have the following.[24]

Proposition 11. *Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a preference structure, let $\mathcal{B}^*$ be the σ-algebra generated by $\mathcal{B}$, and assume $\mathcal{B}$ is either a countable algebra or a countably generated σ-algebra. Let $E \in \mathcal{B}^*$ and let $\mathcal{B}_E$ be the relative algebra, i.e. the algebra of all subsets of E of the form $E \cap A$ where $A \in \mathcal{B}$. Then $(E, \mathcal{B}_E)$ is a Savage component of $(X, \mathcal{B}, \sigma, \vartheta)$ if and only if $(E, \mathcal{B}_E)$ induces a Savage substructure of $(X, \mathcal{B}, \sigma, \vartheta)$.*

Note that, under the notations and assumptions of the proposition, sufficiency is lost if one does not require $E \in \mathcal{B}^*$. In other words, while it is true that a Savage component must induces

[24]The terminology in Proposition 11 is potentially misleading. When we say a structure $(X, \mathcal{B}, \sigma, \vartheta)$ is a *Savage substructure* of a structure $(Y, \mathcal{C}, \varsigma, \theta)$, do we mean (i) that $(X, \mathcal{B}, \sigma, \vartheta)$ is a Savage structure, i.e. that $\vartheta^i(x) \in \Pi_{\mathsf{S}}(X, \mathcal{B})$ for every player i and every $x \in X$, or (ii) that $\theta^i(y) \in \Pi_{\mathsf{S}}(Y, \mathcal{C})$ for every player i and every $y \in X$? The issue is, however, immaterial. It can be easily shown that (i) and (ii) are equivalent. See also the discussion in Subsection 6.1.

a Savage substructure,[25] it is *not* true that a Savage substructure whose underlying set is not in $\mathcal{B}^*$ is induced by a Savage component. This is analogous to the fact that only the simple substructures of a simple structure are induced by its closed components.

5.2. Common Belief and the Canonical Savage Structure

The σ-algebra on Ω generated by $\mathcal{A}$ will be henceforth denoted by $\mathcal{A}^*$. In this subsection we will prove a theorem analogous to Theorem 3, establishing equivalence between certain well-behaved Savage structures — which we will call *prestandard* — and the Savage components of the universal structure. We will also construct a Savage substructure of the universal structure with the property that the Savage components of its associated (by Proposition 10) Savage σ-structure are isomorphic to certain well-behaved Savage σ-structures — which we will call *standard Borel*. In order to better put the results in context, we would like to spend a few more comments on the implications of Proposition 10 above.

In the traditional game-theoretic framework, one often assumes objects of the form

$$\big(X, \mathcal{B}^*, \sigma, \upsilon, \beta\big) \tag{16}$$

where $\mathcal{B}^*$ is a σ-algebra on X, the mapping $\sigma : X \to S$ is assumed to be measurable, and the mappings $\upsilon^i : X \to \mathcal{U}$ and $\beta^i : X \to \Delta_{\mathsf{F}}(X, \mathcal{B}^*)$ are assumed to satisfy

$$\big\{x \in X : \upsilon^i(x) > u\big\} \in \mathcal{B}^* \quad \text{and} \quad \big\{x \in X : \beta^i(x) > p\big\} \in \mathcal{B}^* \tag{17}$$

for all $u, p \in \mathbb{R}$. The latter requirements, equivalent to measurability of υ^i and β^i in the usual sense,[26] coincide with (11) and (12) if the structure assumed in the proposition is in fact Savage (so that $C^i_{\mathsf{S}}(X)$ coincides with X for each player i). Let us call an object of the form (16) with these properties a *subjective expected utility interactive system*, or *SEU system* for short. If one assumes such an object, then one can define $\theta : X \to \Pi^I_{\mathsf{S}}(X, \mathcal{B}^*)$ via υ and β and conclude that the tuple $\big(X, \mathcal{B}^*, \sigma, \theta\big)$ is a Savage σ-structure, because (13) is indeed an obvious consequence of

[25] If $(E, \mathcal{B}_E)$ is a Savage component, then Proposition 10 implies that $E \in \mathcal{B}^*$, thus Proposition 11 implies that $(E, \mathcal{B}_E)$ induces a Savage substructure of $(X, \mathcal{B}, \sigma, \vartheta)$.

[26] The two conditions in (17) are equivalent to measurability of υ^i and β^i with respect to $\mathcal{B}^*$ when $\mathcal{U}$ is seen as a subset of $\mathbb{R}^Z$ and endowed with the usual (relative) product Borel σ-algebra, and $\Delta_{\mathsf{F}}(X, \mathcal{B}^*)$ is endowed with the σ-algebra generated by the sets of the form $\{\mu \in \Delta_{\mathsf{F}}(X, \mathcal{B}^*) : \mu(E) > p\}$ where $E \in \mathcal{B}^*$ and $p \in \mathbb{R}$. The latter σ-algebra has been widely used in the literature on type spaces. As is well known, if $(X, \mathcal{B}^*)$ is a standard Borel space (see Definition 8 below), then this σ-algebra coincides with the (relative) Borel σ-algebra generated by the weak* topology on the set of all beliefs on $\mathcal{B}^*$.

(11) and (12). But Proposition 10 says much more. First, it says that we can go the other way around, that is, assume a Savage σ-structure and *obtain* an associated SEU system, proving (17) from (13). Second, and much more importantly, it says that we can do so *even if all we assume at the beginning is a Savage structure with a countable algebra of events*. This much more elementary structure will be enough to generate both the Savage σ-structure and its associated SEU system. The results below will push these conclusions even further. In the *standard* case it will be enough to start with a *minimal* structure.

Definition 8. An uncertainty space $(X, \mathcal{B})$ is *standard Borel* (resp. *prestandard*) if the algebra $\mathcal{B}$ is (resp. is countable and generates) the Borel σ-algebra generated by a Polish topology on X. A preference structure is *standard Borel* (resp. *prestandard*) if the underlying uncertainty space is standard Borel (resp. prestandard). A structure that is either prestandard or standard Borel will be called *standard*.

Every simple space is prestandard, because every second countable compact Hausdorff topological space is Polish. In particular, $(\Omega, \mathcal{A})$ is prestandard and $(\Omega, \mathcal{A}^*)$ is standard Borel. Note also that if $(X, \mathcal{B})$ is a standard Borel space, then there exists a countable algebra $\mathcal{B}'$ on X such that $\mathcal{B}'$ generates $\mathcal{B}$ and therefore $(X, \mathcal{B}')$ is prestandard; for example, one can take $\mathcal{B}'$ to be the algebra generated by some countable base for the Polish topology generating $\mathcal{B}$. Proposition 12 below shows that the latter remark about standard Borel *spaces* carries over to standard Borel *structures* satisfying non-redundancy. Before stating the result, a few comments are in order.

In general, the minimal structure associated to a structure $(X, \mathcal{B}, \sigma, \vartheta)$ may involve loss of information; its algebra of events need not generate $\mathcal{B}$. In such a case not only do we lose the convenience of working with a much more manageable object (the associated minimal structure), but the generator of $(X, \mathcal{B}, \sigma, \vartheta)$ is not guaranteed to map events in $\mathcal{B}$ into events in $\mathcal{A}$ or even $\mathcal{A}^*$, making the interpretation of an event in $\mathcal{B}$ problematic. While Proposition 10 establishes the important conclusion that a Savage structure has a unique associated Savage σ-structure, it is equally important to establish the converse of this conclusion. Namely, we would like to be assured that if a Savage σ-structure is reduced to a simpler object — a structure with a countable algebra of events — then no information is lost, in the sense that applying Proposition 10 we recover the *same* σ-structure. In particular, we would like this to be true if the σ-structure is reduced to its bare minimum, i.e. to the associated minimal structure, since only with reference to this minimal structure does Theorem 2 apply. Indeed, as the latter theorem makes clear, an event that cannot be constructed from the events in the associated minimal structure is problematic, in the sense that it cannot have an unambiguous interpretation in terms of preference hierarchies, i.e. in terms of events in $\mathcal{A}$. Such a problematic event, however, cannot exist in the non-redundant, standard case;

this is precisely what the following result establishes.

Proposition 12. *Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a non-redundant standard structure and let $\gamma : X \to \Omega$ be its generator. Then the σ-algebra generated by $\mathcal{B}$ is also generated by the algebra of all sets of the form $\gamma^{-1}(A)$ where $A \in \mathcal{A}$. Thus, in particular, $\gamma(E) \in \mathcal{A}^*$ for all $E \in \mathcal{B}$.*

Note that in the latter proposition we do not require $(X, \mathcal{B}, \sigma, \vartheta)$ to be Savage; nevertheless we choose to state the result here, since it has to do with σ-algebras, which we explicitly use only for Savage preferences. The latter result, together with Proposition 10, guarantees that working with a non-redundant standard Borel Savage structure is equivalent to working with the associated minimal (and necessarily also Savage, by Proposition 9) structure. In other words, a preference σ-structure is non-redundant, standard Borel, and Savage if and only if its associated minimal structure is minimal, non-redundant, prestandard, and Savage. We are now ready to prove the main results in this section.

Theorem 4. *A preference structure is minimal, non-redundant, prestandard, and Savage if and only if it is isomorphic to the substructure of the canonical structure induced by a Savage component of the latter.*

Proof. By Theorem 2, a structure $(X, \mathcal{B}, \sigma, \vartheta)$ is minimal, non-redundant, and Savage if and only if it is isomorphic to the Savage substructure of $(\Omega, \mathcal{A}, \varrho_0, \varpi)$ induced by the set of hierarchies generated by $(X, \mathcal{B}, \sigma, \vartheta)$. By Proposition 11, this set is a Savage component if and only if it belongs to $\mathcal{A}^*$. By Proposition 12, this is equivalent to $(X, \mathcal{B}, \sigma, \vartheta)$ being prestandard. □

Now let $\Omega_S = CC_S(\Omega)$. This is the subset of Ω where all players' preferences are Savage and this is common belief. Let $\mathcal{A}_S$ and $\mathcal{A}_S^*$ denote the relative algebra and σ-algebra on Ω_S, respectively. Namely, $\mathcal{A}_S$ is the algebra of sets of the form $A \cap \Omega_S$, where $A \in \mathcal{A}$, and $\mathcal{A}_S^*$ is the σ-algebra generated by $\mathcal{A}_S$, which is the same as the σ-algebra of sets of the form $E \cap \Omega_S$, where $E \in \mathcal{A}^*$. Since

$$\Omega_S = MC_{S,1}(\Omega) \cap CC_S(MC_{S,1}(\Omega)),$$

the space $(\Omega_S, \mathcal{A}_S)$ is a Savage component of the universal structure and thus induces, by Proposition 11, a Savage substructure of the universal structure. This Savage substructure is

$$\left(\Omega_S, \mathcal{A}_S, \varrho_{0S}, \varpi_S\right) \tag{18}$$

where $\varrho_{0S} : \Omega_S \to S$ is the restriction of ϱ_0 to Ω_S and $\varpi_S^i : \Omega_S \to \Pi_S(\Omega_S, \mathcal{A}_S)$ maps

$$\omega \mapsto \left\{(f \circ \iota, g \circ \iota) : (f, g) \in \varpi^i(\omega)\right\}$$

(where $\iota : \Omega_{\mathsf{S}} \to \Omega$ is the inclusion mapping). By Proposition 10, the structure (18) in turn induces a Savage σ-structure, namely

$$\left(\Omega_{\mathsf{S}}, \mathcal{A}_{\mathsf{S}}^*, \varrho_{0\mathsf{S}}, \varpi_{\mathsf{S}}^*\right) \tag{19}$$

where $\varpi_{\mathsf{S}}^{*i} : \Omega_{\mathsf{S}} \to \Pi_{\mathsf{S}}(\Omega_{\mathsf{S}}, \mathcal{A}_{\mathsf{S}}^*)$ is the mapping induced by ϖ_{S}^i — see the observation immediately preceding Proposition 10. We call the preference structures (18) and (19) the *canonical Savage structure* and the *canonical Savage σ-structure*, respectively. Our last theorems show that these structures also feature universality properties, relative to Savage's axioms.

