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Firm Migration and Stock Returns*

Giovanni W. Puopolo†

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Abstract

This paper studies the asset pricing implications of a general equilibrium model in which real investment is reversible at a cost. Firms face higher costs in contracting than in expanding their capital stock and decide to invest when their productive capital is scarce relative to the overall capital of the economy. Positive shocks to the production process of the firm increase the size of the firm and reduce the value of growth options. As a result, the firm is burdened with more unproductive capital and its value lowers with respect to the accumulated capital. The optimal consumption policy alters the optimal allocation of resources and affects firm's value, generating mean-reverting

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dynamics for the M/B ratios. The model (1) captures convergence of price-to-book ratios - negative for growth stocks and positive for value stocks - (firm migration), (2) generates deviations from the classic CAPM in line with the cross-sectional variation in expected stock returns and (3) generates a non-monotone relationship between Tobin's q and conditional volatility consistent with the empirical evidence.

JEL Classifications: G12, D92, D51, D21, D24.

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1 Introduction

Recent empirical studies have shown that firms with high book-to-market ratios earn on average higher returns than firms characterized by low book-to-market ratios (Fama and French (1992)). The standard Capital Asset Pricing Model (CAPM) predicts that market beta captures all the non-diversifiable risk, and thus fails to explain this regularity in the cross-section of stock returns. The existing explanations for these facts still inflame the debate among finance researchers as to whether the value premium is due to security mis-pricing, to beta mis-measurement or to risk premia for omitted state variables.

This paper studies the asset pricing implications of a general-equilibrium model in which real investment is reversible at a cost. To the best of my knowledge, this is the first attempt to rationalize the observed value premium and the convergence of price-to-book ratios (firm migration) using a real-option model in general equilibrium. Firms investment and firms characteristics, in particular the market-to-book ratio, are functions of the state of the economy and, therefore, contain information about the behavior of stock returns. I show that firms are endogenously selected as "value" or "growth" according to the share of their productive capital to the overall capital employed in the economy. The optimal consumption policy alters this concentration of capital and affects the valuation of firms' cash flows. My model, using a "consumption-smoothing argument", (1) captures convergence of price-to-book ratios - negative for growth stocks and positive for value stocks,[1] (2) generates deviations from the classic CAPM in line with the cross-sectional variation in expected stock returns and (3) generates a non-monotone relationship between Tobin's q and conditional volatility consistent with the empirical evidence.

The production side of my economy consists of two industries, each grouping a large number of competitive firms. All firms have identical constant-returns-to scale production technologies, but are subject to industry-specific productivity shocks. Capital is reversible at a cost, implying that firms face higher costs in contracting than in expanding their capital capacity. The rest of the economy, i.e. the consumption side, is characterized by a riskless technology which stores capital. I call it the "pool sector". Agents are constrained to consume only the capital accumulated in the pool sector, which will serve as the numeraire of the economy. The investment/disinvestment decision occurs by means of a transfer of resources between the pool sector and the firm. While investors consume continuously from the rest of the economy, firms adjust their capital size in a lumpy fashion, characterized by periodic episodes of intense investment, spaced out with stretches in which no investment occurs.

The goal of this research, employing the economy just described, is twofold.

First, I explore the link between firm characteristics and stock returns. My model generates a negative relationship between market-to-book ratios and risk premia, consistent with the empirical evidence. The expected returns earned by the firms in states of nature associated with low q are higher

[1] See Fama and French (2007b).

than those earned by the firms in high q states. This relationship remains when controlling for risk, at least according to the traditional CAPM correction for risk. The intuition behind this result resides in the firms' ability to provide "consumption insurance". Value firms are less able to smooth consumption and have to offer a high remuneration to equity holders. In contrast, growth firms are less sensitive to economic conditions, exhibiting lower expected returns.

Another crucial result suggested by my paper is the non-monotone relationship between Tobin's q and conditional volatility consistent with the findings of Kogan (2004). I find that value and growth firms exhibit a higher conditional volatility than neutral firms. In other words, when firms are about to invest or to disinvest, Tobin's q is less sensitive to shocks and stock returns volatility is driven by the volatility σ of the technology process. On the contrary, when firms do not need to alter their capital size, q is more sensitive to shocks and contributes to stock returns by reducing its conditional volatility.

In this paper, the Consumption Capital Asset Pricing model holds since optimal consumption serves to discount future cash flows. On the contrary, a conditional version of the classic CAPM does not hold because the instantaneous stock market return is not perfectly correlated with consumption growth (and hence the pricing kernel). This result distinguishes my paper from Gomes, Kogan and Zhang (2003) and Gala (2006), who do not allow to separate the consumption CAPM from the classic CAPM. In fact, in their models, only one source of risk is priced in equilibrium, generating a perfect correlation between the pricing kernel and the market portfolio, which is instantaneously conditionally mean-variance efficient.

The proper modeling of the intertemporal risk premium is the reason to use a general-equilibrium model, in contrast to the arbitrary pricing kernel postulated by the partial-equilibrium literature.[2] Recently, the literature modeling the investment opportunities as options written on real assets has investigated the asset-pricing implications of the investment decisions, focusing in particular on the cross-sectional variation of expected returns.[3] However, these models do not examine the interaction between optimal consumption and investment policies, because they are set in a partial equilibrium, in which agents are risk-neutral and consumption is completely exogenous. Therefore, all the deviations from the classic CAPM documented by this literature (namely the size anomaly and the value premium) depend on the assumption of an exogenous pricing kernel and not from the intertemporal behavior of risk-averse agents who maximize their lifetime expected utility from consumption.

The second goal of this paper is to study the phenomenon of firm migration across value, investigating the reasons why growth firms become value over time and vice-versa. This convergence, that is the tendency of price-to-book ratios to become less extreme after stocks are placed in value and growth portfolios, has been recently documented by Fama and French (2007b). My paper provides a rationale to explain this evidence, endogenously generating mean reversion in Tobin's q, and, most importantly, capturing the empirical transition probabilities of three portfolios formed on price-to-book ratio.

More precisely, my model uses a consumption-smoothing argument to explain this behavior of price-to-book ratios: shocks to the assets in place alter the distribution of resources available in the economy. In turn, this affects the probability that the firm undertakes an investment decision and the overall value of the firm. The feedback effect on the optimal consumption decision alters the relative importance of productive capital with respect to

[2]See among the others Carlson, Fisher and Giammarino (2004), Cooper (2006) and Zhang (2006).

[3]See Carlson, Fisher and Giammarino (2004), Cooper (2006), and Novy-Marx (2007).

the capital accumulated in the pool, acting as a natural regulator of Tobin's q, pushing it away from its boundaries. The result of endogenous mean-reverting dynamics of Tobin's q distinguishes my paper from the existing literature, in which this "result" was rather imposed in an ad-hoc fashion.

The rest of the paper is organized as follows. Section 2 discusses the related literature. The general-equilibrium model is constructed in Section 3. I present the main findings related to firm migration and stock returns in Section 4. Finally, Section 5 concludes.

2 Related Literature

My work is part of a new and growing line of research, pioneered by Berk, Green, and Naik (1999), which relates the dynamics of stock returns to firm's investment decisions. The partial equilibrium model of Berk et al. (1999) features exogenous project-level cash flows and systematic risk. In their model, multiple sources of risk are used to explain the observed cross-sectional variation of returns. Gomes, Kogan and Zhang (2003) establish an explicit economic relation between firm-level characteristics and expected returns in a dynamic-general equilibrium model. My work differs from these papers along several dimensions. First, in my paper, investment is reversible at a cost, while these authors feature investment irreversibility. Second, I model firms, while they model "projects". In their economy, all "projects" have ex-ante identical productivity, and, once adopted, variation in the project-specific productivity only affects that project capital. As in the standard Q - theory of investment, in my model, variation in the profitability of the assets in place affects the firm investment decisions and its entire stock of capital.

Kogan (2004) develops a single firm two-goods general equilibrium model with investment constraints: real investment is irreversible, as assumed by a strand of the investment literature (e.g. Dixit and Pindyck (1994)), and the investment rate is bounded from above, representing a special case of the standard convex adjustment costs specification. He shows that investment frictions entail time variation in stock returns, and generate high nonlinear patterns between the market-to-book ratio and the first two moments of stock returns. My paper is distinct from his paper along several dimensions. First, mine is a multiple-firms economy with costly reversibility of capital. Second, the objectives behind the two papers are quite different. I am interested in explaining the cross-sectional variation in stock returns and the convergence of price-to-book ratios, while Kogan investigates the effect of investment frictions on equilibrium asset pricing, focusing in particular on the relationship between Tobin's q and conditional volatility.

Zhang (2005) also links expected returns to size and book-to-market in a dynamic-equilibrium model with convex adjustment costs and costly reversibility of capital, using the neoclassical q-theory approach and an exogenous countercyclical market price or risk. He computes the industry equilibrium by applying the "approximate-aggregation" idea of Krusell and Smith (1998). In recent years, Carlson, Fisher and Giammarino (2004) have analyzed the effect of operating leverage on expected returns, while Cooper (2006) has studied the asset pricing implications of non-convex adjustment costs. All these models use a partial-equilibrium framework to explain the value premium, while, in my work, I endogenize the role of consumption, and thus, the pricing kernel. Further differences are that I do not consider operating leverage and that investment is reversible at a cost.

