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Recursive methods for incentive problems*

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Abstract

Many separable dynamic incentive problems have primal recursive formulations in which utility promises serve as state variables. We associate families of dual recursive problems with these by selectively dualizing constraints. We make transparent the connections between recursive primal and dual approaches, relate value iteration under each and give conditions for it to be convergent to the true value function.

1 Introduction

Dynamic incentive models have received widespread application in finance and macroeconomics. They have been used to provide micro-foundations for market incompleteness, firm capital structure and bankruptcy law. In macroeconomics, first Ramsey and later more general Mirrlees models have informed thinking on tax policy and social insurance. In each of these varied cases, the associated dynamic incentive problem recovers equilibrium payoffs and outcomes from a game played by a population of privately informed or uncommitted agents and, often, a committed mechanism designer or principal. Equilibrium restrictions from the game provide the problem's constraints and given additive separability of payoffs over histories, tractable recursive primal and dual formulations

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are available. In contrast to many other problems in economics, however, this recursivity is often implicit and these formulations must be recovered from the payoff/constraint structure via the addition of constraints that define state variables or through the manipulation of a Lagrangian. Recursive formulations of dynamic incentive problems have been developed in different contexts by Kydland and Prescott (1980), Abreu et al (1990), Green (1987), Spear and Srivastava (1987), Fernandes and Phelan (2000), Judd et al (2003) and Marcet and Marimon (1999). Each of these papers uses or develops a particular method and several consider a particular application. Our goal is to provide a unified treatment of recursive primal and dual approaches for dynamic incentive problems. We use basic results from the theory of dynamic programming and duality, especially conjugate function duality, to do so. We emphasize practical issues associated with the application of these methods and identify when particular methods are valid. We relate value iteration under each method and give conditions for it to be convergent to the true value function.

Our starting point is the well known primal recursive formulation in which incentive constraints are re-expressed in terms of utility promises and these promises are used to perturb future constraints via auxiliary "promise-keeping" conditions. The latter ensure consistency of constraints and choices across periods.[1] We show that the promise-keeping formulation is applicable to many problems in which payoffs are separable over histories. The approach can be used to recover the optimal payoff of a principal seeking to motivate a population of uncommitted or privately informed agents or to recover the entire set of equilibrium payoffs available to such a population. In the latter case, we use indicator functions to represent equilibrium payoff sets, permitting a recasting of the set-theoretic treatment of equilibrium payoffs in Abreu et al (1990) in terms of value functions.[2] A difficulty with the primal approach is that the associated value functions are very often non-finite at some points in their domain, i.e. are extended real-valued. For example, the indicator function representation assigns infinite values to points outside of a payoff set. From the point of view of practical computation extended real-valued functions are awkward as they introduce arbitrarily large discontinuities or arbitrarily large steepness at the boundaries of their effective domains, the regions upon which they are finite. This has led some economists to first approximate the effective domain (or "endogenous state

[1]In some settings, for all feasible choices of a principal, agent choice problems are concave, smooth and independent of other agents. Optimal agent choices are then completely characterized by first order conditions and agent shadow values can be used as state variables instead of promises. This can drastically reduce the number of incentive constraints and the dimensionality of the state space. Such first order approaches are commonly used to solve Ramsey models recursively and have been used in some dynamic private information settings. We do not pursue first order approaches in this paper.

[2]The indicator function of $X \subseteq \mathbb{R}^N$ is given by $\delta_X : \mathbb{R}^N \to \{0, \infty\}$ with $\delta_X(x) = 0$ if $x \in X$ and $\delta_X(x) = \infty$ otherwise.

space") and then, in a second step, calculate the relevant value function.[3] However, the former approximation may not be straightforward, the domain may be complicated and the value function may be discontinuous with respect to misspecification of the domain.

Primal problems can be re-expressed using Lagrangians that incorporate some or all constraints. In these re-expressed problems, a sup-inf operation over choices and Lagrange multipliers replaces a sup operation. The sequencing of sup and inf is important. By interchanging them, a dual problem is obtained. Constraints absorbed into the dual problem's Lagrangian are said to be dualized. We show how by selectively dualizing constraints from the recursive primal problem various recursive dual problems may be recovered. In one the current promise-keeping constraint is dualized to give a formulation close to Judd et al (2003). In another the current incentive and promise-keeping constraints are dualized. This second dual problem is related to problems considered by Marcet and Marimon (2011). We elaborate the relationship in Section 2 below.

The different formulations described above introduce duality gaps, differences between optimal primal and dual values. We discuss conditions for these to be zero and show that they are weaker for the Judd et al (2003) formulation. In many cases, recursive dual problems are formulated on state spaces of payoff weights. In these cases, updated weights that perturb the objective encode rewards and penalties for adherence to and violation of past incentive constraints.While the introduction of duality gaps (and the additional assumptions required for their absence) is a disadvantage of recursive dual approaches, the formulation of the problem on state spaces of weights can be useful. Specifcally, it can avoid the non-finite value functions that emerge under the primal approach. Consider again the indicator function representation of an equilibrium payoff set. If this set is closed and convex (and its indicator function lower semicontinuous and convex), then it is represented on the space of weights by its support function.[4] The indicator function can be interpreted as a promise domain value function, the support function as a weight domain one. Importantly, when the equilibrium set is bounded, the indicator function is sometimes infinite-valued, while the support function is everywhere finite.

Primal promise and dual weight domain value functions are tightly connected. Geometrically, the latter gives the family of affine functions minorized by the former. Analytically, modulo a sign change, the weight domain value function is the Legendre-Fenchel transform or conjugate of the promise domain function.[5] Conversely, if (the negative of) the promise domain value function is convex and lower semicontinuous, then it is the

[3] For example, see Abraham and Pavoni (2008).

[4] If $X \subseteq \mathbb{R}^N$, then $\sigma_X : \mathbb{R}^N \to \overline{\mathbb{R}}$ with $\sigma_X(z) = \sup_X \langle z, x \rangle$ is the support function.

[5] If $f : \mathbb{R}^N \to \overline{\mathbb{R}}$, then its Legendre-Fenchel transform or conjugate is $f^*(y) = \sup_{x \in \mathbb{R}^N} \{\langle x, y \rangle - f(x)\}$.

conjugate of its weight domain counterpart. The recursive primal and recursive dual formulations give rise to Bellman operators. We show that these are also related to one another by conjugacy operations. Theorems may be more easily proven in one setting, calculations more easily done in another. Our results makes precise the relations between recursive formulations and the extent to which we can interchange and move between them.

Our main interest in Bellman operators is as devices for recovering value functions. It is standard in economics to consider problems in which the value function belongs to a space of functions that is sup or, more generally, weight-norm bounded and in which the Bellman operator is contractive on this space. In the context of dynamic incentive problems, it is often not possible or obvious to determine spaces of candidate value functions on which the Bellman operator is contractive. In particular this is the case if the value function is extended real-valued. However, we give conditions for the epiconvergence of Bellman operator iterates to the value function in primal promise domain problems. Since the Legendre-Fenchel operator and conjugation is continuous with respect to epiconvergence, our earlier results relating Bellman operators via conjugation ensure that, absent duality gaps, certain dual Bellman iterations also converge to the true value function on a weight domain. We briefly also discuss when dual Bellman operators are contractive on suitable function spaces.

The paper proceeds as follows. Section 2 provides further discussion of the literature and two motivating examples. Section 3 lays out a two period environment and gives economic examples. The essential constraint structure common to many incentive problems is isolated here. Sections 4 and 5 develop the key recursive approaches in a straightforward way in this two period setting. Section 6 describes a framework that accommodates many infinite-horizon problems and that incorporates the necessary constraint and objective structure. Sections 7 to 9 extend and apply the recursive formulations and duality relations from earlier sections to these problems. The additional consideration in the infinite horizon setting concerns the derivation of convergent value iteration procedures. This is taken up in Section 10 where sufficient conditions for epiconvergent promise and weight domain primal value iteration are obtained. Some first results on the contractivity of dual Bellman operators and uniformly convergent value iteration are also provided. Section 11 briefly discusses practical issues relating to the approximation of value functions. Proofs are given in Appendix A, while Appendix B gives background duality results.

2 Literature

In this section, we review the literature and relate our contribution to others. Green (1987) and Spear and Srivastava (1987) provide early applications of the primal promise approach to dynamic incentive problems. Abreu et al (1990) developed a related formulation in the context of repeated games played by privately informed players. Further applications are provided by, inter alia, Fernandes and Phelan (2000), Kocherlakota (1996) and Rustichini (1998). Judd et al (2003) implement a theoretical algorithm proposed by Abreu et al (1990) for finding all of the subgame perfect equilibrium payoffs of an infinitely repeated game. Their implementation dualizes the promise-keeping conditions.

In an important and influential contribution, Marcet and Marimon (1999) (revised: Marcet and Marimon (2011)) develop recursive saddle point methods for a class of dynamic contracting problems. Our approach is quite distinct from theirs. To understand the distinction, let $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ and consider the following simple primal problem:

$$P = \sup_{(a_1,a_2)\in A^2} f(a_1) + f(a_2) \text{ s.t. } g(a_1) + g(a_2) \geq 0. \tag{1}$$

Rewriting (1) in terms of a Lagrangian,

$$P = \sup_{(a_1,a_2)\in A^2} \inf_{\lambda\geq 0} f(a_1) + f(a_2) + \lambda[g(a_1) + g(a_2)], \tag{2}$$

and interchanging the sup and inf operations, the following dual problem is obtained:

$$D = \inf_{\lambda\geq 0} \sup_{(a_1,a_2)\in A^2} f(a_1) + f(a_2) + \lambda[g(a_1) + g(a_2)]. \tag{3}$$

In (3), we say that the constraint $g(a_1) + g(a_2) \geq 0$ is *dualized*. (3) can be decomposed as:

$$D = \inf_{\lambda\geq 0} \sup_{a_1\in A} f(a_1) + \lambda g(a_1) + \sup_{a_2\in A} \{f(a_2) + \lambda g(a_2)\}. \tag{4}$$

Thus, defining $P(\lambda) = \sup_{a_2\in A} \{f(a_2) + \lambda g(a_2)\}$ and assuming conditions for a zero duality gap, $P = D$, we obtain the recursive problem:

$$P = \inf_{\lambda\geq 0} \sup_{a_1\in A} f(a_1) + \lambda g(a_1) + P(\lambda). \tag{5}$$

This combination of decomposition and duality is the essence of our approach. In the remainder of the paper, we present refinements and extensions of it. With respect to refinements, the constraint $g(a_1)+g(a_2) \geq 0$ may be broken down into: $\{(a_1, a_2) : g(a_1) + w \geq 0, g(a_2) = w\}$ and each piece dualized separately. Dualization of only the $g(a_2) = w$ component is done by Judd et al (2003) and, in itself, delivers a key computational advantage. The central extension is to infinite horizon problems with many constraints. There one has many choices about what to dualize. We proceed by dualizing (subsets of) current constraints rather than all constraints. This avoids technical complications by keeping the dual space which houses Lagrange multipliers finite dimensional.

Now Marcet and Marimon (2011) pursue a different approach. They consider the saddle point problem:

$$\underset{(a_1,a_2;\lambda)\in A^2\times\mathbb{R}_+}{\text{maxmin}} f(a_1)+f(a_2)+\lambda[g(a_1)+g(a_2)], \tag{6}$$

where maxmin is the saddle value operation, i.e. $\underset{(x,y)\in X\times Y}{\text{maxmin}} h(x,y) = h(x^*,y^*)$ with $x^* \in \arg\max_X h(x,y^*)$ and $y^* \in \arg\min_Y h(x^*,y)$. They relate this to:

$$\underset{(a_1;\lambda)\in A\times\mathbb{R}_+}{\text{maxmin}} f(a_1)+\lambda g(a_1) + \max_{a_2\in A}\{f(a_2)+\lambda g(a_2)\}. \tag{7}$$

Note that in (6), the minimization over λ is done holding both a_1 and a_2 constant, whereas in (7) only a_1 is held constant. Decomposition of our $\inf_\lambda \sup_a$ operation is more direct and straightforward than decomposition of the $\underset{(a,\lambda)}{\text{maxmin}}$ operation. It is also possible under weaker assumptions. On the other hand, in some situations the recursive saddle approach can give more refined results, see the discussion of policies below.

There are several other differences between Marcet and Marimon (2011) and the current paper. First, there are differences between the sets of incentive problems considered in each paper. Marcet and Marimon explicitly incorporate physical state variables such as capital, we do not. On the other hand, they exclude problems with private information and focus on ones with a committed principal or government. We extend both of these elements. Second, Marcet and Marimon (2011) construct Lagrangians that include incentive constraints (and, implicitly, the law of motion for promises), but exclude the law of motion for physical state variables. As a result, their recursive formulation features a mixture of primal and dual constraints and primal and dual state variables. We keep all laws of motion in either primal or dual form. Thus, we use either primal (promise) or dual (multiplier) state variables, but never a mixture.[6] Third, we spell out the connections

[6] A further difference between us and Marcet and Marimon's original contribution, Marcet and Marimon (1999), is that in this they incorporate incentive constraints from all periods into the Lagrangian and seek a recursive formulation of the resulting problem. This necessitates an explicit treatment of the infinite

between the recursive primal problem and a family of recursive dual problems. Finally, we relax the boundedness conditions necessary for convergent value iteration.

Messner et al (2011) develop recursive primal and dual methods in abstract two and multi-period settings. They consider general separable constraint structures that can accommodate resource or incentive constraints or combinations of both. They associate constraints with periods and derive a recursive primal formulation in which state variables have physical interpretations (e.g. as capital) or accounting interpretations (e.g. as promises) depending upon the setting. They dualize *all* constraints across *all* periods and then seek a recursive formulation. We reverse this, first finding a primal recursive formulation and then dualizing current constraints to obtain new recursive problems. Messner et al (2011) show that primal and recursive dual values are equal if there is a saddle point in the original non-recursive problem, rather than a saddle point after every history. They also consider alternative forms of constraint separability. In contrast to the current paper, they do not focus on infinite horizon problems.

The focus of this paper is on values rather than policies. We invoke assumptions that ensure a zero duality gap between primal and dual problems. Sleet and Yeltekin (2010a) show that if the assumptions are strengthened to ensure strong duality (i.e. a zero duality gap and the existence of a minimizing multiplier), then any solution to the original primal problem attains the suprema in the corresponding recursive dual problem. For example, if (a_1^*, a_2^*) solves (1), $P = D$ and λ^* attains the minimum in (5), then $P = f(a_1^*) + \lambda^* g(a_1^*) + P(\lambda^*)$ and $P(\lambda^*) = f(a_2^*) + \lambda^* g(a_2^*)$. However, as pointed out by Messner and Pavoni (2004), the converse does not hold: strong duality does not guarantee that recursive dual maximizers solve the primal problem. Thus, even if $P = D$, λ^* is minimizing, and for each i, $a_i \in \arg\max_A f(a) + \lambda^* g(a)$, (a_1, a_2) may not solve (1). The recursive saddle approach of Marcet and Marimon (2011) can refine the set of maximizing policies obtained by the recursive dual approach. However, it may still admit maximizers that do not solve the primal problem, see Messner et al (2011). A sufficient condition for either approach to yield a primal solution, if one exists, is uniqueness of the maximizers in the recursive dual, see, for example, Sleet and Yeltekin (2010a) or Messner et al (2011). In recent work Marimon et al (2011) and Cole and Kubler (2010) extend recursive dual/ saddle point methods to permit recovery of optimal solutions in settings when recursive dual maximizers are not unique.

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dimensional dual space. Our sequential dualization procedure avoids this.

3 A two period framework

In the next few sections, we pursue our main ideas in an abstract, but simple two period setting. This allows us to highlight their generality and to avoid cluttering the exposition with the more detailed notation needed for infinite horizon applications. The two period problem we consider incorporates a constraint structure common to many dynamic incentive problems. We illustrate this with examples. We then derive recursive decompositions of this problem by selectively dualizing intertemporal constraints. These decompositions define Bellman operators that have direct application to infinite horizon settings. Indeed our two period formulation may be interpreted as a decomposition of an infinite horizon problem into its first and subsequent periods.

3.1 A two period primal problem

Assume an event tree $\mathcal{Z}(0)$ with a first period node 0 and a set of second period successor nodes $\mathbb{K} = \{1, \cdots, K\}$. The nodes in $\mathbb{K}$ are identified with the aftermath of distinct shocks also indexed by $\mathbb{K}$. A choice $a_k \in A_k$ is made at each node. Let $a = \{a_k\}_{k=0}^K$ denote a profile of choices and $A = \prod_{k=0}^K A_k$ the set of such profiles. The component sets A_k are not further specified. In applications additional mathematical (and economic) structure is placed upon them, but for the general formulation of the problem in this section such structure is not needed.[7] Economic applications and interpretations are provided below.

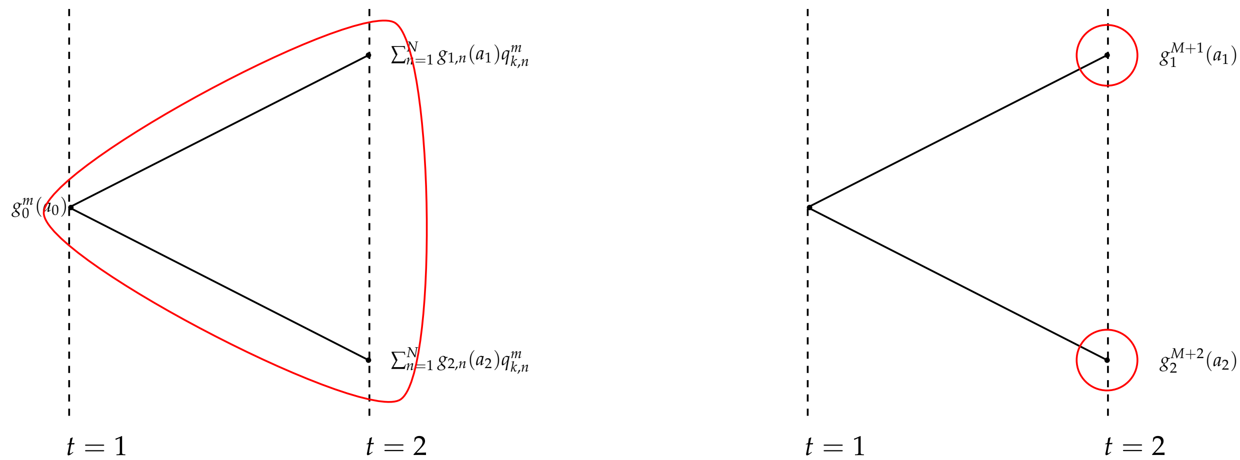
Let $f : A \to \mathbb{R}$ denote an objective function. f is assumed to be additively separable in the components $\{a_k\}$:

$$f(a) = \sum_{k=0}^{K} f_k(a_k) q_k^0, \tag{8}$$

with $f_k : A_k \to \mathbb{R}$ and $q^0 = \{q_k^0\} \in \mathbb{R}_+^{K+1}$ a family of non-negative weights. In applications the weights $\{q_k^0\}$ will incorporate discounting and probabilistic weighting of nodes.

We consider the maximization of the objective (8) subject to a separable constraint structure that is common to many incentive problems. This structure is illustrated in Figure 1. M first period constraints are applied to the entire tree. Anticipating later applications we call them *incentive constraints*. These constraints are indexed by $m \in \mathbb{M}$, where the index set has cardinality M. For example, in hidden information applications constraints are naturally indexed by pairs of shocks (the true shock and a lie). The m-th constraint is constructed from functions $g_0^m : A_0 \to \mathbb{R}$ and $g_{k,n} : A_k \to \mathbb{R}$, $k \in \mathbb{K}$,

[7] A_k is most often a subset of $\mathbb{R}^{p_k}$. Node specific choices are often identified with allocations of consumption, effort or lotteries over these things.



(a) Period 1: $g_0^m(a_0) + \sum_{k=1}^K \sum_{n=1}^N g_{k,n}(a_k)q_{k,n}^m \geq 0$

(b) Period 2: $g_k^{M+k}(a_k) \geq 0$

Figure 1: Common Constraint Structure

$n \in \{1, \cdots, N\}$ and weights $q^m = \{q_{k,n}^m\} \in \mathbb{R}^{KN}$ according to:

$$g_0^m(a_0) + \sum_{k=1}^K \sum_{n=1}^N g_{k,n}(a_k)q_{k,n}^m \geq 0. \tag{9}$$

The function g_0^m and weights q^m are allowed to be constraint specific. However, the functions $g_{k,n}$ are common to all constraints (and, therefore, have no constraint superscript). There are K further constraints that are node-specific. Each describes a restriction on a second period shock-contingent choice that is independent of past choices and of other second period shock-contingent choices. In applications, these will describe incentive and other constraints applied after the first period. For each $k \in \mathbb{K}$, let Y_k be a partially ordered vector space with zero element 0_k and let $g_k^{M+k} : A_k \to Y_k$. The additional node-specific constraints are given by:

$$\forall k \in \mathbb{K}, \quad g_k^{M+k}(a_k) \geq 0_k. \tag{10}$$

The constraint structure in (9) and (10) identifies constraints with nodes and constructs them from functions that are additively separable across histories. In addition, the M first period constraint functions map each future node choice a_k to $\{g_{k,n}(a_k)\}_{n=1}^N \in \mathbb{R}^N$. The latter variables summarize the impact of a_k on each of the first M constraints and do so economically if A_k is of higher dimension than N. A further reduction in the dimension of the summary variables occurs if all of the weights $q_{k,n}^m$ are multiplicatively separable as

$q_{k,n}^m = q_k^m q_{k,n}$ since then (9) becomes:

$$g_0^m(a_0) + \sum_{k=1}^{K} q_k^m \sum_{n=1}^{N} g_{k,n}(a_k) q_{k,n} \geq 0, \tag{11}$$

and $\sum_{n=1}^{N} g_{k,n}(a_k) q_{k,n}$ summarizes the impact of a_k in the first M constraints. These features of the constraint structure are essential for the recursive decompositions that follow.

To ensure a non-trivial problem, the constraint set is assumed to be non-empty:

$$\Omega_1 = \left\{ a \in A \,\middle|\, \forall m \in \mathbb{M},\, g_0^m(a_0) + \sum_{k=1}^{K} \sum_{n=1}^{N} g_{k,n}(a_k) q_{k,n}^m \geq 0 \text{ and } \forall k \in \mathbb{K},\, g_k^{M+k}(a_k) \geq 0 \right\} \neq \varnothing,$$

and the objective $\sum_{k=0}^{K} f_k(a_k) q_k^0$ is assumed to be bounded above on this set. The *sequential primal problem* is then:

$$P = \sup_{\Omega_1} \sum_{k=0}^{K} f_k(a_k) q_k^0 \tag{SP}$$

with P finite.

3.2 A first motivating example

This example is based on Atkeson and Lucas (1992). Suppose there are two periods. An agent receives privately observed taste shocks $\{\theta_t\}_{t=1}^2$. These are described by a probability space $(\Theta \times \Theta, \mathcal{G}, \mathbb{P})$, where for simplicity $\Theta = \{\widehat{\theta}_k\}_{k=1}^K$ is finite and $\mathbb{K} = \{1, \cdots, K\}$. The shocks perturb the agent's utility from consumption $\{c_t\}_{t=1}^2$:[8]

$$\sum_{t=1}^{2} \beta^{t-1} \sum_{\theta^t} \theta_t v(c_t(\theta^t)) \mathbb{P}(\theta^t). \tag{12}$$

In eq. (12), $\beta \in (0,1)$ is a discount factor, $v : \mathbb{R}_+ \to D \subseteq \mathbb{R}$ is a per period utility. v is assumed increasing, concave and continuous with inverse $C = D \to \mathbb{R}_+$.

