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Does Uncertainty Vanish in the Small? The Smooth Ambiguity Case*

Fabio Maccheroni[a], Massimo Marinacci[a], Doriana Ruffino[b]
[a]Department of Decision Sciences, Dondena, and IGIER, Università Bocconi
[b]Department of Finance, University of Minnesota

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Abstract

We study orders of risk and model uncertainty aversion in the smooth ambiguity model proposed by Klibanoff, Marinacci, and Mukerji (2005). We consider a quadratic approximation of their model and we show that both risk and model uncertainty attitudes have at most a second order effect. Specifically, the order depends on the properties of the support of the decision maker's limit prior, which we fully characterize. We find that model uncertainty attitudes have a second order effect unless the support is a singleton, that is, unless model uncertainty fades away in the limit. Special attention is given to the binomial state spaces often used in mathematical finance.

1 Introduction

In this paper we study the order of convergence, as uncertainty becomes smaller and smaller, of risk and model uncertainty attitudes in a quadratic approximation of the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji [6].

We show that, under standard differentiability assumptions, both risk and model uncertainty attitudes have at most a second order effect. The order that they feature depends on the properties of the support of the decision maker's limit prior. In particular, both attitudes have a second order effect when this support is not degenerate. For example, model uncertainty attitudes have a second order effect unless this support is a singleton, that is, unless model uncertainty fades away in the limit.

We illustrate these findings with a binomial example. Consider a binary state space $\Omega = \{1, -1\}$ and the random variable $W(\omega) = \omega$. The net $\{W_t\}_{t>0}$ defined by

$$W_t(\omega) = tW(\omega) = \begin{cases} t & \text{if } \omega = 1 \\ -t & \text{if } \omega = -1 \end{cases}$$

is a small uncertainty as $t \downarrow 0$.[1] For every $t > 0$, if the decision maker knows the probability q_t of state $\omega = 1$ and has wealth w, then she evaluates a perturbation W_t by its certainty equivalent

$$u^{-1}\left(q_t u(w+t) + (1-q_t) u(w-t)\right).$$

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[1] The notion of small uncertainty will be made precise in Definitions 22 and 38. Index t has no specific temporal meaning; the net $\{W_t\}_{t>0}$ should not be regarded as a process, but as a sequence of static settings (see Section 3).

Ignorance of q_t renders this quantity a random monetary amount

$$u^{-1}\left(qu\left(w+t\right)+\left(1-q\right)u\left(w-t\right)\right) \quad \forall q\in\left[0,1\right].$$

The smooth ambiguity model of Klibanoff, Marinacci, and Mukerji [6] (KMM hereafter) evaluates it through its certainty equivalent

$$C\left(W_t,\mu_t\right)=v^{-1}\left(\int_0^1 v\left(u^{-1}\left(qu\left(w+t\right)+\left(1-q\right)u\left(w-t\right)\right)\right)d\mu_t\left(q\right)\right)$$

relative to a prior μ_t on $[0,1]$ and to an index v of model uncertainty aversion. Our Theorem 40 shows that, if u and v are sufficiently regular and μ_t weakly converges to μ, the Taylor polynomial of $C\left(W_t,\mu_t\right)$ with respect to t at zero is

$$C\left(W_t,\mu_t\right)=w+E_\mu\left(E\left(W\right)\right)t-\frac{1}{2}\lambda_u\left(w\right)E_\mu\left(\sigma^2\left(W\right)\right)t^2-\frac{1}{2}\lambda_v\left(w\right)\sigma^2_\mu\left(E\left(W\right)\right)t^2+o\left(t^2\right). \qquad (1)$$

Here the arguments $E\left(W\right):[0,1]\to\mathbb{R}$ and $\sigma^2\left(W\right):[0,1]\to\mathbb{R}$ are, respectively, the average and variance of W with respect to each model $q\in[0,1]$, and so the second order terms $E_\mu\left(\sigma^2\left(W\right)\right)$ and $\sigma^2_\mu\left(E\left(W\right)\right)$ are, respectively, an average of models' variances and a variance of models' averages.

The next result shows that there are four cases to consider according to the properties of the support of the limit prior μ.

Proposition 1 *Suppose μ_t weakly converges to μ. Then*

1. *$E_\mu\left(\sigma^2\left(W\right)\right)>0$ and $\sigma^2_\mu\left(E\left(W\right)\right)>0$ if and only if $\operatorname{supp}\mu$ is neither a singleton nor $\{0,1\}$;*
2. *$E_\mu\left(\sigma^2\left(W\right)\right)>0$ and $\sigma^2_\mu\left(E\left(W\right)\right)=0$ if and only if $\operatorname{supp}\mu$ is a singleton in $(0,1)$;*
3. *$E_\mu\left(\sigma^2\left(W\right)\right)=0$ and $\sigma^2_\mu\left(E\left(W\right)\right)>0$ if and only if $\operatorname{supp}\mu=\{0,1\}$;*
4. *$E_\mu\left(\sigma^2\left(W\right)\right)=\sigma^2_\mu\left(E\left(W\right)\right)=0$ if and only if $\mu=\delta_0$ or $\mu=\delta_1$.*

The simple proof follows from two observations. First,

$$E_\mu\left(\sigma^2\left(W\right)\right)=\int 1-\left(2q-1\right)^2 d\mu\left(q\right)=4\int\left(q-q^2\right)d\mu\left(q\right)$$

and so,

$$E_\mu\left(\sigma^2\left(W\right)\right)=0\iff\int\left(q-q^2\right)d\mu\left(q\right)=0\iff\mu\left(q\in[0,1]:q-q^2=0\right)=1$$
$$\iff\mu\left(\{0,1\}\right)=1\iff\operatorname{supp}\mu\subseteq\{0,1\}.$$

Second,

$$\sigma^2_\mu\left(E\left(W\right)\right)=\int\left(2q-1\right)^2d\mu\left(q\right)-\left(\int\left(2q-1\right)d\mu\left(q\right)\right)^2=4\left(\int q^2d\mu\left(q\right)-\left(\int qd\mu\left(q\right)\right)^2\right)$$

and so, by setting $\bar{q}=\int qd\mu\left(q\right)$,

$$\sigma^2_\mu\left(E\left(W\right)\right)=0\iff\int q^2d\mu\left(q\right)-\left(\int qd\mu\left(q\right)\right)^2=0\iff\int\left(q-\bar{q}\right)^2d\mu\left(q\right)=0$$
$$\iff\mu\left(q\in[0,1]:q=\bar{q}\right)=1\iff\mu\left(\{\bar{q}\}\right)=1\iff\mu=\delta_{\bar{q}}.$$

In view of points 1-4 of Proposition 1, to study the order effects of risk and model uncertainty attitudes we must consider the following four possible cases.

Case 1 Suppose $\operatorname{supp}\mu$ is neither a singleton nor $\{0,1\}$. In this case both risk and model uncertainty attitudes have a second order effect at $\{W_t, \mu_t\}_{t>0}$ as $t \downarrow 0$ and the same relevance in the quadratic approximation.[2]

Case 2 Suppose $\operatorname{supp}\mu$ is a singleton in $(0,1)$. In this case risk attitudes have a second order effect at $\{W_t, \mu_t\}_{t\in(0,1]}$ while model uncertainty attitudes have a negligible effect. In this case a KMM decision maker is eventually indistinguishable from a subjective expected utility one.

Case 3 Suppose $\operatorname{supp}\mu = \{0,1\}$. In this case only model uncertainty attitudes have a second order effect at $\{W_t, \mu_t\}_{t\in(0,1]}$, while risk attitudes have a negligible effect.

Case 4 Suppose $\mu = \delta_0$ or $\mu = \delta_1$. In this case both risk and model uncertainty attitudes have a negligible effect at $\{W_t, \mu_t\}_{t\in(0,1]}$.

Summing up, order effects of risk and model uncertainty attitudes depend on the properties of the support of μ. Case 1 is the "normal" one that features a support not degenerate, that is, neither a singleton nor $\{0,1\}$. In this case, both attitudes have the same asymptotic importance as t goes to zero. Cases 2-4 describe what happens under different cases of a degenerate μ. In particular, unless the limit prior μ concentrates on a single model p – so that model uncertainty eventually vanishes – the attitude toward model uncertainty is relevant in the second order approximation. This is a natural feature of the smooth ambiguity model that Theorem 49 will establish in full generality for the important binomial case.

The rest of the paper investigates in full detail and rigor the meaning of small risks and uncertainties, first in a subjective expected utility setup and then in a more general smooth ambiguity setting. Section 3 considers the notion of small risk in a subjective expected utility setting that suitably extends to a Savagean setting the orders of risk aversion studied by Segal and Spivak [11] in a lottery setup. Section 4 further extends the analysis to the smooth ambiguity model. In this case we deal with small uncertainties and we show when risk and model uncertainty attitudes have a second or higher order effect. The properties of the support of μ will play a key role, along the lines discussed before.

Sections 5 and 6 study order effects for time varying risk and time varying uncertainty, respectively. In the latter case priors μ_t are allowed to depend on index t. This dependence substantially complicates matters and requires a sharper version of the Taylor approximation that we establish in Theorem 40, the main technical result of the paper. Under suitable assumptions, Theorem 40 shows that

$$C_t (w + h_t) = w + E_{\bar{q}_t} (h_t) - \frac{1}{2}\lambda_u (w) E_{\mu_t} \left(\sigma^2 (h_t)\right) - \frac{1}{2}\lambda_v (w) \sigma^2_{\mu_t} (E (h_t)) + o\left(\|h_t\|^2_{\bar{q}_t}\right)$$

where $\{h_t\}_{t>0}$ is a net of random variables. The previous approximation (1) is the special case of this approximation for the net $\{W_t\}_{t>0}$.

Section 8 illustrates the findings of the earlier sections in the important binomial setup. As already mentioned, in this setup Theorem 49 fully characterizes – through the properties of the support of μ – the cases when risk and model uncertainty attitudes have either a second order or a negligible effect. In particular, risk and model uncertainty attitudes have a similar order when this support is not degenerate. In view of the importance in mathematical finance of the binomial case, Theorem 49 can be viewed as the paper main result. Finally, Section 9 considers a special binomial case, related to Skiadas [12]. We show that, at least in a static setting comparable to ours, [12]' findings on the negligible effect of model uncertainty attitudes depend on the singleton nature of the support of μ

[2] To be precise, this conclusion requires that $\lambda_u (w) (\bar{q} - \vartheta) + \lambda_v (w) \left(\vartheta - \bar{q}^2\right) \neq 0$, that is, risk propensity (resp., aversion) does not perfectly compensate model uncertainty aversion (resp., propensity) in the limit.

that [12] considers for its purposes. In a continuous time framework, Hansen and Sargent (2009) propose a continuous-time limit of the smooth model in which model uncertainty attitudes survive (see Gindrat and Lefoll, 2010, for another approach to this issue).

We close with couple of methodological remarks on this paper. First, its focus is on the differentiable case; for this reason, most of our analysis deals with second order and higher effects. Second, the approach of the paper is analytical rather than behavioral, that is, we consider the properties of the functional forms. A natural follow-up of our analysis would carry out a more detailed investigation of first order effects and of the behavioral underpinnings of our exercise.

2 Preliminaries

2.1 Mathematical setup

Throughout the paper we consider a finite state space Ω of cardinality n and a base probability measure $P : 2^{\Omega} \to \mathbb{R}$ defined for simplicity on the power set 2^{Ω} (all our results actually hold in any algebra of subsets of Ω).

The collection of all functions $f : \Omega \to \mathbb{R}$ can be thus identified with $\mathbb{R}^n$. Given an interval $I \subseteq \mathbb{R}$, we set

$$L(I) = \{f \in \mathbb{R}^n : f(\omega) \in I \text{ for almost all } \omega \in \Omega\}.$$

Throughout the paper $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^n$. Recall that in $\mathbb{R}^n$ all norms are equivalent; i.e., given any norm $\|\cdot\|'$ there exist positive constants $c_1, c_2 > 0$ such that

$$c_1 \|f\| \leq \|f\|' \leq c_2 \|f\|, \qquad \forall f \in \mathbb{R}^n. \tag{2}$$

We denote by $E_P(f)$ and $\sigma_P^2(f)$ the expectation and variance of a function $f : \Omega \to \mathbb{R}$, respectively.

Barycenters The set of all probability measures $q : 2^{\Omega} \to [0,1]$ is the simplex of $\mathbb{R}^n$, which we denote by Δ. We endow Δ with the Borel σ-algebra $\mathcal{B}(\Delta)$. Each Borel probability measure $\mu : \mathcal{B}(\Delta) \to [0,1]$ induces a measure $\bar{q} \in \Delta$ given by

$$\bar{q}(\omega) = \int_{\Delta} q(\omega)\, d\mu(q), \tag{3}$$

called the *barycenter* of μ (and also denoted by $\bar{\mu}$). It follows that,

$$\sum_{\omega \in \Omega} f(\omega)\, \bar{q}(\omega) = \int_{\Delta} \left(\sum_{\omega \in \Omega} f(\omega)\, q(\omega) \right) d\mu(q), \qquad \forall f \in \mathbb{R}^n. \tag{4}$$

The barycenter $\bar{q}$ has a natural interpretation in terms of reduction of compound lotteries. In fact, if $\mathrm{supp}\mu = \{q_1, ..., q_m\}$ is finite[3] and $\mu(q_i) = \mu_i$ for $i = 1, ..., m$, then (3) becomes

$$\bar{q}(\omega) = \mu_1 q_1(\omega) + ... + \mu_m q_m(\omega), \qquad \forall \omega \in \Omega.$$

Hence, μ can be seen as a lottery whose outcomes are all possible models q, which in turn can be seen as lotteries that determine the state.

[3] A *carrier* of μ is any Borel subset of Δ having full measure. If the intersection of all closed carriers is a carrier, it is called *support* of μ and denoted by $\mathrm{supp}\mu$.

Orders of convergence Orders of convergence as $t \downarrow 0$ (that is, $t \to 0^+$) are cardinal to our analysis. Given two functions $\varphi, \psi : (0,1] \to \mathbb{R}$,

(i) $\varphi(t) = o(\psi(t))$ if for all $M > 0$ there exist δ such that $|\varphi(t)| \leq M|\psi(t)|$ for $t < \delta$;[4]

(ii) $\varphi(t) = O(\psi(t))$ if there exist $\delta, M > 0$ such that $|\varphi(t)| \leq M|\psi(t)|$ for $t < \delta$;[5]

(iii) $\varphi(t) \asymp \psi(t)$ if $\varphi(t) = O(\psi(t))$ and $\psi(t) = O(\varphi(t))$;[6]

(iv) $\varphi(t) \sim \psi(t)$ if $\lim_{t\downarrow 0} \varphi(t)/\psi(t) = 1$, provided $\psi(t) \neq 0$ for t sufficiently close to 0.

It is important to recall that $\varphi(t) = o(\psi(t))$ and $\psi(t) = O(\varphi(t))$ are abuses for $\varphi(t) \in o(\psi(t))$ and $\psi(t) \in O(\varphi(t))$.

The following property plays a key role in the paper.

Lemma 2 *Given any two functions $\varphi, \psi : (0,1] \to \mathbb{R}$, it holds $\varphi = O(\psi)$ if and only if*

$$\xi = o(\varphi) \Longrightarrow \xi = o(\psi) \tag{5}$$

for any function $\xi : (0,1] \to \mathbb{R}$.

In particular, $\varphi(t) \asymp \psi(t)$ if and only if, for any function $\xi : (0,1] \to \mathbb{R}$,

$$\xi = o(\varphi) \Longleftrightarrow \xi = o(\psi),$$

that is, $o(\varphi) = o(\psi)$ when $o(\varphi)$ (resp., $o(\psi)$) is identified with the set $\{\xi : (0,1] \to \mathbb{R} : \xi = o(\varphi)\}$.

Weak and quadratic convergence Some of our results will use the standard notion of weak convergence of Borel probability measures on $\mathcal{B}(\Delta)$. Specifically, a net $\{\mu_\alpha\}_\alpha$ of Borel probability measures $\mu_\alpha : \mathcal{B}(\Delta) \to [0,1]$ *weakly converges* to some Borel probability measure μ, written $\mu_\alpha \stackrel{w}{\Longrightarrow} \mu$, if

$$\lim_\alpha \int_\Delta \varphi(q)\, d\mu_\alpha(q) = \int_\Delta \varphi(q)\, d\mu(q), \qquad \forall \varphi \in C(\Delta), \tag{6}$$

where $C(\Delta)$ is the space of continuous functions $\varphi : \Delta \to \mathbb{R}$.[7] Weak convergence implies the following moments' convergence.

Lemma 3 *If $\mu_\alpha \stackrel{w}{\Longrightarrow} \mu$, then, for each $\omega \in \Omega$, $\lim_\alpha \int_\Delta q^n(\omega)\, d\mu_\alpha(q) = \int_\Delta q^n(\omega)\, d\mu(q)$ for each $n \geq 1$.*

In the special case in which Ω is binary, the simplex $\Delta = \{(q, 1-q) : q \in [0,1]\}$ can be identified with the unit interval $[0,1]$. In this case, the converse of Lemma 3 is also true.

Lemma 4 *$\mu_\alpha \stackrel{w}{\Longrightarrow} \mu$ on $[0,1]$ if and only if $\lim_\alpha \int_{[0,1]} q^n d\mu_\alpha(q) = \int_{[0,1]} q^n d\mu(q)$ for each $n \geq 1$.*

In view of Lemmas 3 and 4, we give the following definition.

Definition 5 *A net $\{\mu_\alpha\}_\alpha$ of probability measures $\mu_\alpha : \Delta \to [0,1]$ quadratically converges to a probability $\mu : \Delta \to [0,1]$, written $\mu_\alpha \stackrel{sq}{\Longrightarrow} \mu$, if for each $\omega \in \Omega$,*

$$\lim_\alpha \int_\Delta q(\omega)\, d\mu_\alpha(q) = \int_\Delta q(\omega)\, d\mu(q) \quad \textit{and} \quad \lim_\alpha \int_\Delta q^2(\omega)\, d\mu_\alpha(q) = \int_\Delta q^2(\omega)\, d\mu(q).$$

Thus, quadratic convergence only requires convergence of the first two moments. It is straightforward to construct nets of probability measures that quadratically converge, but do not weakly converge.

[4] In other words, $\lim_{t\downarrow 0} \varphi(t)/\psi(t) = 0$, provided $\psi(t) \neq 0$ for t sufficiently close to 0.

[5] In other words, $\limsup_{t\downarrow 0} |\varphi(t)/\psi(t)| < \infty$, provided $\psi(t) \neq 0$ for t sufficiently close to 0.

[6] Equivalently, there exist $\delta, c_1, c_2 > 0$ such that $c_1|\psi(t)| \leq |\varphi(t)| \leq c_2|\psi(t)|$ for $t < \delta$. In other words, $0 < \liminf_{t\downarrow 0} |\varphi(t)/\psi(t)| \leq \limsup_{t\downarrow 0} |\varphi(t)/\psi(t)| < \infty$, provided $\psi(t) \neq 0$ for t sufficiently close to 0.

[7] For details on weak convergence we refer the reader to Aliprantis and Border (2006). Notice that since Δ is a compact metric space, weak convergence coincides with weak* convergence when probability measures on Δ are regarded as continuous linear functionals on $C(\Delta)$.

2.2 Decision theoretic setup

Given any nonsingleton interval $I \subseteq \mathbb{R}$ of monetary outcomes, we consider decision makers who behave according to the smooth ambiguity model. Their preferences $\succsim$ over monetary acts are represented by the preference functional $V : L(I) \to \mathbb{R}$ defined by

$$V(f) = \int_{\Delta} \phi \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right) d\mu(q), \qquad \forall f \in L(I), \tag{7}$$

where μ is a Borel probability measure on Δ, and $u : I \to \mathbb{R}$ and $\phi : u(I) \to \mathbb{R}$ are strictly increasing and continuous functions.

We adopt the same setting of Maccheroni, Marinacci, and Ruffino [7]. In particular, since we consider monetary acts, it is natural to study monetary certainty equivalents. To this end, set $v = \phi \circ u : I \to \mathbb{R}$ (see [6, p. 1859]). In [7] we discuss to what extent v describes attitudes toward model uncertainty. Using v we can rewrite (7) as

$$V(f) = \int_{\Delta} \left(v \circ u^{-1} \right) \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right) d\mu(q), \qquad \forall f \in L(I). \tag{8}$$

The certainty equivalent function $C : L(I) \to \mathbb{R}$ induced by V is defined by $V(C(f)) = V(f)$ for all acts f, that is,

$$C(f) = v^{-1} \left(\int_{\Delta} v \left(u^{-1} \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right) \right) d\mu(q) \right), \qquad \forall f \in L(I). \tag{9}$$

This is the composition of two monetary certainty equivalents,

$$c(f, q) = u^{-1} \left(\int_{\Omega} u(f(\omega)) dq(\omega) \right) \quad \text{and} \quad v^{-1} \left(\int_{\Delta} v(c(f, q)) d\mu(q) \right).$$

Consider the Arrow-Pratt coefficients $\lambda_u(w) = -u''(w)/u'(w)$ and $\lambda_v(w) = -v''(w)/v'(w)$. By [6, Prop. 1], the decision maker is ambiguity averse (as defined in [6, p. 1863]) if and only if ϕ is concave, that is, if and only if $\lambda_v \geq \lambda_u$. Ambiguity neutrality corresponds to $\phi(x) = x$, that is, $u = v$ (up to a normalization). In this case,

$$C(f) = u^{-1} \left(\int_{\Delta} \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right) d\mu(q) \right) = u^{-1} \left(E_{\bar{q}}(u \circ f) \right) = c(f, \bar{q}), \qquad \forall f \in L(I) \tag{10}$$

where $\bar{q}$ is the barycenter of μ given in (3). Under ambiguity neutrality we thus get back to the certainty equivalent of a subjective expected utility decision maker with prior $\bar{q}$. The same happens when the support of μ is a singleton $\{q\}$, that is, when the decision maker does not perceive any ambiguity. In this case it trivially holds $\bar{q} = q$ and again $C(f) = u^{-1}(E_{\bar{q}}(u \circ f))$.

In the ambiguity neutral and risk neutral case $u(x) = v(x) = x$, the certainty equivalent (9) reduces to the expected value $E_{\bar{q}}(f)$ of f under $\bar{q}$. For this reason, the *uncertainty premium* $\pi(f)$ of act f for the decision maker described by (9) is given by

$$\pi(f) = E_{\bar{q}}(f) - C(f), \qquad \forall f \in L(I).$$

Throughout the paper we maintain the following assumption.

Assumption 1 *The functions $u, v : I \to \mathbb{R}$ are continuous, strictly increasing, and concave.*

Under this assumption the certainty equivalent (9) is well defined and $\pi(f) \geq 0$ for each $f \in L(I)$.

2.3 Quadratic approximation

Let $w \in \operatorname{int} I$ be a scalar that we interpret as current wealth. To ease notation, we also denote by w the degenerate random variable $w1_\Omega$. Given $h \in \mathbb{R}^n$ such that $w+h \in L(I)$, the certainty equivalent $C(w+h)$ of $w+h$ is

$$C(w+h) = v^{-1}\left(\int_\Delta v\left(u^{-1}\left(\sum_{\omega\in\Omega} u(w+h)\, q(\omega)\right)\right) d\mu(q)\right). \tag{11}$$

For all $h \in \mathbb{R}^n$, the function

$$E(h) : q \mapsto \sum_{\omega\in\Omega} h(\omega)\, q(\omega)$$

is continuous and bounded on Δ. Its variance with respect to μ is given by

$$\int_\Delta \left(\sum_{\omega\in\Omega} h(\omega)\, q(\omega)\right)^2 d\mu(q) - \left(\int_\Delta \left(\sum_{\omega\in\Omega} h(\omega)\, q(\omega)\right) d\mu(q)\right)^2,$$

and it is denoted by $\sigma^2_\mu(E(h))$. This variance reflects the uncertainty on the expectation $E(h)$ as it is perceived by the decision maker. Thus, higher values of $\sigma^2_\mu(E(h))$ correspond to a higher incidence of model uncertainty in the valuation of $E(h)$.

We now report the second order approximation of the certainty equivalent (11), a special case of a more general approximation derived by [7]. The approximation is based on the Arrow-Pratt coefficients $\lambda_u(w)$ and $\lambda_v(w)$. The Peano remainder $o\left(\|h\|^2_{\bar q}\right)$ is in terms of the $L^2(\bar q)$ norm $\|h\|^2_{\bar q} = E_{\bar q}\left(h^2\right)$.

Proposition 6 *Let $\mu : \mathcal{B}(\Delta) \to [0,1]$ be a Borel probability measure on Δ and $u, v : I \to \mathbb{R}$ be twice continuously differentiable with $u', v' > 0$. Then,*

$$C(w+h) = w + E_{\bar q}(h) - \frac{1}{2}\lambda_u(w)\, \sigma^2_{\bar q}(h) - \frac{1}{2}\left(\lambda_v(w) - \lambda_u(w)\right)\sigma^2_\mu(E(h)) + o\left(\|h\|^2_{\bar q}\right) \tag{12}$$

as $\|h\|_{\bar q} \to 0$.

Thus, the sign and magnitude of the effect of perceived model uncertainty on the certainty equivalent depend on the difference $\lambda_v(w) - \lambda_u(w)$. Since it holds

$$E_\mu\left(\sigma^2(h)\right) = \sigma^2_{\bar q}(h) - \sigma^2_\mu(E(h))$$

for the study of risk and model uncertainty attitudes it is useful to rearrange (12) as

$$\begin{aligned} C(w+h) &= w + E_{\bar q}(h) - \frac{1}{2}\lambda_u(w)\left(\sigma^2_{\bar q}(h) - \sigma^2_\mu(E(h))\right) - \frac{1}{2}\lambda_v(w)\, \sigma^2_\mu(E(h)) + o\left(\|h\|^2_{\bar q}\right) \\ &= w + E_{\bar q}(h) - \frac{1}{2}\lambda_u(w)\, E_\mu\left(\sigma^2(h)\right) - \frac{1}{2}\lambda_v(w)\, \sigma^2_\mu(E(h)) + o\left(\|h\|^2_{\bar q}\right), \end{aligned}$$

where we group terms according to the Arrow-Pratt coefficients. Risk and model uncertainty attitudes are thus weighted by the average (w.r.t. the models) variance $E_\mu\left(\sigma^2(h)\right)$ and the variance (w.r.t. the models) of the averages $\sigma^2_\mu(E(h))$.