Papanikolaou (2008) proposes a two-sector equilibrium model with heterogeneity in the type of firm output and provides evidence that investment-specific technological change is a source of systematic risk that is responsible for some of the cross-sectional variation in risk premia between value and growth firms. My paper is distinct from his paper along several dimensions. First, Papanikolaou focuses on ex-ante firm heterogeneity, that is heterogeneity arising because of differences between capital good and final good producers, rather than differences in productivity or accumulated capital as assumed in my model. Second, in my model, the productive capital is reversible (at a cost), whereas he assumes a fixed level of capital in the investment-good sector.

The literature on investment in general equilibrium using a real-option approach includes Kogan (2001) and Hugonnier, Morellec and Sundaresan (2005). These papers examine mainly the impact of irreversibility on the investment behavior and do not attempt to rationalize the value premium and the convergence of price-to-book ratios.

Additional contributions include the works of Gomes, Yaron and Zhang (2002) and Gala (2006).

Much of the methodology of the present article is borrowed from the literature dealing with portfolio choice under transaction costs. Grossman and Laroque (1990) consider fixed transaction costs, while Dumas and Luciano (1991) consider proportional costs, but allowing for terminal consumption only. Liu (2004) proposes a model of optimal consumption and investment with transactions costs and multiple risky assets. Finally, Dumas (1992) constructs a general-equilibrium model with proportional costs in segmented commodity markets.[4]

3 The Model

I consider an economy populated by a continuum of homogeneous risk-averse agents. The production side of the economy consists of two industries, each grouping a large number of competitive firms. All firms have identical constant-returns-to scale production technologies with expected rate of return μ and standard deviation σ of rate of return, but are subject to

[4] The deterministic model of Black (1973) is a forerunner of this approach.

industry-specific productivity shocks. The rest of the economy, i.e. the consumption side, is characterized by a riskless technology which accumulates capital at the rate r. There exists a single good which can be consumed, invested in the production processes or accumulated in the consumption sector. Investors are constrained to consume only the physical good available in the pool. With this specification, I am implicitly assuming that the qualitative characteristic of the good is different according to the sector in which it is employed. More precisely, once invested in the production processes, it acquires some "technological pecularities", and thus, it cannot be directly consumed. First, it has to lose the "sector-specific component", and only then it becomes available to investors.

The decision to expand the firm capacity requires a transfer of resources from the consumption sector to the firm industry, while the decision to disinvest requires an inverse transfer of physical capital to feed the rest of the economy. I assume that capital is reversible at a cost, implying that firms face higher costs in contracting than in expanding their capacity. These costs are proportional to the amount of capital transferred, being respectively $1-s_i$ in case of investment and $1 - s_d$ for disinvestment, with $0 < s_d < s_i < 1$. Thus, for every unit of capital transferred, the addition to the capital accumulated in the consumption sector (disinvestment) is s_d, while only s_i units are added to the production process in case of investment. Alternatively, $1/s_i$ can be interpreted as the price (in units of capital) at which the firm can purchase one unit of capital, and s_d as the price at which it can sell one unit of capital. Since in each industry there exist a large number of identical firms, all subject to the same shocks, to simplify the notation, I refer to the representative firms of each industry.

While the consumption policy is continuous, the inflows and the outflows from the pool sector are significantly lumpy. In fact, given the nature of the costs considered in the model, there will exist a region of the state space in which no investment/disinvestment occurs.[5] Because of the linear nature of the constraints and the homogeneity of the utility function, the two ratios of firms' capital over the capital stock accumulated in the pool, K^i/K^0 for $i = 1, 2$, are sufficient state variables to characterize the inaction region. Whenever $K^i/K^0 = \underline{\lambda}^i$, a lower edge of the space is reached. The capital stored in the consumption sector is abundant relative to that used in i-th production process, and the firm i finds it optimal to increase its capital size drawing resources from the pool. Therefore, an investment takes place. On the contrary, when the upper threshold is reached, i.e. $K^i/K^0 = \overline{\lambda}^i$, the inverse transfer of resources takes place. Firm i contracts its capital capacity and feeds the pool. Obviously, these thresholds $\underline{\lambda}^i$, and $\overline{\lambda}^i$ are not constant but functions of the state variables of the economy.[6]

I assume that financial markets are complete and that there are no costs or frictions to trade financial claims on the physical assets. This guarantees that agents can achieve a Pareto-optimal allocation of consumption. Accordingly, the central planner's problem is

$$V\left(K^0, K^1, K^2\right) = \max_{\{c, I^1, I^{01}, I^2, I^{02}\}} E\left[\int_0^\infty e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds\right] \tag{1}$$

s.t.

$$dK_t^0 = rK_t^0 - c_t dt + s_d dI_t^1 + s_d dI_t^2 - dI_t^{01} - dI_t^{02}, \tag{2}$$

$$dK_t^1 = \mu K_t^1 dt + \sigma K_t^1 dB_t^1 + s_i dI_t^{01} - dI_t^1, \tag{3}$$

$$dK_t^2 = \mu K_t^2 dt + \sigma K_t^2 dB_t^2 + s_i dI_t^{02} - dI_t^2, \tag{4}$$

[5] See Dumas (1992) for a detailed explanation of the existence and the properties of the no-transfer region.

[6] In his model of optimal consumption and investment with transaction costs, Liu (2004) shows that the no-transaction region is not an ellipse as was suspected before, but rather does have "corners".

$$c_t \geq 0,\ dI_t^{01} \geq 0,\ dI_t^{02} \geq 0,\ dI_t^{1} \geq 0,\ dI_t^{2} \geq 0,\ K_t^0 \geq 0,\ K_t^1 \geq 0,\ K_t^2 \geq 0, \tag{5}$$

where γ is the degree of risk aversion, ρ the rate of impatience and dB_t^1 and dB_t^2 are two standard independent Brownian motions. I_t^i and I_t^{0i} are non-decreasing processes which increase only when, respectively, a disinvestment or an investment involving industry i take place.

When no investment/disinvestment takes place, using the martingale property, I get that

$$-\rho V + \max_c \left\{ \frac{E_t\left[dV_t\right]}{dt} + \frac{c_t^{1-\gamma}}{1-\gamma} \right\} = 0 \tag{6}$$

$$\rho V = \max_c \left\{ \begin{array}{c} V_{K^0}(rK^0 - c) + \mu K^1 V_{K^1} + \mu K^2 V_{K^2} \\ +0.5\left(\sigma K^1\right)^2 V_{K^1K^1} + 0.5\left(\sigma K^2\right)^2 V_{K^2K^2} + \dfrac{c^{1-\gamma}}{1-\gamma} \end{array} \right\}.$$

Substituting the first order condition for consumption, that is,

$$c\left(K^0, K^1, K^2\right) = \left[V_{K^0}\left(K^0, K^1, K^2\right)\right]^{-1/\gamma}, \tag{7}$$

the Hamilton-Jacobi-Bellman equation inside the no-investment region can be written as follows:

$$\begin{aligned} \rho V &= \frac{\gamma\left(V_{K^0}\right)^{\frac{\gamma-1}{\gamma}}}{1-\gamma} + rK^0 V_{K^0} + \mu K^1 V_{K^1} + \mu K^2 V_{K^2} \\ &\quad +0.5\left(\sigma K^1\right)^2 V_{K^1K^1} + 0.5\left(\sigma K^2\right)^2 V_{K^2K^2}. \end{aligned} \tag{8}$$

When an investment takes place, the movement to the target position is instantaneous. Hence, the values of the discounted utility before and after the increase of productive capital must be the same, that is, when $K^i/K^0 = \underline{\lambda}^i$, for $i = 1, 2$,

$$V\left(K^0,K^i,K^j\right)=V\left(K^0-dI^{0i},K^i+s_idI^{0i},K^j\right)\quad\Rightarrow V_{K^0}=s_iV_{K^i}. \tag{9}$$

Value matching must also hold when a disinvestment takes place, that is, when $K^i/K^0=\overline{\lambda}^i$,

$$V\left(K^0,K^i,K^j\right)=V\left(K^0+s_ddI^i,K^i-dI^i,K^j\right)\quad\Rightarrow s_dV_{K^0}=V_{K^i}, \tag{10}$$

for $i=1,2$.

The partial differential equation and the value-matching conditions hold for any arbitrary choice of the investment/disinvestment functions $(\underline{\lambda}^i,\overline{\lambda}^i)$. Smooth-pasting conditions have to be satisfied in order for the barriers to be optimal.[7]

This requires that, in case of investment,

$$\begin{aligned}V_{K^0}\left(K^0,K^i,K^j\right)&=V_{K^0}\left(K^0-dI^{0i},K^i+s_idI^{0i},K^j\right)\quad\Rightarrow V_{K^0K^0}=s_iV_{K^0K^i},\\V_{K^i}\left(K^0,K^i,K^j\right)&=V_{K^i}\left(K^0-dI^{0i},K^i+s_idI^{0i},K^j\right)\quad\Rightarrow V_{K^iK^0}=s_iV_{K^iK^i},\end{aligned} \tag{11}$$

while, in case of disinvestment,

$$\begin{aligned}V_{K^0}\left(K^0,K^i,K^j\right)&=V_{K^0}\left(K^0+s_ddI^i,K^i-dI^i,K^j\right)\quad\Rightarrow s_dV_{K^0K^0}=V_{K^0K^i},\\V_{K^i}\left(K^0,K^i,K^j\right)&=V_{K^i}\left(K^0+s_ddI^i,K^i-dI^i,K^j\right)\quad\Rightarrow s_dV_{K^iK^0}=V_{K^iK^i},\end{aligned} \tag{12}$$

for $i=1,2$.