A planner seeks to insure the agent against different taste shock realizations. She must, however, induce the agent to truthfully report them. Her objective incorporates both the agent's utility and the cost of resources evaluated at (shadow) prices $\{Q_t\}$. She solves:

$$\sup \sum_{t=1}^{2} \beta^{t-1} \sum_{\theta^t} \left[\theta_t v(c_t(\theta^t)) - Q_t c_t(\theta^t)\right] \mathbb{P}(\theta^t) \tag{13}$$

[8] The model is easily extended to one with multiple goods. By labeling one of these goods leisure, it accommodates a Mirrleesian model.

subject to for all $t, \theta^t, c_t(\theta^t) \geq 0$, and the *incentive constraints*, for all $m \in \mathbb{M} := \{(m_1, m_2) \in \mathbb{K}^2, m_1 \neq m_2\}$,

$$\widehat{\theta}_{m_1} v(c_1(\widehat{\theta}_{m_1})) + \beta \sum_{l=1}^{K} \widehat{\theta}_l v(c_2(\widehat{\theta}_{m_1}, \widehat{\theta}_l)) \mathbb{P}(\widehat{\theta}_l | \widehat{\theta}_{m_1}) \tag{14}$$

$$\geq \widehat{\theta}_{m_1} v(c_1(\widehat{\theta}_{m_2})) + \beta \sum_{l=1}^{K} \widehat{\theta}_l v(c_2(\widehat{\theta}_{m_2}, \widehat{\theta}_l)) \mathbb{P}(\widehat{\theta}_l | \widehat{\theta}_{m_1}),$$

and for all $k \in \mathbb{K}$ and $m \in \mathbb{M}$, $\widehat{\theta}_{m_1} v(c_2(\widehat{\theta}_k, \widehat{\theta}_{m_1})) \geq \widehat{\theta}_{m_1} v(c_2(\widehat{\theta}_k, \widehat{\theta}_{m_2}))$. Notice that the incentive constraints are indexed by pairs of shock indices $m = (m_1, m_2)$, where m_1 is the true state and m_2 an alternative state that the agent must be deterred from reporting.

This example can easily be re-expressed along the lines of the abstract two period problem. Let $g_0^m(c_1) = \widehat{\theta}_{m_1} u(c_1(\widehat{\theta}_{m_1})) - \widehat{\theta}_{m_1} u(c_1(\widehat{\theta}_{m_2}))$ and $g_{k,l}(c) = u(c)$. Set the weights $\{q_{k,l}^m\}$ according to: $q_{k,l}^m = \beta \mathbb{P}(\widehat{\theta}_l | \widehat{\theta}_{m_1})$ if $k = m_1$ (the "true shock"), $q_{k,l}^m = -\beta \mathbb{P}(\widehat{\theta}_l | \widehat{\theta}_{m_1})$ if $k = m_2$ (the "lie") and $q_{k,l}^m = 0$ otherwise. The first period constraints may then be re-expressed as:

$$g_0^m(c_1) + \sum_{k=1}^{K} \sum_{l=1}^{K} g_{k,l}(c_2(\widehat{\theta}_k, \widehat{\theta}_l)) q_{k,l}^m \geq 0. \tag{15}$$

The second period incentive constraints may be summarized as:

$$g_k^{M+k}(c_2(\widehat{\theta}_k, \cdot)) = \{\widehat{\theta}_{m_1} v(c_2(\widehat{\theta}_k, \widehat{\theta}_{m_1})) - \widehat{\theta}_{m_1} v(c_2(\widehat{\theta}_k, \widehat{\theta}_{m_2}))\}_{m \in \mathbb{M}} \geq 0. \tag{16}$$

In this example, the future node choice a_k is identified with $\{c_2(\widehat{\theta}_k, \cdot)\} \in \mathbb{R}_+^K$ and the summary variables with $\{g_{k,l}(c_2(\widehat{\theta}_k, \widehat{\theta}_l))\}_{l \in \mathbb{K}} \in \mathbb{R}^K$ The latter offer no reduction in dimension. However, in problems with simpler shock structures (that require lower dimension summary variables) and/or longer time horizons (that involve higher dimension future choices), the summary variables will offer such a reduction. For example, suppose that taste shocks are i.i.d. with per period probability distribution $\mathbb{P}$. In this case, the weights are multiplicatively separable, $q_{k,l}^m = \beta q_k^m \mathbb{P}(\widehat{\theta}_l)$, with $q_k^m = 1$ if $k = m_1$ (k indexes the true shock), $q_k^m = -1$ if $k = m_2$ (k indexes the lie) and $q_k^m = 0$ otherwise. Then, (15) reduces to:

$$g_0^m(c_1) + \sum_{k=1}^{K} q_k^m \sum_{l=1}^{K} g_{k,l}(c_2(\widehat{\theta}_k, \widehat{\theta}_l)) \mathbb{P}(\widehat{\theta}_l) \geq 0, \tag{17}$$

and all constraints map the k-th subtree to the summary variable $\sum_{l=1}^{K} g_{k,l}(c_2(\widehat{\theta}_k, \widehat{\theta}_l)) \mathbb{P}(\widehat{\theta}_l)$.

Problem (13) may be embedded into a family of constraint or objective-perturbed problems. The former augment the constraint set with "promise-keeping" constraints of the form:

$$w_k = \widehat{\theta}_k v(c_1(\widehat{\theta}_k)) + \beta \sum_{l=1}^{K} \widehat{\theta}_l v(c_2(\widehat{\theta}_k, \widehat{\theta}_l))\mathbb{P}(\widehat{\theta}_l|\widehat{\theta}_k),$$

for some $w \in \mathbb{R}^K$ and each $k \in \mathbb{K}$. Letting $\Omega_1(w)$ denote the augmented, promise-perturbed constraint set, the promise-perturbed problem is:

$$S(w) = \begin{cases} \sup_{\Omega_1(w)} \sum_{t=1}^{2} \beta^{t-1} \sum_{\theta^t} \left[\theta_t v(c_t(\theta^t)) - Q_t c_t(\theta^t)\right] \mathbb{P}^t(\theta^t) & \Omega_1(w) \neq \emptyset. \\ -\infty & \text{otherwise} \end{cases} \tag{18}$$

As shown below, this problem has recursive primal and dual formulations that exploit the previously described constraint structure and that use utility promises as state variables.

The objective-perturbed problem attaches a weighted sum of utilities to the objective:

$$\begin{aligned} V(\zeta) = \sup_{\Omega_1} \sum_{t=1}^{2} \beta^{t-1} \sum_{\theta^t} \left[\theta_t v(c_t(\theta^t)) - Q_t c_t(\theta^t)\right] \mathbb{P}^t(\theta^t) \\ + \sum_{k=1}^{K} \zeta_k \left[\widehat{\theta}_k v(c_1(\widehat{\theta}_k)) + \beta \sum_{l=1}^{K} \widehat{\theta}_l v(c_2(\widehat{\theta}_k, \widehat{\theta}_l))\mathbb{P}(\widehat{\theta}_l|\widehat{\theta}_k)\right]. \end{aligned} \tag{19}$$

In (19), Ω_1 is the original (unperturbed) constraint set. This problem has recursive formulations that use weights $\zeta \in \mathbb{R}^K$ as state variables.

3.3 A second motivating example

In our next example, based upon Kocherlakota (1996), the goal is to characterize the incentive-feasible risk sharing arrangements of a group of agents $\mathbb{I} = \{1, \ldots, I\}$ who receive shocks to their endowments of goods and their outside utility options. No agent can be compelled to accept a utility below her outside option.

Again there are two periods, $t = 1, 2$. The publicly observable endowment shock $\theta_t \in \Theta := \{\widehat{\theta}_k\}_{k \in \mathbb{K}}$ determines the aggregate resources available to agents in each period:

$$\sum_{i=1}^{I} c_t^i(\theta^t) \leq Y(\theta_t), \quad Y : \Theta \to \mathbb{R}_+, \tag{20}$$

where c_t^i is the consumption of agent i at t. Agent i values consumption according to:

$$\sum_{t=1}^{2} \beta^{t-1} \sum_{\theta^t} v(c_t^i(\theta^t))\mathbb{P}(\theta^t)$$

and has a date and state-contingent outside utility option: $V_t^i : \Theta \to \mathbb{R}$, $t = 1, 2$. A consumption process $c = \{c_t^i\}$ is *incentive-feasible* if it satisfies (20) and gives each agent m_1 more than their outside option in each first period shock state m_2:

$$u(c_1^{m_1}(\widehat{\theta}_{m_2})) + \beta \sum_{l=1}^{K} u(c_2^{m_1}(\widehat{\theta}_{m_2}, \widehat{\theta}_l))\mathbb{P}(\widehat{\theta}_l|\widehat{\theta}_{m_2}) \geq V_1^{m_1}(\widehat{\theta}_{m_2}), \tag{21}$$

and each second period shock state m_2:

$$u(c_2^{m_1}(\widehat{\theta}_k, \widehat{\theta}_{m_2})) \geq V_2^{m_1}(\widehat{\theta}_{m_2}). \tag{22}$$

It is readily seen that these incentive constraints have the same basic structure as in the last example. Indexing constraints by agent and shock, $m = (m_1, m_2) \in \mathbb{I} \times \mathbb{K}$, the first period constraints (21) apply to the entire associated event tree $\mathcal{Z}(0)$. They may be re-expressed as in (15) where now $g_0^m(c_1) = u^{m_1}(c_1(\widehat{\theta}_{m_2})) - V_1^{m_1}(\widehat{\theta}_{m_2})$, $g_{k,l}(c) = u(c)$ and $q_{k,l}^m = q_k^m \beta \mathbb{P}(\widehat{\theta}_l|\widehat{\theta}_k)$ with $q_k^m = 1$ if $k = m_2$ (k indexes the actual shock received by the m_1 agent) and 0 otherwise. The second period incentive constraints (22) are summarized as:

$$g_k^{M+k}(\{c_2(\widehat{\theta}_k, \cdot)\}) = \{u^{m_1}(c_2(\widehat{\theta}_k, \widehat{\theta}_{m_2})) - V_2^{m_1}(\widehat{\theta}_{m_2})\}_{m \in \mathbb{M}} \geq 0. \tag{23}$$

The set of incentive-feasible payoffs is given by:

$$\mathbb{V}_0 = \left\{ w \in \mathbb{R}^I \,\middle|\, \exists \text{ an incentive-feasible } c \text{ s.t. } \forall i, \quad w^i = \sum_{t=1}^{2} \beta^{t-1} \sum_{\theta^t} v(c_t^i(\theta^t))\mathbb{P}(\theta^t) \right\}.$$

The problem of finding $\mathbb{V}_0$ may be formulated as a constraint-perturbed planning problem. Let $\Omega_1(w)$ denote the set of incentive-feasible consumption processes that deliver the utility w and let $f(\theta, c) := 0$. Define the planning problem:

$$S(w) = \begin{cases} \sup_{\Omega_1(w)} \sum_{t=1}^{2} \beta^{t-1} \sum_{\theta^t} f(\theta_t, c_t(\theta^t))\mathbb{P}(\theta^t) & \text{if } \Omega_1(w) \neq \emptyset, \\ -\infty & \text{otherwise.} \end{cases} \tag{24}$$

Then $S(w) = 0$ if $w \in \mathbb{V}_0$ and $-\infty$ otherwise. Thus $-S$ is simply the *indicator function* for the set $\mathbb{V}_0$.[9] Contrasting (24) with (18) reveals the parallel between this example and the last. Moreover, as for the last example, (24) leads to recursive primal and dual problems formulated in terms of utility promises. These problems now involve indicator functions for incentive-feasible payoff and continuation incentive-feasible payoff sets. As an alter-

[9]The indicator function of $X \subseteq \mathbb{R}^K$ is given by $\delta_X : X \to \overline{\mathbb{R}}$, where $\delta_X(x) = 0$ if $x \in X$ and ∞ otherwise.

native, the following objective-perturbed problem may be formulated:

$$
\begin{aligned}
V(\zeta) &= \sup_{\Omega_1} \sum_{t=1}^{2} \beta^{t-1} \sum_{\theta^t} f(\theta_t, c_t(\theta^t)) \mathbb{P}(\theta^t) \\
&\qquad + \sum_{k=1}^{K} \zeta_k \left[v(c_1(\widehat{\theta}_k)) + \beta \sum_{l=1}^{K} v(c_2(\widehat{\theta}_k, \widehat{\theta}_l)) \mathbb{P}(\widehat{\theta}_l | \widehat{\theta}_k) \right] \\
&= \sup_{\Omega_1} \sum_{k=1}^{K} \zeta_k \left[v(c_1(\widehat{\theta}_k)) + \beta \sum_{l=1}^{K} v(c_2(\widehat{\theta}_k, \widehat{\theta}_l)) \mathbb{P}(\widehat{\theta}_l | \widehat{\theta}_k) \right]. \qquad (25)
\end{aligned}
$$

In this case, V is the *support function* of $\mathbb{V}_0$.[10] Recursive problems are available for (25). These use utility weights as a state variable and involve the support functions of incentive-feasible and continuation incentive-feasible payoff sets.

4 A quartet of recursive decompositions

A recursive formulation of the primal problem (SP) is obtained by introducing supplementary promise-keeping constraints. These are redundant from the point of view of the original problem, but enforce prior constraints in the recursive formulation. Primal optimizations may be expressed as sup-inf problems using Lagrangians. Our subsequent decompositions are obtained by selectively interchanging sup and inf operations to obtain decomposable dual problems. Since there are multiple constraints, there are multiple opportunities for such interchange leading to different decompositions. We focus on three. Their connections to our first primal decomposition are summarized in Figure 2.

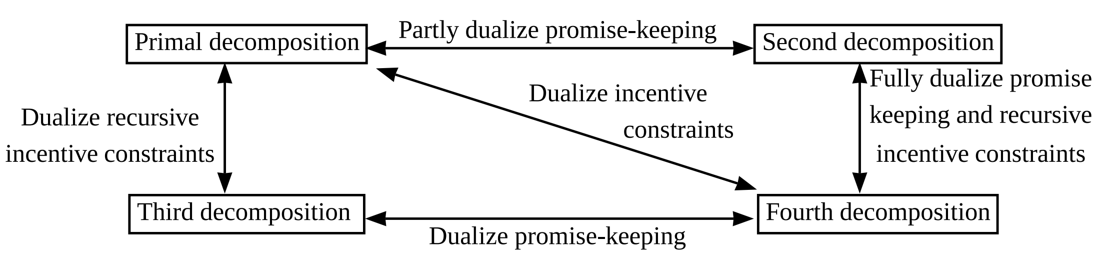


Figure 2: Relations between decompositions

[10]The support function of $X \subseteq \mathbb{R}^K$ is given by $\sigma_X : \mathbb{R}^K \to \overline{\mathbb{R}}$, where $\sigma_X(d) = \sup_{x \in X} \langle x, d \rangle$.

4.1 First decomposition: a primal formulation with promises as states

A recursive decomposition of (SP) is obtained as follows. First define the *promise variables*:

$$w_k = \{w_{k,n}\}_{n\in\mathbb{N}}, \quad w_{k,n} := g_{k,n}(a_k), k \in \mathbb{K}, n \in \mathbb{N}$$

and decompose the constraints (9) into a collection of *recursive incentive constraints* and *promise-keeping constraints*:

$$\forall m \in \mathbb{M}, \quad g_0^m(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n} q_{k,n}^m \geq 0, \qquad \forall k \in \mathbb{K}, n \in \mathbb{N}, \quad w_{k,n} = g_{k,n}(a_k). \tag{26}$$

(SP) may then be rewritten as:

$$P = \sup_{\widehat{\Omega}_1} \sum_{k=0}^{K} f_k(a_k) q_k^0, \tag{27}$$

where

$$\widehat{\Omega}_1 = \left\{ (a,w) \in A \times \mathbb{R}^{KN} \left| \begin{array}{l} \forall m, g_0^m(a_0) + \sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N} w_{k,n} q_{k,n}^m \geq 0, \\ \forall k, n, w_{k,n} = g_{k,n}(a_k), \quad \forall k, g_k^{M+k}(a_k) \geq 0 \end{array} \right. \right\}.$$

Next, define the continuation problems, for all $k \in \mathbb{K}$,

$$S_k(w_k) = \begin{cases} \sup f_k(a_k) & \Phi_k(w_k) \neq \emptyset \\ -\infty & \text{otherwise}, \end{cases} \tag{28}$$

where:

$$\Phi_k(w_k) = \left\{ a_k \in A_k \middle| \forall n \in \mathbb{N}, w_{k,n} = g_{k,n}(a_k), g_k^{M+k}(a_k) \geq 0 \right\}.$$

A routine application of the principal of optimality yields the following decomposition.

Proposition 1 (Primal Decomposition)**.** *The primal value P satisfies:*

$$P = \sup_{\Psi_0} f_0(a_0) q_0^0 + \sum_{k=1}^{K} S_k(w_k) q_k^0, \tag{29}$$

where: $\Psi_0 = \left\{ (a_0, w) \in A_0 \times \mathbb{R}^{KN} \middle| \forall m \in \mathbb{M}, g_0^m(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n} q_{k,n}^m \geq 0 \right\}$.

Proof. See Appendix A. □

The preceding decomposition relies on the association of the first group of incentive constraints with the initial period; the promises w then act as *forward state variables*. They enforce the initial incentive constraints by compelling constraint-consistent first and second period choices. The label "forward" stems from the fact that promises are defined as functions of future second period choices.[11]

In general, not all promise values ensure feasible problems for the second period. Consequently, the functions S_k may be $-\infty$-valued over some subset of their domains. The effective domain of a function $f : X \to \overline{\mathbb{R}}$ is the set upon which it is finite, i.e. Dom $f := \{x \in X | f(x) \in \mathbb{R}\}$. In the context of dynamic incentive problems, the sets Dom S_k are often referred to as endogenous state spaces. Numerical implementation of the primal decomposition is complicated by lack of knowledge of these sets and by their computational representation.

4.2 Second decomposition: weights as states and the dualization of the promise-keeping constraint

Our second decomposition uses an alternative representation of the second period value function and an alternative state space. It is obtained via the partial dualization of the promise-keeping constraints in the preceding primal decomposition.

Returning to (27), the primal problem may be re-expressed in terms of a Lagrangian that incorporates the promise-keeping constraints:

$$P = \sup_{\widehat{\Omega}_1^{PK}} \inf_{\mathbb{R}^{KN}} \sum_{k=0}^{K} f_k(a_k) q_k^0 + \sum_{k=1}^{K} \sum_{n=1}^{N} z_{k,n}[g_{k,n}(a_k) - w_{k,n}],$$

where $z \in \mathbb{R}^{KN}$ is the multiplier on the promise-keeping constraints and $\widehat{\Omega}_1^{PK}$ omits these constraints from $\widehat{\Omega}_1$. Rearrangement gives:

$$P = \sup_{\Psi_0} f_0(a_0) q_0^0 + \sum_{k=1}^{K} \sup_{\{A_k : g_k^{M+k}(a_k) \geq 0\}} \inf_{\mathbb{R}^N} \left\{ f_k(a_k) + \sum_{n=1}^{N} z_{k,n}[g_{k,n}(a_k) - w_{k,n}] \right\} q_0^k. \tag{30}$$

Equivalently, (30) re-expresses the continuation problems in (29) in sup-inf form. Now consider partially dualizing the promise-keeping constraints in (30) by interchanging the

[11] In contrast to "backward" state variables which are defined as functions of past first period ones.

sup and inf operations over a_k and z_k (but not over $(a_0, w) \in \Psi_0$ and z). Defining:

$$V_k(z_k) = \sup_{\{A_k: g^{M+k}(a_k) \geq 0\}} f_k(a_k) + \sum_{n=1}^{K} z_{k,n} g_{k,n}(a_k), \tag{31}$$

we obtain the dual problem:

$$D^{PK} = \sup_{\Psi_0} \inf_{\mathbb{R}^{KN}} f_0(a_0) q_0^0 + \sum_{k=1}^{K} \left\{ V_k(z_k) - \sum_{n=1}^{K} z_{k,n} w_{k,n} \right\} q_0^k. \tag{32}$$

Problem (32) is recursive, but it replaces the promise state variable w with the weight state variable z. In the continuation problem (31), the objective f_k is perturbed by the weighted sum $\sum_{n=1}^{K} z_{k,n} g_{k,n}(a_k)$. In the sequel we refer to such problems as "objective-perturbed". If for each k and all $w_k \in \mathbb{R}^N$,

$$S_k(w_k) = \inf_{\mathbb{R}^N} \left\{ V_k(z_k) - \sum_{n=1}^{K} z_{k,n} w_{k,n} \right\}, \tag{33}$$

then $P = D^{PK}$ and (32) permits the recovery of the optimal primal value. Condition (33) corresponds to the absence of a duality gap in all continuation problems. We consider this absence and the equivalence of (30) and (32) below. Before doing so, we introduce the concept of a conjugate function and recast the discussion in terms of such functions.

4.2.1 Conjugate functions

The *conjugate* of $f : \mathbb{R}^N \to \overline{\mathbb{R}}$ is given by $f^* : \mathbb{R}^N \to \overline{\mathbb{R}}$, where

$$f^*(x^*) = \sup_{x \in \mathbb{R}^N} \{\langle x, x^* \rangle - f(x)\},$$

and $\langle \cdot, \cdot \rangle : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ denotes the usual dot product operation. Geometrically, f^* describes the family of affine functions majorized by f. The conjugate of f^* (i.e. the conjugate of the conjugate) is referred to as the *biconjugate* of f and is denoted f^{**}. Let $\mathbb{F}_0^N$ denote the set of *proper functions* $f : \mathbb{R}^N \to \overline{\mathbb{R}}$ that are nowhere $-\infty$ and are somewhere less than ∞ and let $\mathbb{F}^N$ denote the set of proper, lower semicontinuous and convex functions. A well known result[12] asserts that if $f \in \mathbb{F}_0^N$, then $f^* \in \mathbb{F}^N$ and f^{**} is the lower semicontinuous and convex regularization of f. The *Legendre-Fenchel transform* maps a function to its conjugate and is denoted $\mathcal{C}$. It follows that $\mathcal{C} : \mathbb{F}_0^N \to \mathbb{F}^N$ and $\mathcal{C}$ is a self-

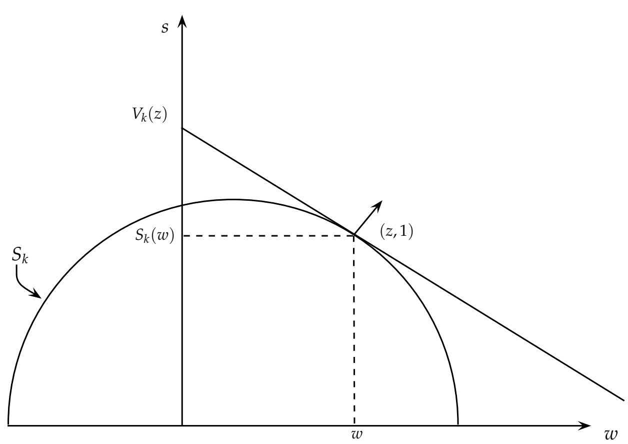
[12] See Rockafellar (1970), p. 103-104.

inverse on $\mathbb{F}^N$.

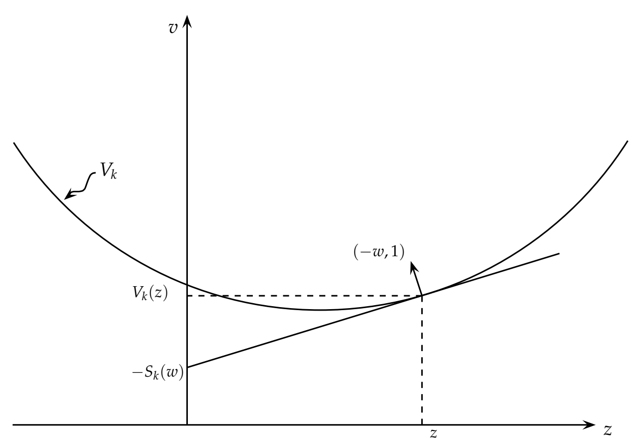
With minor qualifications on the boundaries of effective domains, differentiability dualizes under $\mathcal{C}$ to strict convexity. If $f \in \mathbb{F}^N$ is differentiable on the interior of Dom f, then f^* is strictly convex on the relative interior of Dom f^* and vice versa. There is also an attractive conjugacy between Dom f and Dom f^* which we elaborate below. Finally, conjugacy provides a convenient framework for expressing duality relations between optimization problems. This is elaborated in Appendix B.

