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On the Smooth Ambiguity Model: A Reply*

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Abstract

We find that Epstein (2010)'s Ellsberg-style thought experiments pose, contrary to his claims, no paradox or difficulty for the smooth ambiguity model of decision making under uncertainty developed by Klibanoff, Marinacci and Mukerji (2005). Not only are the thought experiments naturally handled by the smooth ambiguity model, but our reanalysis shows that they highlight some of its strengths compared to models such as the maxmin expected utility model (Gilboa and Schmeidler (1989)). In particular, these examples pose no challenge to the model's foundations, interpretation of the model as affording a separation of ambiguity and ambiguity attitude or the potential for calibrating ambiguity attitude in the model.

1 Introduction

Epstein (2010) describes two Ellsberg (1961)-style thought experiments and argues that they pose difficulties for the smooth ambiguity model of decision making under uncertainty developed by Klibanoff, Marinacci and Mukerji (2005) (henceforth KMM). We revisit these thought experiments and argue that they lend no support to the critical conclusions he draws from them. We demonstrate that the first thought experiment and all its suggested variations

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are handled quite naturally and completely by the smooth ambiguity model if one takes care to formally model the information the decision maker has available. Regarding the second experiment, we elaborate on the behavioral distinction that it provides between the smooth ambiguity model and models such as the maxmin expected utility (MEU) model (Gilboa and Schmeidler (1989)) and explain why the behavior predicted by the smooth ambiguity model is intuitive. Our discussion of these examples highlights and reinforces the relative strengths of the smooth ambiguity model, including the degree of separation between ambiguity attitude and belief it affords and the range of ambiguity attitudes it accommodates.

To fix ideas and remind the reader of the model's functional form, consider in an Anscombe-Aumann setting[1] a state space Ω endowed with an event σ-algebra Σ, a space X of simple lotteries over a set of real outcomes containing $[-1,1]$, and a set Δ of all probability measures $\pi : \Sigma \to [0,1]$ on Ω. Let $\sigma(\Delta)$ be the smallest σ-algebra on Δ that makes the functions $\pi \mapsto \pi(E)$ measurable for all $E \in \Sigma$. The smooth ambiguity model represents preferences $\succsim$ over Σ-measurable simple acts $f : \Omega \to X$ using the following functional:

$$V(f) = \int_{\Delta} \phi\left(\int_{\Omega} u(f(\omega))\, d\pi(\omega)\right) d\mu(\pi), \tag{1.1}$$

where $u : X \to \mathbb{R}$ is nonconstant affine, $\phi : u(X) \to \mathbb{R}$ is strictly increasing, and $\mu : \sigma(\Delta) \to [0,1]$ is a probability measure on Δ.

The model also represents preferences $\succsim^2$ over suitably $\sigma(\Delta)$-measurable second order acts $\mathfrak{f} : \Delta \to X$ using the functional

$$V^2(\mathfrak{f}) = \int_{\Delta} \phi(u(\mathfrak{f}(\pi)))\, d\mu(\pi).$$

Notice that $\succsim$ and $\succsim^2$ agree when restricted to lotteries X (i.e., to constant acts and constant second order acts respectively). Moreover, for ease of exposition we will call Ω the first order state space and Δ the second order state space.

As it will also be useful in what follows, recall that the α-MEU model represents preferences over acts according to

$$U(f) = \alpha \min_{\pi \in C} \int_{\Omega} u(f(\omega))\, d\pi(\omega) + (1-\alpha) \max_{\pi \in C} \int_{\Omega} u(f(\omega))\, d\pi(\omega), \tag{1.2}$$

where $\alpha \in [0,1]$ is a weight and $C \subseteq \Delta$ is a w*-compact set of probabilities. When $\alpha = 1$ we get the MEU model.

[1]Here, as in Epstein (2010), we use an Anscombe-Aumann version of the original KMM model.

2 Thought Experiment 1: State spaces and incorporating information

The experiment takes Ellsberg's (1961) 3-color urn (an urn with 3 balls divided among red (R), blue (B) and green (G)) and adds a construction urn,[2] containing 3 balls each of which has a label r, b or g. The individual is told that exactly one of the balls in the construction urn is labeled r. A draw from this construction urn will determine the composition of the Ellsberg 3-color urn. Specifically, if r is drawn from the construction urn, the Ellsberg urn will contain one ball of each color, denoted $(1R, 1B, 1G)$, and, similarly, draws of b or g result in compositions $(1R, 2B, 0G)$ and $(1R, 0B, 2G)$ respectively in the Ellsberg urn. Apart from the usual bets on the color of a ball drawn from the Ellsberg urn, Epstein (2010) also considers bets on the composition of the Ellsberg urn (equivalently, bets on the type of ball drawn from the construction urn). He argues that the standard ambiguity averse choices over bets about the color drawn from the Ellsberg urn should imply ambiguity averse choices over bets about the color of the ball drawn from the construction urn. He claims that this behavior is incompatible with the smooth ambiguity model. All of his criticisms of the smooth ambiguity model stem from this alleged incompatibility. Below, we show that there is no incompatibility and that this behavior follows from the smooth ambiguity model quite naturally once one adopts, which Epstein (2010) does not, a state space adequate to incorporate the information provided to the individual in the experiment.

2.1 Modeling of the first thought experiment

The only change we make to the description of the thought experiment is to have the construction urn contain six balls rather than three (and thus exactly two balls labeled r rather than one). We do this so as to treat both the basic thought experiment and Epstein's elaborations on it using the same set-up.

In Epstein's interpretation, the first order state space is the set of possible draws from the Ellsberg urn, $\{R, B, G\}$. Thus, his set of second order states must be the set of probability distributions over this first order state space.[3] This state space can incorporate some of the information given to the individual in the experiment – specifically, the information about the possible compositions of the Ellsberg urn. This information rules out all but three such second order states, π_r, π_b and π_g, each corresponding to a possible draw from the

[2] Epstein (2010) calls this the "second-order urn". A referee suggested the term "construction urn" instead. We adopt the latter terminology.

[3] Recall that a second order state space, by definition, is isomorphic to the set of probability distributions over the first order states.

construction urn, r, b or g. These are represented by the columns in Table I, while the rows represent Epstein's first order states. The numbers give the probabilities of the first order states conditional on a given second order state.

[Insert Table I about here.]

Notice, however, that Epstein's state space is too sparse to incorporate the given information about the composition of the construction urn (such as the information that exactly two of the six balls are labeled r). This information does not correspond to an event in Epstein's state space, and therefore beliefs cannot be conditioned on it. This is worrisome, since conditioning behavior on this information is key to the thought experiment.

We now present a state space that is rich enough to incorporate all the given information as events.[4] The first order state space is the set of possible pairs of draws from both the construction urn and the Ellsberg urn, $\{r, b, g\} \times \{R, B, G\}$. The set of second order states is then the set of probability distributions over this first order state space. This state space can incorporate both types of information given to the individual in the experiment: (1) how the distribution over draws from both urns is determined by the composition of the construction urn, and (2) that exactly two of the six balls in the construction urn are labeled r.[5] In particular, this information rules out all but the five second order states $\pi_1, \ldots, \pi_5$ described in the five columns of Table II, each corresponding to a possible composition of the construction urn. The rows in the table represent first order states. As in the previous table, the numbers give the probabilities of the first order states conditional on a given second order state. They are derived by considering the color compositions consistent with the information in (2) and then using (1) to translate those into probabilities of the draws. Notice that the conditional probabilities are all multiples of $\frac{1}{18}$ since there are $6 \times 3 = 18$ possible pairs of drawn balls from the two urns. For example, the probability of observing (b, R) given π_1 is $\frac{4}{6} \times \frac{1}{3} = \frac{4}{18}$.

