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ABSTRACT

Recent evidence on electronic limit order markets shows a growing use of undisclosed orders. This paper offers a theory for the optimal submission strategy in a limit order book where traders simultaneously select price, quantity and exposure, and choose among limit, market, reserve (partially undisclosed) and hidden (totally invisible) orders. Our findings show that to compete for the provision of liquidity in shallow markets relatively patient traders use reserve orders, whilst aggressive traders use hidden pegged orders to undercut depth at the top of liquid books. Undisclosed orders are effective defensive strategies against front running by parasitic traders, whereas they protect against picking-off by scalpers only in slow markets where Fill&Kill orders are not used. Finally, our results show that undisclosed orders increase market depth on the top of the book, but widen the inside spread; as a result they can benefit institutional investors but harm retail traders.

Electronic limit order markets are now the dominant structure for trading financial securities around the World. They are order-driven markets in which traders can either supply liquidity via limit orders or demand liquidity via market orders. Orders posted to the limit order book (LOB) must include instructions specifying sign, size, and possibly their price aggressiveness and degree of disclosure.

Recent empirical evidence¹ about traders' order submission strategies in electronic LOB shows the growing importance of undisclosed orders, which allow traders to limit their exposure by hiding part (reserve orders) or all (hidden orders) of their size. Like limit orders, reserve orders contain an instruction on the price beyond which submitters are not willing to trade; but unlike limit orders, they also contain a further instruction on the fraction of the order that is to remain undisclosed to the market. Hidden orders are instead totally invisible and can be posted at a limit price on the trading grid; they can alternatively be pegged to the best bid (offer) or, more frequently, to the spread midpoint.

In various markets around the World reserve orders account for a surprisingly large proportion of trading volume: more than 44% of Euronext volume, approximately 28% of the Australian Stock Exchange volume, more than 15% of total executions on INET and 16% of executed shares on Xetra. Hidden orders too are widely used: they are allowed on NASDAQ and on the most advanced European trading platforms (e.g. BATS, TradElect, Chi-x and Turquoise). From the still rare accessible data (Hasbrouck and Saar, 2009, and Chakrabarty and Shaw, 2010) we know that they are used on NASDAQ and in secondary markets for treasury bonds.

The introduction of these new order types has brought new challenges for both regulators and practitioners. If they allow undisclosed orders, regulators endogenously reduce pre-trade transparency, thus affecting both liquidity and price informativeness. So it is important for them to understand how the widespread use of undisclosed orders affects market quality. For practitioners as well, it is crucial to know the circumstances under which undisclosed orders constitute an optimal submission strategy.

Despite a growing body of empirical research, there is little theoretical guidance on the optimal choice of order exposure. This paper extends the existing literature on dynamic limit order markets by providing a new theory of optimal order submission strategies, supplementing the standard choice between limit and market orders with the possibility of a choice of order exposure.

Recent empirical evidence has shown that undisclosed orders are broadly used by large uninformed traders.² Accordingly, in our framework, undisclosed orders are amongst the trading strategies available to agents who hold no inside information and differ only in terms of their willingness to trade; they come to the market sequentially and choose their optimal submission strategy contingent on the state of the LOB.

The spectrum of trading strategies considered in the paper is variegated: in addition to market and limit orders, traders can opt for reserve or hidden orders, as well as Fill&Kill orders (F&K).³ They can also choose their degree of price aggressiveness, so they face a simultaneous three-dimensional choice among price, quantity and exposure.

Large traders who are not informed about the future value of the asset use undisclosed orders for three reasons: to compete for the provision of liquidity thus preventing other traders from undercutting their orders; to mitigate the cost of being picked off by fast traders at stale prices in case of an asset value shock; and finally to offset parasitic traders' strategies aimed at exploiting the visibility of large order sizes. To capture all these effects, in our framework both retail and institutional traders select their order placement strategies by taking into consideration the interaction with the two sides of the LOB (Parlour, 1998), as well as the impact of both picking-off (Foucault, 1999) and front running costs.⁴

We mainly concentrate on the first motive and build a model where large uninformed traders compete for the provision of liquidity by submitting undisclosed orders. We start with a framework where traders can use both reserve and standard hidden orders, and then extend it to include a special type of aggressive hidden orders - Mid-Point Peg Orders (MPP) - that are executed at the spread midpoint. We include this order type as it is becoming very popular among market

participants, being offered by exchanges to compete with those dark pools that have a derivative pricing rule linked to the primary market's spread midquote. To discuss the determinants of undisclosed orders and their effects on market quality, the results are compared with those of a benchmark model not allowing for undisclosed orders.

We then extend the model to give a foreword of the other motives that stimulate traders to use dark orders. To this end, we embed picking-off costs by adding to the list of market participants first scalpers who can pick off orders at stale prices, and finally parasitic traders who implement opportunistic strategies that take advantage of the price pressure produced by large orders.

Notice that when undisclosed orders are added to the list of the options available to traders in real financial markets, other market participants react by building trading programs aimed at discovering invisible debt. For this reason, and considering the increasing development of algorithmic trading,⁵ we also solve the model by allowing large traders to identify invisible depth. More precisely, we consider two specifications that differ according to the ability of traders to detect invisible liquidity: in the first one traders can add the Fill&Kill instruction to their orders, in the second one they can use algorithmic trading programs to perfectly detect depth on the opposite side of the LOB. These further extensions to the models with undisclosed orders allow us to discuss how the high frequency trading technology can interact with dark liquidity.

Reserve orders are used by relatively patient traders to compete for the provision of liquidity when the spread is wide, whereas hidden orders, more specifically MPP, are used by aggressive agents in deep markets to undercut existing limit orders at the top of the book. In equilibrium traders maximize the visible part of their reserve orders that still prevents undercutting. The use of reserve orders decreases with relative depth on the opposite side of the LOB, while the use of MPP increases, reverting the standard Parlour (1998) effect on order flow dynamics. Undisclosed orders offer protection against picking-off only in slow markets where scalpers do not use Fill&Kill orders to pick off hidden depth, and they can also be used as defensive strategies against quote matchers. When comparing the benchmark with the undisclosed order model, the results indicate that undis-

closed orders can increase market depth at the BBO as orders become concentrated at a single price; however, they can also widen the inside spread. The conclusion is that in evaluating the performance of undisclosed orders, regulators should consider that they can benefit institutional investors but be detrimental to retail traders.

The model offers several testable empirical predictions ranging from the complementarity of reserve and hidden orders, to the effects of undisclosed orders on market quality, to the relation between dark liquidity and fast trading, that, as explained in detail in Section 6, can be tested empirically by using high frequency data either on executed trades or on undisclosed orders as soon as information providers and exchanges will make them available.

Even more interestingly, the model suggests that empirical investigations of the bid-ask spread should consider a new component that is caused by exposure costs. This is determined by the fact that traders submitting large limit orders sustain “exposure costs” that can arise from the three sources discussed above, i.e. competition for liquidity provision, picking-off by scalpers and front running by parasitic traders. For example, when traders run the risk of a price war, they submit hidden orders to prevent undercutting and therefore widen the inside spread by inducing incoming traders to join the queue at prices away from the best bid-offer.

The remainder of the essay is organized as follows. Section 1 discusses the literature on undisclosed orders, Section 2 describes the structure of the benchmark model and Section 3 presents the model with competition for liquidity provision. Section 4 extends the model to include MPP, Section 5 discusses the effects of picking-off and front running risk, Section 6 presents some empirical and policy implications, and Section 7 concludes. All the proofs are gathered in the Appendix.

1 The Literature on Undisclosed Orders

Most of the literature on undisclosed orders is empirical, with few theoretical works; in addition, most empirical analysis focuses on reserve (or iceberg) orders. Aitken et al. (2001) show that in the Australian stock market there is no difference in the price reaction to disclosed and undisclosed limit

orders and find that traders use reserve orders more intensively when the tick size is smaller, and volatility and order size are greater.⁶ Bessembinder, Panayides and Venkataraman (2009) study the costs and benefits of iceberg orders at Euronext and find that iceberg orders bear smaller implementation shortfall costs⁷ and that patient traders value more than impatient the option to hide. They also show that the presence and magnitude of undisclosed orders can be partly predicted based on orders attributes, firm distinguishing features and market conditions. Furthermore, Bessembinder et al. (2009) and Harris (1996, 1997) show that traders are more likely to hide their orders when competition is intense (i.e. the tick size is small and the trade size is large). Pardo and Pascual (2006) study market reaction to the presence of iceberg orders on the Madrid Stock Exchange and find that hidden volume detection has no significant impact on returns and volatility. De Winne and D'Hondt (2007) show that traders become significantly more aggressive when there is a signal of hidden depth at the best quotes on the opposite side of the market. They also show that traders tend to hide larger amounts when their order is large relative to the displayed depth and conclude that traders use hidden quantity to manage both exposure and picking-off risk. Finally, Frey and Sandas (2009) find that iceberg orders facilitate the search for latent liquidity as they tend to strongly attract market orders when they are discovered by market participants; they also show that the greater the fraction of an iceberg order that is executed, the smaller its price impact. Nevertheless, Tuttle (2006) and Belten (2007) suggest information content of reserve depth. Tuttle looks at the NASDAQ SOES market makers' quotes: she shows that hidden size adds liquidity to the market and that it is used more intensively in stocks with higher probability of informational event; in addition she finds that the presence of hidden depth at the time of a trade is a significant predictor of midquote revision. Using data from the Copenhagen Stock Exchange, Belten shows that hidden depth bears more information content than displayed depth, but trading based on information from both depths does not yield positive returns.

While there exists an extensive empirical literature showing that reserve orders are used both in the US and in the European electronic limit order books, there is barely no empirical evidence on

hidden orders. An exception is Hasbrouck and Saar (2009) who, by using data on orders submissions and cancellations, show that the majority of the orders that are cancelled within two seconds of submission are priced better than the prevailing Island bid or offer -to achieve an execution against hidden depth. Another exception is Chakrabarty and Shaw (2010) who find that, still on INET, hidden orders activity increases around earnings announcements.

Theoretical works on undisclosed orders are few indeed. To our knowledge, only two models explicitly include undisclosed orders. Moinas (2007) proposes a sequential signaling game where reserve orders are used by one insider to trade large volumes without divulging his private information, but the model does not allow uninformed traders to use undisclosed orders, nor informed traders to demand liquidity; moreover, it does not embody the interaction between the two sides of the LOB. Esser and Mönch (2007) extend the literature on optimal liquidation strategies (e.g. Bertsimas and Lo, 1998; Almgren and Chriss, 2000; Mönch, 2004) to include iceberg orders: they determine the optimal limit price and peak size for an iceberg order in a static framework with no strategic interaction among traders.

2 General Framework

Following Bessembinder et al. (2009), Pardo and Pascual (2006), De Winne and D'Hondt (2007) and Frey and Sandas (2009), we build a model where undisclosed orders are chosen by uninformed traders. Three are the reasons why traders who are not informed about the fundamental value of the asset can use undisclosed orders: to compete for the provision of liquidity thus preventing other traders from undercutting their orders; to reduce the probability of being picked off by fast traders in case their order become mispriced following an asset value shock; and finally to avoid both the price impact that their large orders can generate when the top of the book is not sufficiently deep and the quote-matching strategies that can be implemented by parasitic traders attracted by their visible large orders.

After presenting in this Section the most general features that guide the choice of traders' optimal

order submission strategies and that constitutes the benchmark model (B) against which market quality is evaluated, in Section 3 we extend this framework to include undisclosed orders. We mainly concentrate on the “*Competition for the provision of liquidity*” motive that induces traders to use undisclosed orders to prevent undercutting. Within this framework, we allow traders to use reserve and standard hidden orders (R&H),⁸ whereas in the following Section 4 we focus on a special type of more aggressive hidden orders (hidden Mid-Point Peg Orders, MPP) that execute at the spread midpoint. Finally, to give a flavour of how picking-off and quote-matching risk can affect large traders’ choice of undisclosed orders, we extend the R&H protocol to the other two motives that drive uninformed traders to use undisclosed orders (Section 5).

In real financial markets, once traders are allowed to use undisclosed orders, the other market participants react by adopting trading tactics that are aimed at detecting hidden liquidity, and this, in turn, affects traders’ use of undisclosed orders. With the proliferation of hidden liquidity, the technology available to traders to detect invisible depth is becoming highly sophisticated: the development of liquidity-driven tactics is such that now traders can opportunistically use both aggressive orders that seek and cross dark liquidity, and more complex liquidity-seeking algorithms. These are evolutions of the first generation impact-driven algorithms that simply based their decision on the visible order book depth, and are now instead specifically designed to search hidden liquidity. To capture these most recent trends that are shaping actual trading platforms and to embed the market reaction to hidden liquidity, we investigate three specifications of the model with undisclosed orders, that differ according to the types of trading programs available to large traders in search of hidden liquidity. As summarized in Table 1, we consider first a case where traders can use only market orders to search hidden liquidity (M). Second, we give traders access to those orders (F&K) that allow them to walk up or down the book to hit undisclosed depth without being exposed to any signalling risk: as any unfilled part of these orders is immediately cancelled, they do not leave any detectable “footprint” on the LOB. Finally, we consent market participants to trade with the support of the most aggressive algo trading techniques that can perfectly spot hidden depth on

the opposite side of the market (Algo).⁹ This final specification allows us to draw some interesting considerations on the most recent generation of algo trading programs that seek hidden depth by tightly monitoring the order book, and on their interaction with the LOB.

[Insert Table 2 here]

2.1 The Benchmark Market

A market for a risky asset is conducted over a trading day divided into T periods: $t = 1, \dots, T$. The value of the risky asset at time t is v_t . Two categories of risk-neutral agents are active: large institutional traders, who can choose to trade up to j units, with $1 \leq j \leq 10$, and small retail traders, who trade α units, where, as will be clarified later, α is also equal to the equilibrium undisclosed portion of the reserve order. At each trading round nature chooses a large or a small trader with equal probability, and the incoming agent maximizes expected profits by choosing an optimal trading strategy that cannot be modified thereafter; however traders are allowed to cancel their orders. As in Parlour (1998), each agent is characterized by a type β_t that is drawn from the following uniform distribution:¹⁰

$$\beta_t \sim U[\underline{\beta}, \bar{\beta}] \quad \text{where} \quad 0 \leq \underline{\beta} \leq 1 \leq \bar{\beta} \quad (1)$$

Notice that the parameter β_t can be seen as an indication of the willingness to trade for the agent arriving at the market at time t .¹¹ Traders with extreme values of β_t value the asset either very low, or very high, and they are accordingly either the most eager sellers (low β_t) or the most eager buyers (high β_t); traders with a β_t near to 1 have the lowest willingness to trade. We also assume that the distribution of β_t is symmetric around $\beta = 1$.¹²

Each trader arriving at the market observes the LOB, which consists of a grid of six prices, three on the ask and three on the bid side. Hence the prices at which each trader can buy or sell are $A_{1,2,3}$ (ask prices) and $B_{1,2,3}$ (bid prices), with $A_1 < A_2 < A_3$ and $B_1 > B_2 > B_3$; for simplicity we

assume that these prices are symmetric around the common value of the asset, v_t . More precisely, traders can demand liquidity over the whole price grid, but offer it only at the first two levels of the book. This is because at A_3 and B_3 a trading crowd absorbs whatever amount of the risky asset is demanded or offered by the incoming trader. As in Seppi (1997) and Parlour (1998), the trading crowd prevents traders from bidding prices that are too far off the inside spread; in effect, this is only a theoretical shortcut to limit the price grid. It is further assumed that the minimum difference between the ask and the bid price ($A_1 - B_1$) is equal to the tick size, τ , that is the minimum price variation.

The state of the book at each period t , $b_t = [q_2^A, q_1^A, q_1^B, q_2^B]$, is characterized by the number of shares available at each price ($q_{1,2}^A, q_{1,2}^B$). The asset value remains constant between $t = 1$ and $t = T - 1$, but between time $T - 1$ and T a shock may occur¹³ so that v_T can either increase, remain constant, or decrease:

$$\begin{aligned} v_t &= V + \varepsilon_T & t = T \\ v_t &= V & \forall t = 1, \dots, T - 1 \end{aligned} \tag{2}$$

with:

$$\varepsilon_T = \begin{cases} +k \tau & \text{with prob} = x \\ 0 & \text{with prob} = (1 - 2x) \\ -k \tau & \text{with prob} = x \end{cases} \tag{3}$$

where $V > 0$ is constant and assumed for simplicity equal to one; k measures the size of the shock as a multiple of the tick size τ and $2x$ the probability that the shock will occur. Notice that by changing the values of these two parameters one can investigate different volatility specifications. The ask and bid prices after a positive or a negative price change are denoted by A_i^u (A_i^d) and B_i^u (B_i^d) respectively, with $i \in \{1, 2, 3\}$.

2.1.1 Order Types

The market modelled here features a standard limit order book that is regulated by price and time priority rules. Orders that price improve on the existing limit orders gain price priority; whereas the order submitted first in time has time priority on any other limit orders posted at the same price. When a trader arrives at the market, he chooses an order that maximizes his expected profits given his type (β) and the state of the LOB (b_t). Table 2 presents the possible orders that a large trader (Panel A) and a small trader (Panel B) can choose.

[Insert Table 2 here]

An aggressive large trader (Panel A), who wants to sell, can demand liquidity by submitting a market sell order of size j which will match with the limit buy orders with top precedence on the bid side. If the size j of this order is smaller than (or equal to) the number of shares available at the best price (B_i) on the opposite side of the market, we then label this order $MO_j B_i$; ¹⁴ if instead the size j is greater and the order has to walk down the book in search of execution, we then label the strategy $MO_j B$. ¹⁵ A less aggressive trader may choose a limit sell order of size j to either A_1 or A_2 ($LO_j A_{1,2}$). This order will be executed when one or more market buy orders arrive that hit the limit price after all the other orders on the book with either a lower price or a higher time priority have been executed. Finally, the trader can decide not to trade (NTL). Analogous strategies are available to a large trader who wants to buy. In real-world financial markets, traders could also split their limit orders, by submitting them either at different price levels or at different times of the day. We do not consider these strategies here, as they are dominated (a point clarified later). An aggressive small trader (Panel B) who wants to sell will demand liquidity with a market sell order ($MO_\alpha B_i$), and a less aggressive one will act as liquidity supplier by submitting a limit sell order either to the first level of the LOB ($LO_\alpha A_1$), or to the second one ($LO_\alpha A_2$). Finally, if the trader finds no profitable strategies, he can decide to refrain from trading (NTS). Similar strategies can be chosen by a small buyer.

