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Simone Cerreia-Vioglio

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IGIER – Università Bocconi, Via Guglielmo Röntgen 1, 20136 Milano –Italy
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Objective Rationality and Uncertainty Averse Preferences*

Simone Cerreia-Vioglio
Department of Decision Sciences and IGIER, Università Bocconi

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Abstract

We provide a bridge between Bewley preferences [2] and Uncertainty averse preferences [4]. In doing this, we generalize the findings of Gilboa, Maccheroni, Marinacci, and Schmeidler [11]. To exemplify this new framework, we then study a class of preferences that we call Constrained Multiplier preferences and that was first proposed by Wang [19].

1 Introduction

Two of the most successful decision theoretic approaches dealing with Knightian uncertainty consisted in either remove the Completeness of preferences but retain the Independence assumption à la Bewley or to maintain Completeness of preferences but to weaken Independence and consider Uncertainty Aversion à la Gilboa and Schmeidler. The first approach was first proposed by Bewley [2]. The second approach instead was followed by many models in the literature and pioneered by Gilboa and Schmeidler [10]. Recently, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [4] (henceforth, CMMM) provided a general representation result for Uncertainty averse preferences (henceforth, also, UA preferences). This work formally shows what is the link between the two aforementioned approaches toward Knightian uncertainty.

In a similar context, Gilboa, Maccheroni, Marinacci, and Schmeidler [11] (henceforth, GMMS) argued that a Decision Maker (henceforth, DM) is characterized by two binary relations: $(\succsim^*, \hat{\succsim})$.[1] These binary relations are assumed to capture different kinds of the DM's rationality. In particular, the first one is assumed to capture the part of the DM's rankings that appear to him as uncontroversial (*objective rationality*) while the second one is assumed to capture the rankings that the DM express if he has to make a choice (*subjective rationality*). More precisely, $(\succsim^*, \hat{\succsim})$ are two binary relations defined in the classic setting of Anscombe and Aumann [1], that is, over $\mathcal{F}$: the set of all simple Σ-measurable acts from S to X where S is a state space, Σ is an event algebra, and X is a convex set of consequences. In this paper and in [11], the first binary relation is assumed to be a Bewley preference relation, that is,

$$f \succsim^* g \quad \Longleftrightarrow \quad \int u^*(f)\, dp \geq \int u^*(g)\, dp \qquad \forall p \in C \tag{1}$$

where u^* is an affine utility index and C is a closed and convex set of probabilities. $\succsim^*$ represents the rankings that the DM can strongly justify. In fact, whenever $f \succsim^* g$, such a ranking is independent

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[1] For a related approach see also Nehring [15].

of the probabilistic scenario chosen in C. The relation $\succsim^*$ might be also thought as capturing the relevant probabilistic information of the DM.[2] Conversely, the second binary relation is assumed to be an Uncertainty Averse preference relation, that is,

$$f \succsim\hat{} g \quad \Longleftrightarrow \quad \min_{p\in\Delta} G\left(\int u(f)\,dp, p\right) \geq \min_{p\in\Delta} G\left(\int u(g)\,dp, p\right) \tag{2}$$

where u is an affine utility index, Δ is the set of all probabilities over Σ, and G is an index of uncertainty aversion as in [4]. The relation $\succsim\hat{}$ is assumed to represent the DM's rankings if he has to make a choice. The two preferences are considered to represent the same DM's rankings. In fact, the second one is also assumed to be a completion of the first one, that is,

$$f \succsim^* g \quad \Longrightarrow \quad f \succsim\hat{} g.$$

Moreover, in order to make this connection sharper, we axiomatically characterize when the two utility indexes, u^* and u, coincide and when the set of probabilities representing the Bewley preference relation $\succsim^*$ is the same one characterizing $\succsim\hat{}$, that is, when C is also the smallest subset of Δ over which the min in (2) can be considered to be over. In other words, the DM can be seen as acting in the following way. First, he identifies the set of relevant probabilistic scenarios C and a utility index u. The set C and the function u characterize $\succsim^*$ à la Bewley. Then, he chooses an index of uncertainty aversion G. This allows him to consider certain probabilistic scenarios more plausible than others. Finally, he uses these three objects to form consistently his preferences $\succsim\hat{}$. This is done according to the cautious rule in (2) and by just considering the probabilities in C. In other words, the completion procedure considered in this paper still reflect the same probabilistic information of $\succsim^*$ and is consistent with the utility index u^*.

According to this interpretation, GMMS restrict themselves to the case where the index of uncertainty aversion G is the one characterizing the Gilboa and Schmeidler preferences. In this way, they force the DM to consider each probability in C equally plausible. In this paper, we remove this limitation (see Theorems 4 and 12). We are thus able to provide a bridge between Bewley preferences and a vast class of UA preferences. We dub this class *effectively bounded Uncertainty averse preferences* and we show it to be dense in the class of UA preferences (see Proposition 6).

Particular cases of UA preferences are Gilboa and Schmeidler preferences [10] and Multiplier preferences, where the latter were introduced by Hansen and Sargent [12] and axiomatized by Strzalecki [18] as a special class of the Variational preferences of Maccheroni, Marinacci, and Rustichini [14]. In the first case, preferences are represented by $V_{GS} : \mathcal{F} \to \mathbb{R}$ defined by

$$V_{GS}(f) = \min_{p\in C} \int u(f)\,dp \qquad \forall f \in \mathcal{F} \tag{3}$$

where u is an affine utility index and C is a closed and convex subset of Δ. In the second case, Σ is a σ-algebra and preferences are represented by $V_{\theta,HS} : \mathcal{F} \to \mathbb{R}$ defined by

$$V_{\theta,HS}(f) = \min_{p<<q} \left\{\int u(f)\,dp + \theta R(p||q)\right\} \qquad \forall f \in \mathcal{F}. \tag{4}$$

Here, $\theta \in [0,\infty)$, q is a countably additive probability measure on Σ, $R(p||q)$ is the relative entropy of p with respect to q,[3] and $u : X \to \mathbb{R}$ is an affine function.

[2] For a similar interpretation, see also Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [6].

[3] That is,

$$R(p||q) = \begin{cases} \int \log\left(\frac{dp}{dq}\right) dp & p \in \Delta^\sigma(q) \\ \infty & p \notin \Delta^\sigma(q) \end{cases}$$

where $\Delta^\sigma(q) = \{p \in \Delta : p << q \text{ and } p \text{ is countably additive}\}$.

In terms of CMMM' representation (see [14] and [4, Proposition 12]), Gilboa and Schmeidler preferences correspond to a function G_{GS} such that

$$G_{GS}(t,p) = t + \delta_C(p) \qquad \forall (t,p) \in \mathbb{R} \times \Delta \quad \text{and} \quad \delta_C(p) = \begin{cases} 0 & p \in C \\ \infty & p \notin C \end{cases}.$$

Similarly, Multiplier preferences correspond to a function $G_{\theta,HS}$ such that

$$G_{\theta,HS}(t,p) = t + \theta R(p||q) \qquad \forall (t,p) \in \mathbb{R} \times \Delta.$$

To further exemplify the previous procedure we study a preference model that merges the insights of the Gilboa and Schmeidler model to some of the features characterizing Multiplier preferences. Assume that the DM is considering the probability measure q to be the reference model but for robustness arguments he is willing to consider as plausible the following family of perturbed probabilities:

$$C_\eta = \{p \in \Delta^\sigma(q) : R(p||q) \leq \eta\} \text{ where } \eta > 0.$$

Given a utility index u, it is compelling for him to declare

$$f \succsim^* g \iff \int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C_\eta.$$

In fact, if $f \succsim^* g$ then, no matter which probability in C_η he considers, f is weakly better than g. Nevertheless, $\succsim^*$ is incomplete given that typically C_η contains more than one element. Therefore, he might need a decision criterion that always allows him to compare two acts. Moreover, such a criterion should reflect the rankings and the probabilistic information contained in $\succsim^*$. One sensible way to proceed would be to follow the same reasoning that informed the shaping of the DM's set of relevant probability models C_η. That is, the DM might want to evaluate the plausibility of a probabilistic scenario p according to its statistical distance from the reference model q. This leads to the following preferences $\hat{\succsim}$

$$f \,\hat{\succsim}\, g \iff \min_{p \in C_\eta} \left\{ \int u(f)\,dp + \theta R(p||q) \right\} \geq \min_{p \in C_\eta} \left\{ \int u(g)\,dp + \theta R(p||q) \right\}, \tag{5}$$

where $\theta > 0$. We call preferences represented as in (5) Constrained Multiplier preferences. These preferences were first proposed by Wang [19].

2 Preliminaries

2.1 Decision Theoretic Set Up

We consider a nonempty set S of *states of the world*, an algebra Σ of subsets of S called *events*, and a set X of *consequences*. We denote by $\mathcal{F}$ the set of all *(simple) acts*: functions $f : S \to X$ that are Σ-measurable and take finitely many values.

Given any $x \in X$, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. With the usual slight abuse of notation, we thus identify X with the subset of constant acts in $\mathcal{F}$.

We assume additionally that X is a convex subset of a vector space. For instance, this is the case if X is the set of all lotteries on a set of outcomes, as it happens in the classic setting of Anscombe and Aumann [1]. Using the linear structure of X, we define a mixture operation over $\mathcal{F}$. For each $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ is defined to be such that $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s) \in X$ for all $s \in S$.

We model the DM's *preferences* on $\mathcal{F}$ by two binary relations $(\succsim^*, \hat{\succsim})$. Given a binary relation $\succsim$ on $\mathcal{F}$, $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of $\succsim$. Given $f \in \mathcal{F}$, an element $x_f \in X$ is a *certainty equivalent* for f if and only if $f \sim x_f$.

2.2 Mathematical Preliminaries

We denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions so that $u(f) \in B_0(\Sigma)$ whenever $u : X \to \mathbb{R}$ is affine and $f \in \mathcal{F}$. Given an affine function $u : X \to \mathbb{R}$, we denote by $B_0(\Sigma, u(X))$ the set of all real-valued Σ-measurable simple functions that take values in $u(X)$.

As well known, the dual space of $B_0(\Sigma)$ can be identified with the set $ba(\Sigma)$ of all bounded finitely additive measures on (S, Σ). The set of probabilities in $ba(\Sigma)$ is denoted by Δ and is a (weak*) compact and convex subset of $ba(\Sigma)$. Δ is considered to be endowed with the topology inherited from the weak* topology. $\mathbb{R}$ is considered to be endowed with the usual topology. $\mathbb{R} \times \Delta$ is considered to be endowed with the product topology. Elements of Δ are denoted by p and q.

When Σ is a σ-algebra, we denote by Δ^σ the set of all countably additive probabilities in Δ. In particular, given $q \in \Delta^\sigma$, we denote by $\Delta^\sigma(q)$ the set of all probabilities in Δ^σ that are absolutely continuous with respect to q, that is, $\Delta^\sigma(q) = \{p \in \Delta^\sigma : p \ll q\}$.

