Institutional Members: CEPR, NBER and Università Bocconi

WORKING PAPER SERIES

Asking Questions

Nenad Kos

Working Paper n. 405

This Version: July 5, 2011

IGIER – Università Bocconi, Via Guglielmo Röntgen 1, 20136 Milano –Italy
http://www.igier.unibocconi.it

Asking Questions*

Nenad Kos
Department of Economics, IGIER, Bocconi University

July 5, 2011

Abstract

We examine a model of limited communication in which the seller is selling a single good to two potential buyers. Limited communication is modeled as follows: in each of the finite number of periods the seller asks one of the two buyers a binary question. After the final answer, the allocation and the transfers are executed. The model sheds light on the communication protocols that arise in welfare maximizing mechanisms.

Among other things, we show that when the total number of questions is bounded the welfare optimal mechanism requires the seller to start with questioning one of the buyers and conclude with a single last question to the other buyer.

1 Introduction

"*[T]he literature on incentive compatibility is now quite extensive. However, with only a few exceptions, it is assumed that agents can transmit messages that are sufficiently detailed to describe fully all their private information.*" Green and Laffont (1987)

The above comment applies just as well as it did more than two decades ago. A bulk of the literature assumes no restrictions on communication, enabling it to apply the revelation principle and reduce communication to a simple one shot procedure. Observed communication is rarely costless and almost never instantaneous. It tends to proceed through a sequence of exchanges which take both time and effort. Rarely is the private information revealed completely, be it because agents do not want to reveal it, or because they have to carefully choose what they will convey in the limited time they have at disposal. Milgrom (2009) reports several cases where reporting buyers' private information would be too complicated, for example in FCC auctions with many licences.

Our model explores the effects of limited communication in a setup where a seller is selling a single indivisible good to privately informed buyers. More precisely, we seek for an (ex ante) welfare

*Email: nenad.kos@unibocconi.it. This paper is based on the second chapter of my Ph.D. thesis at Northwestern University. I am indebted to Alessandro Pavan and Asher Wolinsky for their continuous guidance and support. I would like to thank Peter Eso and Johannes Hörner for many helpful comments.

maximizing mechanism in which the seller sells an object to one of the two buyers whose valuations are independently distributed over $[0, 1]$. Unlike in the most of mechanism design literature where buyers can fully communicate private information, we assume that communication proceeds through a sequence of binary questions the seller can ask. Binary questions are interpreted as: "Does your valuation belong to a set C?", where C is a Borel measurable set in $[0, 1]$. We call an uninterrupted sequence of questioning of one buyer a round.[1] Such a model was first introduced by Blumrosen, Nisan and Segal (2007). They showed that welfare maximizing mechanisms use threshold questions. I.e., questions asking the agent whether his valuation is above or bellow a certain threshold. Due to results in Fadel and Segal (2009) we know that truthful reporting to threshold questions combined with allocation rules that award the object to the agent with the highest expected value, given the revealed information, can be supported with transfers in a Bayesian Nash equilibrium. In the equilibria we construct all the histories occur with positive probabilities, justifying consideration of Bayesian Nash equilibria.

Novelty of this paper are characterizations of the welfare maximizing protocols. First we consider a framework in which the seller commits to asking buyer i at most k_i questions. In each period the seller asks one buyer a question, possibly depending on the questions and answers in the past. However, before the mechanism starts the seller commits to who will be questioned in which period and what questions will be asked. After the last answer the allocation rule and the transfers are executed. We start the analysis by assuming that all of the buyers report truthfully. This enables us to obtain an upper bound on the welfare to be achieved under limited communication. We show that under truthful reporting, when each buyer has at least two questions, a welfare optimal mechanism entails three rounds. In the first round one of the two buyers is asked all questions assigned to him, but one. In the second round the other buyer is asked all of the questions intended for him, and finally in the last period the first buyer is asked one last question.

Next we consider the setup in which the seller only commits to a total number of questions. An interpretation of such a setup is that questions and answers take time, moreover, they take the same amount of time regardless of who the seller is talking to. In our main result we show that in any welfare optimal equilibrium communication proceeds through two rounds. In the first round all but one question are used on one of the buyers, in the second round the remaining buyer is asked one last question. We characterize questions asked in welfare optimal equilibria, incentives for buyers to report truthfully are then provided by above mentioned results from Fadel and Segal (2009). In particular, the optimal mechanism requires the seller to elicit information from one buyer for a longer period of time and to make a take it or leave it offer, depending on the previously obtained information, to the other buyer in the last period. If the offer is refused, the buyer questioned first receives the object.

We do not lay claim to have modeled any particular occurrence of communication in economic activity, nor do we claim that optimal communication will always have the properties shown to hold

[1] For example, if buyer 1 is asked a question in periods 1 and 2, and buyer 2 in periods 3 to 5 the protocol has two rounds.

in our framework. Our objective was to explore the effects of sequential limited communication in a well understood model and to contrast them with the results obtained when no limits on communication are imposed. When communication is not limited, the revelation principle applies and simultaneous communication is without loss of generality. In our model sequential communication enables one to obtain higher welfare. One might object that numbers are easy to report, and therefore limited communication in our model is forced. We would like to point out, however, that all the words in a language, say English, can be mapped into numbers in the interval $[0, 1]$, as can be all sequences of these words. In such a way any conversation between agents could be boiled down to simultaneous announcement of numbers in $[0, 1]$. Such a 'language' would be rather cumbersome to learn, that is why communication is formed by a sequential exchange of finite sequences of a limited number of words. Our model is a step towards recognition of complexity of communication and formation of optimal communication protocols.

Although we model the bound on communication as exogenously given, the model can be looked at from a somewhat broader perspective. One could specify a more general problem where the buyers' or seller's utility include some cost as a function of the number of questions for each buyer. Consequently, the welfare function would include these costs. The methodology we developed enables one to compute such a welfare for every possible combination of the number of questions for the two buyers. After comparing the welfare over all the combinations one arrives at the optimal number of questions for each buyer; provided the solution exists.

Finally, it should be noted that our approach extends to the problem of revenue maximization. Blumrosen, Nisan and Segal (2007) have shown how a problem of revenue maximization under simultaneous limited communication can be converted into a problem equivalent to welfare maximization with simultaneous limited communication with fictional buyers whose valuations are virtual valuations of the buyers in the original problem. This result, as extended in Kos (2011), allows one to solve the problem of revenue maximization when dynamic communication is allowed by applying the methodology developed in the present paper.