Before stating the results, one more definition is needed. We say a Savage structure $(X, \mathcal{B}, \sigma, \vartheta)$ is *belief complete* if the mapping from X into $S \times \Pi_{\mathsf{S}}^I(X, \mathcal{B})$ such that $x \mapsto (\sigma(x), \vartheta(x))$ is onto. If, in addition, the inverse of this mapping exists and is measurable, we write

$$(X, \mathcal{B}, \sigma, \vartheta) \;\cong_{\mathsf{S}}\; S \times \Pi_{\mathsf{S}}^I(X, \mathcal{B}).$$

Theorem 5. $(\Omega_{\mathsf{S}}, \mathcal{A}_{\mathsf{S}}, \varrho_{0S}, \varpi_{\mathsf{S}}) \;\cong_{\mathsf{S}}\; S \times \Pi_{\mathsf{S}}^I(\Omega_{\mathsf{S}}, \mathcal{A}_{\mathsf{S}})$.

Proof. We only need to prove that the mapping from X into $S \times \Pi_{\mathsf{S}}^I(X, \mathcal{B})$ such that $x \mapsto (\sigma(x), \vartheta(x))$ is injective and onto, as measurability and measurability of its inverse are immediate consequences of Theorem 1. Let $\iota : \Omega_{\mathsf{S}} \to \Omega$ denote inclusion. Being a substructure of the universal structure, the canonical Savage structure satisfies (7), i.e.

$$\left\{(f, g) \in F(\Omega, \mathcal{A}) \times F(\Omega, \mathcal{A}) \;:\; f \circ \iota = g \circ \iota\right\} \subseteq \varpi^i(\omega) \quad \forall i = 1, \ldots, I,\ \forall \omega \in \Omega_{\mathsf{S}}. \tag{20}$$

If ω and ω' are distinct elements of Ω_{S}, then, by Theorem 1, either $\varrho_{0\mathsf{S}}(\omega) \neq \varrho_{0\mathsf{S}}(\omega')$, or $\varpi^i(\omega) \neq \varpi^i(\omega')$ for some player i, or both. In the latter case, by (20), we have $(f, g) \in \varpi^i(\omega) \cap \varpi^i(\omega')$ for all $f, g \in F(\Omega, \mathcal{A})$ that coincide on Ω_{S}, hence $\varpi_{\mathsf{S}}^i(\omega) \neq \varpi_{\mathsf{S}}^i(\omega')$. This establishes injectivity. To prove ontoness, pick any $s \in S$ and any $(\pi_{\mathsf{S}}^1, \ldots, \pi_{\mathsf{S}}^I) \in \Pi_{\mathsf{S}}^I$, and for each player i define

$$\pi^i = \left\{(f, g) \in F(\Omega, \mathcal{A}) \times F(\Omega, \mathcal{A}) \;:\; (f \circ \iota, g \circ \iota) \in \pi_{\mathsf{S}}^i\right\}.$$

This relation is Savage, and by reflexivity of π_{S}^i it satisfies $(f, g) \in \pi^i$ for all $f, g \in F(\Omega, \mathcal{A})$ that coincide on Ω_{S}. Since the universal structure is complete, there exists $\omega \in \Omega$ such that $\varrho_0(\omega) = s$ and $\varpi(\omega) = (\pi^1, \ldots, \pi^I)$. But, for all $m \geq 1$, the set $\Omega \setminus MC_{\mathsf{S},m}(\Omega)$ is null according to the extension of π^i to $F(\Omega, \mathcal{A}^*)$, therefore $\omega \in \Omega_{\mathsf{S}}$, thus ontoness is established. $\square$

The main results obtained so far in this section can be then summarized as follows:

$$\left(\Omega_{\mathsf{S}}, \mathcal{A}_{\mathsf{S}}, \varrho_{0\mathsf{S}}, \varpi_{\mathsf{S}}\right) \quad \longrightarrow \quad S \times \Pi_{\mathsf{S}}^I\left(\Omega_{\mathsf{S}}, \mathcal{A}_{\mathsf{S}}\right) \quad \longrightarrow \quad S \times \mathcal{U}^I \times \Delta_{\mathsf{FR}}^I\left(\Omega_{\mathsf{S}}, \mathcal{A}_{\mathsf{S}}\right)$$

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ (\Omega_{\mathsf{S}}, \mathcal{A}^*_{\mathsf{S}}, \varrho_{0\mathsf{S}}, \varpi^*_{\mathsf{S}}) & \longrightarrow & S \times \Pi^I_{\mathsf{S}}(\Omega_{\mathsf{S}}, \mathcal{A}^*_{\mathsf{S}}) & \longrightarrow & S \times \mathcal{U}^I \times \Delta^I_{\mathsf{CR}}(\Omega_{\mathsf{S}}, \mathcal{A}^*_{\mathsf{S}}) \end{array}$$

This diagram commutes, and indeed every arrow denotes a bijection. The upper-left and bottom-left horizontal arrows denote, in fact, measurable bijections with measurable inverses; the upper-left one is the isomorphism established Theorem 5, the bottom-left one, by Proposition 10, becomes measurable with a measurable inverse, provided that each $\Pi_{\mathsf{S}}(\Omega_{\mathsf{S}}, \mathcal{A}^*_{\mathsf{S}})$ is endowed with the σ-algebra generated by the sets of the form

$$\big\{ \pi^* \in \Pi_{\mathsf{S}}(\Omega_{\mathsf{S}}, \mathcal{A}^*_{\mathsf{S}}) \ : \ (f, g) \in \pi^* \big\}, \tag{21}$$

where $f, g \in F(\Omega_{\mathsf{S}}, \mathcal{A}^*)$. This means that the canonical Savage σ-structure is also belief complete, and indeed satisfies the analogous of Theorem 5. The upper-right and bottom-right horizontal arrows in the diagram refer to the equivalence established by Savage's Theorem; by Proposition 10, the bottom-right one also denotes a measurable bijection with measurable inverse, provided that (i) the set $\Pi_{\mathsf{S}}(\Omega_{\mathsf{S}}, \mathcal{A}^*_{\mathsf{S}})$ has the σ-algebra generated by the sets (21), and (ii) the sets $\mathcal{U}$ and $\Delta^I_{\mathsf{CR}}(\Omega_{\mathsf{S}}, \mathcal{A}^*_{\mathsf{S}})$ are endowed, respectively, with the σ-algebras generated by the sets of the form

$$\big\{ u \in \mathcal{U} \ : \ u(z) > a \big\} \quad \text{and} \quad \big\{ \mu^* \in \Delta_{\mathsf{CR}}(\Omega_{\mathsf{S}}, \mathcal{A}^*_{\mathsf{S}}) \ : \ \mu^*(E) > p \big\},$$

where $a, p \in [0, 1]$ and $E \in \mathcal{A}^*_{\mathsf{S}}$. Finally, the bijections denoted by the left, middle, and right vertical arrows follow from Proposition 10, Proposition 9, and Lemma 7, respectively.

Theorem 6. *A preference structure is minimal, non-redundant, prestandard, and Savage if and only if it is isomorphic to the substructure of the canonical Savage structure induced by a Savage component of the latter. A preference σ-structure is non-redundant, standard Borel, and Savage if and only if it is isomorphic to the σ-substructure of the canonical Savage σ-structure induced by a Savage component of the latter.*

Proof. The second claim is an obvious consequence of the first — see the observation immediately preceding Theorem 4. By the latter theorem and by Proposition 11, a preference structure is minimal, non-redundant, prestandard, and Savage if and only if its generated set of hierarchies Ω' belongs to $\mathcal{A}^*$ and induces a Savage substructure of the universal structure. Again by Proposition 11, a subspace Ω'_{S} of the universal Savage structure is a Savage component of the latter if and only if it satisfies $\Omega'_{\mathsf{S}} \in \mathcal{A}^*_{\mathsf{S}}$ and induces a Savage substructure of the universal Savage structure. Thus, in order to prove the first claim, it suffices to show that every $\Omega' \in \mathcal{A}^*$ inducing a Savage substructure of the universal structure must also induce a substructure of the Savage universal structure, and for this it suffices to prove $\Omega' \subseteq \Omega_{\mathsf{S}}$. Now by (7) we have $\Omega' \subseteq MC_{\mathsf{S},1}(\Omega')$ and, therefore, $\Omega' \subseteq CC_{\mathsf{S}}(\Omega')$. Since $\Omega' \subseteq \Omega$, we also have $CC_{\mathsf{S}}(\Omega') \subseteq CC_{\mathsf{S}}(\Omega)$. Thus, $\Omega' \subseteq \Omega_{\mathsf{S}}$. ☐

Together with Proposition 10 and Theorem 4, the result above provides Savage-like foundations for *non-redundant, standard Borel SEU systems*. These are tuples of the form (16), where the underlying space is a standard Borel space such that distinct points are separated by the smallest σ-algebra satisfying (17). Such objects can be interpreted in several ways. One can think of them (Theorem 4) as the Savage components of the universal structure, obtained imposing (Savage's axioms, common belief of Savage's axioms, and) common belief of some event $E \in \mathcal{A}^*$. This is certainly the more basic interpretation, as it takes the space of coherent hierarchies as primitive. But one can also think of them (Theorem 6) as common belief components of the canonical Savage structure, or of the canonical Savage σ-structure, obtained imposing common belief of some event $E \in \mathcal{A}^*_{\mathsf{S}}$. Indeed, the canonical Savage structure is itself induced by a common belief component of the universal structure, hence can be regarded as an uncertainty space in its own right — the space that is in fact more convenient to use, as is done in traditional game-theory,[27] in all cases where the axioms of Savage are assumed to hold.

6. Discussion and Extensions

6.1. Closed Sets of Axioms and Relative Universality

Consider the universal structure $(\Omega, \mathcal{A}, \varrho_0, \varpi)$ and pick an arbitrary set of axioms for a preference relation on $F(\Omega, \mathcal{A})$. Say that this set of axioms is *closed* if the subset of $\Pi(\Omega, \mathcal{A})$ where the axioms are satisfied is closed. If we choose a closed set of axioms for each player (possibly different sets of axioms for different players), then the results in Subsection 4.1 guarantee that a suitably constructed closed component of the universal structure will feature both the axioms and common certainty of them.[28] Let $(\Omega_{\mathsf{A}}, \mathcal{A}_{\mathsf{A}}, \varrho_{0\mathsf{A}}, \varpi_{\mathsf{A}})$ denote the substructure of the universal structure induced by this closed component, and say a preference structure $(X, \mathcal{B}, \sigma, \vartheta)$ is *consistent* with the chosen axioms if, for every player i and for all $x \in X$, the preference relation $\varpi^i(\gamma(x))$ induced on $F(\Omega, \mathcal{A})$ satisfies the set of axioms chosen for player i. (As usual, γ denotes the generator

[27] The universal beliefs space in Mertens and Zamir [21] includes, of course, atomic beliefs, whereas our universal Savage σ-structure does not admit them. Except for this, and except for the fact that our basic uncertainty space S is finite (theirs is only assumed compact Hausdorff), the universality properties are exactly the same.

[28] By definition of closed set of axioms, using Theorem 1, the set of all $\omega \in \Omega$ such that, for every player i, the preference relation $\varpi^i(\omega)$ satisfies the set of axioms chosen for player i, is a closed subset of Ω, i.e. an intersection of events in $\mathcal{A}$.

of the structure.) One claim that is certainly true is then the following: every simple and non-redundant structure consistent with the chosen sets of axioms must be isomorphic to a substructure of $(\Omega_{\mathsf{A}}, \mathcal{A}_{\mathsf{A}}, \varrho_{0\mathsf{A}}, \varpi_{\mathsf{A}})$. Thus, the latter is universal *relative* to the chosen sets of axioms.