Similarly, the quadratic approximation of the uncertainty premium $\pi(w+h)$ of $w+h$ is

$$\pi(w+h) = \frac{1}{2}\lambda_u(w)\, E_\mu\left(\sigma^2(h)\right) + \frac{1}{2}\lambda_v(w)\, \sigma^2_\mu(E(h)) + o\left(\|h\|^2_{\bar q}\right). \tag{13}$$

Since

$$C(w+h) = w + E_{\bar q}(h) - \pi(w+h),$$

the effects of risk and model uncertainty attitudes on the second order approximation (12) can be studied directly via the premium (13). For this reason in what follows we focus on (13), although our main interest is ultimately in the certainty equivalent approximation (12).

2.4 First remarks on attitudes

In view of the quadratic approximation (13) of the uncertainty premium $\pi(w+h)$ we can make a few preliminary remarks on risk and model uncertainty attitudes. Specifically, according to the properties of the support of μ we consider four basic cases:

Case 1: The support of μ is neither a singleton nor a collection of Dirac measures. In this case both risk and model uncertainty attitudes typically matter.

Case 2: The support of μ is a singleton; i.e., $\operatorname{supp}\mu = \{p\}$ for some $p \in \Delta$. In this case, $\bar{q} = p$ and only risk attitudes may matter. Specifically,

$$\pi(w+h) = \frac{1}{2}\lambda_u(w)\,\sigma_p^2(h) + o\left(\|h\|_p^2\right).$$

Case 3: The support of μ consists of Dirac measures, that is,

$$\mu = \sum_{i=1}^{n} \mu_i \delta_{\delta_{\omega_i}}$$

with $(\mu_1, ..., \mu_n) \in \Delta$. Simple computation delivers $(\bar{q}_1, ..., \bar{q}_n) = (\mu_1, ..., \mu_n)$ and hence $\sigma_\mu^2(E(h)) = \sigma_{\bar{q}}^2(h)$. In this case only model uncertainty attitudes may matter. In fact, we can write

$$\pi(w+h) = \frac{1}{2}\lambda_v(w)\,\sigma_\mu^2(E(h)) + o\left(\|h\|_{\bar{q}}^2\right) = \frac{1}{2}\lambda_v(w)\,\sigma_{\bar{q}}^2(h) + o\left(\|h\|_{\bar{q}}^2\right).$$

Case 4: The support of μ is a singleton consisting of a Dirac measure; i.e., $\operatorname{supp}\mu = \{\delta_\omega\}$. Neither risk nor model uncertainty attitudes matter in this case (which is the intersection of the two previous ones) and actually $\pi(w+h) = 0$.

As argued in the Introduction, Case 1 is the normal one, while Cases 2-4 are special. They are all based on the properties of the support of μ, an information trait. In the asymptotic analysis that we will carry out in the paper the limit behavior of h will also matter, along with that of μ. This will give rise to the Limit Cases 1-4 of Sections 7 and 8.

3 Small risks

3.1 Notion and characterization

The notion of *small risk* is key to this paper. Since a definition of absolute risk smallness is elusive, we follow Pratt [10] in studying it via limit behavior. We first consider the case of a given probability measure $q \in \Delta$. Our analysis will be based on nets $\{h_t, q\}_{t\in(0,1]}$ where each $h_t : \Omega \to \mathbb{R}$ is a monetary act. We call them *risky monetary nets* when $\limsup_{t\downarrow 0} E_q(|h_t|) < \infty$ and h_t is never q-a.e null. This limit condition put some discipline on how outcomes $h_t(\omega)$ can diverge as t goes to zero.

In a coin toss example with $\Omega = \{H, T\}$, the net $\{h_t, q\}_{t\in(0,1]}$ consists of acts whose outcomes $h_t(H)$ and $h_t(T)$ depend on whether heads or tails come up, with probabilities $q(H)$ and $q(T)$. As observed in the Introduction, index t has no temporal meaning. The net $\{h_t, q\}_{t\in(0,1]}$ should not be regarded as a process, but as a sequence of static settings – "snapshots" – that at each t feature an act h_t whose "riskiness" depends on q.[8]

[8]Few remarks on notation: (i) we choose the letter h because small risks will be relevant for the approximations (12) and (13); (ii) the reason why we mention explicitly q in the net $\{h_t, q\}_{t\in(0,1]}$ will become clear later in the paper.

In this risk setting the certainty equivalent (9) reduces to

$$C(f) = u^{-1}\left(\sum_{\omega\in\Omega} u(f(\omega))\, q(\omega)\right). \tag{14}$$

In terms of the smooth model, this can be due to either $\phi(x) = x$ (ambiguity neutrality) or a singleton support for μ (no perceived ambiguity). Either way, here we are in a standard subjective expected utility setting.

We can now present the key notion of small risk.

Definition 7 *A risky monetary net $\{h_t, q\}_{t\in(0,1]}$ is a small risk if $E_q\left(h_t^2\right) \to 0$ as $t \downarrow 0$.*

We illustrate this notion with few important examples.

Example 8 A net $\{h_t\}_{t\in(0,1]}$ such that

$$\lim_{t\downarrow 0} h_t(\omega) = 0, \qquad \forall \omega \in \Omega \tag{15}$$

or, equivalently, such that $\lim_{t\downarrow 0} \|h_t\| = 0$, forms a small risk when paired with any $q \in \Delta$. In view of (2), it does not matter what norm of $\mathbb{R}^n$ is used to measure outcomes' sizes. ▲

Example 9 Adapted to our setting, Pratt [10]'s analysis considers a net $\{h_t\}_{t\in(0,1]}$ of monetary acts that are actuarially neutral with respect to q and whose variances vanish; i.e., $E_q(h_t) = 0$ and $\lim_{t\downarrow 0} \sigma_q^2(h_t) = 0$. In other words, the net $\{h_t\}_{t\in(0,1]}$ goes to zero in $L^2(q)$ norm. The net $\{h_t, q\}_{t\in(0,1]}$ is a small risk. ▲

Example 10 Adapted to our setting, Segal and Spivak [11]'s analysis is based on a fixed monetary act h that determines a net $\{h_t\}_{t\in(0,1]}$ so that each h_t has the same distribution under q as th, for all $t \in (0,1]$. That is,

$$q(h_t \in E) = q(th \in E) \tag{16}$$

for all $t \in (0,1]$ and all Borel subsets $\mathcal{B}$ of $\mathbb{R}$. Since $E_q\left(h_t^2\right) = t^2 E_q\left(h^2\right)$, then the net $\{h_t, q\}_{t\in(0,1]}$ is a small risk. ▲

Example 11 A special case of the previous example is

$$h_t = th \qquad q\text{-a.e.}$$

for some monetary act h. We call *directional small risks* the nets $\{h_t, q\}_{t\in(0,1]}$ with $h_t = th$. ▲

The next result – a version for nets of a known result for sequences – characterizes small risks by considering three types of limit behavior and showing their equivalence to the notion of small risk. In particular, in (ii) limit behavior is in the sense of convergence in probability and acts' values eventually vanish with probability one according to q. In (iii) limit behavior is in the sense of $L^n(p)$ convergence and acts' moments eventually vanish.[9] Finally, in (iv) limit behavior is in the sense of almost sure convergence and acts' outcomes eventually vanish in each state that belongs to the support of q.

Proposition 12 *For a risky monetary net $\{h_t, q\}_{t\in(0,1]}$, the following conditions are equivalent:*

(i) $\{h_t, q\}_{t\in(0,1]}$ is a small risk;

[9] The condition $\limsup_{t\to 0} E_q(|f_t|) = \lim_{\varepsilon\downarrow 0} \sup_{t\in(0,\varepsilon)} E_q(|f_t|) < \infty$ only requires that the expectation $E_q(|f_t|)$ be eventually bounded.

(ii) $\lim_{t\downarrow 0} q\left(\omega : |h_t(\omega)| < \varepsilon\right) = 1$ *for each* $\varepsilon > 0$*;*

(iii) $\lim_{t\downarrow 0} E_q\left(|h_t|^n\right) = 0$ *for all* $n \geq 1$*;*

(iv) $\lim_{t\downarrow 0} h_t(\omega) = 0$ *for all* $\omega \in \operatorname{supp} q$*.*

Condition (iv) easily implies (iii), which implies (i). In turn, (i) easily implies (ii). By the Chebyshev inequality, for all $\varepsilon > 0$

$$0 \leq q\left(\omega : |h_t(\omega)| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} E_q\left(h_t^2\right) \to 0 \text{ as } t \downarrow 0.$$

Showing that (ii) implies (iv) is less immediate and is proved in the Appendix.

3.2 Order effects

Small risks can be used to model small risky deviations h_t from a sure prospect w and to study their effects on the risk premium

$$\pi\left(w + h_t\right) = w + E_q\left(h_t\right) - C\left(w + h_t\right). \tag{17}$$

Consider the following classification: risk attitudes have a

(i) *first order effect at* $\{h_t, q\}_{t\in(0,1]}$ if $\pi\left(w + h_t\right) \asymp \|h_t\|_q$;

(ii) *second order effect at* $\{h_t, q\}_{t\in(0,1]}$ if $\pi\left(w + h_t\right) \asymp \|h_t\|_q^2$;

(iii) *(quadratically) negligible effect at* $\{h_t, q\}_{t\in(0,1]}$ if $\pi\left(w + h_t\right) = o\left(\|h_t\|_q^2\right)$.

This classification revisits in our Savagean setup the orders of risk aversion that Segal and Spivak [11] studied in a lottery setting (see Montesano [9] for related ideas and Machina [8] for a comprehensive analysis). It is straightforward to see that (i)-(iii) are mutually exclusive alternatives, though not exhaustive.

The next result is related with [11, Proposition 1] and provides a behavioral characterization of these effects. To this end, set

$$k_* = \liminf_{t\downarrow 0} \frac{E_q^2\left(h_t\right)}{E_q\left(h_t^2\right)} \quad \text{and} \quad k^* = \limsup_{t\downarrow 0} \frac{E_q^2\left(h_t\right)}{E_q\left(h_t^2\right)}.$$

Since it always holds $E_q^2\left(h_t\right) \leq E_q\left(h_t^2\right)$, we have $0 \leq k_* \leq k^* \leq 1$. For instance, for the small risk of Example 10 it holds $k_* = k^* = E_q^2(h)/E_q\left(h^2\right)$.

Proposition 13 *Consider a small risk* $\{h_t, q\}_{t\in(0,1]}$ *with* $E_q\left(h_t\right) > 0$ *for all* t*.*

(i) If $\pi\left(w + h_t\right) \sim k\|h_t\|_q$ *and* $k > \sqrt{k^*}$*, then for all* t *small enough*

$$C\left(w + h_t\right) < w.$$

(ii) If $\pi\left(w + h_t\right) \sim k\|h_t\|_q$ *and* $0 < k < \sqrt{k_*}$*, then for all* t *small enough*

$$C\left(w + h_t\right) > w.$$

(iii) If $\pi\left(w + h_t\right) = o\left(\|h_t\|_q\right)$*, then for all* t *small enough*

$$C\left(w + h_t\right) > w.$$

Points (i) and (ii) consider first order effects, while point (iii) considers second or higher order effects. Point (i) shows that when k is high enough (e.g., $k \geq 1$), then $\pi(w+h_t) \sim k \|h_t\|_q$ implies that for all t small enough the decision maker strictly prefers the sure amount w over the risky one $w+h_t$. Since $E_q(h_t) > 0$, a risk neutral decision maker would prefer the opposite. Hence, in this case even for small risks the decision maker is not risk neutral.

Points (ii) and (iii) show that this is no longer the case when either k is small enough or $\pi(w+h_t) = o\left(\|h_t\|_q\right)$, that is, the effect is of second or higher order. In these cases, for t small enough the decision maker compares $w+h_t$ and w as if he were risk neutral. That is, risk attitudes do not matter.

In sum, Proposition 13 shows that risk attitudes that have a first order effects may or may not matter when comparing $w+h_t$ and w; with second or higher order effects they do not matter.

3.3 Differential case

Throughout this subsection we make the following assumption.

Assumption 2 *The utility function u is twice continuously differentiable with $u' > 0$ and $\lambda_u(w) \neq 0$.*

By Proposition 6, under this assumption a small risk $\{h_t, q\}_{t\in(0,1]}$ has the quadratic approximation of its premium $\pi(w+h_t)$ given by

$$\pi(w+h_t) = \frac{1}{2}\lambda_u(w)\,\sigma_q^2(h_t) + o\left(\|h_t\|_q^2\right). \tag{18}$$

Remarkably, this approximation makes it possible to study risk attitudes' order effects through the variance $\sigma_q^2(h_t)$, an insight that goes back to Pratt (1964) and Arrow (1970). In particular, these attitudes have a first order effect at $\{h_t, q\}_{t\in(0,1]}$ if and only if $\sigma_q^2(h_t) \asymp \|h_t\|_q$, a second order effect if and only if $\sigma_q^2(h_t) \asymp \|h_t\|_q^2$, and a negligible effect if and only if $\sigma_q^2(h_t) = o\left(\|h_t\|_q^2\right)$.

Since $\sigma_q^2(h_t) \leq \|h_t\|_q^2$, we have $\sigma_q^2(h_t) = o\left(\|h_t\|_q\right)$, and so risk attitudes do not have a first order effect under Assumption 2. They can have a second or higher effects, as we show next.

Lemma 14 *It holds*

$$\sigma_q^2(h_t) \asymp \|h_t\|_q^2 \Longleftrightarrow k^* < 1$$

and

$$\sigma_q^2(h_t) = o\left(\|h_t\|_q^2\right) \Longleftrightarrow k_* = 1.$$

Hence, risk attitudes have a second order effect at $\{h_t, q\}_{t\in(0,1]}$ when $k^* < 1$ and a negligible one when $k_* = 1$. In contrast, there are no clear cut effects when $k_* < k^* = 1$.

Since $k_* \leq k^* \leq 1$, condition $k_* = 1$ is equivalent to $k_* = k^* = 1$, that is,

$$\lim_{t\downarrow 0} \frac{E_q^2(h_t)}{E_q(h_t^2)} = 1. \tag{19}$$

This condition holds when $\{h_t, q\}_{t\in(0,1]}$ is *asymptotically constant*, that is,

$$h_t(\omega) \sim h_t(\omega') \qquad \forall \omega, \omega' \in \operatorname{supp} q. \tag{20}$$

Intuitively, small risks that satisfy this condition are asymptotically riskless (notice that condition (20) trivially holds when $\operatorname{supp} q$ is a singleton). Under a regularity condition, the asymptotic condition (20) is also necessary for (19).

Proposition 15 *Let $\{h_t, q\}_{t\in(0,1]}$ be a small risk.*

(i) If $\{h_t, q\}_{t\in(0,1]}$ is asymptotically constant, then (19) holds and risk attitudes are negligible at $\{h_t, q\}_{t\in(0,1]}$.

(ii) Conversely, $\{h_t, q\}_{t\in(0,1]}$ is asymptotically constant provided (19) and at least one of the following conditions hold:

(a) $\operatorname{supp} q = \{\omega_1, \omega_2\}$ *and* $h_t(\omega_1) = O(h_t(\omega_2))$*;*

(b) $\operatorname{supp} q = \{\omega_1, \omega_2, ..., \omega_s\}$ *and* $h_t(\omega_i) \sim k_{ij} h_t(\omega_j)$ *for all* $\omega_i, \omega_j \in \operatorname{supp} q$*.*

In other words, risk attitudes are negligible when risk asymptotically vanishes, that is, for small risks that are asymptotically constant, and so asymptotically riskless. In this case,

$$\pi(w + h_t) = o\left(\|h_t\|_q^2\right).$$

Instead, when a small risk is not asymptotically constant (and we are not in the indeterminate case $k_* < k^* = 1$) we have

$$\pi(w + h_t) \sim \frac{1}{2}\lambda_u(w)\,\sigma_q^2(h_t),$$

that is, when risk attitudes have a second order effect, then the risk premium is asymptotically equivalent to $2^{-1}\lambda_u(w)\,\sigma_q^2(h_t)$. In this term, risk attitudes captured by $\lambda_u(w)$ are multiplicatively separated from the riskiness of the net $\{h_t, q\}_{t\in(0,1]}$ captured by $\sigma_q^2(h_t)$.

Example 16 (i) Consider a small risk $\{h_t, q\}_{t\in(0,1]}$ defined by (16); for example, a directional small risk. It holds $k^* = E_q^2(h)/E_q(h^2) < 1$ if h is not constant q-a.e. In this case $\sigma_q^2(h_t) \asymp \|h_t\|_q^2$ and so risk attitudes have a second order effect. If, instead, h is constant q-a.e., then $k_* = E_q^2(h)/E_q(h^2) = 1$ and so risk attitudes have a negligible effect (it actually holds $\pi(w + h_t) = 0$ for each $0 < t \leq 1$).

(ii) Consider a space $\Omega = \{1, 2\}$ and a risky monetary net $\{h_t, q\}_{t\in(0,1]}$ with $0 < q(1) < 1$ and

$$h_t(\omega) = \begin{cases} t + t^2 & \text{if } \omega = 1 \\ t & \text{if } \omega = 2 \end{cases}$$

This net is an asymptotically constant small risk (i.e., $k_* = 1$). Hence, $\pi(w + h_t) = o\left(\|h_t\|_{\bar{q}}^2\right)$ and risk attitudes have a negligible effect at this small risk. ▲

Till now we studied order effects at some small risk $\{h_t, q\}_{t\in(0,1]}$. It is a "directional" standpoint. A global one would require that the order be consistent across all small risks. For this reason, we say that risk attitudes have a

(i) *first order effect* if $\pi(w + h) \asymp \|h\|_q$ as $\|h\|_q \to 0$;

(ii) *second order effect* if $\pi(w + h) \asymp \|h\|_q^2$ as $\|h\|_q \to 0$;

(iii) *(quadratically) negligible effect* if $\pi(w + h) = o\left(\|h\|_q^2\right)$ as $\|h\|_q \to 0$.

Clearly, if risk attitudes have a first, second, or negligible order effect, the same applies to all small risks. Indeed, any small risk $\{h_t, q\}_{t\in(0,1]}$ is such that $\|h_t\|_q \to 0$ as $t \downarrow 0$.

The next result shows that, unless q is trivial, there is always some small risk at which risk attitudes are not negligible.

Proposition 17 *Risk attitudes have a negligible effect if and only if q is trivial, i.e., $q = \delta_\omega$ for some $\omega \in \Omega$.*

3.4 Scales of risks

Some classic scales can be used to benchmark the rate at which $E_q\left(h_t^2\right)$ converges to zero (see, e.g., Hardy [5]). The simplest is the power scale

$$..., t^{\frac{1}{n}}, ..., t^{\frac{1}{2}}, t, t^2, ..., t^n, ...$$

Let us consider the case of quadratic speed t^2 (while similar considerations would apply to other powers).

Definition 18 *A risky monetary net $\{h_t, q\}_{t\in(0,1]}$ is a quadratic small risk if $E_q\left(h_t^2\right) = O\left(t^2\right)$.*

In other words, quadratic small risks feature second moments $E_q\left(h_t^2\right)$ that go to zero with at least quadratic speed.

Example 19 The net $\{h_t, q\}_{t\in(0,1]}$ a la Segal and Spivak defined by (16) is a quadratic small risk since $E_q\left(h_t^2\right) = t^2 E_q\left(h^2\right)$. In particular, this is the case for directional small risks. ▲

By Lemma 2, for quadratic small risks the quadratic approximation (18) holds with a Peano remainder $o\left(t^2\right)$ in place of $o\left(\|h_t\|_q^2\right)$, that is,

$$\pi\left(w + h_t\right) = \frac{1}{2}\lambda_u\left(w\right)\sigma_q^2\left(h_t\right) + o\left(t^2\right).$$

Next we characterize quadratic small risks.

Proposition 20 *Given a monetary net $\{h_t\}_{t\in(0,1]}$, the following conditions are equivalent:*

(i) $\{h_t\}_{t\in(0,1]}$ *is a quadratic small risk;*

(ii) $E_q\left(|h_t|^n\right) = O\left(t^n\right)$ *for some* $n \geq 1$*;*

(iii) $E_q\left(|h_t|^n\right) = O\left(t^n\right)$ *for all* $n \geq 1$*;*

(iv) $h_t\left(\omega\right) = O\left(t\right)$ *for all* $\omega \in \operatorname{supp} q$*.*

Example 21 For the net $\{h_t, q\}_{t\in(0,1]}$ defined by (16) we have $E_q\left(|h_t|\right) = tE_q\left(|h|\right)$. ▲

Points (iv) of Propositions 12 and 20 show that the difference between small risks and quadratic ones can be seen by looking at the limit behavior of $h_t\left(\omega\right)$ for each $\omega \in \operatorname{supp} q$. Small risks correspond to their convergence to zero, i.e., $h_t\left(\omega\right) \to 0$, while quadratic small risks require that this convergence occurs at least at linear speed, i.e., $h_t\left(\omega\right) = O\left(t\right)$.

4 Small uncertainties

The earlier risk analysis can be extended to account for ambiguity. We call *uncertain monetary nets*, indicated $\{h_t, \mu\}_{t\in(0,1]}$, nets of monetary acts $h_t : \Omega \to \mathbb{R}$ and probability distributions μ on Δ, with $\limsup_{t\downarrow 0} E_{\bar{q}}\left(|h_t|\right) < \infty$. In a coin toss example, μ is the decision maker's prior on the probability models q that determine how likely heads and tails are.

Now the relevant certainty equivalent is (9), that is,

$$C\left(f\right) = v^{-1}\left(\int_\Delta v\left(u^{-1}\left(\sum_{\omega\in\Omega} u\left(f\left(\omega\right)\right)q\left(\omega\right)\right)\right)d\mu\left(q\right)\right)$$

As in (3), we set

$$\bar{q}(\omega) = \int_{\Delta} q(\omega)\, d\mu(q).$$

Hence, an uncertain monetary net $\{h_t, \mu\}_{t\in(0,1]}$ induces a risky monetary net $\{h_t, \bar{q}\}_{t\in(0,1]}$. Using these induced nets we can extend to uncertain monetary nets some of the notions that we previously established for risky ones.

Definition 22 *An uncertain monetary net $\{h_t, \mu\}_{t\in(0,1]}$ is a small uncertainty if and only if the induced risky monetary net $\{h_t, \bar{q}\}_{t\in(0,1]}$ is a small risk.*

This fact is important to extend to small uncertainties the analysis carried out for small risks. A first consequence is that Proposition 12 holds verbatim with $\bar{q}$ in place of q. This means that, for example, $\{h_t, \mu\}_{t\in(0,1]}$ is a small uncertainty if and only if $\lim_{t\downarrow 0} h_t(\omega) = 0$ for all $\omega \in \operatorname{supp} \bar{q}$.

Regarding order effects, we now have uncertainty attitudes in place of just risk ones. In particular, uncertainty attitudes have a

(i) *first order effect at* $\{h_t, \mu\}_{t\in(0,1]}$ if $\pi(w + h_t) \asymp \|h_t\|_{\bar{q}}$;

(ii) *second order effect at* $\{h_t, \mu\}_{t\in(0,1]}$ if $\pi(w + h_t) \asymp \|h_t\|_{\bar{q}}^2$;

(iii) *(quadratically) negligible effect at* $\{h_t, \mu\}_{t\in(0,1]}$ if $\pi(w + h_t) = o\left(\|h_t\|_{\bar{q}}^2\right)$.

Proposition 13 holds verbatim with $\bar{q}$ in place of q. Hence, the order effects of uncertainty attitudes depend on the limit behavior of the ratio $E_{\bar{q}}\left(h_t^2\right) / \|h_t\|_{\bar{q}}^2$.

More can be said assuming differentiability. Throughout the rest of the section we assume the following version of Assumption 1 (in which strict inequalities simplify the analysis).

Assumption 3 *The functions $u, v : I \to \mathbb{R}$ are twice differentiable with $u', v' > 0$, $\lambda_u(w), \lambda_v(w) \neq 0$ and $\lambda_u(w) \neq \lambda_v(w)$.*

Under Assumption 3 we can consider the quadratic approximation

$$\pi(w + h_t) = \frac{1}{2}\lambda_u(w)\, E_\mu\left(\sigma^2(h_t)\right) + \frac{1}{2}\lambda_v(w)\, \sigma_\mu^2(E(h_t)) + o\left(\|h_t\|_{\bar{q}}^2\right) \tag{21}$$

of the premium $\pi(w + h_t)$ of a small uncertainty $\{h_t, \mu\}_{t\in(0,1]}$. This makes it possible to separate the effects of risk and model uncertainty attitudes. In the present differential setting first order effects never arise with small uncertainties.

Lemma 23 *It holds $E_\mu\left(\sigma^2(h_t)\right) = o\left(\|h_t\|_{\bar{q}}\right)$ and $\sigma_\mu^2(E(h_t)) = o\left(\|h_t\|_{\bar{q}}\right)$ for each small uncertainty $\{h_t, \mu\}_{t\in(0,1]}$.*

In view of this lemma, in the differential case we only need to consider second and higher order effects. In particular, risk attitudes have a

(i) *second order effect at* $\{h_t, \mu\}_{t\in(0,1]}$ if $E_\mu\left(\sigma^2(h_t)\right) \asymp \|h_t\|_{\bar{q}}^2$;

(ii) *(quadratically) negligible effect at* $\{h_t, \mu\}_{t\in(0,1]}$ if $E_\mu\left(\sigma^2(h_t)\right) = o\left(\|h_t\|_{\bar{q}}^2\right)$.

A similar classification holds for model uncertainty attitudes, with $\sigma_\mu^2 (E(h_t))$ in place of $E_\mu \left(\sigma^2 (h_t)\right)$.

If risk and model uncertainty attitudes both have a second order effect, then overall uncertainty attitudes have a second order effect. If, instead, risk and model uncertainty attitudes have effects of different orders, the overall uncertainty attitudes have effects of the lower order. In particular, uncertainty attitudes have a second order effect either if risk attitudes have a second order effect and model uncertainty attitudes have a higher order effect or if the opposite is true. In this case we have

$$\pi (w + h_t) = \frac{1}{2}\lambda_u (w) E_\mu \left(\sigma^2 (h_t)\right) + o\left(\|h_t\|_{\bar{q}}^2\right) \quad \text{or} \quad \pi (w + h_t) = \frac{1}{2}\lambda_v (w) \sigma_\mu^2 (E(h_t)) + o\left(\|h_t\|_{\bar{q}}^2\right),$$

respectively. Finally, $\pi (w + h_t) = o\left(\|h_t\|_{\bar{q}}^2\right)$ if both risk and model uncertainty attitudes have negligible effects.