1.1 Related Literature

The importance of limited communication in trading environments was recognized by Green and Laffont (1987). They consider a model with an agent and a central resource-allocation unit. The decision of a central unit is a vector in $\mathbb{R}^n$, parameters relevant to both players' objectives are a vector in $\mathbb{R}^m$ and the message the sender can send is a vector in $\mathbb{R}^l$. Limited communication is modeled by assuming that $l < \min\{m, n\}$. The problem with modeling limited communication this way is that $\mathbb{R}^m$ can be bijectively encoded into $\mathbb{R}^l$ and the latter can be decoded into $\mathbb{R}^n$. To prevent such coding and decoding and to get limited communication, the authors had to assume that the central resource-allocation unit may only use differentiable mechanisms. We, on the other hand, use a model of limited communication in which message spaces are of smaller cardinality than the spaces of private information. In this way, limited communication arises without a need to impose extraneous assumptions on the allocation rules or transfers.

The closest papers to ours are Blumrosen, Nisan and Segal (2007) and Kos (2011). They study an auction environment in which buyers can only report one of the finite number of messages although their private information is in a compact interval in $\mathbb{R}$. Most of their analysis is concerned with simultaneous communication. Blumrosen, Nisan and Segal (2007), however, provide an example showing that with the same amount of communication one can achieve a higher level of welfare by using sequential communication. Furthermore, they show that restricting communication to simultaneous reporting, at most doubles communication complexity in the here studied environment. Our result implies that their bound is essentially tight.

The problem of how much information (measured in bits) needs to be transmitted between the agents to perform a task is studied in the computer science field of Communication Complexity. For an excellent survey see Kushilevitz and Nisan (1997). The main difference between our paper and the literature on communication complexity is that in our model we fix the amount of communication and solve for the welfare optimal mechanism given the constraint on the communication, while communication complexity literature mostly considers the dual problem where the objective is known and one asks what is the smallest number of bits needed to be communicated between the agents to accomplish the objective. A prominent, and for our paper very relevant, paper on communication complexity in economics is Fadel and Segal (2009). They compute how many bits need to be exchanged to implement a decision rule in an incentive compatible mechanism. We use their Proposition 6 to argue that certain mechanisms derived in our environment can be incentivized.

Lately limited (costly) communication started attracting more attention in contract theory, for example Battigalli and Maggi (2002) and Mookherjee and Tsumagari (2007). For more references see the latter paper.

Next, we point out the link between our paper and the revelation principle; see Myerson (1979) and for a more general version Myerson (1986). In the light of our paper the revelation principle can be seen as stating that when buyers are able to fully communicate their private information in several periods one loses nothing by letting them simultaneously communicate in one period. The amount of information one is able to convey does not change. When communication resources are limited, however, it is of great benefit to make communication sequential. Sequentiality enables one to convey the most relevant information given what has been previously disclosed.

Last, our framework is related to the framework of Arrow, Pesotchinsky and Sobel (1981). They considered a problem of searching for t largest observations in a random sample of size n by asking binary questions. They considered two different criteria of optimality: (a) minimizing the expected number of questions required and (b) maximizing the probability of terminating the search in at most r questions for specified r. Our model, instead, fixes the number of binary questions and maximizes the welfare.

2 Framework

A seller is selling a single good to two buyers, $I = \{1, 2\}$, who have independently distributed valuations over $[0, 1]$. The corresponding distribution functions F_i are assumed to have positive density on all of $[0, 1]$. Each buyer maximizes quasilinear utility function of the form

$$q_i v_i - m_i,$$

where v_i is the privately known valuation, q_i the probability of obtaining a good and m_i the expected transfer the buyer pays. While the preceding assumptions are well established and populate most of the output in mechanism design, and particularly auctions, the following is a stark departure.

Communication proceeds through a sequence of binary questions the seller can ask. The seller commits to asking each buyer at most k_i questions, with $K = (k_1, k_2)$, specifies the sequence in which the buyers are questioned, the actual questions, the allocations and the transfers.

Formally, the horizon is finite with time indexed by $t \in T = \{1, ..., k_1 + k_2\}$. The mechanism specifies who is asked a question in each period[2]:

$$\iota : T \to \{1, 2\},$$

with $\left|\iota^{-1}(i)\right| = k_i$, and the question the buyer is asked given the history:

$$\eta_t : H^{t-1} \to B,$$

where B is the Borel sigma algebra on $[0, 1]$. After each history the specified buyer is asked a question of a type: "Does your valuation belong to a set C?", where C is a Borel set in $[0, 1]$. H^{t-1} is the set of histories at the beginning of period t, with a generic element $h^{t-1} = (h_1, h_2, ..., h_{t-1})$ and the convention $H^0 = \emptyset$. History in this context is a sequence of answers, thus with our notation $h_t \in \{0, 1\}$. When the final history $h^{k_1+k_2}$ is realized the allocation rule Q and the transfer rule M are executed:

$$\begin{aligned} Q &: H^{k_1+k_2} \to [0, 1]^3, \\ M &: H^{k_1+k_2} \to \mathbb{R}^2_+, \end{aligned}$$

where Q_0 is interpreted as the probability that the seller keeps the object. A mechanism in our setup is a tuple (K, ι, η, Q, M). Often we will be interested in a set of mechanism with a fixed set of parameters; $\{(K), (K, \iota), (K, \iota, \eta)\}$ covers those of greater importance. For example, we denote the set of all mechanisms for which communication is fully determined and given by K, ι and η by $G_{K,\iota,\eta}$. These mechanisms differ only in allocation and transfer rules. We use the letter g to denote

[2] We assume that a single person is asked a question in each period. It is easy to see that our characterizations would remain unaltered if we allowed for simultaneous questioning.

a particular mechanism.

Buyer i's pure strategy[3] $\left(\mu_t^i\right)_{t\in\iota^{-1}(T)}$:

$$\mu_t^i : H^{t-1} \times [0,1] \to \{0,1\},$$

is to answer 1 or 0, yes or no respectively, when he is asked a question, i.e. for t such that $\iota(t) = i$, given his valuation and the history. History h^t describes all the answers up to period $t-1$, inclusive. We are assuming that all communication is observed by all the participants in the mechanism. This is purely for expositional purposes.

Throughout most of the paper the objective will be maximization of ex ante welfare

$$E\left[\sum_i Q_i v_i\right];$$

i.e. the mechanism should award the object to a buyer with the highest value.