It is important to observe that, while this universality property resembles the one established in the first part of Theorem 6 for the universal Savage structure $(\Omega_{\mathsf{S}}, \mathcal{A}_{\mathsf{S}}, \varrho_{0\mathsf{S}}, \varpi_{\mathsf{S}})$ relative to Savage's axioms, the two properties are conceptually and technically different. The hypotheses (resp. conclusions) of the necessity (resp. sufficiency) statement in the first part of Theorem 6 say that $(X, \mathcal{B}, \sigma, \vartheta)$ is Savage, *not* that it is consistent with Savage's axioms, although the latter is indeed one of the conclusions (resp. hypotheses) in that statement.[29] Saying that for some player i and some $x \in X$ the relation $\vartheta^i(x)$ on $F(X, \mathcal{B})$ satisfies a certain set of axioms is, in general, *not* the same as saying that the preference relation $\varpi^i(\gamma(x))$ induced on $F(\Omega, \mathcal{A})$ also satisfies those axioms,[30] although for Savage's P1–P6 the two claims are indeed equivalent. This implies that, depending on the specific closed sets of axioms used to construct the structure $(\Omega_{\mathsf{A}}, \mathcal{A}_{\mathsf{A}}, \varrho_{0\mathsf{A}}, \varpi_{\mathsf{A}})$, the latter's relative universality property stated in the previous paragraph may fail to hold if "consistent with the chosen sets of axioms" is replaced by "such that the players' preferences satisfy the chosen sets of axioms". The upshot of this whole discussion is that, in applications of our model, if one assumes a structure $(X, \mathcal{B}, \sigma, \vartheta)$ such that $\vartheta^i(x)$ satisfies a certain axiom for every $x \in X$, then one cannot always (i.e. regardless of the chosen axiom) interpret this structure as a situation where player i's preferences (on the space $F(\Omega, \mathcal{A})$) satisfy that axiom and this is common certainty among all players.[31]

Finally, it would be desirable to know just what sets of axioms are closed. While giving even a partial list goes beyond the scope of this paper, a perfunctory investigation already reveals a few facts. Roughly, all finite or countable sets of axioms — where each axiom involves only the quantifier "for all" applied to events, outcomes, acts, finite partitions of the space into events,

[29]Requiring that $(X, \mathcal{B}, \sigma, \vartheta)$ be Savage means imposing that $\vartheta^i(x)$ satisfies P1–P6 for every player i and all $x \in X$. Requiring that $(X, \mathcal{B}, \sigma, \vartheta)$ be consistent with Savage's axioms means imposing that $\varpi^i(\gamma(x))$ satisfies P1–P6 for every player i and all $x \in X$. These are logically distinct requirements and may fail to be equivalent if P1–P6 are replaced by another set of axioms.

[30] Note that, by (4), the relation $\varpi^i(\gamma(x))$ is the same as $\widehat{\gamma}(\vartheta^i(x))$, where $\widehat{\gamma} : \Pi(X, \mathcal{B}) \to \Pi(\Omega, \mathcal{A})$ is the mapping induced by γ. This makes it easy to show that, for example, *completeness* is preserved, i.e. that $\varpi^i(\gamma(x))$ is complete if $\vartheta^i(x)$ is complete. Indeed, let $f, g \in F(\Omega, \mathcal{A})$. Since γ is measurable $\mathcal{B}/\mathcal{A}$, both $f \circ \gamma$ and $g \circ \gamma$ belong to $F(X, \mathcal{B})$. Thus, if $\vartheta^i(x)$ is complete, then we have either $(f \circ \gamma, g \circ \gamma) \in \vartheta^i(x)$, or $(g \circ \gamma, f \circ \gamma) \in \vartheta^i(x)$, or both, hence (by definition of the induced mapping $\widehat{\gamma}$) we have either $(f, g) \in \widehat{\gamma}(\vartheta^i(x))$, or $(g, f) \in \widehat{\gamma}(\vartheta^i(x))$, or both. For an example showing an axiom that is not preserved, assume $I = 2$ and $Z = \{z, z'\}$ and fix $s \in S$. Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a preference structure where $X = \{x\}$ is a singleton — such a structure is automatically simple and non-redundant, hence also minimal — with $\sigma(x) = s$ and $\vartheta^1(x) = \vartheta^2(x) = \{(z, z'), (z, z), (z', z')\}$. The latter preference relations clearly satisfy *antisymmetry*, i.e. for each player i and for all $f, g \in F(X)$ we have $f = g$ whenever $(f, g) \in \vartheta^i(x)$ and $(g, f) \in \vartheta^i(x)$. However, the preference relations $\widehat{\gamma}(\vartheta^1(x))$ and $\widehat{\gamma}(\vartheta^2(x))$ violate antisymmetry, because, by (7), they both contain all pairs $(f, g) \in F(\Omega, \mathcal{A}) \times F(\Omega, \mathcal{A})$ such that $f(\gamma(x)) = g(\gamma(x))$.

[31]By contrast, for Savage's axioms this interpretation is perfectly legitimate (first part of Theorem 6).

etc. — are closed. For example, all of Savage's axioms except P6 are closed (when each is seen as a set of axioms) and therefore any set of axioms comprising one or more of P1,...,P5 is closed. Indeed, when dealing with structures where the family of events is countable, every axiom involving conditions of the form "for every event A", "for all acts f and g", etc. can be written as a countable intersection of events. On the other hand, P6 does not have this form. Indeed, P6 corresponds to a countable intersection of countable unions ("for every outcome and every pair of acts, there exists a finite partition such that ..."). As a result, there is no substructure of the universal structure that is universal relative to P6 only.

6.2. Own Strategies and Preferences

It is easy to construct a closed component of the universal structure where (i) every player is certain of his own strategy and preferences, and (ii) there is common certainty of (i). It suffices to note that, for all $i = 1, \ldots, I$, the singletons in $S^i \times \Pi(\Omega, \mathcal{A})$ are closed, so the set

$$D^i = \left\{\omega \in \Omega \,:\, \text{player } i \text{ is certain at } \omega \text{ of } (\varrho_0^i)^{-1}(\varrho_0^i(\omega)) \cap (\varpi^i)^{-1}(\varpi^i(\omega))\right\}$$

(where $\varrho_0^i : \Omega \to S^i$ is the mapping induced by ϱ_0 in the obvious way) is well defined and closed. In other words, player i's certainty of player i's own strategy and preferences is a closed set of "axioms" for player i. Thus, the set $D = \cap_{i=1}^{I} D^i$ is also closed, hence $\Omega_{\mathsf{O}} = D \cap CC(D)$ is a closed component of the universal structure. The associated substructure is in fact itself universal, relative to the property that all players are certain of their own strategies and preferences and this is common certainty.[32] Observe also that the set $\Omega_{\mathsf{S}} \cap \Omega_{\mathsf{O}}$ belongs to $\mathcal{A}^*$, therefore

$$\Omega_{\mathsf{SO}} = \left(\Omega_{\mathsf{S}} \cap \Omega_{\mathsf{O}}\right) \cap CC_{\mathsf{S}}\left(\Omega_{\mathsf{S}} \cap \Omega_{\mathsf{O}}\right)$$

induces a Savage substructure of the universal structure. This structure will have (and will be in fact universal relative to) the property that all players' preferences satisfy P1–P6, all players are certain of their own strategies and preferences, and this is common belief.

[32]Contrary to certain other sets of axioms (like, for instance, antisymmetry — see Footnote 30), certainty of one's own strategy and preferences is a property that *is* preserved under the generator of a structure. In other words, take a structure $(X, \mathcal{B}, \sigma, \vartheta)$ and let γ be its generator. Suppose that, at every $x \in X$, every player i is $\vartheta^i(x)$-certain of $(\sigma^i)^{-1}(\sigma^i(x))$ and $(\vartheta^i)^{-1}(\vartheta^i(x))$. (Here $\sigma^i : X \to S^i$ is the mapping induced by σ in the obvious way.) Then, for every $x \in X$, every player i is $\varpi^i(\gamma(x))$-certain at $\gamma(x)$ of both $(\varrho^i)^{-1}(\varrho^i(\gamma(x)))$ and $(\varpi^i)^{-1}(\varpi^i(\gamma(x)))$. Note that, more generally, one can consider a subset of players $J \subseteq I$, define D as $\cap_{i \in J} D^i$ instead, and obtain a substructure where all players in J are certain of their own strategies and preferences, and this is common certainty among all players; this substructure will be again universal, relative to the latter property.

6.3. Rationality

Recall that in Section 2, among the other basic ingredients for our analysis, we have assumed an outcome function $\zeta : S \to Z$. This was necessary in order to view a player's strategy as an act, although it played no role in our analysis.[33] It is also necessary in order to discuss *rationality*. Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a preference structure and let $\zeta : S \to Z$ be an outcome function. For every player i, identify each $s^i \in S^i$ with the act mapping S into Z such that $(\tilde{s}^i, s^{-i}) \mapsto \zeta(s^i, s^{-i})$ for every $\tilde{s}^i \in S^i$ and every $s^{-i} \in S^{-i}$. Player i is *rational at* $x \in X$ if there does not exist $s^i \in S^i$ such that $(s^i, \sigma^i(x)) \dot{\in} \vartheta^i(x)$. It is clear that i's rationality, seen as a set of "axioms", is closed, i.e. the set

$$R^i = \left\{\omega \in \Omega \,:\, \text{player } i \text{ is rational at } \omega\right\}$$

is closed. Thus, $R = \cap_{i=1}^{I} R^i$ is also closed, and $\Omega_{\mathsf{R}} = R \cap CC(R)$ is a closed component of the universal structure. This is the set of hierarchies featuring all players' rationality and common certainty of rationality.

6.4. On the Role of Reflexivity, Transitivity, and Monotone Continuity

While the main results in the paper depend heavily on reflexivity, transitivity, and monotone continuity, the first use of these properties (reflexivity) appears in the proof of Lemma 6. Indeed, transitivity and monotone continuity appear for the first time even later, in the proof of Proposition 6. In particular, the analogue of Proposition 3 — and indeed the analogue of every result appearing before Lemma 6 — could be proved even if we were to define preference relations as arbitrary binary relations. One needs reflexivity to establish the crucial fact that a subspace of a structure induces a substructure if and only if it satisfies (2), whereas transitivity and monotone continuity are (mainly) needed to talk about certainty of closed subsets, i.e. to prove Proposition 7. But we assumed monotone continuity at the outset for another important reason as well, namely, because monotone continuity (seen as a set of axioms) is *not* closed in the sense of Subsection 6.1 above.[34]

[33] Specifying the outcome function since the beginning is also important from a conceptual and methodological point of view; presumably, a player's beliefs about the other players' strategies and beliefs (more generally, the player's preferences and higher-order preferences) depend not just on the other players' strategy sets, but also, and more importantly, on how strategy profiles translate into outcomes.

[34] Even though it has the "for all" form, monotone continuity (as a set of axioms) is not closed if the family of events is not finite. This is because requiring that some property holds "for all n large", as monotone continuity does, means considering a countable union of countable intersections. In fact, we suspect that in the Savage case there is an even more serious problem when not assuming monotone continuity. Namely, we conjecture that given a measurable (say, standard Borel) space X, the set of all countably additive probability measures on X is not a measurable subset of

In other words, while monotone continuity is crucial to guarantee uniqueness of the extension of a preference relation (both in the simple case, as in Proposition 7, and in the Savage case, as in Proposition 9), it is a property that cannot be imposed *a posteriori*, even if we were willing to define certainty of closed subsets anyway (just saying, as we did, that certainty of an intersection of events in a simple structure means that the complements of these events are all null), that is, without the justification provided by Proposition 7.

6.5. Coherency and Common Certainty of Coherency

The results in Section 4 suggest an alternative route to the construction of the universal structure, following the approach of Brandenburger and Dekel [7] to coherency. Specifically, one can construct the space of all hierarchies of preference relations (including those violating coherency) and then impose coherency and common certainty of coherency, thus obtaining a preference structure isomorphic to the universal structure.

Define recursively $W_0 = S$ and $W_{n+1} = W_n \times \Pi^I(W_n)$ for all $n \geq 0$. Let d_{n+1} denote the projection of W_{n+1} on $\Pi^I(W_n)$. Let $W = W_0 \times W_1 \times \cdots$ be endowed with the algebra of sets of the form $r_n^{-1}(A)$ where $n \geq 0$, $A \subseteq W_n$ is an event, and $r_n : W' \to W_n$ denotes the natural projection. By the same arguments as in the proofs of Propositions 2 and 3, the space W is simple, and each *coherent hierarchy* — i.e. sequence $w \in W$ such that $(d_{n-1}^i \circ r_n)(w)$ is the preference relation on $F(W_{n-1})$ induced by $(d_n^i \circ r_{n+1})(w)$ — maps into a unique element of $\Pi^I(W)$. Thus, using Proposition 7, a player's certainty of a closed subset of W can be defined at every coherent hierarchy w. But the set W_{c} of all coherent hierarchies in W is itself a closed subset of W, and therefore so is

$$\overline{W_{\mathsf{c}}} = MC_1(W_{\mathsf{c}}) \cap MC_1(MC_1(W_{\mathsf{c}})) \cap \cdots,$$

where $MC_1(E)$ is the closed subset of W defined for every closed subset E of W as

$$MC_1(E) = \{ w \in W_{\mathsf{c}} : \text{every player is certain of } E \text{ at } w \}.$$

It is then easy to construct a bijection $\chi : \Omega \to \overline{W_{\mathsf{c}}}$ that is measurable $\mathcal{A}/\mathcal{C}$, where $\mathcal{C}$ is the relative algebra on $\overline{W_{\mathsf{c}}}$ inherited from W. Moreover, χ^{-1} will be also measurable, and there will be a function $\varpi_{\mathsf{c}} : \overline{W_{\mathsf{c}}} \to \Pi^I(\overline{W_{\mathsf{c}}})$ such that $\varpi_{\mathsf{c}}^i \circ \chi = \widehat{\chi} \circ \varpi^i$ for all $i = 1, \ldots, I$, where

the set of all *finitely additive* probability measures on X, when the latter has the σ-algebra generated by the sets of the form $\{\mu : \mu(E) > p\}$. In other words, we suspect that one cannot even assume a SEU system without countable additivity, because the latter would not be an event and thus could not be imposed (if so desired) by enclosing it in a suitably constructed common belief subspace.