Example 24 Consider the net $\{h_t, \mu\}_{t\in(0,1]}$ such that the induced net $\{h_t, \bar{q}\}_{t\in(0,1]}$ is defined by (16). For example, a directional small uncertainty with $h_t = th$, for a given a monetary act h. We have $\sigma_{\bar{q}}^2 (h_t) = t^2 \sigma_{\bar{q}}^2 (h)$ and $\sigma_\mu^2 (E(h_t)) = t^2 \sigma_\mu^2 (E(h))$. Hence,

$$E_\mu \left(\sigma^2 (h_t)\right) = t^2 E_\mu \left(\sigma^2 (h)\right) \quad \text{and} \quad \sigma_\mu^2 (E(h_t)) = t^2 \sigma_\mu^2 (E(h)).$$

Suppose that h is such that $\sigma_{\bar{q}}^2 (h) > 0$ and $\sigma_\mu^2 (E(h)) > 0$. Since $\|h_t\|_{\bar{q}}^2 = t^2 \|h\|_{\bar{q}}^2$, both risk and model uncertainty attitudes have second order effects if $\sigma_{\bar{q}}^2 (h) \neq \sigma_\mu^2 (E(h))$ while only uncertainty attitudes have a second order effect if $\sigma_{\bar{q}}^2 (h) = \sigma_\mu^2 (E(h))$. ▲

We defer to Sections 7 and 8 a detailed analysis (for the binomial case) of the asymptotic versions of the four cases discussed in Section 2.4. Here we consider global effects. In particular, in the differential case risk attitudes have a

(i) *second order effect* if $E_\mu \left(\sigma^2 (h_t)\right) \asymp \|h\|_{\bar{q}}^2$ as $\|h\|_{\bar{q}}^2 \to 0$;

(ii) *(quadratically) negligible effect* if $E_\mu \left(\sigma^2 (h_t)\right) = o\left(\|h\|_{\bar{q}}^2\right)$ as $\|h\|_{\bar{q}}^2 \to 0$.

A similar classification holds for model uncertainty attitudes, with $\sigma_\mu^2 (E(h_t))$ in place of $E_\mu \left(\sigma^2 (h_t)\right)$. The following result extends Proposition 17 to uncertainty.

Proposition 25 *(i) Model uncertainty attitudes have a negligible effect if and only if* $\operatorname{supp}\mu = \{p\}$ *for some* $p \in \Delta$. *(ii) Risk attitudes have a negligible effect if and only if* $\operatorname{supp}\mu$ *consists of Dirac measures. (iii) Both risk and model uncertainty attitudes have a negligible effect if and only if* $\operatorname{supp}\mu = \{\delta_\omega\}$ *for some* $\omega \in \Omega$.

In other words, risk attitudes are "globally" negligible if and only if the support of μ consists of Dirac measures on Ω, while model uncertainty attitudes are globally negligible if and only if μ is itself a Dirac measure on Δ (i.e., it gives full weight to a single model $p \in \Delta$). Finally, both attitudes are negligible when μ is a Dirac measure on a Dirac measure, that is, $\operatorname{supp}\mu = \{\delta_\omega\}$. These cases are asymptotic global versions of Cases 2-4 discussed in Section 2.4.

Finally, we can define quadratic small uncertainties when $E_{\bar{q}} \left(h_t^2\right) = O\left(t^2\right)$. The analysis is similar, up to obvious modifications, to the one carried out for the risk case. For brevity we thus omit it.

5 Risk varying setting

5.1 Small risks

We extend our analysis to a risk varying setting in which q may depend on t. Thus, we now focus on nets $\{h_t, q_t\}_{t\in(0,1]}$ of monetary acts $h_t : \Omega \to \mathbb{R}$ and probability distributions $q_t \in \Delta$ that we call *risky monetary nets* provided h_t is never q_t-a.e null and $q_t \to q$ for some $q \in \Delta$; that is, $\lim_{t\downarrow 0} q_t(\omega) = q(\omega)$ for all $\omega \in \Omega$. These two conditions put some discipline on how outcomes and probabilities can vary as t goes to zero.

This notion of risky monetary nets $\{h_t, q_t\}_{t\in(0,1]}$ reduces to the earlier one when $q_t = q$ for all $t \in (0,1]$. Here, each act h_t delivers outcomes under possibly different risk conditions determined by q_t. We therefore relax the assumption that each static situation t features the same risk conditions. In a coin toss example, the probability $q_t(H)$ and $q_t(T)$ with which heads and tails come up may depend on the index t.[10]

The notion of small risk, introduced in Definition 7, is readily extended to the present more general setting.

Definition 26 *A risky monetary net $\{h_t, q_t\}_{t\in(0,1]}$ is a small risk if* $\lim_{t\downarrow 0} E_{q_t}\left(h_t^2\right) = 0$.

There is a key novelty in this notion of small risk for the risk varying case: now there are two (complementary) sources of smallness, outcomes and probabilities. That is, we may have $E_{q_t}\left(h_t^2\right) \to 0$ because either outcomes $h_t(\omega)$ or their probabilities $q_t(\omega)$ (or both) are becoming smaller and smaller as t goes to zero. For example, consider $\Omega = \{1,2\}$ and

$$h_t(\omega) = \begin{cases} \varphi(t) & \text{if } \omega = 1 \\ \psi(t) & \text{if } \omega = 2 \end{cases}$$

Set $q_t = q_t(1) \to q = q(1)$. If $q \in (0,1)$, then $E_{q_t}\left(h_t^2\right) \to 0$ provided $\psi(t) \to 0$ and $\varphi(t) \to 0$. In contrast, if $q = 0$ then $E_{q_t}\left(h_t^2\right) \to 0$ provided only $\psi(t) \to 0$, without any requirement on $\varphi(t)$.

To characterize small risks we extend Proposition 12 to the risk varying setting. To this end, set[11]

$$S = \bigcap_{t\in(0,1]} \bigcup_{\tau\le t} \operatorname{supp} q_\tau,$$

that is, $S = \limsup_{t\downarrow 0} (\operatorname{supp} q_t)$. In words, $\omega \in S$ if and only if for all $t \in (0,1]$ there exists $\tau \le t$ such that $q_\tau(\omega) > 0$. It is easy to see that $\operatorname{supp} q \subseteq \limsup_{t\downarrow 0} (\operatorname{supp} q_t)$. In addition, set

$$\Gamma_q = S - \operatorname{supp} q,$$

that is, Γ_q is defined as the difference between the sets $\limsup_{t\downarrow 0} (\operatorname{supp} q_t)$ and $\operatorname{supp} q$. Clearly, $\Gamma_q = \emptyset$ when $q_t = q$ for all $t \in (0,1]$. In other words, the possible nonemptiness of Γ_q is peculiar to the risk varying setting. Indeed $\omega \in \Gamma_q$ if and only if for all $t \in (0,1]$ there exists $\tau \le t$ such that $q_\tau(\omega) > 0$ and $0 = q(\omega) = \lim_{\tau\downarrow 0} q_\tau(\omega)$.

Lemma 27 *It holds $\omega \notin S$ if and only if $q_t(\omega) = 0$ eventually.*

[10] The risk varying setting is more general than the previous one, even in terms of induced distributions on outcomes. For example, given $\Omega = \{1,2\}$, let $\{h_t, q_t\}_{t\in(0,1]}$ be such that $q_t(1) \neq q_{t'}(1)$ for each $t \neq t'$ and $h_t(1) \neq h_t(2)$ for each t. Given any $q \in \Delta$, it holds $q(h_t = x) \in \{q(1), 1 - q(1)\}$ for all t. Hence, there is no $q \in \Delta$ such that $q(h_t = x) = q_t(h_t = x)$ for all t.

[11] The limit is meaningful since $\bigcup_{\tau\le t} \operatorname{supp} q_\tau \subseteq \bigcup_{\tau\le t'} \operatorname{supp} q_\tau$ if $t < t'$ (see, e.g., [**?**, p. 23] and [**?**, p. 39]).

A direct implication of this lemma is $q_t(S) = 1$ for each t small enough.

We now characterize small risks by generalizing Proposition 12, which is the special case $q_t = q$ for all $t \in (0, 1]$.

Proposition 28 *For a risky monetary net $\{h_t, q_t\}_{t\in(0,1]}$ such that $\limsup_{t\downarrow 0} |h_t(\omega)| < \infty$ for all $\omega \in \Gamma_q$, the following conditions are equivalent:*

(i) $\lim_{t\downarrow 0} E_{q_t}(|h_t|^n) = 0$ *for all* $n \geq 1$*;*

(ii) $\{h_t, q_t\}_{t\in(0,1]}$ *is a small risk;*

(iii) $\lim_{t\downarrow 0} E_{q_t}(|h_t|^n) = 0$ *for some* $n \geq 1$*;*

(iv) $\lim_{t\downarrow 0} q_t(\omega : |h_t(\omega)| \geq \varepsilon) = 0$ *for each* $\varepsilon > 0$*;*

(v) $\lim_{t\downarrow 0} h_t(\omega) = 0$ *for each* $\omega \in \operatorname{supp} q$*.*

Clearly, (i) implies (ii) and (ii) implies (iii). Moreover, (iii) is easily seen to imply (iv) since, by the Chebyshev inequality, for each $\varepsilon > 0$,

$$0 \leq q_t(\omega : |h_t(\omega)| \geq \varepsilon) \leq \frac{1}{\varepsilon^n} E_{q_t}(h_t^n) \to 0$$

as $t \downarrow 0$. The remaining implications are less straightforward and are proved in the Appendix.

Let us adapt Examples 8-10 to the present setting.

Example 29 (i) A net $\{h_t\}_{t\in(0,1]}$ that statewise converges as in (15) forms a small risk $\{h_t, q_t\}_{t\in(0,1]}$ along with any net $\{q_t\}_{t\in(0,1]}$ that converges to some $q \in \Delta$.

(ii) In a risk varying setting, Pratt [10]'s analysis amounts to consider a risky monetary net $\{h_t, q_t\}_{t\in(0,1]}$ with $E_{q_t}(h_t) = 0$ and $\lim_{t\downarrow 0} E_{q_t}\left(h_t^2\right) = 0$.

(iii) As to Segal and Spivak [11], let $\{q_t\}_{t\in(0,1]}$ be any net such that $q_t \to q$. Fix a monetary act $h_1 \in \mathbb{R}^n$ and define a risky monetary net $\{h_t, q_t\}_{t\in(0,1]}$ such that, at each $t \in (0, 1]$, h_t has the same distribution under q_t as th_1 under q_1. That is,

$$q_t(h_t \in E) = q_1(th_1 \in E), \qquad \forall t \in (0, 1], \tag{22}$$

for all Borel subsets $\mathcal{B}$ of $\mathbb{R}$. Since $E_{q_t}\left(h_t^2\right) = t^2 E_{q_1}\left(h_1^2\right)$, this net is a small risk. For example, given $\Omega = \{1, 2\}$ let (h_1, q_1) be such that $h_1(1) = -h_1(2) = 1$ and $q_1(1) = 1/3$. Then, setting

$$q_t(h_t = x) = q_1(th_1 = x) = \begin{cases} \frac{1}{3} & \text{if } x = t \\ \frac{2}{3} & \text{if } x = -t \end{cases}$$

If h_t is such that $h_t(1) = -h_1(2) = t$, then $\{h_t, q_t\}_{t\in(0,1]}$ is a risky monetary net that satisfies (22).

(iv) In the spirit of the previous example are the directional small risks $\{h_t, q_t\}_{t\in(0,1]}$, where $h_t = th$, q_t-a.e. and $q_t \to q$, for a given monetary act $h \in \mathbb{R}^n$. In fact, $E_{q_t} h_t^2 = E_{q_t} t^2 h^2 = t^2 E_{q_t} h^2 \to 0$ because $E_{q_t} h^2 \to E_q h^2$. ▲

5.2 Order effects

To study the order effects of small risks in the present risk varying setting we need to consider the indexed certainty equivalent

$$C_t(f) = u^{-1}\left(\sum_{\omega\in\Omega} u(f(\omega))\, q_t(\omega)\right). \tag{23}$$

Relative to (14), the model q_t may now change with t. In particular, the risk premium (17) now takes the form

$$\pi_t(w+h_t) = w + E_{q_t}(h_t) - C_t(w+h_t).$$

We can define order effects in the usual way: risk attitudes have a

(i) *first order effect at* $\{h_t, q_t\}_{t\in(0,1]}$ if $\pi_t(w+h_t) \asymp \|h_t\|_{q_t}$;

(ii) *second order effect at* $\{h_t, q_t\}_{t\in(0,1]}$ if $\pi_t(w+h_t) \asymp \|h_t\|_{q_t}^2$;

(iii) *(quadratically) negligible effect at* $\{h_t, q_t\}_{t\in(0,1]}$ if $\pi_t(w+h_t) = o\left(\|h_t\|_{q_t}^2\right)$.

If we define

$$k_* = \liminf_{t\downarrow 0} \frac{E_{q_t}^2(h_t)}{E_{q_t}(h_t^2)} \quad \text{and} \quad k^* = \limsup_{t\downarrow 0} \frac{E_{q_t}^2(h_t)}{E_{q_t}(h_t^2)} \tag{24}$$

we can extend Proposition 13 to the present setting, up to obvious changes (i.e., we ought to add the index t whenever needed). Therefore, provided $E_{q_t}(h_t) > 0$ eventually, for all t small enough it holds

$$C_t(w+h_t) < w$$

if $\pi_t(w+h_t) \sim k\|h_t\|_{q_t}$ and $k > \sqrt{k^*}$, while

$$C_t(w+h_t) > w$$

if either $\pi_t(w+h_t) \sim k\|h_t\|_{q_t}$ and $k < \sqrt{k_*}$ or $\pi_t(w+h_t) = o\left(\|h_t\|_{q_t}\right)$. As in the constant risk case, risk attitudes with first order effects may or may not matter when comparing $w+h_t$ and w; risk attitudes with second or higher order do not matter.

Substantially more delicate is the extension of the differential case to this risk varying setting. Indeed, because of the dependence on t, we need to improve the quadratic approximation (18) to control for variations in t. To see why this is the case, consider a risky monetary net $\{h_t, q\}_{t\in(0,1]}$ in which q_t does not depend on t. In this case $\lim_{t\downarrow 0} E_q\left(h_t^2\right) = 0$ and so the quadratic approximation of the uncertainty premium is given by (18), that is,

$$\pi(w+h_t) = \frac{1}{2}\lambda_u(w)\,\sigma_q^2(h_t) + o\left(\|h_t\|_q^2\right). \tag{25}$$

When q_t depends on t, we need an approximation based on the time varying certainty equivalent $C_t(w+h_t)$ that at each t may feature a different model q_t. This complication of the risk varying case has to be carefully addressed. To this end the next notion is key.

Definition 30 *A small risk* $\{h_t, q_t\}_{t\in(0,1]}$ *is controllable if* $\max_{\omega\in\operatorname{supp} q_t} |h_t(\omega)| = O\left(\|h_t\|_{q_t}\right)$ *as* $t\downarrow 0$*.*

The following proposition collects some useful features of controllable small risks.

Proposition 31 *Let $\{h_t, q_t\}_{t\in(0,1]}$ be a small risk.*

1. *$\{h_t, q_t\}_{t\in(0,1]}$ is controllable if $\Gamma_q = \emptyset$, that is, $\operatorname{supp} q_t = \operatorname{supp} q$ eventually.*
2. *$\{h_t, q_t\}_{t\in(0,1]}$ is controllable if and only if $E_{q_t}\left(h_t^2\right) \asymp \sum_{\omega\in\operatorname{supp} q_t} h_t^2(\omega)$.*
3. *$\{h_t, q_t\}_{t\in(0,1]}$ is controllable if and only if $\liminf_{t\downarrow 0} \frac{E_{q_t}\left(h_t^2\right)}{\sum_{\omega\in\operatorname{supp} q_t} h_t^2(\omega)} > 0$.*
4. *If $\liminf_{t\downarrow 0} \frac{E_{q_t}\left(h_t^2\right)}{\sum_{\omega\in S} h_t^2(\omega)} > 0$, then $\{h_t, q_t\}_{t\in(0,1]}$ is controllable.*

Example 32 (i) Let $\Omega = \{1, 2\}$ and

$$h_t(\omega) = \begin{cases} \varphi(t) & \text{if } \omega = 1 \\ \psi(t) & \text{if } \omega = 2 \end{cases}$$

with $\varphi(t) \sim \psi(t)$. If $0 < q_t < 1$ (where $q_t = q_t(1)$ for all $t \in (0,1]$), then

$$\frac{E_{q_t}\left(h_t^2\right)}{\sum_{\omega\in\operatorname{supp} q_t} h_t^2(\omega)} = \frac{q_t\varphi^2(t) + (1-q_t)\psi^2(t)}{\varphi^2(t) + \psi^2(t)} \sim \frac{1}{2}$$

and so this risky monetary net is controllable. If, instead, $\psi(t) = o(\varphi(t))$, then

$$\frac{E_{q_t}\left(h_t^2\right)}{\sum_{\omega\in\operatorname{supp} q_t} h_t^2(\omega)} = \frac{q_t + (1-q_t)\,o(1)}{1 + o(1)} \to q$$

and so, in this case, $\{h_t, q_t\}_{t\in(0,1]}$ is controllable if and only if $q > 0$.

(ii) Directional small risks are controllable. For example, fix h and consider a risk pattern $\{q_t\}_{t\in(0,1]}$ with $q_t \to q$. Let $\{h_t, q_t\}_{t\in(0,1]}$ be such that $h_t = th$, q_t-a.e., for each $t \in (0,1]$. Then, eventually

$$\begin{aligned} \frac{E_{q_t}\left(h_t^2\right)}{\sum_{\omega\in\operatorname{supp} q_t} h_t^2(\omega)} &= \frac{E_{q_t}\left(h^2\right)}{\sum_{\omega\in\operatorname{supp} q_t} h^2(\omega)} \\ &\geq \frac{E_{q_t}\left(h^2\right)}{\sum_{\omega\in S} h^2(\omega)} \to \frac{E_{q}\left(h^2\right)}{\sum_{\omega\in S} h^2(\omega)} \end{aligned}$$

and so $\{h_t, q_t\}_{t\in(0,1]}$ is controllable when $E_q\left(h^2\right) > 0$, i.e., $h \neq 0$ q-a.e. ▲

The next result, a special case of Theorem 40 below, shows that in the controllable case we can suitably extend the quadratic approximation (25) to risky monetary nets.

Theorem 33 *Let $\{h_t, q_t\}_{t\in(0,1]}$ be a controllable small risk. If Assumption 2 holds with u three times continuously differentiable, then*

$$C_t(w + h_t) = w + E_{q_t}(h_t) - \frac{1}{2}\lambda_u(w)\,\sigma_{q_t}^2(h_t) + o\left(\|h_t\|_{q_t}^2\right) \tag{26}$$

for all $w \in \mathbb{R}$.

In other words, the risk premium takes the form

$$\pi_t(w + h_t) = \frac{1}{2}\lambda_u(w)\,\sigma_{q_t}^2(h_t) + o\left(\|h_t\|_{q_t}^2\right), \tag{27}$$

that is, as $t \downarrow 0$,

$$\frac{\pi_t(w+h_t) - \frac{1}{2}\lambda_u(w)\,\sigma^2_{q_t}(h_t)}{\|h_t\|^2_{q_t}} \to 0,$$

where $\|h_t\|^2_{q_t} \to 0$ because $\{h_t, q_t\}_{t\in(0,1]}$ is a small risk. Approximation (25) is the special case of (27) where $q_t = q$ for all t.

Thanks to quadratic approximation (26) we can study order effects through the variances $\sigma^2_{q_t}(h_t)$. Since Lemma 14 is easily extended to the risk varying case, with k_* and k^* given by (24), we have:

(i) risk attitudes do not have a first order effect at $\{h_t, q_t\}_{t\in(0,1]}$ since $\sigma^2_{q_t}(h_t) \leq \|h_t\|^2_{q_t}$ implies $\sigma^2_{q_t}(h_t) = o\left(\|h_t\|_{q_t}\right)$;

(ii) risk attitudes have a second order effect at $\{h_t, q_t\}_{t\in(0,1]}$ if and only if $k^* < 1$;

(iii) risk attitudes have a negligible effect at $\{h_t, q_t\}_{t\in(0,1]}$ if and only if $k_* = 1$, that is,

$$\lim_{t\downarrow 0} \frac{E^2_{q_t}(h_t)}{E_{q_t}(h_t^2)} = 1. \tag{28}$$

We defer a detailed analysis of these cases to Sections 7 and 8, where we study the binomial case in the more general setting of varying uncertainty that we will introduce in the next section. Now we extend Proposition 15 to show that risk attitudes are negligible when risk asymptotically vanishes.

Proposition 34 *Under Assumption 2 with u three times continuously differentiable, risk attitudes are negligible at controllable small risks $\{h_t, q_t\}_{t\in(0,1]}$ that are asymptotically constant, that is,*

$$h_t(\omega) \sim h_t(\omega') \qquad \forall \omega, \omega' \in S. \tag{29}$$

In words, risk eventually vanishes and risk attitudes do not matter.

5.3 Quadratic small risks

The notion of quadratic small risks, introduced in Definition 18, readily extends to the present setting.

Definition 35 *A risky monetary net $\{h_t, q_t\}_{t\in(0,1]}$ is a quadratic small risk if $E_{q_t}\left(h_t^2\right) = O\left(t^2\right)$.*

Next we characterize quadratic small risks: relative to Proposition 20, here the case of $n = 2$ plays a special role.

Proposition 36 *Given a risky monetary net $\{h_t, q_t\}_{t\in(0,1]}$ the following conditions are equivalent:*

(i) $\{h_t, q_t\}_{t\in(0,1]}$ is a quadratic small risk;

(ii) $h_t(\omega) = O(t)$ for all $\omega \in \operatorname{supp} q$ and $h_t^2(\omega)\, q_t(\omega) = O\left(t^2\right)$ for all $\omega \in \Gamma_q$.

Given $\omega \in \Gamma_q$, sufficient conditions for $h_t^2(\omega)\, q_t(\omega) = O\left(t^2\right)$ are $q_t(\omega) = O\left(t^2\right)$ and $\limsup_{t\downarrow 0} |h_t(\omega)| < \infty$. That is, (ii) holds – and so $\{h_t, q_t\}_{t\in(0,1]}$ is a quadratic small risk – if $h_t(\omega) = O(t)$ for all $\omega \in \operatorname{supp} q$ and if $q_t(\omega)$ goes to zero fast enough, with $|h_t(\omega)|$ bounded for all $\omega \in \Gamma_q$.

The next example shows that, unlike Proposition 20, condition (ii) in Proposition 36 does not hold for any n.

Example 37 (i) Directional small risks are quadratic. For

$$\frac{E_{q_t}\left(h_t^2\right)}{t^2} = E_{q_t}\left(h^2\right) \to E_q\left(h^2\right)$$

(ii) Given $\Omega = \{1, 2\}$, let $\{h_t, q_t\}_{t\in(0,1]}$ be given by $h_t(1) = 1$, $h_t(2) = t$, and $q_t(1) = t^3$. Then, $E_{q_t}\left(h_t^2\right) = O\left(t^2\right)$, but $E_{q_t}\left(h_t^4\right) \neq O\left(t^4\right)$. Hence, $\{h_t, q_t\}_{t\in(0,1]}$ is a quadratic small risk, although it is not true that $E_{q_t}\left(|h_t|^n\right) = O\left(t^n\right)$ for all $n \geq 1$. ▲

Finally, when $\{h_t, q_t\}_{t\in(0,1]}$ is a (controllable) quadratic small risk, we can replace $o\left(\|h_t\|_{\bar{q}_t}^2\right)$ with $o\left(t^2\right)$ in the risk premium approximation (27), that is,

$$\pi_t\left(w + h_t\right) = \frac{1}{2}\lambda_u\left(w\right)\sigma_{q_t}^2\left(h_t\right) + o\left(t^2\right).$$

6 Uncertainty varying setting

In this section we extend our earlier risk analysis to account for ambiguity, as modeled by the smooth ambiguity model (7). We consider a net of probability measures $\{\mu_t\}_{t\in(0,1]}$ on Δ and we allow μ to depend on the index t. As in (3), we set

$$\bar{q}_t\left(\omega\right) = \int_\Delta q\left(\omega\right) d\mu_t\left(q\right)$$

We call *uncertain monetary nets*, indicated $\{h_t, \mu_t\}_{t\in(0,1]}$, nets of monetary acts $h_t : \Omega \to \mathbb{R}$ and probability distributions μ_t on Δ such that h_t is never $\bar{q}_t$-a.e null and $\mu_t \overset{sq}{\Longrightarrow} \mu$ for some μ on Δ. In a coin toss example, μ_t is the decision maker's prior on the probability models q that determine how likely heads and tails are. This prior now can vary with t.

Set $\bar{q}\left(\omega\right) = \int_\Delta q\left(\omega\right) d\mu\left(q\right)$. By Definition 5, $\lim_{t\downarrow 0} \bar{q}_t\left(\omega\right) = \bar{q}\left(\omega\right)$ for each $\omega \in \Omega$ if $\mu_t \overset{sq}{\Longrightarrow} \mu$. Hence, an uncertain monetary net $\{h_t, \mu_t\}_{t\in(0,1]}$ induces a risky monetary net $\{h_t, \bar{q}_t\}_{t\in(0,1]}$ such that $\bar{q}_t \to \bar{q}$. Using these induced nets we can readily extend to uncertain monetary nets what we have previously established for risky ones. Let us begin with the notions of small uncertainties.

Definition 38 *An uncertain monetary net* $\{h_t, \mu_t\}_{t\in(0,1]}$ *is a small uncertainty if and only if the induced risky monetary net* $\{h_t, \bar{q}_t\}_{t\in(0,1]}$ *is a small risk.*

This definition extends the scope of Definition 26 to uncertainty. In particular, an uncertain monetary net $\{h_t, \mu_t\}_{t\in(0,1]}$ is a a small uncertainty if and only if $\lim_{t\downarrow 0} E_{\bar{q}_t}\left(h_t^2\right) = 0$. For this reason, Proposition 28 holds with $\bar{q}_t$ and $\bar{q}$ in place of q_t and q, respectively.

To study the behavior of small uncertainties we consider the indexed certainty equivalent

$$C_t\left(f\right) = v^{-1}\left(\int_\Delta v\left(u^{-1}\left(\sum_{\omega\in\Omega} u\left(f\left(\omega\right)\right) q\left(\omega\right)\right)\right) d\mu_t\left(q\right)\right),$$

Relative to the certainty equivalent (14), the prior μ_t may now change with t.