2.1.2 Equilibrium Submission Strategies

A trader decides his optimal order submission strategy by simultaneously choosing the *sign*, the *size* and the *aggressiveness* of his order. Formally, the risk-neutral large trader chooses the optimal strategy, o_{L,β_t,b_t} , that maximizes his expected profits conditional on the state of the LOB, b_t , and his type, β_t . A large trader submits the order that maximizes the profits from all the available strategies:

$$\begin{aligned} & \max_{o_{L,\beta_t,b_t} \in [\Omega_{seller}^L, \Omega_{buyer}^L, NTL]} E[\pi_t(o_{L,\beta_t,b_t})] & (4) \\ \Omega_{seller}^L &= \{MO_j^z B_i^z, MO_j B^z, LO_j A_i\} \\ \Omega_{buyer}^L &= \{MO_j^z A_i^z, MO_j A^z, LO_j B_i\} \end{aligned}$$

where Ω_{seller}^L are the strategies available to large seller and Ω_{buyer}^L those available to a large buyer. Profits from not trading equal zero, $\pi_t(NTL) = 0$; profits from a market sell order of size $j \in [1, 10]$ that hits the quantity available at B_i^z are equal to $\pi_t(MO_j B_i^z) = j(B_i^z - \beta_t v_t)$, with $i \in [1, 3]$, $B_i^z = B_i$ for $t \neq T$ and $B_i^z \in \{B_i^u, B_i^d, B_i\}$ for $t = T$; profits from a j -market order that walks down the book are: $\pi_t(MO_j B^z) = \sum_i f_i(B_i^z - \beta_t v_t)$, where f_i is the number of shares executed at B_i^z with $\sum_i f_i = j$. Finally, expected profits from a limit sell order of size j are given by:

$$\begin{aligned} E[\pi_t(LO_j A_i)] &= E \left\{ (A_i - \beta_t \tilde{v}_{t+1}) \sum_{w_{t+1}=1}^j w_{t+1} \Pr(A_i | b_{t+1}, v_{t+1}) + \right. \\ & \left. I_t \times \left[\sum_{l=t+2}^T (A_i - \beta_t \tilde{v}_l) \sum_{W=0}^{j-1} \sum_{w_l=1}^{j-W} w_l \Pr(A_i | b_l, v_l) \Pr\left(\sum_{m=t+1}^{l-1} w_m = W | b_{l-1}, v_{l-1}\right) \right] \right\} \end{aligned}$$

where $\Pr_{w_l}(A_i | b_l, v_l)$ is the probability that w_l shares will be executed at $t = l$, W is the number of shares executed up to $t = l - 1$, and I_t is an indicator function equal to 0 for $t = T - 1$ and 1 otherwise. Notice that in this formula the first term indicates profits from shares executed in

the period immediately following the order submission; the second term, instead, denotes expected profits from execution in the subsequent periods. Profits for the buyer's strategies are computed in a similar way and hence omitted.

The small trader solves an analogous problem:

$$\begin{aligned} & \max_{o_{S,\beta_t,b_t} \in [\Omega_{seller}^S, \Omega_{buyer}^S, NTS]} E[\pi_t(o_{S,\beta_t,b_t})] \\ \Omega_{seller}^S &= \{MO_\alpha B_i^z, LO_\alpha A_i\} \\ \Omega_{buyer}^S &= \{MO_\alpha A_i^z, LO_\alpha B_i\} \end{aligned} \tag{5}$$

where, for example, profits for the sellers' strategies are given by:

$$\begin{aligned} E[\pi_t(LO_\alpha A_i)] &= E \left\{ (A_i - \beta_t \tilde{v}_{t+1}) \sum_{w_{t+1}=1}^{\alpha} w_{t+1} \Pr(A_i | b_{t+1}, v_{t+1}) + \right. \\ & \quad \left. I_t \times \left[\sum_{l=t+2}^T (A_i - \beta_t \tilde{v}_l) \sum_{W=0}^{\alpha-1} \sum_{w_l=1}^{\alpha-W} w_l \Pr(A_i | b_l, v_l) \Pr\left(\sum_{m=t+1}^{l-1} w_m = W | b_{l-1}, v_{l-1}\right) \right] \right\} \\ \pi_t(MO_\alpha B_i^z) &= \alpha(B_i^z - \beta_t v_t) \end{aligned}$$

with $i \in [1, 3]$, $B_i^z = B_i$ for $t \neq T$ and $B_i^z \in \{B_i^u, B_i^d, B_i\}$ for $t = T$; $\Pr_{w_l}(A_i | b_l, v_l)$ is the probability that w_l shares will be executed at $t = l$. As before, I_t is an indicator function equal to 0 for $t = T - 1$ and 1 otherwise, and W indicates the shares executed before $t = l$.

Equilibrium definition *An equilibrium of the trading game is a set of orders o_{L,β_t,b_t}^* and o_{S,β_t,b_t}^* that solve Program (4) and (5), when the expected execution probabilities, $\Pr_{w_{T-l}}(A_i | b_{T-l}, v_{T-l})$, are computed assuming that traders submit the orders o_{L,β_t,b_t}^* and o_{S,β_t,b_t}^* .*

We solve the model by backward induction, assuming that the tick size is equal to $\tau = 0.1$,¹⁶ and we focus on the last three periods of the trading game. To obtain numerical values for the equilibrium probabilities, we assume that β is uniformly distributed with support $[0, 2]$. The equilibrium strategies resulting from the benchmark model are of crucial relevance as they are compared from

next Section onwards with those obtained from the protocols with undisclosed orders

3 Competition for the Provision of Liquidity

We now extend the model to include undisclosed orders and we focus on traders' willingness to compete for the provision of liquidity, that is one of the motives that move traders to use these orders. Traders compete on prices when there is room in the book that allows undercutting; hence to enforce competition we assume that at $T - 2$ the LOB opens empty. We also use the simplest possible framework with $x = \frac{1}{2}$ and $k = 1$. This means that at time T the asset value goes up or down by one tick with equal probability, as shown in Figure 1.

[Insert Figure 1 here]

3.1 Equilibrium Submission Strategies

A large trader now decides his optimal order submission strategy by simultaneously choosing not only the *sign*, the *size* and the *aggressiveness* of his order, but also the *degree of exposure*. Indeed the trader has the additional option to hide the quantity he wants to submit to the LOB. He can choose a j -reserve sell order ($RO_j A_{1,2}$) and in this case he will have to decide which part of the order to disclose and which not to, bearing in mind that the hidden part of the reserve order loses time priority with respect to the other limit orders submitted at the same level of the book. He can also opt for a hidden order ($HO_j A_{1,2}$), and in this case the entire order is not visible to market participants. Hence, to determine his optimal trading strategy he solves the following program that, compared to the previous one, includes undisclosed orders:

$$\begin{aligned} & \max_{o_L, \beta_t, b_t \in [\Omega_{seller}^L, \Omega_{buyer}^L, NTL]} E[\pi_t(o_L, \beta_t, b_t)] & (6) \\ \Omega_{seller}^L &= \{MO_j^z B_i^z, MO_j B^z, LO_j A_i, RO_j A_i, HO_j A_i\} \\ \Omega_{buyer}^L &= \{MO_j^z A_i^z, MO_j A^z, LO_j B_i, RO_j B_i, HO_j B_i\} \end{aligned}$$

Notice first that now profits from a j -market order that walks down the book become uncertain, as there could be hidden liquidity available on the book. So the trader will rationally compute the probability $\Pr_{f_i}(B_i^z|b_i^z)$ that f_i shares are available at B_i^z . Profits from this order are now equal to: $E[\pi_t(MO_j B^z)] = \sum_i f_i(B_i^z - \beta_t v_t)\Pr_{f_i}(B_i^z|b_i^z)$, with $\sum_i f_i = j$. The profit formula for limit orders are equal to those discussed for the benchmark framework even though here traders, when rationally computing $\Pr_{w_l}(A_i|b_l, v_l)$, will have to take into account the possible presence of hidden depth. The profit formula for a reserve or a hidden order is similar to the one for the limit order presented in Section 2.1.2 and is hence omitted. Clearly, this does not mean that the two strategies, limit versus either reserve or hidden, will return the same profits, as the execution probability of the hidden part of the undisclosed order differs from the corresponding visible part of a limit order posted at the same price.

The small trader still solves program (5), however, as discussed for the large trader's optimization program, his profits from market orders are no longer certain as, depending on the state of the LOB, he may suspect the existence of hidden liquidity.

As for the benchmark, we solve the model by backward induction, assuming that the tick size is equal to $\tau = 0.1$. To obtain numerical values for the equilibrium probabilities, we assume again that β is uniformly distributed with support $[0, 2]$.

It should be noticed that to simplify the algebra, we restrict small traders to choose orders of only one size that we set equal to the equilibrium large traders' reserve peak, i.e. that part of the reserve order that is disclosed. That the size of the orders submitted by small traders is indeed the same as the equilibrium peak size chosen by traders who submit reserve orders is crucial: to prevent other market participants from easily detecting undisclosed depth, when choosing the peak size of their reserve orders, large traders seek camouflage behind small traders. Hence, first we determine the optimal disclosed part of a reserve order (α^*), by assigning different values to α (with $0 < \alpha < j$) and choosing the one that maximizes their profits; then, to simplify the analysis, we set the small traders' order size precisely equal to α^* . To determine the optimal visible size of a reserve order,

α^* , optimization problems (5) and (6) are solved for all the possible values of $\alpha \in [1, 9]$.

3.2 Traders' Strategies: an example

Figure 2 shows an example of the extensive form of the game with $\alpha = 3$ and $j = 10$. Assume that at $T - 2$ the market opens with an empty book, $b_{T-2} = [0000]$, and from period $T - 2$ onwards traders' orders gradually fill the LOB.

[Insert Figure 2 here]

Suppose, for example, that nature selects a large trader at $T - 2$ who decides to submit a $LO_{10}A_2$. In this case his unitary profits are equal to the difference between the price at which he sells and his evaluation of the asset, multiplied by the probability that the order will be executed, and therefore his total expected profits are equal to:

$$E[\pi_{T-2}(LO_{10}A_2)] = \Pr_{10}(A_2|b_{T-1}, v_{T-1}) \times 10 \times (A_2 - \beta_{T-2}v_{T-1}) + \\ \Pr_3(A_2|b_{T-1}, v_{T-1}) \left\{ 3 \times (A_2 - \beta_{T-2}v_{T-1}) + E[(A_2 - \beta_{T-2}\tilde{v}_T) \times [3\Pr_3(A_2|b_T, v_T) + 7\Pr_7(A_2|b_T, v_T)]] \right\} + \\ \Pr_0(A_2|b_{T-1}, v_{T-1}) E[(A_2 - \beta_{T-2}\tilde{v}_T) \times [3\Pr_3(A_2|b_T, v_T) + 10\Pr_{10}(A_2|b_T, v_T)]]$$

Notice that in this formula, the three terms on the RHS of the equation refer respectively to the following possible execution paths at $T - 1$: first, a large incoming trader who buys the whole order of size 10 at A_2 with probability $\Pr_{10}(A_2|b_{T-1}, v_{T-1})$, second, a small trader buying 3 units, and finally, no one hitting the order at $T - 1$. Clearly the unfilled part of the order will be executed at T , provided a market order arrives from the opposite side of the market that hits A_2 . This means that there is no certainty about the execution of a limit order.

If instead the large trader chooses a market sell order ($MO_{10}B_3$), then his order is executed with certainty and his payoff is equal to:

$$E[\pi_{T-2}(MO_{10}B_3)] = 10 \times (B_3 - \beta_{T-2}v_{T-2})$$

Consequently, our model embeds one of the most basic trade-off between market and limit orders in that market orders are executed with certainty but at the most aggressive price on the opposite side of the book, whereas limit orders obtain better prices but at the expenses of an uncertain execution.

If the incoming trader at $T - 2$ actually decides to submit a $LO_{10}A_2$, then at $T - 1$ the book will open with ten shares on A_2 ($b_{T-1} = [(10)000]$), and if the trader arriving at $T - 1$ chooses to undercut this order with a $LO_{10}A_1$, his expected profits are:

$$E[\pi_{T-1}(LO_{10}A_1)] = E[(A_1 - \beta_{T-1} \tilde{v}_T) \times [3 \Pr_3(A_1|b_T, v_T) + 10 \Pr_{10}(A_1|b_T, v_T)]]$$

Given this sequence of orders, the resulting strategies available to the trader who arrives at the market at T (in the event, say, of a positive asset value shock) will be $MO_{10}B_3$, NTL and $MO_{10}A_1$ if he is a large trader, and MO_3B_3 , MO_3A_1 and NTS if he is small. Indeed at time T the market closes and traders only submit market orders as the execution probability of limit orders is zero.

If instead traders choose reserve or hidden orders, the book's depth becomes uncertain. For example, if at $T - 2$ the large trader elects a reserve order to sell ($RO_{10}A_2$), then the book will open at $T - 1$ as $b_{T-1} = [(3 + 7)000]$; however if alternatively at $T - 2$ there happens to be a small trader who selects LO_3A_2 , the opening book will be $b_{T-1} = [3000]$. In both cases the LOB at $T - 1$ shows three units on A_2 and the incoming trader will be uncertain on whether the book has any undisclosed depth: he will accordingly rationally compute the probability of each possible state of the LOB and trade accordingly. Similarly, if a large trader chooses a hidden order to sell ($HO_{10}A_2$), the book will open with 10 undisclosed shares on A_2 ($b_{T-1} = [(0 + 10)000]$) and the next trader will have to estimate the available depth.

Assume now that a large trader wants to weigh the pros and cons of selling the asset by using undisclosed orders at A_1 or A_2 . Given that the difference between A_1 and B_1 is equal to the tick size, orders on the top of the book are not exposed to price competition. Therefore undisclosed orders posted to A_1 have no advantage over limit orders as they cannot be undercut. Moreover,

as they lose time priority on the hidden part, they have lower execution probability and therefore they are dominated strategies. An undisclosed order on A_2 , on the other hand, presents advantages and disadvantages compared with a 10-share limit order on A_2 or A_1 , which are the other two alternatives available to non-aggressive traders. Compared with $LO_{10}A_2$, an undisclosed order might induce the next trader to refrain from undercutting by submitting an order at A_1 ; compared with $LO_{10}A_1$, the undisclosed order gains the tick size but pays the cost of lower execution probability. This example suggests that when traders strategically choose an undisclosed order or any other order, they compute the execution probabilities up to time T and then compare the expected profits associated with all the available orders, conditional on the state of the LOB and, of course, their type.

3.3 Optimal Undisclosed Orders and Market Reaction

Considering that traders solve programs (5) and (6), we find the solution of this game by backward induction, starting from the end-nodes to compute the probabilities of market orders at time T . These are the execution probabilities of limit orders placed at $T - 1$ that allow us to compute the equilibrium order submission strategies in that period. Similarly we compute the equilibrium order submission strategies at $T - 2$. We then solve the game for the possible values of α to determine the optimal visible size of reserve orders (the “reserve peak”). As the model is basically symmetric, we present results only for the branch of the trading game that starts with a seller at $T - 2$.¹⁷

Up to here (M framework), the reaction of market participants to the introduction of undisclosed orders is that they constantly monitor the state of the book and estimate how the probability of hidden depth at each price level affects the execution probability of their market orders. However, in real financial markets traders react to the presence of hidden depth not only by resorting to market orders, but also by choosing more sophisticated order types. These orders contain a fill instruction named Fill&Kill according to which any unfilled part of the order is immediately cancelled.¹⁸ To incorporate this more sophisticated reaction by market participants to the intro-

duction of undisclosed orders, we extend the model with R&H to include F&K orders: aggressive large sellers can now submit orders of size j and limit price B_i ($F\&K_j B_i$) that hit the bid side of the market (analogous strategy is available to large buyers). In this case if there are fewer than j shares available (both visible and invisible) up to B_i , then the unexecuted part of the order will be cancelled. Technically, we just add this new option to the list of orders available to large traders -see program (6)- and compute the profits from this new trading strategy as follows:

$$E[\pi_t(F\&K_j B_i^z)] = \sum_{i \leq \bar{i}} f_i (B_i^z - \beta_t v_t) \Pr(B_i^z | b_t^z)$$

where f_i is the number of shares executed at B_i^z with $\sum_i f_i = j$.

Notice that to discuss the impact of undisclosed orders in a framework where traders can also access F&K orders, we do not need to change the initial benchmark as it is neutral to the inclusion of these orders. The F&K instruction is used by traders to spot hidden liquidity on the opposite side of the book, and therefore it is not an equilibrium strategy in the benchmark model where there is no hidden depth.

The introduction of F&K orders allows us to comment on the effect of the use of add-hoc trading strategies aimed at discovering undisclosed liquidity. Markets, however, are evolving at a great velocity and with the advent of the most advanced trading technology, the limit between transparency and opacity is blurring. We therefore also consider the limiting case in which traders can perfectly spot hidden liquidity on the opposite side of the market. Of course if traders could spot hidden liquidity on both sides, there wouldn't be any incentive left to use undisclosed orders; however this extreme case is rather unrealistic as in order to do so traders should take a position contrary to their trading interest. An example may be clarifying: a seller can use algo trading programs to spot hidden pools of liquidity on the buy side, but doing the same on the sell side would imply that during the discovery process the trader would be buying instead of selling and hence incur losses. Notice also that the cost of searching hidden liquidity on the own side is nowadays amplified by the anti-gaming features - like 'minimum execution size'- that are generally associated with most

undisclosed facilities. We therefore consider the case with traders only being able to spot liquidity on the opposite side of the LOB and we name it “Algo” protocol.

The following Proposition summarizes the model’s result for the three different specifications considered, that, as discussed above, differ according to traders’ ability to spot undisclosed liquidity.

Proposition 1

When undisclosed orders are used by traders to compete for liquidity provision, and traders do not have access either to Fill&Kill orders or to algorithmic trading programs,

- *reserve orders are equilibrium strategies at $T - 2$ and are posted to prevent undercutting by traders arriving at $T - 1$;*
- *traders choose the maximum disclosed size of reserve orders that still prevents undercutting.*

When, all else equal, Fill&Kill orders are added to the list of the possible trading strategies, traders use reserve orders more intensively. Moreover, the probability to observe Fill&Kill orders increases with the probability that traders assign to undisclosed liquidity.

Finally, when traders have also access to algorithmic trading programs, the probability that traders use reserve orders further increases.

Reserve orders are optimal submission strategies with probability .258 (Table 3) and are selected by relatively patient traders who come to the market at time $T - 2$ with a β close to 1.¹⁹ Within this context, reserve are preferred to hidden orders as they allow traders to hold time priority on the visible part of the order, while still preventing undercutting. Indeed one can observe from Table 3 that when for example a 10-unit limit order is submitted at A_2 , the next large trader will undercut it at A_1 with probability .130 ($LO_{10}A_1$), while when a 10-unit reserve order is posted at the same price level, he will join the queue at A_2 with probability .136 (LO_7A_2). Clearly at $T - 1$ traders anticipate that at time T there will be no undercutting and hence do not use either reserve or hidden orders: they lose time priority vis-à-vis limit orders, with no countervailing advantage.

When opting for a reserve order, a trader must choose the optimal disclosed and undisclosed portions. On the one hand, he would like the largest possible part of the order to be visible, as this increases execution probability; however, by increasing the visible size at A_2 , he also increases the incentive for next traders to undercut at A_1 . Our results show that the optimal proportion of visible to undisclosed size is 3 to 7 shares.

Our three-period framework implicitly assumes that all traders have a relatively short time horizon to execute their trade. For this reason they highly value the possibility of maintaining the peak of their undisclosed order visible and hence when faced with the option of choosing between reserve and totally invisible orders, they opt for the former. Indeed when choosing the degree of visibility, traders face the trade-off between execution costs and exposure costs: by increasing their order's visibility, they minimize execution costs, yet they increase exposure costs as the probability of being undercut by incoming traders increases. The length of the trading horizon influences this trade-off as the longer the horizon the smaller execution costs compared to exposure costs. Hence, in a framework where traders had a time horizon longer than three periods, we would expect to observe both reserve and hidden orders as equilibrium strategies, the latter being chosen by particularly patient traders. Future research may tackle this issue by focusing on this specific feature.

[Insert Table 3 here]

We now consider the more realistic extended framework where traders can also access to Fill&Kill orders: traders use F&K orders when they suspect the existence of hidden depth at some level of the book.²⁰ More precisely in our framework rational traders anticipate that reserve orders will be used at A_2 . When the probability of hidden depth is large enough to ensure that the expected profits from a F&K order are sufficient to offset the risk of walking up or down the book in search of complete execution through a market order, they choose the former. This is shown for example in Table 3 when the book opens at $T - 1$ with 3 visible shares at A_2 . More interestingly, Table 3 shows that when at $T - 2$ traders anticipate that in the following period market participants will choose F&K rather than market buy orders, they internalize the increased execution probability of

hidden depth and use reserve orders more intensively.