[Insert Table II about here.]

To see that with this fuller state space, ambiguity aversion in the smooth ambiguity model implies the behavior posited by Epstein in this thought experiment, take ϕ strictly concave,

[4] Such a construction was not given as much prominence in an earlier version of this reply. We thank an anonymous referee, Bob Nau (see Nau (2010)) and Ben Polak for emphasizing the importance of a more detailed treatment (and working out many of the details).

[5] This explains why, for example, considering the first order state space $\{R, G, B\}$ together with a putative second order state space {composition of construction urn $\times \{r, g, b\}$} would not be an adequate state space. In terms of the probability of the first order states, this putative second order space collapses to Epstein's second order space, $\{r, g, b\}$, and therefore suffers from the same inability to handle the information that two of the six balls in the construction urn are labeled r.

let μ be any strictly positive probability distribution over $\pi_1, \ldots, \pi_5$ and normalize u so that $u(100) = 1$ and $u(0) = 0$. Consider bets with stakes 100 if win and 0 if lose. Suppose, as seems reasonable given symmetry of the situation, that betting on π_1 (i.e., betting that the construction urn has composition $(2r, 4b, 0g)$) is indifferent to betting on π_5, and, similarly, that betting on π_2 is indifferent to betting on π_4. These indifferences imply $\mu(\pi_1) = \mu(\pi_5)$ and $\mu(\pi_2) = \mu(\pi_4)$. Then, according to the smooth ambiguity model, betting on R is strictly preferred to betting on B while betting on $B \cup G$ is strictly preferred to betting on $R \cup G$ (i.e., $f_1 \succ f_2$ and $f_4 \succ f_3$ in Epstein's (2010, pp. 2088-89) notation) and, betting on r is strictly preferred to betting on b while betting on $b \cup g$ is strictly preferred to betting on $r \cup g$ (i.e., $F_1 \succ F_2$ and $F_4 \succ F_3$ in Epstein's notation). Furthermore, again as Epstein suggests is intuitive, the preferences are stronger in the case of bets on the color drawn from the construction urn compared to those on the Ellsberg urn since less is known about the composition of the construction urn.[6]

The larger lesson is that in decision models with a state space (whether Savage (1954) or others) properly incorporating information requires that the information be modeled as an event in the state space, i.e., a subset of states. Marschak and Radner (1972), in their classic book, which shaped the way information is modeled in economics, write:

> "... an information signal represents a subset of the states of the environment; in the formulation of a decision problem, the states of the environment must be described in sufficient detail to cover not only those aspects relevant to the payoff function, but also those aspects relevant to the type of information on which the decisions may be based." (p. 48)

Often, in practice, this is done implicitly, with the "full state space" in the background and reduced form updating used in calculating the change in beliefs. This is perfectly fine as a shortcut as long as it leads to the same conclusions as an analysis using the full model. Epstein's analysis is an illustration of how this shortcut can lead one astray – with his chosen reduced form modeling, one obtains different results than when one uses the full model. With a full state space, the information in the thought experiment about the composition of the construction urn must correspond exactly to ruling out some states. Notice that with Epstein's choice of first and second order state spaces, this fails to hold: the fact that exactly two of the six balls in the construction urn are labelled r is consistent with all possible outcomes of draws from the construction urn and Ellsberg urn.

[6] For the calculations behind the claims in this paragraph as well as the ones in the next subsection see Section 5.1 in the Appendix.

2.2 Variations on the first thought experiment

Next, consider Epstein's (2010, Section 2.4) extension of the first thought experiment (Scenario I) to consider a new scenario (Scenario II) in which the subject is additionally told that there is at least one b and at least one g ball in the construction urn. This extra information is easily captured in our state space: second order states π_1 and π_5 become null events. How does behavior compare across the two scenarios according to the smooth ambiguity model? Take $u_I = u_{II} = u$ and again normalize so that $u(100) = 1$ and $u(0) = 0$. Take $\phi_I = \phi_{II} = \phi$ strictly concave. Let μ_I be any strictly positive probability distribution over $\pi_1, \ldots, \pi_5$. Let μ_{II} be the Bayesian update of μ_I reflecting the new information (so π_1 and π_5 are given zero weight and the rest maintain the same relative weights as in Scenario I). Epstein asks for the following intuitive rankings to be satisfied: (1) a bet on b is indifferent to a bet on g in each scenario; (2) a bet on r has the same certainty equivalent in each scenario; (3) a bet on R is strictly preferred to a bet on B in each scenario; and (4) the certainty equivalent of a bet on B is higher in Scenario II than in Scenario I. One can calculate that given our assumptions above, all of these rankings follow. This shows that to accommodate the difference in behavior between the two scenarios, all that needs to change is μ, and furthermore, the required change is a natural reflection of exactly the information difference between the two situations. Therefore, this example reinforces our interpretation that in the smooth ambiguity model there is a separation of beliefs and attitudes (towards ambiguity and towards risk), and that μ reflects information/belief. Epstein used these scenarios to argue that the change in information required changing ϕ to get plausible behavior, as that was true using Epstein's state space, which, as noted, cannot incorporate information of the kind given. On this basis, he challenges the interpretation of ϕ as reflecting ambiguity attitude and μ as reflecting beliefs or information. This leads him to claim that efforts to calibrate an individual's ϕ in a context of interest (e.g., financial markets), by examining the behavior of that individual in another environment (e.g., real or hypothetical Ellsberg experiments), have no justification. Our discussion demonstrates that this, and similar examples, provide no such basis.

Epstein (2010, Section 2.5) uses a final variation on the first thought experiment to argue that nonreduction of objective compound lotteries is implicit in the smooth ambiguity model. To support this, he compares Scenario I above to a scenario (call it Scenario III) in which complete information about the composition of the construction urn is given to the individual. If this change in information were modeled (as Epstein suggests) by leaving μ unchanged but informally interpreting it as objective, then the individual would be facing an objective two-stage lottery and, Epstein argues, would be forced by the smooth ambiguity model to treat it just as he did when it was ambiguous and therefore differently than the corresponding reduced lottery. We find this analysis is flawed in the same way as Epstein's analysis of the comparison between Scenarios I and II above. Specifically, he carries out his analysis in a setting too sparse to incorporate the change in information (i.e., going from partial to full information about the composition of the construction urn). Given the state space we use above, such a change is seen to correspond to μ going from a non-degenerate to a degenerate distribution – there is no longer any uncertainty about the composition of the construction urn. In such a scenario, the smooth ambiguity model treats all events as unambiguous, reduces all uncertainty to risk, and becomes a standard expected utility preference. Thus, no nonreduction of objective probabilities is implied.[7]

2.3 Testability

Epstein (2010, Section 2.3) partially anticipates our resolution of the first thought experiment and claims that such a reformulation of the state space would render our assumption of expected utility over second order acts (KMM, Assumption 2) unfalsifiable when the construction urn exists only "in the mind of the decision-maker". We have several responses to this. First, it seems to us that there is no reason to dismiss a model simply because some of its implications might not be testable in a particular environment. It is clear that there are environments, such as the first thought experiment with two physical urns, where implications regarding second order acts are testable. Furthermore, implications regarding (first order) acts are testable even in situations where implications for second order acts might not be.