$\widehat{\chi} : \Pi(\Omega) \to \Pi(\varpi_{\mathsf{c}})$ is the mapping induced by χ. This will make $(\overline{W_{\mathsf{c}}}, \mathcal{C}, r_0, \varpi_{\mathsf{c}})$ into a well defined structure, isomorphic to the universal structure.

This approach, the one that Epstein and Wang [9] in fact adopted, is essentially equivalent to ours. In this paper coherency is built-in, whereas in the alternative construction it is imposed *a posteriori* and is what guarantees the construction closes. Indeed, we followed a modeling strategy similar to the latter when dealing with Savage substructures of the universal structure. In the construction sketched above, coherency guarantees existence of the mapping ϖ_{c}, thus coherency itself, together with the fact that $\overline{W_{\mathsf{c}}}$ is closed in W, ensures that we can meaningfully speak of common certainty of coherency. Similarly, as shown in Section 5, axioms P1–P6 ensure an extension result (Proposition 9) analogous to Proposition 7, the extension concerning all sets in the σ-algebra generated by the events; moreover, the set of hierarchies $\omega \in \Omega$ such that $\varpi^i(\omega)$ satisfies P1–P6 for every player i belongs to the σ-algebra generated by $\mathcal{A}$. Thus Savage's axioms themselves guarantee we can meaningfully talk about common certainty of them.

6.6. Complete Information and Subjective Correlated Equilibrium

Fix a utility function $u^i : Z \to \mathbb{R}$ for every player i. By (12) in Proposition 10, the set

$$D = \left\{\omega \in \Omega : \omega \in \Omega_{\mathsf{SO}} \text{ and } \varpi^i(\omega) \text{ induces } u^i \text{ for all } i = 1, \ldots, I\right\},$$

where Ω_{SO} is as in Subsection 6.2 above, belongs to $\mathcal{A}^*$. Thus, $\Omega_{\mathsf{SOC}} = D \cap CC_{\mathsf{S}}(D)$ is a Savage component of the universal structure, hence it induces a Savage substructure and (by the last claim in Proposition 10) has a unique associated Savage σ-structure, where the family of events is the σ-algebra $\mathcal{A}^*_{\mathsf{SOC}}$ of all sets of the form $\Omega_{\mathsf{SOC}} \cap E$, where $E \in \mathcal{A}^*$. Now let υ_{SOC} and β_{SOC} denote the associated mappings

$$\upsilon_{\mathsf{SOC}} : \Omega_{\mathsf{SOC}} \to \mathcal{U}^I \qquad \text{and} \qquad \beta_{\mathsf{SOC}} : \Omega_{\mathsf{SOC}} \to \Delta^I_{\mathsf{F}}\left(\Omega_{\mathsf{SOC}}, \mathcal{A}^*_{\mathsf{SOC}}\right)$$

as in Proposition 10. Then

$$\left(\Omega_{\mathsf{SOC}}, \mathcal{A}^*_{\mathsf{SOC}}, \varrho_{0\mathsf{SOC}}, \upsilon_{\mathsf{SOC}}, \beta_{\mathsf{SOC}}\right)$$

is a *complete information SEU system*, i.e. a SEU system where players may be uncertain about other players' beliefs, but are certain of their own strategies, beliefs, and utilities, certain of the other players' utilities, and commonly certain of these certainties. *Such a SEU system is essentially identical to the mathematical object assumed in the original definition of (a posteriori) sub-*

jective correlated equilibrium. In particular, Aumann [2] explicitly imposes non-atomicity, which in Savage's context (P6) and ours is precisely what guarantees existence and uniqueness of beliefs. While without a doubt an interesting exercise, a full exploration of the precise connections between Aumann's (and also Forges's [10]) theory and ours goes, however, beyond the scope of this paper.

6.7. *States of Nature and Strategies as Simple Spaces*

The results in this paper — all of them, as stated and none excluded — remain valid if, instead of assuming that S and Z are finite, we assume that S^0 and $S^1, \dots, S^I$ are simple spaces and Z is countable, provided that acts are restricted to be *simple*, i.e. finite-valued. Indeed, all proofs, taken almost verbatim, still work in this case. This is because the family of finite-valued, measurable mappings from a simple space into a countable set (where the latter is endowed with the algebra of all its subsets) is countable, and this is all one actually needs. We assumed S and Z are finite, *not* because this makes the analysis simpler, but rather because it seems inappropriate to assume S^i is simple — hence possibly uncountable — when the set of acts mapping S into Z, which should *include* the set S^i, turns out to be countable.[35]

7. Proofs

Proof of Proposition 1. Pick an injective function $\nu : F(X) \times F(X) \to \mathbb{N}$ such that $\nu^{-1}(n) \neq \varnothing$ for every $n \geq 1$ such that $\nu^{-1}(n+1) \neq \varnothing$. Such function clearly exists, as X is simple and thus $F(X)$ and $F(X) \times F(X)$ are countable. Now let $\overline{\Pi}(X)$ denote the set of all subsets of $F(X) \times F(X)$, and define a metric on $\overline{\Pi}(X)$ by letting $d(\pi, \pi') = 0$ if $\pi = \pi'$ and $d(\pi, \pi') = 1/n$ if n is the smallest $k \in \mathbb{N}$ such that $\nu^{-1}(k) \in \pi \smallsetminus \pi'$ or $\nu^{-1}(k) \in \pi' \smallsetminus \pi$. The induced topology clearly makes $\overline{\Pi}(X)$ either discrete — this is if Z and X are actually finite, in which case $F(X)$ is finite and so $\overline{\Pi}(X)$ is also finite — or homeomorphic to the Cantor set $\{0,1\}^{\mathbb{N}}$. Thus, $\overline{\Pi}(X)$ is

[35] Here is a somewhat more detailed explanation. Assume Z is countable and S^0 and each S^i are simple; denote the product algebra on S by $\mathcal{S}$. The outcome function $\zeta : S \to Z$ should then satisfy $\zeta^{-1}(z) \in \mathcal{S}$ for all $z \in Z$. Since each element of $\mathcal{S}$ is both closed and open and S is compact, there exists a finite subset $Z' \subseteq Z$ such that $\{\zeta^{-1}(z) : z \in Z'\}$ is a finite partition of S. A preference relation π^i on the set of measurable maps $f : S \to Z$ (by the same proof given for ζ, each such map is automatically finite-valued) does induce a preference relation on S^i, since each $s^i \in S^i$ can be seen — as explained in Subsection 6.3 — as such a map. However, this involves considerable identification among i's strategies; it implies existence of a *finite* partition of S^i such that any two strategies in the same element of the partition correspond to the same act and thus are, by reflexivity of π^i, indifferent to i. In other words, while not more complicated at the technical level, these more general assumptions do not go much farther.

compact Hausdorff, and every d-open ball is also closed. Let

$$\overline{\Pi}_{f,g}(X) = \{\overline{\pi} \in \overline{\Pi}(X) : (f,g) \in \overline{\pi}\}$$

for every $f, g \in F(X)$ and note that, if $\nu(f,g) = n$ and P_n denotes the set of all mappings $p : \{1,\dots,n\} \to \{0,1\}$ such that $p(n) = 1$, one has

$$\overline{\Pi}_{f,g}(X) = \bigcup_{p\in P_n} \bigcap_{k=1}^{n} \{\overline{\pi} \in \overline{\Pi}(X) : \nu^{-1}(k) \in \overline{\pi} \text{ if and only if } p(k) = 1\}.$$

This shows that $\overline{\Pi}_{f,g}(X)$ is a finite union of d-open balls, hence both closed and open. Conversely, every d-open ball can be written as

$$\bigcap_{k=1}^{n} \{\overline{\pi}' \in \overline{\Pi}(X) : \nu^{-1}(k) \in \overline{\pi}' \text{ if and only if } \nu^{-1}(k) \in \overline{\pi}\}$$

for some $\overline{\pi} \in \overline{\Pi}(X)$ and some n, thus every d-open ball is an event in the algebra on $\overline{\Pi}(X)$ generated by the sets of the form $\overline{\Pi}_{f,g}(X)$. We conclude that $\overline{\Pi}(X)$ equipped with this algebra is a simple space. It remains to prove $\Pi(X)$ is d-closed. This is indeed immediate, as $\Pi(X)$ is an intersection of closed subsets of $\overline{\Pi}(X)$, namely, the intersection of all sets having either the form $\overline{\Pi}_{f,f}(X)$ where $f \in F(X)$ — this guarantees reflexivity — or the form

$$\overline{\Pi}_{f,h}(X) \cup \left(\overline{\Pi}(X) \setminus \overline{\Pi}_{f,g}(X)\right) \cup \left(\overline{\Pi}(X) \setminus \overline{\Pi}_{g,h}(X)\right)$$

where $f, g, h \in F(X)$ — this guarantees transitivity. (Monotone continuity is automatically satisfied by every binary relation in $\overline{\Pi}(X)$, since X is simple.) □

Proof of Lemma 4. The third statement follows from the first, using Lemma 3 and the fact that each Ω_n is finite, hence simple. The second statement clearly implies the first and, since Z has at least two elements, it is obviously true for $n = 0$. Let $n \geq 1$ and suppose (induction hypothesis) that, for all $\omega_{n-1} \in \Omega_{n-1}$, the set $\rho_{n-1}^{-1}(\omega_{n-1})$ has two or more distinct elements. Fix $\omega_n \in \Omega_n$ and consider, for each player i, the preference relations

$$\pi^i = \{(f,f) : f \in F(\Omega_n)\} \cup \{(f \circ \rho_{n-1}, g \circ \rho_{n-1}) : (f,g) \in \delta_n^i(\omega_n)\}, \quad \pi'^i = \pi^i \cup \{(f', g')\},$$

where f' and g' are arbitrarily chosen, distinct acts in $F(\Omega_n)$ such that $f' \neq f \circ \rho_{n-1} \neq g'$ for all $f \in F(\Omega_{n-1})$. (Such f' and g' exist due to the induction hypothesis.) These are distinct preference relations, and clearly $\widehat{\rho}_{n-1}(\pi^i) = \widehat{\rho}_{n-1}(\pi'^i) = \delta_n^i(\omega_n)$. Thus, $(\omega_n, (\pi^1, \ldots, \pi^I))$ and $(\omega_n, (\pi'^1, \ldots, \pi'^I))$ are distinct elements of Ω_{n+1}. Thus, for all $\omega_n \in \Omega_n$, the set $\rho_n^{-1}(\omega_n)$ has at least two distinct elements. □

Proof of Proposition 3. As a preliminary step, we prove that for every $f \in F(\Omega)$ there exist $n \geq 0$ and $g \in F(\Omega_n)$ such that $f = g \circ \varrho_n$. Indeed, by definition of $\mathcal{A}$, for every $z \in Z$ there exist $n_z \geq 0$ and an event $E_z \subseteq \Omega_{n_z}$ such that $f^{-1}(z) = \varrho_{n_z}^{-1}(E_z)$. Since Z is finite, $n = \max_{z \in Z} n_z$ is a well defined finite number, and the collection of sets of the form $\rho_n^{-1}(\cdots(\rho_{n_z}^{-1}(E_z)))$, where $z \in Z$, is a finite partition of Ω_n. Thus, $f = g \circ \varrho_n$, where $g \in F(\Omega_n)$ maps each $\omega_n \in \Omega_n$ into the unique $z \in Z$ whose corresponding element of the partition contains ω_n. The proof of the preliminary step is complete. Now note that, as an immediate consequence of our definitions, for all $i = 1, \ldots, I$ and all $n \geq 1$ we have $\delta_n^i \circ \varrho_n = \widehat{\rho}_{n-1} \circ \delta_{n+1}^i \circ \varrho_{n+1}$. Furthermore, by induction using the latter, for all $i = 1, \ldots, I$ and $m \geq n \geq 1$ we have

$$\delta_n^i \circ \varrho_n = \widehat{\rho}_{n-1} \circ \cdots \circ \widehat{\rho}_{m-1} \circ \delta_{m+1}^i \circ \varrho_{m+1}. \tag{22}$$