The equilibrium concept we apply is Bayesian Nash equilibrium.[4] The analysis will proceed somewhat unconventionally, though. At first we will neglect any kind of incentives on the side of buyers; i.e. buyers will be assumed to report truthfully whenever called upon:

$$\mu_t^i\left(h^{t-1}, v_i\right) = \mathbf{1}_{\left[v_i\in\eta_t\left(h^{t-1}\right)\right]},$$

for t such that $\iota(t) = i$. We call a mechanism that achieves the highest welfare under truthful reporting **informationally optimal**.[5] In the second stage of the analysis we will argue that such a mechanism can be incentivized. The welfare achieved in mechanism g, when bidders always report truthfully, is denoted

$$w^{io}(g).$$

Furthermore, given the set of parameters $P \in \{(K), (K,\iota), (K,\iota,\eta), (K,\iota,Q)\}$ we denote the highest welfare achieved in a mechanism with these parameters under truthful reporting by

$$w_P^{io*} = \sup_{g\in G_P} w^{io}(g).$$

2.1 Alternative Interpretation

We make an explicit assumption that questions asked are part of a mechanism. Instead of talking about abstract message spaces spanning through several periods, we have the seller facilitating communication through directed questioning. This is, in a sense, without loss of generality since

[3] In the Discussion section we argue that considering only pure strategies is without loss of generality for our results.

[4] Welfare maximizing equilibria we characterize in the paper are such that every history occurs with positive probability, justifying consideration of Bayesian Nash equilibria in a dynamic game.

[5] The term was coined by Blumrosen and Feldman (2007).

any equilibrium of the model with questions can be embedded as a Bayesian equilibrium in the model without questions (and vice versa) where bidders choose one of the two messages when they are called upon to report.

Results from *the mechanisms with limited simultaneous communication* will be of great benefit to us here. In such mechanisms buyers report simultaneously, each buyer choosing one of the finite number of messages in his message space, although the valuation space is the interval $[0, 1]$. Upon reports the allocation and the transfers are executed. We denote optimal welfare achieved in such a mechanism by

$$w^{s*}_{k_1,k_2},$$

where k_1 and k_2 stand for cardinalities of buyer 1 and 2's message space, respectively. As a reminder we restate Proposition 1 from Kos (2011) showing that in a simultaneous reporting mechanism with limited communication and 2 buyers welfare cannot be increased by increasing the cardinality of the buyer with the higher cardinality.

Lemma 1 *Let $k_1 \geq k_2$ and $k = \min\{k_1, k_2 + 1\}$, then*

$$w^{s*}_{k_1,k_2} = w^{s*}_{k,k_2}.$$

The above Lemma shows that the lower of the two cardinalities is crucial for welfare when it comes to limited communication with simultaneous reporting. For example, suppose $(k_1, k_2) = (7, 3)$, so that buyer 1 can choose among 7 messages and buyer 2 among 3. Thus, Lemma states $w^{s*}_{7,3} = w^{s*}_{4,3}$. That is, the highest welfare that can be achieved in a mechanism with simultaneous communication with bounds on communication given by K is equal to the highest welfare that can be achieved in a simultaneous communication mechanism with bounds given by $(4, 3)$.

On the other hand, if the cardinality of the message space of the buyer with the smaller cardinality is increased or if cardinalities of both bidders are raised, strictly higher welfare can be achieved. The next result follows from the proofs of Proposition 1 and Theorem 2 in Kos (2011).

Lemma 2 *Let $K = (k_1, k_2)$, $K' = (k'_1, k'_2)$, $k = \min\{k_1, k_2\}$ and $k' = \min\{k'_1, k'_2\}$. If $k > k'$ then*

$$w^{s*}_{k_1,k_2} > w^{s*}_{k'_1,k'_2}.$$

The mechanism with the higher lowest cardinality achieves the higher welfare. For example, if $K = (6, 3)$ and $K' = (2, 7)$ then $w^{s*}_{6,3} > w^{s*}_{2,7}$. One applies Lemma 1 to obtain the equalities $w^{s*}_{6,3} = w^{s*}_{4,3}$ and $w^{s*}_{2,7} = w^{s*}_{2,3}$, after which it is easy to see that $w^{s*}_{4,3} > w^{s*}_{2,3}$. A somewhat stronger result than the one stated in Lemma 2 holds; the following example is not covered. Suppose $K = (6, 3)$ and $K' = (3, 3)$, then $w^{s*}_{6,3} = w^{s*}_{4,3} > w^{s*}_{3,3}$. Yet differently, as soon as cardinality of the messages space of a buyer with the lowest cardinality is increased a higher welfare can be achieved.

3 Welfare Maximization

First we show that a mechanism in $G_{K,\iota,\eta}$ that uses an allocation rule awarding the object to a buyer with the highest expected value given the questions and the answers is informationally optimal in $G_{K,\iota,\eta}$.

Lemma 3 *Let $g^* \in G_{K,\iota,\eta}$ be a mechanism that allocates the object to a buyer with the highest expected value given the questions and the answers. Then g^* achieves the welfare $w^{io*}_{K,\iota,\eta}$.*

We provide a simple example to demonstrate the above Lemma.

Example 1 *Suppose buyers 1 and 2 have valuations distributed according to the uniform distribution on the interval $[0,1]$. Furthermore, suppose there are only two periods. In the first period buyer 1 is asked whether his valuation is at least 0.5. If he answeres with yes, buyer 2 is asked whether his valuation is at least 0.75, otherwise whether his valuation is at least 0.25. With our notation this means $K = (1,1)$, $\iota = (1,2)$ and $\eta = (0.5, 0.75, 0.25)$. Clearly among all the mechanisms with these parameters any welfare maximizing mechanim allocates the object to buyer 2 if both of them answer positively, to buyer 1 if both of them answered negatively and so forth.*

We omit a formal proof. By the definition of $G_{K,\iota,\eta}$, all the mechanisms in it have the same sequence of questioning and the same questions. Remember, in each period one of the buyers is asked whether his valuation belongs to a certain set. At the end of questioning, the seller knows each buyer's valuation is in the intersection of the sets the buyer claimed his valuation belongs to.[6] On the technical side, the intersection will be nonempty since the buyers are assumed to report truthfully. To maximize welfare, one is merely left to compute the expected value corresponding to the deduced set for each of the buyers and awarding the object to a buyer with the highest expected value. Since buyers are assumed to report truthfully, the design of the mechanism has no effect on incentives, but solely on welfare. Ties can therefore be broken arbitrarily. Finally, one can prove a somewhat stronger lemma, stating that a mechanism achieves the welfare $w^{io*}_{K,\iota,\eta}$ if and only if it allocates the object to a bidder with the highest valuation with ex ante probability 1. Potentially, an ex ante welfare maximizing mechanism could award the object to the buyer with the lower valuation, as long as that event has probability zero.