4.2.2 Relating the decompositions

Suppose that each value function $V_k : \mathbb{R}^N \to \overline{\mathbb{R}}$ is finite at 0 in which case $-S_k \in \mathbb{F}_0^N$.[13] It is an immediate consequence of Proposition B0 in the appendix, that $V_k = \mathcal{C}[-S_k]$ and that it is in $\mathbb{F}^N$, i.e. is proper, convex and lower semicontinuous. However, except at 0 by assumption, V_k may be ∞-valued over some part of its domain. If $-S_k \in \mathbb{F}^N$ as well, then $-S_k = \mathcal{C}[V_k]$. In this case, S_k and V_k provide alternative representations of the upper surface of the k-th continuation incentive-feasible payoff set. The geometric implications of these relations are illustrated below in Figure 3 (for the case $N = 1$).



(a) $V_k = \mathcal{C}(-S_k)$



(b) $-S_k = \mathcal{C}(V_k)$

Figure 3: Conjugacy between value functions

Since:

$$\mathcal{C}[V_k] = \inf_{\mathbb{R}^N} \left\{ V_k(z_k) - \sum_{n=1}^{K} z_{k,n} w_{k,n} \right\},$$

the preceding discussion implies the absence of a duality gap in the continuation problem when $-S_k \in \mathbb{F}^N$ and, hence, the following result.

[13] If $V_k(0)$ is finite, S_k cannot be ∞ anywhere. Since Ω_1 is non-empty, S_k is more than $-\infty$ somewhere.

Proposition 2. *Assume that each $-S_k \in \mathbb{F}^N$, then:*

$$P = \sup_{\Psi_0} \inf_{\mathbb{R}^{KN}} f_0(a_0)q_0^0 + \sum_{k=1}^{K} \left\{ V_k(z_k) - \sum_{n=1}^{N} w_{k,n} z_{k,n} \right\} q_k^0. \tag{34}$$

In Section 4.1, we emphasized that the potential infinite-valuedness of the functions $\{-S_k\}$ creates difficulties for the primal decomposition. Similar concerns potentially apply to the second decomposition (34) and the value functions V_k. However, the effective domains of V_k and S_k are related by conjugacy arguments. This relation is described in Proposition 3 and leads to the identification of an important case in which the functions V_k are finite-valued.

Recall again that the support function of a set $X \subseteq \mathbb{R}^N$ is given by $\sigma_X(d) = \sup_X \langle x, d \rangle$. Also note the following definition.

Definition 1. If $f \in \mathbb{F}^N$, then its asymptotic function $f^\infty : \mathbb{R}^N \to \overline{\mathbb{R}}$ is given by: $f^\infty(y) = \lim_{\lambda \to \infty} \frac{f(x+\lambda y) - f(x)}{\lambda}$ for any $x \in \mathbb{R}^N$.

Proposition 3. *Let $-S_k \in \mathbb{F}^N$. The support function of Dom $(-S_k)$ equals the asymptotic function of V_k and vice versa.*

Proof. Follows from the previous discussion and Rockafellar (1970), p. 116. □

The following corollary is of particular use. Recall that the epigraph of a function $f : \mathbb{R}^N \to \overline{\mathbb{R}}$, epi f, is given by epi $f = \{(x, r) \in \mathbb{R}^{N+1} | f(x) \leq r\}$.

Corollary 1. *Assume that $-S_k \in \mathbb{F}^N$ and epi $(-S_k)$ contains no non vertical half lines, then Dom $V_k = \mathbb{R}^N$. In particular, if f_k is bounded above and each $g_{k,n}$ is bounded, then Dom $V_k = \mathbb{R}^N$.*

It follows that when $g_{k,n}$ represents the bounded continuation utility of an agent and the objective is bounded above, then the effective domain ("endogenous state space") of the objective-perturbed value function V_k is immediately given as all of $\mathbb{R}^N$. Boundedness above of the objective is quite common. It occurs in most applications with a committed principal and in applications in which a value set is encoded as an indicator function.[14]

For comparison with the results from later decompositions it is useful to recast (32) using a Lagrangian that incorporates the first period incentive constraints. Letting η denote

[14]Every point x in $\mathbb{R}^N$ can be identified with a point on the unit hemisphere $H_n = \{(\gamma, \lambda) \in \mathbb{R}^N \times \mathbb{R} | \lambda > 0, \|\gamma\|^2 + \lambda^2 = 1\}$ in $\mathbb{R}^{N+1}$ by the mappings $\gamma(x) = x/(\|x\|^2 + 1)$ and $\lambda(x) = 1/(\|x\|^2 + 1)$. Thus, in this case, the effective domain can alternatively be identified with this set.

the multiplier on these constraints, we obtain:

$$D^{PK} = \sup_{A_0 \times \mathbb{R}^{KN}} \inf_{\mathbb{R}^M_+ \times \mathbb{R}^{KN}} f_0(a_0)q_0^0 + \sum_{m=1}^{M} \eta_m g_0^m(a_0) + \sum_{k=1}^{K} \left\{ V(z_k) + \sum_{n=1}^{N} \left[\zeta_{k,n}(\eta) - z_{k,n}\right] w_{k,n} \right\} q_k^0, \tag{35}$$

where: $\zeta_{k,n}(\eta) := \sum_{m=1}^{M} \eta_m \frac{q^m_{k,n}}{q^0_k}, \quad \forall k \in \mathbb{K}, n \in \mathbb{N}.$

4.3 Third decomposition: promises as states and the dualization of the incentive constraints

We return to (27) and instead of (partially) dualizing the promise-keeping constraints, we dualize the recursive incentive constraints. Using a Lagrangian that incorporates these constraints, (27) may be re-expressed as:

$$P = \sup_{\widehat{\Omega}_1^{IC}} \inf_{\mathbb{R}^M_+} \sum_{k=0}^{K} f_k(a_k)q_k^0 + \sum_{m=1}^{M} \eta_m \left\{ g_0^m(a_0)q_0^m + \sum_{n=1}^{N} w_{k,n}q^m_{k,n} \right\}, \tag{36}$$

where $\widehat{\Omega}_1^{IC}$ omits the recursive incentive constraints from $\widehat{\Omega}_1$ and $\eta \in \mathbb{R}^M_+$ is the multiplier upon them. Interchanging the sup and inf operations gives the dual problem:

$$D^{IC} = \inf_{\mathbb{R}^M_+} \sup_{\widehat{\Omega}_1^{IC}} \sum_{k=0}^{K} f_k(a_k)q_k^0 + \sum_{m=1}^{M} \eta_m \left\{ g_0^m(a_0)q_0^m + \sum_{n=1}^{N} w_{k,n}q^m_{k,n} \right\}. \tag{37}$$

Breaking apart the inner supremum optimization and substituting for S_k gives our next decomposition.

Proposition 4. *The dual value D^{IC} satisfies the following condition:*

$$D^{IC} = \inf_{\mathbb{R}^M_+} \sup_{A_0 \times \mathbb{R}^{KN}} f_0(a_0)q_0^0 + \sum_{m=1}^{M} \eta_m g_0^m(a_0)q_0^m + \sum_{k=1}^{K} \left\{ \sum_{m=1}^{M} \eta_m \sum_{n=1}^{N} w_{k,n}q^m_{k,n} + S_k(w_k)q_k^0 \right\}. \tag{38}$$

As for the first decomposition, (38) uses promises as state variables and employs the constraint-perturbed value functions S_k.

Problem (38) permits the recovery of the optimal primal value if there is no duality gap between (36) and (37). This absence can be expressed in terms of the conjugacy of value functions, but now these functions involve perturbations of first period incentive constraints. Specifically, (SP) may be embedded into the family of incentive-perturbed problems:

$$P(\delta) := \sup_{a \in \Omega_1(\delta)} \sum_{k=0}^{K} f_k(a_k) q_k^0, \tag{39}$$

where

$$\Omega_1(\delta) = \left\{ a \in A \,\middle|\, \text{each } g_0^m(a_0) + \sum_{k=1}^{K} \sum_{n=1}^{N} g_{k,n}(a_k) q_{k,n}^m \geq -\delta \text{ and each } g_k^{M+k}(a_k) \geq 0 \right\}.$$

$P(\cdot)$ is the value function associated with perturbations of the first period incentive constraints. Evidently, the optimal primal value from (SP) equals $P(0)$. By Theorem B1 in the appendix, after the elimination of the promises,[15] $-D^{IC} = \mathcal{C}^2[-P(\cdot)](0)$ and a zero duality gap occurs if $-P(\cdot) \in \mathbb{F}^M$.

4.4 A fourth decomposition: weights as states and the dualization of the incentive constraints

Our final decomposition uses weights as state variables and the objective-perturbed value function. It is obtained by dualizing the recursive incentive and promise-keeping constraints . Problem (27) may be re-expressed as:

$$P = \sup_{\widehat{\Omega}_1^{PKIC}} \inf_{\mathbb{R}_+^M \times \mathbb{R}^{KN}} \sum_{k=0}^{K} f_k(a_k) q_k^0 + \sum_{m=1}^{M} \eta_m \left\{ g_0^m(a_0) q_0^m + \sum_{k=1}^{K} \sum_{n=1}^{N} w_{k,n} q_{k,n}^m \right\} + \sum_{k=1}^{K} q_k^0 \sum_{n=1}^{N} z_{k,n} [g_{k,n}(a_k) - w_{k,n}], \tag{40}$$

where η and z are the multipliers on the recursive incentive and promise-keeping constraints and both are omitted from $\widehat{\Omega}_1^{PKIC}$. The associated dual is:

$$D^{PKIC} = \inf_{\mathbb{R}_+^M \times \mathbb{R}^{KN}} \sup_{\widehat{\Omega}_1^{PKIC}} \sum_{k=0}^{K} f_k(a_k) q_k^0 + \sum_{m=1}^{M} \eta_m \left\{ g_0^m(a_0) q_0^m + \sum_{k=1}^{K} \sum_{n=1}^{N} w_{k,n} q_{k,n}^m \right\} + \sum_{k=1}^{K} q_k^0 \sum_{n=1}^{N} z_{k,n} [g_{k,n}(a_k) - w_{k,n}]. \tag{41}$$

[15] The promises and the promise-keeping constraints deliver the recursive formulation which is not used in establishing conditions for a zero duality gap.

Rearrangement of the supremum component gives:

$$D^{PKIC} = \inf_{\mathbb{R}_+^M \times \mathbb{R}^{KN}} \sup_{A_0} \left\{ f_0(a_0)q_0^0 + \sum_{m=1}^{M} g_0^m(a_0)q_0^m \right\} + \sum_{k=1}^{K} \sup_{\{A_k: g^{M+k}(a_k)\geq 0\}} \left\{ f_k(a_k) + \sum_{n=1}^{N} z_{k,n} g_{k,n}(a_k) \right\} q_k^0$$
$$+ \sup_{\mathbb{R}^{KN}} \sum_{k=1}^{K} q_k^0 \sum_{n=1}^{N} w_{k,n} \left\{ \sum_{m=1}^{M} \eta_m \frac{q_{k,n}^m}{q_k^0} - z_{k,n} \right\}. \tag{42}$$

Substituting for V_k and using the earlier definition of $\zeta_{k,n}$, (42) reduces to:

$$D^{PKIC} = \inf_{\mathbb{R}_+^M \times \mathbb{R}^{KN}} \sup_{A_0 \times \mathbb{R}^{KN}} f_0(a_0)q_0^0 + \sum_{m=1}^{M} \eta_m g_0^m(a_0) + \sum_{k=1}^{K} \left\{ V(z_k) + \sum_{n=1}^{N} \left[\zeta_{k,n}(\eta) - z_{k,n}\right] w_{k,n} \right\} q_k^0. \tag{43}$$

In (42), the sup value is ∞ unless each $z_{k,n}$ is chosen to equal $\sum_{m=1}^{M} \eta_m \frac{q_{k,n}^m}{q_k^0}$. Substituting these values into (42) and using the definition of V_k implies the following result.

Proposition 5. *The dual value D^{PKIC} satisfies:*

$$D^{PKIC} = \inf_{\mathbb{R}_+^M} \sup_{A_0} f_0(a_0)q_0^0 + \sum_{m=1}^{M} \eta_m g_0^m(a_0)q_0^m + \sum_{k=1}^{K} V_k(\zeta_k(\eta))q_k^0, \tag{44}$$

where $\zeta(\eta) = \{\zeta_k(\eta)\}_k = \left\{ \sum_{m=1}^{M} \eta_m \frac{q_{k,n}^m}{q_k^0} \right\}_{k,n} \in \mathbb{R}^{KN}$.

In this context, $\zeta(\eta) \in \mathbb{R}^{KN}$ may be interpreted as a *backwards state variable* for the dual problem. It penalizes first period constraint violations via perturbations of the continuation objective. The label "backward" stems from the fact that ζ is a function of the first period choice η.

Remark 1. If each $A_0 = \mathbb{R}^P$ and the current constraint functions g_0^m are affine, i.e., $\{g_0^m(a_0)\} = b_0 + B_1 a_0$, with B_1 an $M \times P$ matrix, then the value function from the inner supremum operation equals $f_0^*(h'B_1)$, where $h' = (\eta_1 q_1^0/q_0^0 \ldots \eta_M q_M^0/q_0^0)$. Affine constraint functions are quite common in applications in which the action variables a_0 are identified with agent utilities. If $A_0 = \mathbb{R}^P$ and f_0 is additively separable, then the inner supremum is itself decomposable and each component of a_0 can be solved for separately. If both of the preceding assumptions hold and f_0 is an additive sum of standard functional forms, e.g. polynomial or exponential, then the conjugate f_0^* is often immediately available and no maximization needs to be done explicitly. Again, this is often the case in applied economic problems.

The decomposition (44) is readily related to the preceding three. Relative to the second, it dualizes the incentive and fully dualizes the promise-keeping constraint (compare (43) to (35)). Relative to the third, it dualizes the promise-keeping constraint. However, it is straightforward to check that in the latter case, this additional dualization does not introduce a duality gap and that $D^{PKIC} = D^{IC}$. In addition, if $P = D^{IC}$, then $P = D^{PKIC}$ and so conditions for the absence of a duality gap between the primal and the third decomposition ensure the absence of such a gap between the primal and the fourth decomposition.

4.5 Value function properties

We have seen that duality gaps are absent and the various decompositions give the same optimal value when the value functions obtained by perturbing "intertemporal" constraints are proper, lower semicontinuous and convex. In this case these functions and their conjugates give alternative, but equivalent representations of a relevant payoff surface. We are thus led to consider when value functions inherit properness, lower semicontinuity and convexity from the primitives of a problem. This issue is taken up in general terms in Appendix B. Here we relate the results from Appendix B to the present setting. Properness is a mild condition. We have previously assumed that Ω_1 is non-empty. This ensures the constraint sets for the continuation promise-perturbed problems (31) and the incentive-perturbed problem (39) are non-empty for some parameters. Provided the objective functions in these problems are bounded above on all constraint sets, properness of the relevant primal value functions is obtained. A well known condition for convexity of $-S_k$ or $-P(\cdot)$ is that the constraint correspondences Φ_k and Ω_1 have convex graphs and the problem objectives are concave. For the continuation problems (31), this requires that the functions $g_{k,n}$, g_k^{M+k} and f_k are, respectively, affine, quasiconcave and concave. For the problems (39), concavity of g_0^m, $g_{k,n}(\cdot)q_{k,n}^m$ and f_k along with quasiconcavity of g_k^{M+k} is needed. Some dynamic incentive problems satisfy these conditions, see for example Atkeson and Lucas (1992) or Kocherlakota (1996). Others such as Thomas and Worrall (1990) do not, but, instead satisfy weaker conditions that are sufficient for convexity of the value function. The following example and Appendix B provide further discussion. The latter also provides discussion of conditions for lower semicontinuity of the value function.

Example. Consider the class of hidden information problems in which the planner has an objective of the form:

$$\sum_{\theta^t} f(a_t(\theta^t))\mathbb{P}^t(\theta^t),$$

a single agent has preferences:

$$\sum_{t=1}^{2} \beta^{t-1} \sum_{\theta^t} r(\theta_t, a_t(\theta^t))\mathbb{P}^t(\theta),$$

and shocks θ_t are i.i.d.. The planner maximizes her objective subject to each $a_t(\theta^t) \in A$, first period incentive constraints, $\forall (m_1, m_2) \in \mathbb{M}$,

$$r(\widehat{\theta}_{m_1}, a_1(\widehat{\theta}_{m_1})) + \beta \sum_{l=1}^{K} r(\widehat{\theta}_l, a_2(\widehat{\theta}_{m_1}, \widehat{\theta}_l))\mathbb{P}(\widehat{\theta}_l)$$
$$\geq r(\widehat{\theta}_{m_1}, a_1(\widehat{\theta}_{m_2})) + \beta \sum_{l=1}^{K} r(\widehat{\theta}_l, a_2(\widehat{\theta}_{m_2}, \widehat{\theta}_l))\mathbb{P}(\widehat{\theta}_l),$$

and second period constraints, , $\forall (m_1, m_2) \in \mathbb{M}$, $r(\widehat{\theta}_{m_1}, a_2(\widehat{\theta}_k, \widehat{\theta}_{m_1})) \geq r(\widehat{\theta}_{m_1}, a_2(\widehat{\theta}_k, \widehat{\theta}_{m_2}))$. Suppose that the agent's action a can be decomposed into two components $(a^1, a^2) \in A = A^1 \times A^2$, where each A^k is an interval of $\mathbb{R}$ and that:

$$f(a) = f_1(a^1) + f_2(\theta, a^2), \quad a = (a^1, a^2),$$

and

$$r(\theta, a) = r_1(a^1) + r_2(\theta, a^2), \quad a = (a^1, a^2).$$

This covers many cases. For example, if $f_1 = r_1 = 0$, $f_2(a^2) = -a^2$ and $r_2(\theta, a^2) = u(a^2 + \theta)$ the hidden endowment model of Thomas and Worrall (1990) is obtained. If $f_1(a^1) = -a^1$, $r_1(a^1) = u(a^1)$, $f_2(a^2) = a^2$ and $r_2(\theta, a_2) = v(a^2/\theta)$, a two period Mirrlees model is derived. In many applications, it is natural to assume that f_1, f_2, r_1 and each $r_2(\theta, \cdot)$ are concave. However, unless r_1 and $r_2(\theta, \cdot)$ are linear, as they are, for example, in Atkeson and Lucas (1992), the constraint correspondence does not have a convex graph. Fortunately, alternative weaker assumptions are sufficient for concavity/convexity of the relevant value functions. For example, suppose, in addition to concavity, that f_1 and f_2 are decreasing, r_1 and $r_2(\theta, \cdot)$ are increasing, and r_2 has decreasing differences, i.e., for all $k = 1, \ldots, K-1$, $r_2(\widehat{\theta}_{k+1}, \cdot) - r_2(\widehat{\theta}_k, \cdot)$ is decreasing. These conditions are quite standard. Suppose, in addition, r_2 satisfies: for each $\delta \in (0, 1)$, $k = 1, \ldots, K-1$ and pair a_2 and a_2', let $\tilde{a}^2$ satisfy $r_2(\widehat{\theta}_k, \tilde{a}^2) = \delta r_2(\widehat{\theta}_k, a^2) + (1-\delta) r_2(\widehat{\theta}_k, a^{2'})$, then $r_2(\widehat{\theta}_{k+1}, \tilde{a}^2) > \delta r_2(\widehat{\theta}_{k+1}, a^2) + (1-\delta) r_2(\widehat{\theta}_{k+1}, a^{2'})$. Under this assumption, each $-S_k$ is convex. For example, in Thomas and Worrall (1990)'s hidden endowment model, the agent's utility function u is assumed to be increasing, strictly concave and to satisfy NIARA. This is sufficient to imply that the analogues of $-S_k$ are strictly convex. For further results see Sleet (2011).

5 A quartet of Bellman operators

The decompositions of the preceding sections made use of families of constraint or objective perturbed *continuation* problems. We now consider perturbing the *original* problem in analogous ways. This leads to a fully recursive formulation. Moreover, the initial perturbing parameter often has an economic interpretation as a utility promise or Pareto weight.

Constraint-perturbed problems Given functions $\{g_{0,h}\}_{h=1}^{N}$, $g_{0,h} : A_0 \to \mathbb{R}$, and weights $p = \{p^h_{k,n}\}$, a family of auxiliary "promise constraints" is defined according to, for $h \in \mathbb{N} = \{1, \cdots, N\}$,

$$w_h = g_{0,h}(a_0) + \sum_{k=1}^{K} \sum_{n=1}^{N} g_{k,n}(a_k) p^h_{k,n}. \tag{45}$$

Like the initial period incentive constraints, these depend on second period actions a_k, $k \in \mathbb{K}$, via weighted sums of the functions $g_{k,n}$. By augmenting the constraint set in (SP), the equations (45) define a family of *constraint-perturbed problems* parameterized by $w \in \mathbb{R}^N$:

$$S_0(w) = \begin{cases} \sup_{\Phi_0(w)} \sum_{k=0}^{K} f_k(a_k) q^0_k & \Phi(w) \neq \emptyset \\ -\infty & \text{otherwise,} \end{cases} \tag{46}$$

where

$$\Phi_0(w) = \left\{ a \in \Omega_1 \,\middle|\, \forall h \in \mathbb{N}, w_h = g_{0,h}(a_0) + \sum_{k=1}^{K} \sum_{n=1}^{N} g_{k,n}(a_k) p^h_{k,n} \right\}.$$

Of course, $P = \sup_{\mathbb{R}^N} S_0(w) = \mathcal{C}[-S_0](0)$.

Constraint-perturbed problems have a recursive formulation in terms of promises which parallels the first decomposition from the previous section. This formulation associates the new promise keeping constraints (45) and the initial tree-wide incentive constraints with the initial period. Elements in the range of the functions $g_{k,n}$ which appear in both sets of constraints act as state variables. The formulation relates a first period constraint-perturbed problem to a family of second period constraint-perturbed problems (28) with value functions S_k. It is expressed in terms of a Bellman operator on the space of functions $\mathbb{W}_0 = \{\{W_k\}_{k \in \mathbb{K}} | W_k \in \mathbb{F}^N_0\}$.

Proposition 6. *Define the primal constraint-perturbed Bellman operator* $T^S : -\mathbb{W}_0 \to -\mathbb{F}^N_0$, *according to:*

$$T^S(W)(w) = \sup_{\Psi_0(w)} f_0(a_0) + \sum_{k=1}^{K} W_k(w_k) q^0_k,$$

where : $\Psi_0(w) = \left\{ (a_0, \{w'_k\}) \in \Psi_0 \,\middle|\, \forall h \in \mathbb{N}, w_h = g_{0,h}(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n} p^h_{k,n} \right\}$.

Then:

$$S_0 = T^S(\{S_k\}). \tag{47}$$

The proof is a straightforward extension of Proposition 1 and is omitted. Equation (47) is the type of Bellman equation most commonly encountered in dynamic incentive problems. In these the functions f_k are often interpreted as the per period payoffs of a committed principal. However, if the functions $f_k = 0$, then the value functions S_k are indicator functions for the sets of incentive-feasible promises at each node of the event tree. The Bellman equation (47) is then closely related to the B-operator considered by Abreu et al (1990). Properties of value sets emphasized by Abreu et al (1990) such as monotonicity (in the set inclusion ordering) and closure, then translate into monotonicity and lower semicontinuity of the value functions $-S_k$.