Define the mapping $\varpi : \Omega \to \Pi^I(\Omega)$ as follows: for every $i = 1, \ldots, I$ and every $\omega \in \Omega$,

$$\varpi^i(\omega) = \big\{ (f \circ \varrho_n, g \circ \varrho_n) \ : \ n \geq 0, \ (f, g) \in \delta_{n+1}^i(\varrho_{n+1}(\omega)) \big\}.$$

We must prove the latter is a well defined set and an element of $\Pi(\Omega)$. To verify it is well defined, note that for every $m \geq n \geq 0$, every $f, g \in F(\Omega_n)$, and every $f', g' \in F(\Omega_m)$ such that $f \circ \varrho_n = f' \circ \varrho_m$ and $g \circ \varrho_n = g' \circ \varrho_m$, one has $f' = \rho_n \circ \cdots \circ \rho_{m-1} \circ f$ and $g' = \rho_n \circ \cdots \circ \rho_{m-1} \circ g$, (these follow from ontoness of ϱ_m and from $\varrho_n = \rho_n \circ \cdots \circ \rho_{m-1} \circ \varrho_m$) hence, by (22), one has $(f, g) \in \delta_{n+1}^i(\varrho_{n+1}(\omega))$ if and only if $(f', g') \in \delta_{m+1}^i(\varrho_{m+1}(\omega))$. To verify it is an element of $\Pi(\Omega)$, just note that reflexivity and transitivity are an immediate consequence of the preliminary step and of the corresponding properties of $\delta_{n+1}^i(\varrho_{n+1}(\omega))$ for all $n \geq 0$, whereas monotone continuity is automatically satisfied (since Ω is simple). The preliminary step also guarantees that for any two distinct preference relations in $\Pi(\Omega)$ there exist $n \geq 0$ and $f, g \in F(\Omega_n)$ such that $(f \circ \varrho_n, g \circ \varrho_n)$ is an element of one relation but not of the other. Thus property (1), which ϖ^i satisfies by definition, uniquely identifies ϖ^i. Moreover, by definition of ϖ^i, for all $n \geq 1$ and all $f, g \in F(\Omega_{n-1})$ one has

$$\big\{\omega \in \Omega \ : \ (f \circ \varrho_{n-1}, g \circ \varrho_{n-1}) \in \varpi^i(\omega)\big\} = \big\{\omega \in \Omega \ : \ (f, g) \in \delta_n^i(\varrho_n(\omega))\big\}.$$

The latter proves, again by the preliminary step, that ϖ is measurable, and will clearly also prove the last statements, once we show that the mapping $\omega \mapsto (\varrho_0(\omega), \varpi(\omega))$ is a bijection. Choose any $s \in S$ and any $(\pi^1, \ldots, \pi^I) \in \Pi^I(\Omega)$, and define $\omega = (\omega_0, \omega_1, \ldots) \in \Omega$ by letting $\omega_0 = s$ and recursively defining $\omega_{n+1} = (\omega_n, (\widehat{\varrho}_n(\pi^1), \ldots, \widehat{\varrho}_n(\pi^I)))$ for all $n \geq 0$. Clearly, $\varpi^i(\omega) = \pi^i$ for all $i = 1, \ldots, I$, thus ontoness is established. To prove injectivity, note that for every two distinct $\omega, \omega' \in \Omega$ there exists $n \geq 0$ such that $\varrho_n(\omega) \neq \varrho_n(\omega')$, hence either $\varrho_0(\omega) \neq \varrho_0(\omega')$ or there exist some $n \geq 1$ and some player i such that $\delta_n^i(\varrho_n(\omega)) \neq \delta_n^i(\varrho_n(\omega'))$, and in the latter case $\varpi(\omega) \neq \varpi(\omega')$ by the preliminary step. □

Proof of Proposition 4. Let $(X, \mathcal{B}, \sigma, \vartheta)$ be a simple, complete structure, and let γ be its generator. For all $n \geq 0$, following our earlier notation, let $\gamma_n = \varrho_n \circ \gamma$ and let $\widehat{\gamma}_n : \Pi(X) \to \Pi(\Omega_n)$ be the induced mapping. Since the structure is complete, both γ_0 and ϑ are onto. Thus, by induction, γ_n is onto for every n, because if γ_n is onto, then $\widehat{\gamma}_n$ is onto by Lemma 3 (as the structure is simple), hence $\widehat{\gamma}_n \circ \vartheta^i$ is onto for all $i = 1, \ldots, I$, hence γ_{n+1} is onto. This implies that γ is onto. Indeed, if this were not the case, then we would reach the contradiction — since $(X, \mathcal{B})$ is simple — that, for some $\omega \in \Omega$, the strictly decreasing sequence of nonempty events $\gamma_0^{-1}(\varrho_0(\omega)), \gamma_1^{-1}(\varrho_1(\omega)), \ldots$ in the algebra $\mathcal{B}$ has empty intersection. □

Proof of Proposition 5. Let γ be the generator of $(X, \mathcal{B}, \sigma, \vartheta)$. Let $\mathcal{B}' = \{\, \gamma^{-1}(A) : A \in \mathcal{A} \,\}$. We must prove that $\mathcal{B}'$ is the smallest algebra of subsets of X satisfying (5) and (6). Now (5) is obvious, since $\sigma = \varrho_0 \circ \gamma$. To prove (6), let $\widehat{\gamma}$ denote the mapping from $\Pi(X, \mathcal{B})$ to $\Pi(\Omega, \mathcal{A})$ induced by γ. By (4), for all $f, g \in F(\Omega, \mathcal{A})$ we have

$$\big\{x \in X : (f \circ \gamma, g \circ \gamma) \in \vartheta^i(x)\big\} = \gamma^{-1}\big(\big\{\omega \in \Omega : (f, g) \in \varpi^i(\omega)\big\}\big).$$

This proves (6), as by definition of $\mathcal{B}'$ every act in $F(X, \mathcal{B}')$ can be written as $f \circ \gamma$ for some $f \in F(\Omega, \mathcal{A})$. If $\mathcal{B}''$ is another algebra satisfying (5) and (6), then $\gamma^{-1}(\varrho_n^{-1}(A)) \in \mathcal{B}''$ for all $n \geq 0$ and all $A \subseteq \Omega_n$. For $n = 0$ this follows at once from (5), hence by induction using (6) it is true for $n > 0$ as well. Thus, $\mathcal{B}' \subseteq \mathcal{B}''$. This proves the first claim. To prove the second claim, just observe that, by the first claim in this proposition and by definition of $\mathcal{A}$, non-redundancy holds if and only if for all distinct $x, x' \in X$ there exists $n \geq 0$ such that $\varrho_n(\gamma(x)) \neq \varrho_n(\gamma(x'))$, and such n exists if and only if $\gamma(x) \neq \gamma(x')$. □

Proof of Lemma 6. Let $\iota : X \to Y$ denote inclusion, let $\sigma = \varsigma \circ \iota$, and let $\vartheta : X \to \Pi^I(X)$ be defined by (3). The mapping ϑ is well defined because, by reflexivity of $\eta^i(w)$, we have $(f \circ \varphi, g \circ \varphi) \in \eta^i(w)$ for all $i = 1, \ldots, I$, all $w \in W$, and all $f, g \in F(Y)$ that coincide on X.

Thus, X satisfies (2) and so it induces a substructure. Now we must prove that

$$\sigma \circ \phi = \alpha \circ \tau \quad \text{and} \quad \widehat{\phi} \circ \eta^i = \vartheta^i \circ \phi \tag{23}$$

for every $i = 1, \ldots, I$, where $\widehat{\phi}$ is the mapping from $\Pi(X)$ to $\Pi(Y)$ induced by ϕ. The former equality is immediate: indeed, $\sigma \circ \phi = \varsigma \circ \iota \circ \phi = \varsigma \circ \varphi = \alpha \circ \tau$, where the first equality follows from the fact that X induces a substructure, the second by definition of ϕ, and the third from the fact that (α, φ) is a morphism. For the second equality in (23), since the mapping $\widehat{\iota} : \Pi(X) \to \Pi(Y)$ induced by ι is injective, it will suffice to prove that $\widehat{\iota} \circ \widehat{\phi} \circ \vartheta^i = \widehat{\iota} \circ \vartheta^i \circ \phi$. Indeed,

$$\begin{aligned}
\widehat{\iota} \circ \widehat{\phi} \circ \vartheta^i &= \widehat{\varphi} \circ \vartheta^i && \text{because } \varphi = \iota \circ \phi \text{, hence } \widehat{\varphi} = \widehat{\iota} \circ \widehat{\phi}; \\
&= \theta^i \circ \varphi && \text{because } \varphi \text{ is a morphism;} \\
&= \theta^i \circ \iota \circ \phi && \text{again because } \varphi = \iota \circ \phi; \\
&= \widehat{\iota} \circ \vartheta^i \circ \phi && \text{because } X \text{ induces a substructure.}
\end{aligned}$$

□

In all proofs below, the set of outcomes Z will be identified with the set $\{1, \ldots, |Z|\}$.

Proof of Proposition 6. Necessity is obvious. For sufficiency, suppose each A_n is π-null but A is not. Then there exist $f, g, h \in F(X)$ such that

$$(fAh, gAh) \dot{\in} \pi. \tag{24}$$

Let $B_n = A_1 \cup \cdots \cup A_n$ for all $n \geq 1$. Let $K = \max\{z \in Z \ : \ g^{-1}(z) \neq \varnothing\}$. For all $1 \leq k \leq K$ and all $n \geq 1$, let

$$C_k = \left\{x \in X : 1 \leq j \leq k,\ g(x) = j\right\} \quad \text{and} \quad D_k^n = C_k \cap (A \setminus B_n)$$

and write g_k^n for the act that coincides with g on D_k^n and with fAh everywhere else. Note that g_K^n and gAh only differ on the π-null event B_n, so

$$(gAh, g_K^n) \in \pi \qquad \forall n \geq 1. \tag{25}$$

As $D_1^n \downarrow \varnothing$, by (24) and monotone continuity of π we have $(g_1^{n_1}, gAh) \dot{\in} \pi$ for some $n_1 \geq 1$. Proceeding inductively, let $1 \leq k < K$ and assume we have found $n_k \geq 1$ such that $(g_k^{n_k}, gAh) \dot{\in} \pi$.

Since $D^n_{k+1} \downarrow \varnothing$, again by monotone continuity of π there exists $n_{k+1} > n_k$ such that

$$\big(g^{n_{k+1}}_{k+1} D^{n_{k+1}}_{k+1} g^{n_k}_k, gAh\big) \dot{\in} \pi. \tag{26}$$

But $g^{n_{k+1}}_{k+1} D^{n_{k+1}}_{k+1} g^{n_k}_k$ and $g^{n_{k+1}}_{k+1}$ only differ on the π-null event $D^{n_k}_k \setminus D^{n_{k+1}}_{k+1}$, so

$$\big(g^{n_{k+1}}_{k+1}, g^{n_{k+1}}_{k+1} D^{n_{k+1}}_{k+1} g^{n_k}_k\big) \in \pi. \tag{27}$$

By (26), (27), and transitivity of π, we get $(g^{n_{k+1}}_{k+1}, gAh) \dot{\in} \pi$, contradicting (25) for $k = K-1$. □

Proof of Proposition 7. We prove the proposition in four steps. In the first step we construct the binary relation π^+ and show it satisfies (8). In the second step we prove π^+ is in fact a preference relation. In the third step we prove the second part of the proposition. In the fourth step we show π^+ is the unique preference relation on $F(X, \mathscr{B}^+)$ satisfying (8).