Through the induced small risk $\{h_t, \bar{q}_t\}_{t\in(0,1]}$ we can easily extend to uncertainty all results on order effects established in Section 5.2 by setting

$$k_* = \liminf_{t\downarrow 0} \frac{E_{\bar{q}_t}^2\left(h_t\right)}{E_{\bar{q}_t}^2\left(h_t^2\right)} \quad \text{and} \quad k^* = \limsup_{t\downarrow 0} \frac{E_{\bar{q}_t}^2\left(h_t\right)}{E_{\bar{q}_t}^2\left(h_t^2\right)}.$$

For brevity, we omit the details.

As in the risk varying case, the dependence on t requires a substantial improvement of the quadratic approximation (12) to control variations in t. In order to do so we first extend the notion of controllability to uncertainty.

Definition 39 *A small uncertainty $\{h_t, \mu_t\}_{t\in(0,1]}$ is controllable if* $\max_{\omega\in\operatorname{supp}\bar{q}_t} |h_t(\omega)| = O\left(\|h_t\|_{\bar{q}_t}\right)$ *as $t \downarrow 0$.*

That is, $\{h_t, \mu_t\}_{t\in(0,1]}$ is controllable if and only if $\{h_t, \bar{q}_t\}_{t\in(0,1]}$ is. In particular, Proposition 31 has a natural counterpart and directional small uncertainties are controllable provided $h \neq 0$, $\bar{q}$-a.e..

We can now generalize the quadratic approximation (12). The dependence on t of the priors μ_t substantially complicates the derivation of the following quadratic approximation, which we prove in Appendix B. It is the main technical contribution of the paper.

Theorem 40 *Let $\{h_t, \mu_t\}_{t\in(0,1]}$ be a controllable small uncertainty. If Assumption 3 holds with u and v three-times continuously differentiable, then*

$$C_t(w + h_t) = w + E_{\bar{q}_t}(h_t) - \frac{1}{2}\lambda_u(w) E_{\mu_t}\left(\sigma^2(h_t)\right) - \frac{1}{2}\lambda_v(w)\sigma^2_{\mu_t}(E(h_t)) + o\left(\|h_t\|^2_{\bar{q}_t}\right) \tag{30}$$

for all $w \in \mathbb{R}$.[12]

Thus, the uncertainty premium takes the form

$$\pi_t(w + h_t) = \frac{1}{2}\lambda_u(w) E_{\mu_t}\left(\sigma^2(h_t)\right) + \frac{1}{2}\lambda_v(w)\sigma^2_{\mu_t}(E(h_t)) + o\left(\|h_t\|^2_{\bar{q}_t}\right). \tag{31}$$

Approximation (21) is the special case of (31) where $\mu_t = \mu$ for all t. Approximation (31) allows to study separately the order effects of risk and model uncertainty attitudes through the mean of variances $E_{\mu_t}\left(\sigma^2(h_t)\right)$ and the variance of means $\sigma^2_{\mu_t}(E(h_t))$.

Lemma 23 is readily extended to the present setting, so that $E_{\mu_t}\left(\sigma^2(h_t)\right) = o\left(\|h_t\|_{\bar{q}_t}\right)$ and $\sigma^2_{\mu_t}(E(h_t)) = o\left(\|h_t\|_{\bar{q}_t}\right)$ for each small uncertainty $\{h_t, \mu_t\}_{t\in(0,1]}$. As a result, in the differential case we can have only second and higher order effects. In particular, risk attitudes have a

(i) *second order effect at* $\{h_t, \mu_t\}_{t\in(0,1]}$ if $E_{\mu_t}\left(\sigma^2(h_t)\right) \asymp \|h_t\|^2_{\bar{q}_t}$;

(ii) *(quadratically) negligible effect at* $\{h_t, \mu_t\}_{t\in(0,1]}$ if $E_{\mu_t}\left(\sigma^2(h_t)\right) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$.

A similar classification holds for model uncertainty attitudes, with $\sigma^2_{\mu_t}(E(h_t))$ in place of $E_{\mu_t}\left(\sigma^2(h_t)\right)$. The next two sections will illustrate these notions in the important binomial case. In particular, Theorem 49 will show that the order effect of $E_{\mu_t}\left(\sigma^2(h_t)\right)$ and $\sigma^2_{\mu_t}(E(h_t))$ critically depends on the limit properties of both outcomes, h_t, and beliefs, μ_t. The latter dependence is especially interesting: in a nutshell, when the support of μ_t is trivial, model uncertainty attitudes are negligible; when it is not trivial, model uncertainty persists as t goes to zero. As a matter of fact, both risk and model uncertainty attitudes have a second order effect when the support is not trivial.

Finally, we can define quadratic small uncertainties by proceeding as in Section 5.3, mutatis mutandis. Briefly, quadratic small uncertainties are defined by $E_{\bar{q}_t}\left(h_t^2\right) = O\left(t^2\right)$. In this case in (31) we can replace $o\left(\|h_t\|^2_{\bar{q}_t}\right)$ with $o\left(t^2\right)$.

[12] Here $E_{\mu_t}\left(\sigma^2(h_t)\right) = \sigma^2_{\bar{q}_t}(h_t) - \sigma^2_{\mu_t}(E(h_t))$.

7 A binomial illustration

Consider a binomial model $\Omega = \{1, 2\}$. Let $\{\mu_t\}_{t\in(0,1]}$ be any net of priors that quadratically converges to some μ, and let $W_t : \Omega \to \mathbb{R}$ be given by $W_t = tW$ with

$$W(\omega) = \begin{cases} 1 & \text{if } \omega = 1 \\ -1 & \text{if } \omega = 2 \end{cases}$$

In this section we consider directional small uncertainties $\{W_t, \mu_t\}_{t\in(0,1]}$.

Lemma 41 $\{W_t, \mu_t\}_{t\in(0,1]}$ *is a controllable and quadratic small uncertainty.*

Given $q \in \Delta$, to ease notation we set $q = q(1)$. In particular, each μ_t can be viewed as a Borel probability measure on $[0, 1]$. Set $\vartheta_t = \int q^2 d\mu_t(q)$ for each $t > 0$. Then,

$$0 \le \bar{q}_t^2 \le \vartheta_t \le \bar{q}_t \le 1, \tag{32}$$

and

$$\begin{aligned} \sigma^2_{\bar{q}_t}(W_t) &= 4t^2 \bar{q}_t(1 - \bar{q}_t) \\ \sigma^2_{\mu_t}(E(W_t)) &= 4t^2 \left(\vartheta_t - \bar{q}_t^2\right) \\ E_{\mu_t}\left(\sigma^2(W_t)\right) &= 4t^2(\bar{q}_t - \vartheta_t). \end{aligned}$$

Hence, for all w the quadratic approximation (31) of the uncertainty premium $\pi_t(w + W_t)$ takes the form

$$\pi_t(w + W_t) = 2\lambda_u(w)(\bar{q}_t - \vartheta_t)t^2 + 2\lambda_v(w)\left(\vartheta_t - \bar{q}_t^2\right)t^2 + o\left(t^2\right). \tag{33}$$

Our goal is to study the limit behavior of $\pi_t(w + W_t)$ as t goes to zero, thereby determining the impact of risk and model uncertainty attitudes on the certainty equivalent $C_t(w + W_t)$. To this end, throughout this section we suppose that Assumption 3 holds with u and v three times continuously differentiable and $\lambda_u(w), \lambda_v(w) > 0$.

In particular, risk attitudes have a first order, second order, negligible effect at $\{W_t, \mu_t\}_{t\in(0,1]}$ if, respectively, $(\bar{q}_t - \vartheta_t)t^2 \asymp t$, $(\bar{q}_t - \vartheta_t)t^2 \asymp t^2$, and $(\bar{q}_t - \vartheta_t)t^2 = o\left(t^2\right)$. As usual, a similar classification holds for model uncertainty attitudes, with $\left(\vartheta_t - \bar{q}_t^2\right)t^2$ in place of $(\bar{q}_t - \vartheta_t)t^2$.

By (32) it holds $0 \le \bar{q}_t - \vartheta_t, \vartheta_t - \bar{q}_t^2 \le 1$, and so both $E_{\mu_t}\left(\sigma^2(W_t)\right)$ and $\sigma^2_{\mu_t}(E(W_t))$ are $O\left(t^2\right)$. This, in turn, implies that risk and model uncertainty attitudes never have a first order effect. Moreover, $\lim_{t\downarrow 0}(\bar{q}_t - \vartheta_t) = \bar{q} - \vartheta$ and $\lim_{t\downarrow 0}\left(\vartheta_t - \bar{q}_t^2\right) = \vartheta - \bar{q}^2$ exist finite since μ_t quadratically converges to some μ. As a result, $E_{\mu_t}\left(\sigma^2(W_t)\right)$ and $\sigma^2_{\mu_t}(E(W_t))$ have a well-defined limit behavior. Summing up:

Lemma 42 *Suppose* $\mu_t \overset{sq}{\Longrightarrow} \mu$. *Then*

$$0 \le \lim_{t\downarrow 0} \frac{E_{\mu_t}\left(\sigma^2(W_t)\right)}{t^2} = 4(\bar{q} - \vartheta) \le 4$$

$$0 \le \lim_{t\downarrow 0} \frac{\sigma^2_{\mu_t}(E(W_t))}{t^2} = 4\left(\vartheta - \bar{q}^2\right) \le 4$$

and

(i) $E_{\mu_t}\left(\sigma^2(W_t)\right) \sim 4(\bar{q} - \vartheta)t^2$ *if* $\bar{q} \neq \vartheta$, *while* $E_{\mu_t}\left(\sigma^2(W_t)\right) = o\left(t^2\right)$ *otherwise;*

(ii) $\sigma^2_{\mu_t}(E(W_t)) \sim 4\left(\vartheta - \bar{q}^2\right)t^2$ *if* $\vartheta \neq \bar{q}^2$, *while* $\sigma^2_{\mu_t}(E(W_t)) = o\left(t^2\right)$ *otherwise.*

Thus, risk and model uncertainty attitudes have either a second order or a negligible effect at $\{W_t, \mu_t\}_{t\in(0,1]}$. Depending on which effect prevails, the following four possible cases have to be considered.

Limit Case 1 $\bar{q}^2 < \vartheta < \bar{q}$**:** Then (33) becomes

$$\begin{aligned}\pi_t(w+W_t) &= 2\lambda_u(w)(\bar{q}-\vartheta)t^2 + 2\lambda_v(w)\left(\vartheta-\bar{q}^2\right)t^2 + o\left(t^2\right)\\ &= \frac{1}{2}\lambda_u(w)E_\mu\left(\sigma^2(W)\right)t^2 + \frac{1}{2}\lambda_v(w)\sigma_\mu^2(E(W))t^2 + o\left(t^2\right),\end{aligned}$$

that is, both risk and model uncertainty attitudes have a second order effect at $\{W_t,\mu_t\}_{t\in(0,1]}$ and the same relevance in the quadratic approximation.[13]

Limit Case 2 $\bar{q}^2 = \vartheta < \bar{q}$**:** Then (33) becomes

$$\begin{aligned}\pi_t(w+W_t) &= 2\lambda_u(w)(\bar{q}-\vartheta)t^2 + o\left(t^2\right)\\ &= \frac{1}{2}\lambda_u(w)E_\mu\left(\sigma^2(W)\right)t^2 + o\left(t^2\right),\end{aligned}$$

that is, risk attitudes have a second order effect at $\{W_t,\mu_t\}_{t\in(0,1]}$ while model uncertainty attitudes have a negligible effect. In this case a KMM decision maker is eventually indistinguishable from a subjective expected utility one.

Limit Case 3 $\bar{q}^2 < \vartheta = \bar{q}$**:** Then (33) becomes

$$\begin{aligned}\pi_t(w+W_t) &= 2\lambda_v(w)\left(\vartheta-\bar{q}^2\right)t^2 + o\left(t^2\right)\\ &= \frac{1}{2}\lambda_v(w)\sigma_\mu^2(E(W))t^2 + o\left(t^2\right),\end{aligned}$$

that is, model uncertainty attitudes have a second order effect at $\{W_t,\mu_t\}_{t\in(0,1]}$, while risk attitudes have a negligible effect.

Limit Case 4 $\bar{q}^2 = \vartheta = \bar{q}$**:** Then (33) becomes

$$\pi_t(w+W_t) = o\left(t^2\right),$$

that is, both risk and model uncertainty attitudes have a negligible effect at $\{W_t,\mu_t\}_{t\in(0,1]}$.

These four fundamental cases are asymptotic versions of the four cases discussed in Section 2.4 and they show that, in any case,

$$\begin{aligned}\pi_t(w+W_t) &= 2\lambda_u(w)(\bar{q}-\vartheta)t^2 + 2\lambda_v(w)\left(\vartheta-\bar{q}^2\right)t^2 + o\left(t^2\right) \qquad (34)\\ &= \frac{1}{2}\lambda_u(w)E_\mu\left(\sigma^2(W)\right)t^2 + \frac{1}{2}\lambda_v(w)\sigma_\mu^2(E(W))t^2 + o\left(t^2\right).\end{aligned}$$

Next we essentially restate Proposition 1 of the Introduction, which characterizes the four cases through the properties of the limit prior μ.

Proposition 43 *Suppose* $\mu_t \overset{sq}{\Longrightarrow} \mu$. *Then*

1. $\bar{q}^2 < \vartheta < \bar{q}$ *if and only if* $\operatorname{supp}\mu$ *is neither a singleton nor* $\{0,1\}$*;*
2. $\bar{q}^2 = \vartheta < \bar{q}$ *if and only if* $\operatorname{supp}\mu$ *is a singleton in* $(0,1)$*;*
3. $\bar{q}^2 < \vartheta = \bar{q}$ *if and only if* $\operatorname{supp}\mu = \{0,1\}$*;*

[13] Notice that to derive this conclusions we do not actually need $\lambda_u(w), \lambda_u(w) > 0$. It is enough to exclude $\lambda_u(w)(\bar{q}-\vartheta) + \lambda_v(w)\left(\vartheta-\bar{q}^2\right) = 0$, that is the case in which risk propension (resp. aversion) perfectly compensates model uncertainty aversion (resp. proprension) in the limit.

4. $\bar{q}^2 = \vartheta = \bar{q}$ *if and only if* $\mu = \delta_0$ *or* $\mu = \delta_1$.

The proof follows from the two simple observations mentioned in the Introduction. By now, the interpretation of these four cases in terms of the properties of the support of μ should be clear. For this reason we move directly to illustrate these cases with couple of examples.

Example 44 Let $0 \leq a_t < b_t \leq 1$, for each t. Then, for each Borel subset $\mathcal{B}$ of $[0,1]$, set

$$\mu_t(\mathcal{B}) = \lambda_{[a_t,b_t]}(\mathcal{B}) = \frac{\lambda(\mathcal{B} \cap [a_t,b_t])}{b_t - a_t}, \tag{35}$$

where λ is the Lebesgue measure on $[0,1]$. The support of μ_t is $\operatorname{supp}\mu_t = [a_t,b_t]$ and it is easy to show that if $a_t \to a$ and $b_t \to b$ as $t \downarrow 0$, then

$$\mu_t \xrightarrow{w} \mu = \begin{cases} \lambda_{[a,b]} & \text{if } a < b, \\ \delta_a & \text{if } a = b. \end{cases}$$

Moreover, notice that in this case,

$$q = \frac{b+a}{2} \text{ and } \vartheta = \frac{b^2+ab+a^2}{3}$$

and the approximation can be explicitly written as

$$\pi_t(w+W_t) = \lambda_u(w)\frac{3b+3a-2b^2-2ab-2a^2}{3}t^2 + \lambda_v(w)\frac{(b-a)^2}{6}t^2 + o\left(t^2\right). \tag{36}$$

▲

Example 45 Consider the net $\{W_t,\mu_t\}_{t\in(0,1]}$. Suppose that $\mu_t = \lambda_{[(1/3)+t,(2/3)-t]}$ for each $t \in (0,1/3]$. Then, by (36),

$$\pi_t(w+h_t) = \left(\frac{13}{27}\lambda_u(w) + \frac{1}{54}\lambda_v(w)\right)t^2 + o\left(t^2\right).$$

If $\mu_t = \lambda_{[2^{-1}-t,2^{-1}+t]}$ for each $t \in (0,1/3]$. Then, by (36),

$$\pi_t(w+W_t) = \frac{1}{2}\lambda_u(w)\,t^2 + o\left(t^2\right).$$

If $\mu_t = (1/3)\,\lambda_{[0,t]} + (2/3)\,\lambda_{[1-t,1]}$ for each $t \in (0,1/3]$. Then, $\mu_t \xrightarrow{w} (1/3)\,\delta_0 + (2/3)\,\delta_1$. In particular, $\vartheta = \bar{q} = 2/3$ and (34) becomes

$$\pi_t(w+W_t) = \frac{4}{9}\lambda_v(w)\,t^2 + o\left(t^2\right).$$

Finally, if $\mu_t = \lambda_{[0,t]}$ for each $t \in (0,1/3]$. Then, $\pi_t(w+W_t) = o\left(t^2\right)$ by (36). ▲

8 General binomial analysis

In this section we extend our analysis to a general $h_t : \Omega = \{1,2\} \to \mathbb{R}$ such that, for each $t > 0$,

$$h_t(\omega) = \begin{cases} \varphi(t) & \text{if } \omega = 1 \\ \psi(t) & \text{if } \omega = 2 \end{cases}$$

where φ and ψ are nonzero real-valued functions defined on $(0,1]$. If $\varphi(t) = -\psi(t) = t$, then $h_t = W_t$.

Throughout this section we consider an uncertain monetary net $\{h_t,\mu_t\}_{t\in(0,1]}$ such that μ_t is quadratically convergent to μ and $\bar{q}_t \in (0,1)$ for all $t \in (0,1]$. Next we show that such net is a controllable small uncertainty when $\mu \neq \delta_p$ with $p \in \{0,1\}$.

Lemma 46 *If $\mu \neq \delta_p$ with $p \in \{0,1\}$, then $\{h_t, \mu_t\}_{t\in(0,1]}$ is a small uncertainty if and only if $\lim_{t\downarrow 0} \varphi(t) = \lim_{t\downarrow 0} \psi(t) = 0$. In this case, $\{h_t, \mu_t\}_{t\in(0,1]}$ is always controllable.*

Next we consider the degenerate case $\mu = \delta_0$ (a similar result holds when $\mu = \delta_1$).

Lemma 47 *If $\mu = \delta_0$ and $\lim_{t\downarrow 0} \psi(t) = 0$, then $\{h_t, \mu_t\}_{t\in(0,1]}$ is a controllable small uncertainty if and only if $\varphi(t) = O(\psi(t))$.*

It is convenient to define $\gamma : (0,1] \to \mathbb{R}$ by

$$\gamma(t) = \varphi(t) - \psi(t). \tag{37}$$

For example, $\gamma(t) = 2t$ when $h_t = W_t$. Some algebra shows that

$$\begin{aligned}
\sigma^2_{\bar{q}_t}(h_t) &= \gamma^2(t)\,\bar{q}_t(1-\bar{q}_t)\\
\sigma^2_{\mu_t}(E(h_t)) &= \gamma^2(t)\left(\vartheta_t - \bar{q}_t^2\right)\\
E_{\mu_t}\left(\sigma^2(h_t)\right) &= \gamma^2(t)\left(\bar{q}_t - \vartheta_t\right)
\end{aligned}$$

Therefore, if $\{h_t, \mu_t\}_{t\in(0,1]}$ is a controllable small uncertainty, (31) becomes

$$\pi_t(w+h_t) = \frac{1}{2}\lambda_u(w)\left(\bar{q}_t - \vartheta_t\right)\gamma^2(t) + \frac{1}{2}\lambda_v(w)\left(\vartheta_t - \bar{q}_t^2\right)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) \tag{38}$$

for all w. When $\{h_t, \mu_t\}_{t\in(0,1]}$ is a quadratic small uncertainty, approximation (38) becomes

$$\pi_t(w+h_t) = \frac{1}{2}\lambda_u(w)\left(\bar{q}_t - \vartheta_t\right)\gamma^2(t) + \frac{1}{2}\lambda_v(w)\left(\vartheta_t - \bar{q}_t^2\right)\gamma^2(t) + o\left(t^2\right)$$

which reduces to (33) when $h_t = W_t$.

We now study the order effects of risk and model uncertainty attitudes by extending Limit Cases 1-4 to the present general binomial setting.

Proposition 48 *Let $\{h_t, \mu_t\}_{t\in(0,1]}$ be a controllable small uncertainty, where $\mu_t \overset{sq}{\Longrightarrow} \mu$. Then $\gamma(t) = O\left(\|h_t\|_{\bar{q}_t}\right)$ and*

$$\pi_t(w+h_t) = \frac{1}{2}\lambda_u(w)\left(\bar{q} - \vartheta\right)\gamma^2(t) + \frac{1}{2}\lambda_v(w)\left(\vartheta - \bar{q}^2\right)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right). \tag{39}$$

Notice that we already observed that $(\bar{q} - \vartheta)$ and $\left(\vartheta - \bar{q}^2\right)$ are never negative and we characterized their positivity in Proposition 43. Condition $\gamma(t) = O\left(\|h_t\|_{\bar{q}_t}\right)$ shows that first order effects never arise at a controllable small uncertainty.

Thus, only second order and negligible effects are relevant . We study them through the analysis of Limit Cases 1-4, as the next result shows. Since it fully characterizes the binomial case, often used in mathematical finance, it can be viewed as the paper main result.

Theorem 49 *Let $\{h_t, \mu_t\}_{t\in(0,1]}$ be a controllable small uncertainty, where $\mu_t \overset{sq}{\Longrightarrow} \mu$. Then:*

1. *$E_{\mu_t}\left(\sigma^2(h_t)\right) \asymp \|h_t\|^2_{\bar{q}_t}$ and $\sigma^2_{\mu_t}(E(h_t)) \asymp \|h_t\|^2_{\bar{q}_t}$ if and only if $\operatorname{supp}\mu$ is neither a singleton nor $\{0,1\}$, and $\|h_t\|_{\bar{q}_t} = O(\gamma(t))$;*

2. *$E_{\mu_t}\left(\sigma^2(h_t)\right) \asymp \|h_t\|^2_{\bar{q}_t}$ and $\sigma^2_{\mu_t}(E(h_t)) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$ if and only if $\operatorname{supp}\mu$ is a singleton in $(0,1)$ and $\|h_t\|_{\bar{q}_t} = O(\gamma(t))$;*

3. $E_{\mu_t}\left(\sigma^2\left(h_t\right)\right)=o\left(\left\|h_t\right\|_{\bar{q}_t}^2\right)$ *and* $\sigma_{\mu_t}^2\left(E\left(h_t\right)\right)\asymp\left\|h_t\right\|_{\bar{q}_t}^2$ *if and only if* $\operatorname{supp}\mu=\{0,1\}$ *and* $\left\|h_t\right\|_{\bar{q}_t}=O\left(\gamma\left(t\right)\right)$*;*

4. $E_{\mu_t}\left(\sigma^2\left(h_t\right)\right)=o\left(\left\|h_t\right\|_{\bar{q}_t}^2\right)$ *and* $\sigma_{\mu_t}^2\left(E\left(h_t\right)\right)=o\left(\left\|h_t\right\|_{\bar{q}_t}^2\right)$ *if and only if* $\mu\in\{\delta_0,\delta_1\}$ *or* $\gamma\left(t\right)=o\left(\left\|h_t\right\|_{\bar{q}_t}\right)$*.*

In Limit Case 1 both risk and model uncertainty attitudes have a second order effect at $\{h_t,\mu_t\}_{t\in(0,1]}$. This case is characterized by a μ with nonsingleton support different from $\{0,1\}$.

In Limit Case 2 approximation (39) becomes

$$\pi_t\left(w+h_t\right)=\frac{1}{2}\lambda_u\left(w\right)\bar{q}\left(1-\bar{q}\right)\gamma^2\left(t\right)+o\left(\left\|h_t\right\|_{\bar{q}_t}^2\right), \tag{40}$$

where risk attitudes have a second order effect at $\{h_t,\mu_t\}_{t\in(0,1]}$, while model uncertainty attitudes have a negligible effect.

In Limit Case 3 approximation (39) becomes

$$\pi_t\left(w+W_t\right)=\frac{1}{2}\lambda_v\left(w\right)\bar{q}\left(1-\bar{q}\right)\gamma^2\left(t\right)+o\left(\left\|h_t\right\|_{\bar{q}_t}^2\right), \tag{41}$$

where model uncertainty attitudes have a second order effect at $\{h_t,\mu_t\}_{t\in(0,1]}$, while risk attitudes have a negligible effect.

Finally, in Limit Case 4 approximation (39) trivially becomes

$$\pi_t\left(w+W_t\right)=o\left(\left\|h_t\right\|_{\bar{q}_t}^2\right). \tag{42}$$

Here, neither u nor v play any role in the approximation since both model uncertainty and risk attitudes have a negligible effect at $\{h_t,\mu_t\}_{t\in(0,1]}$.

Example 50 (i) If $\varphi\sim k\psi$ with $k\neq 0,1$, then $\gamma\left(t\right)\asymp\left\|h_t\right\|_{\bar{q}_t}$. In this case the properties of μ determine which one, among Limit Cases 1-4, arises. (ii) If $\varphi\sim\psi$, then $\gamma\left(t\right)=o\left(\left\|h_t\right\|_{\bar{q}_t}\right)$ and so Limit Case 4 arises. ▲

9 A special case

Our analysis so far shows that both risk and model uncertainty attitudes are relevant to describe the limit behavior of $C_t\left(w+W_t\right)$ and $\pi_t\left(w+W_t\right)$ as $t\downarrow 0$, unless μ_t quadratically converges to a Dirac probability measure δ_p or has support $\{0,1\}$. In particular, the attitude toward model uncertainty is relevant in the second order approximation unless $\mu_t\overset{sq}{\Longrightarrow}\delta_p$, that is, unless the decision maker's priors concentrate on a single model p, so that model uncertainty eventually vanishes in terms of quadratic convergence. This is a natural feature of the KMM model that Theorem 49 establishes in full generality for the binomial case.

Intuitively, an instance when model uncertainty eventually vanishes is when the support of the probability μ_t becomes more and more concentrated. The next result shows that this is indeed the case.

Proposition 51 *Suppose that the collection of convex hulls* $\{\operatorname{co}\left(\operatorname{supp}\mu_t\right)\}_{t>0}$ *has the finite intersection property.*[14] *If* $\operatorname{diam}\left(\operatorname{supp}\mu_t\right)\to 0$ *as* $t\downarrow 0$*, then* $\bigcap_t\operatorname{co}\left(\operatorname{supp}\mu_t\right)$ *contains one point* $p\in[0,1]$ *and* $\mu_t\overset{w}{\Longrightarrow}\delta_p$*.*

[14] That is, it holds $\bigcap_{t\in T}\operatorname{co}\left(\operatorname{supp}\mu_t\right)\neq\emptyset$ given any finite index set T. For example, this property trivially holds if $\operatorname{co}\left(\operatorname{supp}\mu_t\right)\subseteq\operatorname{co}\left(\operatorname{supp}\mu_{t'}\right)$ when $t<t'$.