[7]See Dumas (1991) and Dixit (1991) for a discussion of the value-matching and the smooth-pasting conditions.

The optimal solution to the central planner problem is obtained by solving the differential equation (8) subject to the boundaries conditions (9-12). As far as I know, there exists no closed-form solution to the value function V, therefore I apply a numerical technique based on the finite-difference method.

4 Equilibrium Behavior

The knowledge of the function $V\left(K^0, K^1, K^2\right)$ allows one to characterize the equilibrium behavior of the economy. The solution to the Pareto-planner problem, the shape of the inaction region, and especially the asset pricing implications are analyzed in this section.

Considering the linear nature of the constraints and the isoelastic property of the utility function, the value function $V\left(K^0, K^1, K^2\right)$ is homogeneous of degree γ. Therefore, the two variables ω_1 and ω_2, defined by

$$\omega_1 \equiv \log \frac{K^1}{K^0} \text{ and } \omega_2 \equiv \log \frac{K^2}{K^0},$$

suffice to fully characterize the state of the economy.

Exploiting the homogeneity of the value function and using the new state variables, I introduce the following (transformed) value function I,

$$(1-\gamma)\log(K^0) + I(\omega_1, \omega_2) \equiv \log V(K^0, K^1, K^2),$$

which will be useful to compute the price-to-book ratios and stock returns.[8]

4.1 Optimal Investment

Here I analyze the optimal investment policy and discuss the key role played by the optimal choice of consumption in my general-equilibrium framework.[9]

[8] See Appendix 7.1

[9] I discuss the choice of parameter values at the end of Section 4.

According to the new state variables, the inaction region is a subspace of $\mathbb{R}^2$ expressed in terms of the logarithm of ratios of firm's capital over the capital stored, i.e $\omega_i = \log(K^i/K^0)$, and is delimitated by the functions $\underline{\omega}_i(\omega_j)$ and $\overline{\omega}_i(\omega_j)$. When the ratio ω_i reaches the lower edge $\underline{\omega}_i$, firm i acquires new resources from the physical capital stored in the pool, and thus, an instantaneous investment takes place, keeping ω_i between $\underline{\omega}_i(\omega_j)$ and $\overline{\omega}_i(\omega_j)$. On the other side, when the concentration of productive resources in industry i is very high, i.e. when the ratio ω_i is in correspondence of the upper edge $\overline{\omega}_i(\omega_j)$, the capital K^i involved in the production process is abundant relative to the pool K^0. Thus, for firm i it is optimal to disinvest and the excess resources are transferred to the consumption sector. To summarize, all firms decide not to alter their capital capacity when, for $i = 1, 2$, $\underline{\omega}_i(\omega_j) < \omega_i < \overline{\omega}_i(\omega_j)$.[10]

Figure 1 shows the optimal position of the boundaries.

FIGURE 1 GOES HERE

The interior of "ABCD" represents the no investment/disinvestment region. Firms in industry 1 (respectively 2) find optimal to increase their size in correspondence of the line AC (AB), while disinvestment takes place close to the segment BD (CD). The shape of the inaction region confirms the results obtained by Liu (2004) in a portfolio-choice problem with transaction costs: it is not an ellipse as it was suspected by the previous literature,[11] rather it does have "corners". The assumption of identical production technologies for the two industries implies the symmetry around the 45° degree line (i.e. the Merton line). Any deviations from this line are based on a diversification argument and driven by the existence of proportional investment/disinvestment

[10] In the Appendix 7.3 I address the issue of the optimality of this investment/disinvestment policy.

[11] See Morton and Pliska (1995).

costs.

In standard models of investment decisions under uncertainty, the investor has no choice over the consumption stream. On the contrary, in my general-equilibrium model, agents are risk averse and choose their consumption sequence. In turn, this affects the price of financial securities through the endogenous stochastic discount factor, which is defined, as in Kogan (2001), as

$$\Lambda_{t,s} = e^{-\rho(s-t)} \frac{U'(c_s^*)}{U'(c_t^*)}.$$

In the next sections, I will show that the optimal consumption policy plays a crucial role in the paper: it alters the relative distribution of resources available in the economy (thus affecting the relative price of capital), and generates an endogenous mean-reverting process for firms market-to-book ratios. Therefore, consumption serves as a natural regulator of the Tobin's q, pushing it away from its boundaries.

4.2 Tobin q and Firm Migration

In this section, I link firm characteristics, particularly the market-to-book ratio, to the state of the economy, and examine their properties.

I show that firms with high B/M ratios are endogenously selected as the ones with high relative capital concentration ω, while "growth firms" (low B/M ratios), instead, are associated with lower levels of productive capital relative to the pooled capital. The model generates mean-reverting dynamics for the Tobin's q and predicts that firms tend naturally to migrate from value to growth and vice-versa, because the optimal consumption policy alters the relative scarcity of resources and impacts the price of capital. Moreover, I show that the model captures the empirical transition probabilities of firm migration based on three portfolios formed on price-to-book ratio. The fundamental result of convergence of M/B ratios distinguishes my article from most of the existing literature. In fact, in my model, the pricing kernel is completely endogenous and the mean-reverting process for the book-to-market ratio is the result of a general-equilibrium framework in which production processes are characterized by constant-returns-to-scale technologies. On the contrary, other papers are set in partial equilibrium (exogenous consumption) and impose some mean-reverting properties for aggregate and idiosyncratic state variables, thus, forcing the cross-sectional distribution of firms' q.

I start by specifying the numeraire I use to price all financial assets. I choose the physical capital stored in the pool to be the numeraire, because investors are constrained to consume only the good physically available in their sector.

Let q^i denote the (shadow) price of the output involved in the i-th production process in terms of the capital pooled. Then,

$$q^i\left(K^0, K^1, K^2\right) = \frac{V_{K^i}\left(K^0, K^1, K^2\right)}{V_{K^0}\left(K^0, K^1, K^2\right)}. \tag{13}$$

The value of firm i is given by the product of the relative price q^i and its stock of productive capital, that is $S^i = q^i K^i$. Therefore, the relative price q^i coincides with the Tobin's average q^i of the firm, being the ratio of its market value to the replacement cost of its capital.

As anticipated in previous sections, the optimal investment/disinvestment of the firm is zero when its market-to-book ratio is in the interior of the interval $[s_d, 1/s_i]$. The firm expands its capacity to prevent the Tobin's q from rising above the upper trigger value $1/s_i$ and cuts its capital to prevent the Tobin's q from falling below the lower trigger value s_d. The values of q at the boundaries, $\underline{\omega}_i = \log(\underline{\lambda}^i)$ and $\overline{\omega}_i = \log(\overline{\lambda}^i)$, are given by the value-matching conditions

$$q^i(\underline{\omega}_i) = 1/s_i \quad \text{and} \quad q^i(\overline{\omega}_i) = s_d. \tag{14}$$

The smooth-pasting conditions guarantee the optimality of the trigger functions, and imply that

$$\frac{dq^i}{dK^i} = \frac{dq^i}{dK^0} = 0 \qquad \text{and} \qquad \frac{dq^i}{dK^i} = \frac{dq^i}{dK^0} = 0. \tag{15}$$

These conditions ensure that the partial derivatives of the firm q^i with respect to the own capital and the capital stored in the pool are zero when the boundaries are reached.

Figure 2 displays the typical behavior of market-to-book ratio within the inaction region.

FIGURE 2 GOES HERE

In my model, firms with high book-to-market ratios are endogenously selected as the ones with high relative capital concentration. On the contrary, "growth firms" (low B/M ratios) are associated with lower levels of capital ratio K^i/K^0. Recalling the shape of the no-investment region shown in Figure 1, it is easy to locate value and growth firms on the graph: when the state variables are close to the line AB (respectively AC), firms in industry two (one) are growth, while in correspondence of BD (CD), firms in industry one (two) are value. This means that the distribution of existing resources in the economy identifies automatically the position of the firms within the distribution of the book-to-market ratio.

I find that the expected rate of variation of Tobin's q changes sign according to the position of the state variables. In particular, it exhibits mean-reverting dynamics consistent with the convergence of price-to-book ratios.