Step 1. Every element of $\mathscr{B}^+$ has the form $(B \cap A) \cup (C \setminus A)$ where $B, C \in \mathscr{B}$. The latter is immediate, since $\mathscr{B}^+$ must clearly include the family of sets of such form, and this family is easily seen to be an algebra. Thus, for each $f_+ \in F(X, \mathscr{B}^+)$ we can pick two acts ψ_{f_+} and ϕ_{f_+} in $F(X, \mathscr{B})$ such that ψ_{f_+} coincides with f_+ on $X \setminus A$ and ϕ_{f_+} coincides with f_+ on A.[36] Now let

$$\pi^+ = \big\{(f_+, g_+) \in F(X, \mathscr{B}^+) \times F(X, \mathscr{B}^+) : (\psi_{f_+}, \psi_{g_+}) \in \pi\big\}.$$

To prove (8), first note that if $f, g \in F(X, \mathscr{B})$ coincide on $X \setminus A$, then the event

$$A_{f,g} = \big\{x \in X : f(x) \neq g(x)\big\}$$

satisfies $A_{f,g} \subseteq (A_1 \cup \cdots \cup A_n)$ for some n and is thus null according to π. Indeed, suppose to the contrary that $A_{f,g} \not\subseteq (A_1 \cup \cdots \cup A_n)$ for every n. Then $A_{f,g} \setminus (A_1 \cup \cdots \cup A_n)$ is a strictly decreasing sequence of events. Since X is simple, this sequence has nonempty intersection, contradicting the assumption that f and g coincide on $X \setminus A$, since the latter implies $A_{f,g} \subseteq A$. We have thus shown that for all $f, g \in F(X, \mathscr{B})$ that coincide on $X \setminus A$ we have $(f, g) \in \pi$. This has two implications.

[36] One way to choose ψ_{f_+} and ϕ_{f_+} is as follows. Let $N = \max\{n \in Z : {f_+}^{-1}(n) \neq \varnothing\}$ and, for every $1 \leq n \leq N$, choose $G_n, G'_n \in \mathscr{B}$ such that ${f_+}^{-1}(n) = (G_n \cap A) \cup (G'_n \setminus A)$. Define recursively the inverse images under ψ_{f_+} and ϕ_{f_+} by

$$\psi^{-1}_{f_+}(1) = G'_1 \cup \big[X \setminus (G'_1 \cup \cdots \cup G'_N)\big], \quad \psi^{-1}_{f_+}(n) = G'_n \setminus (G'_1 \cup \cdots \cup G'_{n-1}),$$
$$\phi^{-1}_{f_+}(1) = G_1 \cup \big[X \setminus (G_1 \cup \cdots \cup G_N)\big], \quad \phi^{-1}_{f_+}(n) = G_n \setminus (G_1 \cup \cdots \cup G_{n-1}).$$

The first implication is that A is null according to π^+. Indeed, for all $f_+, g_+ \in F(X, \mathcal{B}^+)$ that coincide on $X \setminus A$ we must have $(\psi_{f_+}, \psi_{g_+}) \in \pi$ and hence also $(f_+, g_+) \in \pi^+$. The second implication is that for all $f, g \in F(X, \mathcal{B})$ we have $(f, \psi_f) \in \pi \ni (\psi_f, f)$ and $(g, \psi_g) \in \pi \ni (\psi_g, g)$, hence, by transitivity of π, $(f, g) \in \pi$ if and only if $(f, g) \in \pi^+$. This proves (8).

Step 2. Reflexivity and transitivity of π^+ directly follow from the corresponding properties of π. To prove monotone continuity, let $z \in Z$, let B_n and C_n be sequences in $\mathcal{B}$ such that the sequence $D_n = (B_n \cap A) \cup (C_n \setminus A)$ has empty intersection, and choose $f_+, g_+ \in F(X, \mathcal{B}^+)$ such that $(f_+, g_+) \dot{\in} \pi^+$. Then $(\psi_{f_+}, \psi_{g_+}) \dot{\in} \pi$, and since the sequence $E_n = C_n \setminus (A_1 \cup \cdots \cup A_n)$ also has empty intersection, by monotone continuity of π we have

$$(zE_n\psi_{f_+}, \psi_{g_+}) \dot{\in} \pi \quad \text{and} \quad (\psi_{f_+}, zE_n\psi_{g_+}) \dot{\in} \pi$$

for all n sufficiently large. But the two acts in each of the four pairs

$$(zD_n f_+, zE_n\psi_{f_+}), \quad (zE_n\psi_{g_+}, zD_n g_+), \quad (\psi_{g_+}, g_+), \quad (f_+, \psi_{f_+})$$

coincide on $X \setminus A$. Thus, since A is null according to π^+, by transitivity of π^+ we obtain

$$(zD_n f_+, g_+) \dot{\in} \pi^+ \quad \text{and} \quad (f_+, zD_n g_+) \dot{\in} \pi^+$$

for all n sufficiently large, thus establishing monotone continuity of π^+.

Step 3. Here we prove that a union of events in $\mathcal{B}$ that belongs to $\mathcal{B}^+$ is π^+-null if and only if each event in the union is π-null. Since π^+ satisfies (8), necessity is obvious. In order to prove sufficiency, by Proposition 6 it suffices to show that if $E \in \mathcal{B}$ is π-null then E is also π^+-null. Thus, let $f_+, g_+, h_+ \in F(X, \mathcal{B}^+)$. Since $E \cap A$ is π^+-null (because so is A, by Step 1) and f_+ and ψ_{f_+} coincide on $E \setminus A$, and since A is π^+-null and h_+ and ψ_{h_+} coincide on $X \setminus A$, we have $(f_+Eh_+, \psi_{f_+}E\psi_{h_+}) \in \pi^+$. Similarly, $(\psi_{g_+}E\psi_{h_+}, g_+Eh_+) \in \pi^+$. But E is π-null, so $(\psi_{f_+}E\psi_{h_+}, \psi_{g_+}E\psi_{h_+}) \in \pi$, hence $(\psi_{f_+}E\psi_{h_+}, \psi_{g_+}E\psi_{h_+}) \in \pi^+$ by (8), so $(f_+Eh_+, g_+Eh_+) \in \pi^+$ by transitivity of π^+. This shows that every event in $\mathcal{B}$ that is null according to π is also null according to π^+.

Step 4. In order to prove uniqueness, by the first part of the proof it is clearly enough to show the following: A is null according to every preference relation π' on $F(X, \mathcal{B}^+)$ that, for all $f, g \in F(X, \mathcal{B})$, satisfies $(f, g) \in \pi'$ if and only if $(f, g) \in \pi$. Suppose by contradiction that π' satisfies the latter but A is not π'-null. Then there exist $f_+, g_+, h_+ \in F(X, \mathcal{B}^+)$ such that

$(f_+Ah_+, g_+Ah_+) \dot{\in} \pi'$, hence also $f, g, h \in F(X, \mathscr{B})$ such that

$$(fAh, gAh) \dot{\in} \pi'. \tag{28}$$

(For instance, let $h = \psi_{h_+}$, $f = \phi_{f_+}$, and $g = \phi_{g_+}$.) Now let $A'_n = A_1 \cup \cdots \cup A_n$ for all $n \geq 1$, define $C'_k = h^{-1}(k)$ and $D^n_k = C'_k \cap (A \setminus A'_n)$ for all $k \in Z$ and all $n \in \mathbb{N}$, and let $K = \max\{ k \in Z : C'_k \neq \varnothing \}$. Since $D^n_1 \downarrow \varnothing$ as $n \to \infty$ and h is constant on D^n_1 for all n, by (28) and monotone continuity of π' there exists $n_1 \geq 1$ such that $(f(A \setminus D^{n_1}_1)h, gAh) \dot{\in} \pi'$. Proceeding inductively for all $1 \leq k < K$, assume that we have found numbers $n_k > \cdots > n_1 \geq 1$ such that, letting $E'_k = A \setminus (D^{n_1}_1 \cup \cdots \cup D^{n_k}_k)$, we have $(fE'_k h, gAh) \dot{\in} \pi'$. Since $D^n_{k+1} \downarrow \varnothing$ as $n \to \infty$ and h is constant on D^n_{k+1} for all n, again by monotone continuity of π' there exists $n_{k+1} > n_k$ such that, letting $E'_{k+1} = E'_k \setminus D^{n_{k+1}}_{k+1}$, we have $(fE'_{k+1}h, gAh) \dot{\in} \pi'$. Now perform another induction as follows. Again because $D^n_1 \downarrow \varnothing$ as $n \to \infty$ and h is constant on D^n_1 for all n, by monotone continuity there exists $m_1 \geq 1$ such that $(fE'_K h, g(A \setminus D^{m_1}_1)h) \dot{\in} \pi'$. Proceeding inductively for all $1 \leq k < K$, assume that we have found numbers $m_k > \cdots > m_1 \geq 1$ such that, letting $E''_k = A \setminus (D^{m_1}_1 \cup \cdots \cup D^{m_k}_k)$, we have $(fE'_K h, gE''_k h) \dot{\in} \pi'$. Again because $D^n_{k+1} \downarrow \varnothing$ as $n \to \infty$ and h is constant on D^n_{k+1} for all n, by monotone continuity there exists $m_{k+1} > m_k$ such that, letting $E''_{k+1} = E''_k \setminus D^{m_{k+1}}_{k+1}$, we have $(fE'_K h, gE''_{k+1}h) \dot{\in} \pi'$. But, since both $E'_K = \cup_{1 \leq k \leq K}(C'_k \cap A'_{n_k})$ and $E''_K = \cup_{1 \leq k \leq K}(C'_k \cap A'_{m_k})$ are events in $\mathscr{B}$, the acts $fE'_K h$ and $gE''_K h$ both belong to $F(X, \mathscr{B})$. We have reached the conclusion $(fE'_K h, gE''_K h) \dot{\in} \pi$. Since $E'_K \subseteq A$ and $E''_K \subseteq A$, this contradicts our earlier conclusion that π contains all pairs of acts that coincide on $X \setminus A$. □

Proof of Proposition 8. For every n, since A_n is closed, there exists a sequence of events $A^n_1, A^n_2, \ldots$ such that $A_n = \cap_k A^n_k$. Now

$$MC_1(A) = \cap^I_{i=1} \cap_n \cap_k C^i(A^n_k) = \cap_n \cap^I_{i=1} \cap_k C^i(A^n_k) = \cap_n MC_1(A_n).$$

Thus (i) holds for $m = 1$, and if it holds up to some $m \geq 1$, then

$$\begin{aligned} MC_{m+1}(A) &= MC_m(A) \cap MC_1\big(MC_m(A)\big) = MC_m(A) \cap MC_1\big(\cap_n MC_m(A_n)\big) \\ &= \cap_n\big[MC_m(A_n) \cap MC_1\big(MC_m(A_n)\big)\big] = \cap_n MC_{m+1}(A_n). \end{aligned}$$

Clearly, (ii) is true whenever both A and B are events. To prove it in general, suppose B is a closed subset of X such that $A \subseteq B$. Take a sequence of events B_n such that $B = \cap_n B_n$. Then $A = \cap_n \cap_k (A^n_k \cap B_n)$. Since (ii) holds for events, $MC_1(A^n_k \cap B_n) \subseteq MC_1(B_n)$ for all n, k, hence $MC_1(A) \subseteq MC_1(B)$ by part (i) of this lemma. An obvious induction using (ii) shows that $A \subseteq MC_1(A)$ implies $A \subseteq MC_m(A)$ for every m, hence (iii) follows. To prove sufficiency in the last claim, suppose E is a simple substructure of X, and take a sequence of events E_n such that $E = \cap_n E_n$. Since E satisfies (2) and $E \subseteq E_n$ for every n, we have $E \subseteq MC_1(E_n)$ for every n, hence $E \subseteq MC_1(E)$ by part (i). By part (iii), $E \subseteq CC(E)$, hence $E = E \cap CC(E)$ and E is a closed component of X. To prove necessity, take a sequence of events E_n and let $E = \cap_n E_n$. We will show that $E \cap CC(E)$ satisfies (2) and thus induces a (simple, by Lemma 2) substructure of X. Pick a bijection

$$\nu : I \times \mathbb{N} \times F(X) \times F(X) \times F(X) \to \mathbb{N}.$$

For all $i = 1, \ldots, I$, all $n \in \mathbb{N}$, and all $f, g, h \in F(X)$, define the event

$$B_1[i, n, f, g, h] = \left\{x \in X \,:\, \left(hE_n f, hE_n g\right) \in \vartheta^i(x)\right\}$$

and then, recursively for all $m \geq 1$, the event

$$B_{m+1}[i, n, f, g, h] = \left\{x \in X \,:\, \left(hB_m[\nu^{-1}(n)]f, hB_m[\nu^{-1}(n)]g\right) \in \vartheta^i(x)\right\}.$$