Second, when the construction urn exists only in the mind, if one were to take observability seriously, the informational assumptions in Epstein's own analysis become unfalsifiable. To see this, recall that some of the informational assumptions used to describe the thought experiment (e.g., that the construction urn contains exactly two r balls) exactly correspond to the kind of events (our second order events) that Epstein complains would be unobservable in this case.

Relatedly, one might worry that there is too much freedom if one is allowed to choose the state space *after* seeing the results of an experiment designed to test the model. However, our guiding principle in choosing the state space does not rely on the results and is the

[7] As Epstein (2010, p. 2094) suggests, "Think of the corresponding exercise for a subjective expected utility agent in an abstract state space setting." Suppose we do think in this way. The only formal sense in which one may learn that some distribution is "true" is through the process of updating beliefs over a full state space that includes all possible observations. This is the standard Bayesian model where the state space is the Cartesian product of parameters and signals. Learning in such a setting corresponds to updating by eliminating states including signals that did not occur. Thus, as more and more observations accumulate, the prior may become concentrated on the "true" parameter. Exactly as we suggest here, the standard modeling of learning the truth corresponds to a prior becoming degenerate.

one prescribed by Marschak and Radner (1972): that it should incorporate any relevant information available to the decision maker. In addition, recall that once the (first order) state space is fixed there is no further freedom, as the second order state space must be isomorphic to the set of probability distributions over the first order states.

2.4 Ambiguity of and ambiguity attitude toward second order events

Having shown that the first thought experiment is readily handled by the smooth ambiguity model, we turn to a more general question raised by the spirit of the example: Given that the smooth ambiguity model allows the individual to view some (first order) events as ambiguous (as evidenced by Ellsberg-type behavior), shouldn't such Ellsberg-type behavior toward (the intuitively more amorphous) second order events also be allowed? Not only is such behavior allowed, but, using a definition of ambiguous event that we proposed in KMM based on Ellsberg's two-color thought experiment, we show that it occurs precisely when one would expect it to.

Specifically, whenever, and only when, an event is ambiguous, the naturally associated second order events are also ambiguous. In Proposition 5.1, stated and proved in Section 5.2 of the Appendix, we show that ambiguity of a first order event E implies that non-null and non-universal second order events concerning the probability of E are treated as ambiguous. This emphasizes the point that the smooth ambiguity model property of expected utility evaluation of second order acts does *not* mean that the decision maker treats these acts as based on unambiguous events.

Moreover, ambiguity aversion for acts and second order acts is tied together: ϕ strictly concave implies strict ambiguity aversion in *both* domains. In particular, this tells us that behavior reflecting, for example, strict ambiguity aversion over (first order) acts and ambiguity neutrality or seeking over second order acts is ruled out by the smooth ambiguity model.[8]

[8] For a formal statement and proof of the result on ambiguity aversion see Section 5.3 of the Appendix. We also note that the smooth ambiguity model satisfies a recently proposed notion of two-stage probabilistic sophistication, see Amarante, Halevy and Ozdenoren (2011). This sophistication is perfectly compatible with Ellsberg-type behavior.

3 Thought Experiment 2: Hedging across sources of ambiguity

Consider the second thought experiment proposed by Epstein (2010, Section 3). There are two urns, each containing 50 balls divided among red (R) and blue (B). An individual is told that the relative proportions of red and blue in each urn are determined independently. One ball is drawn from each urn. The individual considers bets on the colors of the drawn balls with outcomes $c^* > c$ and the 50-50 lottery $\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$. Assume that lotteries are evaluated according to an expected utility function u, normalized so that $u(c^*) = 1$ and $u(c) = 0$. We can then write the acts that Epstein considers with utility payoffs as given in Table III (where $R_1 B_2$ is the event that a red ball is drawn from the first urn while a blue ball is drawn from the second urn, etc.).

[Insert Table III about here.]

Epstein argues that $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2$ and $g_1 \succ g_2$ are natural for a strictly ambiguity averse individual, and shows that these preferences are incompatible with any smooth ambiguity model with a concave ϕ. We agree with the intuition for $g_1 \succ g_2$, but disagree that $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2$ is natural for an ambiguity averse individual and think there is good reason to expect $\frac{1}{2}f_1 + \frac{1}{2}f_2 \succ f_1 \sim f_2$. The evaluation of f_1 depends on the ratio of red to blue in urn 1 but not on the composition of urn 2. Similarly, the evaluation of f_2 depends on only the ratio of red to blue in urn 2 and not on the composition of urn 1. In contrast, the evaluation of $\frac{1}{2}f_1 + \frac{1}{2}f_2$ depends on the color compositions of both urns, but has half the exposure to the uncertainty about the ratio in each urn compared to f_1 and f_2. Recall that the determination of the two urn compositions are viewed as independent. The act $\frac{1}{2}f_1 + \frac{1}{2}f_2$ thus diversifies the individual's exposure across the urns: it provides a hedging of the two independent ambiguities in the same sense as diversifying across bets on independent risks provides a hedging of the risks. To an individual who is averse to ambiguity (i.e., to subjective uncertainty about relative likelihoods), such diversification is naturally valuable.

This value is reflected in the smooth ambiguity model with concave ϕ through the fact that mean-preserving spreads in the subjective distribution of expected utilities generated by an act are disliked.[9] However, preferences such as α-MEU that ignore all except (a

[9] Epstein (2010, p. 2096) remarks that our intuition does not rely on ambiguity and claims it would equally apply to cases where there was an objective distribution over expected utilities (i.e., an "objective μ"). His reasoning ignores the fact that the individual's dislike of variation in expected utility is *only* when the variation comes from an ambiguous source – this is why it is ambiguity aversion. Just as we discussed near the end of Section 2.1, what happens when μ becomes "objective" is that, properly modeled, learning eliminates the ambiguity (μ becomes degenerate) and thus the variation in expected utility coming from an ambiguous source disappears.

fixed weighting of) the minimum and maximum possible expected utilities will miss the diversification aspect of this situation. This is extreme behavior, similar to an infinitely risk averse expected utility individual not valuing diversification across independent risks. The smooth ambiguity model delivers more moderate and, to us, reasonable behavior, as it implies that such diversification is valued by ambiguity averse individuals, while this value may vary in size as ambiguity aversion varies.[10]

The next result formally verifies this difference in behavior between the models. Let $\Omega = \{R_1, B_1\} \times \{R_2, B_2\}$ be the (first order) state space. Consider a set $C \subseteq \Delta$ of probabilities on Ω. Think of C as the set of probabilities in an α-MEU model or the support of μ in a smooth ambiguity model. Denote the set of probabilities of drawing red from urn i by $\Gamma_i = \{p(R_i) : p \in C\}$.[11] Consider the following properties on C:

(1) $\Gamma_1 = \Gamma_2$;

(2) Γ_i nonsingleton; and

(3) if $q \in \Gamma_1$ and $q' \in \Gamma_2$, there is $p \in C$ such that $p(R_1) = q$ and $p(R_2) = q'$.