In this regard, when we extend the model to the limiting case where, by means of algo programs, large traders can perfectly detect undisclosed depth on the opposite side of the book, the probability that in equilibrium incoming agents at $T - 2$ will choose a reserve order (.282) increases even more compared to the framework with only F&K instructions (.266); this is due to the increased execution probability of the invisible part of the reserve order.

These results have a very interesting empirical implication as they imply a positive correlation between reserve orders used to compete for the provision of liquidity and fast trading facilities such as Fill&Kill orders and/or algorithmic programs aimed at discovering hidden liquidity.²¹ Indeed market participants interpret the use of these programs as a signal of dark liquidity, and estimate accordingly the probability that their orders can be executed against undisclosed liquidity at different levels of the book (Bongiovanni et al. , 2006).

A final observation hinges on the widespread practice of splitting orders that do not appear among the available strategies. Given time priority, splitting orders on the same level of the book would always be dominated by reserve orders: the hidden portion is automatically disclosed upon execution, thus gaining priority compared to the second part of the split. Nor would splitting different proportions of the order on two levels of the book ever be optimal, as this would induce competitors to join the queue at the most aggressive price.

3.4 Market Quality

In light of the growing use of undisclosed orders and having shown how they can also be related to high frequency trading, it is relevant for regulators to determine whether the widespread adoption of these orders improves market quality. To this end, we compare the model with undisclosed orders to a benchmark model where, all else equal, traders are not allowed to hide liquidity. With reference to Table 1, we evaluate the effects on expected depth and volume, as well as semi-inside spread (effective and weighted)²² of the introduction of undisclosed orders (R&H) under the three

regimes considered so far: no specific tools to detect dark liquidity (M), partial detection by means of the F&K instruction (F&K), and perfect detection via algo trading (Algo). It is worth reminding the reader that the benchmark model doesn't change under the three regimes.

When traders are allowed to use undisclosed orders, we expect a clustering of depth at one price, and hence an increase in depth at the BBO; we also expect a wider spread as undisclosed orders prevent traders from engaging in a price war, and a decrease in trading volume due to the lower visibility of standing liquidity. The following Proposition summarizes the results:

Proposition 2

When undisclosed orders are used by traders to compete for liquidity provision, depth increases, the inside spread widens and volumes decrease.

When, all else equal, traders have the additional option to search hidden liquidity by using F&K or algo programs, depth further increases and the effects on spread and volume are smaller.

By looking at Table 4, we notice that the results obtained are indeed consistent with previous conjectures so that empirically we anticipate wider spread, greater depth and smaller volume associated with an increasing use of undisclosed orders. Notice however that, as shown in Proposition 1, with F&K or algo programs traders use reserve orders more intensively, which explains the further increase in depth at the BBO. Moreover the increased attractiveness of reserve orders due their higher execution probability induces traders to provide liquidity hence reducing the negative effect on spread.

As for volume undisclosed orders introduce uncertainty on the state of the book and hence on traders' execution price. However, when Fill&Kill orders are available, traders can reduce this uncertainty by fixing a price threshold beyond which their order will be cancelled if not completely executed. As a result volume still decreases (compared to the benchmark) but less than in the case without Fill&Kill orders. Finally, with algo programs traders can spot liquidity on the opposite side and they can also take advantage of the depth enhanced by undisclosed orders with the result that,

despite the wider spread, volume increases. Clearly this result crucially depends on the effectiveness of algo trading programs in detecting hidden depth.

[Insert Table 4 here]

The results presented on depth at the BBO are consistent with the ones obtained by Anand and Weaver (2004) on the introduction of reserve orders at the Toronto Stock Exchange, namely that the depth at the inside increases significantly when traders are allowed to use reserve orders. Similarly, Bessembinder and Venkataraman (2004) find for the Paris Bourse that reserve orders augment depth and lower the implicit transaction costs of block trades.

Our results carry significant regulatory implications. Since undisclosed orders enhance market depth, their widespread use may be beneficial to institutional investors, and therefore it can be promoted for wholesale markets. However, our results also suggest that undisclosed orders widen the inside spread, and so could be detrimental to retail traders.

3.5 Discussion

The main purpose of this paper is to investigate the role of exposure costs in securities trading. Its main new contribution, in fact, is to show that these costs can be reduced by using undisclosed orders. To this end, it is crucial to build a framework in which traders can submit orders of different sizes: without trades of at least two different sizes, the detection of hidden quantities would be straightforward, so reserve orders would always be dominated by limit orders. We accordingly model the market as a trading game that finishes at T , and can be solved by backward induction; we use this methodology as the existing models with stationary equilibrium are not suitable for incorporating this essential feature. As Rosu (2009) suggests, his stationary Markov equilibrium would possibly allow multiple submission of 1-unit orders, but not block trading. Similarly, neither Foucault (1999) nor Foucault et al. (2005) would be adequate to model undisclosed orders. In the former, not only are different sizes of order not envisaged, but traders cannot even compete to provide liquidity, as the book is always either empty or full: in the period following its submission,

a limit order is either executed or cancelled. For the latter, the crucial assumption necessary to find a stationary solution is that traders always improve the price when submitting their 1-unit orders, precluding the possibility for an incoming trader to join the queue; thus by construction it eliminates all the potential benefits of using undisclosed orders to reduce competition.

Our finite-horizon model that is solved by backward induction allows us to find a closed-form solution for a market in which traders' strategies include orders of different sizes, undisclosed orders and freedom to choose between price improving and joining the queue. Moreover, in this framework traders not only condition their order submission decisions to the current state of the LOB, but also strategically take into account the effects of their own orders on the dynamic of the book.

4 Competing aggressively for liquidity: Hidden Mid-Point Peg Orders

So far we have shown that when traders use undisclosed orders to prevent undercutting, they prefer reserve to hidden orders as, due to their relatively short trading horizon, they highly value the time priority of the visible part of their orders. There is however an order type that even though totally undisclosed, can still attract traders who compete for the provision of liquidity in the short run, being perceived as relatively aggressive. This is a hidden order that can be pegged to the midpoint of the NBBO (Mid-Point Peg Order) and that is nowadays offered by most electronic limit order books around the world (e.g. BATS Europe, Chi-X, TradElect and Turquoise).²³ Notice that in light of the fierce competition taking place in today's financial markets, the advantage of MPP is that they can aggressively compete with the liquidity supply from both the lit markets, and those dark pools that, as Liquidnet, Pipeline and ITG, execute at the inside spread midquote.²⁴

We now modify the model by allowing traders to choose MPP -to sell (HOS_jM) or to buy (HOB_jM)- rather than reserve or hidden orders. As within this framework the size of small

orders is not relevant, for generality we assume that $\alpha = 1$. Formally, we assume that a large trader arriving at the market at time t chooses the optimal order submission strategy, o_{L,β_t,b_t} , that solves the following problem:

$$\max_{o_{L,\beta_t,b_t} \in [\Omega_{seller}^L, \Omega_{buyer}^L, NTL]} E[\pi_t(o_{L,\beta_t,b_t})] \quad (7)$$

where now the seller's strategies are $\Omega_{seller}^L \in \{MO_j B_i^z, MO_j B^z, LO_j A_i, HOS_j^z M\}$; profits for all orders are unchanged compared to problem (4), and for example profits from a MPP order, $HOS_j M$, are :

$$E[\pi_t(HOS_j^z M)] = E \left\{ (M_t^z - \beta_t v_t^z) \sum_{w_t=1}^j w_t \Pr_{w_t}(M_t^z | b_t, v_t) + I_t \times \left[\sum_{l=t+1}^T (\widetilde{M}_l - \beta_t \widetilde{v}_l) \sum_{W=0}^{j-1} \sum_{w_l=1}^{j-W} w_l \Pr_{w_l}(\widetilde{M}_l | b_l, v_l) \Pr(\sum_{m=t}^{l-1} w_m = W | b_{l-1}, v_{l-1}) \right] \right\}$$

where \widetilde{M}_t is the midquote at time t that will depend on the state of the book b_t and on the asset value v_t , $\Pr_{w_l}(\widetilde{M}_l | b_l, v_l)$ is the probability that, at $t = l$, w_l shares are executed at \widetilde{M}_l , W is the number of shares executed up to $t = l - 1$, and I_t is an indicator function equal to 0 for $t = T - 2$ and 1 otherwise. Notice also that, differently from other limit orders, MPP can have immediate execution, provided another MPP of opposite sign is standing in the LOB. The small trader still solves program (5). As for the previous framework, we solve the model by backward induction considering three different scenarios, with and without Fill&Kill orders, and with algo trading. Notice that MPP can be attractive strategies not only when the book is empty, but especially when there is no room to compete on prices in the LOB. So we will consider two initial states of the book at $T - 2$, an empty LOB, $b_{T-2} = [0000]$, and a deep one, $b_{T-2} = [(10)00(10)]$. The results are summarized in the following Proposition.

Proposition 3

Traders use of Mid-Point Peg Orders to compete for the provision of liquidity depends on the state

of the book. When the book is empty on traders' own side, they choose Mid-Point Peg Orders to compete for the provision of liquidity only if they suspect that someone else submitted a MPP in the previous period. When the book is deep on traders' own side, MPP are equilibrium strategies.

Traders use MPP more intensively when:

- depth moves to the top of the own side of the book, or decreases on the other side;
- volatility increases and time to shock approaches;
- algorithmic programs are used to discover hidden liquidity.

Fill&Kill orders have no effect on the use of MPP.

Following the introduction of MPP, spread and depth increase, while volume decreases.

In terms of aggressiveness, MPP are in between market and limit orders as they seek execution at prices that are less aggressive than the best opposite quote, and at the same time they are ready to wait for an order to arrive with an opposite sign.²⁵ Noticeably, when choosing their order strategy, traders face the standard trade-off between price risk and execution risk. This means that when they decide to supply liquidity, even at a very aggressive price such as the spread midquote, they waive certainty of execution for better prices: when the trade-off becomes too expensive in terms of opportunity costs, they switch to market orders.

Clearly, when the book opens empty at $T - 2$, traders have room to undercut the existing liquidity by submitting limit orders either at A_1 or at A_2 and hence there is no need to undercut existing orders using MPP (Table 5). However, when the book opens empty at $T - 1$, yet traders suspect that someone else posted a MPP on the other side of the market as they observed no change in the book's depth at $T - 2$, they indeed submit $HOS_{10}M$ with probability .104. Traders also use MPP when the book is deep on their own side: if at $T - 1$ they observe 10 shares on A_2 , they use Mid-Point Peg Orders with probability .079, that increases to .119 as liquidity moves to the top of the book on A_1 . Indeed, when at $T - 1$ traders observe 10 shares on A_1 , $HOS_{10}M$ are used more

extensively to aggressively undercut standing limit orders. Notice also that the visible depth must be substantial as 1 share on A_2 or A_1 does not trigger any MPP.

[Insert Table 5 here]

That an increase in depth enhances MPP is confirmed at $T - 2$: when the book opens deep with 10 shares on both A_2 and B_2 , traders post Mid-Point Peg Orders on both sides (Table 6). And once again by looking at $T - 1$, one can notice that when liquidity moves to A_1 , $b_{T-1} = [(10)(10)0(10)]$, traders use MPP more intensively. This latter case is particularly interesting as it shows that when traders observe depth also on B_2 , market orders increase and crowd out MPP. Indeed when comparing this book with the one considered in Table 5, with still 10 shares on A_1 but no depth on A_2 and B_2 , $b_{T-1} = [0(10)00]$, one can observe an increase in market orders from .331 to .402. So we can conclude that aggressive limit orders as MPP become less attractive for traders when they can get certain execution at better prices by submitting market orders.

[Insert Table 6 here]

To summarize the results presented in Table 5 and 6, we expect to see hidden liquidity increasing with market depth on the own side of the book as well as with expected hidden liquidity. Conversely, we expect to observe hidden orders decreasing when the other side of the market becomes deeper, as traders switch to more aggressive market orders. Considering that for limit orders this effect is just the opposite, as own side depth reduces the use of limit orders, whilst depth on the other side increases it (Parlour, 1998), our model offers a new empirical implication for the dynamic pattern of order flow. Empirically it should be possible to disentangle the interaction of depth with MPP and limit orders respectively, and in turn verify their effect on the probability of continuation and reversal.

As MPP are pegged to the spread midpoint, one could argue that traders actually choose them to avoid mispricing rather than to compete for liquidity provision. To verify this conjecture, we have solved the model by assuming that the asset value does not vary at time T : if traders still use MPP,

we can safely conclude that they are used to compete for the provision of liquidity. Interestingly, the difference between the probabilities to observe $HOS_{10}M$ under the two regimes gives a proxy of the degree of protection of MPP against mispricing. Results reported in Table 7 show that both when the book opens empty at $T-2$ and when it opens with 10 shares on the second level, MPP are still equilibrium strategies at $T-1$. Yet, the reason why the equilibrium probabilities of $HOS_{10}M$ are smaller without volatility is that limit orders at A_1 are more convenient due to the lower price risk.

Notice also that MPP are used more intensively as time to shock approaches: for example, if we compare the book $[(10)00(10)]$ at $T-2$ with the same book at $T-1$, we observe that the probability of $HOB_{10}M$ increases from .016 to .043 (Table 6).

[Insert Table 7 here]

As for the previous framework with reserve and hidden orders, we consider the effect of the introduction of Fill&Kill and algorithmic programs on the equilibrium strategies. Actually here the introduction of Fill&Kill orders has no effects as they are dominated strategies: MPP have the same price risk but higher execution probability as they stay on the book in the following periods if unexecuted. However, if we introduce hidden orders detection by traders on the opposite side using algorithmic programs, the probability of hidden liquidity increases: for example, when the book opens as $b_{T-1} = [(10)00(10)]$ and traders suspect the existence of hidden liquidity on the sell side, the probability of observing $HOB_{10}M$ increases from .065 to .490 (Table 6).

Finally, comparison between the benchmark model and the protocol with MPP (Table 8) allows us to investigate the effect of the latter on the standard indicators of market quality. Notice that, according to the practice generally followed by Exchanges around the world, to measure the inside spread we only consider the liquidity which is visible to market participants, whereas for computing the inside depth, we separate disclose from undisclosed limit orders. We observe that the inside spread worsens as traders switch from limit orders posted to the first level to MPP. Accordingly, if we only consider visible limit orders, inside depth falls, whereas if we add the liquidity offered via

MPP to the limit orders posted to the first level, total depth increases. Lastly, volume decreases as a consequence of the increased spread and the smaller visible depth that clearly attract fewer market orders.

[Insert Table 8 here]

5 High Frequency Trading and Undisclosed Orders

Up to here we have focused on one motive that drives traders to use undisclosed orders, namely competition for the provision of liquidity, that can be thought off as a fair game where all participants face a trade-off between execution risk and price risk; clearly when traders opt for undisclosed orders, they renounce a certain degree of execution certainty for better prices. We have also discussed how, within this fair game, the growing use of algorithmic programs in search of liquidity is beneficial as it replenishes the execution probability lost by traders opting for undisclosed orders. We now move to the two other possible sources of exposure costs - and hence motives for the use of undisclosed orders by uninformed traders - that have been considered by the literature (e.g. Harris, 2003 and SEC, 2010). It should be stressed that, due to the nature of these motives, it is here particularly interesting to investigate, as we did for the previous case, how these new sources of exposure costs can be affected by electronic trading.

Consider first the so called picking-off risk that traders face whenever they post a limit order waiting for execution on the book: if the asset value changes, such order can become mispriced, and can be picked off by fast traders, named scalpers, before cancellation. Noticeably, this risk increases with the widespread use of sophisticated algorithmic trading programs aimed at exploiting small profits opportunities.

Traders willing to execute blocks also face another exposure risk that increases with the use of high frequency trading. This risk arises from the adverse price impact that the submission of a large

order can generate over a short period of time and is well known in the financial literature (e.g. Hendershott and Menkveld, 2010). On electronic trading platforms a price impact can be generated either by the lack of liquidity demand, or by opportunistic trading strategies implemented on other markets (ITG, 2010) as well as on derivative securities. From the seminal paper of Kraus and Stoll (1972), that provided the first evidence on how block trading can cause price pressure, other papers have offered empirical proof for the temporary price impact that can arise when the number of potential liquidity providers is not large enough to absorb the block.²⁶ Extensive resources have also been allocated to developing trading strategies aimed at minimizing price impact.²⁷ Theoretically, Gabaix et al. (2006) have shown how price pressure can impact volatility, and Brunnermeier and Pedersen (2009) have analyzed how it can be related to margin requirements, but no attempt has been made so far to model price pressure within the context of a limit order book.

To reduce the exposure costs generated by either volatility and/or price impact, exchanges generally offer undisclosed orders. However, the recent development of high frequency devices have made order anticipation strategies more erudite. Considering the SEC's concerns²⁸ and in light of the actual recent upsurge of high frequency trading techniques, the question that has lately arisen is whether undisclosed orders are still valuable instruments that traders can use to reduce this type of exposure costs. We now investigate this issue and present two examples that extend the previous model to embed both picking-off risk due to unexpected asset value changes, and exposure risk due to adverse short run price impact of large orders.

5.1 Scalpers and Quote Matchers

To investigate the exposure costs generated by volatility and price pressure, we modify the model presented so far in two directions. First, we introduce two new categories of traders that are respectively named “scalpers” and “parasitic traders”, and, second, we revise the distribution of the asset value shock.

Investigation of picking-off risk requires that, as in real markets, the model embeds agents like

scalpers who trade on their own account and usually do not hold a position for more than a few minutes (Harris, 2003). These agents mainly make profits from prices that are no longer right, which they quickly track down from the book. In our model scalpers are arbitrageurs, interested in exploiting the free option offered by limit order submitters on the occasion of an asset value shock. Scalpers are much quicker than the other market participants, so that when there is a shock they can pick off visible outdated prices before limit order traders cancel them.

To discuss undisclosed orders as anti-scalper defensive strategies, we also need to allow orders to be possibly mispriced on both the first and the second level of the LOB ($k = 2$): with a small asset value shock ($k = 1$), orders on the second level would never be mispriced and would bear no exposure costs. Finally, we assume that v_T , the asset value at T , increases, decreases or holds constant with equal probability ($x = \frac{1}{3}$). This is necessary as the asset value shock has to be uncertain: if traders know that the shock will occur (probability 1), they lose the incentive to submit limit orders, because of the certain losses against scalpers in case of mispricing. Figure 3 shows the evolution of the price grid over time for the new asset value shock. Ask and bid prices after a positive (or a negative) price change are denoted as A_i^U (A_i^D) and B_i^U (B_i^D) respectively, $i \in [1, 3]$.

[Insert Figure 3 here]

Parasitic traders are in the market only to front run those traders who offer liquidity via limit orders (i.e. passive traders), they value the asset v_t and are also called (Harris, 2003) quote matchers in that they extract the option value of large limit orders.²⁹ If, as an example, a large trader posts a limit order at A_2 and the quote matcher undercuts it at A_1 , he makes unbounded profits if the price of the stock goes down and limited losses if the price moves against him; in fact, should the price move up, he could use the initial limit sell order as an insurance by immediately buying back his shares. Clearly, the parasitic traders' strategy discussed in Harris (2003) is implemented if traders expect that the initial block posted by the limit seller indeed produces a downward pressure on the asset value. In terms of the price dynamic, we assume -for example- that when a large seller submits his order at time $T - 2$, it has a price impact so that at time $T - 1$ the price grid moves

down by 1 tick.³⁰ To investigate exposure costs we also have to assume that the variance of the asset value is increased to $k = 3$. This is appropriate as we have to assign both passive and parasitic traders' orders the same probability of being mispriced. Figure 4 shows the price dynamic for the case of a large sell order.