Let $D = E \cap CC(E)$. Then

$$D = \cap_n \cap_m \left(E_n \cap B_m[\nu^{-1}(n)]\right) \tag{29}$$

and, moreover, $D \subseteq MC_1(E_n)$ and $D \subseteq MC_1(B_m[\nu^{-1}(n)])$ for all n, m, hence using part (i) also

$$D \subseteq MC_1\left(E_n \cap B_m[\nu^{-1}(n)]\right)$$

for all n, m. In other words, the event $X \setminus (E_n \cap B_m[\nu^{-1}(n)])$ must be $\vartheta^i(x)$-null for every player i and every $x \in D$. By (29) and the last claim in Proposition 7, we conclude that if $f, g \in F(X)$ coincide on D, then $\{x \in X : f(x) \neq g(x)\}$ must be $\vartheta^i(x)$-null for every $i = 1, \ldots, I$ and every $x \in D$. This means D satisfies (2). □

Proof of Lemma 7. The first claim is a restatement of Carathéodory's extension theorem.[37] From the proof of the latter, we know that $\mu^*(A) = \inf \sum_{n \geq 1} \mu(A_n)$ for every $A \in \mathcal{B}^*$, where the infimum is taken over all sequences of events $A_1, A_2, \ldots$ in $\mathcal{B}$ such that $A \subseteq \cup_{n \geq 1} A_n$. It is then clear that for every $A \in \mathcal{B}^*$ and every $\epsilon > 0$ there exists $B \in \mathcal{B}$ such that $\mu^*(A \cap B) \geq \mu^*(A) - \epsilon$

[37] See, for example, Theorem 3.1 in Billingsley [6].

and $\mu^*(B \setminus A) \leq \epsilon$. By Lyapunov's theorem,[38] μ^* is non-atomic if and only if it is convex ranged. Now suppose that μ^* is non-atomic, let $C \in \mathcal{B}$, and let $0 \leq p \leq \mu(C)$. Then there exists $A \in \mathcal{B}^*$ such that $A \subseteq C$ and $\mu^*(A) = p$. Now choose $B \in \mathcal{B}$ such that $\mu^*(B \cap A) \geq p - \epsilon$ and $\mu^*(B \setminus A) \leq \epsilon$. Then $B \cap C$ is an event in $\mathcal{B}$, and moreover

$$p - \epsilon \leq \mu^*(B \cap A) \leq \mu^*(B \cap C) \leq \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A) \leq p + \epsilon,$$

hence $p - \epsilon \leq \mu(B \cap C) \leq p + \epsilon$. Thus, μ is dense ranged. Conversely, suppose μ^* has an atom, that is, suppose there exists $A \in \mathcal{B}^*$ such that $\mu^*(A) = q > 0$ and, for every $B \in \mathcal{B}^*$ such that $B \subseteq A$, either $\mu^*(B) = 0$ or $\mu^*(B) = q$. Choose $0 < \epsilon < q/2$ and $B \in \mathcal{B}$ such that $\mu^*(B \cap A) \geq q - \epsilon$ and $\mu^*(B \setminus A) \leq \epsilon$. Then, for every $C \in \mathcal{B}$ such that $C \subseteq B$, either $\mu^*(C) \geq q$ or $\mu^*(C) \leq \epsilon$. Thus, there is no event $C \in \mathcal{B}$ such that $C \subseteq B$ and $\epsilon - q/2 < \mu(C) - q/2 < q/2 - \epsilon$. Since $\mu(B) \geq q - \epsilon \geq q/2$, this proves that μ is not dense ranged. □

Proof of Proposition 9. Assume there exists $\pi^* \in \Pi_S(X, \mathcal{B}^*)$ satisfying (10). By Savage's Theorem, there exist a function u and a convex ranged belief μ^* on $\mathcal{B}^*$ such that (u, μ^*) induces π^*. Let μ denote the restriction of μ^* to $\mathcal{B}$. Then μ is a dense ranged belief by Lemma 7. Thus, by (10), the pair (u, μ) represents π, hence $\pi \in \Pi_S(X, \mathcal{B})$. Conversely, suppose $\pi \in \Pi_S(X, \mathcal{B})$. By Savage's Theorem, there exist a utility function u and a dense ranged belief μ on $\mathcal{B}$ such that (u, μ) represents π. By Lemma 7, μ has a unique extension to a convex ranged belief μ^* on $\mathcal{B}^*$. Obviously, the preference relation π^* on $F(X, \mathcal{B}^*)$ induced by (u, μ^*) satisfies (10), hence $\pi^* \in \Pi_S(X, \mathcal{B}^*)$. There only remains to prove that π^* is unique. By the uniqueness in Savage's Theorem, every preference relation in $\Pi_S(X, \mathcal{B}^*)$ other than π^* has a Savage representation (u', μ') such that either u' is not a positive affine transformation of u, or $\mu' \neq \mu^*$, or both. In any case, by the uniqueness in Lemma 7, u' and the restriction of μ' to $\mathcal{B}$ do not constitute a Savage representation of π. □

Proof of Lemma 8. Fix a player i and write X^i_{P1} (resp. $X^i_{\mathrm{P2}}, \ldots, X^i_{\mathrm{P6}}$) for the set of all $x \in X$ such that $\vartheta^i(x)$ satisfies P1 (resp. P2,...,P6). Define

$$X^i_{f,g} = \{x \in X : (f, g) \in \vartheta^i(x)\}, \qquad Y^i_{f,g} = X \setminus X^i_{f,g}$$

for all $f, g \in F(X, \mathcal{B})$. By definition of structure, these sets belong to $\mathcal{B}$. Thus, since $\mathcal{B}$ is

[38]See, for example, Lindenstrauss [19].

countable and hence so is $F(X,\mathscr{B})$, the sets

$$X^i_{\mathrm{P1}} = \bigcap_{f,g\in F(X,\mathscr{B})} \big(X_{f,g}\cup X_{g,f}\big), \quad X^i_{\mathrm{P2}} = \bigcap_{f,g,h,h'\in F(X,\mathscr{B});\ A\in\mathscr{B}} \big(X^i_{fAh,gAh}\cup Y^i_{fAh',gAh'}\big)$$

belong to $\mathscr{B}^*$, and so does the set

$$X^i_{\mathrm{P5}} = \bigcup_{z,z'\in Z} \big(X_{z,z'}\cap Y_{z',z}\big).$$

Now let

$$X^i_A = \big\{x\in X : A \text{ is } \vartheta^i(x)\text{-null}\big\} = \bigcap_{f,g,h\in F(X,\mathscr{B})} X^i_{fAh,gAh}$$

for every $A\in\mathscr{B}$ and observe that $X_A\in\mathscr{B}^*$. Then the set

$$X^i_{\mathrm{P3}} = \bigcap_{A\in\mathscr{B};\ z,z'\in Z;\ f\in F(X,\mathscr{B})} \Big[X^i_A\cup\Big[\big(Y^i_{z,z'}\cup X^i_{zAf,z'Af}\big)\cap\big(X^i_{z,z'}\cup Y^i_{zAf,z'Af}\big)\Big]\Big]$$

belongs to $\mathscr{B}^*$ as well. The set X^i_{P4} can be written as

$$\bigcap_{z,z',z'',z'''\in Z;\ A,B\in\mathscr{B}} \Big(Y^i_{z,z'}\cup X^i_{z',z}\cup Y^i_{z'',z'''}\cup X^i_{z''',z''}\cup Y_{zAz',zBz'}\cup X_{z''Az''',z''Bz'''}\Big)$$

and thus belongs to $\mathscr{B}^*$. Finally, writing $\mathscr{P}_N$ for the family of all partitions of X into N events in $\mathscr{B}$, the set X_{P6} can be written as

$$\bigcap_{z\in Z;\ f,g\in F(X,\mathscr{B})}\ \bigcup_{\{A_1,\ldots,A_N\}\in\mathscr{P}_N}\ \bigcap_{1\le n\le N} \big[X_{\mathrm{P1}}\cap\big(X_{g,f}\cup\big(X'_{zA_nf,g}\cap X'_{f,zA_ng}\big)\big)\big]$$

and thus belongs to $\mathscr{B}^*$, too, since $\mathscr{B}$ is countable and hence so is $\mathscr{P}_N$ for every N. $\square$

Proof of Proposition 10. Let $\widetilde{\mathscr{B}}$ be the family of all $E\in\mathscr{B}^*$ satisfying

$$\big\{x\in X :\ x\in C^i_{\mathrm{S}}(X)\,;\ \beta^i(E)>p\big\}\in\mathscr{B}^* \qquad \forall p\in\mathbb{R}. \tag{30}$$

We prove the result in three steps. In the first step, we prove that there exists a countable subalgebra $\mathscr{B}'$ of $\mathscr{B}$ that generates $\mathscr{B}^*$ and satisfies $\mathscr{B}'\subseteq\widetilde{\mathscr{B}}$. In the second step, we prove that $\widetilde{\mathscr{B}}$ is closed under the formation of complements and countable monotone unions (hence also countable monotone intersections). By Halmos's monotone class theorem,[39] these two steps together imply

[39]See, for instance, Theorem 3.4 in Billingsley [6].

$\widetilde{\mathscr{B}} = \mathscr{B}^*$ and hence (11). In the third step, we prove υ^i satisfies (12). Then (13) will clearly follow from (11), (12), and Savage's Theorem, so the proof of the proposition will be indeed complete.

Step 1. By definition of preference structure,

$$\big\{x \in X \ : \ (f, g) \in \vartheta^i(x)\big\} \in \mathscr{B} \qquad \forall f, g \in F(X, \mathscr{B}). \tag{31}$$

Since λ^i is induced by ϑ^i,

$$\vartheta^i(x) = \lambda^i(x) \cap \big[F(X, \mathscr{B}) \times F(X, \mathscr{B})\big] \qquad \forall x \in C^i_{\mathrm{S}}(X). \tag{32}$$

If $\mathscr{B}$ is a countable algebra, let $\mathscr{B}' = \mathscr{B}$. Otherwise, let $\mathscr{B}'$ be any countable algebra that generates $\mathscr{B}$. In any case, since $\mathscr{B}' \subseteq \mathscr{B} \subseteq \mathscr{B}^*$, we have $F(X, \mathscr{B}') \subseteq F(X, \mathscr{B})$ and thus, using (31), (32), and Lemma 8,

$$\big\{x \in X : \ x \in C^i_{\mathrm{S}}(X) \ ; \ (f, g) \in \lambda^i(x)\big\} \in \mathscr{B}^* \qquad \forall f, g \in F(X, \mathscr{B}'). \tag{33}$$

By Proposition 9, for all $x \in C^i_{\mathrm{S}}(X)$ the restriction of $\beta^i(x)$ to $\mathscr{B}'$ represents, together with some utility function, the Savage preference relation on $F(X, \mathscr{B}')$ defined as

$$\lambda^i(x) \cap \big[F(X, \mathscr{B}') \times F(X, \mathscr{B}')\big]. \tag{34}$$

For all $E \in \mathscr{B}'$ and all $n \geq 1$, write $\mathscr{P}^n_E$ for the family of all partitions of E into n events in $\mathscr{B}'$. Note that, since $\mathscr{B}'$ is countable, $\mathscr{P}^n_E$ is countable. Now fix $E \in \mathscr{B}'$ and $p \in \mathbb{R}$. By the proof of Savage's theorem for algebras — Theorem 3.1 in Kopylov [17],

$$\beta^i(x)(E) = \sup\left\{\sum_{A \in P} \frac{1}{\xi^i(x)(A)} \ : \ n \geq 1, \ P \in \mathscr{P}^n_E\right\} \qquad \forall x \in C^i_{\mathrm{S}}(X),$$

where for every $A \in \mathscr{B}'$ and every $x \in C^i_{\mathrm{S}}(X)$ the integer $\xi^i(x)(A)$ is defined as follows. Using the fact that the relation (34) satisfies P5, choose any $\overline{z}, \underline{z} \in Z$ such that $(\overline{z}, \underline{z}) \in \lambda^i(x) \not\ni (\underline{z}, \overline{z})$. Then $\xi^i(x)(A)$ is the smallest $m \geq 2$ such that $(\overline{z}A\underline{z}, \overline{z}A'\underline{z}) \in \lambda^i(x) \not\ni (\overline{z}A'\underline{z}, \overline{z}A\underline{z})$ for some $P \in \mathscr{P}^m_X$ and every $A' \in P$.[40] Thus,

$$\big\{x \in X : \ x \in C^i_{\mathrm{S}}(X) \ ; \ \beta^i(x)(E) > p\big\}$$

[40] If there is no such integer, let $\xi^i(x)(A) = +\infty$. In fact, $\xi^i(x)(A)$ is also the unique (when it exists, that is, when $\beta^i(x)(A) > 0$) integer $m \geq 1$ such that $1/m < \beta^i(x)(A) \leq 1/(m-1)$.

is the same as

$$\bigcup_{n\geq 1}\bigcup_{P\in\mathcal{P}_E^n}\left\{x\in X:\ x\in C_{\mathrm{S}}^i(X)\,;\ \sum_{A\in P}\frac{1}{\xi^i(x)(A)}>p\right\}.$$

It is therefore clear that, in order to prove $\mathcal{B}'\subseteq\widetilde{\mathcal{B}}$, it suffices to show that

$$\left\{x\in X:\ x\in C_{\mathrm{S}}^i(X)\,;\ \xi^i(x)(A)\leq M\right\}\in\mathcal{B}^*\qquad\forall A\in\mathcal{B}',\ \forall M\geq 2. \tag{35}$$

Thus, fix $A\in\mathcal{B}'$ and $M\geq 2$. For all $z,z'\in Z$, write $A_{z,z'}$ for the set

$$\left\{x\in X:\ x\in C_{\mathrm{S}}^i(X)\,;\ (z,z')\dot{\in}\lambda^i(x)\right\}$$

and $B_{z,z'}$ for the set

$$\bigcup_{1\leq m\leq M}\ \bigcup_{P\in\mathcal{P}_X^m}\ \bigcap_{A'\in P}\left\{x\in X:\ x\in C_{\mathrm{S}}^i(X)\,;\ (zAz',zA'z')\dot{\in}\lambda^i(x)\right\}.$$

By (33), these sets belong to $\mathcal{B}^*$. Moreover, by definition of $\xi^i(x)(A)$,

$$\left\{x\in X:\ x\in C_{\mathrm{S}}^i(X)\,;\ \xi(x)(A)\leq M\right\}=\bigcup_{z,z'\in Z}(A_{z,z'}\cap B_{z,z'}),$$

hence (35) follows.