Property (1) reflects symmetry across the urns as it says that the same set of compositions are considered for each urn. Without it, there is no reason to expect $f_1 \sim f_2$. Note that (1) corresponds to the concept of the urns being indistinguishable (as proposed by Walley (1991) and used e.g., in Epstein and Schneider (2003)) but not necessarily identical, which would require $p \in C$ implies $p(R_1) = p(R_2)$. Property (2) says there is ambiguity about the color composition of the urns. Without it, all of the acts in the example are unambiguous. Property (3) seems a necessary condition for independence of the urn compositions as it says that any color composition of urn 1 could be combined with any composition of urn 2.

We can now state the following result, which is proved in the Appendix. Part (i) of the result references the condition

$$\mu(p \in C : p(R_1) \in D) = \mu(p \in C : p(R_2) \in D) \quad \text{for all Borel sets } D \subseteq [0,1], \tag{3.1}$$

[10] The smooth ambiguity model (and its close relatives Nau (2006), Ergin and Gul (2009), Seo (2009) and Neilson (2010)) is not the only model capturing these intuitive choices. Many other models in the ambiguity aversion literature – e.g., invariant biseparable preferences (Ghirardato, Maccheroni and Marinacci (2004) and Amarante (2009)), variational preferences (Maccheroni, Marinacci and Rustichini (2006)), and vector expected utility preferences (Siniscalchi (2009)) – have cases compatible with the choices that we claim are intuitive.

[11] For convenience, we use p and q for probabilities here rather than π. Note the use of $p(R_i)$ in place of the more formal $p(R_i \times \{R_{j \neq i}, B_{j \neq i}\})$.

which is meant to further reflect, in the smooth ambiguity model, the perceived symmetry across urns.[12]

Proposition 3.1 *Suppose $C \subseteq \Delta$ is nonempty, closed, and satisfies properties (1)-(3). Then,*

(i) Any smooth ambiguity preference with ϕ strictly concave and μ with support C and such that condition (3.1) holds,[13] will have

$$\frac{1}{2}f_1 + \frac{1}{2}f_2 \succ f_1 \sim f_2 \quad \text{and} \quad g_1 \succ g_2.$$

(ii) Any α-MEU preference with set of probabilities C will have

$$\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2,$$

while $g_1 \succ g_2$ if and only if $\alpha > 1/2$.

In the above result, properties (1)-(3) ensure that there is some ambiguity that $\frac{1}{2}f_1 + \frac{1}{2}f_2$ hedges against. Suppose, for example, *unlike* in this thought experiment, the two urns are known to have *identical* color compositions. Then the events R_1B_2 and B_1R_2 would have unambiguously equal likelihoods, meaning that, however ambiguity resolves (i.e., whichever $p \in C$ governs the draws), it resolves the same way for each (i.e., $p(R_1B_2) = p(B_1R_2)$). In this case, $\frac{1}{2}f_1 + \frac{1}{2}f_2$ would not be expected to provide a valuable hedge as it diversifies only across these two events when compared to f_1 and f_2. Proposition 3.1 does not apply to this iid case, since the restriction to identical color compositions violates the conjunction of properties (2) and (3). It may be shown that the smooth ambiguity model (as well as α-MEU) indeed delivers $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2$ in the iid case.

To summarize our respective arguments regarding this interesting thought experiment and its implications for the smooth ambiguity model: Epstein argues that $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2$ and $g_1 \succ g_2$ are natural for a strictly ambiguity averse individual, leading to a seeming inconsistency in the modelling of ambiguity attitude in the smooth ambiguity model through ϕ. We argue that under strict ambiguity aversion, $\frac{1}{2}f_1 + \frac{1}{2}f_2 \succ f_1 \sim f_2$ is the more natural behavior. In this case, there is no conflict at all with $g_1 \succ g_2$, since both strict preferences are generated by a strictly concave ϕ in the smooth ambiguity model. Hence, we conclude, contrary to Epstein (2010), that the intuitive ambiguity averse choices

[12] The sets $\{p \in C : p(R_i \times \{R_{j\neq i}, B_{j\neq i}\}) \in D\}$ belong, for all Borel sets $D \subseteq [0,1]$, to the Borel σ-algebra of Δ (see, e.g., Aliprantis and Border (2006, Theorem 15.13)).

[13] Here the support of μ is defined as $\operatorname{supp}\mu = \bigcap \{D \text{ closed} : \mu(D) = 1\}$.

in thought experiment 2 are indeed captured by the smooth ambiguity model, whereas they are not captured by the MEU (or α-MEU) model. Beyond the specific issue of compatibility with the smooth ambiguity model, this discussion and thought experiment highlights a point we feel is fundamental in thinking about ambiguity aversion – hedging across independent but possibly non-identical sources of ambiguity makes a lot of sense. Recently, moreover, Cubitt, van de Kuilen and Mukerji (2011) have investigated experimentally whether strict ambiguity aversion is associated with preference for the act $\frac{1}{2}f_1 + \frac{1}{2}f_2$ over its components, finding evidence that it is.

4 Concluding Remarks

Our analysis of Epstein's first thought experiment shows that his results are due to the failure to use a state space allowing the incorporation of the key information defining the experiment. When one analyzes the thought experiment and the suggested variations using a full state space, the "paradox", the counter intuitive results claimed in Epstein's analyses, all go away. The criticisms Epstein draws from his results (about foundations, interpretation, separation and calibration) similarly disappear. A significant way in which the smooth ambiguity model adds to older frameworks is the ability to do meaningful comparative statics in ambiguity and ambiguity aversion while allowing great flexibility in the ambiguity of (first order) events and in ambiguity attitude and also a quite tractable functional form. This ability stems in part from the degree of separation of beliefs and taste attributes in the representation; a separation that is, as was demonstrated in our analysis, not challenged by Epstein's (2010) first thought experiment.

In analyzing the second thought experiment, we clarify the differences in behavior across models that the experiment illustrates and tie these differences to the intuitive idea that an ambiguity averse individual would want to hedge across separate sources of ambiguity unless their ambiguity attitude were extreme or the sources were guaranteed to have identical realizations of the ambiguity. The smooth ambiguity model delivers this behavior while α-MEU models cannot. In the latter, ambiguity aversion is modeled entirely through preference kinks. The smooth ambiguity model allows us to explore implications of ambiguity aversion that do not have their source in preference kinks. Kinks are not implied by ambiguity averse or Ellsbergian behavior (and, indeed, may be present without such behavior, see e.g. Segal and Spivak (1990)), yet they are what drive behavior in many applications of models like MEU or Choquet expected utility (Schmeidler (1989)) to economics and finance. Such kinks may indeed be important, but are a conceptually separate phenomenon from ambiguity attitude *per se,* and it is valuable to have models that separate the two.

All models have strengths and weaknesses, and the smooth ambiguity model is no exception. However, this reply has shown that the thought experiments at the heart of Epstein (2010) justify none of the criticisms he offers of the model.