By dualizing the first period incentive and promise constraints, the following problem may be associated with (46):

$$S^D_0(w) = \inf_{\mathbb{R}^M_+ \times \mathbb{R}^N} \sup_{\Omega_2} \sum_{k=0}^{K} f_k(a_k) q^0_k + \sum_{m=1}^{M} \eta_m \left[g^m_0(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} g_{k,n}(a_k) q^m_{k,n} \right]$$
$$+ \sum_{h=1}^{N} z_h \left[g^h_0(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} g_{k,n}(a_k) p^h_{k,n} - w^h \right].$$

Along the lines of the third decomposition, this admits the recursive formulation:

$$S^D_0(w) = \inf_{\mathbb{R}^M_+ \times \mathbb{R}^N} \sup_{A_0 \times \mathbb{R}^{KN}} f_0(a_0) q^0_0 + \sum_{k=1}^{K} S_k(w_k) q^0_k + \sum_{m=1}^{M} \eta_m \left[g^m_0(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n} q^m_{k,n} \right]$$
$$+ \sum_{h=1}^{N} z_h \left[g^h_0(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n} p^h_{k,n} - w^h \right].$$

Defining the dual Bellman operator for the constraint perturbed problem as:

$$T^{S,D}(W)(w) = \inf_{\eta \in \mathbb{R}^M_+, z \in \mathbb{R}^N} \sup_{A_0 \times \mathbb{R}^{KN}} f_0(a_0) q^0_0 + \sum_{k=1}^{K} W_k(w_k) q^0_k$$
$$+ \sum_{m=1}^{M} \eta_m \left[g^m_0(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n} q^m_{k,n} \right] + \sum_{h=1}^{N} z_h \left[g^h_0(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n} q^h_{k,n} - w^h \right],$$

we obtain: $S^D_0 = T^{S,D}(\{S_k\})$.

Objective-perturbed problems An objective-perturbed problem is parameterized by $z \in \mathbb{R}^N$ and is given by:

$$V_0(z) = \sup_{\Omega_1} \sum_{k=0}^{K} f_k(a_k)q_k^0 + \sum_{h=1}^{N} z_h \left[g_{0,h}(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} g_{k,n}(a_k)p_{k,n}^h \right]. \tag{48}$$

Thus, $P = V_0(0)$. By extending our first decomposition, a recursive formulation involving weights may be associated with (48). This formulation relates a first period objective-perturbed problem to a family of second period objective-perturbed problems (31) with value functions V_k. The next proposition states the formulation; it uses the definition $\mathbb{W} = \{\{W_k\} | W_k \in \mathbb{F}^N\} \subset \mathbb{W}_0$.

Proposition 7. *Define the objective-perturbed Bellman operator $T^V : \mathbb{W}_0 \to \mathbb{F}_0^N$,*

$$\begin{aligned} T^V(W)(z) = \sup_{A_0 \times \mathbb{R}^{KN}} \inf_{\mathbb{R}_+^M \times \mathbb{R}^{KN}} & f_0(a_0)q_0^0 + \sum_{h=1}^{N} z_h g_0^h(a_0) + \sum_{m=1}^{M} \eta_m g_0^m(a_0) \\ & + \sum_{k=1}^{K} W_k(z_k)q_k^0 + \sum_{k=1}^{K}\sum_{n=1}^{N} \left[\zeta'_{k,n}(z,\eta) - z_{k,n}\right] q_k^0 w_{k,n}, \end{aligned} \tag{49}$$

where: $\zeta'_{k,n}(z,\eta) = \sum_{h=1}^{N} z_h \frac{p_{k,n}^h}{q_k^0} + \sum_{m=1}^{M} \eta_m \frac{q_{k,n}^m}{q_k^0}$. If $\{-S_k\} \in \mathbb{W}$, then $V_0(z) = T^V(\{V_k\})(z)$.

Proof. See Appendix. □

The proof involves the dualization of the second period promise-keeping constraint. The assumption in the proposition ensures the absence of a duality gap. Note that the law of motion for the weight ζ' now incorporates the initial weight z.

Although the Bellman operator (49) may initially appear unfamiliar, it is close to the one used by Judd et al (2003) to compute the payoff surfaces of equilibrium value sets in repeated games. To see this consider the case in which each $f_k = 0$. Then, as before, the value functions $-S_k$ are indicator functions for the sets of incentive-feasible promises (or payoffs) at each node of the event tree. The functions V_k are support functions for these sets. In this case, $T^V(\{V_k\})(z)$ may be rearranged to give:

$$\begin{aligned} T^V(\{V_k\})(z) = \sup_{A_0 \times \mathbb{R}^{KN}} \inf_{\mathbb{R}_+^M \times \mathbb{R}^{KN}} & \sum_{h=1}^{N} z_h \left[g_0^h(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}p_{k,n}^h \right] \\ & + \sum_{m=1}^{M} \eta_m \left[g_0^m(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}q_{k,n}^m \right] + \sum_{k=1}^{K} q_k^0 \left[V_k(z_k) - \sum_{n=1}^{N} z_{k,n}w_{k,n} \right]. \end{aligned} \tag{50}$$

Equivalently, using the fact that V_k is a support function and, hence, homogenous of degree 1,

$$T^V(\{V_k\})(z) = \sup_{A_0\times\mathbb{R}^{KN}} \sum_{h=1}^{N} z_h \left[g_0^h(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n} p_{k,n}^h \right]$$
$$\text{s.t. } \forall m \in \mathbb{M},\ g_0^m(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n} q_{k,n}^m \geq 0, \quad \forall k \in \mathbb{K},\ \inf_{\|z_k\|=1} V_k(z_k) - \sum_{n=1}^{N} z_{k,n} w_{k,n} \geq 0.$$

Judd et al (2003) consider an extension of this problem in which the consequence of a player defection is endogenized. Here, this consequence is absorbed into the incentive constraint functions g_0^m.

In (49), the second period promise-keeping constraint was dualized. By dualizing the first period incentive constraint, the following alternative problem may also be associated with (48).[16]

$$V_0^D(z) = \inf_{\mathbb{R}^M} \sup_{\Omega_2} \sum_{k=0}^{K} f_k(a_k) q_k^0 + \sum_{h=1}^{N} z_h \left[g_{0,h}(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} g_{k,n}(a_k) p_{k,n}^h \right] + \sum_{m=1}^{M} \eta_m \left[g_0^m(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} g_{k,n}(a_k) q_{k,n}^m \right]. \tag{51}$$

Following the fourth decomposition and the line of argument used to derive Proposition 5, we obtain our final Bellman equation.

Proposition 8. *Define the Bellman operator $T^{V,D} : \mathbb{W}_0 \to \mathbb{F}_0^N$, according to:*

$$T^{V,D}(W)(z) = \inf_{\mathbb{R}_+^M} \sup_{A_0} f_0(a_0) q_0^0 + \sum_{h=1}^{N} z_h g_0^h(a_0) + \sum_{m=1}^{M} \eta_m g_0^m(a_0) + \sum_{k=1}^{K} W_k(\zeta'_{k,n}(z,\eta)) q_k^0,$$

where ζ' is as before. Then: $V_0^D = T^{V,D}(\{V_k\})$.

The terms z and $\zeta'_{k,n}(z,\eta)$ may be interpreted as initial and updated weights on, respectively, the function $g_{0,h}(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} g_{k,n}(a_k) p_{k,n}^h$ and its continuations $g_{k,n}(a_k)$. The conditions for $T^{V,D}(\{V_k\}) = V_0$ are stronger than those required for $T^V(\{V_k\}) = V_0$. Roughly, the optimization $T^{V,D}(\{V_k\})$ dualizes more of the constraint structure than $T^V(\{V_k\})$. Weak duality arguments imply: $T^{V,D}(V)(z) \geq T^V(\{V_k\})(z) \geq V_0(z)$ with

[16] As the fourth decomposition makes, this is essentially equivalent to dualizing the first period recursive incentive constraint and the second period promise-keeping constraint, i.e. dualizing all constraints that link the first and second periods.

$T^{V,D}(\{V_k\})(z) = V_0(z)$ if and only if $T^V(\{V_k\})(z) = V_0(z)$ *and* there is a zero duality gap between the values $T^V(\{V_k\})(z)$ and $T^{V,D}(\{V_k\})(z)$.

In the preceding analysis, the first period incentive constraints were dualized, whereas the second period incentive constraints were not. Consequently, in deriving (51), detailed specification of the image spaces of the second period incentive functions, Y_k, and their duals was unnecessary. In contrast, if all constraints, first and second period, were simultaneously dualized such specification would be necessary. In this case, the dual of each Y_k contains the Lagrange multipliers on the k-th second period incentive constraint. Application of the above approach to infinite horizon settings allows us to work with sequences of finite dimensional dual spaces, each housing current constraint Lagrange multipliers, rather than a single infinite dimensional dual space housing the multipliers from all constraints. Technical complications stemming from an explicit treatment of the latter are avoided.[17]

The Bellman operator $T^{V,D}$ is close to that derived by Marcet and Marimon (2011) except that they leave some state variables and constraints in primal form. In addition, their derivation is quite different, relying on the recursive decomposition of a saddle point rather than a dual problem.

Conjugacy relations for perturbed problems We round this section off by stating conjugacy relations between the value functions S_0, V_0, S_0^D and V_0^D. These are combined with similar relations between S and V and the definitions of the Bellman operators to obtain conjugacy relations between Bellman operators.

Exactly paralleling our discussion of the conjugacy of $-S_k$ and V_k, we have the following results.

Proposition 9. *Assume that $-S_0 \in \mathbb{F}_0^N$, then 1) $V_0 = \mathcal{C}[-S_0]$ and $V_0 \in \mathbb{F}^N$, 2) $-S_0 = \mathcal{C}[V_0]$ if $-S_0 \in \mathbb{F}^N$, and 3) Dom $V_0 = \mathbb{R}^N$ if and only if epi $(-S_0)$ contains no non-vertical half-lines. In particular, this is true if $\sum_{k=0}^K f_k(a_k)q_k^0$ is bounded above and each $g_{0,h}(a_0) + \sum_{k=1}^K \sum_{n=1}^N g_{k,n}(a_k)p_{k,n}^h$ is bounded on Ω_1.*

Proposition 10. *Assume that $V_0^D \in \mathbb{F}_0^N$, then 1) $-S_0^D = \mathcal{C}[V_0^D]$ and $-S_0^D \in \mathbb{F}^N$, 2) $V_0^D = \mathcal{C}[-S_0^D]$ if $V_0^D \in \mathbb{F}^N$ and 3) Dom $S_0^D = \mathbb{R}^N$ if and only if epi (V_0^D) contains no non-vertical half-lines.*

We exploit the connections between problems implied by Propositions 9 and 10 in infinite-horizon settings later in the paper. As in the discussion of properties of $-S_k$, the

[17]These complications include the explicit imposition of topological and linear structure on the infinite dimensional image space explicitly and the placing of further restrictions on the problem to ensure that multipliers are summable, i.e. lie in the sub-space of bounded additive sequences.

assumed properness of $-S_0$ in Proposition 9 is mild, while convexity and lower semi-continuity are significantly stronger. Concavity of each f_k and convexity of Graph Φ_0 is sufficient for convexity of $-S_0$. Quasiconcavity of each g_0^m and g_k^{M+k} coupled with affineness of each $g_{0,h}$ and $g_{k,n}$ is sufficient of convexity of Graph Φ_0. Weaker convex-like conditions also suffice.

Conjugacy relations for Bellman operators The various Bellman operators defined above are related by conjugacy arguments.

Proposition 11. *Let $W \in \mathbb{W}_0$, then 1) $T^V(W) = \mathcal{C}[-T^S(-\mathcal{C}[W])] \in \mathbb{W}$ and 2) $-T^{S,D}(-W) = \mathcal{C}[T^{V,D}(\mathcal{C}[W])] \in \mathbb{W}$.*

Proof. See Appendix □

Corollary 2. *1) If $-S \in \mathbb{W}$, then $V_0 = T^V(V) = \mathcal{C}[-T^S(-\mathcal{C}[V])]$; 2) $S_0 = T^S(S)$ and if $-S \in \mathbb{W}$ and $-S_0 \in \mathbb{F}^N$, then $-S_0 = \mathcal{C}[T^V(\mathcal{C}[-S])]$.*

Proof. If $-S \in \mathbb{W}$, then $-S = \mathcal{C}^2[-S] = \mathcal{C}[V]$ and $V_0 = \mathcal{C}[-S_0] = \mathcal{C}[-T^S(S)] = \mathcal{C}[-T^S(-\mathcal{C}[V])] = T^V(V)$. If $-S_0 \in \mathbb{F}^N$, then $-S_0 = \mathcal{C}^2[-S_0] = \mathcal{C}[V_0]$. Hence, by the last result, if $-S \in \mathbb{W}$, then $-S_0 = \mathcal{C}[V_0] = \mathcal{C}[T^V(V)] = \mathcal{C}[T^V(\mathcal{C}[-S])]$. □

Policies Our focus so far has been on optimal values. It is well known that even if strong duality obtains (i.e. primal and dual problems have equal optimal values and a minimizing multiplier exists for the dual problem), the set of actions that attain the supremum in the dual problem may still be a strict superset of those that attain the optimum in the primal problem. This point was made and elaborated in the context of dynamic incentive problems by Messner and Pavoni (2004). We briefly discuss additional conditions for dual problem maximizers to solve the primal problem in Appendix B.

6 Infinite horizon

The remainder of the paper extends the recursive formulations and dual relations from the previous section to dynamic incentive problems in infinite horizon settings. To keep the exposition relatively simple, the focus is on problems with time-homogenous objectives and constraints and a finite number of per period shock realizations. Time varying problems and a continuum of shock realizations may be introduced at the cost of additional notation and an explicit treatment of measure-theoretic details. In infinite horizon settings, a given problem is embedded within a family of perturbed problems. The associated primal or dual Bellman operator is then used to recover the true value function (with an appeal to strong duality in the latter case). Policies are obtained subject to the caveats elaborated above. Establishing that a value function from a problem satisfies an associated Bellman equation proceeds along the lines described above. The recovery of such value functions from the Bellman equation raises additional challenges as they may be extended real valued. Convergent value function iteration requires function convergence concepts for such settings and/or further refinement of the set of candidate value functions. We turn to these issues next.

6.1 Infinite horizon framework

A framework that accommodates many infinite horizon dynamic incentive problems is now specified. In these problems the objective is often explicitly identified as a social payoff and we label it as such. The auxiliary variables that are used to perturb constraints in the objective are identified with private agent payoffs.

6.1.1 Shocks and agents.

Let $\mathbb{I} = \{1, \ldots, I\}$ denote a finite set of agents and $\Theta = \{\widehat{\theta}_k\}_{k=1}^K \subset \mathbb{R}^S$ a finite set of shocks with $\mathbb{K} = \{1, \cdots, K\}$ the corresponding set of shock indices. In specific contracting problems these shocks may include components that affect a group of agents and/or components that are idiosyncratic to a specific agent. In the latter case, these components may be common knowledge or privately observed by the affected agent. The shock process is described by a probability space $(\Theta^\infty, \mathcal{G}, \mathbb{P})$. This process is assumed either to be Markov with transition matrix $\pi = \{\pi_{k,l}\}$ and initial seed shock $\widehat{\theta}_j \in \Theta$ or i.i.d. with per period distribution $\pi = \{\pi_l\}$. The random variables describing t-period shocks and t-period histories of shocks are denoted θ_t and θ^t respectively. The corresponding probability distribution for the latter is denoted $\mathbb{P}_j(\theta^t)$ (or just $\mathbb{P}(\theta^t)$ in the i.i.d. case).

6.1.2 Action plans.

Let $A : \Theta \twoheadrightarrow \mathbb{R}^L$ denote a correspondence mapping states to action sets. A period t action profile is a function $a_t : \Theta^t \to \mathbb{R}^L$ with $a_t(\theta^t) \in A(\theta_t)$. A *plan* is a sequence of action profiles $\alpha = \{a_t\}_{t=1}^\infty$ belonging to a set Ω. It is often convenient to express plans in the form: $\alpha = \{a_k, \alpha'_k\}_{k=1}^K$, where $a_k = a_1(\widehat{\theta}_k)$ and $\alpha'_k = \{a_{t+1}(\widehat{\theta}_k, \cdot)\}_{t=1}^\infty$. Ω is assumed to satisfy the following condition.

Assumption 1. *Ω is a non-empty subset of $\{\{a_t\}_{t=1}^{\infty},\ a_t : \Theta^t \to \mathbb{R}^N,\ a_t(\theta^t) \in A(\theta_t)\}$. If $\alpha \in \Omega$, then each continuation α'_k is also in Ω.*

6.1.3 Social and private payoffs.

Let f : Graph $A \to \mathbb{R}$ denote the per period social payoff and $\beta_P \in (0,1)$ the social discount factor. The following is assumed.

Assumption 2. *β_P, f, π and Ω are such that for all k and $\alpha \in \Omega$,*

$$\mathcal{F}_k(\alpha) := E\Big[\sum_{t=1}^{\infty} \beta_P^{t-1} f(\theta_t, a_t(\theta^t))\Big|\theta_1 = \widehat{\theta}_k\Big]$$

is well defined and finite.

$\mathcal{F}_k(\alpha)$ gives the social payoff conditional on the period 1 shock. The social objective is identified with the unconditional payoff $\sum_{k\in\mathbb{K}} \mathcal{F}_k(\alpha)\pi_{j,k}$.

Remark 2. Dynamic incentive problems can be formulated as games and incentive constraints derived as equilibrium restrictions. In many settings the social objective may be identified with that of a committed mechanism designer and equilibria that are best from her perspective found. Alternatively, with an appropriate specification of the social payoff, the entire set of equilibrium payoffs. See Example 3 below.

The private payoff to the i-th agent conditional on shock $\widehat{\theta}_k$ is defined by a tuple (r^i, β_A, π), with r^i : Graph $A \to \mathbb{R}$ and $\beta_A \in (0,1)$, according to:

$$\mathcal{R}_k^i(\alpha) = E\Big[\sum_{t=1}^{\infty} \beta_A^{t-1} r^i(\theta_t, a_t(\theta^t))\Big|\theta_1 = \widehat{\theta}_k\Big].$$

β_A is a common private agent discount factor. The following assumption is made.

Assumption 3. *$\beta_A, \{r^i\}, \pi$ and Ω are such that for all i, k and $\alpha \in \Omega$, $\mathcal{R}_k^i(\alpha)$ is well defined and finite.*

For $\alpha = \{a_k, \alpha'_k\}_{k\in\mathbb{K}}$, the definition of $\mathcal{R}_k^i$ implies:

$$\mathcal{R}_k^i(\alpha) = r^i(\widehat{\theta}_k, a_k) + \beta_A \sum_{l\in\mathbb{K}} \mathcal{R}_l^i(\alpha'_k)\pi_{k,l}.$$

6.1.4 Constraints.

Incentive constraints ensure that it is in the interests of agents to take prescribed courses of actions. These constraints are expressed in terms of private agent payoffs. Let $\mathbb{M}$ denote a finite set of constraint indices with cardinality M. Call $G : \Omega \to \mathbb{R}^M$ the *current incentive constraint mapping*, where:

$$G(\alpha) = \left\{ \sum_{k \in \mathbb{K}} g^m_{0,k}(a_k) + \beta_A \sum_{i \in \mathbb{I}} \sum_{k \in \mathbb{K}} \sum_{l \in \mathbb{K}} \mathcal{R}^i_l(\alpha'_k) q^m_{i,k,l} \right\}_{m \in \mathbb{M}}. \tag{52}$$

G is a specialized version of the constraint functions considered in Section 4. The continuation constraint functions previously denoted $g_{k,n}$ are now identified with sums of agent continuation payoffs and denoted accordingly. Explicit examples are given below.

An important special case of eq. (52) occurs when the coefficients $q^m_{i,k,l}$ can be decomposed as:

$$q^m_{i,k,l} = q^m_{i,k} \pi_{k,l} \ \ (k,l) \in \mathbb{K}^2. \tag{53}$$

Then,

$$\sum_{k \in \mathbb{K}} \sum_{l \in \mathbb{K}} q^m_{i,k,l} \mathcal{R}^i_l(\alpha'_k) = \sum_{k \in \mathbb{K}} q^m_{i,k} \sum_{l \in \mathbb{K}} \mathcal{R}^i_l(\alpha'_k) \pi_{k,l}$$

and, for each k, the relative weighting of the continuation payoffs $\{\mathcal{R}^i_l(\alpha'_k)\}_{l \in \mathbb{K}}$ coincides with the probability distribution $\{\pi_{k,l}\}_{l \in \mathbb{K}}$. As will become clear this constraint structure affords a considerable simplification of the analysis and is quite common in applications.

Let $\alpha_t(\theta^{t-1})$ denote the continuation of α after θ^{t-1}. Define the set of incentive-constrained allocations $\Omega_1 \subset \Omega$ according to:

$$\Omega_1 = \{\alpha \in \Omega | \forall t, \theta^{t-1}, G(\alpha_t(\theta^{t-1})) \geq 0\}. \tag{54}$$

In addition, let $\Omega_2 = \{\{a_k, \alpha'_k\} \in \Omega | \text{each } \alpha'_k \in \Omega_1\}$ be the set of plans that satisfy the incentive constraints from $t = 2$ onwards.

Assumption 4. Ω_1 *is non-empty.*

6.1.5 Societal choice problem.

The remainder of the paper considers choice problems of the form:

$$\sup_{\alpha \in \Omega_1} \sum_{k \in \mathbb{K}} \mathcal{F}_k(\alpha) \pi_{j,k}. \tag{55}$$

and perturbations thereof. The following is assumed.

Assumption 5. $\beta_A, f, \{r^i\}, \pi$ *and* Ω *are such that for all* j, $\sup_{\alpha \in \Omega_1} \sum_{k \in \mathbb{K}} \mathcal{F}_k(\alpha) \pi_{j,k} < \infty$.

6.2 Examples

Example 1. Atkeson-Lucas component planner with i.i.d shocks. As in Section 2, an agent receives privately observed, i.i.d. taste shocks $\{\theta_t\}$. These perturb the agent's utility from consumption $\{c_t\}$:

$$\liminf_T \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^t} \theta_t v(c_t(\theta^t)) \mathbb{P}^t(\theta^t). \tag{56}$$

In eq. (56), $\beta \in (0,1)$ is a discount factor and $v : \mathbb{R}_+ \to A \subseteq \mathbb{R}$ is a per period utility. v is assumed increasing, concave and continuous with inverse $C = A \to \mathbb{R}_+$. Attention is restricted to allocations such that $\lim_T \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^t} \theta_t v(c_t(\theta^t))\mathbb{P}^t(\theta^t)$ exists and is finite.