[Insert Figure 4 here]

5.2 Picking-off risk

As in the previous case, to choose an optimal trading strategy, agents compare the expected profits from all the feasible orders and solve programs (5) and (6). When exposed to picking-off risk, differently from the model with competition for liquidity provision, traders can find it optimal to submit undisclosed orders not only at $T - 2$, but also at $T - 1$. Hence, market participants will have to take into account the possible effects on the state of the book of undisclosed orders submitted in both periods. This dual uncertainty makes the model technically very complicated and therefore we solve it separately for reserve and hidden orders. In Table 9 we refer to the model with reserve and the one with hidden orders by “R” and “H” respectively. The following Proposition summarizes the results.

Proposition 4 *When traders are concerned by picking-off risk, undisclosed orders are equilibrium strategies both at $T - 1$ and at $T - 2$. Yet, they are not equilibrium strategies when scalpers use Fill&Kill orders or algorithmic trading programs.*

The type of orders available to scalpers is crucial in determining the effectiveness of undisclosed orders to reduce picking-off risk, as when traders can switch from market to Fill&Kill orders, undisclosed orders are no longer profitable strategies.

As scalpers only look for riskless profitable opportunities and do not wish to take a position, when using market orders to hunt down mispriced depth, they select a size equal to the visible mispriced quantity and a limit price equal to the highest outdated price. With such orders they cannot

generally hit invisible mispriced shares, so that traders can effectively select either hidden or reserve orders to reduce picking-off risk.³¹ Notice also that in the case of reserve orders, traders prefer to hide the largest possible amount ($\alpha^* = 1$) to pursue greater protection against scalpers.³² One could wonder whether this implies that hidden orders could dominate reserve orders in a protocol where traders could use both orders when looking for protection from picking-off risk. It would be certainly interesting to check this intuition by extending our model to include both orders. Another interesting effect that emerges from Table 9 is that, consistently with the results of Bessembinder et al. (2009), we find that the use of reserve orders increases with own-side depth, and, as reported by Pardo and Pascual (2006) and by De Winne and D'Hondt (2007), it also increases when the book is full or partially full on the other side.

[Insert Table 9 here]

Results change however, when the type of orders available to scalpers includes not only market, but also Fill&Kill orders. Indeed if scalpers systematically use Fill&Kill to detect undisclosed liquidity, they will be able to pick off all the invisible mispriced shares so that patient traders lose any incentive to submit either reserve or hidden orders. Clearly, if this is the case, equilibrium strategies coincide with those of the benchmark model and therefore we do not present them separately. Analogous conclusions can be drawn for the case with algo trading where reserve and hidden orders are never equilibrium strategies and hence we are back to the benchmark case.

The interesting implication that emerges from this result is that the more widespread the use of fast trading tools, the stronger these effects should be, and we can predict that undisclosed orders submitted to prevent picking-off risk are used less frequently with the development of high frequency trading techniques.

One can now wonder whether traders could resort to MPP which, as discussed in Section 4, offer defence against mispricing. Unfortunately, it is straightforward to show that also MPP are ineffective strategies against fast trading programs as these can actually pick them off before the midquote value updates.

5.3 Front Running Risk

As in this setting the advantage of undisclosed liquidity is to prevent price pressure and the consequent aggressive undercutting by quote matchers, traders choose to disclose the largest possible size that does not produce any price pressure. In this simplified example where the price impact at T is exogenous, we have arbitrarily assumed that such size is equal to 1 unit so that traders will have an incentive to use reserve orders with a visible peak of that size. Hidden orders could be equilibrium strategies only if any visible order of whatever size would generate price pressure. However, we prefer to focus on a more realistic setting where very small orders do not create any price pressure. In a more sophisticated framework with endogenous price impact, that we leave for future research, when selecting the undisclosed part of their order, traders should balance the benefits of visibility -and hence increased execution probability- with the costs of a higher price pressure.

We solve the model under three different specifications: with and without a price impact (models “P” and “B”), and with a price impact and reserve orders (“P&R” model). If large orders do have a price impact, quote matchers can place their orders ahead of them with the goal of capturing the price movement. By doing so they could attain positive profits (if the price moves in a favorable direction or stays constant) and view the large trader’s limit order as a free option to trade against (if the price moves contrary to their position that in our model happens with probability $1/3$). For simplicity we focus only on the case with Fill&Kill orders as in this framework, similarly to the the case with competition for liquidity provision presented in Section 3, the detectability of undisclosed shares just increases their execution probability.

The following Proposition summarizes the results.

Proposition 5 *When traders use undisclosed orders to prevent price pressure, reserve orders are optimally selected to avoid front running by parasitic traders.*

Our results show that in the B framework at $T - 2$ large sellers submit limit orders at A_2 with

probability .203. In the P framework this probability decreases to .139 as parasitic traders optimally undercut the limit orders posted at A_2 . When instead in the P&R framework traders are allowed to choose reserve orders, in equilibrium at $T - 2$ they post reserve orders at A_2 (instead of limit orders) to prevent the swell up of prices.

[Insert Table 10 here]

6 Evaluation and Empirical Implications

Dark liquidity is at the center of the current regulatory debate³³ both in the US and in Europe, and our model allows us to draw some conclusions on the comparative advantage of reserve and hidden orders. First of all, when the book is shallow so that there is room to compete on price for the provision of liquidity, traders use reserve orders to prevent undercutting. When instead the book is increasingly deep, traders use Hidden Mid-Point Peg Orders as the spread midpoint allows them to aggressively undercut any visible limit order standing on the book. Indeed, reserve orders are used as defensive strategies, whereas Hidden Mid-Point Peg Orders are actually used to aggressively compete for liquidity.³⁴

Empirically we expect traders to switch from reserve to hidden orders when liquidity increases or, cross-sectionally, when moving from illiquid to liquid stocks. This effect could also be captured by looking directly at executions, as we expect an increase of trades executed at the spread midquote both when stocks are more liquid and, in a time series, when the book becomes deeper. Furthermore, the empirical evaluation based on trade executions should also consider the interaction between the use of MPP and depth on own and opposite side of the book: we expect an increase in the former to foster executions at the midquote, and an increase in the latter to reduce them. We also expect to observe an increase of midquote executions when depth moves to the top of the book.

Even more interestingly, the interaction of depth with the strategic choice of undisclosed orders has also empirical implications for the systematic pattern of order flows. Our findings on the effects of

own and opposite side depth show that while the use of reserve orders decreases (increases) with depth on own (opposite) side, as it is standard for limit orders (Parlour, 1998), the opposite holds for MPP. Consequently, we expect the probability of continuation to be enhanced by reserve orders, and to be reduced by MPP. This means that empirically one should check whether an increase in liquidity across different stocks, or in the own side depth, is associated with an increase in the probability of reversal, contrary to what observed by Biais, Hillion and Spatt for their 1995 sample of stocks listed at the Paris Bourse. Of course empiricists should also control for the recent increase in order splitting due to high frequency trading. All these empirical implications are readily testable with high frequency intradaily data.

Our findings suggest that when traders use undisclosed orders to minimize exposure costs, they prevent visible undercutting by other traders or by quote-matchers, and hence make the inside spread wider. A ready testable implication of this result is to verify whether the presence of undisclosed orders is positively correlated with the size of the quoted spread. As far as the effective spread is concerned, instead, we still expect to observe an increase associated with the use of reserve and standard hidden orders, but a decrease with the use of MPP that are executed at the spread midquote. This is interesting as it departs from the convergence pattern of quoted and effective spread observed for NYSE and NASDAQ stocks by Bessembinder (2003) and Chordia et al. (2001). Notice further that empirically we expect to observe an increased spread prevailing for small trades, while for large trades the clustering of depth at the BBO could compensate the wider spread. To capture this effect one should use, as a proxy of the semi-spread, a measure of the price impact associated with different trade sizes. We expect the difference in price impact for small and large trades to be decreasing in the use of undisclosed orders.

To conclude, we suggest that the empirical estimation of the bid-ask spread should include a component that is due to exposure costs, and depends on the state of the book, the time of day, the asset volatility, the trading frequency and more importantly the order size. This component differs from the Copeland and Galai (1983) argument that dealers set a spread that is increasing

in the asset volatility to reduce picking-off by insiders, as it focuses on two other elements of exposure costs, namely competition and front running. Furthermore, it can arise independently of the presence of asymmetric information and hence it also applies to bond and currency markets.

We have shown that the execution probability of undisclosed orders increases when trading becomes faster, so we would also expect high frequency trading, pervasive in low priced and liquid stocks, to foster the use of undisclosed orders submitted to compete for the provision of liquidity. When instead traders use undisclosed orders to prevent picking-off risk, then the increased use of high frequency trading programs reduces traders' resort to undisclosed orders. Therefore, in a time series perspective we should observe that when market conditions are such that traders use undisclosed orders to compete for the provision of liquidity, i.e. during less volatile trading periods, the use of undisclosed orders is positively related with fast trading, the opposite holding for more volatile periods. For example, we could capture this effect by comparing low and high volatility periods at the beginning and at the end of the last decade, as algo trading has substantially increased during this time frame (Hendershott et al., 2010).

Yet, when undisclosed orders are used as protection against the risk of front running, their performance depends on how order anticipation strategies are able to detect hidden liquidity. In the extreme case where most market participants have access to highly sophisticated algo trading programs that allow them to estimate hidden liquidity even on their own side, traders have to resort to those platforms, as dark pools, that are specifically structured to trade blocks safely. Certainly with the development of algo trading aimed at tracking the footprints of undisclosed orders, dark pools may become a safer venue for trading blocks. This explains why many European trading platforms (CERS, 2010) contend that the Large In Scale Threshold (LIS)³⁵ for hidden orders should be reviewed by regulators to take into account the recent reduction in the average trade size (e.g. on LSE it decreased by 55% between 2006 and 2009). Indeed if hidden orders are much larger in size than average orders, they become easier to detect following post-trade reporting. Actually institutional traders and broker/dealers have access to real time information on executed volumes,

provided either directly by exchanges or by the Bloomberg's facilities, and therefore, even though hidden orders still remain not visible to retail traders, they have a lower degree of opacity vis-a-vis larger traders.

7 Concluding Remarks

A growing body of empirical evidence shows that undisclosed orders are widely used by uninformed traders in many electronic limit order platforms, but there is no theory on how undisclosed orders can be used to control exposure costs, what factors determine their use, and how they affect market liquidity or traders' profits. In this paper a theory of undisclosed orders is presented to discuss agents' optimal trading strategies in an LOB where traders are allowed to choose between reserve, hidden and a range of other order types. The attractiveness of undisclosed orders is related here to the exposure costs that can arise under three circumstances. Firstly, when traders compete for the provision of liquidity; secondly, in the event of an asset value shock when they run the risk of being picked off by scalpers; and thirdly, when the market is populated by quote matchers who exploit the price pressure generated by large blocks.

Our results indicate that reserve orders are equilibrium strategies for patient uninformed traders who compete for the provision of liquidity in markets where the spread is wide and there is room for undercutting. Hidden Mid-Point Peg Orders are instead chosen by more aggressive traders who wish to compete in deep markets by undercutting the existing liquidity within the BBO. More precisely, the use of MPP increases with depth on own side, which reverts the order flow dynamic (Parlour, 1998) of standard limit orders. Undisclosed orders used to compete for liquidity generally benefit from fast trading in search of hidden depth that increases their execution probability.

Undisclosed orders are also equilibrium defensive strategies against picking-off risk in slow markets where scalpers do not use Fill&Kill orders as well as algorithmic trading techniques to search dark liquidity. Finally, reserve orders can be effectively used by traders wishing to protect their interest from the opportunistic strategies of quote matchers aimed at exploiting large orders' price

impact. However they do not protect traders against order anticipation strategies that rely on very sophisticated algorithmic trading techniques: in this case traders probably have to resort to those dark pools that are precisely designed to trade blocks.

The use of undisclosed orders is not only relevant for traders' optimal order strategies but, perhaps more importantly, it is also an instrument that regulators can use to fine-tune the optimal degree of pre-trade transparency. Permitting undisclosed orders decreases market transparency, as investors, observing the screen, are not necessarily informed of the true depth at the posted quotes. It therefore becomes important to see whether there are any benefits to market quality to validate the authorization of undisclosed orders. We address this important issue in market design by comparing a model with undisclosed orders to a benchmark model without. Our results show that when traders use undisclosed orders, depth at the BBO increases since the orders are concentrated at a single price; however, inside spread widens. The conclusion is that in evaluating the performance of undisclosed orders, regulators should consider that they can benefit institutional investors but be detrimental to retail traders.

Figure Legends

Figure 1 This Figure shows the price grid for $k = 1$. The ask prices are equal to $A_{1,2,3}$ and the bid prices are equal to $B_{1,2,3}$, with $A_1 < A_2 < A_3$ and $B_1 > B_2 > B_3$. These prices are symmetric around the common value of the asset, v , which at time T can take values v^u and v^d .

Figure 2 This Figure shows an example of the extensive form of the game in the case with $j = 10$ and $\alpha = 3$. At $T - 2$ the book opens empty, $b_{T-2} = [0000]$; nature chooses with equal probability a large trader (LT) or a small trader (ST) who decides his optimal submission strategy among all the feasible orders (Table 1). If, for example, at $T - 2$ LT chooses $LO_{10}A_2$, at $T - 1$ the book will be $b_{T-1} = [(10)000]$; if then another LT arrives who, still as an example, chooses $LO_{10}A_1$, then at T the book will open as $b_T = [(10)(10)00]$ so that the next LT will submit either $MO_{10}B_3$, $MO_{10}A_1$, or will not trade (*NTL*); ST instead will choose among MO_3B_3 and MO_3A_1 , or decide not to trade (*NTS*). On the other hand, if at $T - 2$ a LT chooses $RO_{10}A_2$ or $HO_{10}A_2$, traders arriving at time $T - 1$ and T will be uncertain on the actual depth of the book.

Figure 3 This Figure shows the price grid for $k = 2$. The ask and bid prices are equal to $A_{1,2,3}$ and $B_{1,2,3}$ respectively with $A_1 < A_2 < A_3$ and $B_1 > B_2 > B_3$. These prices are symmetric around the common value of the asset, v , which at time T can take values v , v^U and v^D .

Figure 4 This Figure shows the price grid for $k = 3$ and for the case with a negative price impact of 1 tick. The ask and bid prices are equal to $A_{1,2,3}$ and $B_{1,2,3}$ respectively with $A_1 < A_2 < A_3$ and $B_1 > B_2 > B_3$. These prices are symmetric around the common value of the asset, v , which at time T can take values v^d , v^U and $v^{D'}$ and at $T - 1$ falls to $v - \tau$.

Appendix

Proof of Proposition 1

Benchmark Framework³⁶

Due to agents' risk neutrality, profits are increasing in the order size. For example, if we consider the strategy $MO_j B_i$, traders' profits will be $j(B_i - \beta_t v_t)$: the larger j , the larger the profits. Hence, from now onwards we assume that j is equal to its maximum possible value, given the LOB depth.

Period T To compute the equilibrium strategies, we need to compare trader's profits and find the β thresholds such that traders with an asset valuation within this range will submit a certain order type. At time T a small trader will submit a market sell order if the price is higher than his evaluation of the asset ($B_i^z \geq \beta_T v_T^z$, i.e. $\beta_T \leq B_i^z / v_T^z$, where $z = \{u, d\}$), a market buy order in the opposite case ($\beta_T v_T^z \geq A_i^z$, i.e. $\beta_T \geq A_i^z / v_T^z$) and will not trade for intermediate values of β_T . As an example, if $b_T^u = [0\alpha 00]$, the probabilities are:

$$\begin{aligned} \Pr_T(MO_\alpha B_3^u | b_T^u) &= \Pr(S) \frac{\beta_{(MO_\alpha B_3^u, NTS^u)}}{2} &= \frac{2-3\tau}{8(1+\tau)} \\ \Pr_T(NTS^u | b_T^u) &= \Pr(S) \frac{\beta_{(MO_\alpha^s A_1^u, NTS^u)} - \beta_{(MO_\alpha B_3^u, NTS^u)}}{2} &= \frac{3\tau}{4(1+\tau)} \\ \Pr_T(MO_\alpha^s A_1^u | b_T^u) &= \Pr(S) \frac{2 - \beta_{(MO_\alpha^s A_1^u, NTS^u)}}{2} &= \frac{2+\tau}{8(1+\tau)} \end{aligned}$$

where $\Pr(S) = 1/2$ is the probability that a small trader arrives at the market, and, for example, $\beta_{(MO_\alpha B_3^u, NTS^u)}$ is the threshold between a market sell order of size α executed at B_3^u and no trading. Notice that in cases where small and large traders optimally choose the same equilibrium strategy, we add the superscript "s" to indicate the order submitted by small traders.

Considering large traders, if j shares are available at the best bid and ask, the β_T thresholds are the same as those of retail traders, even if they will be trading j shares rather than α . However, if for example only $f_i < j$ shares are available at the best ask A_i and $n \geq j - f_i$ shares are available at $A_l > A_i$, large traders have the option to submit either a market sell order of size f_i at A_i or a larger market order of size j , that will walk up the book in search of execution. So the large trader will submit $MO_j A^z$ if $\beta_T \geq A_l^z / v_T^z$, $MO_{f_i} A_i^z$ if $A_l^z / v_T^z \leq \beta_T < A_i^z / v_T^z$ and prefer not to trade if $1 < \beta_T \leq A_i^z / v_T^z$.

Period $T - 1$ We start by considering the possible opening states of the LOB at $T - 1$ that are summarized in Table A1.

Table A1 - Opening LOBs at $T - 1$							
State of the book	Strategies at $T - 2$	$MO_{10} B_3, MO_3 B_3$	$LO_{10} A_2$	$LO_{10} A_1$	$LO_\alpha A_2$	$LO_\alpha A_1$	NTS, NTL
	A_2	0	10	0	α	0	0
	A_1	0	0	10	0	α	0
	B_1	0	0	0	0	0	0
	B_2	0	0	0	0	0	0

Notice that at $T - 2$ the book opens empty and that we use for presentation the branch of the trading game in which a seller arrives; hence, at $T - 1$ the bid side is always empty. Equilibrium strategies for the other branch of the trading game starting with a buyer arriving at $T - 2$ are basically symmetric and hence not presented in this proof.

We focus on the large trader's problem and present the case with $b_{T-1} = [\alpha 000]$ as an example. The feasible large trader's strategies and the associated profits are:

$$\begin{aligned}\pi_{T-1}(MO_{10}B_3) &= 10(B_3 - \beta_{T-1}v_{T-1}) \\ E[\pi_{T-1}(LO_j A_i)] &= E[(A_i - \beta_{T-1}\tilde{v}_T) \sum_{w_T=1}^j w_T \times \Pr(A_i | b_T, v_T)] \\ E[\pi_{T-1}(MO_\alpha A_2)] &= \alpha(\beta_{T-1}v_{T-1} - A_2) \\ E[\pi_{T-1}(MO_{10}A)] &= 10\beta_{T-1}v_{T-1} - \alpha A_2 - (10 - \alpha)A_3 \\ E[\pi_{T-1}(LO_{10}B_i)] &= E[(\beta_{T-1}\tilde{v}_T - B_i) \sum_{w_T=\alpha}^{10} w_T \times \Pr(B_i | b_T, v_T)]\end{aligned}$$

where in the case of $LO_j A_i$, $j = 10$ for A_1 and $j = 10 - \alpha$ for A_2 . As limit orders can be eventually executed at T after the asset value shock is realized, traders have to formulate expectations on the value of the asset and also compute the order execution probabilities. As an example, we specify the profit formula for $\pi_{T-1}(LO_{10-\alpha}A_2)$:

$$\begin{aligned}E[\pi_{T-1}(LO_{10-\alpha}A_2)] &= \frac{1}{2}(A_2 - \beta_{T-1}v_T^d) [\frac{1}{2}(10 - \alpha)\Pr_T(MO_{10}A_3^d | b_T^d)] \\ &\quad + \frac{1}{2}(A_2 - \beta_{T-1}v_T^u) \times [\frac{1}{2}(10 - \alpha)\Pr_T(MO_{10}A_1^u | b_T^u)] \\ \text{where } b_T^d &= [0000] \text{ and } b_T^u = [0(10)00].\end{aligned}$$

The equilibrium intervals of the β_{T-1} are obtained by comparing the above profits and by finding the ranges of β_{T-1} associated with large trader's optimal strategies. Results are presented in Table 3 for $\alpha = 3$. The small trader's equilibrium strategies are available from the authors upon request.