We define *threshold questions* to be questions of the type $A_i = [a_i, 1]$ for $a_i \in [0,1]$; or $A_i = [0, a_i)$. Threshold strategies, the natural analog to threshold questions, are the crux of the analysis in Blumrosen, Nisan and Segal (2007) and Kos (2011). The origins of the following lemma can be traced to Blumrosen, Nisan and Segal (2007); see the Theorem 6.1 on the page 260 and the discussion preceding it.

[6] Suppose buyer 1 is asked in the first period whether his valuation belongs to the interval $[0.5, 0.6)$ and in the third whether it belongs to the interval $[0.57, 0.62)$. Furthermore, suppose he answered affirmatively in both periods. Then it is clear that his valuation is in the interval $[0.57, 0.6)$.

Lemma 4 *There exists a mechanism with threshold questions in $G_{K,\iota,Q}$ that achieves $w^{io*}_{K,\iota,Q}$, and a mechanism with threshold questions in $G_{K,\iota}$ that achieves $w^{io*}_{K,\iota}$.*
Proof. *Proof of this and the subsequent results can be found in the Appendix.* ■

In the first part of Lemma 4 we fix the number of questions, who is questioned in each period and the allocation rule, then show that considering only threshold questions is without loss of generality. This should be rather intuitive since the seller's objective is to award the object to a bidder with the highest valuation. The second part shows the same for a more general class of mechanisms.

We are still left to determine the optimal sequence of questions. That is, we are left to determine the optimal ι given the fixed vector K. Exploitation of simultaneous communication mechanisms will be of great value here. Before we proceed to the formal analysis we present a simple example.

Example 2 *Let, $k_1 = k_2 = 2$ and $\iota = (1, 2, 2, 1)$. In the first period, buyer 1 is asked a question which can be characterized as a threshold. For each of the two answers of buyer 1, there is a threshold (question) for buyer 2 in period two. By the end of the second period, there are four possible histories and for each of those there is another threshold (question) for buyer 2 in period three. Therefore, there are altogether 6 thresholds (corresponding to threshold questions) for buyer 2 (two in the second period and four in the third). Finally, at the beginning of the third period there are 8 possible histories, thus eight thresholds, which together with the first period threshold yields 9 thresholds for buyer 1. The above analysis implies that whatever expected welfare can be achieved in our sequential mechanism can also be achieved in a mechanism with simultaneous communication in which buyer 1 uses 9 thresholds and buyer 2 uses 6, i.e. in a simultaneous communication mechanism in which cardinalities of the message spaces of buyer 1 and 2 are 10 and 7, respectively. Thus we have $w^{io*}_{(2,2),\iota} \leq w^{s*}_{(10,7)}$. By Lemma 1 $w^{s*}_{(10,7)} = w^{s*}_{(8,7)}$, hence $w^{io*}_{(2,2),\iota} \leq w^{s*}_{(8,7)}$. There exists a profile of thresholds $c^1 = \left(c^1_1, c^1_2, ..., c^1_7\right)$, $c^2 = \left(c^2_1, c^2_2, ..., c^2_6\right)$ for buyers 1 and 2 respectively that together with an appropriate allocation rule and the transfers achieves the welfare $w^{s*}_{(8,7)}$.* [7], [8] *These thresholds can be naturally embedded into the dynamic setup. Let*

$$\begin{aligned}
\eta_1(\emptyset) &= \left[c^1_4, 1\right], \\
\eta_2(1) &= \left[c^2_2, 1\right],\ \eta_2(0) = \left[c^2_6, 1\right], \\
\eta_3(1,1) &= \left[c^2_1, 1\right],\ \eta_3(1,0) = \left[c^2_3, 1\right],\ \eta_3(0,1) = \left[c^2_5, 1\right],\ \eta_3(0,0) = \left[c^2_7, 1\right], \\
\eta_4(1,1,1) &= \left[c^1_1, 1\right], \eta_4(1,1,0) = \left[c^1_2, 1\right], \eta_4(1,0,1) = \left[c^1_3, 1\right], \eta_4(1,0,0) = \left[c^1_4, 1\right], \\
\eta_4(0,1,1) &= \left[c^1_5, 1\right], \eta_4(0,1,0) = \left[c^1_6, 1\right], \eta_4(0,0,1) = \left[c^1_7, 1\right], \eta_4(0,0,0) = \left[c^1_8, 1\right].
\end{aligned}$$

Notice that $(\eta_t)_t$ achieves the expected welfare $w^{s*}_{(8,7)}$; therefore $w^{io*}_{(2,2),\iota} = w^{s*}_{(8,7)}$.

[7] For details see Kos (2011).
[8] We adopt the convention $c^i_j \geq c^i_{j+1}$ for each i. c^i_1 is the highest threshold of buyer i, c^i_2 the second highest, etc.

We call an uninterrupted sequence of questioning of one buyer a *round*.[9] For example, if g is some mechanism with $\iota = (1, 1, 2, 2, 1, 1)$, then the mechanism has three rounds, in each of which a buyer is asked two questions. The following lemma shows it is inefficient to use more than one question in the final round (if there are at least three rounds).

Lemma 5 *Let g be a mechanism in G_K with at least three rounds, and more than one question in the last round. Then there exists an alternative mechanism $g' \in G_K$, with one question in the last round, such that $w^{io}(g) \leq w^{io}(g')$.*

After the penultimate round, the seller holds all necessary information regarding the penultimate buyer (the buyer questioned in the penultimate round). At that point, the seller could compute the set the penultimate buyer's valuation belongs to and the expected value corresponding to it. Since the objective is welfare maximization, the seller only cares about whether the other buyer's valuation is above or below that expected value, a matter that can be settled by a single question. By redistributing the remaining questions from the last round to earlier rounds one can achieve higher welfare.[10]

Next, we show that it is optimal to have three rounds; every additional round decreases welfare. The optimal sequence of questioning ι^* is either of the type $(1, 1, ..., 1, 2, 2, ..., 2, 1)$ or with the roles of buyers 1 and 2 reversed. Furthermore, it is optimal to begin questioning the buyer assigned with the least number of questions.