Functions of the form $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ play a key role in the results of CMMM and ours. We denote

$$dom_\Delta G = \{p \in \Delta : G(t, p) < \infty \text{ for some } t \in \mathbb{R}\}.$$

Borrowing and modifying the notation of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [5], we denote by $\mathcal{L}_n(\mathbb{R} \times \Delta)$ the class of such functions that satisfy the following requirements:

(i) G is quasiconvex and lower semicontinuous on $\mathbb{R} \times \Delta$;

(ii) $G(\cdot, p)$ is increasing for all $p \in \Delta$;

(iii) $\min_{p \in \Delta} G(t, p) = t$ for all $t \in \mathbb{R}$.

Similarly, we say that $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ if and only if $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ and G satisfies

(iv) $\sup_{p \in dom_\Delta G} G(t, p) < \infty$ for all $t \in \mathbb{R}$.

Given a function $c : \Delta \to [0, \infty]$, we say that c is grounded if and only if $\min_{p \in \Delta} c(p) = 0$. We denote

$$dom(c) = \{p \in \Delta : c(p) < \infty\}.$$

It is immediate to see that if c is a grounded, convex, and lower semicontinuous function then the map $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$, defined by

$$G(t, p) = t + c(p) \qquad \forall (t, p) \in \mathbb{R} \times \Delta, \tag{6}$$

belongs to $\mathcal{L}_n(\mathbb{R} \times \Delta)$ and $dom(c) = dom_\Delta G$. Moreover, in this case, $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ if and only if there exists $k \in \mathbb{R}$ such that $dom(c) = \{p \in \Delta : c(p) \leq k\}$, that is, if and only if c is bounded on its *effective domain* $dom(c)$.

Finally, a function $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ is said to be *linearly continuous* if and only if the map

$$\varphi \mapsto \inf_{p \in \Delta} G\left(\int \varphi dp, p\right)$$

from $B_0(\Sigma)$ to $[-\infty, \infty]$ is extended-valued continuous. The function G defined in (6) is linearly continuous.

3 The Axiomatic Framework

3.1 Basic Axioms

In this work, we consider the DM to be endowed with two different binary relations. We interpret the first one, potentially incomplete, as capturing the rankings that appear to the DM as uncontroversial while the second one captures the rankings of the DM if he is forced to make a choice or express a preference. GMMS [11] dub the first one Objective Rationality and the second one Subjective Rationality. Following [11], we denote these two binary relations, respectively, $\succsim^*$ and $\succsim\,\hat{}$. In this subsection, we list the assumptions that we impose on these two binary relations. In Subsection 4.2, we compare these axioms with the ones of [11].

Next, we list a set of axioms that we impose both on $\succsim^*$ and $\succsim\,\hat{}$. We state them for a generic binary relation $\succsim$ on $\mathcal{F}$.

Basic Conditions:

Preorder: $\succsim$ *is reflexive, transitive, and nontrivial.*

Monotonicity: If $f, g \in \mathcal{F}$ *and* $f(s) \succsim g(s)$ *for all* $s \in S$ *then* $f \succsim g$.

Mixture Continuity: If $f, g, h \in \mathcal{F}$ *then the sets* $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim h\}$ *and* $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda) g\}$ *are closed in* $[0,1]$.

Unboundedness: *For each* x *and* y *in* X *such that* $x \succ y$ *there are* $z, z' \in X$ *such that*

$$\frac{1}{2}z + \frac{1}{2}y \succsim x \succ y \succsim \frac{1}{2}x + \frac{1}{2}z'.$$

The Basic Conditions are standard assumptions in an Anscombe and Aumann setting. They are satisfied by a vast class of decision theoretic models discussed in the literature of choice under Knightian uncertainty. On the other hand, Unboundedness is a technical assumption which will result in imposing that there are arbitrarily good and arbitrarily bad consequences. We refer the interested reader to [11] for a more complete discussion of the Basic Conditions as fundamental tenets of rational behavior.

Next, we list the assumptions that are peculiar for $\succsim^*$:

C-Completeness: *If* $x, y \in X$ *either* $x \succsim^* y$ *or* $y \succsim^* x$.

Independence: *If* $f, g, h \in \mathcal{F}$ *and* $\alpha \in (0,1)$

$$f \succsim^* g \iff \alpha f + (1-\alpha) h \succsim^* \alpha g + (1-\alpha) h.$$

The last two assumptions matched with the Basic Conditions imply that the DM has complete preferences over the set of consequences and his preferences on X are further represented by a non-constant and affine utility index $u : X \to \mathbb{R}$. In the original Anscombe and Aumann setting, this is equivalent to say that when the DM faces objective probabilities he is a standard Expected Utility DM. On the other hand, it follows that, under the Basic Conditions, $\succsim^*$ admits a representation à la Bewley [2].

Definition 1 *Let* $\succsim^*$ *be a binary relation on* $\mathcal{F}$. $\succsim^*$ *is a Bewley preference if and only if it satisfies the Basic Conditions, C-Completeness, and Independence.*

The next three assumptions instead are peculiar to $\succsim\,\hat{}$:

Completeness: *If* $f, g \in \mathcal{F}$ *either* $f \succsim\hat{} g$ *or* $g \succsim\hat{} f$.

Risk Independence: *If* $x, y, z \in X$ *and* $\alpha \in (0,1)$

$$x \succsim\hat{} y \iff \alpha x + (1-\alpha) z \succsim\hat{} \alpha y + (1-\alpha) z.$$

Uncertainty Aversion: *If* $f, g \in \mathcal{F}$ *are such that* $f \sim\hat{} g$ *then* $\alpha f + (1-\alpha) g \succsim\hat{} f$ *for all* $\alpha \in (0,1)$.

Given the interpretation we chose for $\succsim\hat{}$ and since a DM might need a complete ranking, Completeness seems to be a natural assumption. Differently, in a problem of choice under Knightian uncertainty, Uncertainty Aversion imposes that the DM responds cautiously to such uncertainty by exhibiting a preference toward hedging.[4] Finally, under the Basic Conditions, Completeness and Risk Independence force the DM to have complete preferences over constant acts where these preferences are further represented by a nonconstant and affine utility index $u : X \to \mathbb{R}$. So even for $\succsim\hat{}$, we take the stance that a DM who faces objective probabilities behaves as a standard Expected Utility DM. Notice that Risk Independence is the assumption of Independence just restricted to constant acts where Knightian uncertainty has no bite.[5]

Definition 2 *Let* $\succsim\hat{}$ *be a binary relation on* $\mathcal{F}$. $\succsim\hat{}$ *is an Uncertainty averse preference if and only if it satisfies the Basic Conditions, Completeness, Risk Independence, and Uncertainty Aversion.*

Theorem 3 (CMMM, Theorems 3 and 5) *Let* $\succsim\hat{}$ *be a binary relation on* $\mathcal{F}$. $\succsim\hat{}$ *is an Uncertainty averse preference that satisfies Unboundedness if and only if there exist an onto and affine function* $u : X \to \mathbb{R}$ *and a linearly continuous* $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ *such that* (u, G) *represent* $\succsim\hat{}$ *as in (2). Moreover,* u *is cardinally unique and, given* u, G *is unique.*

The next two assumptions connect to each other the previous two binary relations.

Consistency: *If* $f \succsim^* g$ *then* $f \succsim\hat{} g$.

Weak Caution: *For each* $x \in X$ *there exists* $y \in X$ *such that* $y \succsim\hat{} x$ *and*

$$f \not\succsim^* x \implies y \succsim\hat{} f.$$

Consistency is a natural assumption. In fact, if for a DM f is clearly/objectively weakly better than g then he should declare f weakly better than g, particularly, when he is forced to make a choice. Weak Caution instead provides the main axiomatic departure of our work from the one of [11]. Notice that Weak Caution amounts to impose that for given x in X there exists a common bound y in X for *all* acts f that are not unambiguously preferred to x. In GMMS' paper, this assumption is trivially satisfied. In fact, the bound y, under $\succsim\hat{}$, is assumed to be x itself. Conversely, Weak Caution is violated if the DM has a family of acts, say $\{f_n\}_{n\in\mathbb{N}}$, where each element is *not* unambiguously preferred to x. But, despite this, in completing his preferences, $\succsim\hat{}$ turns out to be such that the DM can find arbitrarily good acts in $\{f_n\}_{n\in\mathbb{N}}$ where arbitrarily good is meant to be with respect to $\succsim\hat{}$. In light of these observations, notice that Weak Caution is binding and meaningful just in a context where Unboundedness (from above) is satisfied.

[4] For a similar interpretation see also: Debreu [7], Schmeidler [17], and Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [4].

[5] For sake of generality, we could have equivalently chosen a weaker form of Risk Independence, as in [4]. Nevertheless, the actual formulation allows an easier comparison with the axiom of Independence previously imposed on $\succsim^*$.

4 Results

4.1 Uncertainty Averse Preferences

In this subsection, we present the first main result of our paper. It provides a bridge between Bewley preferences and UA preferences. We state the following theorem following the format of [11, Theorem 3] in order to facilitate the comparison.

Theorem 4 *Let $(\succsim^*, \succsim^\wedge)$ be two binary relations on $\mathcal{F}$ and let one of them satisfy Unboundedness. The following are equivalent conditions:*

(i) $\succsim^*$ satisfies the Basic Conditions, C-Completeness, and Independence; $\succsim^\wedge$ satisfies the Basic Conditions, Completeness, Risk Independence, and Uncertainty Aversion; and jointly $(\succsim^*, \succsim^\wedge)$ satisfy Consistency and Weak Caution.

(ii) There exist an onto and affine function $u : X \to \mathbb{R}$, a linearly continuous function $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$, and a closed and convex set $C \subseteq \Delta$ such that $dom_\Delta G = C$ and for each f and g

$$f \succsim^* g \quad \Longleftrightarrow \quad \int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C \tag{7}$$

and

$$f \succsim^\wedge g \quad \Longleftrightarrow \quad \min_{p \in C} G\left(\int u(f)\,dp, p\right) \geq \min_{p \in C} G\left(\int u(g)\,dp, p\right). \tag{8}$$

Moreover, C is unique, u is cardinally unique, and, given u, G is unique.

First, the DM identifies the set of relevant probabilities C and a utility index u. These two objects characterize the rankings, as in (7), that for the DM appear as uncontroversial. For example, in the classic Ellsberg two colors urn experiment, C could be the convex hull of all the possible urn compositions of the unknown urn and u a utility index over all the objective urns. Nevertheless, C and u are not enough for the DM to be able to always rank acts. For this reason, he chooses an index of uncertainty aversion G that is further bounded on C. This allows him to consider certain probabilistic scenarios p in C more plausible than others. Finally, he uses these three objects to form consistently his preferences $\succsim^\wedge$ according to the cautious rule in (8) and thus just using the probabilities in C.