The planner maximizes agent utility net of resource costs:

$$\sum_{t=1}^{\infty} \beta^{t-1} \sum_{\theta^t} \left[\theta_t v(c_t(\theta^t)) - Qc_t(\theta^t)\right] \mathbb{P}^t(\theta^t) \tag{57}$$

subject to the incentive constraints, for all t, θ^{t-1}, $m_1 \neq m_2 \in \mathbb{K}$,

$$\begin{aligned} &\widehat{\theta}_{m_1} v(c_t(\theta^{t-1}, \widehat{\theta}_{m_1})) + \beta \sum_{r=1}^{\infty} \beta^{r-1} \sum_{\theta^r} \theta_r v(c_{t+r}(\theta^{t-1}, \widehat{\theta}_{m_1}, \theta^r))\mathbb{P}^r(\theta^r) \\ \geq \;\; &\widehat{\theta}_{m_1} v(c_t(\theta^{t-1}, \widehat{\theta}_{m_2})) + \beta \sum_{r=1}^{\infty} \beta^{r-1} \sum_{\theta^r} \theta_r v(c_{t+r}(\theta^{t-1}, \widehat{\theta}_{m_2}, \theta^r))\mathbb{P}^r(\theta^r). \end{aligned}$$

It is straightforward to map this model into our general framework. Define the t-th period action $a_t(\theta^t) = v(c_t(\theta^t))$ and let:

$$\Omega = \left\{ \{a_t\} \middle| \text{each } a_t(\theta^t) \in A \text{ and } \lim_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^t} \theta_t a_t(\theta^t) \mathbb{P}^t(\theta^t) \text{ exists and is finite} \right\}.$$

Let the per period social and private payoffs be given by:

$$f(\theta, a) = \theta a - QC(a) \;\text{ and }\; r(\theta, a) = \theta a.$$

Set $\beta_P = \beta_A = \beta$. The aggregators $\mathcal{F}$ and $\mathcal{R}$ are then defined in the obvious ways. Let $\mathbb{M} = \{(m_1, m_2) \in \mathbb{K}^2, m_1 \neq m_2\}$, and collect current constraints together as:

$$G(\alpha) = \left\{ \widehat{\theta}_{m_1} a_{m_1} - \widehat{\theta}_{m_1} a_{m_2} + \beta \sum_{l \in \mathbb{K}} \mathcal{R}_l(\alpha'_{m_1})\pi_l - \beta \sum_{l \in \mathbb{K}} \mathcal{R}_l(\alpha'_{m_2})\pi_l \right\}_{(m_1,m_2)\in\mathbb{M}} \geq 0. \tag{58}$$

Thus, constraints can be written in the form (52) with $g_0^{m_1,m_2}(a) = \widehat{\theta}_{m_1} a_{m_1} - \widehat{\theta}_{m_1} a_{m_2}$ and $q_{k,l}^{m_1,m_2} = q_k^{m_1,m_2}\pi_l$, where $q_k^{m_1,m_2}$ equals 1 if $k = m_1$, -1 if $k = m_2$, and 0 otherwise. □

This example can be extended to accommodate Farhi and Werning (2007)'s formulation by setting the societal discount factor β_P strictly larger than the private one β_A. The aggregator $\mathcal{F}$ incorporates the discount factor β_P, while $\mathcal{R}$ incorporates β_A. The constraint function remains the same.

Example 2. Atkeson-Lucas component planner with Markov shocks. The next example is identical to Example 1, except that shocks evolve according to a Markov process with kernel π. In this case, $q_{k,l}^{m_1,m_2}$ equals $\pi_{m_1,l}$ if $k = m_1$, $-\pi_{m_1,l}$ if $k = m_2$, and 0 otherwise. These q weights do not in general satisfy eq. (53). Thus, the relative weighting of future plans can differ across the objective and constraints. Consider the action path after some history $(\cdots, \widehat{\theta}_k, \widehat{\theta}_l)$. The agent payoff associated with this path receives a weight proportional to $\pi_{k,l}$ in the objective and the (k, m_2)-th $(m_2 \neq k)$ constraint, but a weight proportional to $-\pi_{m_1,l}$ in the (m_1, k)-th $(m_1 \neq k)$ constraint. Such variation in weighting across constraints complicates the history dependence of the optimal allocation and the recursive formulation necessary to find it. □

Example 3. Kocherlakota's model of no commitment. The second example of Section 2 can be extended to an infinite-horizon setting to give the model of Kocherlakota (1996). Everything is as before except that now agents $m_1 \in \mathbb{I}$ live for an infinite number of periods, have an outside utility option $\underline{V}^{m_1} : \Theta \to \mathbb{R}$ in each period and face the per period incentive constraints, for all t, θ^{t-1} and $m_2 \in \mathbb{K}$,

$$v(c_t^{m_1}(\theta^{t-1}, \widehat{\theta}_{m_2})) + \beta \sum_{r=1}^{\infty} \beta^{r-1} \sum_{\theta^r} v(c_t^{m_1}(\theta^{t-1}, \widehat{\theta}_{m_2}, \theta^r))\mathbb{P}(\theta^r | \widehat{\theta}_{m_2}) \geq \underline{V}^{m_1}(\widehat{\theta}_{m_2}).$$

Identifying action plans with consumption allocations and setting $\beta_A = \beta$, the current incentive constraints can be expressed as:

$$G(\alpha) = \left\{ v(a_{m_2}^{m_1}) - \underline{V}^{m_1}(\widehat{\theta}_{m_2}) + \beta \sum_{l \in \mathbb{K}} \mathcal{R}_l^{m_1}(\alpha'_{m_2})\pi_{m_2,l} \right\}_{(m_1,m_2)\in\mathbb{M}} \geq 0, \tag{59}$$

where $\mathbb{M} = \mathbb{I} \times \mathbb{K}$. Thus, constraints can be written in the form (52) with $g_0^{m_1,m_2}(a) = v(a_{m_2}^{m_1}) - \underline{V}^{m_1}(\widehat{\theta}_{m_2})$, $q_{i,k,l}^{m_1,m_2} = q_{i,k}^{m_1,m_2}\pi_{k,l}$ and $q_{i,k}^{m_1,m_2} = 1$ if $i = m_1$ and $k = m_2$ and 0 otherwise. It follows that the weights $q_{i,k,l}^{m_1,m_2}$ satisfy eq. (53).

Let $f(\theta, a) = 0$ and set $\mathcal{F}_k(\alpha) = \sum_{t=1}^{\infty} \sum_{\Theta^t} \beta_P^{t-1} f(\theta_t, a_t(\theta^t))\mathbb{P}(\theta^t | \widehat{\theta}_k) = 0$ for some $\beta_P \in (0, 1)$. Then identifying Ω_1 with the set of plans satisfying (59) one obtains the optimization: $\sup 0$ subject to $\alpha \in \Omega_1$. This is trivial if Ω_1 is (assumed) non-empty. Its constraint-perturbed variation identifies whether a given utility promise $w \in \mathbb{R}^I$ is feasible or not:

$$S_j(w) = \begin{cases} \sup_{\Omega_{1,j}(w)} \sum_{k=1}^K \mathcal{F}_k(\alpha)\pi_{j,k} & \text{if } \Omega_{1,j}(w) \neq \emptyset \\ -\infty & \text{otherwise,} \end{cases}$$

where $\Omega_{1,j}(w)$ denotes the set of plans in Ω_1 which deliver w to agents in shock state j. S_j is then simply an indicator function for the set of incentive-feasible payoffs. The value function from the objective-perturbed version of this problem gives the support function of this payoff set.[18,19] □

7 Constraint-perturbed incentive problems

This section considers incentive-constrained problems perturbed with initial utility promises to private agents.

7.1 Constraint-perturbed problem

Agents are partitioned into two groups: $\mathbb{I}_1$ and $\mathbb{I}_2$. $\mathbb{I}_1$ consists of those agents for whom each q_i^m satisfies eq. (53). $\mathbb{I}_2$ consists of the remaining agents. Let $N = |\mathbb{I}_1| + K|\mathbb{I}_2|$. Define the *constraint-perturbed problem* by:

$$S_j(w) = \begin{cases} \sup_{\alpha \in \Omega_{1,j}(w)} \sum_k \mathcal{F}_k(\alpha)\pi_{j,k} & \text{if } \Omega_{1,j}(w) \neq \emptyset \\ -\infty & \text{otherwise} \end{cases} \tag{60}$$

where $w \in \mathbb{R}^N$ and

$$\Omega_{1,j}(w) = \left\{ \alpha \in \Omega_1 \,\middle|\, w^i = \sum_{k \in \mathbb{K}} \mathcal{R}_k^i(\alpha)\pi_{j,k}, i \in \mathbb{I}_1, \text{and } w_k^i = \mathcal{R}_k^i(\alpha),\ i \in \mathbb{I}_2 \right\}. \tag{61}$$

$w \in \mathbb{R}^N$ is referred to as a promise. Agents $i \in \mathbb{I}_1$ receive ex ante promises that do not depend on the first period shock k; agents $i \in \mathbb{I}_2$ receive ex post utility promises that

[18] An alternative approach maximizes the utility of a player subject to incentive constraints and the delivery of utility promises to the other $I-1$ players. This approach incorporates the continuation value function of the maximized player into the constraint set. See Kocherlakota (1996) and Rustichini (1998).

[19] In the current example, as in Kocherlakota (1996), the outside utility option is exogenously given. Our approaches can be readily extended to accommodate the case in which these are determined endogenously as payoffs from continuation sub-game perfect equilibria. In this case, the B-operator of Abreu et al (1990) is recovered as a Bellman operator on a space of indicator functions.

do.[20] (60) augments the original societal choice problem with additional promise-keeping constraints (contained in (61)). Let Dom $S_j = \{w|S_j(w) > -\infty\} = \{w : \Omega_{1,j}(w) \neq \varnothing\}$ denote the *effective domain* of S_j and let $S = \{S_j\}_{j\in\mathbb{K}}$. We note the following immediate implication of our assumptions.

Lemma 1. *Under Assumptions 2-5, $-S \in \mathbb{W}_0$ and, for each j, Dom $S_j \neq \emptyset$.*

Proof. By Assumptions 3 and 4, there is some w such that $\alpha \in \Omega_{1,j}(w) \neq \emptyset$. By Assumption 2 and the definition of S_j, $-S_j(w) < \infty$. Hence, Dom $S_j \neq \emptyset$. By Assumption 5, $\inf_w -S_j(w) > -\infty$ and so $-S_j$ is proper. Since j was arbitrary, the result follows. □

7.2 Constraint-perturbed Bellman equations

Inspection of (60) reveals that it has essentially the same structure as the constraint perturbed problems considered in Section 5. This is easily seen by defining, for a fixed $j \in \mathbb{K}$, $A_0 = \prod_{k=1}^{\mathbb{K}} A(\widehat{\theta}_k)$, each $A_k = \Omega_1$, $\mathbb{N} = \mathbb{I}_1 \cup (\mathbb{I}_2 \times \mathbb{K})$, $f_0(a) = \sum_{k=1}^{K} f(\widehat{\theta}_k, a_k)\pi_{j,k}$, $f_k(\alpha'_k) = \sum_{l=1}^{K} \mathcal{F}_l(\alpha'_k)\pi_{k,l}$, $q_k^0 = \beta\pi_{j,k}$,

$$g_{0,h}(a) = \begin{cases} \sum_{k=1}^{K} r^i(\widehat{\theta}_k, a_k)\pi_{j,k} & h = i \in \mathbb{I}_1 \\ \\ r^i(\widehat{\theta}_k, a_k) & h = (i,k) \in \mathbb{I}_2 \times \mathbb{K} \end{cases}$$

$$g_{k,n}(\alpha'_k) = \begin{cases} \sum_{k=1}^{K} \mathcal{R}_l^i(\alpha'_k)\pi_{k,l} & n = i \in \mathbb{I}_1 \\ \\ \mathcal{R}_l^i(\alpha'_k) & n = (i,l) \in \mathbb{I}_2 \times \mathbb{K}, \end{cases}$$

$p_{k,i}^i = \pi_{j,k}$ if $i \in \mathbb{I}_1$ and $p_{k,i,l}^{i,k} = \pi_{k,l}$ if $i \in \mathbb{I}_2$. The objective, promise and incentive-constraints may be re-expressed in terms of these functions and weights. Pursuing exactly the line of argument in Section 5, let

$$\Psi_j(w) = \left\{ (a, w') \;\middle|\; \begin{array}{c} w_i = \sum_{\mathbb{K}} [r^i(\widehat{\theta}_k, a_k) + \beta_A w'_{i,k}]\pi_{j,k}, \ i \in \mathbb{I}_1 \\ w_{i,k} = r^i(\widehat{\theta}_k, a_k) + \beta_A \sum_{\mathbb{K}} w'_{i,k,l}\pi_{k,l}, \ i \in \mathbb{I}_2, k \in \mathbb{K} \\ g_0^m(a) + \beta_A \sum_{\mathbb{I}_1,\mathbb{K}} q_{i,k}^m w'_{i,k} + \beta_A \sum_{\mathbb{I}_2,\mathbb{K},\mathbb{K}} q_{i,k,l}^m w'_{i,k,l} \geq 0, \quad m \in \mathbb{M} \end{array} \right\}.$$

Then as a corollary to Proposition 6, we obtain the following result.

[20]In principle any agent could receive an ex ante or ex post utility promise. But the subsequent recursive formulation relies on ex ante promises for $\mathbb{I}_1$ agents and ex post promises for $\mathbb{I}_2$.

Corollary 3. *Define* $T^S : -\mathbb{W}_0 \to -\mathbb{W}_0$ *by* $T^S = \{T_j^S\}_{j\in\mathbb{K}}$ *with for all* $W \in -\mathbb{W}_0$*,*

$$T_j^S(W)(w) = \begin{cases} \sup_{(a,w')\in\Psi_j(w)} \sum_{k\in\mathbb{K}} \left\{ f(\widehat{\theta}_k, a_k) + \beta W_k(w'_k) \right\} \pi_{j,k} & \Psi_j(w) \neq \emptyset \\ -\infty & \textit{otherwise.} \end{cases}$$

Then: $S = T^S(S)$*.*

The first period incentive constraints in (60) may be dualized to give:

$$\begin{aligned} S_j^D(w) = \inf_{\mathbb{R}^M\times\mathbb{R}^N} \sup_{A\times\mathbb{R}^{KN}} \sum_{k\in\mathbb{K}} & \left\{ f(\widehat{\theta}_k, a_k) + \beta_P S_k(w'_k) \right\} \pi_{j,k} \\ & + \sum_{i\in\mathbb{I}_1} \zeta^i \left[\sum_{k\in\mathbb{K}} [r^i(\widehat{\theta}_k, a_k) + \beta_A w'_{i,k}]\pi_{j,k} - w^i \right] \\ & + \sum_{i\in\mathbb{I}_2} \sum_{k\in\mathbb{K}} \zeta_k^i \left[r^i(\widehat{\theta}_k, a_k) + \beta_A \sum_{l\in\mathbb{K}} w'_{i,k,l}\pi_{k,l} - w_k^i \right] \pi_{j,k} \\ & + \sum_m \eta_m \left[g_0^m(a) + \beta_A \sum_{\mathbb{I}_1,\mathbb{K}} q_{i,k}^m w'_{i,k} + \beta_A \sum_{\mathbb{I}_2,\mathbb{K}} q_{i,k,l}^m w'_{i,k,l} \right]. \end{aligned} \tag{62}$$

Then as a corollary to the discussion following Proposition 6 we obtain the following result.

Corollary 4. *Define* $T^{S,D} : -\mathbb{W}_0 \to -\mathbb{W}_0$ *by* $T^{S,D} = \{T_j^{S,D}\}_{j\in\mathbb{K}}$ *with for all* $W \in -\mathbb{W}_0$*,*

$$\begin{aligned} T_j^{S,D}(W)(w) = \inf_{\mathbb{R}^M\times\mathbb{R}^N} \sup_{A\times\mathbb{R}^{KN}} \sum_{k\in\mathbb{K}} & \left\{ f(\widehat{\theta}_k, a_k) + \beta_P W_k(w'_k) \right\} \pi_{j,k} \\ & + \sum_{i\in\mathbb{I}_1} \zeta^i \left[\sum_{k\in\mathbb{K}} [r^i(\widehat{\theta}_k, a_k) + \beta_A w'_{i,k}]\pi_{j,k} - w^i \right] \\ & + \sum_{i\in\mathbb{I}_2} \sum_{k\in\mathbb{K}} \zeta_k^i \left[r^i(\widehat{\theta}_k, a_k) + \beta_A \sum_{l\in\mathbb{K}} w'_{i,k,l}\pi_{k,l} - w_k^i \right] \pi_{j,k} \\ & + \sum_m \eta_m \left[g_0^m(a) + \beta_A \sum_{\mathbb{I}_1,\mathbb{K}} q_{i,k}^m w'_{i,k} + \beta_A \sum_{\mathbb{I}_2,\mathbb{K}} q_{i,k,l}^m w'_{i,k,l} \right]. \end{aligned} \tag{63}$$

Then: $S^D = T^{S,D}(S)$*.*

8 Objective-perturbed incentive problems

This section considers dynamic incentive problems whose objectives are perturbed with weighted utilities of private agents.

8.1 Objective-perturbed problem

Define the *objective-perturbed problem*:

$$V_j(\zeta) = \sup_{\alpha \in \Omega_1} \sum_{k \in \mathbb{K}} \mathcal{F}_k(\alpha)\pi_{j,k} + \sum_{i \in \mathbb{I}_1} \zeta^i \sum_{l \in \mathbb{K}} \mathcal{R}^i_k(\alpha)\pi_{j,k} + \sum_{i \in \mathbb{I}_2} \sum_{k \in \mathbb{K}} \zeta^i_k \mathcal{R}^i_k(\alpha)\pi_{j,k} \tag{64}$$

and let $V = \{V_j\}_{j \in \mathbb{K}}$. The problem in eq. (64) perturbs the objective from the date t societal problem (55) with weighted sums of private payoffs $\mathcal{R}^i_k(\alpha)$. The weights are collected into $\zeta \in \mathbb{R}^N$. For agents $i \in \mathbb{I}_2$, these weights are allowed to depend on the current shock $\widehat{\theta}_k$. Assumptions 1-4 ensure that Ω_1 is non-empty and that the objective in eq. (64) is well defined and real-valued on Ω_1. Hence, $V_j : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$ is also well defined, though possibly infinite-valued. Let Dom $V_j = \{\zeta \in \mathbb{R}^N \mid V_j(\zeta) < \infty\} \subseteq \mathbb{R}^N$ denote the *effective domain* of V_j and Dom $V = \{(\zeta, j) \in \mathbb{R}^N \times \mathbb{K} | V_j(\zeta) < \infty\}$ the effective domain of V.

8.2 Objective-perturbed Bellman equations

Problem (64) has the same structure as the objective-perturbed problems considered in Section 5. The following result is derived as a corollary to Proposition 7.

Corollary 5. *Define the operator $T^V : \mathbb{W}_0 \to \mathbb{W}_0$ by $T^V = \{T^V_j\}_{j \in \mathbb{K}}$ with for all $W \in \mathbb{W}_0$,*

$$T^V_j(W)(\zeta) = \sup_{A \times \mathbb{R}^{KN}} \inf_{\mathbb{R}^M_+ \times \mathbb{R}^{KN}} \Big\{ f(\widehat{\theta}_k, \widehat{u}_k) + \sum_{m \in \mathbb{M}} \frac{\eta_m}{\pi_{j,k}} g^m_{0,k}(\widehat{a}_k) + \sum_{i \in \mathbb{I}_1} \zeta^i r^i(\widehat{\theta}_k, \widehat{a}_k) + \sum_{i \in \mathbb{I}_2} \zeta^i_k r^i(\widehat{\theta}_k, \widehat{a}_k)$$
$$+\beta_P \Big(W(z_k) + \frac{\beta_A}{\beta_P} \sum_{k \in \mathbb{K}} \langle \zeta'_{j,k}(\zeta, \eta) - z_k, w_k \rangle \Big) \Big\} \pi_{j,k}, \tag{65}$$

where $\langle x, y \rangle = \sum_{i \in \mathbb{I}_1} x_i y_i + \sum_{i \in \mathbb{I}_2} \sum_{l \in \mathbb{K}} x_{i,l} y_{i,l} \pi_{k,l}$, $w_k = \{\{w^i_k\}_{\mathbb{I}_1}, \{w^i_{k,l}\}_{\mathbb{I}_2 \times \mathbb{K}}\}$ and

$$\zeta^{i\prime}_{j,k}(\zeta, \eta) = \frac{\beta_A}{\beta_P} \Big[\zeta^i + \sum_{m \in \mathbb{M}} \frac{\eta_m q^m_{i,k}}{\pi_{j,k}} \Big], \ i \in \mathbb{I}_1, j \text{ and } k \in \mathbb{K}, \tag{66}$$
$$\zeta^{i\prime}_{j,k,l}(\zeta, \eta) = \frac{\beta_A}{\beta_P} \Big[\zeta^i_k + \sum_{m \in \mathbb{M}} \frac{\eta_m q^m_{i,k,l}}{\pi_{j,k}\pi_{k,l}} \Big], \ i \in \mathbb{I}_2, j, k \text{ and } l \in \mathbb{K}.$$

Then if $-S \in \mathbb{W}$, $V = T^V(V)$.

Alternatively, dualizing the first period recursive incentive and promise-keeping constraints, we obtain:

$$V_j^D(\zeta) = \inf_{\mathbb{R}_+^M \times \mathbb{R}^{KN}} \sup_{A \times \mathbb{R}^{KN}} \sum_{k \in \mathbb{K}} \Big\{ f(\widehat{\theta}_k, \widehat{a}_k) + \sum_{m \in \mathbb{M}} \frac{\eta_m}{\pi_{j,k}} g_{0,k}^m(\widehat{a}_k) + \sum_{i \in \mathbb{I}_1} \zeta^i r^i(\widehat{\theta}_k, \widehat{a}_k) + \sum_{i \in \mathbb{I}_2} \zeta_k^i r^i(\widehat{\theta}_k, \widehat{a}_k)$$
$$+ \beta_P \Big(V(z_k) + \frac{\beta_A}{\beta_P} \sum_{k \in \mathbb{K}} \langle \zeta'_{j,k}(\zeta, \eta) - z_k, w_k \rangle \Big) \Big\} \pi_{j,k}. \tag{67}$$

Then, as a corollary to Proposition 8 we have the following result.

Corollary 6. *Define the operator $T^{V,D} : \mathbb{W}_0 \to \mathbb{W}_0$ by $T^{V,D} = \{T_j^{V,D}\}_{j \in \mathbb{K}}$ with*

$$T_j^{V,D}(W)(\zeta) = \inf_{\mathbb{R}_+^M} \sup_{A} \sum_{k \in \mathbb{K}} \Big\{ f(\widehat{\theta}_k, \widehat{a}_k) + \sum_{m \in \mathbb{M}} \frac{\eta_m}{\pi_{j,k}} g_{0,k}^m(\widehat{a}_k) + \sum_{i \in \mathbb{I}_1} \zeta^i r^i(\widehat{\theta}_k, \widehat{a}_k)$$
$$+ \sum_{i \in \mathbb{I}_2} \zeta_k^i r^i(\widehat{\theta}_k, \widehat{a}_k) + \beta_P W(\zeta'_{j,k}(\zeta, \eta)) \Big\} \pi_{j,k}, \tag{68}$$

then $V^D = T^{V,D}(V)$.

9 Duality relations for incentive problems

9.1 Duality of value functions and state spaces

Duality results from Section 4 are applicable. The following propositions are immediate consequences of Lemma 1 and Propositions 9 to 10.

Proposition 12. *Given Assumptions 2-5, 1) $V = \mathcal{C}[-S]$ and $V \in \mathbb{W}$; 2) $-S = \mathcal{C}[V]$ if $-S \in \mathbb{W}$ and 3) each Dom V_j is convex and non-empty. Also, Dom $V_j = \mathbb{R}^N$ if and only if epi $(-S_j)$ contains no non-vertical half-lines. A sufficient condition for the latter is that each $\mathcal{F}_k$ is bounded above and each $\mathcal{R}_k^i$ is bounded.*

Proposition 13. *Assume that $V^D \in \mathbb{W}_0$, then 1) $-S^D = \mathcal{C}[V^D]$ and $-S^D \in \mathbb{W}$; 2) $V^D = \mathcal{C}[-S^D]$ if $V^D \in \mathbb{W}$ and 3) each Dom S_j^D is convex and non-empty. Also, Dom $S_j^D = \mathbb{R}^N$ if and only if epi (V_j^D) contains no non-vertical half-lines.*

Earlier remarks on functions and their conjugates are applicable here. V and $-S^D$ are convex and lower semicontinuous (given the properness of $-S$ and V^D). Other properties are inherited via conjugacy arguments. In particular, conditions for strict convexity and differentiability of $-S_j$ have been established in several settings[21] and so, under these

[21]For example, Atkeson and Lucas (1992), Atkeson and Lucas (1995), Farhi and Werning (2007).

conditions, V_j is strictly convex and continuously differentiability on the interior of its domain as well.