Along with Proposition 48, this implies that if $p \in (0,1)$ and h_t is such that $\lim_{t\downarrow 0} \varphi(t) = \lim_{t\downarrow 0} \psi(t) = 0$, then

$$\pi_t(w + h_t) = \frac{1}{2}\lambda_u(w)\, p(1-p)\, \gamma^2(t) + o\left(\|h_t\|_{\bar{q}_t}^2\right). \tag{43}$$

While, if $p = 0$ and h_t is such that $\lim_{t\downarrow 0} \psi(t) = 0$ and $\varphi(t) = O(\psi(t))$ as $t \downarrow 0$, then

$$\pi_t(w + h_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right). \tag{44}$$

A similar result holds when $p = 1$. Hence, under the "shrinking" hypotheses of Proposition 51, only attitudes toward risk may have a second order effect at $\{h_t, \mu_t\}_{t\in(0,1]}$. That is, only Limit Cases 2 and 4 apply. As we emphasized, this is not surprising given that model uncertainty itself vanishes because $\mu_t \overset{w}{\Longrightarrow} \delta_p$. When $p = 0$ (or $p = 1$) risk vanishes too; as (44) shows, in this case both risk and model uncertainty attitudes are negligible.

We now provide some examples to show how the properties of the KMM model established in Theorems 4 and 5 of Skiadas [12] may represent a special case of (43) and (44). As a consequence, [12]' findings that attitudes toward model uncertainty become irrelevant as t goes to 0 may be attributed to the fact that, in the special case it considers for its purposes, model uncertainty itself vanishes ($\mu_t \overset{w}{\Longrightarrow} \delta_p$).

Example 52 Given a scalar $\rho \in \mathbb{R}$, let

$$q_t = \frac{1}{2}(1 + \rho t). \tag{45}$$

Then, $1 - q_t = 2^{-1}(1 - \rho t)$ and

$$q_t \in [0,1] \Longleftrightarrow \rho \in \left[-\frac{1}{2t}, \frac{1}{2t}\right]. \tag{46}$$

Consider a finite collection of scalars $\rho_1 < ... < 0 < ... < \rho_K$. By (46), eventually $\left\{q_t^k\right\}_{k=1}^K \subseteq [0,1]$. Let μ_t be a discrete probability measure on $[0,1]$ with $\operatorname{supp}\mu_t = \left\{q_t^k\right\}_{k=1}^K$.[15] Since

$$\operatorname{diam}(\operatorname{supp}\mu_t) = \frac{1}{2}(\rho_K - \rho_1)\, t,$$

we have $\operatorname{diam}(\operatorname{supp}\mu_t) \to 0$ as $t \downarrow 0$. Moreover, $\operatorname{co}(\operatorname{supp}\mu_{t'}) \subseteq \operatorname{co}(\operatorname{supp}\mu)$ for each $t' < t$. By Proposition 51, $\mu_t \overset{w}{\Longrightarrow} \delta_{1/2}$ so that model uncertainty vanishes and approximation (43) applies. For instance, consider the uncertain monetary net $\{h_t, \mu_t\}_{t\in(0,1]}$, where $h_t = W_t + O\left(t^2\right)$ as in [12, Theorem 4]. It is straightforward to verify that this is a controllable quadratic small uncertainty. Hence, (43) takes the form

$$\pi_t(w + h_t) = \frac{1}{2}\lambda_u(w)\, t^2 + o\left(t^2\right)$$

because $\gamma^2(t) = 4t^2 + O\left(t^3\right)$ and $\|h_t\|_{\bar{q}_t}^2 = t^2 + O\left(t^3\right)$. Only risk attitudes have a second order effect at $\{h_t, \mu_t\}_{t\in(0,1]}$. ▲

Example 53 Given a scalar $\rho \in \mathbb{R}$ and a function $o\left(t^2\right)$, let

$$q_t = (1 + \rho)\, t^2 + o\left(t^2\right). \tag{47}$$

[15] Skiadas uses Q, $\sqrt{h}$, and π instead of q, t, and μ, respectively. This same observation also applies to Example 53.

Then, $1 - q_t = 1 - (1+\rho)\, t^2 + o\left(t^2\right)$ and eventually $q_t \in (0,1)$ if $\rho > -1$. Consider a finite collection of scalars $-1 < \rho_1 < ... < \rho_K$. Then, define through (47) a finite collection of probabilities $\left\{q_t^k\right\}_{k=1}^K$. Let μ_t be a discrete probability measure on $[0,1]$ with $\operatorname{supp} \mu_t = \left\{q_t^k\right\}_{k=1}^K$. Then, by Lemma 4, $\mu_t \xrightarrow{w} \delta_0$ and model uncertainty vanishes.

That said, as in [12, Theorem 5] consider the uncertain monetary net $\{h_t, \mu_t\}_{t \in (0,1]}$, where $h_t = h + O\left(t^2\right)$ and

$$h(\omega) = \begin{cases} 1 & \text{if } \omega = 1 \\ 0 & \text{if } \omega = 2 \end{cases}$$

This net is a quadratic small uncertainty since $\|h_t\|_{\bar{q}_t}^2 = O\left(t^2\right)$. It is not, however, controllable since $h_t(1)$ does not converge to zero. Hence, we do not know whether it admits the quadratic approximation (38). However, if it did, then it would hold

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w)\, \bar{q}_t + o\left(t^2\right)$$

because $\vartheta_t = O\left(t^4\right)$, $\bar{q}_t^2 = O\left(t^4\right)$, and $\gamma^2(t) = 1 + O\left(t^2\right)$. Risk attitudes only would have a second order effect at $\{h_t, \mu_t\}_{t \in (0,1]}$. ▲

A Proofs and related Analysis

Proof of Lemma 2. Suppose $\varphi \in O(\psi)$, then there are $\eta, M > 0$ such that

$$|\varphi(t)| \le M |\psi(t)| \qquad \forall t < \eta.$$

Let $\xi \in o(\varphi)$. For all $\varepsilon > 0$ there exists $0 < \delta < \eta$ such that

$$|\xi(t)| \le \frac{\varepsilon}{M} |\varphi(t)| \qquad \forall t < \delta.$$

Hence

$$|\xi(t)| \le \frac{\varepsilon}{M} |\varphi(t)| \le \varepsilon |\psi(t)| \qquad \forall t < \delta$$

and $\xi \in o(\psi)$.

Conversely, suppose $\varphi \notin O(\psi)$. Then for all $\eta, M > 0$ there is $t_{\eta,M} < \eta$ such that

$$|\varphi(t_{\eta,M})| > M |\psi(t_{\eta,M})|.$$

Hence, for all $n \in \mathbb{N}$ (choosing $\eta = 1/n$ and $M = n$) there exists $t_n < 1/n$ such that

$$|\varphi(t_n)| > n |\psi(t_n)|.$$

Without loss of generality, assume that $\{t_n\}$ is strictly decreasing. Set $\xi(t_n) = \psi(t_n)$ for all $n \in \mathbb{N}$ such that $\psi(t_n) \neq 0$, set $\xi(t_n) = \frac{1}{n}\varphi(t_n)$ for all $n \in \mathbb{N}$ such that $\psi(t_n) = 0$, and set $\xi(t) = 0$ otherwise. Let $\varepsilon > 0$ and choose N so that $1/N < \varepsilon$. Set $\delta = t_N$. Now for $t < \delta$ consider the two following cases:

- $t = t_n$ for some $n \in \mathbb{N}$, then $t < t_N$ implies $n > N$, and hence $|\xi(t)| \le \frac{1}{n} |\varphi(t_n)| \le \frac{1}{N} |\varphi(t_n)| \le \varepsilon |\varphi(t)|$;
- $t \neq t_n$ for all $n \in \mathbb{N}$, then $|\xi(t)| = 0 \le \varepsilon |\varphi(t)|$.

In conclusion, $\xi \in o(\varphi)$. But, letting $M = 1/2$,

$$|\xi(t_n)| > M |\psi(t_n)|$$

for all $n \in \mathbb{N}$. Since $t_n \downarrow 0$ as $n \to \infty$, there cannot exist $\delta > 0$ such that

$$|\xi(t)| \leq M |\psi(t)| \qquad \forall t < \delta.$$

Thus $\xi \in o(\varphi)$ and $\xi \notin o(\psi)$. ∎

Proof of Lemma 4 We prove the "if" part, the converse being trivial. Given $\varphi \in C([0,1])$, let $\varepsilon > 0$. By the Weierstrass Approximation Theorem, there is some polynomial φ_ε on $[0,1]$ with $\|\varphi - \varphi_\varepsilon\|_\infty \leq \varepsilon$. By hypothesis, there is α_ε such that $\left|\int_{[0,1]} \varphi_\varepsilon(q)\, d\mu_\alpha(q) - \int_{[0,1]} \varphi_\varepsilon(q)\, d\mu(q)\right| < \varepsilon$ for all $\alpha \geq \alpha_\varepsilon$. Hence, for all $\alpha \geq \alpha_\varepsilon$,

$$\left|\int_{[0,1]} \varphi(q)\, d\mu_\alpha(q) - \int_{[0,1]} \varphi(q)\, d\mu(q)\right| \leq \|\varphi - \varphi_\varepsilon\|_\infty + \left|\int_{[0,1]} \varphi_\varepsilon(q)\, d\mu_\alpha(q) - \int_{[0,1]} \varphi_\varepsilon(q)\, d\mu(q)\right| + \|\varphi_\varepsilon - \varphi\|_\infty \leq 3\varepsilon$$

This implies $\lim_\alpha \int_{[0,1]} \varphi(q)\, d\mu_\alpha(q) = \int_{[0,1]} \varphi(q)\, d\mu(q)$. ∎

Proof of Proposition 12. We refer to the proof of the more general Proposition 28. ∎

Proof of Proposition 13 Consider a small risk $\{h_t, q\}_{t\in(0,1]}$. Let $\pi(w + h_t) \sim k \|h_t\|_q$ for some $k > 0$. Then

$$w + E_q(h_t) - C(w + h_t) \sim k \|h_t\|_q$$

and so for every $\varepsilon > 0$ there is t_ε such that

$$(1-\varepsilon) k \|h_t\|_q \leq w + E_q(h_t) - C(w + h_t) \leq (1+\varepsilon) k \|h_t\|_q \qquad \forall t \in (0, t_\varepsilon]$$

that is

$$E_q(h_t) - (1+\varepsilon) k \|h_t\|_q \leq C(w + h_t) - w \leq E_q(h_t) - (1-\varepsilon) k \|h_t\|_q \qquad \forall t \in (0, t_\varepsilon]$$

(i) Suppose $k > \sqrt{k^*} = \sqrt{\limsup_{t\downarrow 0} E_q^2(h_t) / \|h_t\|_q^2}$. Take ε small enough so that $k(1-\varepsilon) > \sqrt{k^*}$. Since $k^* = \lim_{s\downarrow 0} \sup_{t\in(0,s)} \frac{E_q^2(h_t)}{\|h_t\|_q^2}$, there is $t'_\varepsilon \in (0, t_\varepsilon]$ such that

$$\frac{E_q(h_t)}{\|h_t\|_q} < k(1-\varepsilon) \qquad \forall t \in (0, t'_\varepsilon]$$

and hence

$$C(w + h_t) - w \leq E_q(h_t) - (1-\varepsilon) k \|h_t\|_q < 0 \qquad \forall t \in (0, t'_\varepsilon].$$

(ii) Suppose $k < \sqrt{k_*} = \sqrt{\liminf_{t\downarrow 0} E_q^2(h_t) / \|h_t\|_q^2}$. Take ε small enough so that $k(1+\varepsilon) < \sqrt{k^*}$. Since $k^* = \lim_{s\downarrow 0} \inf_{t\in(0,s)} \frac{E_q^2(h_t)}{\|h_t\|_q^2}$, there is $t''_\varepsilon \in (0, t_\varepsilon]$ such that

$$\frac{E_q(h_t)}{\|h_t\|_q} > k(1+\varepsilon) \qquad \forall t \in (0, t''_\varepsilon]$$

and hence

$$C(w + h_t) - w \geq E_q(h_t) - (1+\varepsilon) k \|h_t\|_q > 0 \qquad \forall t \in (0, t''_\varepsilon].$$

(iii) Finally, $\pi(w+h_t) = o\left(\|h_t\|_q\right)$ and $E_q(h_t) > 0$ imply $\pi(w+h_t) = o(E_q(h_t))$. Therefore

$$C(w+h_t) - w = E_q(h_t) - \pi(w+h_t) = E_q(h_t)(1+o(1)).$$

The proof is concluded dividing both sides by $E_q(h_t) > 0$. ■

Proof of Lemma 14 By contrapositive, we first show that $\sigma_q^2(h_t) \not\asymp \|h_t\|_q^2$ if and only if $k^* = 1$.

For all t, it holds

$$0 \le \liminf_{t\downarrow 0} \sigma_q^2(h_t) / \|h_t\|_q^2 \le \limsup_{t\downarrow 0} \sigma_q^2(h_t) / \|h_t\|_q^2 \le 1.$$

Hence

$$\begin{aligned}
\sigma_q^2(h_t) \not\asymp \|h_t\|_q^2 &\iff \liminf_{t\downarrow 0} \frac{\sigma_q^2(h_t)}{\|h_t\|_q^2} = 0 \\
&\iff \liminf_{t\downarrow 0}\left(1 - \frac{(E_q(h_t))^2}{\|h_t\|_q^2}\right) = 0 \\
&\iff 1 - \limsup_{t\downarrow 0} \frac{(E_q(h_t))^2}{\|h_t\|_q^2} = 0 \\
&\iff \limsup_{t\downarrow 0} \frac{(E_q(h_t))^2}{\|h_t\|_q^2} = 1.
\end{aligned}$$

Hence, $\sigma_q^2(h_t) \asymp \|h_t\|_q^2$ if and only if $k^* < 1$.

On the other hand,

$$\begin{aligned}
\sigma_q^2(h_t) = o\left(\|h_t\|_q^2\right) &\iff \limsup_{t\downarrow 0} \frac{\sigma_q^2(h_t)}{\|h_t\|_q^2} = 0 \\
&\iff \liminf_{t\downarrow 0} \frac{(E_q(h_t))^2}{\|h_t\|_q^2} = 1
\end{aligned}$$

and so $\sigma_q^2(h_t) = o\left(\|h_t\|_q^2\right)$ if and only if $k_* = 1$. ■

Proof of Proposition 15 We only prove point (ii). Set $\operatorname{supp} q = \{\omega_1, ..., \omega_s\}$.

If $s = 2$, set $x_t = h_t(\omega_1)$ and $y_t = h_t(\omega_2)$ and $q(\omega_1) = q$. Recall that $x_t = O(y_t)$, that is, there exist $\delta, M > 0$ such that $|x_t| \le M|y_t|$ for $t < \delta$. The assumption $\|h_t\|_q \ne 0$ for all t, implies that $|y_t| > 0$ for all $t < \delta$. As $t \downarrow 0$, (19) amounts to

$$\frac{(qx_t + (1-q)y_t)^2}{qx_t^2 + (1-q)y_t^2} \to 1,$$

that is, $q^2x_t^2 + (1-q)^2 y_t^2 + 2q(1-q)x_t y_t = qx_t^2 + (1-q)y_t^2 + o\left(qx_t^2 + (1-q)y_t^2\right)$. Simple computations deliver $(x_t - y_t)^2 = o\left(qx_t^2 + (1-q)y_t^2\right)$, and, dividing by y_t^2,

$$\frac{\left(\frac{x_t}{y_t} - 1\right)^2}{q\frac{x_t^2}{y_t^2} + (1-q)} \to 0.$$

Finally, $x_t^2 \le M^2 y_t^2$ for $t < \delta$, implies

$$0 \le \frac{\left(\frac{x_t}{y_t} - 1\right)^2}{qM^2 + (1-q)} \le \frac{\left(\frac{x_t}{y_t} - 1\right)^2}{q\frac{x_t^2}{y_t^2} + (1-q)} \to 0.$$

Else, if $s > 2$, then $h_t(\omega_i) \sim k_{i1} h_t(\omega_1)$ for $i = 1, 2, ..., s$. Write k_i rather than k_{i1} and notice that $k_i \in \mathbb{R} \setminus \{0\}$ for all $i = 1, ..., s$. Then,

$$\begin{aligned}
\frac{E_q^2(h_t)}{\|h_t\|_q^2} &= \frac{\left([k_1 h_t(\omega_1) + o(h_t(\omega_1))] q(\omega_1) + \cdots + [k_s h_t(\omega_1) + o(h_t(\omega_1))] q(\omega_s)\right)^2}{[k_1 h_t(\omega_1) + o(h_t(\omega_1))]^2 q(\omega_1) + \cdots + [k_s h_t(\omega_1) + o(h_t(\omega_1))]^2 q(\omega_s)} \\
&= \frac{(k_1 q(\omega_1) + \cdots + k_s q(\omega_s) + o(1))^2}{(k_1^2 q(\omega_1) + \cdots + k_s^2 q(\omega_s) + o(1))}.
\end{aligned}$$

Set $\kappa^2 = k_1^2 q(\omega_1) + \cdots + k_s^2 q(\omega_s) > 0$ and $\varkappa = k_1 q(\omega_1) + \cdots + k_s q(\omega_s)$. Now

$$\frac{(\varkappa + o(1))^2}{(\kappa^2 + o(1))} \to 1 \text{ and } \kappa \neq 0 \Longrightarrow \frac{\varkappa^2}{\kappa^2} = 1 .$$

Therefore $E_q(k^2) = (E_q(k))^2$ and k is constant. The observation that $k_1 = 1$ concludes the proof.■

Proof of Proposition 17. Let us prove the "only if", the converse being trivial. Let $h \in \mathbb{R}^n$. Since $\|th\|_q \to 0$ as $t \downarrow 0$,

$$\sigma_q^2(h) = \frac{\sigma_q^2(th)}{t^2} = \frac{o\left(\|th\|_q^2\right)}{t^2} = \frac{o(t^2)}{t^2} \to 0$$

and so $\sigma_q^2(h) = 0$. Since this holds for any $h \in \mathbb{R}^n$, there is some $\omega \in \Omega$ such that $q = \delta_\omega$. ■

Proof of Proposition 20 For convenience we add two points that will turn out to be equivalent to (i)-(iv):

(v) $\left|\left|\left|\frac{h_t}{t}\right|\right|\right| = O(1)$ for some norm on $L(q)$.

(vi) $\left|\left|\left|\frac{h_t}{t}\right|\right|\right| = O(1)$ for every norm on $L(q)$.

Clearly, (i) implies (ii) and (iii) implies (i).

Next we show that (ii) implies (v). Let $n \in \mathbb{N}$ be such that $E_q(|h_t|^n) = O(t^n)$, there exist $M, \delta > 0$ such that $E_q(|h_t|^n) \leq M t^n$ for all $t \in (0, \delta)$. Then,

$$\sqrt[n]{E_q\left(\left|\frac{h_t}{t}\right|^n\right)} \leq \sqrt[n]{M}$$

for all $t \in (0, \delta)$, that is, $\left\|\frac{h_t}{t}\right\|_{L^n(q)} = O(1)$, so that (v) holds.

By (v), there exists a norm $|||\cdot|||'$ on $L(q)$ and $M, \delta > 0$ such that

$$\left|\left|\left|\frac{h_t}{t}\right|\right|\right|' \leq M$$

for all $t \in (0, \delta)$. Since Ω is finite, all norms on $L(q)$ are equivalent. Therefore for every norm $|||\cdot|||$ on $L(q)$, there is $c > 0$ such that $|||\cdot||| \leq c\, |||\cdot|||'$. Finally,

$$\left|\left|\left|\frac{h_t}{t}\right|\right|\right| \leq c \left|\left|\left|\frac{h_t}{t}\right|\right|\right|' \leq cM$$

for all $t \in (0, \delta)$, that is, $\left|\left|\left|\frac{h_t}{t}\right|\right|\right| = O(1)$. Thus (vi) holds.

(vi) implies that $\left\|\frac{h_t}{t}\right\|_{L^n(q)} = O(1)$ for all $n \in \mathbb{N}$, that is, there exist $M = M(n), \delta = \delta(n) > 0$ such that

$$\sqrt[n]{E_q\left(\left|\frac{h_t}{t}\right|^n\right)} \leq M.$$

This is clearly equivalent to (iii). Therefore (i), (ii), (iii), (v), (vi) are equivalent.

(vi) implies (iv). In fact, considering the $L^{\infty}(q)$ norm, there exist $M, \delta > 0$ such that $\max_{\omega \in \operatorname{supp} q} \left|\frac{h_t(\omega)}{t}\right| \leq M$ for all $t \in (0, \delta)$.

(iv) implies (v). In fact, for all $\omega \in \operatorname{supp} q$, there exist $M_\omega, \delta_\omega > 0$ such that $\left|\frac{h_t(\omega)}{t}\right| \leq M_\omega$ for all $t \in (0, \delta_\omega)$. Let $\delta = \min_{\omega \in \operatorname{supp} q} \delta_\omega$ and $M = \sum_{\omega \in \operatorname{supp} q} q(\omega) M_\omega$. Clearly, for all $t \in (0, \delta)$,

$$\left\|\frac{h_t}{t}\right\|_{L^1(q)} = \sum_{\omega \in \operatorname{supp} q} q(\omega) \left|\frac{h_t(\omega)}{t}\right| \leq \sum_{\omega \in \operatorname{supp} q} q(\omega) M_\omega = M,$$

that is $\left\|\frac{h_t}{t}\right\|_{L^1(q)} = O(1)$. ∎

Proof of Lemma 23 Since $\sigma_{\bar{q}}^2(h_t) \leq \|h_t\|_{\bar{q}}^2$, it holds $\sigma_{\bar{q}}^2(h_t) = o\left(\|h_t\|_{\bar{q}}\right)$. Notice that

$$\sigma_\mu^2(E(h_t)) = \sigma_\mu^2(\Sigma_{\omega \in \Omega} h_t(\omega) \langle e_\omega, \cdot\rangle)$$

where e_ω is the ω-th vector of the canonical base of $\mathbb{R}^\Omega$ (hence $\langle e_\omega, \cdot\rangle : \Delta \to \mathbb{R}$ is the projection on the ω-th component $q \mapsto q(\omega)$).

The usual variance-covariance decomposition delivers

$$\sigma_\mu^2(\Sigma_{\omega \in \Omega} h_t(\omega) \langle e_\omega, \cdot\rangle) = h_t^T \Xi h_t$$

where

$$\begin{aligned}
\Xi(\omega, \omega') &= \int_\Delta \langle e_\omega, \cdot\rangle \langle e_{\omega'}, \cdot\rangle \, d\mu - \int_\Delta \langle e_\omega, \cdot\rangle \, d\mu \int_\Delta \langle e_{\omega'}, \cdot\rangle \, d\mu \\
&= \int_\Delta q(\omega) q(\omega') \, d\mu(q) - \int_\Delta q(\omega) \, d\mu(q) \int_\Delta q(\omega') \, d\mu(q) \\
&= \int_\Delta q(\omega) q(\omega') \, d\mu(q) - \bar{q}(\omega) \bar{q}(\omega') \in [-1, 1]
\end{aligned}$$

for all $\omega, \omega' \in \Omega$. Moreover notice that $\omega \notin \operatorname{supp} \bar{q}$ is equivalent to $\int_\Delta \langle e_\omega, \cdot\rangle \, d\mu = \bar{q}(\omega) = 0$, which in turn amounts to $\langle e_\omega, \cdot\rangle = 0$ μ-almost everywhere. In particular $\Xi(\omega, \omega') = 0$ if $(\omega, \omega') \notin \operatorname{supp} \bar{q} \times \operatorname{supp} \bar{q}$. Therefore

$$\begin{aligned}
\frac{\sigma_\mu^2(E(h_t))}{\|h_t\|_{\bar{q}}} &= \frac{\sum_{\omega, \omega' \in \Omega} h_t(\omega) h_t(\omega') \Xi(\omega, \omega')}{\|h_t\|_{\bar{q}}} = \frac{\sum_{\omega, \omega' \in \operatorname{supp} \bar{q}} h_t(\omega) h_t(\omega') \Xi(\omega, \omega')}{\|h_t\|_{\bar{q}}} \\
&\leq \frac{\sum_{\omega, \omega' \in \operatorname{supp} \bar{q}} |h_t(\omega)| |h_t(\omega')| |\Xi(\omega, \omega')|}{\|h_t\|_{\bar{q}}} \leq \frac{\sum_{\omega, \omega' \in \operatorname{supp} \bar{q}} |h_t(\omega)| |h_t(\omega')|}{\|h_t\|_{\bar{q}}} \\
&\leq \frac{\sum_{\omega, \omega' \in \operatorname{supp} \bar{q}} \|h_t\|_{L^\infty(\bar{q})}^2}{\|h_t\|_{\bar{q}}} \leq \frac{|\Omega|^2 \|h_t\|_{L^\infty(\bar{q})}^2}{\|h_t\|_{\bar{q}}} \leq c \frac{\|h_t\|_{\bar{q}_t}^2}{\|h_t\|_{\bar{q}_t}} = c \|h_t\|_{\bar{q}_t}
\end{aligned}$$

because all norms on $L(q)$ are equivalent. Since $\|h_t\|_{\bar{q}_t} \to 0$, then $\sigma_\mu^2(E(h_t)) = o\left(\|h_t\|_{\bar{q}}\right)$. This concludes the proof. ∎

Proof of Proposition 25 Set $\Lambda_u(h, \mu) = E_\mu\left(\sigma^2(h)\right)$ and $\Lambda_v(h, \mu) = \sigma_\mu^2(E(h))$. We prove the "only if", the converses being trivial.