Theorem 1 *An informationally optimal mechanism in G_K exists and entails at most three rounds. If $k_i > k_{-i} \geq 2$, for some $i \in \{1, 2\}$, then it is informationally optimal to first ask buyer $-i$, $k_{-i} - 1$ questions, then ask buyer i k_i questions and finally ask buyer $-i$ one last question. If $\min\{k_1, k_2\} = 1$, then an informationally optimal mechanism has two rounds, whereby buyer with $\min\{k_1, k_2\}$ is questioned in the second round.*

The later a question is asked, the more preceding histories it has. Since for each history a question creates a threshold, more preceding histories translates into more thresholds. More thresholds, in turn, result in a better idea of the buyer's valuation, thus the possibility of achieving a higher welfare. Consequently, one would like to ask questions assigned to the penultimate buyer as late as possible, meaning in the penultimate round. Optimally, the penultimate buyer will only be questioned in the penultimate round. Since Lemma 1 implies that the other buyer is asked only one question in the last round, his remaining questions have to be asked in the first round, delivering three rounds altogether. Turning to the case where each buyer is asked at least two questions and $k_1 \neq k_2$, we show that the buyer with the larger number of questions should be questioned in the second round. Thus, the number of thresholds for the penultimate buyer is maximized. Finally,

[9] This corresponds to the definition of a round in a protocol introduced in Kushilevitz and Nisan (1997); see Definition 4.23 on page 49.

[10] This is formally shown in the following Theorem.

reducing the number of rounds to two would be inefficient. As we showed in Lemma 5, all the relevant information for welfare maximization in the last round can be obtained by one question. The other questions, would therefore be better utilized if asked at the beginning.

We present two examples to clarify the rather abstract analysis above. First, one presents the welfare maximizing equilibrium of a mechanism with limited communication and simultaneous reporting. The second shows how strategies from a mechanism with simultaneous communication can be embedded into the framework with questions.

Example 3 *Two buyers have valuations independently and uniformly distributed over the interval* $[0,1]$. *Buyer 1's cardinality of the message space* k_1 *is 8 and buyer 2's,* k_2*, is 7. After buyers observe their private valuations they simultaneously report one of the messages in their message space after which the allocation and the transfers are executed. For the details of the mechanism and the analysis see Blumrosen, Nisan and Segal (2011). Optimal reporting strategies are threshold strategies, and indeed they are mutually centered. Threshold strategy of the buyer with cardinality of the message space* k *can be described by* $k-1$ *thresholds. Buyer 1's threshold strategy is denoted* $c^1 = \left(c_1^1, .., c_{k_1-1}^1\right)$ *with* $c_1^1 \geq c_2^1 \geq ... \geq c_{k_1-1}^1$*, and buyer 2's* $c^2 = \left(c_1^2, c_2^2, ..., c_{k_2-1}^2\right)$ *with* $c_1^2 \geq ... \geq c_{k_2-1}^2$*. In the welfare optimal equilibrium the inequalities between thresholds are strict. To be more precise, thresholds are mutually centered:*

$$\begin{aligned} c_j^1 &= E\left[V \mid c_{j-1}^2 \geq V \geq c_j^2\right], \ j \in \{1, 2, ..., k_1 - 1\}, \\ c_j^2 &= E\left[V \mid c_j^1 \geq V \geq c_{j+1}^1\right], \ j \in \{1, 2, ..., k_2 - 1\}, \end{aligned}$$

with the convention $c_j^i = 1$ *for* $j \leq 0$ *and* $c_j^i = 0$ *for* $j \geq k_i$ *and* $i = 1, 2$*. For our case of uniform distribution and* $k_1 = k_2 + 1 = 8$ *the above system of equations yields a unique solution:*

$$\begin{aligned} c_k^1 &= 1 - \frac{2k-1}{14}, \ k = 1, ..., 7, \\ c_k^2 &= 1 - \frac{2k}{14}, \ k = 1, ..., 6. \end{aligned}$$

With simultaneous communication, the above threshold strategies coupled with the allocation rule that awards the object to a buyer with the highest expected value given the strategies and the reports yields the informationally optimal mechanism.

Next we show how to deal with the sequential binary questions.

Example 4 *Revisiting Example 1, assume two buyers have valuations distributed independently and uniformly over* $[0,1]$ *and* $k_1 = k_2 = 2$*. An informationally optimal mechanism entails* $\iota = (1, 2, 2, 1)$ *and achieves the welfare of* $w_{8,7}^{s*}$*. This level of welfare, while assuming that buyers report truthfully, can be achieved by embedding the thresholds that achieve the highest welfare in the simultaneous communication mechanism with* $k_1 = k_2 + 1 = 8$ *into the sequential model as was done in Example 2.*

So far, we were concerned with the question of how to organize communication in order to maximize welfare when the seller commits to a certain number of questions for each buyer under the assumption that buyers report truthfully. However, in some instances the real constraint for the seller will be the total number of questions asked, say if the seller is time constrained and questioning each buyer is equally time costly. Therefore, the natural next step is to ask what can be done when the seller is only restricted by the total number of questions. In particular, what are the welfare optimal mechanisms when the restriction on k_1 and k_2 is $k_1 + k_2 \leq k^*$, for $k^* \geq 2$, and $k_1, k_2 \geq 1$.[11]

Let G_{k^*} be the set of all mechanisms with $k_1+k_2 \leq k^*$, and let $w^{io*}_{k^*}$ be the lowest upper bound on welfare achieved by mechanisms in G_{k^*} under truthful reporting, i.e. $w^{io*}_{k^*} = \sup_{g \in G_{k^*}} w^{io}(g)$. We call a mechanism g^* informationally optimal in G_{k^*} if it achieves welfare $w^{io*}_{k^*}$ under truthful reporting. That is, if $w^{io}(g^*) = w^{io*}_{k^*}$.

Theorem 2 *Let $k^* \geq 2$, $k^* \in \mathbb{N}$. An informationally optimal mechanism in G_{k^*} exists. Moreover, for any such informationally optimal mechanism g^* in G_{k^*} there exists an $i \in \{1, 2\}$ such that $k_i = k^* - 1$, $k_{-i} = 1$ and $\iota = (i, ..., i, -i)$.*

The above theorem provides existence of an informationally optimal mechanism in G_{k^*}, and shows that an informationally optimal mechanism has only two rounds. In the first round one of the buyers is asked all the questions but one and in the last period the other buyer is asked one last question. Using more than two rounds is wasteful, because the more thresholds the penultimate buyer has, the higher the welfare that can be achieved. The highest number of thresholds he could possibly have is achieved by allotting him all the questions but the last one.

One of our underlying assumptions is that buyers observe all the previous questions and answers. This assumption is not necessary for the mechanism constructed in the proof of Theorem 2. Namely, in each period and at each history, all relevant information for the buyer can be deduced from the question. Clearly, when we assume truthful reporting, observing what the other buyer reports is redundant. The same, however, applies under the equilibrium analysis. The questions reflect all the relevant information revealed in the past.