In summary, each value function (and effective domain) can be characterized directly or via the conjugacy properties of the other. In a given application, the most convenient route may be chosen.

9.2 Conjugacy of T^V and T^S.

Proposition 11 has immediate application to dynamic incentive problems. It implies that, for $W \in \mathbb{W}_0$,

$$T^V(W) = \mathcal{C}[-T^S(-\mathcal{C}[W])] \text{ and } T^{S,D}(W) = -\mathcal{C}[T^{V,D}(\mathcal{C}[-W])]. \tag{69}$$

If $T^S : \mathbb{W} \to \mathbb{W}$, then, for $W \in \mathbb{W}$, $-T^S(-W) = \mathcal{C}[T^V(\mathcal{C}[W])]$ and the Bellman operators T^S and T^V are conjugate in the sense that (modulo sign changes) applying the Legendre-Fenchel transform, then one of the operators and then the transform again is equivalent to applying the other operator. If, in addition, $T^{V,D} = T^V$ on $\mathbb{W}$, then $T^{V,D}$ and T^S are conjugate as well.

9.3 Conjugacy in value iteration

Ultimately our goal in deriving Bellman equations is to use them to solve problems. This section begins our discussion of this by relating value iteration under the various Bellman operators introduced in the preceding discussion. The corollary is an immediate implication of eq. (69) and the above discussion.

Corollary 7. *Suppose that $-S_0 \in \mathbb{W}$ and for $r = 0, 1, \ldots$, $-S_{r+1} = -T^S S_r \in \mathbb{W}$. Then for $r = 0, 1, \ldots$, $\mathcal{C}[-S_{r+1}] = T^V \mathcal{C}[-S_r]$. If, in addition, $T^{V,D} = T^V$ on $\mathbb{W}$, then for $r = 0, 1, \ldots$, $\mathcal{C}[-S_{r+1}] = T^{V,D}\mathcal{C}[-S_r]$. Suppose that $V_0 \in \mathbb{W}$ and for $r = 0, 1, \ldots$, $V_{r+1} = T^{V,D} V_r \in \mathbb{W}$. Then for $r = 0, 1, \ldots$, $-\mathcal{C}[V_{r+1}] = T^{S,D}(-\mathcal{C}[S_r])$. If, in addition, $T^S = T^{S,D}$ on $\mathbb{W}$, then for $r = 0, 1, \ldots$, $-\mathcal{C}[V_{r+1}] = T^S(-\mathcal{C}[V_r])$.*

Proof. If $-S_r \in \mathbb{W}$, then $T^V(\mathcal{C}[-S_r]) = \mathcal{C}[-T^S(-\mathcal{C}^2[-S_r])] = \mathcal{C}[-T^S(S_r)] = \mathcal{C}[-S_{r+1}]$. The first part of the result then follows by induction from $r = 0$. The second part of the result is immediate. The remainder follows by an analogous argument. □

Consequently, if repeated application of T^S to S_0 induces a sequence of functions whose negatives are in $\mathbb{W}$, then repeated application of T^V to $\mathcal{C}[-S_0]$ induces the corresponding sequence of conjugates. If $T^{V,D} = T^V$ on $\mathbb{W}$, then of, course, the same sequence of conjugates is generated by repeated application of $T^{V,D}$. Similar relations hold with respect to iteration on $T^{V,D}$ from V_0 and $T^{S,D}$ from $-\mathcal{C}[V_0]$.

We are led to consider whether convergent-T^S (resp. -$T^{V,D}$) value iteration implies convergent-T^V (resp. -$T^{S,D}$) value iteration.

Definition 2. A sequence of functions $\{f^r\}_{r=1}^{\infty}$, $f^r : \mathbb{R}^N \to \overline{\mathbb{R}}$ is said to *epiconverge* to $f : \mathbb{R}^N \to \overline{\mathbb{R}}$ written $f^r \xrightarrow{e} f$ if for any x, 1) $\forall x^r \to x$, $\liminf_r f^r(x^r) \geq f(x)$ and 2) $\exists x^r \to x$, $\limsup_r f^r(x^r) \leq f(x)$. f is called the *epi-limit* of $\{f^r\}_{r=1}^{\infty}$.

Extending this definition slightly, we will say that a sequence $\{W_r\}$ in $\mathbb{W}_0$ or $\mathbb{W}$ epiconverges to W (in $\mathbb{W}_0$ or $\mathbb{W}$) if each $W_{r,k} \xrightarrow{e} W_k$. We will write $W_r \xrightarrow{e} W$. Epiconvergent sequences of functions have several desirable properties. In particular, the limit of a sequence of minimizers for an epiconvergent function sequence is the minimizer of the epi-limit.[22] Epiconvergence also relates well to the conjugation operation. Specifically, by a well known theorem of Wijsman, the Fenchel transform is epi-continuous on $\mathbb{W}$, i.e. if the functions W_r and W belong to $\mathbb{W}$, then $W_r \xrightarrow{e} W \iff \mathcal{C}[W_r] \xrightarrow{e} \mathcal{C}[W]$. This result has the following implication for our setting.

Proposition 14. *Let $-S_0 \in \mathbb{W}$ and for $r = 0, 1, \ldots$, $S_{r+1} = T^S(S_r)$. Suppose that each $-S_r \in \mathbb{W}$ and that $-S_r \xrightarrow{e} -S_\infty$. Let $V_r = \mathcal{C}[-S_r]$. Then $V_{r+1} = T^V(V_r)$ and $V_r \xrightarrow{e} V_\infty = \mathcal{C}[-S_\infty]$.*

Let $V_0 \in \mathbb{W}$ and for $r = 0, 1, \ldots$ $V_{r+1} = T^{V,D}(V_r)$. Suppose that each $V_r \in \mathbb{W}$ and that $V_r \xrightarrow{e} V_\infty$. Let $S_r = -\mathcal{C}[V_r]$. Then $S_{r+1} = T^{S,D} S_r$ and $-S_r \xrightarrow{e} -S_\infty = \mathcal{C}[V_\infty]$.

In summary, epiconvergent T^S (resp. $T^{V,D}$) value iteration from some $W_0 \in \mathbb{W}$, implies epiconvergent T^V (resp. $T^{S,D}$) value iteration from $\mathcal{C}[-W_0]$ (resp. $-\mathcal{C}[W_0]$). Modulo sign changes, the limit of the latter iteration is the conjugate of the limit of the former.

Contractive relations Recall that the Fenchel transform is a one-to-one mapping on $\mathbb{W}$ with $\mathcal{C}[W] = \mathcal{C}^{-1}[W]$, $W \in \mathbb{W}$. Let (D, ρ) with $D \subseteq \mathbb{W}$ be a metric space. Then $(\mathcal{C}(D), \phi)$ with $\phi(W', W'') = \rho(\mathcal{C}^{-1}(W'), \mathcal{C}^{-1}(W'')) = \rho(\mathcal{C}(W'), \mathcal{C}(W''))$ is isometric to (D, ρ). In particular, if (D, ρ) is complete, then so is $(\mathcal{C}(D), \phi)$. Suppose that $S \in D$, (D, ρ) is complete, $T^S : D \to D$ and T^S is contractive on (D, ρ). Then, by the contraction mapping theorem, S is the unique fixed point of T^S on D and for any sequence $\{S_r\}$ with $S_0 \in D$ and $S_{r+1} = T^S S_r$, $S_r \xrightarrow{\rho} S$. By Corollary 7, the sequence of functions $V_{r+1} = T^V V_r$, with $V_0 = \mathcal{C}[-S_0]$, corresponds to the sequence of conjugates of $\{S_r\}$. Hence, $\phi(V_r, \mathcal{C}[-S]) = \rho(S_r, S)$ and so iterative application of T^V to $\mathcal{C}[-S_0]$ induces a

[22] For this reason, epiconvergence is more useful than pointwise convergence which relates poorly to maximization and minimization. On the other hand, uniform convergence (or weighted norm convergence) which relates well is stronger and more restrictive.

sequence that ϕ-converges to $\mathcal{C}[-S]$. If $T^{V,D} = T^V$ on $\mathcal{C}[D]$, then iterative application of $T^{V,D}$ to $\mathcal{C}[-S_0]$ converges to $\mathcal{C}[-S]$.

10 Dynamic programming

The previous section related convergent value iteration across cases. This section gives sufficient conditions for such convergence. Section 10.1 considers T^S-value iteration and provides sufficient conditions for it to generate an epiconvergent sequence (with limit $-S$) on $\mathbb{W}$. Epiconvergence of T^V-value iteration (to V) then follows from the results in the preceding section. The key condition is the existence of a bounding function S_0 for S that has bounded level sets and that becomes arbitrarily negative as expected future utilities become arbitrarily large. This condition subsumes the stronger requirement that agents have bounded utilities used in, for example, Abreu et al (1990).

Epiconvergence is well suited to handling the convergence of possibly extended real-valued functions. However, when the effective domain of the relevant value function is known, it is sometimes possible to refine the set of candidate value functions so that the Bellman operator is a contractive upon it. In particular, we have previously noted that the effective domain of the objective-perturbed value function is all of $\mathbb{R}^N$ when the social objective is bounded above and agent utilities are bounded. Section 10.2 develops contractive arguments for this case.

10.1 Level bounded problems

Let $\overline{\mathbb{W}} = \{W = \{W_k\} | W_k : \mathbb{R}^N \to \overline{\mathbb{R}}\}$ denote the set of extended real valued functions with domain $\mathbb{K} \times \mathbb{R}^N$. For W and $W' \in \overline{\mathbb{W}}$, let $W \geq W'$ denote for all k, w, $W_k(w) \geq W'_k(w)$. For $W \in \overline{\mathbb{W}}$, define $U(W) = \{U_j(W)\}_{j \in \mathbb{K}}$, where:

$$U_j(W)(w, a, w') := \begin{cases} \sum_{k \in \mathbb{K}} \left\{ f(\widehat{\theta}_k, a_k) + \beta_P W_k(w'_k) \right\} \pi_{j,k} & \text{if } \Psi_j(w) \neq \emptyset \\ -\infty & \text{otherwise.} \end{cases}$$

Given $u \in \overline{\mathbb{R}}$, $\text{lev}_u U_j(W)(w, \cdot) := \{(a, w') | U_j(W)(w, a, w') \geq u\}$ is called the u-level set of $U_j(W)(w, \cdot)$. $U_j(W)(w, \cdot)$ is said to be *level bounded* if for every $u \in \mathbb{R}$ the set $\text{lev}_u U_j(W)(w, \cdot)$ is bounded (and possibly empty). $U_j(W)$ is said to be *level bounded locally in w* if for each $\widehat{w} \in \mathbb{R}^N$ and $u \in \mathbb{R}$ there is a neighbourhood V of $\widehat{w}$ and a bounded set B such that $\text{lev}_u U_j(W)(w, \cdot) \subset B$ for each $w \in V$. If each $U_j(W)$, $j \in \mathbb{K}$ is level bounded locally in w, then $U(W)$ is said to be locally level bounded in w. Clearly, if each $f(\widehat{\theta}_k, \cdot)$

and W_k are level bounded, then $U_j(W)$ is level bounded locally in w.

Lemma 2. *Let $W \geq W' \in \overline{\mathbb{W}}$. If $U(W)$ is level bounded locally in w, then $U(W')$ is level bounded locally in w.*

Proof. See Appendix. □

Lemma 3. *Assume that $U_j(W)$ 1) belongs to $-\mathbb{W}_0$, 2) is upper semicontinuous and 3) is level-bounded locally in w. Then $T_j^S(W)$ belongs to $-\mathbb{W}_0$ and is upper semicontinuous. In addition, if $T_j^S(W)(w) > -\infty$, then $\Gamma_j(w) = \arg\min U_j(W)(w, \cdot)$ is non-empty and compact, otherwise it is empty. If $(a^v, w'^v) \in \Gamma_j(w^v)$, $w^v \to w$ and $T_j^S(w^v) \to T_j^S(w) > -\infty$, then (a^v, w'^v) is bounded and its cluster points lie in $\Gamma_j(w)$.*

Proof. See Appendix. □

The following assumption is placed on primitives in the remainder of the section.

Assumption 6. *Each function $f(\widehat{\theta}_k, \cdot)$ is upper semicontinuous and bounded above on $A(\widehat{\theta}_k)$.*

Remark 3. Much of the dynamic incentive literature assumes continuity of the social objective functions $f(\widehat{\theta}_k, \cdot)$. It is also standard to identify the latter with a bounded sum of agent utilities or a sum of agent utilities net of a resource costs that is bounded above. All of the previously described examples satisfy Assumption 6.

Theorem 8. *Let Assumption 6 holds and let $S_0 \in -\mathbb{W}_0$ be upper semicontinuous. Suppose that 1) $U_j(S_0)$ is level bounded locally in w, 2) $S_0 \geq T^S(S_0)$, and 3) $S_0 \geq S$. For $n = 0, 1, 2, \ldots$, let $S_{n+1} = T^S S_n$. Then the sequence $\{S_n\}$ has a pointwise limit $S_\infty \geq S$ with $S_\infty = T^S S_\infty$.*

Proof. See Appendix. □

Given $W \in \overline{\mathbb{W}}$, let $W(\widehat{\theta}_k, w) := W_k(w)$. We will call a function $W \in \overline{\mathbb{W}}$ *coercive in expectation* if for all $j \in \mathbb{K}$ and $\{w_{t+1} : \Theta^t \to \mathbb{R}^N\}$ such that $\lim \beta_A^t E[\|w_{t+1}\| | \theta_0 = \widehat{\theta}_j] \neq 0$, $\beta_P^t E[W(\theta_t, w_{t+1}(\theta^t)) | \theta_0 = \widehat{\theta}_j] \to \infty$.

Remark 4. Coercivity in expectation is satisfied if each W_k is finite on a bounded set and is otherwise ∞. The negative of the true value function S satisfies this condition when agent utilities are bounded. Alternatively, coercivity in expectation is satisfied if each W_k is proper, lower semicontinuous, level bounded and convex (in which case it is level coercive, i.e. $\liminf_{\|w\|} W_k(w)/\|w\| > 0$, see Rockafellar and Wets (1998), p. 92).

Theorem 9. *Let the assumptions of Theorem 8 hold. Assume additionally that $-S_0 \in \mathbb{W}_0$ is coercive in expectation. For $n = 0, 1, \ldots$, let $S_{n+1} = T^S S_n$. Then the sequence of functions of $\{S_n\}$ epiconverges to S.*

Proof. See Appendix. □

We say that the problems defined by $T^S W$ are *concave-like* if for each j, $(w^0, a^0, w^{0\prime})$, $(w^1, a^1, w^{1\prime})$ and $\lambda \in [0,1]$, there is a $(a^\lambda, w^{\lambda\prime})$ such that $U_j(W)(\lambda w^0 + (1-\lambda)w^1, a^\lambda, w^{\lambda\prime}) \geq \lambda\, U_j(W)(w^0, a^0, w^{0\prime}) + (1-\lambda)\, U_j(W)(w, a^1, w^{1\prime})$. Note that concave-likeness does not require that a^λ and $w^{\lambda\prime}$ are convex combinations of a^0 and a^1 and $w^{0\prime}$ and $w^{1\prime}$. It is easy to see that if the problems defined by $T^S(W)$ are concave-like, then $T^S(W)$ is concave.

Assumption 7. *If $-W \in \mathbb{W}$, then the family of problems defined by $T^S(W)$ are concave-like.*

The following result is then immediate.

Theorem 10. *Under Assumption 7 and the assumptions of Theorem 9, the sequence of functions $\{-S_n\}$ epiconverges to $-S$ and $\{S_n\} \cup \{S\} \subset \mathbb{W}$.*

Proof. Follows from preceding discussion, Theorem 9 and the fact that set of convex functions is closed under pointwise convergence. □

Hence, from our earlier results, under the conditions of Theorem 10 the sequence $\{V_n\}$ with $V_0 = \mathcal{C}[-S_0]$ and, for $n = 0, 1, \ldots$, $V_{n+1} = T^V V_n$ epiconverges to $\mathcal{C}[-S]$. Moreover, following the argument from Section 4, $V = \mathcal{C}[-S]$ is the true objective perturbed value function. Also, if $T^{V,D} = T^V$, then V may be obtained by iterating on $T^{V,D}$ from $\mathcal{C}[-S]$.

10.2 Contraction mapping based approaches (Preliminary)

It is sometimes possible to refine the set of candidate value functions so that the Bellman operator is contractive upon it. Sharper characterizations of the true value function and rates of convergence for value iteration are then available.

As had been emphasized, the optimal value functions from dynamic incentive problems are often unbounded and extended real valued. A further difficulty concerns the fact that the conventional theorem of the maximum is often not available because the constraint correspondence is not compact-valued or upper hemicontinuous. In this section tighter restrictions are made on the problem to alleviate these difficulties.

The following assumptions are made.

Assumption 8. *(Bounds) The functions $r^i(\widehat{\theta}_k, \cdot)$ have finite infima and suprema given by $\underline{r}^i_k$ and $\overline{r}^i_k$. For each $\widehat{\theta}_k$, f is bounded above on Graph A.*

Assumption 9. *(Convexity) The problem (64) is convex-like with respect to perturbations of the incentive constraints, for example $f(\theta, \cdot)$ is concave for all θ, $r^i(\theta, \cdot)$ is affine and G is concave.*

Assumption 10. *(Slater) There is an $\alpha \in \Omega$ such that $G(\alpha) > 0$.*

Assumption 8 ensures that each $V_j : \mathbb{R}^N \to \mathbb{R}$ is convex and continuous. Assumptions 9 and 10 guarantee that $V = T^{V,D}(V)$. Thus, use can be made of the objective-perturbed, dual Bellman operator $T^{V,D}$. The convenient feature of this operator is that it can be written as $T^{V,D} = \{T_j^{V,D}\}_{j \in \mathbb{K}}$,

$$T_j^{V,D}(W)(\zeta) = \inf_{\mathbb{R}_+^M} \sum_{k \in \mathbb{K}} \left\{ J_k(\zeta, \eta) + \beta_P W(\zeta'_{j,k}(\zeta, \eta)) \right\} \pi_{j,k}, \tag{70}$$

where:

$$J_k(\zeta, \eta) = \sup_{A(\widehat{\theta}_k)} \left\{ f(\widehat{\theta}_k, \widehat{a}_k) + \sum_{m \in \mathbb{M}} \frac{\eta_m}{\pi_{j,k}} g_{0,k}^m(\widehat{a}_k) + \sum_{i \in \mathbb{I}_1} \zeta^i r^i(\widehat{\theta}_k, \widehat{a}_k) + \sum_{i \in \mathbb{I}_2} \zeta_k^i r^i(\widehat{\theta}_k, \widehat{a}_k) \right\}.$$

Consequently, the "supremum" operation is cleanly distributed across periods and the dynamic link is purely through the weight ζ' and the multiplier choice η. If $W = \{W_j\}$ with each W_j convex, real-valued and, hence, continuous on $\mathbb{R}^N$, then $T_j^{V,D}(W)$ is easily shown to be convex, real-valued and continuous on $\mathbb{R}^N$ also.

A remaining difficulty is that the function V is unbounded, even if real-valued. One approach to dealing with this is to enlarge the space of candidate value functions to include those that are norm-bounded with respect to a weighting function and then to show that the growth of optimal multipliers is bounded with respect to this norm. This is the approach taken by Marcet and Marimon (2011). An alternative approach is to consider candidate value functions contained between two bounding functions. If the bounds are tight enough, then the relevant space of candidate value functions may a complete metric space with respect to the usual sup-metric. We sketch this approach below.[23]

Assume the existence of constraint functions $\underline{G}$ and $\overline{G}$ such that:

$$\{\alpha \in \Omega | \underline{G}(\alpha) \geq 0\} \subset \{\alpha \in \Omega | G(\alpha) \geq 0\} \subset \{\alpha \in \Omega | \overline{G}(\alpha) \geq 0\}.$$

Let $\Omega(\alpha)$ denote all continuations of a given plan α and let

$$\underline{\Omega} = \{\alpha \in \Omega | \forall \alpha' \in \Omega(\alpha), \underline{G}(\alpha') \geq 0\}.$$

$\overline{\Omega}$ is defined analogously from $\overline{G}$. Then $\underline{\Omega} \subset \Omega_1 \subset \overline{\Omega}$. For each j, define the corresponding

[23] The two approaches may be combined, see Sleet and Yeltekin (2010a). In addition, Sleet and Yeltekin (2010a) show that Marcet and Marimon's approach is equivalent to restricting attention to candidate value functions within $\{W : \mathbb{R}^N \to \mathbb{R}, \sup_{\zeta \in \mathbb{R}^N} |W(\zeta)| / \|(1, \zeta)\| < \infty\}$ with metric induced by the weight function $\tau(\zeta) = \|(1, \zeta)\|$. Sleet and Yeltekin (2010a) use more general and flexible weight functions.

more and *less constrained problems*:

$$\underline{V}_j(\zeta) = \sup_{\underline{\Omega}} \sum_{k\in\mathbb{K}} \mathcal{F}_k(\alpha)\pi_{j,k} + \sum_{i\in\mathbb{I}_1} \zeta^i \sum_{k\in\mathbb{K}} \mathcal{R}^i_k(\alpha)\pi_{j,k} + \sum_{i\in\mathbb{I}_2}\sum_{k\in\mathbb{K}} \zeta^i_k \mathcal{R}^i_k(\alpha)\pi_{j,k}$$
$$\overline{V}_j(\zeta) = \sup_{\overline{\Omega}} \sum_{k\in\mathbb{K}} \mathcal{F}_k(\alpha)\pi_{j,k} + \sum_{i\in\mathbb{I}_1} \zeta^i \sum_{k\in\mathbb{K}} \mathcal{R}^i_k(\alpha)\pi_{j,k} + \sum_{i\in\mathbb{I}_2}\sum_{k\in\mathbb{K}} \zeta^i_k \mathcal{R}^i_k(\alpha)\pi_{j,k}.$$

Suppose that $\underline{V}_j$ and $\overline{V}_j$ are convex and real-valued on $\mathbb{R}^N$. Define:

$$\mathbb{V} := \{W = \{W_k\} | \text{each } W_k : \mathbb{R}^N \to \mathbb{R} \text{ is convex and } \underline{V} \leq W \leq \overline{V}\}.$$

It is evident that $V \in \mathbb{V}$. Also,

$$\underline{V}_j \leq T^{V,D}_j(\underline{V}) \leq T^{V,D}_j(V) \leq T^{V,D}_j(\overline{V}) \leq \overline{V}_j, \tag{71}$$

(see Lemma A1 in the Appendix for the proof), so that given the preceding discussion: $T^{V,D} : \mathbb{V} \to \mathbb{V}$.

Assumption 11. *There is an* $L \in \mathbb{R}_+$ *such that:* $\sup_{(j,\zeta)\in Z} \overline{V}_j(\zeta) - \underline{V}_j(\zeta) < L$.

Given a pair of bounding problems with convex value functions satisfying Assumption 11, define $\mathbb{V}$ as above and let $d : \mathbb{V} \times \mathbb{V} \to \mathbb{R}_+$ be such that $d(W, W') = \sup_Z |W_j(\zeta) - W'_j(\zeta)|$, $W, W' \in \mathbb{V}$. $(\mathbb{V}, d)$ is then a complete metric space. A mapping $T : \mathbb{V} \to \mathbb{V}$ is contractive if for any W and W' in $\mathbb{V}$, $d(T(W), T(W')) \leq \delta d(W, W')$, $\delta \in (0, 1)$. The following result is now routine.

Proposition 15. *Assume the existence of a pair of more and less constrained problems with convex value functions satisfying Assumption 11. Define* $\mathbb{V}$ *as above. Then* $T^{V,D}$ *is contractive on* $(\mathbb{V}, d)$ *and if* $W_0 \in \mathbb{V}$*, the sequence* $\{W_s\}$ *with* $W_{s+1} = T^{V,D}W_s$ *converges in the d-metric to* V*. Also,* V *is the unique function in* $\mathbb{V}$ *satisfying* $V = T^{V,D}(V)$*.*

The remaining question concerns the determination of good bounding problems. This issue is taken up in Sleet and Yeltekin (2010a).