I propose a simple story, driven by a consumption-smoothing argument, to explain this behavior. A sequence of positive technological shocks generates an excess supply of (un)productive capital with respect to the capital stored in the pool, which translates into a higher probability that the firm reduces its size. In turn, this reduces the value of growth options available to the firm and the (shadow) price of its productive capital relative to the consumption capital. As a result, the value of the firm decreases with respect to the accumulated capital, and the firm is more a “value” firm than before. The feedback effect on the optimal consumption decision alters the distribution of existing resources leading to an increase in the firm book-to-market ratio, therefore mitigating the effects of these positive shocks to the technology of the firm. In contrast, negative shocks to the production function of the firm reduce its accumulated capital, increase not only the value of the assets in the place, but also its growth options, thus, increasing the probability that the firm invests. The firm is more a “growth” firm than before. Again, the optimal consumption reduces the effects of these negative shocks for growth firms, pushing back the market-to-book ratio. In other words, consumption acts as a natural regulator of Tobin’s q, pushing it away from its extreme values, meaning that firms with low book-to-market ratios tend to lose their growth opportunities, migrating from growth to value, while the reverse happens to value firms.

As mentioned in previous sections, one of the goal of this paper is to capture the empirical probabilities of migration across value. Recently, Fama and French (2007a) provided a better understanding of the firm migration phenomenon by quantifying the speed of convergence of market-to-book ratios. Using their methodology, I construct the average transition frequencies given by the data, Table 1, and generated by my model, Table 2, of three portfolios formed on price-to-book ratios.

INSERT TABLE 1 HERE

INSERT TABLE 2 HERE

Notice that the migration probabilities shown in table 2, obtained using 200 artificial panels each with 3000 firms, capture quite well the average transition densities found in the data.

At this point, it is worthwhile to remind that my model consists of two industries, each characterized by a large number of identical firms. Therefore, Table 2 describes the migration of two representative firms. However, since the distribution of firms in the real world is stationary, it is possible to assimilate the transition densities that have been observed in the empirical evidence, and shown in Table 1 above, with the probability of migration of these two industries.

4.3 The cross section of stock returns

Here I study the implications of costly reversibility of capital on expected returns. I find a negative relationship between market-to-book ratios and expected returns. More precisely, the expected returns earned by firms when they are "value" are higher than those earned when they are "growth".

Let S_t^i denote the market value of firm i. Then, it can be computed as the product of the Tobin's q^i and its stock of capital, that is,

$$S_t^i = q_t^i K_t^i. \tag{16}$$

Within the no-investment region, the dynamics of the cumulative return to the firm's owners can be written as

$$dR_t^i = \frac{dS_t^i + \pi_t^i dt}{S_t^i} = \frac{q_t^i dK_t^i + K_t^i dq_t^i + dq_t^i dK_t^i + \pi_t^i dt}{q_t^i K_t^i}, \tag{17}$$

where π_t^i denotes the rate of cash flows from the sales of firm i's output.

It is convenient to rewrite the previous dynamics as

$$dR_t^i = \mu_{R^i}(\omega_1, \omega_2)\, dt + \sigma_{R^i}^1(\omega_1, \omega_2)\, dB_t^1 + \sigma_{R^i}^2(\omega_1, \omega_2)\, dB_t^2, \tag{18}$$

to show the dependence of the expected rate of return μ_{R^i} and the instantaneous volatilities $\sigma_{R^i}^1$ and $\sigma_{R^i}^2$ on the state variables (ω_1, ω_2).

I start by comparing the model-implied unconditional moments of equity returns with the corresponding empirical estimates.

INSERT TABLE 3 HERE

Table 3 shows the model's ability to reproduce key features of aggregate data. In fact, it seems appropriate to capture the historical levels of the equity premium and consumption growth. The risk free rate generated by the model is constant and equal to r.[12] This is not a surprise since the pool sector has the characteristics of a risk-less technology with rate of return r.

It is worthwhile to remember that, at each point in time, there exist only two different types of firms. More precisely, depending on the concentration of capital in the production technologies, one firm can be value and the other growth, or both neutral, and so on. This means that the relationship between market-to-book ratios and risk premia is not a real cross-section of stock returns, but is evidence of the expected returns earned by portfolios formed with the two industries at ten different states of nature.

Table 4 summarizes this pseudo cross-section of expected returns.

INSERT TABLE 4 HERE

The mean-excess returns found in the data are taken from Santos and Veronesi (2006). Notice that the magnitudes of the risk premia implied by my model are much in line with their empirical counterparts, especially for growth portfolios, whereas the differences among these values slightly increase for value portfolios. I would like to underline that this higher spread depends exclusively on the source of empirical data I have used. In fact, compared to the evidence provided by Gala (2006) and Zhang (2006), the risk premia generated by my model would be close to the actual risk premia

[12] See the Appendix 7.2.

for value firms, but much higher for growth firms. In Table 4, I decided to show the results from Santos and Veronesi (2006) and not from other authors because the overall spread for the ten portfolios is the lowest possible.

In my model, a sequence of positive shocks to the technology of the firm increases its capital, thus, decreasing the value of the firm with respect to the accumulated capital. As a result, the firm migrates towards value and, as shown in Table 4, earns on average higher expected returns. On the contrary, value firms hit by adverse technological shocks, increase their growth opportunities and migrate towards growth firms, earning on average lower expected returns. These mean excess returns still persists when controlling for risk, according to the traditional CAPM correction for risk, suggesting that a unique factor model based on the market portfolio is not enough to explain the cross-sectional variation in stock returns.[13]

The economic forces driving these results are pretty intuitive. The general-equilibrium analysis provides a "consumption insurance" explanation for the relationship between risk and return. In fact, risk averse investors look at "consumption smoothing" over time and states of nature. Therefore, the value of a firm is directly linked with its ability to provide consumption insurance: the more able a firm is and the less risky it will be. Obviously, the price of the firm will be high and the expected return low. The possibility to use capital investment in response to shocks in the current state of nature affects the ability of the firm to provide consumption insurance. In fact, because of costly reversibility of capital, which is clearly the main impediment to smooth consumption, a value firm is sensitive to economic conditions, exhibits a high covariance with future consumption, and has to offer a high remuneration to its equity holders. In contrast, a growth firm is less sensitive to economic conditions, exhibiting lower expected returns.

[13] See next sections.

4.4 Tobin's q and Conditional Volatility

The implications of costly reversibility of capital on the conditional volatility are analyzed here.

My model generates a non-monotone relationship between market-to-book ratios and conditional volatility consistent with the finding of Kogan (2004). More precisely, value and growth firms, i.e. firms that, respectively, are about to invest or to disinvest, exhibit a higher volatility. On the contrary, neutral firms, i.e. firms that do not need to alter their capital size, show a lower sensitivity to economic innovations. To illustrate this characteristic of stock returns, I plot the equity conditional volatility of returns as a function of q. The graph is based on a panel of 40000 firms.

INSERT FIGURE 3 HERE

Qualitatively, two main areas can be located on the graph. The first region is characterized by the M/B ratio close to the boundaries s_d and $1/s_i$. There, firms find it optimal to alter their capital size. Tobin's q is less sensitive to shocks, and, since the market-to-book ratio is a decreasing function of the capital concentration, stock returns are more volatile, depending almost exclusively on the volatility σ of the technology process. In the other region, firms do not invest or disinvest. Tobin's q is more sensitive to shocks, but, because of its negative sign, it lowers the volatility of stock returns.

In light of this non-monotone relationship, I follow Kogan (2004) and estimate the following time series models using a panel of 200 simulations, each with 300 firms and 50 years:

$$\left|R_t - \overline{R}\right| = a_0 + a_1(M/B)_{t-1} + a_2(M/B)^2_{t-1} + \varepsilon_t, \tag{19}$$

$$\left|R_t - \overline{R}\right| = a_0 + a_1(M/B)^-_{t-1} + a_2(M/B)^+_{t-1} + \varepsilon_t, \tag{20}$$

where R_t denotes the excess monthly return obtained by subtracting the risk-free rate from the portfolio return, $\overline{R}$ denotes the sample mean, and

M/B is the natural logarithm of the the market-to-book ratio measured as a deviation from its mean. In Equations (19) and (20), I study the dependence between the conditional volatility, captured by the conditional expectation of the absolute value of return, and Tobin's q. In Equation (19), I allow for a second-order term in the dependence on M/B while in Equation (20), I consider a piece-wise linear specification of conditional volatility, where the terms $(M/B)^-$ and $(M/B)^+$ denote the negative and positive parts of M/B, respectively. Consistent with Figure 3 and the finding of Kogan (2004), the coefficient a_2 should be positive in both equations.

INSERT TABLE 5 HERE

Table 5 reports the estimates of the coefficients of the time-series models (19) and (20). The estimates of the coefficients a_2 are positive in both cases. Thus, the relation between the conditional volatility and the M/B ratio indeed appears to be nonlinear in the way suggested by Figure 3.

4.5 Stock Returns and Capital Asset Pricing Models

The risk-return relation is studied in this section. As said in previous sections, the Consumption Capital Asset Pricing Model holds since optimal consumption serves to discount future cash flows. Therefore, stock returns can be perfectly described using aggregate consumption returns as a single risk factor. On the contrary, they cannot be described using market returns as a single risk factor, as in the conditional CAPM, because the market portfolio is not perfectly correlated with the pricing kernel.

I start by showing the risk-return relation driven by consumption smoothing.