(ii) Let $h \in \mathbb{R}^n \setminus \{0\}$. Notice that,

$$\sigma_{\bar{q}}^2(h) - \sigma_\mu^2(E(h)) = \frac{\sigma_{\bar{q}}^2(th)}{t^2} - \frac{\sigma_\mu^2(E(th))}{t^2} = \frac{\Lambda_u(th, \mu)}{\|th\|_{\bar{q}}^2} \|h\|_{\bar{q}}^2$$

for all $t>0$. Since $\Lambda_u(k,\mu)=o\left(\|k\|_{\bar{q}}^2\right)$ as $k \overset{\|\cdot\|_{\bar{q}}}{\to} 0$, then

$$\lim_{t\downarrow 0}\frac{\Lambda_u(th,\mu)}{\|th\|_{\bar{q}}^2}\|h\|_{\bar{q}}^2=0.$$

Therefore, $\sigma_{\bar{q}}^2(h)=\sigma_\mu^2(E(h))$ for all $h\in\mathbb{R}^\Omega$. In turn, this is equivalent to $\int_\Delta E_q\left(h^2\right)d\mu(q)=\int_\Delta E_q^2(h)\,d\mu(q)$. Since $E_q\left(h^2\right)\geq E_q^2(h)$ for all $q\in\Delta$, we conclude that $E_q\left(h^2\right)=E_q^2(h)$ μ-a.e.. In particular, $E_q\left(h^2\right)=E_q^2(h)$ for all $q\in\operatorname{supp}\mu$,[16] and this is true for all $h\in\mathbb{R}^\Omega$. This implies that each $q\in\operatorname{supp}\mu$ is a Dirac measure.

(i) Similarly, for all $h\in\mathbb{R}^n\setminus\{0\}$,

$$\sigma_\mu^2(E(h))=\frac{\sigma_\mu^2(E(th))}{t^2}=\frac{\Lambda_v(th,\mu)}{\|th\|_{\bar{q}}^2}\|h\|_{\bar{q}}^2$$

for all $t>0$. Since $\Lambda_v(k,\mu)=o\left(\|k\|_{\bar{q}}^2\right)$ as $k \overset{\|\cdot\|_{\bar{q}}}{\to} 0$, then

$$\lim_{t\downarrow 0}\frac{\Lambda_v(th,\mu)}{\|th\|_{\bar{q}}^2}\|h\|_{\bar{q}}^2=0$$

and $\sigma_\mu^2(E(h))=0$ for all $h\in\mathbb{R}^n$. Therefore, for each $\omega\in\Omega$, setting $h=e_\omega$ it follows that

$$\begin{aligned}0&=\int_\Delta\left(E_q(e_\omega)-\left(\int_\Delta E_q(e_\omega)\,d\mu(q)\right)\right)^2d\mu(q)\\&=\int_\Delta\left(q(\omega)-\bar{q}(\omega)\right)^2d\mu(q).\end{aligned}$$

Then $q(\omega)=\bar{q}(\omega)$ μ-a.e.. In particular, $q(\omega)=\bar{q}(\omega)$ for all $q\in\operatorname{supp}\mu$,[17] and this is true for all $\omega\in\not\leq$. That is, $\operatorname{supp}\mu=\{\bar{q}\}$.

(iii) Follows from (i) and (ii). ∎

Proof of Lemma 27 By definition

$$\omega\notin\bigcap_{t\in(0,1]}\bigcup_{\tau\leq t}\operatorname{supp}q_\tau\iff\omega\in\left(\bigcap_{t\in(0,1]}\bigcup_{\tau\leq t}\operatorname{supp}q_\tau\right)^c\iff\omega\in\bigcup_{t\in(0,1]}\bigcap_{\tau\leq t}\left(\operatorname{supp}q_\tau\right)^c$$

if and only if there exists $t\in(0,1]$ such that $\omega\in\bigcap_{\tau\leq t}\left(\operatorname{supp}q_\tau\right)^c$, if and only if there exists $t\in(0,1]$ such that $\omega\in\left(\operatorname{supp}q_\tau\right)^c$ for all $\tau\leq t$, if and only if there exists $t\in(0,1]$ such that $q_\tau(\omega)=0$ for all $0<\tau\leq t$. ∎

Proof of Proposition 28 We already observed that (i) implies (ii), which implies (iii), which implies (iv).

(iv) implies (v). Assume per contra that there exists $\omega\in\operatorname{supp}q$ such that there is a sequence $\{t_n\}$, with $t_n\downarrow 0$, such that $h_{t_n}(\omega)$ does not converge to 0. Then, for some $\varepsilon>0$, $|h_{t_n}(\omega)|>\varepsilon$ for infinitely many n. Passing to a subsequence, we can assume $|h_{t_n}(\omega)|>\varepsilon$ for all n. Setting $\delta=\min\{\varepsilon/2,q(\omega)/2\}$, we have $q(\omega),\varepsilon,|h_{t_n}(\omega)|>\delta$ for all n. Since $\lim_{n\to\infty}q_{t_n}(\omega)=q(\omega)$, for n large enough it holds $q_{t_n}(\omega)\geq q(\omega)-\delta$. For all such n,

$$q_{t_n}(\omega':|h_{t_n}(\omega')|\geq\delta)\geq q_{t_n}(\omega)\geq q(\omega)-\delta>0,$$

[16] Notice that $q\mapsto E_q\left(h^2\right)$ and $q\mapsto E_q^2(h)$ are continuous functions on Δ.

[17] Notice that $q\mapsto q(\omega)$ and $q\mapsto\bar{q}(\omega)$ are continuous functions on Δ.

which contradicts (iv).

Notice that, for the chain of implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), we did **not** use the assumption $\limsup_{t\downarrow 0} |h_t(\omega)| < \infty$ for all $\omega \in \Gamma_q$, which will be crucial in showing that

(v) implies (i). Let $n \geq 1$.

Let $\omega \in \operatorname{supp} q$. Then $0 \leq |h_t(\omega)|^n q_t(\omega) \leq |h_t(\omega)|^n$, for all t, implies $\lim_{t\downarrow 0} |h_t(\omega)|^n q_t(\omega) = 0$.

Let $\omega \in \Gamma_q$, then $q_t(\omega) \to 0$ as $t \downarrow 0$. Moreover, $|h_t(\omega)|^n$ is eventually bounded so that $\lim_{t\downarrow 0} |h_t(\omega)|^n q_t(\omega) = 0$.

If $\omega \notin S$, by Lemma 27 there is $t_\omega \in (0,1]$ such that $q_t(\omega) = 0$ for all $t \leq t_\omega$. Hence, $\lim_{t\downarrow 0} |h_t(\omega)|^n q_t(\omega) = 0$. Summing up:

$$\lim_{t\downarrow 0} E_{q_t}(|h_t|^n) = \lim_{t\downarrow 0}\left(\sum_{\omega\in\Omega} |h_t|^n q_t(\omega)\right) = \sum_{\omega\in\Omega} \lim_{t\downarrow 0} |h_t|^n q_t(\omega) = 0,$$

as desired. This completes the proof. ■

Proof of Theorem 33 See the more general proof of Theorem 40. ■

Proof of Proposition 34 Let $\{h_t, q_t\}_{t\in(0,1]}$ be a controllable small risk that is asymptotically constant. Since eventually $\operatorname{supp} q \subseteq \operatorname{supp} q_t \subseteq S$, then choosing $\omega' \in \operatorname{supp} q$,

$$\sum_{\omega\in\operatorname{supp} q_t} h_t(\omega) q_t(\omega) = \sum_{\omega\in\operatorname{supp} q_t} h_t(\omega')(1+o(1)) q_t(\omega) \sim h_t(\omega')$$

then

$$\left(\sum_{\omega\in\operatorname{supp} q_t} h_t(\omega) q_t(\omega)\right)^2 \sim h_t^2(\omega')$$

Analogously,

$$\sum_{\omega\in\operatorname{supp} q_t} h_t^2(\omega) q_t(\omega) = \sum_{\omega\in\operatorname{supp} q_t} h_t^2(\omega')(1+o(1)) q_t(\omega) \sim h_t^2(\omega').$$

Therefore, $E_{q_t}^2(h_t) \sim E_{q_t}\left(h_t^2\right)$. ■

Proof of Proposition 36 To see that (i) implies (ii) it is enough to observe that there is $a > 0$ such that, for each t small enough,

$$\frac{h_t^2(\omega) q_t(\omega)}{t^2} \leq \frac{E_{q_t}\left(h_t^2\right)}{t^2} \leq a, \qquad \forall \omega \in \Omega.$$

(ii) implies (i). For, there are $a, b > 0$ such that, for t small enough,

$$0 \leq \frac{E_{q_t}\left(h_t^2\right)}{t^2} = \sum_{\omega\in\operatorname{supp} q} \frac{h_t^2(\omega)}{t^2} q_t(\omega) + \sum_{\omega\in\Gamma_q} \frac{h_t^2(\omega)}{t^2} q_t(\omega) \leq a + b.$$

Hence, $E_{q_t}\left(h_t^2\right) = O\left(t^2\right)$. ■

Proof of Theorem 40 Since $\{h_t, \mu_t\}_{t\in(0,1]}$ is a small uncertainty, then $\lim_{t\downarrow 0} E_{\bar{q}_t}\left(h_t^2\right) = 0$. By controllability, eventually $\max_{\omega\in\operatorname{supp}\bar{q}_t} |h_t(\omega)| \leq M\sqrt{E_{\bar{q}_t}\left(h_t^2\right)}$ for some $M > 0$. Then $\max_{\omega\in\operatorname{supp}\bar{q}_t} |h_t(\omega)| \to 0$. As observed immediately after the proof of Corollary 59, this implies

$$C_t(w+h_t) - \left[w + E_{\bar{q}_t}(h_t) - \frac{1}{2}\lambda_u(w)\sigma_{\bar{q}_t}^2(h_t) - \frac{1}{2}(\lambda_v(w) - \lambda_u(w))\sigma_{\mu_t}^2(\langle h_t, \cdot\rangle)\right] = o\left(\left(\max_{\omega\in\operatorname{supp}\bar{q}_t} |h_t(\omega)|\right)^2\right)$$

then, $\left(\max_{\omega\in\operatorname{supp}\bar{q}_t} |h_t(\omega)|\right)^2 = O\left(\|h_t\|_{\bar{q}_t}^2\right)$ and Lemma 2 deliver (30). ■

Proof of Lemma 46 It follows from Lemmas 54 and 55 below. ■

Lemma 54 *An uncertainty monetary net* $\{h_t, \mu_t\}_{t\in(0,1]}$ *is a small uncertainty if and only if:*

(i) $\lim_{t\downarrow 0} \varphi(t) = \lim_{t\downarrow 0} \psi(t) = 0$ *when* $\mu \neq \delta_p$ *with* $p \in \{0,1\}$*;*

(ii) $\lim_{t\downarrow 0} \varphi^2(t)\,\bar{q}_t = \lim_{t\downarrow 0} \psi(t) = 0$ *when* $\mu = \delta_0$*;*

(iii) $\lim_{t\downarrow 0} \varphi(t) = \lim_{t\downarrow 0} \psi^2(t)(1-\bar{q}_t) = 0$ *when* $\mu = \delta_1$*.*

Moreover, $\{h_t, \mu_t\}_{t\in(0,1]}$ *is a quadratic small uncertainty if and only if:*

(a) $\varphi(t) = O(t)$ *and* $\psi(t) = O(t)$ *when* $\mu \neq \delta_p$ *with* $p \in \{0,1\}$*;*

(b) $\varphi^2(t)\,\bar{q}_t = O(t^2)$ *and* $\psi(t) = O(t)$ *when* $\mu = \delta_0$*;*

(c) $\varphi(t) = O(t)$ *and* $\psi^2(t)(1-\bar{q}_t) = O(t^2)$ *when* $\mu = \delta_1$*.*

Proof It follows from Propositions 28, 36, and their uncertainty counterparts. ■

Lemma 55 *If* $\mu \neq \delta_p$ *with* $p \in \{0,1\}$*, then any small uncertainty* $\{h_t, \mu_t\}_{t\in(0,1]}$ *is controllable.*

Proof In fact, $\mu \neq \delta_0, \delta_1$ is equivalent to $\bar{q} \neq 0, 1$. Then $\bar{q}_t, \bar{q} \in (0,1)$ for all $t \in (0,1]$ which implies controllability by Proposition 31. ■

Proof of Lemma 47 If $\{h_t, \mu_t\}_{t\in(0,1]}$ is a controllable small uncertainty, there exists $\delta, M > 0$ such that, for all $t \in (0,\delta)$

$$\varphi^2(t) + \psi^2(t) \leq M\left(\varphi^2(t)\,\bar{q}_t + \psi^2(t)(1-\bar{q}_t)\right).$$

Since $\bar{q}_t \to 0$, there is $\varepsilon > 0$ such that $M\bar{q}_t < 1/2$ for all $t \in (0,\varepsilon)$. Then, for all $t \in (0, \delta \wedge \varepsilon)$,

$$\varphi^2(t)(1 - Mq_t) \leq \psi^2(t)(M(1-\bar{q}_t) - 1),\ 1 - Mq_t > \frac{1}{2}, \text{ and } M(1-\bar{q}_t) - 1 \leq M$$

so that

$$\frac{1}{2}\varphi^2(t) \leq \varphi^2(t)(1 - Mq_t) \leq \psi^2(t)(M(1-\bar{q}_t) - 1) \leq M\psi^2(t)$$

hence, $\varphi(t) = O(\psi(t))$.

Conversely, if $\varphi(t) = O(\psi(t))$, then there exist $\delta, M > 0$ such that

$$|\varphi(t)| \leq M|\psi(t)|$$

for all $t \in (0,\delta)$. This implies $\lim_{t\downarrow 0} \varphi(t) = 0$ and

$$\lim_{t\downarrow 0} \varphi^2(t)\,\bar{q}_t + \psi^2(t)(1-\bar{q}_t) = 0$$

that is, $\{h_t, \mu_t\}_{t\in(0,1]}$ is a small uncertainty. Moreover,

$$\frac{\varphi^2(t)\,\bar{q}_t + \psi^2(t)(1-\bar{q}_t)}{\varphi^2(t) + \psi^2(t)} = \frac{\bar{q}_t \frac{\varphi^2(t)}{\psi^2(t)} + 1 - \bar{q}_t}{\frac{\varphi^2(t)}{\psi^2(t)} + 1} \geq \frac{1-\bar{q}_t}{M^2+1} \to \frac{1}{M^2+1} > 0$$

and so $\{h_t, \mu_t\}_{t\in(0,1]}$ is a controllable small uncertainty by Proposition 31. ■

Proof of Proposition 48 For all $t \in (0,1]$,

$$\frac{\gamma^2(t)}{\varphi^2(t) + \psi^2(t)} \leq 2$$

hence $\gamma^2(t) = O\left(\varphi^2(t) + \psi^2(t)\right)$. But, by controllability, $\varphi^2(t) + \psi^2(t) = O\left(\|h_t\|^2_{\bar{q}_t}\right)$, thus $\gamma^2(t) = O\left(\|h_t\|^2_{\bar{q}_t}\right)$ and $\gamma(t) = O\left(\|h_t\|_{\bar{q}_t}\right)$.

Since $\mu_t \overset{sq}{\Longrightarrow} \mu$, then

$$\begin{aligned}\bar{q}_t - \vartheta_t &= \bar{q} - \vartheta + o(1)\\ \vartheta_t - \bar{q}_t^2 &= \vartheta - \bar{q}^2 + o(1)\end{aligned}$$

and so

$$\begin{aligned}\frac{1}{2}\lambda_u(w)(\bar{q}_t - \vartheta_t)\gamma^2(t) &= \frac{1}{2}\lambda_u(w)(\bar{q} - \vartheta)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right)\\ \frac{1}{2}\lambda_v(w)\left(\vartheta_t - \bar{q}_t^2\right)\gamma^2(t) &= \frac{1}{2}\lambda_v(w)\left(\vartheta - \bar{q}^2\right)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right)\end{aligned}$$

which together with (38) deliver (39). ■

Proof of Theorem 49 By Proposition 48 $\gamma(t) = O\left(\|h_t\|_{\bar{q}_t}\right)$, and, as observed in the proof of the same proposition,

$$\begin{aligned}E_{\mu_t}\left(\sigma^2(h_t)\right) &= (\bar{q}_t - \vartheta_t)\gamma^2(t) = (\bar{q} - \vartheta)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right)\\ \sigma^2_{\mu_t}(E(h_t)) &= \left(\vartheta_t - \bar{q}_t^2\right)\gamma^2(t) = \left(\vartheta - \bar{q}^2\right)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right).\end{aligned}$$

1. *If.* By Proposition 43, if $\operatorname{supp}\mu$ is neither a singleton nor $\{0,1\}$, then $\bar{q}^2 < \vartheta < \bar{q}$. Moreover, $\|h_t\|_{\bar{q}_t} = O(\gamma(t))$ together with $\gamma(t) = O\left(\|h_t\|_{\bar{q}_t}\right)$, implies $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$. Therefore $\gamma(t)^2 \asymp \|h_t\|^2_{\bar{q}_t}$, whence

$$\begin{aligned}(\bar{q} - \vartheta)\gamma^2(t) &\asymp \|h_t\|^2_{\bar{q}_t}\\ \left(\vartheta - \bar{q}^2\right)\gamma^2(t) &\asymp \|h_t\|^2_{\bar{q}_t}\end{aligned}$$

and so $(\bar{q} - \vartheta)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) \asymp \|h_t\|^2_{\bar{q}_t}$ and $\left(\vartheta - \bar{q}^2\right)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) \asymp \|h_t\|^2_{\bar{q}_t}$.
1. *Only if.* The assumption $E_{\mu_t}\left(\sigma^2(h_t)\right) \asymp \|h_t\|^2_{\bar{q}_t}$ and $\sigma^2_{\mu_t}(E(h_t)) \asymp \|h_t\|^2_{\bar{q}_t}$ implies

$$(\bar{q} - \vartheta)\gamma^2(t) \asymp \|h_t\|^2_{\bar{q}_t} \text{ and } \left(\vartheta - \bar{q}^2\right)\gamma^2(t) \asymp \|h_t\|^2_{\bar{q}_t}. \tag{48}$$

Therefore, $\bar{q}^2 < \vartheta < \bar{q}$ and by Proposition 43, $\operatorname{supp}\mu$ is neither a singleton nor $\{0,1\}$.

Moreover, (48) implies $\gamma^2(t) \asymp \|h_t\|^2_{\bar{q}_t}$, whence $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$.

2. *If.* By Proposition 43, $\bar{q}^2 = \vartheta < \bar{q}$, then $\sigma^2_{\mu_t}(E(h_t)) = \left(\vartheta - \bar{q}^2\right)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$. Moreover, $\|h_t\|_{\bar{q}_t} = O(\gamma(t))$ together with $\gamma(t) = O\left(\|h_t\|_{\bar{q}_t}\right)$, implies $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$. Therefore $\gamma(t)^2 \asymp \|h_t\|^2_{\bar{q}_t}$, whence

$$(\bar{q} - \vartheta)\gamma^2(t) \asymp \|h_t\|^2_{\bar{q}_t}$$

and so $E_{\mu_t}\left(\sigma^2(h_t)\right) \asymp \|h_t\|^2_{\bar{q}_t}$.
2. *Only if.* The assumption $E_{\mu_t}\left(\sigma^2(h_t)\right) \asymp \|h_t\|^2_{\bar{q}_t}$ implies

$$(\bar{q} - \vartheta)\gamma^2(t) \asymp \|h_t\|^2_{\bar{q}_t}.$$

Therefore, $\vartheta < \bar{q}$ and $\gamma^2(t) \asymp \|h_t\|^2_{\bar{q}_t}$, so that $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$.

Since $\sigma^2_{\mu_t}(E(h_t)) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$, it cannot be the case that $\bar{q}^2 < \vartheta$, because it would deliver

$$o\left(\|h_t\|^2_{\bar{q}_t}\right) = \sigma^2_{\mu_t}(E(h_t)) = \left(\vartheta - \bar{q}^2\right)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) \asymp \|h_t\|^2_{\bar{q}_t}$$

which is absurd. Then $\bar{q}^2 = \vartheta < \bar{q}$ and, by Proposition 43, $\operatorname{supp}\mu$ is a singleton in $(0,1)$.

3. *If.* By Proposition 43, $\bar{q}^2 < \vartheta = \bar{q}$, then $E_{\mu_t}\left(\sigma^2(h_t)\right) = (\bar{q} - \vartheta)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$. Moreover, $\|h_t\|_{\bar{q}_t} = O(\gamma(t))$ together with $\gamma(t) = O\left(\|h_t\|_{\bar{q}_t}\right)$, implies $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$. Therefore $\gamma(t)^2 \asymp \|h_t\|^2_{\bar{q}_t}$, whence

$$\left(\vartheta - \bar{q}^2\right)\gamma^2(t) \asymp \|h_t\|^2_{\bar{q}_t}$$

and so $\sigma^2_{\mu_t}(E(h_t)) \asymp \|h_t\|^2_{\bar{q}_t}$.

3. *Only if.* The assumption $\sigma^2_{\mu_t}(E(h_t)) \asymp \|h_t\|^2_{\bar{q}_t}$ implies

$$\left(\vartheta - \bar{q}^2\right)\gamma^2(t) \asymp \|h_t\|^2_{\bar{q}_t}.$$

Therefore, $\bar{q}^2 < \vartheta$ and $\gamma^2(t) \asymp \|h_t\|^2_{\bar{q}_t}$, so that $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$.

Since $E_{\mu_t}\left(\sigma^2(h_t)\right) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$, it cannot be the case that $\vartheta < \bar{q}$, because it would deliver

$$o\left(\|h_t\|^2_{\bar{q}_t}\right) = E_{\mu_t}\left(\sigma^2(h_t)\right) = (\bar{q} - \vartheta)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) \asymp \|h_t\|^2_{\bar{q}_t}$$

which is absurd. Then $\bar{q}^2 < \vartheta = \bar{q}$ and, by Proposition 43, $\operatorname{supp}\mu = \{0,1\}$.

4. *If.* If $\mu = \delta_0$ or $\mu = \delta_1$, by Proposition 43, $\bar{q}^2 = \vartheta = \bar{q}$, then $E_{\mu_t}\left(\sigma^2(h_t)\right) = (\bar{q} - \vartheta)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$ and $\sigma^2_{\mu_t}(E(h_t)) = \left(\vartheta - \bar{q}^2\right)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$. Else $\gamma(t) = o\left(\|h_t\|_{\bar{q}_t}\right)$, implies $k\gamma(t)^2 = o\left(\|h_t\|^2_{\bar{q}_t}\right)$ for all $k \geq 0$, whence $E_{\mu_t}\left(\sigma^2(h_t)\right) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$ and $\sigma^2_{\mu_t}(E(h_t)) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$.

4. *Only if.* The assumption $E_{\mu_t}\left(\sigma^2(h_t)\right) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$ and $\sigma^2_{\mu_t}(E(h_t)) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$ implies

$$\begin{aligned} o\left(\|h_t\|^2_{\bar{q}_t}\right) &= E_{\mu_t}\left(\sigma^2(h_t)\right) = (\bar{q} - \vartheta)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) \\ o\left(\|h_t\|^2_{\bar{q}_t}\right) &= \sigma^2_{\mu_t}(E(h_t)) = \left(\vartheta - \bar{q}^2\right)\gamma^2(t) + o\left(\|h_t\|^2_{\bar{q}_t}\right) \end{aligned}$$

whence

$$\begin{aligned} (\bar{q} - \vartheta)\gamma^2(t) &= o\left(\|h_t\|^2_{\bar{q}_t}\right) \\ \left(\vartheta - \bar{q}^2\right)\gamma^2(t) &= o\left(\|h_t\|^2_{\bar{q}_t}\right). \end{aligned}$$

Suppose, per contra, $\mu \neq \delta_0$ and $\mu \neq \delta_1$, and $\gamma(t) \neq o\left(\|h_t\|_{\bar{q}_t}\right)$. By Proposition 43, it cannot be the case that $\bar{q}^2 = \vartheta = \bar{q}$. Then either $\vartheta - \bar{q}^2 \neq 0$ or $\bar{q} - \vartheta \neq 0$, in any case $\gamma^2(t) = o\left(\|h_t\|^2_{\bar{q}_t}\right)$ and $\gamma(t) = o\left(\|h_t\|_{\bar{q}_t}\right)$, which is absurd. ■

Proof of Proposition 51 For all $t \in (0,1]$, $\operatorname{supp}\mu_t$ being closed in the compact $[0,1]$ is compact. Let $a_t = \min\operatorname{supp}\mu_t$ and $b_t = \max\operatorname{supp}\mu_t$. Then clearly $[a_t, b_t] \subseteq \operatorname{co}(\operatorname{supp}\mu_t)$, but also the converse inclusion is true since $[a_t, b_t]$ is convex and contains $\operatorname{supp}\mu_t$. Thus $[a_t, b_t] = \operatorname{co}(\operatorname{supp}\mu_t)$ and $\operatorname{diam}(\operatorname{supp}\mu_t) = \operatorname{diam}(\operatorname{co}(\operatorname{supp}\mu_t)) = b_t - a_t$.

By the compactness of $[0,1]$ and the finite intersection property,

$$\bigcap_{t\in(0,1]} [a_t, b_t] \neq \emptyset$$

and $b_t - a_t \to 0$ (as $t \downarrow 0$) implies that $\bigcap_{t\in(0,1]} [a_t, b_t]$ is actually a singleton $\{p\}$.[18]

Then $a_t \leq p \leq b_t$ and $b_t - a_t \to 0$ (as $t \downarrow 0$), imply $\lim_{t\downarrow 0} a_t = \lim_{t\downarrow 0} b_t = p$. For all $f \in C([0,1])$ and all $t > 0$, there exists $c_t \in [a_t, b_t]$ such that

$$\int_0^1 f d\mu_t = \int_{a_t}^{b_t} f d\mu_t = f(c_t).$$

Then $a_t \leq c_t \leq b_t$ implies $c_t \to p$ (as $t \downarrow 0$) and continuity of f delivers $f(c_t) \to f(p)$, that is,

$$\int f d\mu_t \to \int f d\delta_p.$$

Finally, $\mu_t \overset{w}{\Longrightarrow} \delta_p$. ■

B Taylor

B.1 Preliminaries

Let $\Omega = \{1, 2, ..., N\}$ be a finite state space $\Delta = \Delta(\Omega)$ the probability simplex. If μ is a Borel probability measure on Δ,

$$\bar{\mu} = \int_\Delta q d\mu(q) \in \Delta$$

is called *barycenter* of μ.