11 Approximation

Convex/concave functions lend themselves to piecewise linear *inner* and *outer* approximations.[24] As emphasized by Judd et al (2003), such approximation of iterates interacts

[24] An outer (resp. inner) approximation to a convex function lies everywhere below (resp. above) this function. An outer (resp. inner) approximation to a concave function lies everywhere above (resp. below) this function.

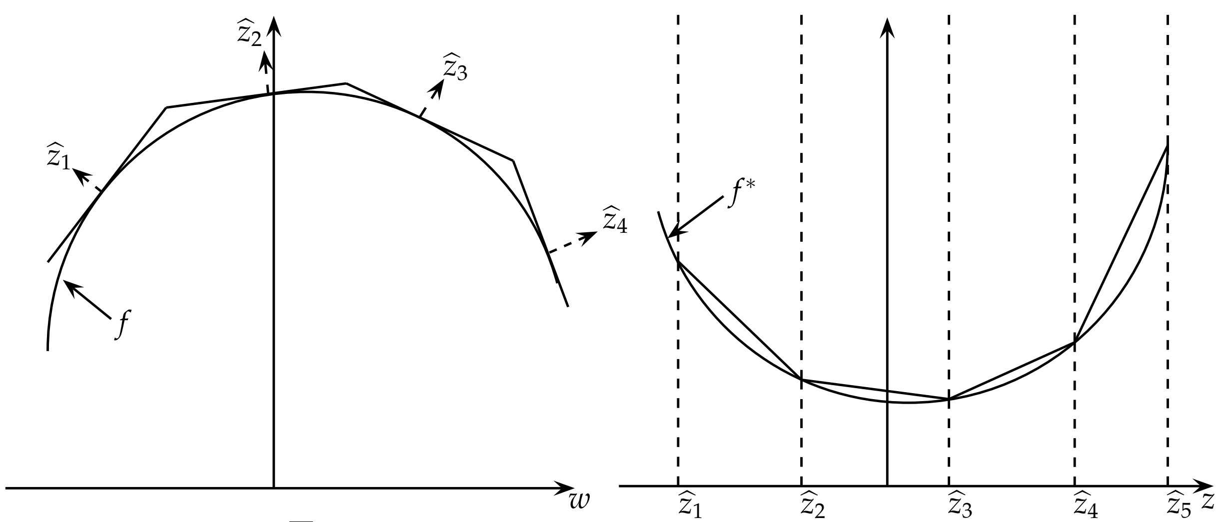
with the monotone nature of the Bellman operator to yield inner or outer limiting approximations to the true value function (given concavity or convexity).

Let $-f \in \mathbb{F}^N$ and $f^* = \mathcal{C}[-f]$. A piecewise linear *outer approximation* $\widehat{f}$ to the concave function f can be obtained by fixing a finite set of points $\{\widehat{z}_q\}_{q\in\mathbb{Q}}$ in $\mathbb{R}^N$ and forming the "approximate conjugate":

$$-\widehat{f}(w) = \sup_{\mathbb{Q}}\{\langle w, \widehat{z}_q\rangle - f^*(\widehat{z}_q)\} \leq \sup_{\mathbb{R}^N}\{\langle w, z\rangle - f^*(z)\} = -f(w).$$

Each $h_q(w) = \langle w, \widehat{z}_q\rangle - f^*(\widehat{z}_q)$ may be interpreted as a hyperplane with normal $\widehat{z}_q$. Thus, $-\widehat{f}(w)$ is the convex hull formed from these hyperplanes. Alternatively, substituting for f^*, $\widehat{f}$ may be determined as: $\widehat{f}(w) = -\overline{\mathcal{D}}[-f](w)$, where $\overline{\mathcal{D}}(-f)(w) = \sup_{\mathbb{Q}} \inf_{\mathbb{R}^N}\{\langle w - w', \widehat{z}_q\rangle - f(w')\}$. The outer approximation operator, $\overline{\mathcal{D}}$, is a composition of the approximate conjugate and conjugate operators and is monotone increasing on $\mathbb{F}^N$ (when this set is ordered in the usual pointwise way).

A piecewise linear *inner approximation* $\widehat{f^*}$ to the convex function f^* may be obtained by forming the conjugate of the approximation $\widehat{f}$, $\widehat{f}^*(z) = \mathcal{C}[-\widehat{f}](z) = \underline{\mathcal{D}}[f^*](z) = \sup_{\mathbb{R}^N} \inf_{\mathbb{Q}}\{\langle w, (z - \widehat{z}_q)\rangle + f^*(\widehat{z}_q)\} \geq f^*(z)$. Thus, the inner approximation operator $\underline{\mathcal{D}}$ reverses the composition of approximate conjugate and conjugate operators relative to the outer approximation operator $\overline{\mathcal{D}}$. $\widehat{f}^*$ also has a geometric interpretation: its epigraph is the convex hull of the union of halflines $\{(\widehat{z}_q, v) | f^*(\widehat{z}_q) \leq v\}$. Of course, by applying $\underline{\mathcal{D}}$ to $-f$ and $\overline{\mathcal{D}}$ to f^*, inner approximations to f and outer approximations to f^* may be obtained. Figure 4 illustrates the application of $\overline{\mathcal{D}}$ and $\underline{\mathcal{D}}$.



(a) $-\overline{\mathcal{D}}(-f)$ (b) $\underline{\mathcal{D}}(f^*)$

Figure 4: Outer and inner approximation

The operators $\overline{\mathcal{D}}$ and $\underline{\mathcal{D}}$ may be combined with the Bellman operators described in previous sections. Iteration of the resulting approximate Bellman operators can yield an epiconvergent value iteration whose limit is an outer or inner approximation to the true value function. We sketch an example in which $\underline{\mathcal{D}}$ is combined with $T^V = \{T^V_j\}$. Let $\{\widehat{z}_q\}_Q$ denote a finite set of weights and consider a piecewise linear function $W^0 \geq V$ satisfying for each j, $W^0_j = \underline{\mathcal{D}}[W^0_j] \geq T^V_j(W^0) = W^1_j$. Then, the monotonicity of $\underline{\mathcal{D}}$, T^V and the fixed point property of V implies: $W^0_j = \underline{\mathcal{D}}[W^0_j] \geq \underline{\mathcal{D}}[T^V_j(W^0)] = \widehat{W}^1_j \geq V_j$, where $\widehat{W}^p_j$, $p = 1, 2, \ldots$, is the p-th iterate of the approximate Bellman operator. The monotonicity of the approximate operator $\mathcal{D}T^V$ then implies that the sequence of approximations $\{\widehat{W}^p\}$ satisfies: $W \geq \widehat{W}^p \geq \widehat{W}^{p+1} \geq \ldots \geq V$. Thus, the pointwise (and epigraphical) limit of the sequence $\lim_p \widehat{W}^p$ is an inner approximation (upper bound) to V.

12 Conclusion

This paper derives and relates alternative recursive formulations of dynamic incentive problems. Under appropriate separability assumptions, primal formulations that use promises as state variables are available. These formulations incorporate promise-keeping and recursive incentive constraints either or both of which may be dualized to obtain related recursive dual problems. To recover primal values, from a recursive dual problem additional assumptions are required (e.g. concave or convex-likeness). Still further assumptions (e.g. strict concavity) or an extension of the methods described here[25] are needed to obtain optimal policies. These represent disadvantages of the recursive dual approaches. On the other hand, recursive dual problems are less likely to involve value functions that are extended real valued.

Our results indicate the extent to which recursive primal and recursive dual approaches can be interchanged and clarify the connections between approaches taken elsewhere in the literature. In addition, our analysis of epiconvergent value iteration relaxes boundedness assumptions needed for contraction mapping based analyses of value iteration.

References

Abraham, A. and N. Pavoni. 2008. Efficient allocations with morak hazard and hidden borrowing and lending: A recursive formulation. *Review of Economic Dynamics* 11: 781-803.

[25] E.g. Marimon et al (2011) or Cole and Kubler (2010)

Abreu, D., D. Pearce and E. Stacchetti. 1990. Towards a theory of discounted games with imperfect information. *Econometrica* 58: 1041-1063.

Atkeson, A. and R. Lucas. 1992. Efficient distributions with private information. *Review of Economic Studies* 59: 427-453.

Atkeson, A. and R. Lucas. 1995. Efficiency and equality in a simple model of unemployment insurance. *Journal of Economic Theory* 66: 64-88.

Borwein, J. and A. Lewis. 2006. *Convex analysis and nonlinear optimization* 2nd edition. Springer Verlag.

Chien, Y. and H. Lustig. 2009. The market price of risk and the aggregate wealth distribution. Forthcoming *Review of Economic Studies*.

Farhi, E. and I. Werning. 2007. Inequality and social discounting. *Journal of Political Economy* 115: 365-402.

Cole, H. and F. Kubler. 2010. Recursive contracts, lotteries and weakly concave Pareto sets. Mimeo. UPenn.

Fernandes, A. and C. Phelan. 2000. A recursive formulation for repeated agency with history dependence. *Journal of Economic Theory.* 91: 223-247.

Green, E. 1987. Lending and the smoothing of uninsurable income. In *Contractual Arrangements for Intertemporal Trade.* E. Prescott and N. Wallace, Eds., U. of Minnesota Press.

Judd, K., S. Yeltekin and J. Conklin. 2003. Computing supergame equilibria. *Econometrica* 71: 1239-1254.

Kocherlakota, N. 1996. Implications of efficient risk sharing without commitment. *Review of Economic Studies* 63:595-609.

Kydland, F. and E. Prescott. 1980. Dynamic optimal taxtion, rational expectations and optimal control. *Journal of Economic Dynamics and Control* 2:79-91.

Luenberger, D. 1969. *Optimization by Vector Space Methods.* Wiley.

Mele, A. 2009. Repeated moral hazard and recursive Lagrangians. Mimeo. UPF.

Marcet, A. and R. Marimon. 1999. Recursive contracts. Mimeo. UPF.

Marcet, A. and R. Marimon. 2011. Recursive contracts. Mimeo.

Messner, M. and N. Pavoni. 2004. On the recursive saddle point method. Mimeo. UCL.

Marimon, R., M. Messner and N. Pavoni. 2011. Solving recursive contracts with non-unique saddle-points. Mimeo EUI.

Messner, M., N. Pavoni, C. Sleet. 2011. On the dual approach to recursive optimization. Mimeo. Bocconi.

Osambela, E. 2009. Fear of default and volatility in a dynamic financial-market equilibrium. Mimeo. CMU.

Rockafellar, R.T. 1970. *Convex analysis.* Princeton University Press.

Rockafellar, R.T. 1974. *Conjugate duality and optimization.* SIAM

Rockafellar, R.T. and R. Wets. 1998. *Variational analysis.* Springer-Verlag.

Rustichini, A. 1998. Dynamic programming solution of incentive constrained problems. *Journal of Economic Theory* 78:329-354.

Sleet, C. 2011. Convex-like dynamic agency. Mimeo. Carnegie Mellon University.

Sleet, C. and Ş. Yeltekin. 2010a. The recursive Lagrangian method for solving incentive-constrained problems: Discrete time. Mimeo. Carnegie Mellon University.

Sleet, C. and Ş. Yeltekin. 2010b. The recursive Lagrangian method for solving incentive-constrained problems: Continuous time. Mimeo. Carnegie Mellon University.

Spear, S. and S. Srivastava. 1987. On repeated moral hazard with discounting. *Review of Economic Studies* 54: 599-617.

Thomas, J. and T. Worrall. 1990. Income fluctuation and asymmetric information: An example of a repeated principal-agent problem. *Journal of Economic Theory* 51: 367-390.

Appendix A: Proofs

Proof of Prop. 1 Let $a' = (a'_0, \{a'_k\}) \in \Omega_1$ and for $k \in \mathbb{K}$ define $w'_k = \{g_{k,n}(a'_k)\}$. Then since $a' \in \Omega_1$ implies for $k \in \mathbb{K}$, $g_k^{M+k}(a'_k) \geq 0$, $a'_k \in \Phi_k(w'_k)$. Hence, $S_k(w'_k) \geq f_k(a'_k)$. Also, since for each m, $g_0^m(a'_0) + \sum_{k=1}^K \sum_{n=1}^N w'_{k,n} q^m_{k,n} = g_0^m(a'_0) + \sum_{k=1}^K \sum_{n=1}^N g_{k,n}(a'_k) q^m_{k,n} \geq$

0, $(a'_0, \{w'_k\}) \in \Psi_0$ and so $\sup_{\Psi_0} f_0(a_0) + \sum_{k=1}^K S_k(w_k)q_k^0 \geq f_0(a'_0) + \sum_{k=1}^K S_k(w'_k)q_k^0 \geq \sum_{k=0}^K f_k(a'_k)q_k^0$. Since a' was an arbitrary element of Ω_1, $\sup_{\Phi_0} f_0(a_0) + \sum_{k=1}^K S_k(w_k)q_k^0 \geq P$.

Conversely, if $(a'_0, \{w'_k\}) \in \Psi_0$, then either $\Phi_k(w'_k) = \emptyset$ for some $k \in \mathbb{K}$ in which case $f_0(a_0) + \sum_{k=1}^K S_k(w_k)q_k^0 = -\infty \leq P$ or $\Phi_k(w'_k) \neq \emptyset$ for all $k \in \mathbb{K}$. In the latter case, for each $k \in \mathbb{K}$, let $a'_k \in \Phi_k(w'_k)$, then $(a'_0, \{a'_k\}) \in \Omega_1$ and so $P \geq f(a'_0)q_0^0 + \sum_{k=0}^K f_k(a'_k)q_k^0$. Since the a'_k are arbitrary elements of $\Phi_k(w'_k)$, $P \geq f(a'_0)q_0^0 + \sum_{k=0}^K S_k(w'_k)q_k^0$. Since $(a'_0, \{w'_k\})$ is an arbitrary element of Ψ_0, $P \geq \sup_{\Psi_0} f(a_0)q_0^0 + \sum_{k=0}^K S_k(w_k)q_k^0$. □

Proof of Prop. 7 Following a similar argument to that used in proving Proposition 1,

$$V_0(z) = \sup_{\Psi_0} f_0(a_0)q_0^0 + \sum_{k=1}^K S_k(w_k)q_k^0 + \sum_{h=1}^N z_h \left[g_0^h(a_0) + \sum_{k=1}^K \sum_{n=1}^N w_{k,n} p_{k,n}^h \right].$$

Incorporating the first period incentive constraints into a Lagrangian and rearranging:

$$\begin{aligned} V_0(z) = &\sup_{A_0 \times \mathbb{R}^{KN}} \inf_{\mathbb{R}_+^M} f_0(a_0)q_0^0 + \sum_{k=1}^K S_k(w_k)q_k^0 + \sum_{h=1}^N z_h \left[g_0^h(a_0) + \sum_{k=1}^K \sum_{n=1}^N w_{k,n} p_{k,n}^h \right] \\ &+ \sum_{m=1}^M \eta_m \left[g_0^m(a_0) + \sum_{k=1}^K \sum_{n=1}^N w_{k,n} q_{k,n}^m \right] \\ = &\sup_{A_0 \times \mathbb{R}^{KN}} \inf_{\mathbb{R}_+^M} f_0(a_0)q_0^0 + \sum_{h=1}^N z_h g_0^h(a_0) + \sum_{m=1}^M \eta_m g_0^m(a_0) \\ &+ \sum_{k=1}^K S_k(w_k)q_k^0 + \sum_{k=1}^K \sum_{n=1}^N \left[\sum_{m=1}^M \eta_m q_{k,n}^m + \sum_{h=1}^N z_h p_{k,n}^h \right] w_{k,n} \end{aligned}$$

Since $-S \in \mathbb{W}$, for each $k \in \mathbb{K}$, $S_k(w_k) = \inf_{\mathbb{R}^N} -\sum_{n=1}^N w_{k,n} z_{k,n} + V_k(z_k)$ and so, after substitution and rearrangement,

$$\begin{aligned} V_0(z) = &\sup_{A_0 \times \mathbb{R}^{KN}} \inf_{\mathbb{R}_+^M \times \mathbb{R}^{KN}} f_0(a_0)q_0^0 + \sum_{h=1}^N z_h g_0^h(a_0) + \sum_{m=1}^M \eta_m g_0^m(a_0) \\ &+ \sum_{k=1}^K V_k(z_k)q_k^0 + \sum_{k=1}^K \sum_{n=1}^N \left[\zeta'_{k,n}(z, \eta) - z_{k,n} \right] q_k^0 w_{k,n} \end{aligned}$$

as required. □

Proof of Prop. 11. Fix $W \in \mathbb{W}_0$. Using the definitions of $T^{V,D}$ and ζ' and rearranging:

$$
\begin{aligned}
T^{V,D}(\mathcal{C}[W])(z) &= \inf_{\mathbb{R}_+^M}\Big[\sup_{a_0}\left\{f_0(a_0)q_0^0 + \sum_{m=1}^{M}\eta_m g_0^m(a_0)q_0^m + \sum_{h=1}^{N} z_h g_{0,h}(a_0)\right\} \\
&\quad + \sum_{k\in\mathbb{K}}\sup_{\mathbb{R}^N}\{\langle\zeta_k'(z,\eta),w\rangle + W_k(w)\}q_k^0\Big] \\
&= \inf_{\mathbb{R}_+^M}\sup_{A_0\times\mathbb{R}^{KN}}\left\{\left[f_0(a_0)q_0^0 + \sum_{k=1}^{K} W_k(w_k)q_k^0\right] + \sum_{h=1}^{N} z_h\left[g_{0,h}(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}p_{k,n}^h\right]\right. \\
&\quad \left. + \sum_{m=1}^{M}\eta_m\left[g_0^m(a_0)q_0^m + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}q_{k,n}^m\right]\right\}.
\end{aligned}
$$

Combining this with $\mathcal{C}[T^{V,D}(\mathcal{C}[-W])](w) = -\sup_{\mathbb{R}^N}\left\{\sum_{h=1}^{N} z_h w_h - T^{V,D}(\mathcal{C}[-W])(z)\right\}$ and rearranging yields:

$$
\begin{aligned}
-\mathcal{C}[T^{V,D}(\mathcal{C}[-W])](w) &= \inf_{\mathbb{R}^N}\left\{-\sum_{h=1}^{N} z_h w_h + \inf_{\mathbb{R}_+^M}\sup_{A_0\times\mathbb{R}^{KN}}\left\{\left[f_0(a_0)q_0^0 + \sum_{k=1}^{K} W_k(w_k)q_k^0\right]\right.\right. \\
&\quad \left.\left. + \sum_{h=1}^{N} z_h\left[g_{0,h}(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}p_{k,n}^h\right] + \sum_{m=1}^{M}\eta_m\left[g_0^m(a_0)q_0^m + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}q_{k,n}^m\right]\right\}\right\}. \\
&= \inf_{\mathbb{R}^N\times\mathbb{R}_+^M}\sup_{A_0\times\mathbb{R}^{KN}}\left\{\left[f_0(a_0)q_0^0 + \sum_{k=1}^{K} W_k(w_k)q_k^0\right]\right. \\
&\quad \left. + \sum_{h=1}^{N} z_h\left[g_{0,h}(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}p_{k,n}^h - w_h\right] + \sum_{m=1}^{M}\eta_m\left[g_0^m(a_0)q_0^m + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}q_{k,n}^m\right]\right\} \\
&= T^{S,D}(W)(w).
\end{aligned}
$$

Now, $T^S(-\mathcal{C}[W])(w) = \sup_{\Phi(w)} f_0(a_0)q_0^0 - \sum_{k=1}^{K}\sup_{\mathbb{R}^N}\left\{\sum_{n=1}^{N} w_{k,n}z_{k,n} + W(z_k)\right\}$ and so,

$$
\mathcal{C}[-T^S(-\mathcal{C}[W])](z) = \sup_{\mathbb{R}^N}\left(\sum_{h=1}^{N} z_h w^h + \sup_{\Phi_0(w)}\left[f_0(a_0)q_0^0 - \sum_{k=1}^{K}\sup_{\mathbb{R}^N}\left\{\sum_{n=1}^{N} w_{k,n}z_{k,n} + W(z_k)\right\}q_k^0\right]\right).
$$

Collecting the supremum operations together and using the definition of Φ_0 and Ψ_0 gives:

$$
\begin{aligned}
\mathcal{C}[-T^S(-\mathcal{C}[W])](z) = \sup_{\Psi_0}\Bigg(&\sum_{h=1}^{N} z_h\left[g_{0,h}(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}p_{k,n}^h\right] + f_0(a_0)q_0^0 \\
&- \sum_{k=1}^{K}\sup_{\mathbb{R}^N}\left\{\sum_{n=1}^{N} w_{k,n}z_{k,n} + W(z_k)\right\}q_k^0\Bigg).
\end{aligned}
$$

Hence, bringing the incentive constraints into the Lagrangian:

$$\mathcal{C}[-T^S(-\mathcal{C}[W])](z) = \sup_{A_0\times\mathbb{R}^{KN}} \inf_{\mathbb{R}^M_+} \left(f_0(a_0)a_0^0 + \sum_{h=1}^{N} z_h \left[g_{0,h}(a_0) + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}p^h_{k,n} \right] \right.$$
$$\left. - \sum_{k=1}^{K} \sup_{\mathbb{R}^N} \left\{ \sum_{n=1}^{N} w_{k,n}z_{k,n} - W(z_k) \right\} q_k^0 + \sum_{m=1}^{M} \eta_m \left[g_0^m(a_0)q_0^m + \sum_{k=1}^{K}\sum_{n=1}^{N} w_{k,n}q^m_{k,n} \right] \right).$$

Grouping infimum operations together and rearranging gives:

$$\begin{aligned}\mathcal{C}[-T^S(-\mathcal{C}[W])](z) &= \sup_{A_0\times\mathbb{R}^{KN}} \inf_{\mathbb{R}^M_+\times\mathbb{R}^{KN}} \left[\left(f_0(a_0)q_0^0 + \sum_{h=1}^{N} z_h g_{0,h}(a_0) + \sum_{m=1}^{M} \eta_m g_0^m(a_0) q_0^m \right) \right. \\ &\quad \left. + \sum_{k=1}^{K} \left(\sum_{n=1}^{N} w_{k,n} \left(\sum_{h=1}^{N} z^h \frac{p^h_{k,n}}{q_k^0} + \sum_{m=1}^{M} \eta_m \frac{q^m_{k,n}}{q_k^0} - z_{k,n} \right) + W(z_k) \right) q_k^0 \right] \\ &= T^V(W)(z).\end{aligned}$$

The remaining equalities in the proposition are immediate. □

Proof of Lemma 2. If $W \geq W'$ and $\text{lev}_u U_j(W)(w,\cdot) \subset B$ for all $w \in V$ a neighborhood of $\widehat{w}$, then $\text{lev}_u U_j(W')(w,\cdot) \subset \text{lev}_u U_j(W)(w,\cdot) \subset B$. Thus, if $U_j(W)(w,\cdot)$ is level bounded locally in w, so is $U_j(W')(w)$. □

Proof of Lemma 3. Follows from Rockafellar and Wets (1998), Theorem 1.17, p. 16. □

Proof of Theorem 8. T^S is readily shown to be monotone on $-\mathbb{W}_0$ (i.e. $S' \geq S''$ implies $T^S(S') \geq T^S(S'')$). Consider the interval of functions $I := \{S' | S_0 \geq S' \geq S\} \subset -\mathbb{W}_0$. $T^S : I \to I$ since, using 1) and 2) in the proposition, the monotonicity of T^S and the fact that S is a fixed point of T^S, $S_0 \geq T^S S_0 \geq T^S S' \geq T^S S = S$. Thus, $\{S_n\} \subset I$. Moreover, the sequence of functions $\{S_n\}$ is a decreasing sequence since $S_0 \geq T^S S_0 = S_1$ and, by monotonicity of T^S, $S_n \geq S_{n+1}$ implies $S_{n+1} = T^S S_n \geq T^S S_{n+1} = S_{n+2}$. For each j and w, $\{S_{n,j}(w)\}$ is a decreasing sequence bounded below by $S_j(w) \in \mathbb{R} \cup \{-\infty\}$ and so $\lim_n S_{n,j}(w) = S_{\infty,j}(w) \geq S_j(w)$. It follows that S_n converges pointwise to the function $S_\infty = \{S_{\infty,j}(w)\}$. Also, $S_n \geq S_\infty$ implies that for all n, $S_{n+1} = T^S S_n \geq T^S S_\infty$. Hence, $S_\infty = \lim S_{n+1} \geq T^S S_\infty$.