Period $T - 2$ For period $T - 2$, we compute and compare the profits associated with trader's strategies on the sell side, assuming that the initial book is empty. Strategies on the bid side are qualitatively similar, given the symmetry of the model. We focus again on the large trader's strategies:

$$\begin{aligned}\pi_{T-2}(MO_{10}B_3) &= 10(B_3 - \beta_{T-2}v_{T-2}) \\ E[\pi_{T-2}(LO_{10}A_i)] &= E[(A_i - \beta_{T-2}v_{T-1}) \sum_{w_{T-1}=\alpha}^{10} w_{T-1} \Pr(A_i | b_{T-1}, v_{T-1}) + \\ &\quad + (A_i - \beta_{T-2}\tilde{v}_T) \sum_{W=0}^{\alpha} \sum_{w_T=1}^{10-W} w_T \Pr(A_i | b_T, v_T) \Pr(w_{T-1} = W | b_{T-1}, v_{T-1})]\end{aligned}$$

The results are reported for $\alpha = 3$ in Table 3.

M Framework

Period T Two possible cases: with no uncertainty on available depth we are back to the benchmark framework; with uncertainty due to undisclosed orders, traders have to rationally estimate the probability of hidden depth and hence compute the expected execution prices.

As small traders' order size is equal to the peak size of reserve orders, they face uncertainty only in the case of hidden orders. Indeed, if the best observable liquidity is at A_l^z , but they suspect the existence of hidden depth at a better price, $A_i^z < A_l^z$, they will rationally compute their execution price as a weighted average of the two possible prices:

$$\Lambda_m^z = \sum_m \Pr_\alpha(A_m^z | b_T^z) A_m^z$$

with $m = \{i, l\}$, where the weights $\Pr_\alpha(A_m^z | b_T^z)$ are the probabilities that the α shares will be executed at price A_m^z . As an example, if a small trader comes to the market at time T and observes no visible trading at $T - 2$, $b_{T-1} = [0000]$, and a small limit order of α shares at B_2 at $T - 1$, $b_T = [000\alpha]$, then in case of a positive shock the value of Λ_m^u is:

$$\Lambda_m^u = \frac{A_1^u \Pr_{T-2}(HO_{10}A_2 | b_T^u) + A_3^u [\Pr_{T-2}(HO_{10}B_2 | b_T^u) + \Pr_{T-2}(NTL | b_T^u) + \Pr_{T-2}(NTS | b_T^u)]}{\Pr_{T-2}(HO_{10}A_2 | b_T^u) + \Pr_{T-2}(HO_{10}B_2 | b_T^u) + \Pr_{T-2}(NTL | b_T^u) + \Pr_{T-2}(NTS | b_T^u)}$$

where for example $\Pr_{T-2}(HO_{10}A_2 | b_T^u)$ is the probability that a hidden sell order was submitted at A_2 at time $T - 2$, given the current state of the LOB. So the small trader will submit a market sell order if $\beta_T \leq B_3^u / v_T^u$, a market buy order if $\beta_T \geq \Lambda_m^u / v_T^u$, and will not trade for intermediate values of β_T . Turning to the large trader, if he suspects the existence of a reserve or hidden order, he will also compute the execution price as a weighted average of all the possible prices.

Period $T - 1$ We consider as an example the visible book $b_{T-1} = [\alpha 000]$ where traders suspect the existence of a reserve order and focus on the large traders' problem. Profits from those feasible strategies that differ from the benchmark are:

$$\begin{aligned} E[\pi_{T-1}(LO_j A_i)] &= E[(A_i - \beta_{T-1} \tilde{v}_T) \sum_{w_T=1}^j w_T \times \Pr(A_i | \tilde{b}_T, v_T)] \\ E[\pi_{T-1}(MO_{10}A)] &= [\alpha + (10 - \alpha) \frac{\Pr(A_2 | b_{T-1})}{10 - \alpha}] (\beta_{T-1} v_{T-1} - A_2) \\ &\quad + (10 - \alpha) [1 - \frac{\Pr(A_2 | b_{T-1})}{10 - \alpha}] (\beta_{T-1} v_{T-1} - A_3) \end{aligned}$$

where in the case of $LO_j A_i$ $j = 10$ for A_1 and $j = 10 - \alpha$ for A_2 . We specify the profit formula for $\pi_{T-1}(MO_{10}A)$:

$$\begin{aligned} \pi_{T-1}(MO_{10}A) &= (\beta_{T-1} v_{T-1} - A_2) \left[\alpha + (10 - \alpha) \frac{\Pr_{T-2}(RO_{10}A_2 | b_{T-1})}{\Pr_{T-2}(RO_{10}A_2 | b_{T-1}) + \Pr_{T-2}(LO_\alpha A_2 | b_{T-1})} \right] \\ &\quad + (\beta_{T-1} v_{T-1} - A_3) (10 - \alpha) \frac{\Pr_{T-2}(LO_\alpha A_2 | b_{T-1})}{\Pr_{T-2}(RO_{10}A_2 | b_{T-1}) + \Pr_{T-2}(LO_\alpha A_2 | b_{T-1})} \end{aligned}$$

The probabilities associated with the equilibrium strategies for the case with $\alpha = 3$ are presented in Table 3.

Period $T - 2$ The large trader solves again problem (6). We do not report the general profit formulas as they only differ from the benchmark model for the uncertainty that characterizes the state of the book. We only specify as an example the profit formula for $\pi_{T-2}(RO_{10}A_2)$:

$$\begin{aligned}
E[\pi_{T-2}(RO_{10}A_2)] &= \frac{1}{2}(A_2 - \beta_{T-2}v_{T-1})10 \Pr_{T-1}(MO_{10}A | \tilde{b}_{T-1}) + \frac{1}{2} \Pr_{T-1}(MO_{\alpha}A_2 | \tilde{b}_{T-1}) \\
&+ [\alpha(A_2 - \beta_{T-2}v_{T-1}) + \frac{1}{2}\zeta_{b_T^u, 10-\alpha}^1 + \frac{1}{2}\zeta_{b_T^d, 10-\alpha}^2] + \{[\frac{1}{2}(\Pr_{T-1}(MO_{10}B_3 | \tilde{b}_{T-1}) + \Pr_{T-1}(LO_{10}B_2 | \tilde{b}_{T-1})) \\
&\quad + \frac{1}{2}(\Pr_{T-1}(MO_{\alpha}B_3 | \tilde{b}_{T-1}) + \Pr_{T-1}(LO_{\alpha}B_2 | \tilde{b}_{T-1}))] \times [\frac{1}{2}\zeta_{b_T^u, 10}^1 + \frac{1}{2}\zeta_{b_T^d, 10}^2] \\
&+ \frac{1}{2}\Pr_{T-1}(LO_{10-\alpha}A_2 | \tilde{b}_{T-1}) \times [\frac{1}{2}\alpha(A_2 - \beta_{T-2}v_T^u)[\frac{1}{2}\Pr_T(MO_{10}A_1^u | \tilde{b}_T^u) + \frac{1}{2}\Pr_T(MO_{\alpha}A_1^u | \tilde{b}_T^u)] + \frac{1}{2}\zeta_{b_T^d, \alpha}^2] \\
&+ \frac{1}{2}\Pr_{T-1}(LO_{\alpha}A_1 | \tilde{b}_{T-1})[\frac{1}{2}\zeta_{b_T^u, 10}^1 + \frac{1}{2}(A_2 - \beta_{T-1}v_T^d)[\frac{1}{2}((10 - \alpha)\Pr_T(MO_{10}A^d | \tilde{b}_T^d) + \alpha\Pr_T(MO_{2\alpha}A^d | \tilde{b}_T^d))] \\
&\quad + \frac{1}{2}\Pr_{T-1}(LO_{\alpha}A_2 | \tilde{b}_{T-1})[\frac{1}{2}\zeta_{b_T^d, 10-\alpha}^2 + \frac{1}{2}(A_2 - \beta_{T-1}v_T^u) \\
&\quad \quad [\frac{1}{2}((10 - \alpha)\Pr_T(MO_{10}A^u | \tilde{b}_T^u) + \alpha\Pr_T(MO_{2\alpha}A_1^u | \tilde{b}_T^u)] + \frac{1}{2}\alpha\Pr_T(MO_{\alpha}A_1^u | \tilde{b}_T^u)]\}
\end{aligned}$$

where $b_{T-1} = [\alpha 000]$ or $b_{T-1} = [(\alpha + (10 - \alpha))000]$, $\zeta_{b_T^u, j}^1$ and $\zeta_{b_T^d, j}^2$ are defined as follows:

$$\begin{aligned}
\zeta_{b_T^u, j}^1 &= (A_2 - \beta_{T-2}v_T^u)\{\frac{1}{2}[j\Pr_T(MO_{10}A^u | \tilde{b}_T^u) + \alpha\Pr_T(MO_{\alpha}A_1^u | \tilde{b}_T^u)] + \frac{1}{2}\alpha\Pr_T(MO_{\alpha}^s A_1^u | \tilde{b}_T^u)\} \\
\zeta_{b_T^d, j}^2 &= (A_2 - \beta_{T-2}v_T^d)[\frac{1}{2}j\Pr_T(MO_{10}A_3^d | \tilde{b}_T^d) + \frac{1}{2}\alpha\Pr_T(MO_{\alpha}A_3^d | \tilde{b}_T^d)]
\end{aligned}$$

Results for $\alpha = 3$ are presented in Table 3.

Optimal exposure size for reserve orders (α^*) We solve the model for different values of α . When α shares are visible at A_2 , we find that for $\alpha > 3$ incoming traders at $T - 1$ prefer to undercut at A_1 meaning that reserve orders do not protect against price competition. For $\alpha \leq 3$, incoming traders at $T - 1$ join the queue at A_2 . As time priority is preserved for the visible shares, reserve orders' profits increase with the size of the visible part; hence, the optimal disclosed size is the largest compatible with traders joining the queue at $T - 1$: $\alpha^* = 3$.

F&K Framework

Notice that even with the introduction of this new order type, there is no need to compute a new benchmark. It is straightforward to show that profits from a F&K order at A_i (B_i) are equivalent to those from a market order of size equal to the liquidity available at A_i (B_i).

Period T Notice that, as only large traders are allowed to use F&K orders, small traders' strategies at T are unchanged compared to the M framework. F&K orders are used by large traders only

when they suspect the existence of hidden liquidity. We differentiate two cases for large sellers, similar strategies applies to large buyers:

1. If j shares available at A_i^z , but traders suspect the existence of a hidden order at $A_i^z < A_l^z$, they will have the option to submit either a market order of size j or a F&K order of the same size and limit price A_i^z . In equilibrium the large seller submits $MO_j A^z$ if $\beta_T \geq (A_l^z/v_T^z)$, $F\&K_{f_i} A_i^z$ if $(A_i^z/v_T^z) \leq \beta_T < (A_l^z/v_T^z)$ and does not trade if $1 < \beta_T \leq (A_i^z/v_T^z)$.
2. If $f_i < j$ shares possibly shadowing a reserve order are visible at A_i^z and $n \geq j - f_i$ shares are available at $A_l^z > A_i^z$, traders will have the option of submitting a market order of size f_i or a F&K order of size j with limit price A_i^z . Equilibrium strategies are the same as in the previous case.

Period $T-1$ and $T-2$ The only cases that differ from the M framework are those where traders suspect the existence of hidden depth. We consider again the book $b_{T-1} = [\alpha 000]$ as an example. As the trader suspects the existence of a reserve order on the ask side, only $F\&K_{10} A_2$ is a possible equilibrium strategy; profits from this order type are equal to:

$$\pi_{T-1}(F\&K_{10} A_2) = (\beta_{T-1} v_{T-1} - A_2) \left[\alpha + (10 - \alpha) \frac{\Pr_{T-2}(RO_{10} A_2 | b_{T-1})}{\Pr_{T-2}(RO_{10} A_2 | b_{T-1}) + \Pr_{T-2}(LO_{\alpha} A_2 | b_{T-1})} \right]$$

The same logic is followed to obtain equilibrium strategies at $T-2$. The optimal disclosed size for reserve orders is again the largest one compatible with traders joining the queue at $T-1$: $\alpha^* = 3$. Results are presented in round brackets in Table 3.

Algo Framework

At T traders' strategies are the same as in the F&K framework. At $T-1$, as an example, we consider again the visible book $b_{T-1} = [\alpha 000]$. The large trader solves the same problem as in the F&K framework, however now large buyers can determine whether the α shares visible on A_2 were originated by a reserve order or by a small trader's limit sell order. As a result profits from $F\&K_{10} A_2$ and $MO_{10} A$ depend now on the actual state of the LOB.

If $b_{T-1} = [(\alpha + (10 - \alpha)) 000]$, then:

$$\pi_{T-1}(F\&K_{10} A_2) = E[\pi_{T-1}(MO_{10} A)] = 10 (\beta_{T-1} v_{T-1} - A_2)$$

if instead $b_{T-1} = [\alpha 000]$, then:

$$\begin{aligned} \pi_{T-1}(F\&K_{10} A_2) &= \alpha (\beta_{T-1} v_{T-1} - A_2) \\ \pi_{T-1}(MO_{10} A) &= \alpha (\beta_{T-1} v_{T-1} - A_2) + (10 - \alpha) (\beta_{T-1} v_{T-1} - A_3) \end{aligned}$$

So, compared with the F&K framework, when large traders observe undisclosed liquidity, they obtain higher profits from market and F&K orders. This explains why in equilibrium they use these orders more aggressively, thus increasing the execution probability of reserve (or hidden)

orders submitted at $T - 2$. Results for both $T - 2$ and $T - 1$ are presented in Table 3 in square brackets.

Proof of Proposition 2

The expected values of the inside spread at the opening of period $t + 1$ is computed by weighting the period t equilibrium order submission probabilities associated with each possible state of the book by the inside semi-spread S_{t+1} that characterizes that particular state:

$$E[S_{t+1}] = \sum_{a=S,L} \Pr(a) E_{b_t} \left[\int_0^2 (A_{t+1}^*(o_{a,b_t}^*|\beta_t) - v_{t+1}) \times f(\beta_t) d\beta_t \right]$$

where o_{a,b_t}^* is the optimal trading strategy of agent a , conditional on b_t , and $A_{t+1}^*(o_{a,b_t}^*|\beta_t)$ is the best ask price available at time $t + 1$ as a function of the equilibrium strategies of the traders. The expected value of the weighted inside semi-spread WS_{t+1} is computed in a similar way, the only difference being that now spreads are multiplied by the quantity available at the best ask A^* , $q_{t+1}^{A^*}$:

$$E[WS_{t+1}] = \sum_{a=S,L} \Pr(a) E_{b_t} \left[\int_0^2 q_{t+1}^{A^*}(o_{a,b_t}^*|\beta_t) \times S_{t+1}(o_{a,b_t}^*|\beta_t) \times f(\beta_t) d\beta_t \right]$$

Similarly, the expected value of market depth on the first level of the book at the opening of period $t + 1$ is computed as follows:

$$E[D_{t+1}] = \sum_{a=S,L} \Pr(a) E_{b_t} \left[\int_0^2 q_{t+1}^{A^*}(o_{a,b_t}^*|\beta_t) f(\beta_t) d\beta_t \right]$$

Expected LOB semi-volume is estimated in each period t by averaging the equilibrium probabilities associated with market buy orders hitting the ask side of the LOB, adequately weighted by their size:

$$E[V_t] = \sum_{a=S,L} \Pr(a) E_{b_t} \left[\int_0^2 q_t^A(o_{a,b_t}^*|\beta_t) \times f(\beta_t) d\beta_t \right]$$

where $q_t^A(o_{a,b_t}^*)$ is the traded quantity on the ask side of the market, which is a function of both the agent type a and the state of LOB. The results, reported in Table 4, are derived by comparing the values of these market quality indicators for the four different frameworks presented in the proof of Proposition 1.

Proof of Proposition 3

Notice that the benchmark model is the same as in the proof of Proposition 1 and is omitted, the only difference being that in the Tables we present results for $\alpha = 1$ instead of $\alpha = 3$.

M Framework - Empty Book at $T - 2$

Period T We refer to the benchmark for the case with no uncertainty. As an example of LOB state uncertainty we present the case with no trading observed at $T - 2$, $b_{T-2} = [0000]$, and a 1-unit limit buy order at $T - 1$, $b_{T-1} = [0001]$. In case of a positive shock of the asset value, the small trader's expected execution price on the ask, $\Lambda_m^u(A)$, is the following:

$$\Lambda_m^u(A) = \frac{M_{T-2}^u \Pr_{T-2}(HOS_{10}M|b_T^u) + A_3^u [\Pr_{T-2}(HOB_{10}M|b_T^u) + \Pr_{T-2}(NTL|b_T^u) + \Pr_{T-2}(NTS|b_T^u)]}{\Pr_{T-2}(HOS_{10}M|b_T^u) + \Pr_{T-2}(HOB_{10}M|b_T^u) + \Pr_{T-2}(NTL|b_T^u) + \Pr_{T-2}(NTS|b_T^u)}$$

where M_t indicates the spread mid-quote in period t ; $\Lambda_m^u(B)$ is derived similarly. So the small trader will submit $MO_1A_3^u$ if $\beta_T \geq (\Lambda_m^u(A)/v_T)$, $MO_1B_3^u$ if $\beta_T \leq (\Lambda_m^u(B)/v_T)$, and not trade if $(\Lambda_m^u(B)/v_T) < \beta_T < (\Lambda_m^u(A)/v_T)$. The large trader has the additional option of submitting MPP and hence his thresholds are as follows: submit $MO_{10}A^z$ if $\beta_T \geq (A_3^z/v_T^z)$, $HOB_{10}M$ if $(M_T^z/v_T^z) \leq \beta_T < (A_3^z/v_T^z)$, $HOS_{10}M$ if $(B_3^z/v_T^z) \leq \beta_T < (M_T^z/v_T^z)$ and $MO_{10}B^z$ if $\beta_T < (B_3^z/v_T^z)$.