Let $I \subseteq \mathbb{R}$ be a nonsingleton interval and consider two functions functions $u : I \to \mathbb{R}$ and $v : I \to \mathbb{R}$ that are continuous on I and thrice continuously differentiable on its interior, with $u', v' > 0$. Set

$$C(\mathbf{x}, \mu) = v^{-1}\left(\int_\Delta v\left(u^{-1}\left(\sum_{\omega\in\Omega} u(x_\omega) q_\omega\right)\right) d\mu(q)\right) \qquad \forall \mathbf{x} \in I^\Omega.$$

Notice that for each $\mathbf{x} \in I^\Omega$, $u(\mathbf{x}) = \begin{bmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{bmatrix} \in u(I)^\Omega$ (and $u(I)$ is an interval since u is continuous), and

$$\begin{array}{rccl} \langle u(\mathbf{x}), \cdot\rangle : & \Delta & \to & \mathbb{R} \\ & q & \mapsto & \sum_{\omega\in\Omega} u(x_\omega) q_\omega = \langle u(x), q\rangle \end{array}$$

is affine, hence it is continuous, bounded, and measurable, with range in $u(I)$. Therefore,

$$\begin{array}{rccl} \left(v \circ u^{-1}\right) \circ \langle u(x), \cdot\rangle : & \Delta & \to & \mathbb{R} \\ & q & \mapsto & v\left(u^{-1}\left(\sum_{\omega\in\Omega} u(x_\omega) q_\omega\right)\right) \end{array}$$

is continuous, bounded, and measurable too, with range in $v(I)$. Hence,

$$\int_\Delta v\left(u^{-1}\left(\langle u(x), q\rangle\right)\right) d\mu(q) \in v(I)$$

[18] If $p', p'' \in \bigcap_{t\in(0,1]} [a_t, b_t]$, then $|p'' - p'| \leq b_t - a_t$ for all $t \in (0,1]$. Passing to the limit delivers $|p'' - p'| = 0$.

is well defined, and so is

$$v^{-1}\left(\int_{\Delta} v\left(u^{-1}\left(\langle u(x), q\rangle\right)\right) d\mu(q)\right) \in I.$$

The next two lemmas on derivatives will play a key role in the derivation of the Taylor approximation.

Lemma 56 *Let O be an open subset of a finite dimensional euclidean space, (S,Σ) be a measurable space, and $f : O \times S \to \mathbb{R}$ be a function with the following properties:*

(a) for each $\mathbf{x} \in O$, $s \mapsto f(\mathbf{x}, s)$ is Σ-measurable;

(b) for each $s \in S$, $\mathbf{x} \mapsto f(\mathbf{x}, s)$ is k-times continuously differentiable on O;

(c) the functions f and $\partial_\alpha f$ are bounded (hence uniformly bounded by M) on $O \times S$ for every multi-index α such that $|\alpha| \leq k$.

Then, for each probability measure P on Σ:

(i) the function defined on O by

$$F_P(\mathbf{x}) = \int_S f(\mathbf{x}, s)\, dP(s)$$

is k-times continuously differentiable;

(ii) the functions $s \mapsto \partial_\alpha f(\mathbf{x}, s)$ are Σ-measurable for all $\mathbf{x} \in O$, with

$$\partial_\alpha F_P(\mathbf{x}) = \int_S \partial_\alpha f(\mathbf{x}, s)\, dP(s) \tag{49}$$

for all $\mathbf{x} \in O$ and α such that $|\alpha| \leq k$.

Moreover, as $\mathbf{y}$ ranges in O, Q ranges in the set of all probability measures on Σ, β ranges in the set of all multi-indexes of length not greater than k, $\sup_{\mathbf{y},Q,\beta} |\partial_\beta F_Q(\mathbf{y})| \leq M$.

Proof. First observe that, by (a) and (c), for each $\mathbf{x} \in O$, $f(\mathbf{x}, \cdot)$ is bounded and measurable, then F_P is well defined on O for every P in the set $\Delta^\sigma(S,\Sigma)$ of all probability measures on Σ.

Let $k = 1$, and arbitrarily choose a multi-index α with $|\alpha| \leq 1$. All components of α except one, say the j-th, are zero, and $\alpha_j = 1$. Denote by $\mathbf{e}^j$ the j-th element of the canonical basis of the euclidean space.

For each $\mathbf{x} \in O$, there is $\eta = \eta_{\mathbf{x}} > 0$ such that $\mathbf{x}+t\mathbf{e}^j \in O$ for all $t \in (-\eta, \eta)$. For each vanishing sequence $\{t_n\}_{n\in\mathbb{N}} \in (-\eta, \eta) \setminus \{0\}$ of real numbers

$$\partial_j f(\mathbf{x}, s) = \lim_{n\to\infty} \frac{f\left(\mathbf{x}+t_n\mathbf{e}^j, s\right) - f(\mathbf{x}, s)}{t_n}$$

for all $s \in S$. Thus, for each $\mathbf{x} \in O$, $\partial_j f(\mathbf{x}, \cdot)$ is the statewise limit of a sequence $\left\{t_n^{-1}\left[f\left(\mathbf{x}+t_n\mathbf{e}^j, \cdot\right) - f(\mathbf{x}, \cdot)\right]\right\}_{n\in\mathbb{N}}$ of measurable functions, and it is measurable.

For each $\mathbf{x} \in O$, $t \in (-\eta_{\mathbf{x}}, \eta_{\mathbf{x}}) \setminus \{0\}$, and $s \in S$, by the Mean Value Theorem on the segment $[x_j, x_j + t]$ (if $t > 0$, or $[x_j + t, x_j]$ if $t < 0$), there is $\lambda \in (0, 1)$ such that

$$\frac{\left|f\left(\mathbf{x}+t\mathbf{e}^j, s\right) - f(\mathbf{x}, s)\right|}{|t|} = \left|\partial_j f\left(\mathbf{x}+\lambda t\mathbf{e}^j, s\right)\right| \leq M.$$

Then, for each $\mathbf{x} \in O$, and each vanishing sequence $\{t_n\}_{n\in\mathbb{N}} \in (-\eta_{\mathbf{x}}, \eta_{\mathbf{x}}) \setminus \{0\}$, the sequence of functions $\left\{t_n^{-1}\left[f\left(\mathbf{x}+t_n\mathbf{e}^j, \cdot\right) - g\left(\mathbf{x}, \cdot\right)\right]\right\}_{n\in\mathbb{N}}$ is uniformly bounded by M. The Dominated Convergence Theorem implies

$$\begin{aligned}\int \partial_j f\left(\mathbf{x}, s\right) dP\left(s\right) &= \int \lim_{n\to\infty} \frac{f\left(\mathbf{x}+t_n\mathbf{e}^j, s\right) - f\left(\mathbf{x}, s\right)}{t_n} dP\left(s\right) \\ \text{DCT} \quad &= \lim_{n\to\infty} \int \frac{f\left(\mathbf{x}+t_n\mathbf{e}^j, s\right) - f\left(\mathbf{x}, s\right)}{t_n} dP\left(s\right) \\ &= \lim_{n\to\infty} \frac{\int f\left(\mathbf{x}+t_n\mathbf{e}^j, s\right) dP\left(s\right) - \int f\left(\mathbf{x}, s\right) dP\left(s\right)}{t_n} \\ &= \lim_{n\to\infty} \frac{F_P\left(\mathbf{x}+t_n\mathbf{e}^j\right) - F_P\left(\mathbf{x}\right)}{t_n}\end{aligned}$$

for every $P \in \Delta^\sigma\left(S, \Sigma\right)$. Since this is true for all vanishing sequences $\{t_n\}_{n\in\mathbb{N}} \in (-\eta_{\mathbf{x}}, \eta_{\mathbf{x}}) \setminus \{0\}$, F_P has partial derivative at $\mathbf{x}$, in direction j, given by

$$\partial_j F_P\left(\mathbf{x}\right) = \int \partial_j f\left(\mathbf{x}, s\right) dP\left(s\right). \tag{50}$$

Summing up, $\partial_\alpha f\left(\mathbf{x}, \cdot\right)$ is measurable (and bounded) for all $\mathbf{x} \in O$, and, for each $P \in \Delta^\sigma\left(S, \Sigma\right)$, the function $F_P : O \to \mathbb{R}$ has α-th partial derivative given by (49) at every $\mathbf{x} \in O$. Moreover,

$$\sup_{x\in O, P\in\Delta^\sigma(S,\Sigma)} \left|\partial_\alpha F_P\left(\mathbf{x}\right)\right| = \sup_{x\in O, P\in\Delta^\sigma(S,\Sigma)} \left|\int \partial_\alpha f\left(\mathbf{x}, s\right) dP\left(s\right)\right| \leq \sup_{x\in O, P\in\Delta^\sigma(S,\Sigma)} \int \left|\partial_j f\left(\mathbf{x}, s\right)\right| dP\left(s\right) \leq M$$

and the generality of α implies $\sup_{x\in O, P\in\Delta^\sigma(S,\Sigma), |\alpha|\leq 1} \left|\partial_\alpha F_P\left(\mathbf{x}\right)\right| \leq M$.

Let $\mathbf{x}_n \to \mathbf{x}$ in O. For each $s \in S$, by (b),

$$\lim_{n\to\infty} \partial_\alpha f\left(\mathbf{x}_n, s\right) = \partial_\alpha f\left(\mathbf{x}, s\right)$$

then the sequence of measurable functions $\{\partial_\alpha f\left(\mathbf{x}_n, \cdot\right)\}_{n\in\mathbb{N}}$ statewise converges to $\partial_\alpha f\left(\mathbf{x}, \cdot\right)$ and, by (c), it is uniformly bounded, another application of the Dominated Convergence Theorem yields

$$\begin{aligned}\lim_{n\to\infty} \partial_\alpha F_P\left(\mathbf{x}_n\right) &= \lim_{n\to\infty} \int \partial_\alpha f\left(\mathbf{x}_n, s\right) dP\left(s\right) \\ \text{DCT} \quad &= \int \partial_\alpha f\left(\mathbf{x}, s\right) dP\left(s\right) = \partial_\alpha F_P\left(\mathbf{x}\right).\end{aligned}$$

This shows the continuity of $\partial_\alpha F_P$.

Assume the result is true for $k \in \mathbb{N}$. Next we show it holds for $k+1$. The induction hypothesis guarantees that points (i) and (ii) hold verbatim for all the derivatives of order not greater than k and $\sup_{x\in O, P\in\Delta^\sigma(S,\Sigma), |\alpha|\leq k} \left|\partial_\alpha F_P\left(\mathbf{x}\right)\right| \leq M$. Let $|\beta| = k+1$ and j be the first non-zero component of β, then $\beta = \alpha + \mathbf{e}^j$ where α is a multi-index of length k. Set

$$g\left(\mathbf{x}, s\right) = \partial_\alpha f\left(\mathbf{x}, s\right) \quad \forall \left(\mathbf{x}, s\right) \in O \times S.$$

By the induction hypothesis, and the assumptions on f:

- for each $\mathbf{x} \in O$, $s \mapsto g\left(\mathbf{x}, s\right) = \partial_\alpha f\left(\mathbf{x}, s\right)$ is measurable;
- for each $s \in S$, $\mathbf{x} \mapsto g\left(\mathbf{x}, s\right)$ is 1-time continuously differentiable on O, since $\mathbf{x} \mapsto f\left(\mathbf{x}, s\right)$ is $k+1$-times continuously differentiable on O;

- the function $g = \partial_\alpha f$ and all its first partial derivatives are uniformly bounded by M on $O \times S$.

Then (by the initial step), for each probability measure P on Σ, the function defined on O by

$$G_P(\mathbf{x}) = \int g(\mathbf{x}, s)\, dP(s) = \partial_\alpha F_P(\mathbf{x})$$

is 1-time continuously differentiable and the function $s \mapsto \partial_j g(\mathbf{x}, s) = \partial_\beta f(\mathbf{x}, s)$ is measurable for all $\mathbf{x} \in O$, with

$$\partial_j G_P(\mathbf{x}) = \int \partial_j g(\mathbf{x}, s)\, dP(s) \ \text{ i.e. } \partial_\beta F_P(\mathbf{x}) = \int \partial_\beta f(\mathbf{x}, s)\, dP(s)$$

for all $\mathbf{x} \in O$. Moreover, again by the initial step,

$$\sup_{x \in O, P \in \Delta^\sigma(S,\Sigma)} |\partial_\beta F_P(\mathbf{x})| = \sup_{x \in O, P \in \Delta^\sigma(S,\Sigma)} |\partial_j G_P(\mathbf{x})| \leq M$$

which concludes the proof, thanks to the generality of β. ■

Set $\phi = v \circ u^{-1} : u(I) \to v(I)$ and $\psi = v^{-1} : v(I) \to I$, clearly they are strictly increasing and continuous. Set $W = \operatorname{int} I$.

Lemma 57 *The functions $\phi : u(W) \to v(W)$ and $\psi : v(W) \to W$ are thrice continuously differentiable.*

In particular, for each $w \in W$,

$$\phi'(u(w)) = \frac{v'(w)}{u'(w)}, \quad \phi''(u(w)) = \frac{v''(w)}{u'(w)^2} - v'(w)\frac{u''(w)}{u'(w)^3}, \tag{52a}$$

$$\psi'(\phi(u(w))) = \frac{1}{v'(w)}, \quad \psi''(\phi(u(w))) = -\frac{v''(w)}{v'(w)^3}, \tag{52b}$$

and there exist $\varepsilon > 0$ such that $[w-\varepsilon, w+\varepsilon] \subseteq W$ and $M > 1$ such that the absolute values of u, v, ϕ, and ψ – as well as their first, second, and third derivatives – are bounded by M on $[w-\varepsilon, w+\varepsilon]$, $[w-\varepsilon, w+\varepsilon]$, $u([w-\varepsilon, w+\varepsilon])$, and $v([w-\varepsilon, w+\varepsilon])$, respectively.

Proof. Since u and v are strictly increasing and continuous, then $u(\operatorname{int} I) = \operatorname{int} u(I)$ and $v(\operatorname{int} I) = \operatorname{int} v(I)$, that is

$$u(W) = \operatorname{int} u(I) \ \text{ and } v(W) = \operatorname{int} v(I).$$

Moreover, $u, v \in \mathcal{C}^3(W)$ and $u', v' > 0$ imply $u^{-1} \in \mathcal{C}^3(u(W))$ and $(u^{-1})' > 0$, $\phi = v \circ u^{-1} \in \mathcal{C}^3(u(W))$ and $\phi' > 0$, $\psi = v^{-1} \in \mathcal{C}^3(v(W))$ and $\psi' > 0$. The relations (52a) and (52b) descend from the usual calculus of the derivatives of inverse functions and composite functions. The final part descends from $\mathcal{C}^3$ differentiability and the Weierstrass Theorem. ■

B.2 Approximation

Theorem 58 *Let $\mathbf{x} \in (-\varepsilon, \varepsilon)^\Omega$, then*

$$\left| C(w + \mathbf{x}, \mu) - \left[w + E_{\bar{\mu}}(\mathbf{x}) - \frac{1}{2}\lambda_u(w)\sigma^2_{\bar{\mu}}(\mathbf{x}) - \frac{1}{2}(\lambda_v(w) - \lambda_u(w))\sigma^2_\mu(\langle \mathbf{x}, \cdot \rangle) \right] \right| \leq L|\mathbf{x}|^3 \tag{53}$$

for all Borel probability measures μ on Δ, where ε and M are as in Lemma 57 and

$$L = \frac{125M^{13} + 75M^9 + 5M^5}{6} |\Omega|^{3/2}.$$

Before entering the proof's details notice that

$$\begin{aligned}
&-\frac{1}{2}\lambda_u(w)\,\sigma^2_{\bar{\mu}}(\mathbf{x})-\frac{1}{2}(\lambda_v(w)-\lambda_u(w))\,\sigma^2_{\mu}(\langle\mathbf{x},\cdot\rangle)\\
&=-\frac{1}{2}\lambda_u(w)\,\mathbf{x}^{\intercal}\left[\sigma_{\bar{\mu}}\left(\mathbf{e}^k,\mathbf{e}^j\right)\right]\mathbf{x}-\frac{1}{2}(\lambda_v(w)-\lambda_u(w))\,\mathbf{x}^{\intercal}\left[\sigma_{\mu}\left(\left\langle\mathbf{e}^k,\cdot\right\rangle,\left\langle\mathbf{e}^j,\cdot\right\rangle\right)\right]\mathbf{x}\\
&=\frac{1}{2}\mathbf{x}^{\intercal}\left[-\lambda_u(w)\,\sigma_{\bar{\mu}}\left(\mathbf{e}^k,\mathbf{e}^j\right)-(\lambda_v(w)-\lambda_u(w))\,\sigma_{\mu}\left(\left\langle\mathbf{e}^k,\cdot\right\rangle,\left\langle\mathbf{e}^j,\cdot\right\rangle\right)\right]\mathbf{x}\\
&=\frac{1}{2}\mathbf{x}^{\intercal}\left[(\lambda_u(w)-\lambda_v(w))\,\sigma_{\mu}\left(\left\langle\mathbf{e}^k,\cdot\right\rangle,\left\langle\mathbf{e}^j,\cdot\right\rangle\right)-\lambda_u(w)\,\sigma_{\bar{\mu}}\left(\mathbf{e}^k,\mathbf{e}^j\right)\right]\mathbf{x}=\maltese
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{\mu}\left(\left\langle\mathbf{e}^k,\cdot\right\rangle,\left\langle\mathbf{e}^j,\cdot\right\rangle\right)&=\int_{\Delta}\left\langle\mathbf{e}^k,q\right\rangle\left\langle\mathbf{e}^j,q\right\rangle d\mu(q)-\left(\int_{\Delta}\left\langle\mathbf{e}^k,q\right\rangle d\mu(q)\right)\left(\int_{\Delta}\left\langle\mathbf{e}^j,q\right\rangle d\mu(q)\right)\\
&=\int_{\Delta}q_kq_j d\mu(q)-\bar{\mu}_k\bar{\mu}_j\\
\sigma_{\bar{\mu}}\left(\mathbf{e}^k,\mathbf{e}^j\right)&=\sum_{i=1}^{N}\mathbf{e}^k_i\mathbf{e}^j_i\bar{\mu}_i-\left(\sum_{i=1}^{N}\mathbf{e}^k_i\bar{\mu}_i\right)\left(\sum_{i=1}^{N}\mathbf{e}^j_i\bar{\mu}_i\right)=\begin{cases}-\bar{\mu}_k\bar{\mu}_j & k\neq j\\ \bar{\mu}_j-\bar{\mu}_k\bar{\mu}_j & k=j\end{cases}=\delta_{kj}\bar{\mu}_j-\bar{\mu}_k\bar{\mu}_j
\end{aligned}$$

finally

$$\begin{aligned}
\maltese&=\frac{1}{2}\mathbf{x}^{\intercal}\left[(\lambda_u(w)-\lambda_v(w))\left(\int_{\Delta}q_kq_jd\mu(q)-\bar{\mu}_k\bar{\mu}_j\right)-\lambda_u(w)\left(\delta_{kj}\bar{\mu}_j-\bar{\mu}_k\bar{\mu}_j\right)\right]\mathbf{x}\\
&=\frac{1}{2}\mathbf{x}^{\intercal}\left[\lambda_u(w)\int_{\Delta}q_kq_jd\mu(q)-\lambda_u(w)\,\bar{\mu}_k\bar{\mu}_j-\lambda_v(w)\int_{\Delta}q_kq_jd\mu(q)+\lambda_v(w)\,\bar{\mu}_k\bar{\mu}_j-\lambda_u(w)\,\delta_{kj}\bar{\mu}_j+\lambda_u(w)\,\bar{\mu}_k\bar{\mu}_j\right]\mathbf{x}\\
&=\frac{1}{2}\mathbf{x}^{\intercal}\left[(\lambda_u(w)-\lambda_v(w))\int_{\Delta}q_kq_jd\mu(q)+\lambda_v(w)\,\bar{\mu}_k\bar{\mu}_j-\lambda_u(w)\,\delta_{kj}\bar{\mu}_j\right]\mathbf{x}
\end{aligned}$$

Proof. Notice that, for all $\mathbf{x}\in(-\varepsilon,\varepsilon)^{\Omega}=O$, $w\mathbf{1}+\mathbf{x}\in(w-\varepsilon,w+\varepsilon)^{\Omega}\subseteq I^{\Omega}$.

Define

$$\begin{aligned}
g:\ &O\times\Delta &&\to\ \mathbb{R}\\
&(\mathbf{x},q) &&\mapsto\ \langle u(w+\mathbf{x}),q\rangle=\textstyle\sum_{i=1}^{N}u(w+x_i)\,q_i\in(u(w-\varepsilon),u(w+\varepsilon))
\end{aligned}$$

next we show that g satisfies assumptions (a), (b), (c) of Lemma 56 for 3-times continuously differentiable functions.

(a) For each $\mathbf{x}\in O$, $q\mapsto g(\mathbf{x},q)$ is continuous, bounded, and measurable.

(b) For each $q\in\Omega$, $\mathbf{x}\mapsto g(\mathbf{x},q)$ is thrice continuously differentiable on O. In fact, given $q\in\Delta$, for all $\mathbf{x}\in O$ and all $j,k,l\in\{1,2,...,N\}$

$$\begin{aligned}
\partial_j g(\mathbf{x},q)&=q_ju'(w+x_j)\\
\partial_{kj} g(\mathbf{x},q)&=\begin{cases}0 & k\neq j\\ q_ju''(w+x_j) & k=j\end{cases}=q_ju''(w+x_j)\,\delta_{kj}\\
\partial_{lkj} g(\mathbf{x},q)&=\begin{cases}0 & k\neq j\\ 0 & l\neq k=j\\ q_ju'''(w+x_j) & l=k=j\end{cases}=q_ju'''(w+x_j)\,\delta_{lkj}
\end{aligned}$$

and the functions defined by the above equations (for fixed q and j,k,l) are continuous on O.

(c) The functions g, $\partial_j g$, $\partial_{kj} g$, and $\partial_{lkj} g$ are bounded on $O \times \Omega$ for all $j,k,l \in \{1,2,...,N\}$; in fact, given $j,k,l \in \{1,2,...,N\}$, for all $(\mathbf{x},q) \in O \times \Delta$ (choosing M like in Lemma 57)

$$\begin{aligned}
|g(\mathbf{x},q)| &= \left|\sum_{i=1}^N u(w+x_i)\, q_i\right| \leq \sum_{i=1}^N |u(w+x_i)|\, q_i \leq M \\
|\partial_j g(\mathbf{x},q)| &= |q_j u'(w+x_j)| \leq M \\
|\partial_{kj} g(\mathbf{x},q)| &= |q_j u''(w+x_j)\, \delta_{kj}| \leq M \\
|\partial_{lkj} g(\mathbf{x},q)| &= |q_j u'''(w+x_j)\, \delta_{lkj}| \leq M.
\end{aligned}$$

Consider

$$\begin{array}{rccl} f = \phi \circ g: & O \times \Delta & \to & \mathbb{R} \\ & (\mathbf{x},q) & \mapsto & \phi \circ g(\mathbf{x},q) \in (v(w-\varepsilon), v(w+\varepsilon)) \end{array}$$

next we show that f satisfies assumptions (a), (b), (c) of Lemma 56 for 3-times continuously differentiable functions.

(a) For each $\mathbf{x} \in O$, $q \mapsto \phi \circ g(\mathbf{x},q)$ is continuous, bounded, and measurable.

(b) For each $q \in \Omega$, $\mathbf{x} \mapsto \phi \circ g(\mathbf{x},q)$ is thrice continuously differentiable on O; in fact, given $q \in \Delta$, for all $\mathbf{x} \in O$ and all $j,k,l \in \{1,2,...,N\}$

$$\begin{aligned}
\partial_j(\phi\circ g)(\mathbf{x},q) &= \phi'(g(\mathbf{x},q))\, \partial_j g(\mathbf{x},q) \\
\partial_{kj}(\phi\circ g)(\mathbf{x},q) &= \phi''(g(\mathbf{x},q))\, \partial_k g(\mathbf{x},q)\, \partial_j g(\mathbf{x},q) + \phi'(g(\mathbf{x},q))\, \partial_{kj} g(\mathbf{x},q) \\
\partial_{lkj}(\phi\circ g)(\mathbf{x},q) &= \phi'''(g(\mathbf{x},q))\, \partial_l g(\mathbf{x},q)\, \partial_k g(\mathbf{x},q)\, \partial_j g(\mathbf{x},q) + \phi''(g(\mathbf{x},q))\, (\partial_{lk} g(\mathbf{x},q)\, \partial_j g(\mathbf{x},q) + \partial_k g(\mathbf{x},q)\, \partial_{lj} g(\mathbf{x},q)) \\
&\quad + \phi''(g(\mathbf{x},q))\, \partial_l g(\mathbf{x},q)\, \partial_{kj} g(\mathbf{x},q) + \phi'(g(\mathbf{x},q))\, \partial_{lkj} g(\mathbf{x},q) \\
&= \phi'''(g(\mathbf{x},q))\, \partial_l g(\mathbf{x},q)\, \partial_k g(\mathbf{x},q)\, \partial_j g(\mathbf{x},q) + \phi'(g(\mathbf{x},q))\, \partial_{lkj} g(\mathbf{x},q) \\
&\quad + \phi''(g(\mathbf{x},q))\, [\partial_j g(\mathbf{x},q)\, \partial_{lk} g(\mathbf{x},q) + \partial_k g(\mathbf{x},q)\, \partial_{lj} g(\mathbf{x},q) + \partial_l g(\mathbf{x},q)\, \partial_{kj} g(\mathbf{x},q)]
\end{aligned}$$

and the functions defined by the above equations (for fixed q and j,k,l) are continuous on O.