Step 2. Choose $E\in\widetilde{\mathcal{B}}$ and $p\in\mathbb{R}$. Then

$$\left\{x\in X:\ x\in C_{\mathrm{S}}^i(X)\,;\ \beta^i(x)(X\setminus E)>p\right\}$$

can be written as

$$\bigcup_{k\geq 1}\left\{x\in X:\ x\in C_{\mathrm{S}}^i(X)\,;\ \beta^i(x)(E)\leq 1-p-1/k\right\}.$$

Since $E\in\widetilde{\mathcal{B}}$, each set in the union above is the complement of a set of the form (30). Thus, the union is itself an event in $\mathcal{B}^*$, so $X\setminus E\in\widetilde{\mathcal{B}}$ and therefore $\widetilde{\mathcal{B}}$ is closed under the formation of complements. Next, take a sequence of elements A_n of $\widetilde{\mathcal{B}}$ such that $A_n\subseteq A_{n+1}$ for every n, and let $A=\cup_n A_n$. Since $\beta^i(x)$ is countably additive for every $x\in C_{\mathrm{S}}^i(X)$, we have

$$\left\{x\in X:\ x\in C_{\mathrm{S}}^i(X)\,;\ \beta^i(x)(A)>p\right\}=\bigcup_{n\geq 1}\left\{x\in X:\ x\in C_{\mathrm{S}}^i(X)\,;\ \beta^i(x)(A_n)>p\right\}.$$

Since $A_n \in \widetilde{\mathcal{B}}$ for all n, the right-hand side is a countable union of elements of $\mathcal{B}^*$, hence itself an element of $\mathcal{B}^*$. Thus $A \in \widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}$ is also closed under the formation of countable monotone unions.

Step 3. Fix a player i. Pick a bijection $\nu : Z \times Z \to \{1, \dots, |Z|^2\}$. For all $n = 1, \dots, |Z|^2$, write z_n and z'_n for the two outcomes satisfying $\nu^{-1}(n) = (z_n, z'_n)$ and define

$$A^i_n = \big\{x \in X : x \in C^i_{\mathrm{S}}(X) \,;\, (z_n, z'_n) \,\dot{\in}\, \lambda^i(x)\big\}.$$

Define recursively $B^i_1 = A^i_1$ and $B^i_{n+1} = A^i_{n+1} \smallsetminus (B^i_1 \cup \dots \cup B^i_n)$ for all $n = 1, \dots, |Z|^2$. Note that, since $\mathcal{B} \subseteq \mathcal{B}^*$, by (31) and (32) we have

$$B^i_n \in \mathcal{B}^* \qquad \forall n = 1, \dots, |Z|^2. \tag{36}$$

Then, since $\lambda^i(x)$ satisfies Savage's P5 for all $x \in C^i_{\mathrm{S}}(X)$, the sequence of events B^i_n is a partition of $C^i_{\mathrm{S}}(X)$. To establish (12), fix $z \in Z$ and $a \in \mathbb{R}$. Clearly, we may assume $0 \le a < 1$. We have

$$\big\{x \in X : x \in C^i_{\mathrm{S}}(X) \,;\, \upsilon^i(x)(z) > a\big\} = \bigcup_{1 \le n \le |Z|^2} \big\{x \in X : x \in B^i_n,\ \upsilon^i(x)(z) > a\big\}. \tag{37}$$

We will prove that the set on the right-hand side of the latter belongs to $\mathcal{B}^*$, using the fact that for all $x \in C^i_{\mathrm{S}}(X)$, by Lemma 7, the restriction of $\beta^i(x)$ to the algebra $\mathcal{B}'$ above (Step 1) is finely ranged. For every $n = 1, \dots, |Z|^2$ and every $x \in B^i_n$, the inequality $\upsilon^i(x)(z) > a$ is true if, and only if,[41] there exists $A \in \mathcal{B}'$ such that $\upsilon^i(x)(z) > \beta^i(x)(A) > a$. The first inequality is equivalent to $(z, z_n A z'_n) \,\dot{\in}\, \lambda^i(x)$ by Savage's Theorem, so the right-hand side of (37) is

$$\bigcup_{1 \le n \le |Z|^2} \bigcup_{A \in \mathcal{B}'} \big\{x \in X : x \in B^i_n,\ \beta^i(x)(A) > a,\ (z, z_n A z'_n) \,\dot{\in}\, \lambda^i(x)\big\}.$$

Thus, (36), (11), and (33) imply that the right-hand side of (37) belongs to $\mathcal{B}^*$. ☐

Proof of Proposition 11. If $(E, \mathcal{B}_E)$ induces a substructure then it satisfies (2), so $E \subseteq MC_{\mathrm{S},1}(E)$ and therefore $E = E \cap CC_{\mathrm{S}}(E)$. Conversely, assume $(E, \mathcal{B}_E)$ is a Savage component, say $D \in \mathcal{B}^*$ and $E = D \cap CC_{\mathrm{S}}(D)$. Then $E \subseteq MC_{\mathrm{S},m}(D)$ for all m, hence $E \subseteq MC_{\mathrm{S},1}(E)$ by (15). Thus, E induces a substructure, with the maps $\vartheta^i_E : E \to \Pi(E, \mathcal{B}_E)$ defined by

$$x \mapsto \big\{(f \circ \iota, g \circ \iota) : (f, g) \in \vartheta^i(x)\big\},$$

[41] The fact that the restriction of $\beta^i(x)$ to $\mathcal{B}'$ is finely ranged is precisely what guarantees necessity here.

where $\iota : E \to X$ denotes inclusion. It remains to prove that this substructure is Savage, i.e. that $\vartheta^i_E(x)$ satisfies P1–P6 for every player i and all $x \in E$. Thus, fix a player i and $x \in X$. Let π be the unique extension of $\vartheta^i(x)$ to $F(X, \mathcal{B}^*)$ and let (u, μ) be a Savage representation of π. Since μ is convex ranged and $\mu(E) = 1$, the restriction μ_E of μ to E is a convex ranged belief on $\mathcal{B}_E$. Since (u, μ_E) clearly represents $\vartheta^i_E(x)$, the proof is complete. □

Proof of Proposition 12. Let $\mathcal{B}^*$ be the σ-algebra generated by $\mathcal{B}$. The spaces $(X, \mathcal{B}^*)$ and $(\Omega, \mathcal{A}^*)$ are standard Borel, and the mapping γ is injective (by non-redundancy and Proposition 5) and measurable $\mathcal{B}^*/\mathcal{A}^*$ (by measurability $\mathcal{B}/\mathcal{A}$). Since the image of an event under an injective and measurable function between standard Borel spaces is an event,[42] we have $\gamma(E) \in \mathcal{A}^*$ for all $E \in \mathcal{B}^*$. Thus (by injectivity of γ) a subset of X belongs to $\mathcal{B}^*$ if and only if it has the form $\gamma_X^{-1}(A)$ for some $A \in \mathcal{A}^*$. The latter statement is equivalent to the statement to be proved. □

References

[1] K. J. Arrow, *Essays in the Theory of Risk-Bearing*, North-Holland, Amsterdam (1970).

[2] R. J. Aumann, Subjectivity and Correlation in Randomized Strategies, J. Math. Econ. 1 (1974) 67–96.

[3] R. J. Aumann, Correlated Equilibrium as an Expression of Bayesian Rationality, Econometrica 55 (1987) 1–18.

[4] P. Battigalli, Comment to ‘On the Decision Theoretic Foundation of Game Theory’ by Marco Mariotti, in K. J. Arrow, E. Colombatto, M. Perlman, C. Schmidt, editors, *The Foundations of Rational Economic Behavior*, McMillan, London (1996).

[5] P. Battigalli, M. Siniscalchi, Strong Belief and Forward Induction Reasoning, J. Econ. Theory 106 (2002) 356–391.

[6] P. Billingsley, *Probability and Measure*, John Wiley & Sons (1995).

[7] A. Brandenburger, E. Dekel, Hierarchies of Beliefs and Common Knowledge, J. Econ. Theory 59(1) (1993) 189–198.

[8] A. Brandenburger, A. Friedenberg, H. J. Keisler, Admissibility in Games (2004), unpublished.

[42] See, for instance, Corollary 15.2 in Kechris [15].

[9] L. G. Epstein, T. Wang, 'Beliefs about Beliefs' without Probabilities, Econometrica 64 (1996) 1343–1373.

[10] F. Forges, Five Legitimate Definitions of Correlated Equilibrium in Games with Incomplete Information, Theory and Decision 35 (1993) 277–310.

[11] F. Gul, Savage's Theorem with a Finite Number of States, Journal of Economic Theory 57 (1992) 99–110. See also the "Erratum", *Journal of Economic Theory*, 61, p.184.

[12] J. C. Harsanyi, Games with Incomplete Information Played by 'Bayesian' Players, Parts I–III, Manage. Sci. 14 (1967-68) 159–182, 320–334, 486–502.

[13] A. Heifetz, D. Samet, Topology-Free Typology of Beliefs, Journal of Economic Theory 82 (1998) 324–341.

[14] J. B. Kadane, P. D. Larkey, Subjective Probability and the Theory of Games, Manage. Sci. 28 (1982) 113–120.

[15] A. S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York (1995).

[16] S. Koppelberg, *Handbook of Boolean Algebras, Vol. 1,* J. D. Monk and R. Bonnet eds., North-Holland (1989).

[17] I. Kopylov, Subjective Probabilities on "Small" Domains, J. Econ. Theory Forthcoming.

[18] C. H. Kraft, J. W. Pratt, A. Seidenberg, Intuitive Probability on Finite Sets, Ann. Math. Statist. 30 (1959) 408–419.

[19] J. Lindenstrauss, A Short Proof of Liapounoff's Convexity Theorem, Journal of Mathematics and Mechanics 15 (1966) 971–972.

[20] M. Mariotti, On the Decision Theoretic Foundation of Game Theory, in K. J. Arrow, E. Colombatto, M. Perlman, C. Schmidt, editors, *The Foundations of Rational Economic Behavior*, McMillan, London (1996).

[21] J.-F. Mertens, S. Zamir, Formulation of Bayesian Analysis for Games with Incomplete Information, Int. J. Game Theory 14 (1985) 1–29.

[22] D. Monderer, D. Samet, Approximating Common Knowledge with Common Beliefs, Games Econ. Behav. 1 (1989) 170–190.

[23] L. J. Savage, *The Foundations of Statistics*, John Wiley & Sons, New York. Revised and enlarged version, Dover Publications, New York, 1972 (1954).

[24] C. Villegas, On Qualitative Probability σ-Algebras, Ann. Math. Statist. 35 (1964) 1787–1796.

[25] J. von Neumann, O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press. Second Edition, 1947. Third Edition, 1953. (1944).