5 Appendix

5.1 Calculations supporting Sections 2.1 and 2.2

Acts are real valued functions defined on $\Omega = \{r, b, g\} \times \{R, B, G\}$. For example, bet f_1 on R is given by

$$f_1(\omega) = \begin{cases} 100 & \text{if } \omega \in \{(r, R), (g, R), (b, R)\} \\ 0 & \text{else} \end{cases}$$

To see that $f_1 \succ f_2$, $f_4 \succ f_3$, $F_1 \succ F_2$ and $F_4 \succ F_3$ observe that

$$\begin{aligned}
f_1 &\succ f_2 \Leftrightarrow \phi\left(\frac{1}{3}\right) > \mu(\pi_1)\phi\left(\frac{5}{9}\right) + \mu(\pi_2)\phi\left(\frac{4}{9}\right) + \mu(\pi_3)\phi\left(\frac{1}{3}\right) + \mu(\pi_4)\phi\left(\frac{2}{9}\right) + \mu(\pi_5)\phi\left(\frac{1}{9}\right) \\
f_4 &\succ f_3 \Leftrightarrow \phi\left(\frac{2}{3}\right) > \mu(\pi_1)\phi\left(\frac{4}{9}\right) + \mu(\pi_2)\phi\left(\frac{5}{9}\right) + \mu(\pi_3)\phi\left(\frac{2}{3}\right) + \mu(\pi_4)\phi\left(\frac{7}{9}\right) + \mu(\pi_5)\phi\left(\frac{8}{9}\right) \\
F_1 &\succ F_2 \Leftrightarrow \phi\left(\frac{1}{3}\right) > \mu(\pi_1)\phi\left(\frac{2}{3}\right) + \mu(\pi_2)\phi\left(\frac{1}{2}\right) + \mu(\pi_3)\phi\left(\frac{1}{3}\right) + \mu(\pi_4)\phi\left(\frac{1}{6}\right) + \mu(\pi_5)\phi(0) \\
F_4 &\succ F_3 \Leftrightarrow \phi\left(\frac{2}{3}\right) > \mu(\pi_1)\phi\left(\frac{1}{3}\right) + \mu(\pi_2)\phi\left(\frac{1}{2}\right) + \mu(\pi_3)\phi\left(\frac{2}{3}\right) + \mu(\pi_4)\phi\left(\frac{5}{6}\right) + \mu(\pi_5)\phi(1).
\end{aligned}$$

Since $\mu(\pi_1) = \mu(\pi_5)$ and $\mu(\pi_2) = \mu(\pi_4)$, each of the four inequalities hold because the subjective distribution of expected utilities on the right-hand side is a mean-preserving spread of the (degenerate) distribution of expected utilities on the left-hand side and ϕ is strictly concave.

That the differences in evaluations are larger for the bets on the draws from the construction urn follows from strict concavity and the fact that the subjective distributions of expected utilities from F_2 and F_3 are mean-preserving spreads of those from f_2 and f_3 respectively given that $\mu(\pi_1) = \mu(\pi_5)$ and $\mu(\pi_2) = \mu(\pi_4)$.

The four behaviors Epstein suggests as desirable in the two scenarios may be verified as follows: The symmetry of μ_I is inherited by μ_{II} through Bayes' rule and together they assure (1); $u_I = u_{II} = u$ ensures (2); strict concavity of ϕ plus symmetry of μ_I and μ_{II} (which ensures that the induced distribution of expected utilities from betting on B is a mean-preserving spread of the distribution of expected utilities from betting on R in each scenario) implies (3); and (4) follows from the fact that the induced distribution of expected utilities from betting on B in Scenario I is a mean-preserving spread of that in Scenario II together with strict concavity of ϕ.

5.2 Results supporting Section 2.4

Here we show formally that ambiguity/unambiguity of first order events results in ambiguity/unambiguity of naturally associated second order events. To discuss ambiguity of second order events, recall from KMM that (adapted here to the Anscombe-Aumann setting) a second order act f^2 associated with an act f is defined as

$$f^2(\pi) = l_f(\pi) \qquad \forall \pi \in \Delta$$

where $l_f(\pi) \in X$ is the reduced lottery generated by f together with π. We now use this notion to define associated second order events:

Definition 5.1 *Given any $E \in \Sigma$, let I_E be the second order act associated with the act 1_E.*[14] *The collection of* associated second order events *is the sub σ-algebra $\sigma(I_E)$ of $\sigma(\Delta)$ generated by I_E.*

Observe that for any $\pi \in \Delta$, $I_E(\pi)$ is the lottery assigning probability $\pi(E)$ to the outcome 1 and the remaining probability to the outcome 0. Therefore, given E, the associated second order events are events like $\{\pi : \pi(E) \in D\}$ where D is a Borel subset of $[0, 1]$. We next write down the immediate adaptation to events in $\sigma(\Delta)$ of our (KMM, Definition 7) definition of unambiguous events in Ω.

Definition 5.2 *An event $A \in \sigma(\Delta)$ is* unambiguous *if, for each $p \in [0, 1]$ and each $x, y \in X$ such that $x \succ y$, either $[xAy \succ^2 px + (1-p)y$ and $py + (1-p)x \succ^2 yAx]$, $[xAy \prec^2 px + (1-p)y$ and $py + (1-p)x \prec^2 yAx]$ or $[xAy \sim^2 px + (1-p)y$ and $py + (1-p)x \sim^2 yAx]$. An event is* ambiguous *if it is not unambiguous.*

Notice that this definition declares an event to be ambiguous if it is impossible to calibrate the likelihood of the event against lotteries. The following results relate formally, within the smooth ambiguity model, the ambiguity/unambiguity of events in Ω with the ambiguity/unambiguity of their associated second order events.

Proposition 5.1 *Fix a smooth ambiguity model with ϕ that has some open interval of utility values over which it is strictly concave or strictly convex. An event $E \in \Sigma$ is unambiguous if and only if all the associated second order events are unambiguous.*

[14] 1_E is the indicator function for E. In this regard, note that throughout this section we adopt the normalization $u(0) = 0$ and $u(1) = 1$.

The proof makes use of the following two lemmas.

Lemma 5.1 *Let $(S, \mathcal{S}, P)$ be any probability space. A $\mathcal{S}$-measurable function $\xi : S \to \mathbb{R}$ is constant P-a.e. if and only if $P(A) \in \{0,1\}$ for all $A \in \sigma(\xi)$.*

Proof Suppose $\xi : S \to \mathbb{R}$ is constant P-a.e., i.e., there is $\bar{t} \in \mathbb{R}$ such that $P(\xi = \bar{t}) = 1$. Set $E_t = (\xi \leq t)$ for $t \in \mathbb{R}$. The σ-algebra $\sigma(\xi)$ is generated by the chain $\{E_t\}$ of all lower contour sets. Since $P(\xi = \bar{t}) = 1$, we have $P(E_t) \in \{0,1\}$ for $t \in \mathbb{R}$. Moreover, the collection $\Lambda = \{A \in \mathcal{S} : P(A) \in \{0,1\}\}$ is a λ-class. By the Dynkin Lemma, $\sigma(\xi) \subseteq \Lambda$.