Thus far, the analysis was conducted under the assumption that both buyers report truthfully in all periods. This yields an upper bound on what could possibly be achieved under limited communication. Now we can apply Proposition 6 of Fadel and Segal (2009). Using their terminology, the allocation rule in the optimal protocol of our Theorem 2 is EPIC-implementable (each agent's allocation is nondecreasing in his value) therefore Bayesian incentivizing transfers can be computed as shown by Fadel and Segal (2009). Even more, there is no extra substance to requiring concept

[11] The last assumption is for convenience. It is easy to see that it cannot be welfare maximizing to question only one buyer. Indeed, under our assumptions, it would be welfare optimal to award the object to this buyer without any questions. But then the welfare could be raised by adding an additional buyer and asking him whether his valuation is larger than the first buyer's expected valuation.

like Perfect Bayesian Equilibrium rather than Bayesian Nash Equilibrium since all histories have positive probabilities.

Through the definition of ι, we assumed that in each period the same buyer is questioned for all the possible histories. Here we argue why this assumption is without loss of generality for Theorem 2. Suppose one allows for more general mechanisms in which who is asked a question depends on the history. It still follows from a result in Blumrosen, Nisan and Segal (2007) that the optimal welfare is achieved in threshold strategies. Furthermore, in any mechanism with k^* questions there are at most $2^{k^*}-1$ distinct thresholds. I.e., in the first period there is one threshold, in the second two (one for each history), etc. Fix a mechanism g. Let x_i be the number of thresholds created by questions to agent i, clearly $x_i+x_{-i}=2^{k^*}-1$. For the sake of the argument let $x_1>x_2$. At least as high a welfare as in the sequential mechanism could be achieved by a mechanism with simultaneous communication in which agent 1 has x_1+1 messages and agent 2, x_2+1, or $w^{io}(g)\leq w^{s*}_{x_1+1,x_2+1}$. By Lemma 1 $w^{io}(g)\leq w^{s*}_{x_2+2,x_2+1}$. One would like to make this upper bound as large as possible, i.e. one would want x_2 as large as possible. This together with $x_i+x_{-i}=2^{k^*}-1$ and $x_2<x_1$ yields $x_2=2^{k^*-1}-1$ and the upper bound $w^{io}(g)\leq w^{s*}_{2^{k^*-1}+1,2^{k^*-1}}$. More generally one obtains $w^{io}(g)\leq \max\left\{w^{s*}_{2^{k^*-1}+1,2^{k^*-1}}, w^{s*}_{2^{k^*-1},2^{k^*-1}+1}\right\}$. The proof of Theorem 2 constructs a mechanism which achieves precisely this upper bound and questions only one agent in the first k^*-1 periods (along all the histories) after which the other agent is asked one last question in the last period.

Finally, Blumrosen, Nisan and Segal (2007) show that restriction to simultaneous communication can at most double communication complexity. More precisely, to achieve the same welfare as in a sequential mechanism with k^* bits of communication in a mechanism with simultaneous reporting one needs less than $2k^*+1$ bits.[12] Our proof of Theorem 2 provides the tight bound. In particular, to achieve the same expected welfare as in an informationally optimal mechanism provided by Theorem 2 in a mechanism with simultaneous reporting one needs 2^{k^*-1} messages for agent i and $2^{k^*-1}+1$ messages for agent $-i$. Differently, if k^* bits are transmitted in the informationally optimal mechanism one needs $2k^*-1$ bits to achieve the same welfare in a mechanism with simultaneous reporting.

4 Discussion

Several assumptions in our model warrant an explanation. The assumption of questions being asked was commented on in the text above. One could dispense with questions altogether and assume that buyers report one of two messages in each period.

We assume that questions are binary, for two purposes. First, since we are restricting communication, the results should be most striking in the most restrictive case, the one in which buyers can only answer with yes or no. Second, binary questions can be reinterpreted as bits which are

[12] There is a slight imprecision in the statment of Theorem 6.1 in Blumrosen, Nisan and Segal (2007). The theorem makes a claim "... [w]ith a communication requirement smaller than nm." Their proof, however, gives a bound $nm-\frac{n(n-3)}{2}$. Thus for $n=2$ the statement should read communication requirement smaller than $2m+1$.

well established units of information transmission in information theory; see for example Cover and Thomas (1991) and Kushilevitz and Nisan (1997).

Some of the assumptions we made are there for ease of exposition. They would not change the main results if we dropped or suitably relaxed them. We assume that a buyer is asked a single question in each period. We could allow for both buyers to be asked a question in one period, or even both buyers to be asked several questions simultaneously within a period. It is easy to see that this would not enable one to achieve higher welfare. Such simultaneous communication would prevent one from conditioning on the information being reported in the same period. In addition, we assume that buyers use pure strategies. Again, the more general case would not help to achieve a higher welfare. For a buyer to mix between two messages for some positive measure of states he would have to be indifferent in all those states, meaning that for those two messages he would win with the same expected probability and have to pay the same transfer. This would in turn be a waste of a question.

Finally, we take into account the number of periods, or questions, to which the seller commits. In effect, this means that the cost of communication originates in the seller committing up front to a certain amount of questions and therefore time. For example, if the seller commits to five questions, he incurs the cost corresponding to five questions even if for some history he knows which of the two buyers has the highest valuation after two questions. Future research could explore how the optimal mechanism changes when one takes into account only the actual number of questions used to achieve the objective along each history.

5 Conclusion

We explore a model of limited communication in a setup where a seller is selling a single indivisible object to one of two buyers. We show how results from models with simultaneous limited communication can be used to solve the dynamic problem. In our main result, we show that when the seller commits to the total number of questions the welfare maximizing protocol requires only two rounds: in the first round one of the buyers is sequentially asked all the questions but one, while in the second round the other buyer is asked one last question. The welfare maximizing allocation rule awards the object to the buyer with the highest expected value computed from revealed information.

Our analysis proceeds by providing a bound on the welfare achieved by a mechanism with questions by a mechanism with simultaneous limited communication. We show that for any mechanism with questions, there exists a mechanism with simultaneous limited communication, and an equilibrium of that mechanism that achieves at least as high a welfare. Furthermore, the bound we provide is tight. That is, for a welfare maximizing equilibrium of a mechanism with questions there exists a mechanism with simultaneous limited communication and an equilibrium of the latter, achieving the same welfare. This can be seen as an analogue of the revelation principle. In the mechanism design without limits on communication the revelation principle implies it is without loss of generality to use simultaneous reporting. Simultaneous reporting when communication is limited, however, requires a larger message space to achieve the same welfare, corresponding to the idea that if one introduced rich enough language, then every story, question, and answer could be described by a single word. Consequently, simultaneous reporting of messages would be without loss of generality. How people would manage such a language is a different question altogether.