The price of financial securities, discounted with the pricing kernel Λ, follows a martingale process, i.e.

$$0 = \Lambda D^i + \mathcal{D}\left[\Lambda S^i\right], \tag{21}$$

where $\mathcal{D}[\Lambda S^i]$ is the infinitesimal generator of the discounted firm value. Dividing both sides of the previous relationship by ΛS^i, and rewriting $\mathcal{D}[\Lambda S^i]$ as $E_t[d\Lambda S^i]/dt$, yields the more familiar expression:

$$0 = \frac{D^i}{S^i} + E_t\left[\frac{d\Lambda S^i}{\Lambda S^i}\right].$$

A straightforward application of Ito's formula to the discounted value of the firm leads to the fundamental asset pricing relation:

$$E_t\left[dR_t^i\right] - rdt = -E_t\left[\frac{d\Lambda}{\Lambda}\frac{dS^i}{S^i}\right], \tag{22}$$

where $r = -\frac{1}{dt}E_t\left[\frac{d\Lambda}{\Lambda}\right]$ denotes the instantaneous risk-free rate. Finally,

$$E_t\left[dR_t^i\right] - rdt = \beta_{ic}\left[E_t(dR_t^c) - rdt\right], \tag{23}$$

where dR_t^c is the cumulative return on any portfolio whose total dividend is equal to the optimal consumption, and β_{ic} is the beta of asset i with respect to that portfolio paying the total consumption.

In this economy, the market-based multi-period CAPM does not hold, as shown by Merton (1973). In fact, substituting R_t^M everywhere for R_t^c, where R_t^M is the return on the market portfolio, is not correct since the two are not perfectly correlated. To better understand the last point, suppose that uncertainty was generated only by a one-dimensional Brownian motion. Then, the market-based CAPM would be in principle correct, since the marginal utility of the optimal consumption of the representative agent would depend on just one source of risk, and so, every Ito process would be instantaneously perfectly correlated with every other Ito process. On the contrary, my model consists of two industries, each affected by a specific source of risk and both are priced in equilibrium. Therefore, the market portfolio is not perfectly correlated with the equilibrium consumption growth (and hence with the pricing kernel).

INSERT TABLE 6 HERE

INSERT FIGURE 4 HERE

Table 6 shows the market beta generated by my model for each of the 10 portfolios sorted on book-to-market ratio. I find a monotone relationship between B/M ratios and systematic risk (β) implying that growth firms are associated to lower betas, while value firms exhibit higher systematic risk. Figure 4 shows the deviations from the market-based multi-period CAPM for each of these 10 portfolios. Notice that growth firms lie below the security market line and, according to the CAPM, should be over-priced, while the reverse happens to value firms. CAPM deviations are precisely the reason why I do need two industries in my model. In fact, an equivalent economy with only one aggregate firm does not generate any deviation from the security market line suggesting that the value premium would entirely be explained by market betas.

4.6 Quasi-Fixed Investment Costs

Proportional investment and disinvestment costs are not the sole type of cost studied by the investment-based asset pricing literature. In fact, the literature proposes three main categories of investment costs: i) proportional, ii) quadratic, and iii) fixed (and quasi-fixed) costs. In this section, I discuss about the main drawbacks and limitations associated with these alternative specifications.

One of the main advantages of purely proportional investment/disinvestment costs in my general equilibrium framework is the resulting homogeneity property of the value function, which allows me to reduce the number of state variables. On the contrary, solving the same problem of Section 3 under the assumption of either quadratic or fixed costs would be a much harder task because this homogeneity property would not apply and I would not be able to reduce the dimensionality of the problem.[14] However, the assumption of quasi-fixed costs preserves the benefit of proportional costs of exploiting the homogeneity property but, as will be shown in the remaining of the section, its asset pricing implications are not always in line with the empirical evidence.

I assume the same model seen in Section 3, but with a different investment costs structure: I add a quasi-fixed component. Whenever the firm decides to invest (respectively to disinvest), the cost incurred is not just the one proportional to the amount bought (sold); there is an extra component which is proportional to the capital available in that sector, capturing the idea of foregone output due to the investment/disinvestment decision (quasi-fixed component).

Here is the corresponding central planner's problem:

$$V\left(K^0, K^1, K^2\right) = \max_{\{c, I^1, I^{01}, I^2, I^{02}\}} E\left[\int_0^\infty e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds\right]$$

s.t.

$$dK_t^0 = rK_t^0 dt - c_t dt + X^{1d} dI_t^{1d} + X^{2d} dI_t^{2d} - (X^{1i}/s_i + \theta_i K_t^0) dI_t^{1i} - (X^{2i}/s_i + \theta_i K_t^0) dI_t^{2i},$$

[14] This does not mean that equilibrium models assuming quadratic or fixed costs cannot be solved in closed form or using numerical techniques, but simply that these cost specifications require a different model to be handled. In fact, in recent years the investment-based literature has proposed several works featuring quadratic and fixed adjustment costs. However, none of these papers is set in general equilibrium using a real-option approach.

$$dK_t^1 = \mu K_t^1 dt + \sigma K_t^1 dB_t^1 - (X^{1d}/s_d + \theta_d K_t^1) dI_t^{1d} + X^{1i} dI_t^{1i},$$
$$dK_t^2 = \mu K_t^2 dt + \sigma K_t^2 dB_t^2 - (X^{2d}/s_d + \theta_d K_t^2) dI_t^{2d} + X^{2i} dI_t^{2i},$$

$$c_t \geq 0,\ K_t^0 \geq 0,\ K_t^1 \geq 0,\ K_t^2 \geq 0,\ X^{1i} \geq 0,\ X^{2i} \geq 0,\ X^{1d} \geq 0,\ X^{2d} \geq 0,$$

where X^{1i} (X^{2i}) is the amount invested in industry one (two) and X^{1d} (X^{2d}) is the amount disinvested by firm one (two). $\theta_i > 0$ and $\theta_i > 0$ are the quasi fixed costs, proportional to the capital available in that sector. Finally, $I_t^{1i}(I_t^{2i})$ denotes the investment time indicator of industry one (two), i.e. $dI_t^{1i} = 1$ if firms in industry one decide to invest at date t and 0 else, while $I^{1d}(I^{2d})$ denotes the disinvestment time indicator of industry one (two).

The asset pricing implications of this model, as shown by Goswami, Shrikhande and Wu (2001) and Casassus, Collin-Dufresne and Routledge (2005), are not always in line with the empirical evidence. First, both returns and consumption (more precisely the rate of consumption per unit of time) are characterized by jumps. Second, the Tobin's q^i is no longer a monotone function of the ratio ω_i, implying a cross-sectional variation of expected returns not consistent with the data. Third, the model does not generate the convergence of price-to-book ratios, implying that it would not be able to capture the migration probabilities.

4.7 Calibration

Parameter values were obtained from the investment-based asset pricing literature. Two groups of parameters must be chosen. The first group includes parameters belonging to the investor's preferences: the rate of impatience, ρ, and the degree of relative risk aversion (RRA) γ. The second set of values refers to firms' technologies: the risk-free rate r, the expected rate of return on capital μ, the standard deviation σ of the production process, and the investment costs, s_i and s_d. The discount rate ρ is set to 0.02 as is typically done in macroeconomic studies, while the risk aversion coefficient is 15. The investment-cost parameter s_i is 0.9 (which implies a purchase price of about 1.1),[15] while the disinvestment s_i is 0.75 (in line with the value reported by Novy-Marx (2007)). Finally, for the technological parameters I have chosen $r = 0.01$, $\mu = 0.08$, and $\sigma = 0.14$, which are close to the values used by Kogan (2001) since the two production technologies modeled in my general-equilibrium model share more the characteristics of productive sectors rather than the features of individual firms. All returns are expressed as yearly returns.

5 Conclusion

I propose a simple general-equilibrium model of real options to study the migration of firms across their book-to-market ratios and the cross-sectional distribution of stock returns. The production side of my economy consists of two industries, each grouping a large number of competitive firms using identical production technologies with constant-returns-to-scale and facing higher costs in contracting than in expanding their capital capacity. The consumption side is characterized by a risk-less technology which stores capital. Investors are constrained to consume only the capital accumulated in the pool sector,

I show how firms with low M/B ratios are endogenously selected as the ones with high relative capital concentration ω, while "growth firms" (high M/B ratios) are instead associated with lower levels of productive capital relative to the storage capital. The optimal consumption policy plays a cru-

[15] Whited (1992) documents that adjustment costs are about 10% of investment expenditures.

cial role in the model: it alters the distribution of resources available in the economy, affects the price of the firms, and mitigates the effects of positive (negative) capital shock for value (growth) firms, therefore generating convergence of price-to-book ratios.

My model generates a negative relationship between market-to-book ratios and risk premia, consistent with the empirical evidence. The expected returns earned by the firms in states of nature associated with low q are higher than those earned by the firms in high q states. I show that the value premium is driven by the firm's ability to provide consumption insurance. Value firms are less able to smooth consumption because the costly reversibility of capital restricts the use of capital investment to smooth consumption, and have to offer a high remuneration to equity holders. In contrast, growth firms are less sensitive to economic conditions, exhibiting lower expected returns.

Moreover, the model suggests a non-monotone relationship between Tobin's q and conditional volatility consistent with the findings of Kogan (2004). Firms' conditional volatility is higher for value and growth firms and lower for neutral firms. Finally, I show that the asset pricing implications of quasi-fixed costs are in line with the empirical evidence. Returns and consumption (the rate of consumption per unit of time) are characterized by jumps, the Tobin's q^i is no longer a monotone function of the ratio ω_i, and the model is not able to capture the convergence of price-to-book ratios.