Mathematically, the above result descends from the following arguments. The axioms on $\succsim^*$ deliver that $\succsim^*$ is represented à la Bewley with a set of probabilities C and a utility index u^*. The axioms on $\succsim^\wedge$ deliver that $\succsim^\wedge$ is an UA preference relation. Thus, by [4], there exist a nonconstant and affine $u : X \to \mathbb{R}$ and a linearly continuous $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \min_{p \in \Delta} G\left(\int u(f)\,dp, p\right) \qquad \forall f \in \mathcal{F}, \tag{9}$$

represents $\succsim^\wedge$. Consistency delivers that u^* can be chosen to be equal to u while Weak Caution delivers that G further belongs to $\mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ and the min in (9) can be taken to be over $C = dom_\Delta G$. It is not hard to show that C is the smallest closed and convex set over which the min in (9) can be restricted to. This latter fact confirms that the perceived uncertainty of $\succsim^\wedge$ is the same characterizing $\succsim^*$. Moreover, in the appendix, we show that C also characterizes the *Unambiguous preference relation* of Ghirardato, Maccheroni, and Marinacci [9].

Finally, given Theorem 3, it should be noticed that the class of functions $\mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ characterizes a subset of UA preferences that we call *effectively bounded* UA preferences and that we next define:

Definition 5 *Let $\succsim\hat{}$ be a binary relation on $\mathcal{F}$. $\succsim\hat{}$ is an effectively bounded Uncertainty averse preference if and only if there exist an onto and affine function $u : X \to \mathbb{R}$ and a linearly continuous $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ such that (u, G) represent $\succsim\hat{}$ as in (2).*[6]

In order to better understand this class of preferences, consider the case in which G is as in (6), that is, the DM's preferences $\succsim\hat{}$ are Variational. The condition

$$\sup_{p \in dom(c)} \{t + c(p)\} = \sup_{p \in dom_{\Delta} G} G(t, p) < \infty \qquad \forall t \in \mathbb{R}$$

amounts to say that c is bounded over C. In other words, in completing his preferences $\succsim^*$, the DM might not be willing to consider all the probabilistic scenarios in C to be equivalent, as in [11]. But, at the same time, he does not want to penalize these alternative probabilities in an arbitrary unbounded fashion.

In light of the previous discussion and Theorem 3, it is then natural to ask how big is the class of *effectively bounded* UA preferences within the class of UA preferences. The next result shows that the former is "dense" in the latter.

Proposition 6 *Let $\succsim$ be a binary relation on $\mathcal{F}$ that satisfies Unboundedness. If $\succsim$ is an Uncertainty averse preference then there exists a sequence of effectively bounded Uncertainty averse preference relations $\{\succsim_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_n V_n(f) = V(f) \qquad \forall f \in \mathcal{F},$$

where $V, V_n : \mathcal{F} \to \mathbb{R}$ represent $\succsim$ and $\succsim_n$ as in (2) and for all $n \in \mathbb{N}$.

4.2 Main Departures from GMMS' Work

The departures from GMMS' work are three. Before discussing them, we start by listing two assumptions we dispense with but that play a major role in [11]:

C-Independence: *If $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0, 1)$*

$$f \succsim\hat{} g \iff \alpha f + (1-\alpha)x \succsim\hat{} \alpha g + (1-\alpha)x.$$

Caution: *If $f \in \mathcal{F}$ and $x \in X$*

$$f \not\succsim^* x \implies x \succsim\hat{} f.$$

The first departure consists in restricting ourselves to the case in which preferences satisfy Unboundedness, that is, the case in which there are arbitrarily good consequences and arbitrarily bad ones.[7] For example, this is the case if $X = \mathbb{R}$ and the DM satisfies the Basis Conditions as well as C-Completeness and Risk Independence.[8] The second departure consists in weakening C-Independence to Risk Independence and in explicitly assuming Uncertainty Aversion. This allows us to consider

[6] It is also possible, within the single preference framework adopted by [4], to provide an axiomatic foundation for this class of UA preferences. This can be achieved by requiring the Unambiguous preference relation of [9] to satisfy Weak Caution.

[7] An inspection of the proof of Theorem 4 reveals that we could state a slightly stronger version of it. Indeed, we could impose the assumption of Unboundedness in point (i) instead of making it a premise of the result. Nevertheless, we opted for this format so to facilitate the comparison with [11]. A similar observation applies for all the other results in the paper.

[8] In this case, the DM can be interpreted as being risk neutral.

preferences $\succsim^\wedge$ that are Variational, as in [14], or Uncertainty averse as in [4].[9] Finally, we connect $\succsim^*$ and $\succsim^\wedge$ by using a weaker form of Caution.

In order to justify Weak Caution and the introduction of Uncertainty Aversion, we first propose a stronger version of [11, Theorem 3] (see also [11, Theorem 4]). In comparison to [11, Theorem 3], here we just weaken C-Independence to Risk Independence in (i) but we still obtain the same functional characterization in (ii).

Definition 7 *Let $\succsim^\wedge$ be a binary relation on $\mathcal{F}$. $\succsim^\wedge$ is a Rational preference if and only if it satisfies the Basic Conditions, Completeness, and Risk Independence.*

Proposition 8 *Let $(\succsim^*, \succsim^\wedge)$ be two binary relations on $\mathcal{F}$. The following are equivalent conditions:*

(i) $\succsim^*$ is a Bewley preference; $\succsim^\wedge$ is a Rational preference; and jointly $(\succsim^*, \succsim^\wedge)$ satisfy Consistency and Caution.

(ii) There exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a closed and convex set $C \subseteq \Delta$ such that for each f and g

$$f \succsim^* g \iff \int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C$$

and

$$f \succsim^\wedge g \iff \min_{p\in C} \int u(f)\,dp \geq \min_{p\in C} \int u(g)\,dp.$$

Moreover, C is unique and u is cardinally unique.

It is worth noticing that, on $\succsim^\wedge$, no assumption regarding attitudes toward uncertainty or of independence involving uncertain acts is made. Thus, as it emerges also from the proof of Proposition 8, it is mainly Caution to force the DM to complete his preferences $\succsim^*$ according to the worst expected utility criterion. That is, not just Caution forces the perceived uncertainty revealed by $\succsim^\wedge$ to coincide with the one of $\succsim^*$ but it also forces the DM to answer to it in the most pessimistic way. In other words, Caution is not just an assumption on how $\succsim^*$ and $\succsim^\wedge$ are related in terms of perceived uncertainty but it is also an assumption on how $\succsim^\wedge$ reacts to Knightian uncertainty.

In our case, this latter feature is captured by Uncertainty Aversion, while the former is arguably captured by Weak Caution. It is immediate to see that Weak Caution is a mathematical weakening of Caution. Moreover, given next proposition, it is arguable that Weak Caution provides a common structure where to study UA completions as well as Gilboa and Schmeidler completions. Indeed, Proposition 10 clarifies why, *under unboundedness*, Weak Caution provides a common structure to study the connection between Bewley preferences and Uncertainty averse ones, as in this paper, as well as the connection between Bewley preferences and Gilboa and Schmeidler ones as done in [11].

Definition 9 *Let $\succsim^\wedge$ be a binary relation on $\mathcal{F}$. $\succsim^\wedge$ is an Invariant biseparable preference if and only if it satisfies the Basic Conditions, Completeness, and C-Independence.*

Proposition 10 *Let $(\succsim^*, \succsim^\wedge)$ be two binary relations on $\mathcal{F}$ and let one of them satisfy Unboundedness. Moreover, let $\succsim^*$ be a Bewley preference; $\succsim^\wedge$ be an Invariant biseparable preference; and jointly $(\succsim^*, \succsim^\wedge)$ satisfy Consistency. The following conditions are equivalent:*

[9] See also Maccheroni, Marinacci, and Rustichini [14, p. 1454] for a positive/normative discussion justyfing an axiomatic departure from C-Independence.

(i) jointly $(\succsim^*, \hat{\succsim})$ *satisfy Weak Caution;*

(ii) jointly $(\succsim^*, \hat{\succsim})$ *satisfy Caution.*

Notice that, as a corollary, we could prove again [11, Theorem 3], in this case, by replacing Caution with Weak Caution and by retaining C-Independence. In other words, in terms of Theorem 4 and its interpretation, in the GMMS' case, the DM chooses $G \in \mathcal{L}_{bd}(\mathbb{R}\times\Delta)$ to be such that

$$G(t,p) = t + \delta_C(p) \qquad \forall (t,p) \in \mathbb{R} \times \Delta.$$

In the following subsection, we propose a different form of Weak Caution which is more natural in the setting of Variational preferences. This form of Weak Caution, called Weak C(onstant)-Caution, allows us to carry our exercise with a weaker form of Unboundedness.

4.3 Variational Preferences

An important case of UA preferences are Variational preferences, being Multiplier preferences an important subcase. In this subsection, we provide a result similar to Theorem 4 for the Variational preferences case. This result is not just a mere corollary of Theorem 4. In fact, we require a weaker form of Unboundedness and in order to do so we propose a slightly stronger version of Weak Caution.

One Side Unboundedness: *For each* x *and* y *in* X *such that* $x \succ y$ *there is* $z \in X$ *such that*

$$\frac{1}{2}z + \frac{1}{2}y \succsim x.$$

Weak C-Caution: *For each* $x \in X$ *there exists* $y \in X$ *such that* $y \hat{\succsim} x$,

$$f \not\succsim^* x \quad \Longrightarrow \quad y \hat{\succsim} f$$

and

$$\frac{1}{2}x + \frac{1}{2}z \hat{\sim} \frac{1}{2}y + \frac{1}{2}z' \text{ for some fixed } z, z' \in X.$$

In the classic Anscombe and Aumann setting, the first weakening allows to consider more risk attitudes than possible under Unboundedness while Weak C-Caution is obviously a strengthening of Weak Caution. Indeed, for each $x \in X$ the upper bound y for x is required to move in a constant fashion. In other words, since $\hat{\succsim}$ on X is represented by an affine function u, the extra condition required in the axiom of Weak C-Caution amounts to impose that there exist z and z' in X such that for each $x \in X$ there exists $y \hat{\succsim} x$ and

$$u(y) - u(x) = u(z) - u(z'). \tag{10}$$

Condition (10) is in line with the extra assumption that characterizes Variational preferences among the class of UA preferences, that is,

Weak C-Independence: *If* $f, g \in \mathcal{F}$, $x, y \in X$, *and* $\alpha \in (0,1)$

$$\alpha f + (1-\alpha)x \hat{\succsim} \alpha g + (1-\alpha)x \quad \Longrightarrow \quad \alpha f + (1-\alpha)y \hat{\succsim} \alpha g + (1-\alpha)y.$$

This latest assumption coincides with independence relative to mixing with constant acts. In turn, functionally, this assumption coincides with the invariance of preferences to "translations".