Period $T - 1$ and $T - 2$ We focus on the only case that differs from the benchmark: $b_{T-1} = [0000]$. Traders here are uncertain whether at $T - 2$ a large trader submitted a $HOS_{10}M$ ($b_{T-1} = [0000, +10]$), a $HOB_{10}M$ ($b_{T-1} = [0000, -10]$), or refrained from trading (NTL), $b_{T-1} = [0000, 0]$, or if a small trader decided not to trade (NTS), $b_{T-1} = [0000, 0]$. The profits from the large trader's strategies are (we omit $LO_{10}A_i$ and $LO_{10}B_i$ as they have no relevant differences in the general formulas):

$$\begin{aligned} E[\pi_{T-1}(MO_{10}B_3)] &= 10f_B(M_{T-1} - \beta_{T-1}v_{T-1}) + 10(1 - f_B)(B_3 - \beta_{T-1}v_{T-1}) \\ E[\pi_{T-1}(MO_{10}A_3)] &= 10f_S(\beta_{T-1}v_{T-1} - M_{T-1}) + 10(1 - f_S)(\beta_{T-1}v_{T-1} - A_2) \\ E[\pi_{T-1}(HOS_{10}M)] &= E[(M_{T-1} - \beta_{T-1}v_{T-1}) \sum_{w_{T-1}=1}^j w_{T-1} \times \Pr_{w_{T-1}}(M_{T-1}|b_{T-1}, v_{T-1}) \\ &\quad + (\tilde{M}_T - \beta_T \tilde{v}_T) \sum_{W=0}^{j-1} \sum_{w_T=1}^{j-W} w_T \Pr_{w_T}(\tilde{M}_T|b_T, v_T) \Pr(w_{T-1} = W|b_{T-1}, v_{T-1})] \\ E[\pi_{T-1}(HOB_{10}M)] &= E[(\beta_{T-1}v_{T-1} - M_{T-1}) \sum_{w_{T-1}=1}^j w_{T-1} \times \Pr_{w_{T-1}}(M_{T-1}|b_{T-1}, v_{T-1}) \\ &\quad + (\beta_T \tilde{v}_T - \tilde{M}_T) \sum_{W=0}^{j-1} \sum_{w_T=1}^{j-W} w_T \Pr_{w_T}(\tilde{M}_T|b_T, v_T) \Pr(w_{T-1} = W|b_{T-1}, v_{T-1})] \end{aligned}$$

where:

$$\begin{aligned} f_S &= \frac{\Pr_{T-2}(HOS_{10}M|b_T^u)}{\Pr_{T-2}(HOS_{10}M|b_T^u) + \Pr_{T-2}(HOB_{10}M|b_T^u) + \Pr_{T-2}(NTL|b_T^u) + \Pr_{T-2}(NTS|b_T^u)} \\ f_B &= \frac{\Pr_{T-2}(HOB_{10}M|b_T^u)}{\Pr_{T-2}(HOS_{10}M|b_T^u) + \Pr_{T-2}(HOB_{10}M|b_T^u) + \Pr_{T-2}(NTL|b_T^u) + \Pr_{T-2}(NTS|b_T^u)} \end{aligned}$$

As an example, we specify the profit formula for $\pi_{T-1}(HOS_{10}M)$:

$$\begin{aligned}
E[\pi_{T-1}(HOS_{10}M)] &= f_B 10(v - \beta_{T-1}v) + f_S \times 0 + (1 - f_S - f_B) \\
&\quad \left\{ \frac{1}{2}(v_T^d - \beta_{T-1}v_T^d) \left[\frac{1}{2} 10 \left(\Pr(MO_{10}A_3^d | \tilde{b}_T^d) + \Pr(HOB_{10}M^d | \tilde{b}_T^d) \right) + \frac{1}{2} \Pr(MO_1A_3^d | \tilde{b}_T^d) \right] \right. \\
&\quad \left. + \frac{1}{2}(v_T^u - \beta_{T-1}v_T^u) \times \left[\frac{1}{2} 10 \left(\Pr(MO_{10}A_3^u | \tilde{b}_T^u) + \Pr(HOB_{10}M^u | \tilde{b}_T^u) \right) + \frac{1}{2} \Pr(MO_1A_3^u | \tilde{b}_T^u) \right] \right\} \\
&\text{where } \tilde{b}_T^d = \tilde{b}_T^u = [0000, 0], \text{ or } [0000, +10] \text{ or } [0000, -10]
\end{aligned}$$

Notice that in this case $HOS_{10}M$ will never be executed if another MPP sell order is already standing on the book, as the former has lower time priority. But it will be executed immediately if hidden liquidity at the midquote is available on the opposite side of the market. Alternatively, the order can be executed at T against a market or a MPP buy order. Equilibrium strategies are reported in Table 5. Equilibrium strategies for $T - 2$ are derived similarly and results are presented in Table 5. For the computation of the market quality measures, we refer to the proof of Proposition 2 and results are presented in Table 8.

F&K Framework - Empty Book at $T - 2$

To show that the introduction of F&K orders does not change the equilibrium strategies, we analyze the same examples presented for the M framework. We consider only periods T and $T - 1$, as at $T - 2$ there is no hidden liquidity available on the LOB, so F&K orders are never used.

Period T We consider again the case $b_{T-2} = [0000]$ and $b_{T-1} = [0001]$. Notice that the profits from a $HOS_{10}M^z$ and a $F\&K_{10}B_i^z$, with $\tilde{i} \in \{1, 2\}$, perfectly coincide:

$$\pi_T(HOS_{10}M^z) = 10 (\beta_T v_T^z - M_T^z) \frac{\Pr_{T-2}(HOB_{10}M | b_T^u)}{\Pr_{T-2}(HOS_{10}M | b_T^u) + \Pr_{T-2}(HOB_{10}M | b_T^u) + \Pr_{T-2}(NTL | b_T^u) + \Pr_{T-2}(NTS | b_T^u)} = \pi_T(F\&K_{10}B_i^z)$$

So the large trader's thresholds are the same as in the M framework since F&K orders provide the same profits as MPP orders.

Period $T - 1$ We consider again the case where the visible book opens as $b_{T-1} = [0000]$ and specify the profit formula of a $F\&K_{10}B_i^z$, with $\tilde{i} \in \{1, 2\}$:

$$E[\pi_{T-1}(F\&K_{10}B_i^z)] = f_B 10(v - \beta_{T-1}v)$$

It is straightforward to show that $E[\pi_{T-1}(F\&K_{10}B_i^z)] < E[\pi_{T-1}(HOS_{10}M)]$ (the profit formula for $HOS_{10}M$ is specified in the M framework and is omitted here). Indeed, if hidden liquidity is available at the midquote on the buy side, profits from a $F\&K_{10}B_i^z$ and a $HOS_{10}M$ coincide. When instead the order is not immediately executed, MPP has the additional option of being executed at T . So the M and F&K frameworks coincide.

Algo Framework - Empty Book at $T - 2$

We analyze the introduction of algo trading for the same cases presented in the M framework. At T we focus again on the case with $b_{T-2} = [0000]$ and $b_{T-1} = [0001]$. Large traders can differentiate among $b_{T-1} = [0000, +10]$, $b_{T-1} = [0000, -10]$, or $b_{T-1} = [0000, 0]$, and select their optimal strategies accordingly. If $b_{T-1} = [0000, 0]$, thresholds will be as follows: submit $MO_{10}A_3^z$ if $\beta_T \geq (A_3^z/v_T^z)$, $MO_{10}B_3^z$ if $\beta_T < (B_3^z/v_T^z)$, and not trade if $(B_3^z/v_T^z) \leq \beta_T < (A_3^z/v_T^z)$. If instead, for example, $b_{T-1} = [0000, +10]$, traders will be more aggressive on the buy side since they know that a market buy order will be executed at the midquote. Optimal thresholds are as follows: submit $HOS_{10}M^z$ if $\beta_T \geq (M^z/v_T^z)$, $MO_{10}B_3^z$ if $\beta_T < (B_3^z/v_T^z)$, and not trade if $(M^z/v_T^z) \leq \beta_T < (A_3^z/v_T^z)$.

At $T - 1$ we focus on $b_{T-1} = [0000]$: if $b_{T-1} = [0000, +10]$ traders willing to submit market orders will know that $f_S = 1$ and $f_B = 0$, the opposite being true when $b_{T-1} = [0000, -10]$, and if $b_{T-1} = [0000, 0]$ then $f_S = f_B = 0$. Therefore their optimal strategies depend on the actual state of the LOB on the opposite side. Notice however that, when submitting limit or MPP orders, traders do not know the state of the LOB on their own side, so that for example the specification of the profit formula for $\pi_{T-1}(HOS_{10}M)$ is identical to the one presented in the M framework.

At $T - 2$ we assume that the book opens with no hidden liquidity, so profits from market orders are unchanged by the introduction of algo trading tools. Profits from limit orders are very similar to the M framework, the only difference being that now large traders arriving at the market in the following periods will hit the hidden liquidity more aggressively, as they observe the effective state of the LOB. Results are presented in Table 5 in square brackets.

No Volatility Framework - Empty Book at $T - 2$

The analysis presented so far is repeated for the case where there is no volatility shock at T , so that $x = 0$. As the methodology is exactly the same as the one presented for the case with volatility, we directly show the results in Table 7.

MPP - Deep Book at $T - 2$

The three cases considered for the empty $T - 2$ book (M, F&K and Algo) are also solved for the case $b_{T-2} = [(10)00(10), 0]$. We refer to the case with an empty book for an in depth analysis of the solution methodology, and present directly the results in Tables 6, 7 (no volatility) and 8 (market quality).

Proof of Proposition 4

Within the framework with scalpers in equilibrium large traders can submit both α^* and 10-unit limit orders, as when trading α^* units the higher execution probability and the smaller losses in case of mispricing compensate the reduced gains due to the smaller order size. Notice however that for $j \in [\alpha^* + 1, \dots, 9]$ the execution probability of a limit order does not change and the profits from larger order size outweigh the greater losses due to mispricing. Therefore in this case, conditional on the state of the book, traders will choose the maximum order size.

Benchmark Framework

The benchmark is similar to the one presented in Proposition 1, the main difference being that mispriced orders are picked-off by scalpers rather than cancelled. As an example we consider again the book $b_{T-1} = [\alpha 000]$ and specify the profit formula for a $LO_{10-\alpha}A_2$ to highlight the differences with the other benchmark framework previously presented:

$$\begin{aligned} E[\pi_{T-1}(LO_{10-\alpha}A_2)] &= \frac{1}{3}(A_2 - \beta_{T-1}v_T)\frac{1}{2}(10 - \alpha)\Pr_T(MO_{10}A_2 | b_T) + \frac{1}{3}(A_2 - \beta_{T-1}v_T^U) \\ &\quad + \frac{1}{3}\alpha(A_2 - \beta_{T-1}v_T^D) \times 0 \\ \text{where } b_T &= [(10)000] \end{aligned}$$

Similarly, we specify the profit formula for $\pi_{T-2}(LO_{10}A_2)$:

$$\begin{aligned} E[\pi_{T-2}(LO_{10}A_2)] &= \\ &10(A_2 - \beta_{T-2}v_{T-1})\frac{1}{2}\Pr_{T-1}(MO_{10}A_2 | b_{T-1}) + \frac{1}{2}\Pr_{T-1}(LO_{10}A_1 | b_{T-1}) \times \frac{1}{3}10(A_2 - \beta_{T-2}v_T^U) \\ &+ \frac{1}{2}[\Pr_{T-1}(LO_{10}B_1 | b_{T-1}) + \Pr_{T-1}(LO_{10}B_2 | b_{T-1}) + \Pr_{T-1}(LO_{\alpha}B_1 | b_{T-1}) + \Pr_{T-1}(LO_{\alpha}B_2 | b_{T-1})]\zeta_{b_T}^3 \\ &+ \frac{1}{2}[\Pr_{T-1}(MO_{10}B_3 | b_{T-1}) + \Pr_{T-1}(MO_{\alpha}B_3 | b_{T-1}) + \Pr_{T-1}(NTL | b_{T-1}) + \Pr_{T-1}(NTS | b_{T-1})]\zeta_{b_T}^3 \\ &+ \frac{1}{2}\Pr_{T-1}(LO_{\alpha}A_1 | b_{T-1}) \times \left\{ \frac{1}{3}(10 - \alpha)(A_2 - \beta_{T-2}v_T)\frac{1}{2}\Pr_T(MO_{10}A | b_T) + \frac{1}{3}10(A_2 - \beta_{T-2}v_T^U) \right\} \\ &\quad + \frac{1}{2}\Pr_{T-1}(MO_{\alpha}A_2 | b_{T-1})\left\{ \alpha(A_2 - \beta_{T-2}v_{T-1}) + \frac{1}{3}(A_2 - \beta_{T-2}v_T) \right\} \\ &\left\{ \frac{1}{2}[(10 - \alpha)\Pr_T(MO_{10}A_2 | b_T) + \Pr_T(MO_{10-\alpha}A_2 | b_T)] + \frac{1}{2}\alpha\Pr_T(MO_{\alpha}A_2 | b_T) \right\} + \frac{1}{3}(10 - \alpha)(A_2 - \beta_{T-2}v_T^U) \end{aligned}$$

where $b_{T-1} = [(10)000]$, and $\zeta_{b_T}^3$ is defined as follows:

$$\zeta_{b_T}^3 = \frac{1}{3}(A_2 - \beta_{T-2}v_T)\left\{ \frac{1}{2}10\Pr_T(MO_{10}A_2 | b_T) + \frac{1}{2}\alpha\Pr_T(MO_{\alpha}A_2 | b_T) \right\} + \frac{1}{3}10(A_2 - \beta_{T-2}v_T^U)$$

Results are presented for $\alpha = 1$ in Table 9.

Reserve Order Framework

We only present the model with reserve orders; the solution of the model with hidden orders is technically very similar, hence we omit the proof and present directly the results in Table 9. The thresholds and the order placement probabilities at T are derived as in Proposition 1. Notice, however, that we have to consider an additional case, as large traders could submit undisclosed orders both at A_1 and A_2 . So, if for example there are $f_i < j$ visible shares at A_i for both $i = 1$ and $i = 2$, with $f_1 + f_2 < j$, the large trader's β_T thresholds for the ask side will be the following: submit MO_jA^z if $\beta_T \geq \Lambda_y^z/v_T^z$, $MO_{f_1+f_2}A^z$ if $\Lambda_m^z/v_T^z \leq \beta_T < \Lambda_y^z/v_T^z$, $MO_{f_1}A_1^z$ if $A_1^z/v_T^z \leq \beta_T < \Lambda_m^z/v_T^z$ and not trade if $1 \leq \beta_T < A_1^z/v_T^z$, where $\Lambda_m^z = \sum \Pr_{j-f_1}(A_m^z | b_T^z)A_m^z$, with $m = \{1, 2\}$, and $\Lambda_y^z = \sum \Pr_{j-f_2}(A_y^z | b_T^z)A_y^z$, with $y \in \{1, 2, 3\}$.

Differently from Proposition 1, here to gain protection from picking-off risk large traders can optimally select reserve orders also at $T - 1$. We focus again on the book $b_{T-1} = [\alpha 000]$ and present profits only for these additional strategies ($RO_{10}A_1$ and $RO_{10-\alpha}A_2$). We refer to the proof of Proposition 1 for the other strategies.

$$E[\pi_{T-1}(RO_j A_i)] = E[(A_i - \beta_{T-1} \tilde{v}_T) \sum_{w_T=1}^j w_T \Pr_{w_T}(A_i | b_T, v_T)]$$

Equilibrium strategies for $T - 2$ are obtained as shown in the previous proofs and the model is solved for different values of α . Since traders are now mainly concerned about picking-off, it is straightforward to show that it is optimal to hide as much as possible, hence $\alpha^* = 1$. All results are presented in Table 9.

F&K and Algo Frameworks

When scalpers are allowed to use F&K orders, they can hunt down both visible and invisible mispriced liquidity on the opposite side. Hence reserve orders do not offer protection from scalpers anymore and it is trivial to show that they are not equilibrium strategies. Clearly, the same reasoning applies to the case with algo programs.

Proof of Proposition 5

The benchmark case with no hidden liquidity is solved similarly to the one presented in Proposition 1. We directly present the results in Table 10.

Parasitic Framework

Parasitic traders enter the market only when they observe a large visible order and they have enough time to take advantage of the price pressures generated by the order. Hence they just influence the large traders' strategies at $T - 2$. For periods $T - 1$ and T we refer to the proof of Proposition 1 and 4, as equilibrium strategies are obtained following the same methodology.

To understand parasitic traders' strategy, we provide an example. If at $T - 2$ a large trader submits a $LO_j A_2$, parasitic traders will anticipate the following one-tick downward movement of the price grid and, if profitable, immediately undercut the standing limit order (in the new price grid at A_3^d) by either one ($LO_j A_2^d$) or two ticks ($LO_j A_1^d$). Formally, in this case parasitic traders (P) choose their optimal undercutting strategy by solving:

$$\max_{o_P, b_{T-1} \in [NTP, LO_j A_i^d]} E[\pi_{T-1}(o_P, b_{T-1})]$$

We provide the formula of parasitic profits for $LO_{10}A_2^d$ (due to risk neutrality, $j = 10$):

$$\begin{aligned}
E[\pi_{T-1}(LO_{10}A_2^d)] &= \frac{1}{2}[10 \Pr_{T-1}(MO_{10}A_2 | \tilde{b}_{T-1}) + \Pr_{T-1}(MO_1A_2 | \tilde{b}_{T-1})] \\
&\frac{1}{3}[(A_2^d - v_T^d) + (A_2^d - A_3^d) + (A_2^d - v_T^{D'})] + \frac{1}{3}(A_2^d - v_T^d)\frac{1}{2}\{\Pr_{T-1}(MO_1A_2 | \tilde{b}_{T-1}) \\
&\frac{1}{2}[9 \Pr_T(MO_{10}A^d | \tilde{b}_T) + 9 \Pr_T(MO_9A_2^d | \tilde{b}_T) + \Pr_T(MO_1A_2^d | \tilde{b}_T)] \\
&+ [\Pr_{T-1}(MO_{10}B_3 | \tilde{b}_{T-1}) + \Pr_{T-1}(MO_1B_3 | \tilde{b}_{T-1}) + \Pr_{T-1}(NTL | \tilde{b}_{T-1}) + \Pr_{T-1}(NTS | \tilde{b}_{T-1}) \\
&+ \Pr_{T-1}(LO_{10}B_2 | \tilde{b}_{T-1}) + \Pr_{T-1}(LO_1B_2 | \tilde{b}_{T-1})]\frac{1}{2}[10 \Pr_T(MO_{10}A_2^d | \tilde{b}_T) + \Pr_T(MO_1A_2^d | \tilde{b}_T)]\}
\end{aligned}$$

In the formula, the first term refers to the units executed at $T - 1$. If the asset value remains constant, parasitic traders will get $(A_2^d - v_T^d)$; if the asset value decreases, they will take full advantage of the price movement and gain $(A_2^d - v_T^{D'})$; while if the asset value increases, they will protect themselves by trading against the limit order posted on the LOB, limiting their losses to $(A_2^d - A_3^d)$ instead of $(A_2^d - v_T^U)$. The second term refers to the units not executed at $T - 1$: here parasitic traders make profits only if no shock occurs and they are executed at T . Indeed, if a shock occurs, they will be either mispriced and hence cancel their order, or queue behind the trading crowd and not executed.

Provided that parasitic traders optimally decide to undercut, when solving program (4) at $T - 2$ large traders will take into account that the profits from limit orders are reduced. As an example, we provide the profit formula of a $LO_{10}A_2$:

$$\begin{aligned}
E[\pi_{T-2}(LO_{10}A_2)] &= \frac{1}{2} \Pr_{T-1}(MO_1A_2 | \tilde{b}_{T-1})[\frac{1}{3}(A_2^d - \beta_{T-2}v_T^d) \Pr_T(MO_{10}A^d | \tilde{b}_T) + \frac{1}{3}(A_2^d - \beta_{T-2}v_T^U)] \\
&\frac{1}{2} \Pr_{T-1}(MO_{10}A_2 | \tilde{b}_{T-1})\frac{1}{3}\{(A_2^d - \beta_{T-2}v_T^d)\frac{1}{2}[10 \Pr_T(MO_{10}A_2^d | \tilde{b}_T) + \Pr_T(MO_1A_2^d | \tilde{b}_T)] + 10(A_2^d - \beta_{T-2}v_T^U)\}
\end{aligned}$$

Results are presented in Table 10.