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7 Appendix

7.1 The homogeneity property

In this appendix I show how to reduce the dimensionality of the problem from three to two state variables.

As seen in Section 3, the central-planner problem is

$$V\left(K^0, K^1, K^2\right) = \max_{\{c,I^1,I^{01},I^2,I^{02}\}} E\left[\int_0^\infty e^{-\rho s}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right]$$

s.t.

$$dK_t^0 = rK_t^0 dt - c_t dt + s_d dI_t^1 + s_d dI_t^2 - dI_t^{01} - dI_t^{02},$$

$$dK_t^1 = \mu K_t^1 dt + \sigma K_t^1 dB_t^1 + s_i dI_t^{01} - dI_t^1,$$
$$dK_t^2 = \mu K_t^2 dt + \sigma K_t^2 dB_t^2 + s_i dI_t^{02} - dI_t^2,$$

$$c_t \geq 0,\ dI_t^{01} \geq 0,\ dI_t^{02} \geq 0,\ dI_t^1 \geq 0,\ dI_t^2 \geq 0,\ K_t^0 \geq 0,\ K_t^1 \geq 0,\ K_t^2 \geq 0.$$

The HJB fundamental equation is

$$\begin{aligned}\rho V \quad = \quad & \frac{\gamma}{1-\gamma}\left(V_{K^0}\right)^{\frac{\gamma-1}{\gamma}} + rK^0 V_{K^0} + \mu K^1 V_{K^1} + \mu K^2 V_{K^2} \\ & +0.5\left(\sigma K^1\right)^2 V_{K^1K^1} + 0.5\left(\sigma K^2\right)^2 V_{K^2K^2}.\end{aligned}$$

Considering the linear nature of the constraints and the isoelasticity of the period utility function, the value function $V\left(K^0, K^1, K^2\right)$ is homogeneous of degree $1-\gamma$. Exploiting this homogeneity and defining

$$\omega_i \equiv \log\frac{K^i}{K^0},$$

I can rewrite the previous value function V as

$$V(K^0,K^1,K^2) \equiv (K^0)^{1-\gamma}G(\omega^1,\omega^2).$$

Taking the log, I get

$$(1-\gamma)\log(K^0) + I(\omega_1,\omega_2) \equiv \log V(K^0,K^1,K^2).$$

Therefore, the P.D.E. (8) becomes

$$\begin{aligned}\rho \;=\; & \frac{\gamma}{1-\gamma}\left((1-\gamma) - I_{\omega_1} - I_{\omega_2}\right)^{\frac{\gamma-1}{\gamma}}\left(e^I\right)^{\frac{-1}{\gamma}} \\ & + r\left((1-\gamma) - I_{\omega_1} - I_{\omega_2}\right)\left(e^I\right) + \mu\left[I_{\omega_1} + I_{\omega_2}\right] \\ & + 0.5\sigma^2\left[I_{\omega_1\omega_1} + I_{\omega_1}^2 - I_{\omega_1} + I_{\omega_2\omega_2} + I_{\omega_2}^2 - I_{\omega_2}\right].\end{aligned}$$

Recalling the previous sections, an investment takes place when there is an abundance of K^0 with respect to K^i, i.e. when $K^i/K^0 = \underline{\lambda}^i$, or, equivalently, when ω_i reaches $\underline{\omega_i} = \log(\underline{\lambda}^i)$. On the contrary, when ω_i reaches the upper bound $\overline{\omega_i} = \log(\overline{\lambda}^i)$, a disinvestment takes place.

From the value-matching conditions, I have that

$$V\left(K^0,K^i,K^j\right) = V\left(K^0 - dI^{0i}, K^i + s_i dI^{0i}, K^j\right) \quad \Rightarrow V_{K^0} = s_i V_{K^i}\ , \qquad \omega_i = \underline{\omega_i}$$

$$V\left(K^0,K^i,K^j\right) = V\left(K^0 + s_d dI^i, K^i - dI^i, K^j\right) \quad \Rightarrow s_d V_{K^0} = V_{K^i}\ , \qquad \omega_i = \overline{\omega_i}$$

or

$$(1-\gamma) - I_{\omega_1} - I_{\omega_2} = s_i e^{-\omega_i} I_{\omega_i}\ , \qquad \omega_i = \underline{\omega_i}$$

$$s_d\left[(1-\gamma) - I_{\omega_1} - I_{\omega_2}\right] = e^{-\omega_i} I_{\omega_i}\ , \qquad \omega_i = \overline{\omega_i}$$

From the smooth-pasting conditions, I get, when $\omega_i = \underline{\omega_i}$,

$$V_{K^0}\left(K^0, K^i, K^j\right) = V_{K^0}\left(K^0 - dI^{0i}, K^i + s_i dI^{0i}, K^j\right) \quad \Rightarrow V_{K^0K^0} = s_i V_{K^0K^i},$$

$$V_{K^i}\left(K^0, K^i, K^j\right) = V_{K^i}\left(K^0 - dI^{0i}, K^i + s_i dI^{0i}, K^j\right) \quad \Rightarrow V_{K^iK^0} = s_i V_{K^iK^i},$$

and, when $\omega_i = \overline{\omega_i}$,

$$V_{K^0}\left(K^0, K^i, K^j\right) = V_{K^0}\left(K^0 + s_d dI^i, K^i - dI^i, K^j\right) \quad \Rightarrow s_d V_{K^0K^0} = V_{K^0K^i},$$

$$V_{K^i}\left(K^0, K^i, K^j\right) = V_{K^i}\left(K^0 + s_d dI^i, K^i - dI^i, K^j\right) \quad \Rightarrow s_d V_{K^iK^0} = V_{K^iK^i},$$

which becomes, when ω_i=$\underline{\omega_i}$,

$$s_i\left[I_{\omega_i\omega_i} + I_{\omega_i}^2 - I_{\omega_i}\right] = e^{\omega_i}\left\{I_{\omega_i}\left[(1-\gamma) - I_{\omega_1} - I_{\omega_2}\right] - I_{\omega_i\omega_i} - I_{\omega_i\omega_j}\right\}$$

$$s_i\left\{I_{\omega_i}\left[(1-\gamma) - I_{\omega_1} - I_{\omega_2}\right] - I_{\omega_i\omega_i} - I_{\omega_i\omega_j}\right\} =$$
$$e^{\omega_i}\left[\left[(1-\gamma) - I_{\omega_1} - I_{\omega_2}\right]^2 - \left[(1-\gamma) - I_{\omega_1} - I_{\omega_2}\right] + 2I_{\omega_j\omega_i} + I_{\omega_i\omega_i} + I_{\omega_j\omega_j}\right],$$

and, when ω_i=$\overline{\omega_i}$,

$$\left[I_{\omega_i\omega_i} + I_{\omega_i}^2 - I_{\omega_i}\right] = s_d e^{\omega_i}\left\{I_{\omega_i}\left[(1-\gamma) - I_{\omega_1} - I_{\omega_2}\right] - I_{\omega_i\omega_i} - I_{\omega_i\omega_j}\right\}$$

$$\left\{I_{\omega_i}\left[(1-\gamma) - I_{\omega_1} - I_{\omega_2}\right] - I_{\omega_i\omega_i} - I_{\omega_i\omega_j}\right\} =$$
$$s_d e^{\omega_i}\left[\left[(1-\gamma) - I_{\omega_1} - I_{\omega_2}\right]^2 - \left[(1-\gamma) - I_{\omega_1} - I_{\omega_2}\right] + 2I_{\omega_j\omega_i} + I_{\omega_i\omega_i} + I_{\omega_j\omega_j}\right].$$

7.2 The Risk-free rate

Here I show that the instantaneous riskfree rate is constant and equal to r. This is not a surprise since the pool sector has the characteristics of a riskless technology.