Definition 11 *Let* $\hat{\succsim}$ *be a binary relation on* $\mathcal{F}$. $\hat{\succsim}$ *is a Variational preference if and only if it satisfies the Basic Conditions, Completeness, Weak C-Independence, and Uncertainty Aversion.*

In this particular case, we have that:

Theorem 12 *Let $(\succsim^*, \succsim^\wedge)$ be two binary relations on $\mathcal{F}$ and let one of them satisfy One Side Unboundedness. The following are equivalent conditions:*

(i) *$\succsim^*$ is a Bewley preference; $\succsim^\wedge$ is a Variational preference; and jointly $(\succsim^*, \succsim^\wedge)$ satisfy Consistency and Weak C-Caution.*

(ii) *There exist an unbounded from above and affine function $u : X \to \mathbb{R}$, a grounded, convex, and lower semicontinuous function $c : \Delta \to [0, \infty]$, and a closed and convex set $C \subseteq \Delta$ such that $C = dom(c) = \{p \in \Delta : c(p) \leq k\}$ where $k \in \mathbb{R}$ and for each f and g*

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C \tag{11}$$

and

$$f \succsim^\wedge g \iff \min_{p \in C} \left\{ \int u(f)\, dp + c(p) \right\} \geq \min_{p \in C} \left\{ \int u(g)\, dp + c(p) \right\}. \tag{12}$$

Moreover, C is unique, u is cardinally unique, and, given u, c is unique.

Remark 13 *It should be noticed that this representation follows as an easy corollary of Theorem 4 if One Side Unboundedness is replaced by Unboundedness and Weak C-Caution is replaced by Weak Caution.*

5 Constrained Multiplier Preferences

In their seminal paper [12], Hansen and Sargent compare the Gilboa and Schmeidler criterion as in (3) to the decision criterion contained in (4). They do this when

$$C = C_\eta = \{p \in \Delta : R(p||q) \leq \eta\} \text{ and } \eta > 0. \tag{13}$$

For fixed $\eta > 0$, the aforementioned authors show that the previous two preferences deliver the same choices in a consumption problem provided θ is chosen to be big enough. Nevertheless, these two classes of preferences represent two very different approaches toward Knightian uncertainty and they reveal different kind of aversion to it. Indeed, in the Gilboa and Schmeidler criterion (3), just the probabilities p in C are considered to be relevant probabilistic scenarios. On the other hand, in the Hansen and Sargent criterion (4), *all* countably additive probabilities that are absolutely continuous with respect to the reference probability measure q are considered. This difference is particularly striking in the formulation proposed by Hansen and Sargent where $C = C_\eta$. Moreover, a DM that follows the Gilboa and Schmeidler criterion deems, a priori, all the expected utility evaluations induced by C to be equally plausible while a DM that follows the Hansen and Sargent criterion considers a probabilistic scenario less plausible the higher is its distance from q. A natural and intermediate approach would be the one of merging the two previous decision criteria. That is, the DM has a prior of reference q and considers as relevant probabilistic scenarios the set C_η for some $\eta > 0$. Nevertheless, in evaluating an act f, he considers a probability less plausible the higher is its distance from q. In other words, he ranks acts according to

$$V_{\theta,\eta}(f) = \min_{p \in C_\eta} \left\{ \int u(f)\, dp + \theta R(p||q) \right\} \qquad \forall f \in \mathcal{F}. \tag{14}$$

We dub these preferences Constrained Multiplier preferences. These preferences were first proposed by Wang [19].[10]

In light of Theorems 4 and 12, it is immediate to see that the preferences represented by (14) can be seen as arising through a procedure of completion as the ones described and axiomatized in the previous section. Next, we provide an extra *assumption* which characterizes these particular preferences within this framework. We begin by proposing a notion of comparative uncertainty aversion, due to Ghirardato and Marinacci [8]. Given two preferences $\succsim_1$ and $\succsim_2$ on $\mathcal{F}$, we say that $\succsim_1$ is more uncertainty averse than $\succsim_2$ if and only if for each $f \in \mathcal{F}$ and $x \in X$

$$f \succsim_1 x \quad \Longrightarrow \quad f \succsim_2 x.$$

Given $\eta > 0$, we define $\succsim_{\eta,GS}$ to be a nontrivial Gilboa and Schmeidler preference relation with $C = C_\eta$. In particular, as the next remark will clarify, it is irrelevant to specify which forms takes the index u. Same observation applies for $\succsim_{\theta,HS}$ where the latter denotes the Hansen and Sargent preference relation. Similarly, we define $G_{\eta,GS} : \mathbb{R} \times \Delta \to (-\infty, \infty]$ by $(t,p) \mapsto t + \delta_{C_\eta}(p)$ for all $(t,p) \in \mathbb{R} \times \Delta$.

Minimal Maximal Uncertainty Aversion: *There exist $\eta, \theta \in (0,\infty)$ such that for each $f \in \mathcal{F}$ and $x \in X$*

$$f \succsim_{\theta,HS} x \text{ or } f \succsim_{\eta,GS} x \quad \Longrightarrow \quad f \succsim\hat{\ } x \tag{15}$$

and given any other uncertainty averse preference $\succsim'$ that satisfies (15)

$$f \succsim\hat{\ } x \quad \Longrightarrow \quad f \succsim' x.$$

The first part of previous assumption imposes that the preference relation $\succsim\hat{\ }$ is less uncertainty averse than the Hansen and Sargent preference relation with parameter θ and the Gilboa and Schmeidler preference relation with $C = C_\eta$. The second part of the assumption states that $\succsim\hat{\ }$ is the more uncertainty averse among the (uncertainty averse) preferences that satisfy the first part of the requirement.

Remark 14 *It is not hard to show that (15) implies that $\succsim_{\eta,GS}$ and $\succsim_{\theta,HS}$ coincide with $\succsim\hat{\ }$ on X.*

Theorem 15 *Let $(\succsim^*, \succsim\hat{\ })$ be two binary relations on $\mathcal{F}$, let one of them satisfy Unboundedness, and let Σ be a σ-algebra. The following conditions are equivalent:*

(i) $\succsim^*$ is a Bewley preference; $\succsim\hat{\ }$ is an Uncertainty averse preference that satisfies Minimal Maximal Uncertainty Aversion ; and jointly $(\succsim^*, \succsim\hat{\ })$ satisfy Consistency and Weak Caution.

(ii) There exist an onto and affine function $u : X \to \mathbb{R}$ and $\eta, \theta \in (0,\infty)$ such that for each f and g

$$f \succsim^* g \quad \Longleftrightarrow \quad \int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C_\eta$$

and

$$f \succsim\hat{\ } g \quad \Longleftrightarrow \quad \min_{p \in C_\eta} \left\{ \int u(f)\,dp + \theta R(p||q) \right\} \geq \min_{p \in C_\eta} \left\{ \int u(g)\,dp + \theta R(p||q) \right\}.$$

[10] Wang considers preferences over triples (f, C, q). Thus, his modeling is very different from ours. We refer the reader to [19] for further details.

We can interpret a DM with Constrained Multiplier preferences as a DM who is sure that the relevant probabilistic information lies in the set $\{p \in \Delta : R(p||q) \leq \eta\}$. Such information is enough to form an initial objective ranking $\succsim^*$. The Constrained Multiplier preferences complete the former ranking and they complete it in a weakly cautious way. In particular, the DM still considers relevant only the models in $\{p \in \Delta : R(p||q) \leq \eta\}$ but he penalizes each model the farther is from q. In terms of index of uncertainty aversion $G = G_{\theta,\eta}$, Constrained Multiplier preferences are characterized by an index which is the pointwise supremum of the indexes characterizing Multiplier preferences and the index characterizing the Gilboa and Schmeidler preferences,[11] that is,

$$G_{\theta,\eta}(t,p) = \sup\{G_{\theta,HS}(t,p), G_{\eta,GS}(t,p)\} \qquad \forall (t,p) \in \mathbb{R} \times \Delta.$$

It is immediate to see that Constrained Multiplier preferences are characterized by two parameters: η and θ. It is also not hard to show that a DM 1 with Constrained Multiplier preferences given by (η_1, θ_1) is more uncertainty averse than a DM 2 with Constrained Multiplier preferences given by (η_2, θ_2) if $\eta_1 \geq \eta_2$ and $\theta_1 \leq \theta_2$.

A Appendix A

Given a binary relation $\succsim\hat{}$ on $\mathcal{F}$, we define $\succsim^\circ$ by

$$f \succsim^\circ g \quad \Longleftrightarrow \quad \lambda f + (1-\lambda) h \succsim\hat{}\, \lambda g + (1-\lambda) h \qquad \forall \lambda \in (0,1], \forall h \in \mathcal{F}.$$

Notice that $\succsim^\circ$ is the revealed unambiguous preference relation of Ghirardato, Maccheroni, and Marinacci [9]. In the sequel, with a small abuse of notation, given $k \in \mathbb{R}$, we will denote by k both the real number and the constant function on S that takes value k.

In the rest of the paper, we will invoke some of the results of GMMS. Even though all the results in [11] were derived under the hypothesis that X is the set of all simple lotteries over a generic outcome space, their extension to the case when X is a generic convex set is straightforward.

Before proving the main results, we need some extra notation and an ancillary proposition. Given a functional $I : B_0(\Sigma) \to \mathbb{R}$, we define $\succcurlyeq$ to be the binary relation on $B_0(\Sigma)$ such that

$$\varphi \succcurlyeq \psi \quad \Longleftrightarrow \quad I(\varphi) \geq I(\psi).$$

We define $\succcurlyeq^\circ$ to be the binary relation on $B_0(\Sigma)$ such that

$$\varphi \succcurlyeq^\circ \psi \quad \Longleftrightarrow \quad I(\lambda\varphi + (1-\lambda)\phi) \geq I(\lambda\psi + (1-\lambda)\phi) \qquad \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma). \tag{16}$$

Given $C \subseteq \Delta$, we define $\succcurlyeq_C$ to be the binary relation on $B_0(\Sigma)$ such that

$$\varphi \succcurlyeq_C \psi \quad \Longleftrightarrow \quad \int \varphi dp \geq \int \psi dp \qquad \forall p \in C.$$

Finally, given C and I, we say that I is consistent with C if and only if

$$\varphi \succcurlyeq_C \psi \quad \Longrightarrow \quad I(\varphi) \geq I(\psi).$$

Proposition 16 *Let I be a functional from $B_0(\Sigma)$ to $\mathbb{R}$ and let C be a nonempty, closed, and convex subset of Δ. The following conditions are equivalent:*

[11] Maccheroni, Marinacci, and Rustichini [14], as a characterization, suggest the sum of the cost functions, $R(\cdot||q)$ and $\delta_{C_\eta}(\cdot)$, rather than the supremum. Mathematically, in this case, the two operations deliver the same index of uncertainty aversion. Nevertheless, from an economic point of view, the assumption Minimal Maximal Uncertainty Aversion suggests that the supremum is the more appropriate operation.