As to the converse, suppose $P(A) \in \{0,1\}$ for all $A \in \sigma(\xi)$. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(t) = P(E_t)$. The cumulative density function F is increasing and right continuous. Consider the interval $I = \{t \in \mathbb{R} : F(t) = 1\}$. Set $\alpha = \inf I$. The right continuity of F implies $\alpha \in I$. Then, $P(\xi = \alpha) = 1$. For, $P(\xi \leq \alpha) = 1$ and $P(\xi < \alpha) = P\left(\bigcup_n (\xi \leq \alpha - 1/n)\right) = \lim_n P(\xi \leq \alpha - 1/n) = 0$. ■

Lemma 5.2 *Fix a smooth ambiguity model with ϕ strictly concave or strictly convex over some open interval of utility values. An event $A \in \sigma(\Delta)$ is ambiguous if and only if it is such that $0 < \mu(A) < 1$.*

Proof Let $A \in \sigma(\Delta)$ be such that $0 < \mu(A) < 1$. Without loss of generality, assume $\mu(A) \geq 1/2$ (if it is not, simply swap the roles of A and A^c). Let J be an open interval of utility values over which ϕ is strictly concave or strictly convex. For $p \in [0,1]$ and $x, y \in X$ such that $x \succ y$ and $u(x), u(y) \in J$, xAy is evaluated as $\mu(A)\phi(u(x)) + (1 - \mu(A))\phi(u(y))$, while $px + (1-p)y$ is evaluated as $\phi(pu(x) + (1-p)u(y))$. By continuity of ϕ and the fact that $0 < \mu(A) < 1$, there exists a $\hat{p} \in (0,1)$ such that

$$\mu(A)\phi(u(x)) + (1 - \mu(A))\phi(u(y)) = \phi(\hat{p}u(x) + (1 - \hat{p})u(y)).$$

If ϕ is strictly concave on J, this equality implies $\mu(A) > \hat{p}$. Similarly, strict convexity on J implies $\mu(A) < \hat{p}$.

Similarly, there exists a $\hat{q} \in (0,1)$ such that

$$\mu(A)\phi(u(y)) + (1 - \mu(A))\phi(u(x)) = \phi(\hat{q}u(y) + (1 - \hat{q})u(x)).$$

If ϕ is strictly concave on J, this equality implies $1 - \mu(A) > 1 - \hat{q}$, and so $\mu(A) < \hat{q}$. Strict convexity on J similarly implies $\mu(A) > \hat{q}$. Therefore, either $\hat{q} > \hat{p}$ and $yAx \sim^2 \hat{q}y + (1 - \hat{q})x \prec^2 \hat{p}y + (1-\hat{p})x$ (under strict concavity) or $\hat{q} < \hat{p}$ and $yAx \sim^2 \hat{q}y + (1-\hat{q})x \succ^2 \hat{p}y + (1-\hat{p})x$

(under strict convexity). This shows that A is ambiguous since $xAy \sim^2 \hat{p}x + (1-\hat{p})y$ and $yAx \nsim^2 \hat{p}y + (1-\hat{p})x$.

For the other direction, it is enough to observe that $\mu(A) \in \{0,1\}$ implies that A is unambiguous. ■

Proof of Proposition 5.1 Observe that, denoting any lottery between the outcomes 0 and 1 by the probability assigned to 1, we can view I_E as a real-valued function given by $I_E(\pi) = \pi(E)$ for all $\pi \in \Delta$. Since ϕ is strictly concave or strictly convex on some open interval of utility values, by Theorem 3 of KMM an event $E \in \Sigma$ is unambiguous if and only if I_E is constant μ-a.e. By Lemma 5.1, this happens if and only if $\mu(A) \in \{0,1\}$ for all $A \in \sigma(I_E)$. By Lemma 5.2, this is equivalent to requiring that all $A \in \sigma(I_E)$ are unambiguous. We conclude that $E \in \Sigma$ is unambiguous if and only if all $A \in \sigma(I_E)$ are unambiguous, as desired. ■

5.3 Linking ambiguity aversion over acts and second order acts

Observe from the definition of unambiguous event and continuity that for an event E to be ambiguous, there must be some $p \in [0,1]$ and $x, y \in X$ with $x \succ y$ such that the lottery $px + (1-p)y$ is either strictly better than both xEy and yEx, strictly worse than both, or indifferent to one and strictly ranked relative to the other. Note that in any model where preference for the lottery $px + (1-p)y$ is increasing and continuous in p, we can ignore the cases involving indifference, as when they exist one of the strict cases occurs as well. As xEy and yEx involve ambiguity but the lottery does not, it is natural to call strictly ambiguity averse the case where the lottery is strictly better than both, and strictly ambiguity seeking the case where the lottery is strictly worse than both. In this vein, let us call a preference $\hat{\succsim}$ *strictly ambiguity averse* if, given any ambiguous event E and $x, y \in X$ such that $x \hat{\succ} y$, there exists a $p \in (0,1)$ such that

$$xEy \hat{\prec} px + (1-p)y \text{ and } yEx \hat{\prec} py + (1-p)x \tag{5.1}$$

and for no $p \in [0,1]$ is it true that

$$xEy \hat{\succ} px + (1-p)y \text{ and } yEx \hat{\succ} py + (1-p)x. \tag{5.2}$$

The result below implies what we claimed regarding ambiguity aversion in Section 2.4 – ϕ strictly concave implies preferences over acts and over second order acts are strictly ambiguity averse. An analogous proposition holds for strict ambiguity seeking when ϕ is convex rather than concave.

Proposition 5.2 *Fix a smooth ambiguity model with ϕ concave. The following are equivalent:*

(i) $\succsim$ *is strictly ambiguity averse;*

(ii) $\succsim^2$ *is strictly ambiguity averse;*

(iii) ϕ *is strictly concave.*

Proof We prove separately the equivalence of (i) and (iii) and of (ii) and (iii).

(iii) implies (i): Suppose ϕ is strictly concave. We want to show that $\succsim$ is strictly ambiguity averse. Let $x, y \in X$, with $y \prec x$, and $E \in \Sigma$ be an ambiguous event. By setting $p = \int \pi(E)\, d\mu(\pi)$ it is easy to see that

$$xEy \prec px + (1-p)y \quad \text{and} \quad yEx \prec py + (1-p)x$$

It remains to show that there is no $p \in [0,1]$ such that

$$xEy \succ px + (1-p)y \quad \text{and} \quad yEx \succ py + (1-p)x.$$

Suppose *per contra* there is such a p. Since $y \prec x$, by the continuity of ϕ there is $1 \geq p' > p$ such that

$$\int \phi\left(\pi(E)\, u(x) + (1 - \pi(E)\, u(y))\right) d\mu(\pi) = \phi(p'u(x) + (1-p')u(y)).$$

Since ϕ is strictly concave, this equality implies $\int \pi(E)\, d\mu(\pi) > p' > p$.

Similarly, there is $0 < p'' < p$ such that

$$\int \phi\left(\pi(E)\, u(y) + (1 - \pi(E)\, u(x))\right) d\mu(\pi) = \phi(p''u(y) + (1-p'')u(x)).$$

Since ϕ is strictly concave, this equality implies $\int \pi(E)\, d\mu(\pi) < p'' < p$, a contradiction.