Finally, we would like to point out that the analysis in our paper carries over to revenue maximization. Blumrosen, Nisan and Segal (2007) have shown that the problem of revenue maximization with simultaneous limited communication can be transformed into the problem akin of welfare maximization with simultaneous limited communication. After such a transformation, our approach of using bounds applies.

References

[1] K.J.Arrow, L. Pesotchinsky, M. Sobel, On Partitioning a Sample With Binary-Type Questions in Lieu of Collecting Observations, *Journal of American Statistical Association* **76** (1981), 402 - 409

[2] P. Battigalli, G. Maggi, Rigidity, Discretion and the Cost of writing Contracts, *American Economics Review* **92** (2002), 798-817

[3] L. Blumrosen, M. Feldman, Implementation with a Bounded Action Space, (2007), mimeo

[4] L. Blumrosen, N. Nisan, I. Segal, Auctions with Severely Bounded Communication, *Journal of Artificial Intelligence Research* **28** (2007), 233-266

[5] T.M. Cover, J.A.Thomas, Elements of Information Theory, John Wiley and Sons, Inc. (1991)

[6] R. Fadel, I. Segal, The Communication Cost of Selfishness, *Journal of Economic Theory* **144** (2009), 1895 - 1920

[7] N. Kos, Communication and Efficiency in Auctions, (2011), June 2011 mimeo.

[8] Kushilevitz, E., Nisan, N. (1997), Communication Complexity, *Cambridge University Press.*

[9] J. Green, J. Laffont, Limited Communication and Incentive Compatibility, *Information, Incentives and Economic Mechanisms: Essays in Honor of Leonid Hurwicz*, T. Groves, R. Radner and S. Reiter (ed.), Minneapolis: University of Minnesota Press, (1987)

[10] P. Milgrom, Assignment Messages and Exchanges, *American Economic Journal: Microeconomics* (2009)*,95-113*

[11] D. Mookherjee, M. Tsumagari, Mechanism Design With Costly Communication: Implications for Decentralization, (2007), mimeo

[12] R.B. Myerson, Incentive Compatibility and The Bargaining Problem, *Econometrica* **47** (1979), 61-73

[13] R.B. Myerson, Multistage Games with Communication, *Econometrica* **54** (1986), 323-358

Appendix

Proof of Lemma 4. The first part of the claim follows from Blumrosen, Nisan and Segal (2007); see the discussion preceding Theorem 6.1. For the second part observe that to achieve informationally optimal welfare in $G_{K,\iota}$ it is enough to consider deterministic allocation rules, by Lemma 3. Therefore, given the fixed K and ι we only need to consider a finite number of allocation rules for each of which the informationally optimal welfare in $G_{K,\iota}$ can be achieved by threshold questions. ■

Proof of Lemma 5. All that is relevant for welfare in the last round is whether the valuation of the questioned buyer is higher or lower than the expected valuation of the buyer questioned in the penultimate round. This can be settled by a single question, while the remaining questions can be dispensed with or moved to earlier rounds to extract more information. More precisely, suppose that bidder i is questioned in the last round of g. We can construct g' by replacing the questions in the last round of g by a single question asking bidder i whether his valuation is above (or below) the expected value of bidder $-i$ computed from the information gained from his questions and answers. To have g and g' of the same length, same number of periods, questions $[0,1]$ can be added at the beginning of g'. The allocation rule in g' should award the object to the bidder with the highest expected value given the questions and the answers. Clearly $w^{io}(g) \leq w^{io}(g')$.

While the above argument proves the Lemma, clearly the $[0,1]$ questions at the beginning can be used more productively. ■

Proof of Theorem 1. Existence of an informationally optimal mechanism in G_K is easy to establish. By Lemma 4 there exists an informationally optimal mechanism in $G_{K,\iota}$. For any fixed K there are only finitely many ι, therefore there also exists an informationally optimal mechanism in G_K. Moreover, by the same lemma, it is enough to consider only threshold questions.

If there are more than three rounds in a mechanism, one can increase welfare by redistributing the questions from the earlier rounds to the later. Indeed, let g be a mechanism with at least four rounds. By Lemma 5 it is enough to consider a mechanism with one question in the last round. Let the number of rounds be $R = 2n$ for some $n \in \mathbb{N}$, $n \geq 2$ (the case of odd R is handled analogously) and let buyer 1 report in the first round. The number of questions for buyer 2 in the round $2r$ is denoted by m_r, while the number of questions for buyer 1 in the round $2r-1$ is denoted by l_r. The restrictions are $\sum l_r = k_1$ and $\sum m_r = k_2$.

Starting with the first round, in the first period buyer 1 is asked a question, creating a threshold. For each of the two answers in the first period buyer 1 is asked another question in the second period, creating two possibly distinct thresholds. At the beginning of l_1th period there are 2^{l_1-1} histories, for each of which a question is asked. This yields at most $1+2+...+2^{l_1-1} = 2^{l_1}-1$ distinct thresholds for buyer 1 in the first round. At the beginning of the third round there are $2^{l_1+m_1}$ histories, hence at most $2^{l_1+m_1}\left(2^{l_2}-1\right)$ distinct thresholds in round three, etc. Altogether there are at most

$$x = 2^{l_1} - 1 + 2^{l_1+m_1}\left(2^{l_2}-1\right) ... + 2^{l_1+m_1+...+m_{n-1}}\left(2^{l_n}-1\right)$$

thresholds for buyer 1 in the mechanism g. In the last round there are at least x thresholds created for buyer 2, and in the second at least 2. Lemma 1 then implies that we can bound the welfare achieved by g by $w^{s*}_{x+1,x+2}$.

An alternative mechanism, g', constructed by moving the last question from the round $R-3$ to the beginning of the round $R-1$ yields at most

$$\begin{aligned} x' &= x - 2^{l_1+...+m_{n-2}}\left(2^{l_{n-1}}-1\right) + 2^{l_1+...+l_{n-2}+m_{n-2}}\left(2^{l_{n-1}-1}-1\right) - 2^{l_1+m_1+...+m_{n-1}}\left(2^{l_n}-1\right) \\ &\quad +2^{l_1+m_1+...+l_{n-1}-1+m_{n-1}}\left(2^{l_n+1}-1\right) \end{aligned}$$

distinct thresholds for buyer 1. It is easy to verify that $x' > x$. The upper bound on the welfare to be achieved in g' is then $w^{s*}_{x'+1,x'+2}$. Also, by Lemma 2 $w^{s*}_{x'+1,x'+2} > w^{s*}_{x+1,x+2}$. By iterating the process, one arrives at a three rounds mechanism with a higher upper bound on welfare than any mechanism with more than three rounds. Next, we show that the last upper bound can be achieved.