(i) I is normalized, monotone, continuous, quasiconcave, consistent with C, and such that for each $k \in \mathbb{R}$ there exists $h \geq k$ such that

$$\varphi \not\succcurlyeq_C k \quad \Longrightarrow \quad h \geq I(\varphi). \tag{17}$$

(ii) There exists a unique linearly continuous $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ such that

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma)$$

and $dom_\Delta G = C$.

Proof. (i) implies (ii). We proceed by steps. But, first, by construction, observe that $\succcurlyeq$, $\succcurlyeq_C$, and $\succcurlyeq^\circ$ are binary relations over acts in an Anscombe and Aumann setting where S is the state space, Σ is the algebra, and $X = \mathbb{R}$.

Step 1. $\succcurlyeq$ satisfies the Basic Conditions, Completeness, Risk Independence, and Uncertainty Aversion. Moreover, $\succcurlyeq$ restricted to $\mathbb{R}$ is represented by the identity.

Proof of the Step.

Since I is normalized, $\succcurlyeq$ restricted to $\mathbb{R}$ is represented by the identity. By [4, Lemma 57] and since I is normalized, monotone, continuous, and quasiconcave, the statement follows. $\square$

Step 2. There exists a nonempty, closed, and convex set $C^\circ \subseteq \Delta$ such that for each φ and ψ in $B_0(\Sigma)$

$$\varphi \succcurlyeq^\circ \psi \quad \Longleftrightarrow \quad \int \varphi dp \geq \int \psi dp \qquad \forall p \in C^\circ \tag{18}$$

and

$$\varphi \succcurlyeq^\circ \psi \quad \Longrightarrow \quad I(\varphi) \geq I(\psi).$$

Moreover, C° is unique and $\succcurlyeq^\circ = \succcurlyeq_{C^\circ}$.

Proof of the Step.

By definition of $\succcurlyeq^\circ$ and $\succcurlyeq$, we have that

$$\varphi \succcurlyeq^\circ \psi \quad \Longleftrightarrow \quad \lambda\varphi + (1-\lambda)\phi \succcurlyeq \lambda\psi + (1-\lambda)\phi \qquad \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma). \tag{19}$$

By Step 1 and [3, Proposition 2], the first part of the statement follows as well as the uniqueness of C° and $\succcurlyeq^\circ = \succcurlyeq_{C^\circ}$. By taking $\lambda = 1$ in (19) and by definition of $\succcurlyeq$, the second part follows as well. $\square$

Step 3. $C^\circ \subseteq C$.

Proof of the Step.

By the definition of $\succcurlyeq_C$ and $\succcurlyeq^\circ$ and Step 2 and since I is consistent with C, we have that

$$\begin{aligned}
\varphi \succcurlyeq_C \psi \quad &\Longrightarrow \quad \lambda\varphi + (1-\lambda)\phi \succcurlyeq_C \lambda\psi + (1-\lambda)\phi \qquad \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma) \\
&\Longrightarrow I(\lambda\varphi + (1-\lambda)\phi) \geq I(\lambda\psi + (1-\lambda)\phi) \qquad \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma) \\
&\Longrightarrow \varphi \succcurlyeq^\circ \psi.
\end{aligned}$$

By Step 2 and [9, Proposition A.1.], this implies that $C^\circ \subseteq C$. $\square$

Step 4. There exists a unique linearly continuous $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ such that

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma). \tag{20}$$

Moreover, for each $(t,p) \in \mathbb{R} \times \Delta$

$$G(t,p) = \sup\left\{I(\varphi) : \int \varphi dp \leq t\right\} \tag{21}$$

and $cl\,(dom_\Delta G) = C^\circ$*.*

Proof of the Step.

By [4] (see also [5]) and Step 1 and since I is normalized, monotone, continuous, and quasiconcave, there exists a unique linearly continuous $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ such that (20) and (21) hold. By [4, Theorem 10], we further have that $cl\,(dom_\Delta G) = C^\circ$.

Step 5. $C^\circ = C$*.*

Proof of the Step.

We start by making a definition. Given $\phi \in B_0(\Sigma)$, we define $k_\phi = \min_{p\in C} \int \phi dp$. By contradiction, suppose that $C^\circ \neq C$. By Steps 3 and 4, we know that this implies that there exists $q \in C \backslash C^\circ$ and $q \notin dom_\Delta G$. By [16, Theorem 3.4] and since C° is closed and convex, there exists $\psi \in B_0(\Sigma)$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$\min_{p\in C} \int \psi dp \leq \int \psi dq \leq \alpha - \varepsilon < \alpha + \varepsilon \leq \min_{p\in C^\circ} \int \psi dp. \tag{22}$$

Without loss of generality, we can assume that ψ is such that $k_\psi \leq -\varepsilon < 0$ and $\min_{p\in C^\circ} \int \psi dp \geq \varepsilon > 0$. By (22), if we define the sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma)$ to be such that $\varphi_n = n\psi$ for all $n \in \mathbb{N}$ then it follows that

$$k_{\varphi_n} < 0 \quad \text{and} \quad \min_{p\in C^\circ} \int \varphi_n dp = \min_{p\in C^\circ} \int n\psi dp = n \min_{p\in C^\circ} \int \psi dp \geq n\varepsilon > 0 \qquad \forall n \in \mathbb{N}. \tag{23}$$

Recall that I satisfies (17), that is, for each $k \in \mathbb{R}$ there exists $h \geq k$ such that

$$\varphi \not\geq_C k \quad \Longrightarrow \quad h \geq I(\varphi).$$

Take $k = 0$ and h as in (17). By (23), it follows that there exists $\bar{n} \in \mathbb{N}$ such that

$$k_{\varphi_{\bar{n}}} < 0 = k \quad \text{and} \quad \int \varphi_{\bar{n}} dp' \geq \min_{p\in C^\circ} \int \varphi_{\bar{n}} dp > h + 1 \qquad \forall p' \in C^\circ. \tag{24}$$

By Step 2, I is consistent with C°. Thus, the first part of (24) delivers that $\varphi_{\bar{n}} \not\geq_C k$ while the second part delivers that $I(\varphi_{\bar{n}}) > h$, a contradiction. $\square$

Step 6. $\sup_{p\in dom_\Delta G} G(t,p) < \infty$ *for all* $t \in \mathbb{R}$*, that is,* $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$*.*

Proof of the Step.

Before starting recall that, by Step 4, G satisfies (21), that is,

$$G(t,p) = \sup\left\{I(\varphi) : \int \varphi dp \leq t\right\} \qquad \forall (t,p) \in \mathbb{R} \times \Delta.$$

By contradiction, suppose that $\sup_{p\in dom_\Delta G} G(\bar{t},p) = \infty$ for some $\bar{t}$ in $\mathbb{R}$. By working hypothesis, there exists a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq dom_\Delta G$ such that $G(\bar{t},p_n) \geq n$ for all $n \in \mathbb{N}$. By (21) and since $C = C^\circ = cl\,(dom_\Delta G)$, this implies that for each $n \in \mathbb{N}$ there exists $\varphi_n \in B_0(\Sigma)$ such that

$$\min_{p\in C} \int \varphi_n dp \leq \int \varphi_n dp_n \leq \bar{t} < \bar{t} + 1 \quad \text{and} \quad I(\varphi_n) \geq \frac{n}{2}. \tag{25}$$

Since I satisfies (17), consider $k = \bar{t} + 1$ and fix $h \geq k$ to satisfy (17). From the first part of (25), we have that $\varphi_n \not\geq_C k$ for all $n \in \mathbb{N}$. At the same time, by the second part of (25), it is immediate to see that there exists $\bar{n} \in \mathbb{N}$ such that $I(\varphi_{\bar{n}}) \geq \frac{\bar{n}}{2} \geq h$, a contradiction with I satisfying (17). $\square$

Step 7. $cl\left(dom_{\Delta}G\right) = dom_{\Delta}G$.

Proof of the Step.

It is enough to prove that given a generic net $\{p_{\alpha}\}_{\alpha \in A} \subseteq dom_{\Delta}G$ such that $p_{\alpha} \to \bar{p}$ then $\bar{p} \in dom_{\Delta}G$. Fix a generic $t \in \mathbb{R}$. By Step 6, it follows that $G\left(t, p_{\alpha}\right) \leq \sup_{p \in dom_{\Delta}G} G\left(t, p\right) < \infty$ for all $\alpha \in A$. Since $G \in \mathcal{L}_{bd}\left(\mathbb{R} \times \Delta\right)$, we have that

$$\infty > \sup_{p \in dom_{\Delta}G} G\left(t, p\right) \geq \liminf_{\alpha} G\left(t, p_{\alpha}\right) \geq G\left(t, \bar{p}\right).$$

Hence, $\bar{p} \in dom_{\Delta}G$. □

Steps 4 and 6 imply the first part of (ii) while Steps 4, 5, and 7 imply that $C = C^{\circ} = cl\left(dom_{\Delta}G\right) = dom_{\Delta}G$.

(ii) implies (i). By assumption, we have that there exists a linearly continuous $G \in \mathcal{L}_{bd}\left(\mathbb{R} \times \Delta\right)$ such that

$$I\left(\varphi\right) = \min_{p \in \Delta} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0\left(\Sigma\right).$$

By [5], it follows that I is normalized, monotone, and quasiconcave. Since G is linearly continuous, I is continuous. Next, by definition of $dom_{\Delta}G$, we have that

$$I\left(\varphi\right) = \min_{p \in dom_{\Delta}G} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0\left(\Sigma\right). \tag{26}$$

Since G is increasing in the first component and $dom_{\Delta}G = C$, it follows that I is consistent with C. Finally, we show that I satisfies (17). We proceed by arguing by contradiction. Suppose that there exists $k \in \mathbb{R}$ such that for each $h \geq k$ in $\mathbb{R}$ we can find $\varphi_h \in B_0\left(\Sigma\right)$ such that $\varphi_h \not\geq_C k$ and $I\left(\varphi_h\right) > h$. It follows that for each $n \in \{\lfloor k \rfloor + 1, ..., \lfloor k \rfloor + m, ...\}$ there exists $\varphi_n \in B_0\left(\Sigma\right)$ such that $\varphi_n \not\geq_C k$ and $I\left(\varphi_n\right) > n$. Thus, for each $n \in \{\lfloor k \rfloor + 1, ..., \lfloor k \rfloor + m, ...\}$ there exists $p_n \in C = dom_{\Delta}G$ such that $\int \varphi_n dp_n < k$. By (26) and since $G \in \mathcal{L}_{bd}\left(\mathbb{R} \times \Delta\right)$, it follows that

$$\sup_{p \in dom_{\Delta}G} G\left(k, p\right) \geq G\left(k, p_n\right) \geq G\left(\int \varphi_n dp_n, p_n\right) \geq I\left(\varphi_n\right) > n \qquad \forall n \in \{\lfloor k \rfloor + 1, ..., \lfloor k \rfloor + m, ...\},$$

a contradiction with $\sup_{p \in dom_{\Delta}G} G\left(t, p\right) < \infty$ for all $t \in \mathbb{R}$. ■

Proof of Theorem 4. (i) implies (ii). We again proceed by steps.