Note that $\infty > S_{0,j} \geq S_{1,j} = T^S_j(S_0) \geq U_j(S_0)$ and $S_{1,j} \geq S_j(w)$. It follows that

$U_j(S_0)$ is everywhere less than ∞ and is somewhere more than $-\infty$. By assumption, $U_j(S_0)$ is level bounded locally and since each $f(\widehat{\theta}_k, \cdot)$ and $S_{0,k}$ are upper semicontinuous and Graph Φ_j is closed, $U_j(S_0)$ is upper semicontinuous. It follows from Lemma 3 that if $U_j(S_0)(w) \geq S_{\infty,j}(w) > -\infty$ that $\arg\min U_j(S_0)(w, \cdot)$ is nonempty and compact. In addition, $\arg\min U_j(S_0)(w, \cdot) \subset \text{lev}_{S_{\infty,j}(w)} U_j(S_0)(w, \cdot)$, where the latter set is nonempty and, since $U_j(S_0)(w, \cdot)$ is upper semicontinuous and level bounded, compact. Also by Lemma 3, $S_{1,j}$ is upper semicontinuous and, since $S_0 \geq S_1$, by Lemma 2 $U_j(S_1)$ is level bounded locally in w. Since $\infty > S_{1,j} \geq S_{2,j} = T_j^S(S_1) \geq U_j(S_1)$ and $S_{2,j} \geq S_j(w)$ implies that $U_j(S_1)$ is everywhere less than ∞ and is somewhere more than $-\infty$. Repeatedly applying these arguments, each S_n is found to be upper semicontinuous and each $U_j(S_n)$ is found to be upper semicontinuous, level bounded locally, everywhere less than ∞ and somewhere more than $-\infty$. In addition, if $S_{\infty,j}(w) > -\infty$, there is a sequence $(a_n, w'_n) \in \arg\min U_j(S_n)(w) \subset \text{lev}_{S_{\infty,j}(w)} U_j(S_0)(w, \cdot)$. Since $\text{lev}_{S_{\infty,j}(w)} U_j(S_0)(w, \cdot)$ is compact, the sequence $\{a_n, w'_n\}$ admits a convergent subsequence $\{a_{n_v}, w'_{n_v}\}$ with limit $(a_\infty, w'_\infty) \in \text{lev}_{S_{\infty,j}(w)} U_j(S_0)(w, \cdot)$. Now,

$$
\begin{aligned}
T_j^S(S_\infty)(w) &= \sup_{(a,w')} U_j(S_\infty)(w, a, w') \geq U_j(S_\infty)(w, a_\infty, w'_\infty) = \lim_{v \to \infty} U_j(S_{n_v})(w, a_\infty, w'_\infty) \\
&\geq \lim_{v \to \infty} \limsup_{\widetilde{v} \geq v} U_j(S_{n_v})(w, a_{n_{\widetilde{v}}}, w'_{n_{\widetilde{v}}}) \geq \lim_{v \to \infty} \limsup_{\widetilde{v} \geq v} U_j(S_{n_{\widetilde{v}}})(w, a_{n_{\widetilde{v}}}, w'_{n_{\widetilde{v}}}) \\
&= \limsup_{\widetilde{v} \geq v} U_j(S_{n_{\widetilde{v}}})(w, a_{n_{\widetilde{v}}}, w'_{n_{\widetilde{v}}}) = \limsup_{\widetilde{v} \geq v} S_{n_{\widetilde{v}}+1}(w) = S_\infty(w).
\end{aligned}
$$

Combining inequalities $S_\infty = T^S S_\infty$. The sequence $\{S_n\}$ is a decreasing sequence of upper semicontinuous functions with pointwise limit S_∞. Thus, $\{-S_n\}$ is an increasing sequence of lower semicontinuous functions with pointwise limit $-S_\infty$. By Rockafellar and Wets (1998), Proposition 7.4(d), the sequence $\{-S_n\}$ epiconverges to $\sup_n[cl(-S_n)]$. But since each $-S_n$ is proper and lower semicontinuous, $cl(-S_n) = -S_n$ and since the sequence is increasing $\sup_n(-S_n) = -S_\infty$, the result follows. □

Proof of Theorem 9. By Theorem 8, S_∞, the epi-limit of $\{S_n\}$, satisfies $S_\infty \geq S$. It remains only to show the reverse inequality. If $S_{\infty,j}(w) = -\infty$, then immediately $S_{\infty,j}(w) = S_j(w) = -\infty$. Suppose that $S_{\infty,j}(w) > -\infty$. By Theorem 8, $T_j^S S_\infty(w) = S_{\infty,j}(w)$. Also, each $S_{\infty,j}$ is upper semicontinuous (as the pointwise limit of a decreasing sequence of upper semicontinuous functions) and bounded above by $S_{0,j}$. Hence, $U_j(S_\infty)$ is upper semicontinuous and level bounded locally in w. and applying Lemma 3, there is a $(a_1^*, w_2^*) \in$

$\Phi_j(w)$ such that:

$$S_{\infty,j}(w) = \sum_{k=1}^{K} [f(\widehat{\theta}_k, a_1^*(\widehat{\theta}_k)) + \beta_P S_{\infty,k}(w_2^*(\widehat{\theta}_k))]\pi_{j,k}.$$

Since $S_{\infty,j}(w) > -\infty$, each $S_{\infty,k}(w_2^*(\widehat{\theta}_k)) > -\infty$. Repeatedly applying this argument at successive nodes gives a sequence $\{a_t^*, w_{t+1}^*\}$ such that for each θ^{t-1}, $(a_t^*(\theta^{t-1}), w_{t+1}^*(\theta^{t-1})) \in \Phi_{k(\theta_{t-1})}(w_t^*(\theta^{t-1}))$, where $w_1^*(\theta^0) = w$, $k(\theta_0) = j$ and $k(\theta_{t-1})$, $t > 0$, gives the index of the θ_{t-1} shock. Also for each T,

$$S_{\infty,j}(w) = E\left[\sum_{t=1}^{T} \beta_P^{t-1} f(\theta_t, a_t^*(\theta^t))|\theta_0 = \widehat{\theta}_j\right] + \beta_P^{T+1} E\left[S_\infty(w_{T+1}^*(\theta^T))|\theta_0 = \widehat{\theta}_j\right]. \tag{72}$$

Hence, using the fact that S_∞ is bounded above (since it is upper semicontinuous and coercive) gives: $S_{\infty,j}(w) \leq \limsup_{T\to\infty} E\left[\sum_{t=1}^{T} \beta_P^{t-1} f(\theta_t, a_t^*(\theta^t))|\theta_0 = \widehat{\theta}_j\right]$. In addition, since each $(a_t^*(\theta^{t-1}), w_{t+1}^*(\theta^{t-1})) \in \Phi_{k(\theta_{t-1})}(w_t^*(\theta^{t-1})$,

$$w^i = E\left[\sum_{t=1}^{T} r^i(\theta_t, a_t^*(\theta^t))|\theta_0 = \widehat{\theta}_j\right] + \beta_A^{T+1} E[w_{T+1}^{i,*}(\theta^T)|\theta_0 = \widehat{\theta}_j], \quad i \in \mathbb{I}_1$$
$$w_k^i = E\left[\sum_{t=1}^{T} r^i(\theta_t, a_t^*(\theta^t))|\theta_1 = \widehat{\theta}_k\right] + \beta_A^{T+1} E\left[\sum_l w_{T+1,l}^{i,*}(\theta^T)\pi(\widehat{\theta}_l|\theta_T)|\theta_1 = \widehat{\theta}_k\right], \quad i \in \mathbb{I}_2, k \in \mathbb{K}.$$

Since S_0 is coercive in expectation and $S_0 \geq S_\infty$, S_∞ is coercive in expectation as well. Hence, if $\lim \beta_A^T E[\|w_{T+1}^*\||\theta_0 = \widehat{\theta}_j] \neq 0$, then $\lim \beta_P^T E[S_\infty(w_{T+1}^*(\theta^T))|\theta_0 = \widehat{\theta}_j] = -\infty$. But this, eq. (72) and the fact that f is bounded above, contradict $S_{\infty,j}(w) \geq S_j(w) > -\infty$. Thus, we infer that $\limsup_{T\to\infty} \beta_A^T \|E[w_{T+1}^*|\theta_0 = \widehat{\theta}_j]\| = 0$ and so

$$w^i = \lim_{T\to\infty} E\left[\sum_{t=1}^{T} r^i(\theta_t, a_t^*(\theta^t))|\theta_0 = \widehat{\theta}_j\right] \quad \text{and} \quad w_k^i = \lim_{T\to\infty} E\left[\sum_{t=1}^{T} r^i(\theta_t, a_t^*(\theta^t))|\theta_1 = \widehat{\theta}_k\right].$$

By similar logic, each $w_{t+1}^*(\theta^t)$ satisfies: $w_{t+1}^{i,*}(\theta^t) = \lim_{T\to\infty} E\left[\sum_{s=1}^{T} r^i(\theta_{t+s}, a_{t+s}^*(\theta^{t+s}))|\theta^t\right]$ and $w_{t+1,k}^{i,*} = \lim_{T\to\infty} E\left[\sum_{s=1}^{T} r^i(\theta_{t+s}, a_{t+s}^*(\theta^{t+s}))|\theta^t, \theta_{t+1} = \widehat{\theta}_k\right]$. Combining these equalities with $(a_t^*(\theta^{t-1}, \cdot)$, $w_{t+1}^*(\theta^{t-1}, \cdot)) \in \Phi_{k(\theta_{t-1})}(w_{t-1}^*(\theta^{t-1}))$ ensures that $\{a_t^*\} \in \Omega_{1,j}(w)$ and so is feasible for 60. Thus, $S_j(w) \geq S_{\infty,j}(w) \geq S_j(w) > -\infty$. □

Proposition A1 $\underline{V}_j \leq T_j^{V,D}\underline{V} \leq V_j = T_j^{V,D}V \leq T_j^{V,D}\overline{V} \leq \overline{V}_j$.

Proof. First,

$$\begin{aligned}
\underline{V}_j(\zeta) &= \sup_{\underline{\Omega}} \sum_{k\in\mathbb{K}} \mathcal{F}_k(\alpha)\pi_{j,k} + \sum_{i\in\mathbb{I}_1} \zeta^i \sum_{k\in\mathbb{K}} \mathcal{R}^i_k(\alpha)\pi_{j,k} + \sum_{i\in\mathbb{I}_2}\sum_{k\in\mathbb{K}} \zeta^i_k \mathcal{R}^i_k(\alpha)\pi_{j,k} + \sum_{m\in\mathbb{M}} \eta_m \underline{G}_m(\alpha) \\
&\leq \sup_{\{\alpha|G(\alpha)\geq 0\}\cap\underline{\Omega}_2} \sum_{k\in\mathbb{K}} \mathcal{F}_k(\alpha)\pi_{j,k} + \sum_{i\in\mathbb{I}_1} \zeta^i \sum_{k\in\mathbb{K}} \mathcal{R}^i_k(\alpha)\pi_{j,k} + \sum_{i\in\mathbb{I}_2}\sum_{k\in\mathbb{K}} \zeta^i_k \mathcal{R}^i_k(\alpha)\pi_{j,k} \\
&= \inf_{\eta\in\mathbb{R}^M_+} \sup_{\underline{\Omega}_2} \sum_{k\in\mathbb{K}} \mathcal{F}_k(\alpha)\pi_{j,k} + \sum_{i\in\mathbb{I}_1} \zeta^i \sum_{k\in\mathbb{K}} \mathcal{R}^i_k(\alpha)\pi_{j,k} + \sum_{i\in\mathbb{I}_2}\sum_{k\in\mathbb{K}} \zeta^i_k \mathcal{R}^i_k(\alpha)\pi_{j,k} + \sum_{m\in\mathbb{M}} \eta_m G_m(\alpha),
\end{aligned}$$

where $\underline{\Omega}_2 = \{\alpha = (a, \alpha') \in \Omega_0 | \text{ each } \alpha'_k \in \underline{\Omega}\}$. Rearrangements similar to those in the text and the definition of $\underline{V}$ imply that the last expression equals $T^{V,D}_j \underline{V}(\zeta)$. Hence, the first inequality in the proposition holds. The monotonicity of $T^{V,D}_j$, $V \geq \underline{V}$ and $V_j = T^{V,D}_j V$ gives $T^{V,D}_j \underline{V} \leq V_j = T^{V,D}_j V$. The other inequalities follow by similar reasoning. □

Appendix B: Duality

The problems from the preceding sections incorporate first period incentive and, in some cases, auxiliary second period promise-keeping conditions. Relations between these problems can be expressed in terms of the value functions associated with perturbations of these constraints and their conjugates. The results underpinning these relations are classical and are collected into two theorems given below. The first considers a family of optimizations subject to parameterized equality and inequality constraints.[26]

Theorem B0. (Dualizing all constraints) Let Ω be a non-empty subset of a vector space, $f : \Omega \to \mathbb{R}$, $g : \Omega \to \mathbb{R}^N$ and $h : \Omega \to \mathbb{R}^M$. Let:

$$\psi(w, \delta) = \sup_{\{x\in\Omega|g(x)=w,h(x)\geq\delta\}} f(x), \tag{73}$$

and

$$\varphi(z, \eta) = \sup_{x\in\Omega} f(x) + \langle z, g(x)\rangle + \langle \eta, h(x)\rangle, \tag{74}$$

where by convention ψ equals $-\infty$ if the constraint set is non-empty. Assume that $-\psi \in \mathbb{F}^{NM}_0$, then: 1) $\varphi = \mathcal{C}[-\psi] \in \mathbb{F}^{NM}$ and 2) $-\psi = \mathcal{C}[\varphi] = \mathcal{C}^2[-\psi]$ if $-\psi \in F^{NM}$.

It follows from Theorem B0 that, modulo sign changes, the "objective-perturbed" value function φ is the conjugate of the "constraint-perturbed" value function ψ and is in $\mathbb{F}^{NM}$ if $-\psi$ is in $\mathbb{F}^{NM}_0$. Conversely, $-\psi$ is the conjugate of φ if $-\psi \in \mathbb{F}^{NM}$. Writing $\psi(w, \delta)$ in

[26] Theorem B0 is classical. For a related proofs see Borwein and Lewis (2006), p. 90.

terms of the associated Lagrangian gives:

$$\psi(w,\delta) = \sup_{x\in\Omega} \inf_{(z,\eta)\in\mathbb{R}^N\times\mathbb{R}^M_+} f(x) + \langle z, g(x) - w\rangle + \langle \eta, h(x) - \delta\rangle. \tag{75}$$

Writing $-\mathcal{C}^2[-\psi](w,\delta)$ out explicitly:

$$-\mathcal{C}^2[-\psi](w,\delta) = \inf_{(z,\eta)\in\mathbb{R}^N\times\mathbb{R}^M_+} \sup_{x\in\Omega} f(x) + \langle z, g(x) - w\rangle + \langle \eta, h(x) - \delta\rangle, \tag{76}$$

it follows that there is no difference between primal and dual values in (73) if and only if, after a sign change, the associated primal value function ψ equals its biconjugate. This result holds at all (w,δ) if $-\psi \in \mathbb{F}^{NM}$.

Theorem B1 below decomposes the conjugation operations of Theorem B0 in various useful ways. Parts A1)-A2) consider the dualization of the equality constraint only. Part A3) treats the conjugation operation $\mathcal{C}[-\psi]$ recursively, decomposing it into a supremum operation over w and then δ. Part A4) dualizes the inequality constraint only (leaving the equality constraint inside the objective); Part A5) dualizes the inequality constraint after the objective has been perturbed with the equality constraint function g.

Theorem B1. (Dualizing subsets of constraints) Let Ω, f, g, h and ψ be as before and let:

$$\gamma(z,\delta) = \sup_{\{x\in\Omega|h(x)\geq\delta\}} f(x) + \langle z, g(x)\rangle.$$

For given $\delta \in \mathbb{R}^M$, assume that $-\psi(\cdot,\delta) \in \mathbb{F}^N_0$, then:

A1) $\gamma(\cdot,\delta) = \mathcal{C}[-\psi(\cdot,\delta)] \in \mathbb{F}^N$ and $-\psi(\cdot,\delta) = \mathcal{C}[\gamma(\cdot,\delta)]$ if $-\psi(\cdot,\delta) \in \mathbb{F}^N$;

A2) there is no duality gap with respect to the equality constraint if: $-\psi(w,\delta) = -\mathcal{C}[\gamma(\cdot,\delta)](w) = \inf_{z\in\mathbb{R}^N} \sup_{\{x\in\Omega|h(x)\geq\delta\}} f(x) + \langle z, g(x) - w\rangle$;

A3) $\mathcal{C}[-\psi](z,\eta) = \mathcal{C}[\gamma(z,\cdot)](\eta)$;

A4) $\mathcal{C}^2[-\psi(w,\cdot)](\delta) = \inf_{\eta\in\mathbb{R}^M_+} \sup_{\{x\in\Omega|g(x)=w\}} f(x) + \langle \eta, h(x) - \delta\rangle$;

A5) $\mathcal{C}^2[-\gamma(z,\cdot)](\delta) = \inf_{\eta\in\mathbb{R}^M_+} \sup_{x\in\Omega} f(x) + \langle z, g(x)\rangle + \langle \eta, h(x) - \delta\rangle$.

The preceding results lead us to focus on cases in which value functions are proper, convex and lower semicontinuous. In particular, since equality constraints are used to define state variables in our decompositions, we will be interested in cases in which $-\psi(\cdot,0)$ lies in $\mathbb{F}^N$.[27] We briefly review some assumptions on primitives that ensure this.

[27]In other words, $-\psi(\cdot,0) = \mathcal{C}[-\psi(\cdot,0)]$. We will also be interested in situations in which for all w,

Properness Properness of $-\psi$ is ensured if the constraint set is non-empty at some parameter pair (w, δ) and f is bounded on each $\{x \in \Omega | g(x) = w, h(x) \geq \delta\}$.

Convexity/ concavity Convexity of $-\psi(\cdot, 0)$ (concavity of $\psi(\cdot, 0)$) is ensured if (f, g, h, Ω) define a family of *convex-like* problems. Formally, let $\Omega_1(w) = \{x \in \Omega | g(x) = w, h(x) \geq \delta\}$ and define $\widehat{f}$ such that $\widehat{f}(w, x) = f(x)$ if $x \in \Omega_1(w)$ and $\widehat{f}(w, x) = -\infty$ otherwise. The family of problems $\sup_{\Omega_1(w)} f(x) = \sup \widehat{f}(w, x)$ is said to be convex-like if for all pairs (w^1, x^1) and (w^2, x^2) and numbers $\lambda \in [0, 1]$, there is an x^λ such that $\widehat{f}(\lambda w^1 + (1-\lambda) w^2, x^\lambda) \geq \lambda \widehat{f}(w^1, x^1) + (1-\lambda)\widehat{f}(w^2, x^2)$.We say that (f, g, h, Ω) is *concave-like* if $(-f, g, h, \Omega)$ is convex-like. A well known sufficient condition for convex-likeness is that the graph of Ω_1 is a convex set and f is concave. Convexity of Graph Ω_1 is ensured if Ω is convex, g is affine and h is concave. However, weaker conditions for the convexity of $-\psi(\cdot, 0)$ are available see Sleet (2011).

Lower semicontinuity If $-\psi(\cdot, 0)$ is convex and its effective domain is all of $\mathbb{R}^N$, then it is immediately continuous and, hence, lower semicontinuous. If its effective domain is a strict subset of $\mathbb{R}^N$, then additional assumptions are required. These relate to the structure of the objective and constraint function level sets.[28] Following Rockafellar and Wets (1998), $-\psi(\cdot, 0)$ is lower semicontinuous if Ω is a subset of finite dimensional vector space and $\widehat{f}$ is proper, upper semicontinuous and uniformly level bounded in x locally in w.[29] The latter property is ensured if the level sets of the Lagrangian $\lambda_0 f(x) + \langle z, g(x) \rangle + \langle \eta, h(x) \rangle$ are compact for some (λ_0, z, η), see Borwein and Lewis (2006). This in turn is guaranteed if either f, g or h have compact level sets.[30]

Primal and dual attainment The above mentioned compactness conditions on level sets used to obtain lower semicontinuity of $-\psi(\cdot, 0)$ also ensure primal attainment whenever $-\psi(w, 0)$ is finite and the constraint set $\Omega_1(w)$ is non-empty. An alternative route to a zero duality gap is via dual attainment. There is *strong duality* (i.e. a zero duality gap and dual attainment) with respect to inequality constraint perturbations at $(w, 0)$ if $-\psi(w, \cdot)$ is subdifferentiable at 0. In this case, the subdifferentials of the value function are the minimizing multipliers of the dual. Since convex functions are subdifferentiable on the

$-\psi(w, 0) = \mathcal{C}[-\psi(w, \cdot)](0)$ and for all z, $\gamma(z, 0) = \mathcal{C}[\gamma(z, \cdot)](0)$.

[28] The (upper) level set of $f : X \to \overline{\mathbb{R}}$ is given by $\text{lev}_r f\{x \in X | f(x) \geq r\}$.

[29] $\widehat{f}$ is uniformly level bounded in x locally in w if for each $w \in \mathbb{R}^N$ and $r \in \mathbb{R}$, there is a neighborhood $N(w)$ of w and a bounded set $B \subset X$ such that $\{x | \widehat{f}(w', x) \geq r\} \subset B$, $w' \in N(w)$.

[30] That is, if Ω is finite dimensional, are upper semicontinuous and have bounded level sets. If these functions are concave, then boundedness of level sets is equivalent to level coercivity: $\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} < 0$.

interior of their effective domain, a sufficient condition for strong duality at a $(w, 0)$ is that the $-\psi(w, \cdot)$ is convex and 0 lies in the interior of Dom $-\psi(w, \cdot)$. A classical condition for ensuring the latter is the so called *Slater condition* which requires the existence of an $x \in \Omega \cap \{g(x) = w\}$ satisfying $h(x) > 0$.

Coincidence of primal and dual solutions Conditions for the coincidence of optimal primal and dual solutions (as opposed to values) are more stringent. Consider the dual pair (75)-(76) (at some (w, δ)). If strong duality holds, x^* is maximal for (75) and (z^*, η^*) is minimal for (76), then x^* maximizes $L(x; z^*, \eta^*) = f(x) + \langle z^*, g(x) \rangle + \langle \eta^*, h(x) \rangle$ over Ω. However, $L(\cdot; z^*, \eta^*)$ may admit additional maximizers that do not solve (75). A maximizer x^* of $L(\cdot; z^*, \eta^*)$ solves (75) if (z^*, η^*) minimizes $L(x^*; \cdot)$, i.e. (x^*, z^*, η^*) is a saddle point, or if strong duality holds, (z^*, η^*) solves (76) and x^* is the unique maximizer of $L(\cdot, z^*, \eta^*)$ (which is, for example, the case case if $L(\cdot, z^*, \eta^*)$ is strictly concave). These ideas are further developed in Messner et al (2011) and Sleet and Yeltekin (2010a). Recently, Cole and Kubler (2010) and Marimon et al (2011) have shown how the optimal policy might be recovered in non-strictly concave recursive problems (in which checking complementary slackness is not straightforward).