Parasitic&Reserve Framework

Since we have assumed that any visible order of size $j \geq 2$ generates price pressure and hence activates parasitic traders, clearly the optimal visible part of a reserve order is $\alpha^* = 1$. When large traders can use reserve orders, no price pressure is generated and hence parasitic traders don't enter the market. Results are reported in Table 10.

References

- Aitken, M. J., Berkman H., and D. Mak. 2001. The use of undisclosed limit orders on the Australian Stock Exchange. *Journal of Banking and Finance* 25: 1589-1603.
- Almgren, R., and N. Chriss. 2000. Optimal execution of portfolio transaction. *Journal of Risk* 3: 5-39.
- Anand A., and D. Weaver. 2004. Can Order Exposure be Mandated? *Journal of Financial Markets* 7: 405-426.
- Belten K. 2007. Supply and Information Content of Order Book Depth: the case of displayed and hidden depth. Aarhus School of Business. Working Paper.
- Bertsimas, D., and A. W. Lo. 1998. Optimal Control and Execution. *Journal of Financial Markets*, 1: 1-50.
- Bessembinder, H. 2003. Trade Execution Costs and Market Quality after Decimalization. *Journal of Financial and Quantitative Analysis* 38: 747-777.
- Bessembinder, H., M. Panayides, and K. Venkataraman. 2009. Hidden Liquidity: An Analysis of Order Exposure Strategies in Electronic Stock Markets. *Journal of Financial Economics* 94: 361-383.
- Bessembinder, H., and K. Venkataraman. 2004. Does an Electronic Stock Exchange Need an Upstairs Market? *Journal of Financial Economics* 73: 3-36.
- Biais, B., P. Hillion and C. Spatt. 1995. An Empirical Analysis of the Order Flow and Order Book in the Paris Bourse. *Journal of Finance* 50: 1655-89.
- Bongiovanni, S., M. Borkovec and R. D. Sinclair. 2006. Lat's Play Side-and-Seek: The Location and Size of Undisclosed Limit Order Volume. *Journal of Trading*.
- Buti, S., B. Rindi and I. Werner. 2010. Dark Pool Trading Strategies. Fisher College of Business Working Paper.
- Brunnermeier, M. K., and L. H. Pedersen. 2009. Market liquidity and funding liquidity. *Review of Financial Studies* 22: 2201–2238.
- Campbell, J. Y., S. J. Grossman, and J. Wang. 1993. Trading volume and serial correlation in stocks returns. *The Quarterly Journal of Economics* 108: 905–939.
- Chakrabarty, B, and K. Shaw. 2010. Hidden Liquidity: Order Exposure Strategies Around Earnings Announcements, *Journal of Business, Finance and Accounting*, forthcoming.
- Chordia, T., R. Roll and A. Subrahmanyam. 2001. Market Liquidity and Trading Activity. *Journal of Finance* 56: 501-530.
- Committee of European Securities Regulators (CESR). 2010. Call for Evidence on “Micro-structural issues of the European equity markets”. CESR/10-142, www.cesr.eu.
- Copeland, T., and D. Galai. 1983. Information Effects on the Bid-Ask Spread. *Journal of Finance* 38: 1453-69.
- Coval, J., and E. Stafford. 2007. Asset fire sales (and purchases) in equity markets. *Journal of Financial Economics* 86: 479–512.
- De Winne, R., and C. D'Hondt. 2007. Hide-and-Seek in the Market: Placing and Detecting Hidden Orders. *Review of Finance* 11: 663-692.

- Esser, A., and B. Mönch. 2007. The Navigation of an Iceberg: The Optimal Use of Hidden Orders. *Finance Research Letters* 4: 68-81.
- Foucault, T. 1999. Order flow composition and trading costs in a dynamic limit order market. *Journal of Financial Markets* 2: 99-134.
- Foucault, T., Kadan, O., and E. Kandel. 2005. Limit Order Book as a Market for Liquidity. *Review of Financial Studies* 18: 1171-1217.
- Frey, S., and P. Sandas. 2009. The Impact of Hidden Liquidity in Limit Order Books. Working Paper.
- Gabaix, X., P. Gopikrishnan, V. Plerou, and H. Stanley. 2006. Institutional investors and stock market volatility. *Quarterly Journal of Economics* 121: 461–504.
- Goettler, R. L., Parlour, C. A., and U. Rajan. 2005. Equilibrium in a Dynamic Limit Order Market. *Journal of Finance* 60: 2149-2192.
- Goettler, R. L., Parlour, C. A., and U. Rajan. 2009. Informed Traders and Limit Order Markets. *Journal of Financial Economics* 93: 67-87.
- Greenwood, R. 2005. Short- and long-term demand curves for stocks: Theory and evidence on the dynamics of arbitrage. *Journal of Financial Economics* 75: 607–649.
- Harris, L. 1996. Does a Large Minimum Price Variation Encourage Order Exposure? Working Paper.
- Harris, L. 1997. Order Exposure and Parasitic Traders. University of Southern California. Working Paper.
- Harris, L. 2003. *Trading & Exchanges: Market Microstructure for Practitioners*. Oxford University Press.
- Harris, L. and E. Gurel. 1986. Price and volume effects associated with changes in the S&P 500 list: New evidence for the existence of price pressures. *Journal of Finance* 41: 815–829.
- Hasbrouck, J., and G. Saar. 2004. Limit Orders and Volatility in a Hybrid Market: The Island ECN. Working Paper.
- Hasbrouck, J., and G. Saar. 2009. Technology and Liquidity Provision: The Blurring of Traditional Definitions. *Journal of Financial Markets* 12: 143-172.
- Hendershott, T., C. M. Jones, and A. J. Menkveld. 2010. Does Algorithmic Trading Improve Liquidity?, *Journal of Finance*, forthcoming.
- Hendershott, T., and A. J. Menkveld. 2010. Price Pressures. Working Paper.
- Investment Technology Group. 2009. Understanding and avoiding adverse selection in Dark Pools. Working Paper.
- Jiang, G. J., Lo, I., and A. Verdelhan. 2009. Information Shocks, Liquidity Shocks, Jumps, and Price Discovery. Evidence from the U.S. Treasury Market. *Journal of Financial and Quantitative Analysis*, forthcoming.
- Kraus, A. and H. Stoll. 1972. Price impacts of block trading on the New York stock exchange. *Journal of Finance* 27: 569–588.
- Moinas, S. 2007. Hidden Orders and Liquidity on a Limit Order Market. Working Paper.
- Mönch, B. 2004. Optimal Liquidation strategies. *Journal of Business* 74: 1867-1913.
- Obizhaeva, A., and J. Wang. 2005. Optimal Trading Strategy and Supply/Demand Dynamics. Working Paper.

- Pardo, A., and R. Pascual. 2006. On the Hidden Side of Liquidity. Working Paper.
- Parlour, C. A. 1998. Price Dynamics in Limit Order Markets. *Review of Financial Studies* 11: 789-816.
- SEC 2010. 17 Release No. 34-61358; File No. S7-02-10 Concept Release on Equity Market Structure
- Seppi, D. 1997. Liquidity Provision with Limit Orders and a Strategic Specialist. *Review of Financial Studies* 10: 103-150.
- Subrahmanyam, A., and R. Jarrow. 2001. The Liquidity Discount. *Mathematical Finance* 11: 447-474.
- Rosu, I. 2008. Liquidity and Information in Order Driven Markets. Working Paper.
- Rosu, I. 2009. A Dynamic Model of the Limit Order Book. *Review of Financial Studies*: 22, 4601-4641.
- Tuttle, L. 2006. Hidden orders, Trading Costs and Information. Working Paper.
- Ye, M. 2009. A Glimpse into the Dark Pool: Price Formation, Transaction Cost and Market Share of the Crossing Network. Working Paper.

Footnotes

1. See Bessembinder, Panayides and Venkataraman (2009); Aitken, Berkman and Mak (2001); Hasbrouck and Saar (2004 and 2009); Frey and Sandas (2009); Tuttle (2006) and Jiang, Lo and Verdelhan (2009).
2. See for example Aitken et al. (2001), Bessembinder et al. (2009), De Winne and D'Hondt (2007) and Frey and Sandas (2009).
3. Fill&Kill orders, also named Immediate-Or-Cancel (IOC) or Execute and Eliminate (ENE), are immediately executed, partially or fully, depending on the orders available on the opposite side of the book. Any unfilled portion is automatically cancelled by the system.
4. Goettler, Parlour and Rajan (2005, 2009) focus as well on the working of an LOB and extend Parlour's framework to model limit order trading as a stochastic sequential game with private and common value; they also introduce endogenous information acquisition. To examine the resiliency and spread dynamic of the LOB, Foucault, Kadan and Kandel (2005) include traders' waiting costs; Rosu (2009) considers a continuous time model with endogenous undercutting.
5. Two are the most relevant elements that have recently characterized electronic trading platforms: dark liquidity and algorithmic trading. Hendershott, Jones and Menkveld (2010) report that more than one third of the trading volume in U.S. equity markets is algorithmically initiated.
6. As Harris (2003) notes, large uninformed orders risk being picked off by scalpers and quote-matchers, whose profits increase with volatility; quote-matchers' profits are also decreasing in the minimum tick size.
7. These costs are the weighted sum of the price impact (the appropriately signed difference between the fill price and the quote midpoint at the time of order submission) and the opportunity cost (smaller price drifts subsequent to order submission time), where the weights are the filled and unfilled portions of the order, respectively.
8. Orders that allow traders to display only a fraction of the entire order are named either reserve orders (e.g. NASDAQ and BATS) or iceberg (e.g. TradElect and Chi-X). Totally undisclosed orders are instead named hidden orders: these are invisible limit orders that lose time priority with respect to standard observable limit orders.
9. As it will be clarified later, we assume that algo trading techniques can only spot hidden liquidity on the opposite side of the market. This is a natural assumption as otherwise during the discovery process the trader would be acting against his own trading interest and hence incur losses.
10. Alternatively, following Parlour (1998) one can assume that in this economy there are two goods, consumption in day 1 (C_1) and consumption in day 2 (C_2). The agent's preference

over consumption in the two days is given by: $U(C_1, C_2; \beta) = C_1 + \beta C_2$, where β reflects his personal trade-off between current and future consumption. During the trading day (day 1), claims to the asset can be exchanged for C_1 . For example, assume a trader sells a unit of the asset that he values βv ; he will proceed with his transaction only if the price he pays (denominated in units of C_1) is smaller than his asset evaluation: $price - \beta v > 0$.

11. Significantly, given that β_t is not related to the future value of the asset, it cannot be interpreted as a measure of private information.
12. Notice that because the tick size is assumed constant, when the common value of the asset changes due to the shock, the tick size relative to the asset's price also changes. This slightly modifies market order execution probability at T , and hence makes the optimal trading strategies at $t \neq T$ not perfectly symmetric around the asset value. However, the degree of this asymmetry is negligible.
13. This assumption greatly simplifies the algebra and allows us to focus only on the last periods of the game. We could include an asset value shock at each trading round, but this would multiply the possible trading strategies and greatly lengthen the computations.
14. For example, if the best bid price for an order of size j is B_2 , then a market sell j -order will be executed at that price and named $MO_j B_2$.
15. As a market order that walks up or down the book until totally executed generally crosses various prices, we do not use an index for the level of the book as we do for the other order types.
16. Our results are qualitatively robust to other values of τ : what changes with the value of the tick size is the width of the β_t ranges and hence the probability associated with different order types. With a lower tick size traders tend to use more market orders, whereas for larger values of the tick they opt more frequently for limit and undisclosed orders.
17. Please notice that this does not mean that at time $T - 2$ only sellers arrive at the market: incoming traders will act as buyers and sellers depending on their β value.
18. The advantage of F&K orders is that they still aim at seeking hidden liquidity, but they avoid the risk of taking a position in case of incomplete execution. Some market (Euronext, but not for example LSE) allow traders to submit also price contingent orders (i.e. market-to-limit orders) that for the unfilled part eventually convert to a limit order on the own side of the book. These orders however are becoming less popular as with the advent of algorithmic programs aimed at searching hidden liquidity, they leave too evident a footprint on the book. On the contrary trading tactics based on F&K minimize this signalling risk by eliminating any unexecuted part of the order. Following this real market practice and the wider diffusion of F&K, we focus on the latter.
19. For brevity, we only report the probability of each possible order type and not the values of the β ranges, which are available from the authors on request.

20. See Appendix: proof to Proposition 1.
21. Notice, however, that here we are focusing on competition for the provision of liquidity. Should we assume that traders use undisclosed orders to prevent price impact, then the effect of high frequency trading would reverse. We will discuss this issue in Section 5.3.
22. Detailed descriptions and formulas of the market quality measures are provided in the Appendix.
23. According to informal conversations with practitioners active on the major European trading platforms, this is the type of hidden orders that is mostly used by traders who compete for the provision of liquidity. Field data on totally undisclosed orders are still not available for empirical investigations; the only existing evidence is reported by Hasbrouck and Saar (2009) who suggest that hidden orders are executed inside the NBBO.
24. Dark pool trading is not embedded in our model (see Buti et al., 2010, and Ye, 2009) but we expect MPP orders to have a higher execution probability compared to dark pools: indeed they have the advantage of being potentially executed not only against MPP of opposite sign, but also against any market order crossing the spread from the other side of the market. For this reason, undisclosed orders bear a competitive advantage over dark pool orders when merely used to compete for liquidity provision. This could probably explain why fees imposed by exchanges on hidden orders are generally higher than dark pools' fees.
25. Notice also that as in this model traders are risk neutral, they clearly post MPP of the largest possible size.
26. See Harris and Gurel (1986), Campbell, Grossman, and Wang (1993), Greenwood (2005), Coval and Stafford (2007) and more recently Hendershott and Menkveld (2010).
27. See Bertsimas and Lo (1998); Almgren and Chriss (2000), Subrahmanyam and Jarrow (2001) and Obizhaeva and Wang (2005).
28. In the latest concept release on equity market structure, the SEC stressed that “[a]n important issue is whether the current market structure and the availability of sophisticated, high-speed trading tools enable proprietary firms to engage in order anticipation strategies on a greater scale than in the past”, where by order anticipation strategies the SEC means “the employment of sophisticated pattern recognition software to ascertain from publicly available information the existence of a large buyer (seller), or the sophisticated use of orders to “ping” different market centers in an attempt to locate and trade in front of large buyers and sellers”.
29. As documented by Harris (2003) these traders must be faster than passive traders and hence must have very good access to the trading platform. We assume that quote matchers look for profits from large blocks and even if they are fast traders, they do not behave as scalpers. Strictly, should they also exploit small profit opportunities, they would get slightly greater gains from trade, but, as it will be clearer later in this Section, this would only add complexity to the model.

30. This assumption can be modified by allowing the price impact to be related with order size and aggressiveness. In a more focused setting, the price impact could also be made endogenous. Besides, the asset value shock could appear sooner leaving the possibility to react only to very fast quote matchers, in which case at $T - 1$ only parasitic traders could arrive. We leave these extensions for future research.
31. To keep the model tractable, only for the case with reserve orders we postulate that traders coming to the market at T rationally compute the probability of hidden depth for orders submitted at $T - 1$; however, they hold adaptive expectations for orders submitted at $T - 2$, meaning that they assume the probability of hidden liquidity to be the same as at $T - 1$. To check the robustness of this hypothesis, we run numerical simulations with different parameters values and found that results do not qualitatively change.
32. Notice that also in this extended version of the model (Table 9), the use of undisclosed orders reduces competition from incoming limit order traders. For example comparing the two states of the LOB at $T - 1$, with 1 + 9 and 10 shares posted at A_2 respectively, we observe that when a trader submits a reserve order at $T - 2$, the next trader joins the queue at A_2 ; but if he submits a $LO_{10}A_2$, the incoming trader undercuts with a limit order at A_1 .
33. See: SEC 2009/2010 Proposals and Concept Releases for the US markets, and CESR (Committee of European Securities Regulators) recent Call for Evidence on “Micro-structural issues of the European Equity markets” (2010).
34. We have shown that, while reserve orders are equilibrium strategies when the book opens empty at $T - 2$, MPP are not. The opposite holds for a deep book: in this case MPP are optimally selected, while it is straightforward to show that reserve orders can not be equilibrium strategies as joining the queue will not prevent incoming traders from undercutting standing limit orders.
35. For securities subject to the Markets in Financial Instruments Directive (MIFID) regulations, hidden limit orders are only permitted where the order consideration meets the “Large in Scale” qualification as per Article 20 of the MIFID pre-trade transparency regime. Large in Scale values are calculated by CESR in Euros (€) with reference to a security’s Average Daily Turnover (ADT). For securities not subject to MIFID regulations the Exchange will apply a LIS based on a security’s ADT.
36. An extended version of the Appendix is available from the authors upon request.

Table 1 - Model's Extensions							
Framework	Limit Orders	Market Orders	Hidden Orders	Reserve Orders	MPP Orders	F&K Orders	Algo Techniques
Benchmark (B)	yes	yes	no	no	no	indiff.	indiff.
M	yes	yes	yes	yes	no	no	no
F&K	yes	yes	yes	yes	no	yes	no
Algo	yes	yes	yes	yes	no	yes	yes
M	yes	yes	no	no	yes	no	no
F&K	yes	yes	no	no	yes	yes	no
Algo	yes	yes	no	no	yes	yes	yes

Table 1 Model's Extensions This Table lists the three protocols considered in this paper (column 1) that differ depending on the type of undisclosed orders offered to large traders. The benchmark model (B) does not allow for hidden depth, the R&H model includes both reserve and hidden orders and the MPP model considers a special type of hidden orders that are pegged to the spread midpoint. For R&H and MPP we further differentiate three cases depending on the trading facilities available to large traders to spot hidden depth: market orders only (M), Fill&Kill orders (F&K) and algo trading techniques (Algo). By looking at the columns one can check which protocols allow for that specific trading strategy. Notice that in the B framework, as there is no hidden liquidity, equilibrium strategies do not change with the introduction of F&K orders or algo techniques.

Table 2 - Order Submission Strategies - Benchmark		
Panel A: Large Trader ($j \in [1, 10]$, $i \in [1, 3]$ for MO , and $i \in [1, 2]$ for LO)		Panel B: Small Trader ($i \in [1, 3]$ for MO , and $i \in [1, 2]$ for LO)
Market Sell Order	$MO_j B_i$ or $MO_j B$	Market Sell Order
Limit Sell Order	$LO_j A_i$	Limit Sell Order
No Trade	NTL	No Trade
Limit Buy Order	$LO_j B_i$	Limit Buy Order
Market Buy Order	$MO_j A_i$ or $MO_j A$	Market Buy Order
		$MO_\alpha B_i$
		$LO_\alpha A_i$
		NTS
		$LO_\alpha B_i$
		$MO_\alpha A_i$

Table 2: Order Submission Strategies. This Table presents the possible orders that a large trader (Panel A) and a small trader (Panel B) can choose upon arrival at the market. Large traders can submit orders of size up to 10 share whereas small traders can only trade α shares. By assumption, a large trader can submit a market sell order ($MO_j B_i$) of size j at price B_i , or a market sell order that walks down the buy side in search of execution ($MO_j B$). A large trader can also choose to submit a limit sell order of size j to either A_1 or A_2 ($LO_j A_i$), or he can decide not to trade (NTL). On the sell side (the buy side is symmetrical) small traders can submit a market sell order ($MO_\alpha B_i$) that will be executed at the first price level at which liquidity is available (B_i). In addition small traders can opt to submit a limit sell order to the first ($LO_\alpha A_1$) or to the second level of the ask side of the LOB ($LO_\alpha A_2$), and they can also decide not to trade (NTS).