(c) The functions $\phi \circ g$, $\partial_j(\phi\circ g)$, $\partial_{kj}(\phi\circ g)$, and $\partial_{lkj}(\phi\circ g)$ are bounded by $5M^4$ on $O \times \Omega$ for all $j,k,l \in \{1,2,...,N\}$; in fact, given $j,k,l \in \{1,2,...,N\}$, for all $(\mathbf{x},q) \in O \times \Delta$ (choosing M like in Lemma 57 and observing $g(\mathbf{x},q) \in (u(w-\varepsilon), u(w+\varepsilon))$)

$$|(\phi\circ g)(\mathbf{x},q)| = |\phi(g(\mathbf{x},q))| \leq M \leq 5M^4$$

$$|\partial_j(\phi\circ g)(\mathbf{x},q)| = |\phi'(g(\mathbf{x},q))|\, |\partial_j g(\mathbf{x},q)| \leq M^2 \leq 5M^4$$

$$|\partial_{kj}(\phi\circ g)(\mathbf{x},q)| \leq |\phi''(g(\mathbf{x},q))\, \partial_k g(\mathbf{x},q)\, \partial_j g(\mathbf{x},q)| + |\phi'(g(\mathbf{x},q))\, \partial_{kj} g(\mathbf{x},q)| \leq M^3 + M^2 \leq 5M^4$$

$$\begin{aligned}
|\partial_{lkj}(\phi\circ g)(\mathbf{x},q)| &\leq |\phi'''(g(\mathbf{x},q))\, \partial_l g(\mathbf{x},q)\, \partial_k g(\mathbf{x},q)\, \partial_j g(\mathbf{x},q)| + |\phi'(g(\mathbf{x},q))\, \partial_{lkj} g(\mathbf{x},q)| \\
&\quad + |\phi''(g(\mathbf{x},q))|\, [|\partial_j g(\mathbf{x},q)\, \partial_{lk} g(\mathbf{x},q)| + |\partial_k g(\mathbf{x},q)\, \partial_{lj} g(\mathbf{x},q)| + |\partial_l g(\mathbf{x},q)\, \partial_{kj} g(\mathbf{x},q)|] \\
&\leq M^4 + M^2 + 3M^3 \leq 5M^4.
\end{aligned}$$

Then (by Lemma 56), for each Borel probability measure μ on Δ the function defined on O by

$$F_\mu(\mathbf{x}) = F(\mathbf{x}) = \int f(\mathbf{x},q)\, d\mu(q) = \int_\Delta \phi\left(\sum_{i=1}^N u(w+x_i)\, q_i\right) d\mu(q)$$

is thrice continuously differentiable; the functions $q \mapsto \partial_\alpha f(\mathbf{x}, q)$ are measurable for all $\mathbf{x} \in O$, with

$$\partial_\alpha F(\mathbf{x}) = \int \partial_\alpha f(\mathbf{x}, q) \, d\mu(q)$$

for all $\mathbf{x} \in O$ and α such that $|\alpha| \leq 3$; $\sup_{\mathbf{x}\in O, \mu\in\Delta^\sigma(\Delta,\mathcal{B}), |\alpha|\leq 3} |\partial_\alpha F_\mu(\mathbf{x})| \leq 5M^4$.

Finally, for all $\mathbf{x} \in O$ and all $q \in \Delta$, $g(\mathbf{x}, q) \in (u(x-\varepsilon), u(x+\varepsilon))$ implies $f(\mathbf{x}, q) = \phi(g(\mathbf{x}, q)) \in v\left(u^{-1}\left((u(w-\varepsilon), u(w+\varepsilon))\right)\right) = (v(w-\varepsilon), v(w+\varepsilon))$ and $F(\mathbf{x}) \in (v(w-\varepsilon), v(w+\varepsilon))$.[19] Thus

$$c_\mu(\mathbf{x}) = \psi \circ F_\mu(\mathbf{x}) \in (w-\varepsilon, w+\varepsilon) \quad \forall \mathbf{x} \in O$$

is well defined and thrice continuously differentiable on $O = (-\varepsilon, \varepsilon)^N$.

Next we explicitly compute its derivatives, using repeatedly the relations obtained above as well as those provided by Lemma 57. As we did for F we just write c rather than c_μ.

For all $\mathbf{x} \in O$ and all $j, k, l \in \{1, 2, ..., N\}$

$$\begin{aligned}
\partial_j c(\mathbf{x}) &= \partial_j (\psi \circ F)(\mathbf{x}) = \psi'(F(\mathbf{x}))\, \partial_j F(\mathbf{x}) \\
\partial_{kj} c(\mathbf{x}) &= \partial_{kj} (\psi \circ F)(\mathbf{x}) = \psi''(F(\mathbf{x}))\, \partial_k F(\mathbf{x})\, \partial_j F(\mathbf{x}) + \psi'(F(\mathbf{x}))\, \partial_{kj} F(\mathbf{x}) \\
\partial_{lkj} c(\mathbf{x}) &= \partial_{lkj} (\psi \circ F)(\mathbf{x}) = \psi'''(F(\mathbf{x}))\, \partial_l F(\mathbf{x})\, \partial_k F(\mathbf{x})\, \partial_j F(\mathbf{x}) + \psi'(F(\mathbf{x}))\, \partial_{lkj} F(\mathbf{x}) \\
&\quad + \psi''(F(\mathbf{x}))\left[\partial_j F(\mathbf{x})\, \partial_{lk} F(\mathbf{x}) + \partial_k F(\mathbf{x})\, \partial_{lj} F(\mathbf{x}) + \partial_l F(\mathbf{x})\, \partial_{kj} F(\mathbf{x})\right]
\end{aligned}$$

in particular

$$|c(\mathbf{x})| = |\psi(F(\mathbf{x}))| \leq M \leq 205M^{13}$$

$$|\partial_j c(\mathbf{x})| = |\psi'(F(\mathbf{x}))|\, |\partial_j F(\mathbf{x})| \leq 5M^5 \leq 205M^{13}$$

$$|\partial_{kj} c(\mathbf{x})| \leq |\psi''(F(\mathbf{x}))\, \partial_k F(\mathbf{x})\, \partial_j F(\mathbf{x})| + |\psi'(F(\mathbf{x}))\, \partial_{kj} F(\mathbf{x})| \leq 25M^9 + 5M^5 \leq 205M^{13}$$

$$\begin{aligned}
|\partial_{lkj} c(\mathbf{x})| &\leq |\psi'''(F(\mathbf{x}))\, \partial_l F(\mathbf{x})\, \partial_k F(\mathbf{x})\, \partial_j F(\mathbf{x})| + |\psi'(F(\mathbf{x}))\, \partial_{lkj} F(\mathbf{x})| \\
&\quad + |\psi''(F(\mathbf{x}))|\left[|\partial_j F(\mathbf{x})\, \partial_{lk} F(\mathbf{x})| + |\partial_k F(\mathbf{x})\, \partial_{lj} F(\mathbf{x})| + |\partial_l F(\mathbf{x})\, \partial_{kj} F(\mathbf{x})|\right] \\
&\leq 125M^{13} + 5M^5 + 75M^9 \leq 205M^{13}
\end{aligned}$$

by Taylor's inequality (see, e.g., [2, p. 95])

$$\begin{aligned}
\left| c(\mathbf{x}) - \left[c(\mathbf{0}) + \nabla c(\mathbf{0})\, \mathbf{x} + \frac{1}{2} \mathbf{x}^\intercal \nabla^2 c(\mathbf{0})\, \mathbf{x} \right] \right| &\leq \frac{125M^{13} + 75M^9 + 5M^5}{6} \left(\sum_{i=1}^N |x_i| \right)^3 \qquad (55) \\
&\leq \frac{125M^{13} + 75M^9 + 5M^5}{6} N^{3/2} |\mathbf{x}|^3 \\
&= L |\mathbf{x}|^3
\end{aligned}$$

Notice that L does not depend on μ.

Since

$$c(\mathbf{x}) = v^{-1}\left(\int_\Delta v\left(u^{-1}\left(\sum_{i=1}^N u(w + x_i)\, q_i \right) \right) d\mu(q) \right) = C(w + \mathbf{x}, \mu) \quad \forall \mathbf{x} \in O,$$

it only remains to explicitly evaluate $c(\mathbf{0})$, $\nabla c(\mathbf{0})\, \mathbf{x}$, and $\mathbf{x}^\intercal \nabla^2 c(\mathbf{0})\, \mathbf{x}$.

[19] Clearly, $F(\mathbf{x}) \in [v(w-\varepsilon), v(w+\varepsilon)]$, if $F(\mathbf{x}) = v(w-\varepsilon)$, then $\int (f(\mathbf{x}, q) - v(w-\varepsilon))\, d\mu(q) = 0$ and $f(\mathbf{x}, q) = v(w-\varepsilon)$ for μ almost all $q \in \Delta$, a contradiction. A similar contradiction derives from assuming $F(\mathbf{x}) = v(w+\varepsilon)$.

First,

$$\begin{aligned} g\left(\mathbf{0},q\right) &= u\left(w\right) \quad \forall q\in\Delta \\ F\left(\mathbf{0}\right) &= \int \phi\left(g\left(\mathbf{0},q\right)\right) d\mu\left(q\right) = \phi\left(u\left(w\right)\right) \end{aligned}$$

then

$$c\left(\mathbf{0}\right) = v^{-1}\left(\phi\left(u\left(w\right)\right)\right) = w.$$

Second, for all $j\in\{1,2,...,N\}$,

$$\begin{aligned} \partial_j g\left(\mathbf{0},q\right) &= q_j u'\left(w\right) \quad \forall q\in\Delta \\ \partial_j F\left(\mathbf{0}\right) &= \int \partial_j f\left(\mathbf{0},q\right) d\mu\left(q\right) = \int \phi'\left(g\left(\mathbf{0},q\right)\right)\partial_j g\left(\mathbf{0},q\right) d\mu\left(q\right) \\ &= \int \phi'\left(u\left(w\right)\right) q_j u'\left(w\right) d\mu\left(q\right) = \phi'\left(u\left(w\right)\right) u'\left(w\right)\bar{\mu}_j \\ &= v'\left(w\right)\bar{\mu}_j \end{aligned}$$

then

$$\partial_j c\left(\mathbf{0}\right) = \psi'\left(F\left(\mathbf{0}\right)\right)\partial_j F\left(\mathbf{0}\right) = \psi'\left(\phi\left(u\left(w\right)\right)\right) v'\left(w\right)\bar{\mu}_j = \bar{\mu}_j.$$

Finally, for all $j,k\in\{1,2,...,N\}$,

$$\begin{aligned} \partial_{kj} g\left(\mathbf{0},q\right) &= q_j u''\left(w\right)\delta_{kj} \quad \forall q\in\Delta \\ \partial_{kj} F\left(\mathbf{0}\right) &= \int \partial_{kj} f\left(\mathbf{0},q\right) d\mu\left(q\right) = \int \left[\phi''\left(g\left(\mathbf{0},q\right)\right)\partial_k g\left(\mathbf{0},q\right)\partial_j g\left(\mathbf{0},q\right) + \phi'\left(g\left(\mathbf{0},q\right)\right)\partial_{kj} g\left(\mathbf{0},q\right)\right] d\mu\left(q\right) \\ &= \int \left[\phi''\left(u\left(w\right)\right) q_k u'\left(w\right) q_j u'\left(w\right) + \phi'\left(u\left(w\right)\right) q_j u''\left(w\right)\delta_{kj}\right] d\mu\left(q\right) \\ &= \phi''\left(u\left(w\right)\right) u'\left(w\right)^2 \int q_k q_j d\mu\left(q\right) + \phi'\left(u\left(w\right)\right) u''\left(w\right)\delta_{kj}\int q_j d\mu\left(q\right) \\ &= \phi''\left(u\left(w\right)\right) u'\left(w\right)^2 \int q_k q_j d\mu\left(q\right) + \phi'\left(u\left(w\right)\right) u''\left(w\right)\delta_{kj}\bar{\mu}_j \end{aligned}$$

then

$$\begin{aligned} \partial_{kj} c\left(\mathbf{0}\right) &= \psi''\left(F\left(\mathbf{0}\right)\right)\partial_k F\left(\mathbf{0}\right)\partial_j F\left(\mathbf{0}\right) + \psi'\left(F\left(\mathbf{0}\right)\right)\partial_{kj} F\left(\mathbf{0}\right) \\ &= \psi''\left(\phi\left(u\left(w\right)\right)\right) v'\left(w\right)\bar{\mu}_k v'\left(w\right)\bar{\mu}_j + \psi'\left(\phi\left(u\left(w\right)\right)\right)\left(\phi''\left(u\left(w\right)\right) u'\left(w\right)^2 \int q_k q_j d\mu\left(q\right) + \phi'\left(u\left(w\right)\right) u''\left(w\right)\delta_{kj}\bar{\mu}_j\right) \\ &= -\frac{v''\left(w\right)}{v'\left(w\right)^3} v'\left(w\right)^2\bar{\mu}_k\bar{\mu}_j + \frac{1}{v'\left(w\right)}\left(\left(\frac{v''\left(w\right)}{u'\left(w\right)^2} - v'\left(w\right)\frac{u''\left(w\right)}{u'\left(w\right)^3}\right) u'\left(w\right)^2 \int q_k q_j d\mu\left(q\right) + \frac{v'\left(w\right)}{u'\left(w\right)} u''\left(w\right)\delta_{kj}\bar{\mu}_j\right) \\ &= -\frac{v''\left(w\right)}{v'\left(w\right)}\bar{\mu}_k\bar{\mu}_j + \left(\frac{v''\left(w\right)}{v'\left(w\right)} - \frac{u''\left(w\right)}{u'\left(w\right)}\right)\int q_k q_j d\mu\left(q\right) + \frac{u''\left(w\right)}{u'\left(w\right)}\delta_{kj}\bar{\mu}_j \\ &= \left(\lambda_u\left(w\right) - \lambda_v\left(w\right)\right)\int q_k q_j d\mu\left(q\right) + \lambda_v\left(w\right)\bar{\mu}_k\bar{\mu}_j - \lambda_u\left(w\right)\delta_{kj}\bar{\mu}_j \end{aligned}$$

as wanted. In fact

$$\left| c\left(\mathbf{x}\right) - \left[c\left(\mathbf{0}\right) + \nabla c\left(\mathbf{0}\right)\mathbf{x} + \frac{1}{2}\mathbf{x}^{\intercal}\nabla^2 c\left(\mathbf{0}\right)\mathbf{x}\right]\right| \le L\left|\mathbf{x}\right|^3$$

then amounts to

$$\left| C\left(w+\mathbf{x},\mu\right) - \left[w + \left\langle \mathbf{x},\bar{\mu}\right\rangle + \frac{1}{2}\mathbf{x}^{\intercal}\left[\left(\lambda_u\left(w\right) - \lambda_v\left(w\right)\right)\int_{\Delta} q_k q_j d\mu\left(q\right) + \lambda_v\left(w\right)\bar{\mu}_k\bar{\mu}_j - \lambda_u\left(w\right)\delta_{kj}\bar{\mu}_j\right]\mathbf{x}\right]\right| \le L\left|\mathbf{x}\right|^3$$

that is (53). ■

B.3 Other Approximations

For all $\mathbf{x} \in \mathbb{R}^{\Omega}$ such that $w + \mathbf{x} \in I^{\Omega}$ and each Borel probability measure μ on Δ, set

$$R_2(\mathbf{x}, \mu) = C(w + \mathbf{x}, \mu) - \left[w + E_{\bar{\mu}}(\mathbf{x}) - \frac{1}{2}\lambda_u(w)\,\sigma^2_{\bar{\mu}}(\mathbf{x}) - \frac{1}{2}(\lambda_v(w) - \lambda_u(w))\,\sigma^2_{\mu}(\langle \mathbf{x}, \cdot \rangle) \right].$$

Equation (55) in the proof of Theorem 58 shows that if $\mathbf{x} \in (-\varepsilon, \varepsilon)$, then

$$|R_2(\mathbf{x}, \mu)| \leq \frac{125M^{13} + 75M^9 + 5M^5}{6} \left(\sum_{\omega \in \Omega} |x_\omega| \right)^3 = \frac{L}{|\Omega|^{3/2}} \left(\sum_{\omega \in \Omega} |x_\omega| \right)^3. \tag{56}$$

Corollary 59 *For each Borel probability measure μ on Δ, and each $\mathbf{x} \in \mathbb{R}^{\Omega}$ such that $w + \mathbf{x} \in I^{\Omega}$ and $\max_{\omega \in \operatorname{supp} \bar{\mu}} |x_\omega| < \varepsilon$, then*

$$|R_2(\mathbf{x}, \mu)| \leq \frac{L}{\bar{m}^{3/2}} \|\mathbf{x}\|^3_{2,\bar{\mu}} \tag{57}$$

where $\bar{m} = \min\{\bar{\mu}_\omega : \omega \in \operatorname{supp} \bar{\mu}\}$, and

$$|R_2(\mathbf{x}, \mu)| \leq L\,|\Omega|^{3/2} \|\mathbf{x}\|^3_{\infty,\bar{\mu}}. \tag{58}$$

The proof builds on the following fact.

Proposition 60 *Let $\mathbf{y}, \mathbf{z} \in \mathbb{R}^{\Omega}$, then*

1. $\mathbf{y}_{|\operatorname{supp} \bar{\mu}} = \mathbf{z}_{|\operatorname{supp} \bar{\mu}} \Leftrightarrow \mathbf{y} = \mathbf{z}$ *$\bar{\mu}$-a.e. on Ω;*
2. $\mathbf{y}_{|\operatorname{supp} \bar{\mu}} = \mathbf{z}_{|\operatorname{supp} \bar{\mu}} \Rightarrow E_{\bar{\mu}}(\mathbf{y}) = E_{\bar{\mu}}(\mathbf{z})$ *and* $\sigma^2_{\bar{\mu}}(\mathbf{y}) = \sigma^2_{\bar{\mu}}(\mathbf{z})$*;*
3. $\mathbf{y}_{|\operatorname{supp} \bar{\mu}} = \mathbf{z}_{|\operatorname{supp} \bar{\mu}} \Rightarrow \langle \mathbf{y}, \cdot \rangle = \langle \mathbf{z}, \cdot \rangle$ *μ-a.e. on Δ;*
4. $\mathbf{y}_{|\operatorname{supp} \bar{\mu}} = \mathbf{z}_{|\operatorname{supp} \bar{\mu}} \Rightarrow \sigma^2_{\mu}(\langle \mathbf{y}, \cdot \rangle) = \sigma^2_{\mu}(\langle \mathbf{z}, \cdot \rangle)$ *and, provided $w + \mathbf{x} \in I^{\Omega}$ for $\mathbf{x} = \mathbf{y}, \mathbf{z}$, $C(w + \mathbf{y}, \mu) = C(w + \mathbf{z}, \mu)$.*

Proof. 1. and 2. are trivial.

3. For each $\omega \in \Omega$, $\bar{\mu}_\omega = 0$ is equivalent to $\int_\Delta \langle \mathbf{e}^\omega, q \rangle \, d\mu(q) = 0$, which in turn is equivalent to

$$\langle \mathbf{e}^\omega, \cdot \rangle = 0 \quad \mu\text{-a.e. on } \Delta.$$

Thus, $\langle \mathbf{y}, \cdot \rangle = \sum_{\omega \in \Omega} y_\omega \langle \mathbf{e}^\omega, \cdot \rangle = \sum_{\omega \in \operatorname{supp} \bar{\mu}} y_\omega \langle \mathbf{e}^\omega, \cdot \rangle + \sum_{\omega \in \Omega \setminus \operatorname{supp} \bar{\mu}} y_\omega \langle \mathbf{e}^\omega, \cdot \rangle \overset{\mu-\text{a.e.}}{=} \sum_{\omega \in \operatorname{supp} \bar{\mu}} y_\omega \langle \mathbf{e}^\omega, \cdot \rangle$ $= \sum_{\omega \in \operatorname{supp} \bar{\mu}} z_\omega \langle \mathbf{e}^\omega, \cdot \rangle \overset{\mu-\text{a.e.}}{=} \sum_{\omega \in \operatorname{supp} \bar{\mu}} z_\omega \langle \mathbf{e}^\omega, \cdot \rangle + \sum_{\omega \in \Omega \setminus \operatorname{supp} \bar{\mu}} z_\omega \langle \mathbf{e}^\omega, \cdot \rangle = \langle \mathbf{z}, \cdot \rangle$.

4. The equality $\sigma^2_{\mu}(\langle \mathbf{y}, \cdot \rangle) = \sigma^2_{\mu}(\langle \mathbf{z}, \cdot \rangle)$ follows from 3. while, setting $u(w + \mathbf{x}) = [u(w + x_\omega)]_{\omega \in \Omega}$ for $\mathbf{x} = \mathbf{y}, \mathbf{z}$,

$$\begin{aligned}
\mathbf{y}_{|\operatorname{supp} \bar{\mu}} = \mathbf{z}_{|\operatorname{supp} \bar{\mu}} &\Longrightarrow u(w + \mathbf{y})_{|\operatorname{supp} \bar{\mu}} = u(w + \mathbf{z})_{|\operatorname{supp} \bar{\mu}} \\
&\Longrightarrow \langle u(w + \mathbf{y}), \cdot \rangle = \langle u(w + \mathbf{z}), \cdot \rangle \quad \mu\text{-a.e. on } \Delta \\
&\Longrightarrow v\left(u^{-1}(\langle u(w + \mathbf{y}), \cdot \rangle)\right) = v\left(u^{-1}(\langle u(w + \mathbf{z}), \cdot \rangle)\right) \quad \mu\text{-a.e. on } \Delta \\
&\Longrightarrow \int_\Delta v\left(u^{-1}(\langle u(w + \mathbf{y}), q \rangle)\right) d\mu(q) = \int_\Delta v\left(u^{-1}(\langle u(w + \mathbf{z}), q \rangle)\right) d\mu(q) \\
&\Longrightarrow C(w + \mathbf{y}, \mu) = C(w + \mathbf{z}, \mu)
\end{aligned}$$

where the second implication, again follows from 3. ∎

Proof of Corollary 59. The vector $\mathbf{x}^{\bar{\mu}} = \mathbf{x}_{\operatorname{supp}\bar{\mu}}\mathbf{0}$ (which coincides with $\mathbf{x}$ on $\operatorname{supp}\bar{\mu}$ and with $\mathbf{0}$ on $\Omega \setminus \operatorname{supp}\bar{\mu}$) belongs to $(-\varepsilon,\varepsilon)^{\Omega}$, and obviously $\mathbf{x}$ and $\mathbf{x}^{\bar{\mu}}$ coincide on $\operatorname{supp}\bar{\mu}$. Thus Proposition 60 and (56) deliver

$$\begin{aligned}
|R_2(\mathbf{x},\mu)| &= \left|C(w+\mathbf{x},\mu) - \left[w + E_{\bar{\mu}}(\mathbf{x}) - \frac{1}{2}\lambda_u(w)\sigma^2_{\bar{\mu}}(\mathbf{x}) - \frac{1}{2}(\lambda_v(w)-\lambda_u(w))\sigma^2_{\mu}(\langle \mathbf{x},\cdot\rangle)\right]\right| \\
&= \left|C\left(w+\mathbf{x}^{\bar{\mu}},\mu\right) - \left[w + E_{\bar{\mu}}\left(\mathbf{x}^{\bar{\mu}}\right) - \frac{1}{2}\lambda_u(w)\sigma^2_{\bar{\mu}}\left(\mathbf{x}^{\bar{\mu}}\right) - \frac{1}{2}(\lambda_v(w)-\lambda_u(w))\sigma^2_{\mu}\left(\langle \mathbf{x}^{\bar{\mu}},\cdot\rangle\right)\right]\right| \\
&\leq \frac{L}{|\Omega|^{3/2}}\left(\sum_{\omega\in\Omega}\left|x^{\bar{\mu}}_{\omega}\right|\right)^3 = \frac{L}{|\Omega|^{3/2}}\left(\sum_{\omega\in\operatorname{supp}\bar{\mu}}|x_{\omega}|\right)^3.
\end{aligned}$$

In particular,

- setting $\bar{m} = \min\{\bar{\mu}_{\omega} : \omega \in \operatorname{supp}\bar{\mu}\}$,

  $$\sum_{\omega\in\operatorname{supp}\bar{\mu}}|x_{\omega}| \leq |\operatorname{supp}\bar{\mu}|^{1/2}\sqrt{\sum_{\omega\in\operatorname{supp}\bar{\mu}}x^2_{\omega}} \leq |\Omega|^{1/2}\sqrt{\sum_{\omega\in\operatorname{supp}\bar{\mu}}\frac{\bar{\mu}_i}{\bar{m}}x_i^2} \leq \frac{|\Omega|^{1/2}}{\bar{m}^{1/2}}\|\mathbf{x}\|_{2,\bar{\mu}}$$

  thus

  $$|R_2(\mathbf{x},\mu)| \leq \frac{L}{|\Omega|^{3/2}}\frac{|\Omega|^{3/2}}{\bar{m}^{3/2}}\|\mathbf{x}\|^3_{2,\bar{\mu}}$$

  which is (57);

- analogously

  $$\sum_{\omega\in\operatorname{supp}\bar{\mu}}|x_{\omega}| \leq \sum_{\omega\in\operatorname{supp}\bar{\mu}}\max_{\omega'\in\operatorname{supp}\bar{\mu}}|x_{\omega'}| = \|\mathbf{x}\|_{\infty,\bar{\mu}}|\operatorname{supp}\bar{\mu}| \leq |\Omega|\,\|\mathbf{x}\|_{\infty,\bar{\mu}}$$

  thus

  $$|R_2(\mathbf{x},\mu)| \leq \frac{L}{|\Omega|^{3/2}}|\Omega|^3\|\mathbf{x}\|^3_{\infty,\bar{\mu}}$$

  which is (58).

As wanted. ■

Now assume there is a net $(\mathbf{x}_t,\mu_t)_{t>0}$ such that eventually $w+\mathbf{x}_t \in I^{\Omega}$ and $\|\mathbf{x}_t\|_{\infty,\bar{\mu}_t} \to 0$ as $t \downarrow 0$, then eventually $\max_{\omega\in\operatorname{supp}\bar{\mu}}|x_{\omega}| < \varepsilon$, by (58),

$$|R_2(\mathbf{x}_t,\mu_t)| \leq L|\Omega|^{3/2}\|\mathbf{x}_t\|^3_{\infty,\bar{\mu}_t}$$

and $R_2(\mathbf{x}_t,\mu_t) = O\left(\|\mathbf{x}_t\|^3_{\infty,\bar{\mu}_t}\right) = o\left(\|\mathbf{x}_t\|^2_{\infty,\bar{\mu}_t}\right)$.

It is important to observe that (57) cannot be directly used to obtain a similar result, in fact, in this case, $\|\mathbf{x}_t\|_{2,\bar{\mu}_t} \to 0$ as $t \downarrow 0$ and (57) delivers

$$|R_2(\mathbf{x}_t,\mu_t)| \leq \frac{L}{\bar{m}_t^{3/2}}\|\mathbf{x}_t\|^3_{2,\bar{\mu}_t}$$

which does not allow to conclude $R_2(\mathbf{x}_t,\mu_t) = O\left(\|\mathbf{x}_t\|^3_{2,\bar{\mu}_t}\right)$ unless $\bar{m}_t^{3/2}$ is bounded away from 0.

Corollary 61 *Assume the net $(\mathbf{x}_t,\mu_t)_{t>0}$ is such that eventually $w+\mathbf{x}_t \in I^{\Omega}$, $\|\mathbf{x}_t\|_{2,\bar{\mu}_t} \to 0$ and $\|\mathbf{x}_t\|_{\infty,\bar{\mu}_t} = O\left(\|\mathbf{x}_t\|_{2,\bar{\mu}_t}\right)$ as $t \downarrow 0$, then $R_2(\mathbf{x}_t,\mu_t) = O\left(\|\mathbf{x}_t\|^3_{2,\bar{\mu}_t}\right)$.*

The simple proof is omitted.

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