Applying Ito's lemma to $V_{K^0}(K^0, K^1, K^2)$ gives, in the no-investment region,

$$\begin{aligned} dV_{K^0} &= \left[\begin{array}{c} V_{K^0K^0}(rK^0 - c) + \mu K^1 V_{K^0K^1} + \mu K^2 V_{K^0K^2} \\ +0.5\left(\sigma K^1\right)^2 V_{K^0K^1K^1} + 0.5\left(\sigma K^2\right)^2 V_{K^0K^2K^2} \end{array} \right] dt + \\ & \sigma K^1 V_{K^0K^1} dB^1 + \sigma K^2 V_{K^0K^2} dB^2. \end{aligned}$$

Using the martingale property, I get

$$\rho V = \max_c \left\{ \begin{array}{c} V_{K^0}(rK^0 - c) + \mu K^1 V_{K^1} + \mu K^2 V_{K^2} \\ +0.5\left(\sigma K^1\right)^2 V_{K^1K^1} + 0.5\left(\sigma K^2\right)^2 V_{K^2K^2} + \dfrac{c^{1-\gamma}}{1-\gamma} \end{array} \right\}.$$

Differentiating the above equation with respect to K^0, and using the envelope theorem gives:

$$-\rho V_{K^0} + \left\{ \begin{array}{c} V_{K^0K^0}(rK^0 - c) + rV_{K^0} + \mu K^1 V_{K^0K^1} + \mu K^2 V_{K^0K^2} \\ +0.5\left(\sigma K^1\right)^2 V_{K^0K^1K^1} + 0.5\left(\sigma K^2\right)^2 V_{K^0K^2K^2} \end{array} \right\} \equiv 0$$

Therefore,

$$dV_{K^0} = (\rho - r) V_{K^0} dt + \sigma K^1 V_{K^0K^1} dB^1 + \sigma K^2 V_{K^0K^2} dB^2.$$

Since K^0 is used as a numeraire, the price $P_\theta(t)$ of an asset with stochastic dividend stream $\theta(t)$ in consumption units is: $P_\theta(t) = E_t\left[\int_t^\infty e^{-\rho(u-t)} \frac{V_{K^0(u)}}{V_{K^0(t)}} \theta(u) du\right]$. Applying this to price an instantaneously riskless bond yields, as in Cox, Ingersoll, and Ross (1985),

$$\frac{E_t\left[dV_{K^0}\right]}{V_{K^0}} = \left[\rho - r(t)\right] dt.$$

Using the previous result, I get that $r(t) = r$.

7.3 Optimality of the investment policy

Since the problem studied in Section 3 involves continuous consumption and discrete investment at stopping times, this optimal control problem belongs to the class of *combined stochastic control* as studied by Brekke and Øksendal (1998). In this appendix I do not provide a formal proof of the existence of the value function $V(K^0, K^1, K^2)$ satisfying the partial differential equation (8), and of the optimality of the investment/disinvestment policy described in Sections 3 and 4, because the verification theorem provided by Liu (2004)[16] encompasses the model outlined in my paper. In fact, his Lemma 1 applies to any well-behaved utility function $U(c)$, including the power utility considered in my framework, and to the dynamics of capital shown in Equations 2-4.

Here, I simply show that the combined stochastic control implied by the optimal consumption policy and the investment/disinvestment strategy satisfies the conditions of his verification theorem.

Let $\tau_j, j \in \mathbb{N}$ denote the time when the firms invest/disinvest according to the policy specified in Section 4. Since this strategy consists of investing the minimal amount necessary to maintain ω_{it}^* between $\underline{\omega}_i(\omega_j)$ and $\overline{\omega}_i(\omega_j)$, where ω_{it}^* is the relative capital process derived from following the above policy, the investment time is clearly a stopping time, with $0 \leq \tau_j \leq \tau_{j+1}$ a.s., $\forall\ j \in \mathbb{N}$.

For all $j \in \mathbb{N}$, define χ_i^j the amount invested or disinvested at time τ_j by firm i. More precisely,

$$\chi_i^j = \begin{cases} \underline{\omega}_i^* - \omega_{i\tau_j} & \text{if } \omega_{i\tau_j} \leq \underline{\omega}_i(\omega_j) \\ \overline{\omega}_i^* - \omega_{i\tau_j} & \text{if } \omega_{i\tau_j} \geq \underline{\omega}_i(\omega_j) \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, χ_i^j is $\mathcal{F}_{\tau_j}-$measurable. Finally, since $\forall t \in (0, \infty),\ P\left\{\omega_{it} \in [\underline{\omega}_i(\omega_j), \overline{\omega}_i(\omega_j)]\right\} =$

[16] Liu (2004) provides a modified version of the verification theorems of Brekke and Øksendal (1998) and Korn (1998).

1, it follows that $P\left(\lim_{m\to\infty} \tau_m \leq K\right) = 0, \forall K \geq 0$, thus satisfying the conditions stated in Definition 1 of Liu (2004).

Table 1: Empirical transition probabilities of migration

Average transition vectors for stocks that migrate within or exit the group of three B/P portfolios, as a percent of firms in a portfolio and as a percent of the portfolio's market cap in June of the portfolio formation year.

Average Transition Vectors within the Group of Three B/M Portfolios - years 1963-2005

Percent of Portfolio Stock				Percent of Portfolio's Market Cap		
	Growth	Neutral	Value	Growth	Neutral	Value
Growth	74.38	20.00	5.71	79.18	15.67	5.15
Neutral	12.34	67.98	19.68	10.22	70.50	19.28
Value	4.39	17.77	77.84	3.52	16.10	80.38

I form three value weight portfolios, G, N, V, at the end of each June from 1963 to 2005 based growth (G, firms in the top 30% of NYSE P/B), neutral (N, middle 40%), and value (V, bottom 30%). In the P/B sorts for portfolios formed in June of year t, book equity is for the fiscal year ending in calendar year $t-1$ and market equity is for the end of December of $t-1$. The portfolios for year t include NYSE, Amex (after 1963), and Nasdaq (after 1972) stocks with positive book equity in $t-1$. Book equity for 1963 to 2005 is Compustat's total assets (data item 6), minus liabilities (181), plus deferred taxes and investment tax credit (35) if available, minus (as available) liquidating (10), redemption (56), or carrying value (130) of preferred stock. The transition vectors are for the firms assigned to a portfolio in June of year t that are also in one of the three portfolios in $t+1$. I decided to exclude four categories of firms because my model does not generate those: (i) Good Delists, which stop trading between June of t and June of $t+1$ because they are acquired by another firm (CRSP delist codes 200 to 399); (ii) Bad Delists, which stop trading because they no longer meet listing requirements (CRSP delist codes below 200 and above 399), (iii) firms with negative book equity for the fiscal year ending in calendar year t (Neg); and (iv) firms missing book equity for year t or market equity for December of t or June of $t+1$ (NA). The year t transition vector for a portfolio is the fraction of firms in the portfolio or the fraction of the total market cap in the portfolio when formed at the end of June of year t that falls into each of the groups at the end of June of $t+1$. The table reports averages of the annual transition vectors. Each row shows the average transition vector for a particular portfolio. Up to rounding error, the overall sum of the transition percents for a portfolio is 100, both for percents of portfolio stocks and for percents of portfolio market cap.

Table 2: Theoretical transition probabilities of migration

Transition probabilities generated by the model for stocks that migrate within or exit the group of three B/P portfolios, as a percent of firms in a portfolio.

Transition Probabilities

	Growth	Neutral	Value
Growth	75.21	24.72	0.6
Neutral	8.71	66.88	24.41
Value	0.4	5.78	94.2

The migration densities shown in Table 2 above are obtained using 200 artificial panels each with 3000 firms. I followed the same procedure described by Fama and French (2007a) in order to construct the three portfolios based on price-to-book ratio. The riskfree rate is constant and equal to r =0.01. The discount rate ρ is set to 0.02, while the risk aversion γ is 15. The investment cost parameter s_i is 0.9, while the disinvestment s_d is 0.75. Finally, the expected rate of return μ and the standard deviation σ of the productivity process are given by $\mu = 0.08$ and $\sigma = 0.14$.

Table 3: **Unconditional moments of the equity premium**

	Data		Model	
	Mean	Std.	Mean	Std.
Equity Premium	7.71	16.25	10.55	11.63
Consumption Growth	1.72	3.28	3.08	4.87

Table 3 shows the unconditional moments of aggregate returns and consumption growth. The riskfree rate is constant and equal to r =0.01. The discount rate ρ is set to 0.02, while the risk aversion γ is 15. The investment cost parameter s_i is 0.9, while the disinvestment s_d is 0.75. Finally, the expected rate of return μ and the standard deviation σ of the productivity process are given by $\mu = 0.08$ and $\sigma = 0.14$.

Table 4: The cross-section of expected returns

Growth to Value									
Growth									Value
1	2	3	4	5	6	7	8	9	10
Mean Excess Return (% per year): Empirical Data									
6.86	7.77	7.67	7.63	8.53	9.96	8.39	11.00	11.39	12.36
Mean Excess Return (% per year): Implied Returns									
6.95	7.90	8.91	9.80	10.90	11.56	12.42	14.00	15.08	16.16

Table 4 shows the cross sectional variation of expected returns. The riskfree rate is constant and equal to r =0.01. The discount rate ρ is set to 0.02, while the risk aversion γ is 15. The investment cost parameter s_i is 0.9, while the disinvestment s_d is 0.75. Finally, the expected rate of return μ and the standard deviation σ of the productivity process are given by $\mu = 0.08$ and $\sigma = 0.14$.

Table 5: **Conditional volatility and tobin's** q

	a_1	$p-value$	a_2	$p-value$
QUADRATIC SPECIFICATION	-0.066	0.000	0.506	0.000
PIECE-WISE LINEAR SPECIFICATION	-0.161	0.000	0.034	0.010

Table 5 shows the results of regressions (19) and (20) using a panel of 200 simulations each with 300 firms and 50 years.

R_t denotes the excess monthly return obtained by subtracting the risk-free rate from the portfolio return, $\overline{R}$ denotes the sample mean, and M/B is the natural logarithm of the the market-to-book ratio measured as a deviation from its mean. In Equation (19), I allow for a second-order term in the dependence on M/B, while in Equation (20), I consider a piece-wise linear specification of conditional volatility, where the terms $(M/B)^-$ and $(M/B)^+$ denote the negative and positive parts of M/B, respectively. Consistent with Figure 3 and the finding of Kogan (2004), the estimate a_2 is positive and significant in both regressions.