Step 1. $\succsim^{\wedge}$ *coincide to* $\succsim^{*}$ *on* X.

Proof of the Step.

Notice that $\succsim^{*}$ and $\succsim^{\wedge}$ restricted to X satisfy C-Completeness, Mixture Continuity, and Risk Independence. By [13] and since $\succsim^{*}$ and $\succsim^{\wedge}$ satisfy the Basic Conditions, it follows that there exist two nonconstant and affine functions, u^{*} and $u^{\wedge}$, from X to $\mathbb{R}$ that represent, respectively, $\succsim^{*}$ and $\succsim^{\wedge}$ on X. Since jointly $\left(\succsim^{*}, \succsim^{\wedge}\right)$ satisfy Consistency, it follows that for each $x, y \in X$

$$u^{*}\left(x\right) \geq u^{*}\left(y\right) \implies u^{\wedge}\left(x\right) \geq u^{\wedge}\left(y\right).$$

By [9, Corollary B.3.], it follows that u^{*} and $u^{\wedge}$ are equal up to an affine and positive transformation, hence the statement. □

Step 2. There exist an onto and affine function $u^{*} : X \to \mathbb{R}$ *and a nonempty, closed, and convex set* C *such that*

$$f \succsim^{*} g \iff \int u^{*}\left(f\right) dp \geq \int u^{*}\left(g\right) dp \qquad \forall p \in C. \tag{27}$$

Moreover, C is unique.

Proof of the Step.

By assumption, $\succsim^*$ satisfies the Basic Conditions, C-Completeness, and Independence. By [11, Theorem 1] and since, by Step 1 and the premises of Theorem 4, $\succsim^*$ satisfies Unboundedness, the statement follows. $\square$

Step 3. There exist an onto and affine function $u^{\wedge} : X \to \mathbb{R}$ and a normalized, monotone, continuous, and quasiconcave functional $I : B_0(\Sigma) \to \mathbb{R}$ such that

$$f \succsim^{\wedge} g \iff I\left(u^{\wedge}(f)\right) \geq I\left(u^{\wedge}(g)\right).$$

Moreover, $u^{\wedge}$ is cardinally unique and, given $u^{\wedge}$, I is unique.

Proof of the Step.

By assumption, Step 1, and the premises of Theorem 4, $\succsim^{\wedge}$ satisfies the Basic Conditions, Completeness, Risk Independence, Uncertainty Aversion, and Unboundedness. By [4, Lemma 57 and Lemma 59], the statement follows. $\square$

Notice that, by Step 1, we can assume without loss of generality that $u^* = u^{\wedge} = u$.

Step 4. I is consistent with C.

Proof of the Step.

Consider $\varphi, \psi \in B_0(\Sigma)$ and assume that $\varphi \succcurlyeq_C \psi$. It is immediate to see that there exist $f, g \in \mathcal{F}$ such that $\varphi = u(f)$, $\psi = u(g)$, and $f \succsim^* g$. By Steps 2 and 3 and since jointly $(\succsim^*, \succsim^{\wedge})$ satisfy Consistency, we have that

$$\varphi \succcurlyeq_C \psi \implies f \succsim^* g \implies f \succsim^{\wedge} g \implies I(u(f)) \geq I(u(g)) \implies I(\varphi) \geq I(\psi),$$

proving the statement. $\square$

Step 5. I satisfies (17).

Proof of the Step.

We need to show that for each $k \in \mathbb{R}$ there exists $h \geq k$ such that

$$\varphi \not\succcurlyeq_C k \implies h \geq I(\varphi).$$

Fix a generic $k \in \mathbb{R}$. Since $\succsim^{\wedge}$ satisfies Unboundedness, there exists $x \in X$ such that $k = u(x)$. Define $h = u(y)$ where $y \in X$ is such that $y \succsim^{\wedge} x$ and

$$f \not\succsim^* x \implies y \succsim^{\wedge} f.$$

Next, consider $\varphi \in B_0(\Sigma)$ such that $\varphi \not\succcurlyeq_C k$. Given (27), it is immediate to see that there exists $f \in \mathcal{F}$ such that $\varphi = u(f)$ and $f \not\succsim^* x$. Since jointly $(\succsim^*, \succsim^{\wedge})$ satisfy Weak Caution, it follows that $y \succsim^{\wedge} f$. By Step 3, this implies that $h = u(y) = I(u(y)) \geq I(u(f)) = I(\varphi)$, hence the statement. $\square$

Step 6. There exist an onto and affine function $u : X \to \mathbb{R}$, a linearly continuous function $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$, and a closed and convex set $C \subseteq \Delta$ such that $dom_{\Delta} G = C$ and for each f and g

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C. \tag{28}$$

and

$$f \succsim^{\wedge} g \iff \min_{p \in C} G\left(\int u(f)\, dp, p\right) \geq \min_{p \in C} G\left(\int u(g)\, dp, p\right). \tag{29}$$

Proof of the Step.

Define $V:\mathcal{F}\to\mathbb{R}$ by $V(f)=I(u(f))$ for all $f\in\mathcal{F}$ where $u=u^{\wedge}$ and I are as in Step 3. It is immediate to see that V represents $\succsim^{\wedge}$. By Steps 2, 3, 4, and 5 and Proposition 16, it follows that there exists a linearly continuous $G\in\mathcal{L}_{bd}(\mathbb{R}\times\Delta)$ such that $dom_{\Delta}G=C$ where C is closed and convex and

$$V(f)=\min_{p\in\Delta}G\left(\int u(f)\,dp,p\right)=\min_{p\in C}G\left(\int u(f)\,dp,p\right)\qquad\forall f\in\mathcal{F},$$

proving that (29) holds. By Step 2 and since $u^{*}=u$, (28) holds. □

(ii) implies (i). Consider a nonempty, closed, and convex set $C\subseteq\Delta$, an onto and affine function $u:X\to\mathbb{R}$, and a linearly continuous $G\in\mathcal{L}_{bd}(\mathbb{R}\times\Delta)$ such that $C=dom_{\Delta}G$. Suppose further that C and (u,G) satisfy (7) and (8). By [11, Theorem 1], it follows that $\succsim^{*}$ satisfies the Basic Conditions, C-Completeness, and Independence. By [4, Theorem 3], $\succsim^{\wedge}$ satisfies the Basic Conditions, Completeness, Risk Independence, and Uncertainty Aversion (as well as Unboundedness). Define $I:B_0(\Sigma)\to\mathbb{R}$ by

$$I(\varphi)=\min_{p\in\Delta}G\left(\int\varphi dp,p\right)\qquad\forall\varphi\in B_0(\Sigma).$$

By Proposition 16, it follows that I is consistent with C and satisfies (17). Since I composed with u represents $\succsim^{\wedge}$, this implies that jointly $(\succsim^{*},\succsim^{\wedge})$ satisfy Consistency and Weak Caution.

The uniqueness part of the statement follows from routine arguments (see [11] and [4]). ■

Proof of Proposition 6. Let $\succsim$ be a binary relation on $\mathcal{F}$ that satisfies Unboundedness and assume $\succsim$ is an Uncertainty averse preference. By [4, Lemma 57 and Lemma 59], there exist an onto and affine function $u:X\to\mathbb{R}$ and a normalized, monotone, continuous, and quasiconcave functional $I:B_0(\Sigma)\to\mathbb{R}$ such that $f\succsim g$ if and only if $V(f)\geq V(g)$ where $V(f)=I(u(f))$ for all $f\in\mathcal{F}$. For each $n\in\mathbb{N}$ define $J_n:B_0(\Sigma)\to\mathbb{R}$ by $\varphi\mapsto\min_{s\in S}\varphi(s)+n$ and $I_n:B_0(\Sigma)\to\mathbb{R}$ by

$$I_n(\varphi)=\min\{I(\varphi),J_n(\varphi)\}\qquad\forall\varphi\in B_0(\Sigma).$$

It is immediate to verify that I_n is a normalized, monotone, continuous, and quasiconcave functional for all $n\in\mathbb{N}$. For each $n\in\mathbb{N}$ define $\succcurlyeq_n^{\circ}$ as in (16). It follows that there exists a nonempty, closed, and convex set C_n of Δ such that

$$\begin{aligned}&\varphi\succcurlyeq_n^{\circ}\psi\quad\Longleftrightarrow\quad\varphi\succcurlyeq_{C_n}\psi\\&\quad\textit{and}\\&\varphi\succcurlyeq_{C_n}\psi\quad\Longrightarrow\quad I_n(\varphi)\geq I_n(\psi).\end{aligned}$$

We next show that I_n satisfies (17) for all $n\in\mathbb{N}$. Fix $n\in\mathbb{N}$. First, given $k\in\mathbb{R}$ define $h_k=k+n$. Next, consider $\varphi\in B_0(\Sigma)$ such that $\varphi\not\succcurlyeq_{C_n}k$. This implies that $\min_{s\in S}\varphi(s)<k$. It follows that

$$I_n(\varphi)=\min\{I(\varphi),J_n(\varphi)\}\leq J_n(\varphi)<k+n=h_k,$$

proving that I_n satisfies (17). By Proposition 16, it follows that for each $n\in\mathbb{N}$ there exists a unique linearly continuous $G\in\mathcal{L}_{bd}(\mathbb{R}\times\Delta)$ such that

$$I_n(\varphi)=\min_{p\in\Delta}G_n\left(\int\varphi dp,p\right)\qquad\forall\varphi\in B_0(\Sigma).$$

Moreover, notice that

$$\lim_n I_n(\varphi)=I(\varphi)\qquad\forall\varphi\in B_0(\Sigma).\tag{30}$$

For each $n \in \mathbb{N}$ define $V_n : \mathcal{F} \to \mathbb{R}$ and $\succsim_n$ to be such that

$$V_n(f) = \min_{p \in \Delta} G_n\left(\int u(f)\, dp, p\right) \qquad \forall f \in \mathcal{F}$$

$$\textit{and}$$

$$f \succsim_n g \iff V_n(f) \geq V_n(g).$$

It follows that $\succsim_n$ is an effectively bounded Uncertainty averse preference relation for all $n \in \mathbb{N}$. By (30), we further have that

$$\lim_n V_n(f) = V(f) \qquad \forall f \in \mathcal{F},$$

proving the statement. ■

Proof of Proposition 8. (i) implies (ii). By [11, Theorem 1] and since $\succsim^*$ satisfies the Basic Conditions, C-Completeness, and Independence, there exist a nonconstant and affine function $u^* : X \to \mathbb{R}$ and a nonempty, closed, and convex set C such that

$$f \succsim^* g \iff \int u^*(f)\, dp \geq \int u^*(g)\, dp \qquad \forall p \in C. \tag{31}$$