(i) implies (iii): Suppose $\succsim$ is strictly ambiguity averse. Suppose *per contra* that ϕ is not strictly concave. Then there exist $u(x) > u(y)$ such that, for all $\alpha \in [0,1]$,

$$\phi\left(\alpha u(x) + (1-\alpha)\, u(y)\right) = \alpha \phi\left(u(x)\right) + (1-\alpha)\, \phi\left(u(y)\right). \tag{5.3}$$

Let $E \in \Sigma$ be ambiguous. For each $\pi \in \operatorname{supp} \mu$ it holds that

$$\begin{aligned} \phi\left(\pi(E)\, u(x) + (1-\pi(E))\, u(y)\right) &= \pi(E)\, \phi\left(u(x)\right) + (1-\pi(E))\, \phi\left(u(y)\right), \text{ and} \\ \phi\left((1-\pi(E))\, u(x) + \pi(E)\, u(y)\right) &= (1-\pi(E))\, \phi\left(u(x)\right) + \pi(E)\, \phi\left(u(y)\right), \end{aligned}$$

and so, by setting $p = \int \pi(E)\, d\mu(\pi)$,

$$\begin{aligned}\int \phi\left(\pi(E)\, u(x) + (1-\pi(E))\, u(y)\right) d\mu(\pi) &= p\phi(u(x)) + (1-p)\,\phi(u(y)) \\ &= \phi(pu(x) + (1-p)\, u(y))\end{aligned}$$

and

$$\begin{aligned}\int \phi\left((1-\pi(E))\, u(x) + \pi(E)\, u(y)\right) d\mu(\pi) &= (1-p)\,\phi(u(x)) + p\phi(u(y)) \\ &= \phi((1-p)\, u(x) + pu(y))\,.\end{aligned}$$

Hence, both $xEy \sim px + (1-p)y$ and $yEx \sim py + (1-p)x$. Since $\succsim$ is strictly ambiguity averse, there is a $q \in (0,1)$ such that

$$px + (1-p)y \sim xEy \prec qx + (1-q)y \quad \text{and} \quad py + (1-p)x \sim yEx \prec qy + (1-q)x.$$

In turn, this implies $pu(x) + (1-p)\, u(y) < qu(x) + (1-q)\, u(y)$ and $pu(y) + (1-p)\, u(x) < qu(y) + (1-q)\, u(x)$, that is,

$$\begin{aligned}u(y) + u(x) &= p\,(u(x) + u(y)) + (1-p)\,(u(y) + u(x)) \\ &< q\,(u(x) + u(y)) + (1-q)\,(u(y) + u(x)) = u(x) + u(y)\,,\end{aligned}$$

a contradiction. We conclude that ϕ is strictly concave.

(iii) implies (ii) Suppose ϕ is strictly concave. We want to show that $\succsim^2$ is strictly ambiguity averse. Let $x, y \in X$, with $y \prec^2 x$, and $A \in \sigma(\Delta)$ be an ambiguous event, i.e., $0 < \mu(A) < 1$. By setting $p = \mu(A)$ it is easy to see that

$$xAy \prec^2 px + (1-p)y \quad \text{and} \quad yAx \prec^2 py + (1-p)x.$$

It remains to show that there is no $p \in [0,1]$ such that

$$xAy \succ^2 px + (1-p)y \quad \text{and} \quad yAx \succ^2 py + (1-p)x.$$

Suppose *per contra* there is such a p. Since $y \prec^2 x$, by the continuity of ϕ there is $1 \geq p' > p$ such that

$$\mu(A)\phi(u(x)) + (1-\mu(A))\phi(u(y)) = \phi(p'u(x) + (1-p')u(y)).$$

Since ϕ is strictly concave, this equality implies $\mu(A) > p' > p$.

Similarly, there is $0 < p'' < p$ such that

$$\mu(A)\phi(u(y)) + (1 - \mu(A))\phi(u(x)) = \phi(p''u(y) + (1 - p'')u(x)).$$

Since ϕ is strictly concave, this equality implies $\mu(A) < p'' < p$, a contradiction.

(ii) implies (iii) Suppose $\succsim^2$ is strictly ambiguity averse. Suppose *per contra* that ϕ is not strictly concave. Then there exist $u(x) > u(y)$ such that (5.3) holds for all $\alpha \in [0, 1]$. Let $A \in \sigma(\Delta)$ be an ambiguous event. It holds that

$$\begin{aligned}
\phi\left(\mu\left(A\right)u\left(x\right) + \left(1 - \mu\left(A\right)\right)u\left(y\right)\right) &= \mu\left(A\right)\phi\left(u\left(x\right)\right) + \left(1 - \mu\left(A\right)\right)\phi\left(u\left(y\right)\right), \text{ and} \\
\phi\left(\left(1 - \mu\left(A\right)\right)u\left(x\right) + \mu\left(A\right)u\left(y\right)\right) &= \left(1 - \mu\left(A\right)\right)\phi\left(u\left(x\right)\right) + \mu\left(A\right)\phi\left(u\left(y\right)\right)
\end{aligned}$$

and so, by setting $p = \mu\left(A\right)$,

$$px + (1 - p)\,y \sim^2 xAy \quad \text{and} \quad py + (1 - p)x \sim^2 yAx.$$

Since $\succsim^2$ is strictly ambiguity averse, there is a $q \in (0, 1)$ such that

$$qx + (1 - q)y \succ^2 xAy \sim^2 px + (1 - p)y \quad \text{and} \quad qy + (1 - q)x \succ^2 yAx \sim^2 py + (1 - p)x.$$

In turn, this implies $pu\left(x\right) + (1 - p)\,u\left(y\right) < qu\left(x\right) + (1 - q)\,u\left(y\right)$ and $pu\left(y\right) + (1 - p)\,u\left(x\right) < qu\left(y\right) + (1 - q)\,u\left(x\right)$, which, as seen before, leads to a contradiction. We conclude that ϕ is strictly concave. ■

5.4 Proof of Proposition 3.1

Abbreviate $p(R_1 \times \{R_2, B_2\})$ by $p(R_1)$ and so on. Observe that properties (2) and (3) imply that there exist $p \in C$ such that $p\left(R_1\right) \neq p\left(R_2\right)$.

(i) Suppose $\operatorname{supp}\mu = C$ and $\mu\left(p \in C : p\left(R_1\right) \in D\right) = \mu\left(p \in C : p\left(R_2\right) \in D\right)$ for all Borel sets D in $[0, 1]$. Since ϕ is strictly increasing, by (1) we have $\{(\phi \circ p)\left(R_1\right) : p \in C\} = \{(\phi \circ p)\left(R_2\right) : p \in C\}$, and so $\int_\Delta (\phi \circ p)\left(R_1\right) d\mu\left(p\right) = \int_\Delta (\phi \circ p)\left(R_2\right) d\mu\left(p\right)$ because of the assumption on μ. Hence, $f_1 \sim f_2$. On the other hand,

$$\phi\left(\frac{1}{2}p\left(R_1\right) + \frac{1}{2}p\left(R_2\right)\right) \geq \frac{1}{2}\left(\phi \circ p\right)\left(R_1\right) + \frac{1}{2}\left(\phi \circ p\right)\left(R_2\right), \qquad \forall p \in \operatorname{supp}\mu,$$

with strict inequality if $p\left(R_1\right) \neq p\left(R_2\right)$.