Table 3 - Order Submission Probabilities - Competition for Liquidity Provision										
State of LOB	[0000]			R&H	B	[(10)000]	[(3 + 7)000]			[3000]
	B		T - 1				R&H		T - 1	
Protocol	T - 1	T - 2	T - 1	T - 2	T - 1	T - 1	T - 1	T - 1	T - 1	R&H
Period										
$MO_{10}B_3$.302	.224	.302	.242 (.234) [.218]	.346	.341	.341	.341	.341	.341
$RO_{10}A_2$.258 (.266) [.282]						
$LO_{10}A_2$.197	.276	.197							
LO_7A_2							.136	.136	.136	.136
$LO_{10}A_1$.130					
$LO_{10}B_2$.201		.201		.154	.184 [.153]	.202	.184 [.202]	.339 [.321]	
$MO_{10}A$.370			
$MO_{10}A_2$										
$F\&K_{10}A_2$								[.370]		
$MO_{10}A_3$.300		.300							

Table 3: Order Submission Probabilities - Competition for Liquidity Provision. This Table reports large traders' submission probabilities for the orders listed in column 1 for two model specifications: the first one is a benchmark model (B) with no undisclosed orders; the second one introduces reserve and hidden orders (R&H). For the R&H model we consider three specifications: the M framework without fast trading, the F&K framework where we add Fill&Kill orders (in round brackets, if different) and the Algo framework where traders can also access algo trading programs (in square brackets). Execution probabilities are reported for the equilibrium states of the book listed in row 2, and for both period $T - 1$ and period $T - 2$. Notice that when the book opens at $T - 1$ with 3 shares at A_2 , in the M framework traders cannot distinguish between $b_{T-1} = [(3 + 7)000]$ and $b_{T-1} = [3000]$ and use the same trading strategy. When instead algorithmic trading is introduced, buyers are able to differentiate between the two and trade accordingly.

Table 4 - Market Quality - Competition for Liquidity Provision - T - 1

Competition Framework	Depth at Best Ask	Inside Semi-Spread	Weighted Semi-Spread	Volume
Benchmark	1.770	0.0935	0.771	1.039
M	1.875	0.0956	0.808	1.036
R&H F&K	1.902	0.0954	0.806	1.038
Algo	1.939	0.0950	0.804	1.041

Table 4: Estimated Depth, Inside Spread and Volume. This Table reports indicators of market quality for the ask side of the market: estimated depth at best ask price, best semi-spread, quoted and weighted by associated depth, and volume generated by orders hitting the ask side. All indicators are reported respectively for the benchmark model and for the model with undisclosed orders under the three specifications: M (no fast trading), F&K (Fill&Kill orders) and Algo (algorithmic trading).

State of LOB	[0000]			[0(10)00]			[(10)000]			[0100]			[1000]		
	B		MPP	B		MPP	B		MPP	B		MPP	B		MPP
Protocol	$T-1$	$T-2$	$T-1$	$T-2$	$T-1$	$T-2$	$T-1$	$T-2$	$T-1$	$T-2$	$T-1$	$T-2$	$T-1$	$T-2$	$T-1$
Period															
$MO_{10}B_3$.316	.253	.316 (.375)	.256	.350	.331	.351	.335	.323	.323	.330	.330	.330	.330	.330
$HOS_{10}M$			(.104)			.119		.079							
$LO_{10}A_2$.181	.247	.182 (.019)	.244	.115	.015		.170	.170	.170			.162	.162	
LO_9A_2															
$LO_{10}A_1$.125	.063							
$LO_{10}B_2$.186	.187 (.024)		.089	.089	.143	.143	.179	.179	.187	.187	.187	.187	.187
$MO_{10}A_1$.446	.446									
$MO_{10}A$															
$MO_{10}A_2$.381	.381	.328	.328	.322	.322	.322	.322	.322
$HOB_{10}M$			(.103)												
$MO_{10}A_3$.315		.315 (.375)												

Table 5: Order Submission Probabilities: Mid-Point Peg Orders (MPP) - Empty LOB. This Table reports large traders' submission probabilities for the orders listed in column 1 for both the benchmark (B) and the model with MPP. The book starts empty at $T-2$, $b_{T-2} = [0000]$, and values in parenthesis indicate the case where traders suspect the existence of hidden liquidity on the LOB. Order submission probabilities are reported for the equilibrium states of the book listed in row 2. For example, when the book opens at $T-1$ with 10 shares visible at A_1 , e.g. $b_{T-1} = [0(10)00]$, and traders are allowed to use MPP, large sellers submit market orders at B_3 , $MO_{10}B_3$, with probability .331; less aggressive traders submit MPP ($HOS_{10}M$) with probability .119 and limit orders at A_2 ($LO_{10}A_2$) with probability .015.

Table 6 - Order Submission Probabilities: MPP - Deep LOB									
State of LOB	[(10)00(10)]			[(10)(10)0(10)]			[(10)10(10)]		
Protocol	B		MPP	B		MPP	B		MPP
Period	$T-1$	$T-2$	$T-1$	$T-2$	$T-1$	$T-2$	$T-1$	$T-2$	$T-1$
$MO_{10}B_2$.407	.364	.399 (.425) [.425]	.362	.425	.402	.410	.402	.402
$HOS_{10}M$.043 (.065) [.065]	.016		.067			.041
$LO_{10}A_1$.092	.203	.057 (.008) [.008]	.122			.801		
LO_9A_1									.049
NTL					.007				
$LO_{10}B_1$.098		.074 (.012) [.012]		.107	.071	.099	.055	
$MO_{10}A_1$.461	.461			
$MO_{10}A$.409	.401	
$HOB_{10}M$.029 (.065) [.490]						.053
$MO_{10}A_2$.403		.399 (.425) [.000]						

Table 6: Order Submission Probabilities: Mid-Point Peg Orders (MPP) - Deep LOB. This Table reports large traders' submission probabilities for the orders listed in column 1 when the book starts deep at $T-2$, $b_{T-2} = [(10)00(10)]$. Two model specifications are considered: the benchmark model and the model with MPP; results for the latter are presented for both the framework with no fast trading (M) and the one with algorithmic trading (Algo). Values in round brackets indicate the cases where traders suspect the existence of hidden liquidity on the LOB, while results for the Algo framework are reported in square brackets when different. Order submission probabilities are reported for the equilibrium states of the book listed in row 2. Compared with Table 4 all books considered here are deeper as they have at least 10 shares on both A_2 and B_2 . When the book opens at $T-1$ with 10 shares visible at A_1 , e.g. $b_{T-1} = [(10)(10)0(10)]$, and traders are allowed to use MPP, large sellers submit market orders at B_2 , $MO_{10}B_2$, with probability .402, while less aggressive traders submit MPP ($HOS_{10}M$) with probability .067.

Table 7 - Order Submission Probabilities: MPP - No Volatility										
State of LOB	[0000]		[(10)00(10)]		[0(10)00]	[(10)(10)0(10)]	[10000]	[1000]	[0100]	[(10)10(10)]
	$T-1$	$T-2$	$T-1$	$T-2$	$T-2$	$T-1$				
Period	.375	.258								
$MO_{10}B_3$.343		.322	.328		
$MO_{10}B_2$.390 (.425)	.347		.405				.399
$HOS_{10}M$.102		(.066)		.096	.070				
$LO_{10}A_2$.023	.242							.165	
LO_9A_2								.104		
LO_9A_1										.100
$LO_{10}A_1$.110 (.009)	.153			.156	.066		
$LO_{10}B_2$.023				.116		.143	.187	.180	
$LO_{10}B_1$.110 (.009)		.444	.068				.106
$MO_{10}A_1$.457				
$MO_{10}A$.321	.327	.396
$HOB_{10}M$.102		(.066)							
$MO_{10}A_2$.390 (.425)				.379			
$MO_{10}A_3$.375									

Table 7 - Order Submission Probabilities: Mid-Point Peg Orders (MPP) - No Volatility. This Table reports large traders' submission probabilities for the MPP model without an asset value shock at time T . Order submission probabilities are reported for the equilibrium states of the book listed in row 2; values in parenthesis indicate the case where traders suspect the existence of hidden liquidity on the LOB. Probabilities for $T-2$ are reported for the empty book, $b_{T-2} = [0000]$; for example, when the book opens at $T-1$ with 10 shares visible at A_1 , e.g. $b_{T-1} = [0(10)00]$, large sellers submit market orders at B_3 , $MO_{10}B_3$, with probability .343; less aggressive large traders submit MPP ($HOS_{10}M$) with probability .096.

Competition Framework with MPP	Empty LOB		Deep LOB	
	B	MPP	B	MPP
Visible Depth at Best Ask	1.382	1.373	3.415	3.413
Total Depth at Best Ask	1.382	1.421	3.415	3.621
Inside Semi-Spread	0.095	0.095	0.066	0.068
Weighted Semi-Spread	0.743	0.748	0.593	0.607
Volume	0.908	0.907	1.105	1.099

Table 8: Estimated Depth, Inside Spread and Volume. This Table reports indicators of market quality for the ask side of the market: estimated depth at best ask price (visible and total), best semi-spread, semi-spread weighted by associated depth, and volume generated by orders hitting the ask side. All indicators are reported for both the benchmark model (B), and for the model with Mid-Point Peg Orders (MPP).

State of LOB	[0000]										[(10)000]				[1000] & [(1 + 9)000]			
	B		R		H		B	R	H		B	R	H		B	R	H	
Protocol	$T - 1$	$T - 2$	$T - 1$	$T - 2$	$T - 1$	$T - 2$	$T - 1$	$T - 2$	$T - 1$	$T - 2$								
Period	.265	.169	.265	.170	.313	.169	.301	.169	.301	.301	.288	.288	.288	.288	.288	.288	.288	
$MO_{10}B_3$																		
$RO_{10}A_2$.019	.009											.013			
$HO_{10,9}A_2$.012	.014										.009		
$LO_{10,9}A_2$.228	.331	.216	.321	.175	.317					.206	.193	.193					
LO_1A_2	.007																	
$LO_{10}A_1$.151	.151	.149									
LO_1A_1							.01	.005										
$N TL$.032				.001							
LO_1B_2	.009						.015				.014							
$LO_{10}B_2$.178		.158		.131		.102	.081	.080	.170	.150	.149	.149					
$HO_{10}B_2$.018				.081			.041						
$RO_{10}B_2$.029					.073							.034			
$MO_{10}A$.321	.322	.320					
$MO_{10}A_2$.389	.389	.389									
$MO_{10}A_3$.313		.313		.351													

Table 9 Order Submission Probabilities: Picking-off Risk. This Table reports large traders' submission probabilities for the benchmark model (B), the model with reserve orders (R) and the one with hidden orders (H). These probabilities are computed for the four equilibrium states of the book listed in row 2. For example, when the book is empty, $b_{T-1} = [0000]$, large sellers submit market orders at B_3 , $MO_{10}B_3$, with probability .265 at $T - 1$ and .169 at $T - 2$ in the benchmark model. The corresponding probabilities for the R model are .265 at $T - 1$ and .170 at $T - 2$, while for the H model they are respectively .313 and .169. Notice that in the R model, traders cannot differentiate between the books [1000] and [(1 + 9)000], so they use the same trading strategy. The book [(1 + 9)000] does not exist for the B and H model.

Table 10 - Order Submission Probabilities: Front Running Risk										
State of LOB	[0000]		[(10)000]		[1000]	[(1+9)000]		State of LOB	[(10)000] ^d	[0(10)00] ^d
Period	T-2	T-1	T-1	T-1	T-1	T-1	T-1	Period	T-1	T-1
Protocol	B	P	P&R	B	B,P	P&R	P	Protocol	P	P
Parasitic	-	-	-	-	-	-	-	Parasitic	$LO_{10}A_2^d$	$LO_{10}A_1^d$
$MO_{10}B_3$.297	.361	.300	.361	.361	.361	.361	$MO_{10}B_3^d$.345	.361
$LO_{10}A_1$.117	.028	.028		$LO_{10}A_1^d$.130	
$LO_{10}A_2$.203	.139						$LO_{10}A_2^d$		
LO_9A_2					.104	.104		$LO_9A_2^d$		
$RO_{10}A_2$.200					$RO_{10}A_2$		
NT								NT		.056
$LO_{10}B_2$.110	.144	.113		$LO_{10}B_2^d$.121	.120
$MO_{10}A_1$								$MO_{10}A_1^d$.463
$MO_{10}A_2$.412				$MO_{10}A_2^d$.430	
$F\&K_{10}A_2$.019		$F\&K_{10}A_2^d$		
$MO_{10}A$.364	.375		$MO_{10}A^d$		
Parasitic Profits	-	-	-	-	-	-	-	Parasitic Profits	.450	.202

Table 10 Order Submission Probabilities: Front Running Risk. This Table reports the order submission probabilities of large traders, respectively for the benchmark (B), the model with parasitic traders (P), and the model with both parasitic traders and reserve orders (P&R). These probabilities are computed for the states of the book listed in row 2; equilibrium submission probabilities for $T-2$ are reported for the empty book, $b_{T-2} = [0000]$. For example, when the book opens empty, $b_{T-2} = [0000]$, the third column shows that large sellers submit market orders at B_3 , $MO_{10}B_3$, with probability .297, .361 and .300 respectively for B, P, and P&R model. The last two rows show LOB equilibrium order probabilities and parasitic traders' profits at $T-1$ after a one tick downward movement of the LOB price grid due to a large limit sell order posted at A_2 . For example, $[(10)000]^d$ derives from a non aggressive undercutting by parasitic traders ($LO_{10}A_2^d$) of the limit order originally posted at A_2 and now turned into A_3^d .

	T-2	T-1	T	
			A_3^u	
	A_3	A_3	A_2^u	
	A_2	A_2	A_1^u	A_3^d
$v^u = 1 + \tau$ ←	A_1	A_1	B_1^u	A_2^d
$v = 1$ ←	B_1	B_1	B_2^u	A_1^d
$v^d = 1 - \tau$ ←	B_2	B_2	B_3^u	B_1^d
	B_3	B_3		B_2^d
				B_3^d

Figure 1 - Competition for the Provision of Liquidity: Price Dynamic

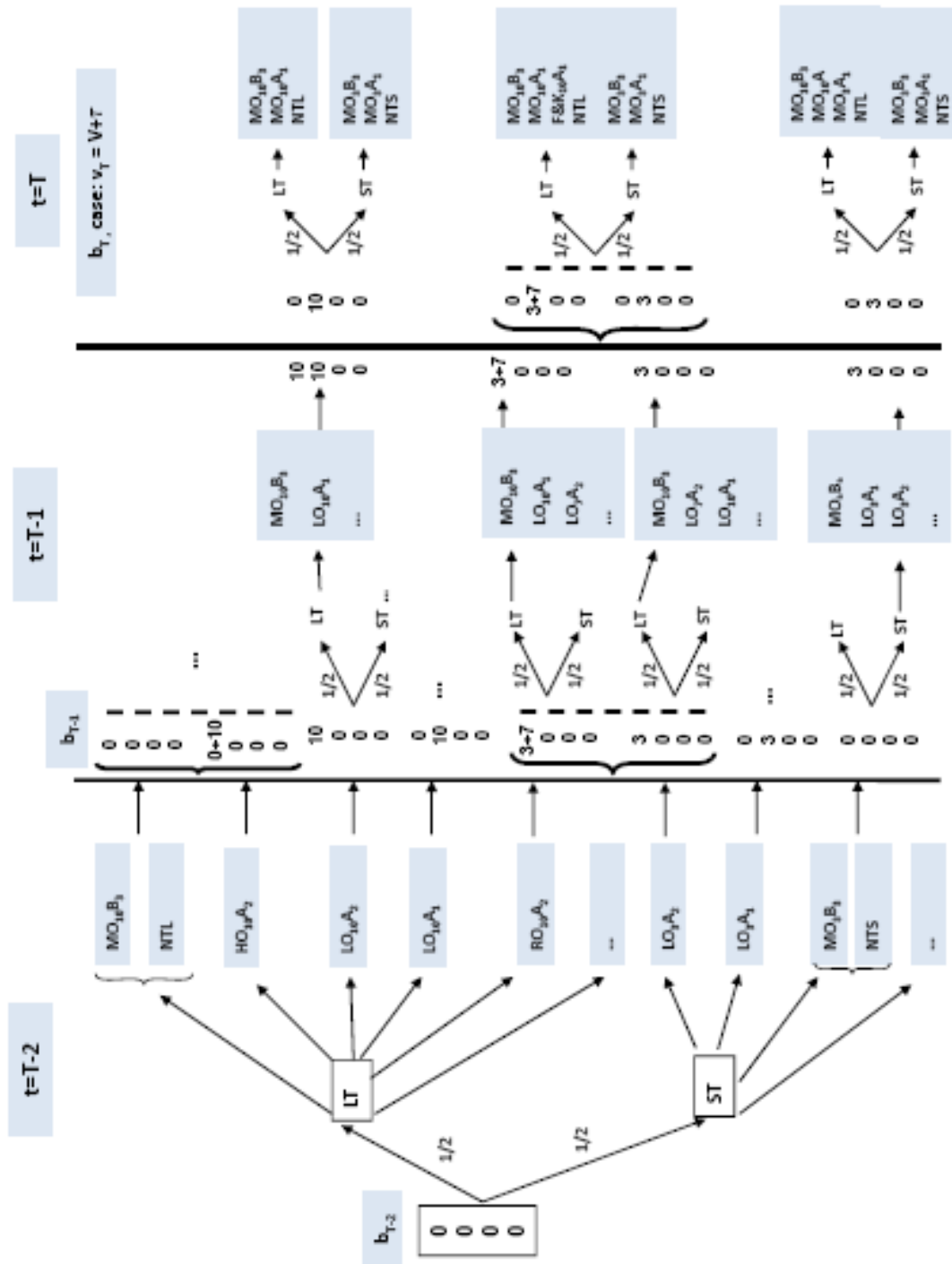


Figure 2 - Competition for the Provision of Liquidity: Example of the Extensive Form of the Game

	$T-2$	$T-1$	T		
				A_3^U	
				A_2^U	
$v^U = 1 + 2\tau$ ←	A_3	A_3	A_3	A_1^U	
	A_2	A_2	A_2	B_1^U	
$v = 1$ ←	A_1	A_1	A_1	B_2^U	A_3^D
	B_1	B_1	B_1	B_3^U	A_2^D
$v^D = 1 - 2\tau$ ←	B_2	B_2	B_2		A_1^D
	B_3	B_3	B_3		B_1^D
					B_2^D
					B_3^D

Figure 3 - Exposure Costs: Price Dynamic with Picking-off Risk

	$T-2$	$T-1$	T	
				A_3^U
				A_2^U
$v^U = 1 + 2\tau$ ←	A_3			A_1^U
	A_2	A_3^d	A_3^d	B_1^U
$v = 1$ ←	A_1	A_2^d	A_2^d	B_2^U
$v^d = 1 - \tau$ ←	B_1	A_1^d	A_1^d	B_3^U
	B_2	B_1^d	B_1^d	A_3^D
	B_3	B_2^d	B_2^d	A_2^D
$v^D = 1 - 4\tau$ ←		B_3^d	B_3^d	A_1^D
				B_1^D
				B_2^D
				B_3^D

Figure 4 - Exposure Costs: Price Dynamic with Negative Price Impact