Since we started with a mechanism with one question in the last round, we have a mechanism with k_1-1 questions for buyer 1 in round 1, k_2 questions for buyer 2 in round 2 and finally one question for buyer 1 in round three. This results in at most $y = 2^{k_1-1}\left(2^{k_2}-1\right)$ distinct thresholds for buyer 2 and an upper bound on the welfare of $w^{s*}_{y+2,y+1}$. Let $c^1 = \left(c^1_1, ..., c^1_{y+1}\right)$ and $c^2 = \left(c^2_1, ..., c^2_y\right)$ be profiles of thresholds that achieve such welfare in a simultaneous communication mechanism. From Kos (2011) we know that these thresholds are mutually centered: $c^1_1 = E\left[X_2 \mid 1 \geq X_2 \geq c^2_1\right]$, $c^2_1 = E\left[X_1 \mid c^1_1 \geq X_1 \geq c^1_2\right]$, etc. In particular $c^1_1 > c^2_1 > c^1_2 > ... > c^1_{y+1}$. The construction of questions proceeds by a bisection on the level of thresholds. One starts with the middle threshold of buyer 1

$$\eta_1 = \left[c^1_{2^{k_1-1-1}\left(2^{k_2}-1\right)+1}, 1\right].$$

Suppose buyer 1 is asked a question in the second period again; otherwise proceed to the next step. If he answered positively in the first period, 1, then the threshold in period two is the middle threshold of the thresholds above and including $c^1_{2^{k_1-1-1}\left(2^{k_2}-1\right)+1}$, i.e.

$$\eta_2(1) = \left[c^1_{2^{k_1-1-2}\left(2^{k_2}-1\right)+1}, 1\right].$$

Similarly

$$\eta_2(0) = \left[c^1_{2^{k_1-1-1}\left(2^{k_2}-1\right)+2^{k_1-1-2}\left(2^{k_2}-1\right)+1}, 1\right].$$

The rest of the thresholds in the first $k_1 - 1$ periods is obtained by the same method. Formally, in the first $k_1 - 1$ periods

$$\eta_t\left(h^{t-1}\right) = \left[c^1_{s_t(h^{t-1})}, 1\right],$$

where

$$s_t\left(h^{t-1}\right) = 2^{k_1-2}\left(2^{k_2}-1\right) + 1 + (1-2h_1)\,2^{k_1-3}\left(2^{k_2}-1\right) + \ldots + (1-2h_{t-1})\,2^{k_1-t-1}\left(2^{k_2}-1\right),$$

for $h^t = (h_1, h_2, ..., h_t)$.

After the first round is over, one knows whether buyer 1's valuation is in $\left[c^1_{2^{k_2}}, 1\right]$, $\left[c^1_{2^{k_2+1}-1}, c^1_{2^{k_2}}\right]$,..., or $\left[0, c^1_{2^{k_1-1}\left(2^{k_2}-1\right)-2^{k_2}}\right]$. In each of these intervals there are $2^{k_2} - 1$ thresholds of buyer 2. For example $\left\{c^2_1, c^2_2, ..., c^2_{2^{k_2}-1}\right\} \subset \left[c^1_{2^{k_2}}, 1\right]$. Now one proceeds by bisection on the set of thresholds for buyer 2 belonging to the identified interval of buyer 1. Finally, let $\left[c^2_l, 1\right]$ be the last question for buyer 2 in the second round. If he answers with 1, then the question for buyer 1 in the last period is $\left[c^1_l, 1\right]$, otherwise $\left[c^1_{l+1}, 1\right]$.

We still need to show that $k_1 > k_2 \geq 2$ implies it is optimal to question buyer 1 in the second round. This is done by a simple computation. If buyer 1 is questioned in rounds 1 and 3 the highest welfare achievable is $w^{s*}_{2^{k_1-1}\left(2^{k_2}-1\right)+2,2^{k_1-1}\left(2^{k_2}-1\right)+1}$. If, on the other hand, buyer 2 is questioned in the second round the highest welfare achievable is $w^{s*}_{2^{k_2-1}\left(2^{k_1}-1\right)+1,2^{k_2-1}\left(2^{k_1}-1\right)+2}$. Clearly

$$2^{k_2-1}\left(2^{k_1}-1\right) + 1 > 2^{k_1-1}\left(2^{k_2}-1\right) + 1.$$

By Lemma 2 it is optimal to question buyer 1 in the second round. It should also be noted that the constructed three round mechanism achieves higher welfare then any two round mechanism, which follows the reasoning of Lemma 5. If there are only two rounds, the highest welfare is achieved by computing the expected value of the buyer questioned in the first round from the available information, and asking the buyer in the second round whether his valuation is above that value. All other questions for the buyer in the last round can be productively used at the beginning of questioning, thus creating three rounds.

Finally, if $\min\{k_1, k_2\} = 1$, one or both buyers are assigned only one question. By Lemma 5 we know that a single question suffices for the last round. Therefore it is optimal to question a buyer who was assigned a single question in the last round. ■

Proof of Theorem 2. First, we assume $k^* \geq 3$. When $k^* = 2$ there can be no more than two rounds, by definition. Fix K such that $k_1 + k_2 = k^*$. By Theorem 1, an informationally optimal mechanism for any fixed K never entails more than three rounds. Take any three round mechanism in which buyer 1 is questioned in the first round. It is without loss of generality to assume there are $k_1 - 1$ questions for buyer 1 in the first round followed by k_2 questions to buyer 2, while in the last round buyer 1 is asked a single question. This gives altogether $x = 2^{k_1-1}\left(2^{k_2} - 1\right)$ thresholds for buyer 2 and an upper bound on welfare of $w^{s*}_{x+2,x+1}$. By first using $k_1 - 1$ questions on buyer 2 rather than buyer 1, one could get $x' = 2^{k_1+k_2-1} > x$ thresholds for buyer 2. This yields an upper bound on welfare of $w^{s*}_{x'+2,x'+1}$. This upper bound can be achieved by the strategies constructed similarly as in the proof of Theorem 1. One looks at the threshold strategies that achieve welfare $w^{s*}_{x'+2,x'+1}$ in the mechanism with simultaneous communication and embeds them into the asking questions setup.

We conclude that in every informationally optimal mechanism, either buyer 1 is asked $k^* - 1$ consecutive questions at the beginning followed by a single question to buyer 2, or the roles of the players are reversed. ■