By [3, Proposition 1] and since $\hat{\succsim}$ satisfies the Basic Conditions, Completeness, and Risk Independence, there exist a nonconstant and affine function $\hat{u} : X \to \mathbb{R}$ and a normalized, monotone, and continuous functional $I : B_0(\Sigma, \hat{u}(X)) \to \mathbb{R}$ such that

$$f \,\hat{\succsim}\, g \iff I(\hat{u}(f)) \geq I(\hat{u}(g)).$$

Moreover, by [3, Proposition 2], it follows that there exists a nonempty, closed, and convex set C° such that

$$f \succsim^\circ g \iff \int \hat{u}(f)\, dp \geq \int \hat{u}(g)\, dp \qquad \forall p \in C^\circ.$$

Since $(\succsim^*, \hat{\succsim})$ jointly satisfy Consistency, it follows that for each $x, y \in X$

$$u^*(x) \geq u^*(y) \implies \hat{u}(x) \geq \hat{u}(y).$$

By [9, Corollary B.3.], it follows that u^* is a positive affine transformation of $\hat{u}$. Wlog, we can assume that $\hat{u} = u^* = u$. By (31), we have that if $f \succsim^* g$ then $\lambda f + (1-\lambda) h \succsim^* \lambda g + (1-\lambda) h$ for all $\lambda \in (0,1]$ and all $h \in \mathcal{F}$. Since $(\succsim^*, \hat{\succsim})$ jointly satisfy Consistency, it follows that

$$\lambda f + (1-\lambda) h \,\hat{\succsim}\, \lambda g + (1-\lambda) h \qquad \forall \lambda \in (0,1], \forall h \in \mathcal{F}$$

which in turn delivers $f \succsim^\circ g$. In other words, we have that if $f \succsim^* g$ then $f \succsim^\circ g$. Since $B_0(\Sigma, u(X)) = \{u(f) : f \in \mathcal{F}\}$ and by [9, Proposition A.1.], this implies that $C^\circ \subseteq C$. By [3, Corollary 3], we have that

$$\min_{p \in C} \int u(f)\, dp \leq \min_{p \in C^\circ} \int u(f)\, dp \leq I(u(f)) \qquad \forall f \in \mathcal{F}. \tag{32}$$

Conversely, fix $f \in \mathcal{F}$ and define $k = \min_{p \in C} \int u(f)\, dp$. Since u is affine and $C \subseteq \Delta$, we have that $k \in u(X)$. Thus, there exists $x \in X$ such that $u(x) = k$. We have two cases:

1. $x \,\hat{\succsim}\, y$ for all $y \in X$. By Monotonicity, this implies that $x \,\hat{\succsim}\, f$, that is,

$$I(u(f)) \leq I(u(x)) = u(x) = \min_{p \in C} \int u(f)\, dp.$$

2. There exists $y \in X$ such that $y \succ^{\wedge} x$. Define $x_\varepsilon = \varepsilon y + (1-\varepsilon)x$ for all $\varepsilon \in (0,1)$. Since u is affine and represents $\succsim^{\wedge}$ on X, we have that

$$u(x_\varepsilon) > u(x) \qquad \forall \varepsilon \in (0,1).$$

This implies that $f \not\succsim^* x_\varepsilon$ for all $\varepsilon \in (0,1)$. Since $(\succsim^*, \succsim^{\wedge})$ jointly satisfy Caution, it follows that $x_\varepsilon \succsim^{\wedge} f$ for all $\varepsilon \in (0,1)$, that is,

$$I(u(f)) \leq I(u(x_\varepsilon)) = u(x_\varepsilon) = \varepsilon u(y) + (1-\varepsilon)u(x) \qquad \forall \varepsilon \in (0,1).$$

This implies that $I(u(f)) \leq u(x) = \min_{p \in C} \int u(f)\, dp$.

In both cases and by (32), we obtain that $I(u(f)) = \min_{p \in C} \int u(f)\, dp$, proving the statement since f was chosen to be generic.

(ii) implies (i). It follows from [11, Theorem 3].

The uniqueness part of the statement follows from routine arguments. ■

Proof of Proposition 10. (i) implies (ii). By contradiction, suppose that jointly $(\succsim^*, \succsim^{\wedge})$ do not satisfy Caution. Therefore, there exist $\bar{x} \in X$ and $\bar{f} \in \mathcal{F}$ such that $\bar{f} \not\succsim^* \bar{x}$ and $\bar{f} \succ^{\wedge} \bar{x}$. By the premises, [9], and [11, Theorem 1] and since one binary relation between $\succsim^*$ and $\succsim^{\wedge}$ satisfies Unboundedness and jointly $(\succsim^*, \succsim^{\wedge})$ satisfy Consistency, it follows that there exist an affine and onto function $u : X \to \mathbb{R}$, a closed and convex set C in Δ, and a normalized, positively homogeneous functional $I : B_0(\Sigma) \to \mathbb{R}$ such that: $u(\bar{x}) = 0$,

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C$$

and

$$f \succsim^{\wedge} g \iff I(u(f)) \geq I(u(g)).$$

Moreover, since jointly $(\succsim^*, \succsim^{\wedge})$ satisfy Consistency, we have that $f \succsim^* g$ implies $I(u(f)) \geq I(u(g))$. Define $x_a, x_b \in X$ to be such that

$$u(x_a) = I\left(u\left(\bar{f}\right)\right) \text{ and } u(x_b) = \min_{p \in C} \int u\left(\bar{f}\right) dp.$$

Since $\bar{f} \not\succsim^* \bar{x}$ and $\bar{f} \succ^{\wedge} \bar{x}$, it follows that $u(x_a) > 0$ and $u(x_b) < 0$. Define now $\{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}$ and $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ to be such that for each $n \in \mathbb{N}$

$$u(f_n) = nu\left(\bar{f}\right) \text{ and } u(x_n) = nu(x_a).$$

This implies that for each $n \in \mathbb{N}$

$$\min_{p \in C} \int u(f_n)\, dp = \min_{p \in C} \int nu\left(\bar{f}\right) dp = n \min_{p \in C} \int u\left(\bar{f}\right) dp = nu(x_b) < 0 = u(\bar{x})$$
$$\textit{and}$$
$$I(u(f_n)) = I\left(nu\left(\bar{f}\right)\right) = nI\left(u\left(\bar{f}\right)\right) = nu(x_a) = u(x_n).$$

That is, we have that $f_n \not\succsim^* \bar{x}$ and $f_n \succsim^{\wedge} x_n$ for all $n \in \mathbb{N}$. Finally, observe that jointly $(\succsim^*, \succsim^{\wedge})$ satisfy Weak Caution. Therefore, it follows that there exists $\bar{y} \succsim^{\wedge} \bar{x}$ such that

$$f \not\succsim^* \bar{x} \implies \bar{y} \succsim^{\wedge} f.$$

Consider then $\bar{n} \in \mathbb{N}$ such that $u(x_{\bar{n}}) = \bar{n}u(x_a) > u(\bar{y})$. By construction, it follows that $f_{\bar{n}} \not\succsim^* \bar{x}$ but $f_{\bar{n}} \succsim^\wedge x_{\bar{n}} \succ^\wedge \bar{y}$, a contradiction.

(ii) implies (i). It is trivial. ■

Proof of Theorem 12. (i) implies (ii). We again proceed by steps. The first two steps can be proved by using the same arguments deployed for their counterparts in the proof of Theorem 4.

Step 1. $\succsim^\wedge$ coincide to $\succsim^*$ on X.

Step 2. There exist an unbounded from above and affine function $u^* : X \to \mathbb{R}$ and a nonempty, closed, and convex set C such that

$$f \succsim^* g \quad \Longleftrightarrow \quad \int u^*(f)\,dp \geq \int u^*(g)\,dp \qquad \forall p \in C.$$

Moreover, C is unique.

Step 3. There exist an unbounded from above and affine function $u^\wedge : X \to \mathbb{R}$ and a normalized, monotone, translation invariant, and concave functional $I : B_0(\Sigma) \to \mathbb{R}$ such that

$$f \succsim^\wedge g \quad \Longleftrightarrow \quad I\left(u^\wedge(f)\right) \geq I\left(u^\wedge(g)\right). \tag{33}$$

Moreover, $u^\wedge$ is cardinally unique and, given $u^\wedge$, I is unique.

Proof of the Step.

By assumption, $\succsim^\wedge$ satisfies the Basic Conditions, Completeness, Weak C-Independence. By [14, Lemma 28], there exist a nonconstant affine function $u^\wedge : X \to \mathbb{R}$ and a normalized niveloid $I : B_0\left(\Sigma, u^\wedge(X)\right) \to \mathbb{R}$ such that (33) holds. By Step 1 and the premises of Theorem 12, $\succsim^\wedge$ satisfies One Side Unboundedness and so $u^\wedge$ is unbounded from above. It is routine to check that $u^\wedge$ is cardinally unique and, given $u^\wedge$, I is unique. Without loss of generality, we can assume that $0 \in int\left(u^\wedge(X)\right)$. By the proof of Theorem 3 of [14], [14, Lemma 25], and since $\succsim^\wedge$ satisfies Uncertainty Aversion, we have that I is normalized, monotone, translation invariant, and concave. By [14, p. 1476], we have that I admits a unique extension to $B_0(\Sigma)$ with the same properties. □

Notice that, by Step 1, we can assume without loss of generality that $u^* = u^\wedge = u$ and that $0 \in int(u(X))$.

Step 4. I is consistent with C.

Proof of the Step.