Table 6: Market betas for portfolios sorted on book-to-market ratio

Growth to Value									
Growth									Value
1	2	3	4	5	6	7	8	9	10
Market Betas									
0.73	0.80	0.88	0.95	1.04	1.13	1.19	1.29	1.37	1.46
Betas implied from the SML									
0.66	0.75	0.84	0.93	1.03	1.10	1.18	1.32	1.43	1.53

Table 6 shows market betas generated by my model for each of the 10 portfolios sorted on book-to-market ratio. The riskfree rate is constant and equal to r =0.01. The discount rate ρ is set to 0.02, while the risk aversion γ is 15. The investment cost parameter s_i is 0.9, while the disinvestment s_d is 0.75. Finally, the expected rate of return μ and the standard deviation σ of the productivity process are given by $\mu = 0.08$ and $\sigma = 0.14$.

Figure 1: The no-investment region

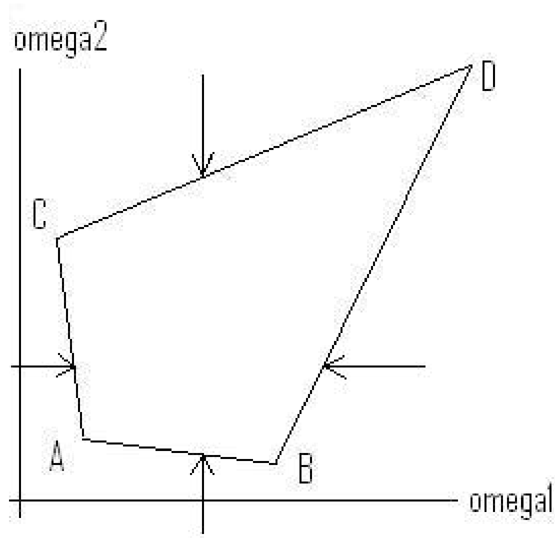

Figure 1 shows the no-investment region in the space (ω_1, ω_2). The coordinates of the corners are: $A = (0.61, 0.61)$, $B = (0.81, 0.08)$, $D = (2.01, 2.01)$ and $C = (0.08, 0.81)$. The riskfree rate is constant and equal to r =0.01. The discount rate ρ is set to 0.02, while the risk aversion γ is 15. The investment cost parameter s_i is 0.9, while the disinvestment s_d is 0.75. Finally, the expected rate of return μ and the standard deviation σ of the productivity process are given by $\mu = 0.08$ and $\sigma = 0.14$. Growth firms in industry two (respectively one) are located close the segment AB (AC). Value firms in industry one (respectively two) are located close to segment BD (CD).

Figure 2: Tobin's q

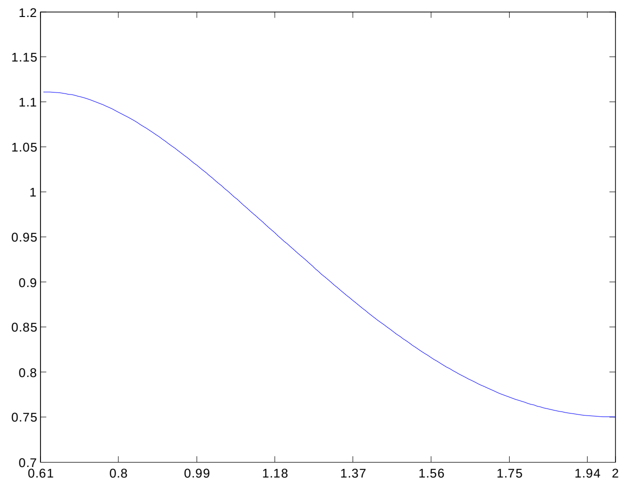

Figure 2 shows the behavior of the firms market-to-book ratios along the line AD of the inaction region. The riskfree rate is constant and equal to r =0.01. The discount rate ρ is set to 0.02, while the risk aversion γ is 15. The investment cost parameter s_i is 0.9, while the disinvestment s_d is 0.75. Finally, the expected rate of return μ and the standard deviation σ of the productivity process are given by $\mu = 0.08$ and $\sigma = 0.14$.

Figure 3: **Conditional volatility and tobin's** q

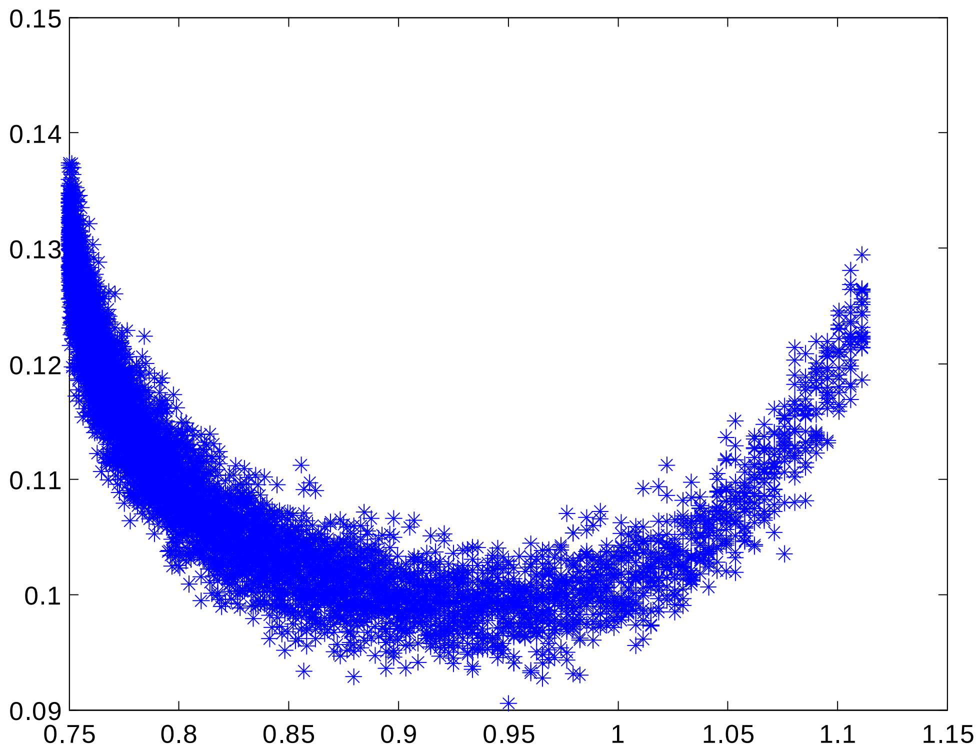

Figure 3 shows the relationship between Tobin's q (horizontal axis) and the conditional volatility (vertical axis) implied by my model using a panel of 40000 firms. The riskfree rate is constant and equal to r =0.01. The discount rate ρ is set to 0.02, while the risk aversion γ is 15. The investment cost parameter s_i is 0.9, while the disinvestment s_d is 0.75. Finally, the expected rate of return μ and the standard deviation σ of the productivity process are given by $\mu = 0.08$ and $\sigma = 0.14$.

Figure 4: Deviations from the CAPM

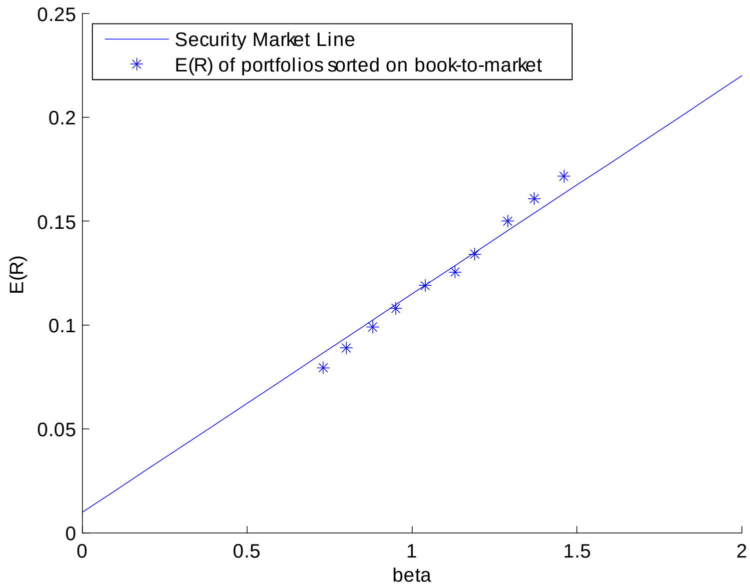

Figure 4 shows the deviations from the market-based multi-period CAPM. For each of the 10 portfolios sorted on book-to-market ratio, I computed the market beta finding a monotone relationship between B/M ratios and systematic risk (β): growth firms are associated to lower betas, while value firms exhibit higher systematic risk (i.e. β). The riskfree rate is constant and equal to r =0.01. The discount rate ρ is set to 0.02, while the risk aversion γ is 15. The investment cost parameter s_i is 0.9, while the disinvestment s_d is 0.75. Finally, the expected rate of return μ and the standard deviation σ of the productivity process are given by $\mu = 0.08$ and $\sigma = 0.14$.