Claim There is a Borel set $A \subseteq \operatorname{supp}\mu$, with $\mu\left(A\right) > 0$, such that $p\left(R_1\right) \neq p\left(R_2\right)$ for all

$p \in A$.

Proof of the Claim As shown at the start of the proof, there is $\overline{p} \in \operatorname{supp} \mu$ such that $\overline{p}(R_1) \neq \overline{p}(R_2)$. Suppose first that $\overline{p}$ is an isolated point in $\operatorname{supp} \mu$. Then, $\mu(\overline{p}) > 0$ and the claim trivially holds. Suppose that $\overline{p}$ is not an isolated point in $\operatorname{supp} \mu$. Then, $B_\varepsilon(\overline{p}) \cap \operatorname{supp} \mu \neq \emptyset$ for every neighborhood $B_\varepsilon(\overline{p})$ of $\overline{p}$. Since $\overline{p}(R_1) \neq \overline{p}(R_2)$, by taking ε small enough there is $B_\varepsilon(\overline{p})$ such that $p(R_1) \neq p(R_2)$ for all $p \in B_\varepsilon(\overline{p})$. By setting $A = B_\varepsilon(\overline{p}) \cap \operatorname{supp} \mu$, this proves the claim since $\mu(A) > 0$ because $B_\varepsilon(\overline{p}) \cap \operatorname{supp} \mu \neq \emptyset$. For, if $\mu(A) = 0$, then $\mu(B_\varepsilon(\overline{p})) = \mu(A) + \mu(B_\varepsilon(\overline{p}) \cap (\operatorname{supp} \mu)^c) = 0$, and so $\operatorname{supp} \mu \subseteq B_\varepsilon(\overline{p})^c$, a contradiction (see Aliprantis and Border (2006, p. 442)). $\triangle$

The Claim implies

$$\int \phi\left(\frac{1}{2}p(R_1) + \frac{1}{2}p(R_2)\right) d\mu(p) > \frac{1}{2}\int (\phi \circ p)(R_1)\, d\mu(p) + \frac{1}{2}\int (\phi \circ p)(R_2)\, d\mu(p),$$

that is, $\frac{1}{2}f_1 + \frac{1}{2}f_2 \succ f_1 \sim f_2$.

Act g_1 is evaluated as $\phi(1/2)$. Act g_2 is evaluated as $\int \phi(1/2 + (p(B_1R_2) - p(R_1B_2))/2)\, d\mu(p)$. Define $\gamma : \Delta \to \mathbb{R}$ by $\gamma(p) = 1/2 + (p(B_1R_2) - p(R_1B_2))/2$. Since $p(B_1R_2) - p(R_1B_2) = p(R_2) - p(R_1)$, the Claim implies $\gamma(p) \neq 1/2$ for all $p \in A$. Therefore, by the Jensen inequality and the assumption on μ, we have

$$\int (\phi \circ \gamma)(p)\, d\mu(p) < \phi\left(\int \gamma(p)\, d\mu(p)\right) = \phi\left(\int \left(\frac{1}{2} + \frac{1}{2}(p(R_2) - p(R_1))\right) d\mu(p)\right) = \phi\left(\frac{1}{2}\right),$$

that is, $g_1 \succ g_2$.

(ii) By properties (1) and (3), $\max_{p\in C} p(R_1) = \max_{p\in C} p(R_2)$ and $\min_{p\in C} p(R_1) = \min_{p\in C} p(R_2)$, as well as

$$\begin{aligned}
\max_{p\in C}\left(\frac{1}{2}p(R_1) + \frac{1}{2}p(R_2)\right) &= \frac{1}{2}\max_{p\in C} p(R_1) + \frac{1}{2}\max_{p\in C} p(R_2) \\
\min_{p\in C}\left(\frac{1}{2}p(R_1) + \frac{1}{2}p(R_2)\right) &= \frac{1}{2}\min_{p\in C} p(R_1) + \frac{1}{2}\min_{p\in C} p(R_2)
\end{aligned}$$

Hence, $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2$. From $\min_{p\in C}(p(R_2) - p(R_1)) = -\max_{p\in C}(p(R_2) - p(R_1))$,

$$\begin{aligned}
&\alpha \min_{p\in C}\left(\frac{1}{2} + \frac{1}{2}(p(R_2) - p(R_1))\right) + (1-\alpha)\max_{p\in C}\left(\frac{1}{2} + \frac{1}{2}(p(R_2) - p(R_1))\right) \\
&= \frac{1}{2} + \frac{1-2\alpha}{2}\max_{p\in C}(p(R_2) - p(R_1)),
\end{aligned}$$

and so $g_1 \succ g_2$ if and only if $1/2 > 1/2 + (1/2 - \alpha) \max_{p \in C} (p(R_2) - p(R_1))$. By properties (2) and (3), $\max_{p \in C} (p(R_2) - p(R_1)) > 0$, so that $g_1 \succ g_2$ if and only if $\alpha > 1/2$. ■

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Table I

Epstein's state space

		2nd order states		
		π_r	π_b	π_g
		Draws from the construction urn		
1st order states	Draws from Ellsberg urn↓	r	b	g
	R	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
	B	$\frac{1}{3}$	$\frac{2}{3}$	0
	G	$\frac{1}{3}$	0	$\frac{2}{3}$

Table II

A full state space

		2nd order states				
		π_1	π_2	π_3	π_4	π_5
		Composition of the construction urn				
1st order states	Draws↓	$(2r, 4b, 0g)$	$(2r, 3b, 1g)$	$(2r, 2b, 2g)$	$(2r, 1b, 3g)$	$(2r, 0b, 4g)$
	(r, R)	$\frac{2}{18}$	$\frac{2}{18}$	$\frac{2}{18}$	$\frac{2}{18}$	$\frac{2}{18}$
	(r, B)	$\frac{2}{18}$	$\frac{2}{18}$	$\frac{2}{18}$	$\frac{2}{18}$	$\frac{2}{18}$
	(r, G)	$\frac{2}{18}$	$\frac{2}{18}$	$\frac{2}{18}$	$\frac{2}{18}$	$\frac{2}{18}$
	(b, R)	$\frac{4}{18}$	$\frac{3}{18}$	$\frac{2}{18}$	$\frac{1}{18}$	0
	(b, B)	$\frac{8}{18}$	$\frac{6}{18}$	$\frac{4}{18}$	$\frac{2}{18}$	0
	(b, G)	0	0	0	0	0
	(g, R)	0	$\frac{1}{18}$	$\frac{2}{18}$	$\frac{3}{18}$	$\frac{4}{18}$
	(g, B)	0	0	0	0	0
	(g, G)	0	$\frac{2}{18}$	$\frac{4}{18}$	$\frac{6}{18}$	$\frac{8}{18}$

Table III

Acts with utility payoffs for Experiment 2

	R_1R_2	R_1B_2	B_1R_2	B_1B_2
f_1	1	1	0	0
f_2	1	0	1	0
$\frac{1}{2}f_1 + \frac{1}{2}f_2$	1	$\frac{1}{2}$	$\frac{1}{2}$	0
g_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
g_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$