Consider $\varphi, \psi \in B_0(\Sigma)$ and assume that $\varphi \succcurlyeq_C \psi$. Since $u(X)$ is unbounded from above, there exists $k \in \mathbb{R}$ such that $\bar{\varphi} = \varphi + k$ and $\bar{\psi} = \psi + k$ belong to $B_0(\Sigma, u(X))$. It is immediate to see that $\bar{\varphi} \succcurlyeq_C \bar{\psi}$. Since $B_0(\Sigma, u(X)) = \{u(f) : f \in \mathcal{F}\}$, there exist $f, g \in \mathcal{F}$ such that $\bar{\varphi} = u(f)$ and $\bar{\psi} = u(g)$. It follows that $f \succsim^* g$. By Step 3 and since $(\succsim^*, \succsim^\wedge)$ jointly satisfy Consistency, this implies that

$$\begin{aligned} \varphi \succcurlyeq_C \psi \quad &\Longrightarrow \quad \bar{\varphi} \succcurlyeq_C \bar{\psi} \quad \Longrightarrow \quad f \succsim^* g \quad \Longrightarrow \quad f \succsim^\wedge g \\ &\Longrightarrow I(u(f)) \geq I(u(g)) \quad \Longrightarrow \quad I(\varphi + k) \geq I(\psi + k) \quad \Longrightarrow \quad I(\varphi) \geq I(\psi), \end{aligned}$$

proving the statement. □

Step 5. There exists $h' \in (0, \infty)$ for each $k \in \mathbb{R}$ such that

$$\varphi \not\succcurlyeq_C k \quad \Longrightarrow \quad k + h' \geq I(\varphi). \tag{34}$$

In particular, I satisfies (17).

Proof of the Step.

By contradiction, assume that for each $h' \in (0,\infty)$ there exists $k_{h'} \in \mathbb{R}$ and $\varphi_{h'} \in B_0(\Sigma)$ satisfying

$$\varphi_{h'} \not\geq_C k_{h'} \quad \text{and} \quad I(\varphi_{h'}) > k_{h'} + h'.$$

Consider $\{\varphi_n\}_{n\in\mathbb{N}}$. Since $\varphi_n \not\geq_C k_n$ for all $n \in \mathbb{N}$ and $C \subseteq \Delta$, we have that for each $n \in \mathbb{N}$

$$\inf_{s\in S}\varphi_n(s) < k_n \Longrightarrow \sup\left\{-\inf_{s\in S}\varphi_n(s), 0\right\} \geq -\inf_{s\in S}\varphi_n(s) > -k_n \Longrightarrow \sup\left\{-\inf_{s\in S}\varphi_n(s), 0\right\} + k_n > 0.$$

Define $\{\psi_n\}_{n\in\mathbb{N}}$ by

$$\psi_n = \varphi_n + \sup\left\{-\inf_{s\in S}\varphi_n(s), 0\right\} \geq 0 \qquad \forall n \in \mathbb{N}.$$

It follows that $\{\psi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, u(X))$. Define $\{x_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}} \subseteq X$ and $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ by

$$\begin{aligned}
u(x_n) &= \sup\left\{-\inf_{s\in S}\varphi_n(s), 0\right\} + k_n \qquad \forall n \in \mathbb{N} \\
u(z_n) &= k_n + n + \sup\left\{-\inf_{s\in S}\varphi_n(s), 0\right\} \geq 0 \qquad \forall n \in \mathbb{N} \\
u(f_n) &= \psi_n \qquad \forall n \in \mathbb{N}.
\end{aligned}$$

By construction and Step 3, we have that $f_n \not\succsim^* x_n$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}
I(\varphi_n) > k_n + n &\Longrightarrow I(\varphi_n) + \sup\left\{-\inf_{s\in S}\varphi_n(s), 0\right\} > k_n + n + \sup\left\{-\inf_{s\in S}\varphi_n(s), 0\right\} \\
&\Longrightarrow I(\psi_n) > k_n + n + \sup\left\{-\inf_{s\in S}\varphi_n(s), 0\right\} \\
&\Longrightarrow I(u(f_n)) > I(u(z_n)) \Longrightarrow f_n \succ\hat{}\, z_n \qquad \forall n \in \mathbb{N}.
\end{aligned}$$

On the other hand, since $(\succsim^*, \succsim\hat{}\,)$ jointly satisfy Weak C-Caution, it follows that there exists $\{y_n\}_{n\in\mathbb{N}} \subseteq X$ such that

$$y_n \succsim\hat{}\, x_n, \quad y_n \succsim\hat{}\, f_n, \quad \frac{1}{2}x_n + \frac{1}{2}z \sim\hat{}\, \frac{1}{2}y_n + \frac{1}{2}z' \text{ for some fixed } z, z' \in X \text{ and } \forall n \in \mathbb{N}. \tag{35}$$

This implies that $y_n \succ\hat{}\, z_n$, that is, $u(y_n) > u(z_n)$ for all $n \in \mathbb{N}$. By (35), we have that for each $n \in \mathbb{N}$

$$\begin{aligned}
\infty > u(z) - u(z') &= u(y_n) - u(x_n) \geq u(z_n) - u(x_n) \\
&= n + k_n + \sup\left\{-\inf_{s\in S}\varphi_n(s), 0\right\} - \sup\left\{-\inf_{s\in S}\varphi_n(s), 0\right\} - k_n \\
&= n,
\end{aligned}$$

a contradiction with z, z' being fixed. □

Step 6. There exist an unbounded from above and affine function $u : X \to \mathbb{R}$, a grounded, convex, and lower semicontinuous function $c : \Delta \to [0,\infty]$, and a closed and convex set $C \subseteq \Delta$ such that $C = dom(c) = \{p \in \Delta : c(p) \leq k\}$ where $k \in \mathbb{R}$ and for each f and g

$$f \succsim^* g \quad \Longleftrightarrow \quad \int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C \tag{36}$$

and

$$f \succsim\hat{}\, g \quad \Longleftrightarrow \quad \min_{p\in C}\left\{\int u(f)\,dp + c(p)\right\} \geq \min_{p\in C}\left\{\int u(g)\,dp + c(p)\right\}. \tag{37}$$

Proof of the Step.

Define $V : \mathcal{F} \to \mathbb{R}$ by $V(f) = I(u(f))$ for all $f \in \mathcal{F}$ where $u = u^{\hat{}}$ and I are as in Step 3. It is immediate to see that V represents $\succsim^{\hat{}}$. By Steps 2, 3, 4, and 5, and Proposition 16, it follows that there exists a linearly continuous $G \in \mathcal{L}_{bd}(\mathbb{R}\times\Delta)$ such that $dom_{\Delta}G = C$ and

$$I(\varphi) = \min_{p\in\Delta} G\left(\int \varphi dp, p\right) = \min_{p\in C} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma)$$

and

$$V(f) = \min_{p\in\Delta} G\left(\int u(f)\, dp, p\right) = \min_{p\in C} G\left(\int u(f)\, dp, p\right) \qquad \forall f \in \mathcal{F}.$$

By Step 2 and since $u^* = u$, (36) holds. By [5] and since I is real valued, normalized, and translation invariant, we have that $G(t,p) = t + c(p)$ for all $(t,p) \in \mathbb{R}\times\Delta$ where $c : \Delta \to [0,\infty]$ is a grounded, convex, and lower semicontinuous function. This proves that (37) holds. Since it is immediate to prove that $dom_{\Delta}G = dom(c)$, the fact that $\sup_{p\in dom_{\Delta}G} G(t,p) < \infty$ for all $t \in \mathbb{R}$ is equivalent to the existence of a constant $k \in \mathbb{R}$ such that $dom(c) = \{p \in \Delta : c(p) \leq k\}$, proving the statement. □

(ii) implies (i). Consider a nonempty, closed, and convex set $C \subseteq \Delta$, an unbounded from above and affine function $u : X \to \mathbb{R}$, and a grounded, convex, and lower semicontinuous $c : \Delta \to [0,\infty]$ such that $C = dom(c) = \{p \in \Delta : c(p) \leq k\}$ for some $k \in \mathbb{R}$. Suppose further that C and (u,c) satisfy (11) and (12). By [11, Theorem 1], it follows that $\succsim^*$ satisfies the Basic Conditions, C-Completeness, and Independence. By [14, Theorem 3], $\succsim^{\hat{}}$ satisfies the Basic Conditions, Completeness, Weak C-Independence, and Uncertainty Aversion (as well as One Side Unboundedness). Define $G : \mathbb{R}\times\Delta \to (-\infty,\infty]$ by

$$G(t,p) = t + c(p) \qquad \forall (t,p) \in \mathbb{R}\times\Delta.$$

It follows that G is linearly continuous and belongs to $\mathcal{L}_{bd}(\mathbb{R}\times\Delta)$ as well as $C = \{p \in \Delta : c(p) \leq k\} = dom(c) = dom_{\Delta}G$ for some $k \in \mathbb{R}$. By [5], if we define $I : B_0(\Sigma) \to \mathbb{R}$ by

$$I(\varphi) = \min_{p\in\Delta} G\left(\int \varphi dp, p\right) = \min_{p\in C} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma)$$

then I is normalized, monotone, continuous, translation invariant, quasiconcave, and consistent with C. By Proposition 16, it follows that I satisfies (17). In particular, since I is translation invariant, we have that I satisfies (34). Since I composed with u represents $\succsim^{\hat{}}$, these facts imply that jointly $(\succsim^*, \succsim^{\hat{}})$ satisfy Consistency and Weak C-Caution.

The uniqueness part of the statement follows from routine arguments (see [14] and [11]). ■

Proof of Theorem 15. (i) implies (ii). By Theorem 4, it follows that there exist an onto and affine function $u : X \to \mathbb{R}$, a linearly continuous function $G \in \mathcal{L}_{bd}(\mathbb{R}\times\Delta)$, and a closed and convex set $C \subseteq \Delta$ such that $dom_{\Delta}G = C$ and for each f and g

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C$$

and

$$f \succsim^{\hat{}} g \iff \min_{p\in C} G\left(\int u(f)\, dp, p\right) \geq \min_{p\in C} G\left(\int u(g)\, dp, p\right).$$

By [4, Proposition 6] and since $\succsim^{\hat{}}$ satisfies the first part of Minimal Maximal Uncertainty Aversion, it follows that there exist $\eta, \theta \in (0,\infty)$ such that $G_{\theta,HS}, G_{\eta,GS} \leq G$. Define $G_{\theta,\eta} : \mathbb{R}\times\Delta \to (-\infty,\infty]$ by

$$G_{\theta,\eta}(t,p) = G_{\theta,HS}(t,p) \vee G_{\eta,GS}(t,p) \qquad \forall (t,p) \in \mathbb{R}\times\Delta.$$

It is not hard to check that $G_{\theta,\eta} \leq G$, $dom_{\Delta} G_{\theta,\eta} = C_{\eta}$, $G_{\theta,\eta} \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$, and

$$G_{\theta,\eta}(t,p) = \begin{cases} t + \theta R(p||q) & p \in C_{\eta} \\ \infty & p \notin C_{\eta} \end{cases} \qquad \forall (t,p) \in \mathbb{R} \times \Delta.$$

By [4, Proposition 6] and since $\succsim^{\wedge}$ satisfies the second part of Minimal Maximal Uncertainty Aversion and $G_{\theta,HS}, G_{\eta,GS} \leq G_{\theta,\eta} \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$, it follows that $G_{\theta,\eta} \geq G$. It follows that $G_{\theta,\eta} = G$, proving the implication.

(ii) implies (i). It is routine. ■

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