Institutional Members: CEPR, NBER and Università Bocconi

WORKING PAPER SERIES

Dark Pool Trading Strategies

Sabrina Buti, Barbara Rindi, Ingrid M. Werner

Working Paper n. 421

This Version: July, 2013

IGIER – Università Bocconi, Via Guglielmo Röntgen 1, 20136 Milano –Italy
http://www.igier.unibocconi.it

Dark Pool Trading Strategies, Market Quality and Welfare*

Sabrina Buti† Barbara Rindi‡

Ingrid M. Werner§

July, 2013

*We thank Hans Degryse, Dan Li, Albert Menkveld, Maureen O'Hara and Gideon Saar for useful comments as well as participants at 6th Early Career Women in Finance Conference (2010), 4th IIROC DeGroote Conference on Market Structure and Market Integrity (2010), Northern Finance Association Meeting (2010), Western Finance Association Meeting (2011), European Finance Association Meeting (2011) and seminar participants at Bocconi University and Toronto University. The usual disclaimer applies. We acknowledge financial support from IGIER, Bocconi University.

†University of Toronto, Rotman School of Management, sabrina.buti@rotman.utoronto.ca

‡Bocconi University and IGIER, barbara.rindi@unibocconi.it

§Fisher College of Business, Ohio State University, werner_47@fisher.osu.edu

Dark Pool Trading Strategies, Market Quality and Welfare

Abstract

We build a model where a dark pool is introduced to a transparent limit order book market. We show that orders are diverted to the dark pool, but more orders are also executed so total volume increases especially when the order book is shallow. A smaller spread, greater depth and larger tick size stimulate order migration to the dark pool. Institutional traders always benefit from having access to the dark pool. Market quality and retail traders' welfare deteriorate when the order book is shallow, but improve when it is deep. These effects are stronger for a continuous than for a periodic dark pool. If pre-trade transparency is required, the effects on market quality and retail traders' welfare are magnified if the dark pool executes periodically but do not change significantly if the dark pool is continuous.

Dark pools are Alternative Trading Systems (ATS) that do not provide their best-priced orders for inclusion in the consolidated quotation data. They offer subscribers venues where anonymous, undisplayed orders interact away from the lit market. This feature is particularly attractive to institutional investors seeking to trade large quantities while minimizing price impact. Dark pools today represent a considerable fraction of volume (Figure 1). In the U.S. there are over 30 dark pools, and the 19 of them for which data is available (from Rosenblatt Securities Inc.) account for more than 14% of consolidated volume. In Europe the 16 dark markets which report to Rosenblatt account for approximately 4.5% of volume, and in Canada they represent 2% of volume.

[**Insert Figure** 1 **here**]

The rising market share of dark trading recently prompted three major U.S. exchanges to publicly urge the Securities and Exchange Commission (SEC) to put rules in place to curb dark pool trading. Exchange officials are concerned that dark pools divert volume away from lit venues, rather than attracting new order flow to the market. With declining trading volumes world-wide, such a diversion of order flow is a real threat to exchanges' bottom line. Consequently, it is important for exchanges to understand which factors cause order flow to go dark, and under what circumstances dark pools are likely to primarily divert volume away from lit venues as opposed to create more opportunities for trades to take place.

Regulators are concerned about the welfare effects of dark trading, the welfare effects of differential pre-trade transparency, and the effects of dark trading on the informational efficiency of prices. Dark trading can affect both total welfare and its distribution between retail and institutional investors. Dark pools may influence total welfare as a reduction in pre-trade transparency impacts the quality of lit markets and hence the trading costs. Dark pools may also affect the distribution of welfare between retail and institutional investors, as dark markets are primarily used by institutional traders. Some dark pools give institutional traders privileged information about the liquidity available in the pool (Securities and Exchange Commission, SEC 2010), and regulators are considering leveling the playing field by raising the pre-trade transparency of dark pools for all market participants. Regulators are also concerned about the effects of the introduction of dark markets on the informational efficiency of the pricing process. If a significant fraction of trading migrates to dark pools, the ability of traders to discover the fundamental value of the asset by looking at quotes and transaction prices on the lit market might be adversely affected.[1]

In this paper we build a theoretical model that captures the salient features of today's equity markets. Specifically, traders in our model can choose to submit orders to a transparent limit order book (LOB) and to a dark pool. The dark pool can either execute orders periodically or it can execute orders continuously, meaning that traders can simultaneously access the lit and the dark market. We use this model to address the concerns raised by exchange officials and regulators about order migration, market quality, welfare and transparency.

We first investigate to what extent orders migrate away from the lit market following the introduction of a dark pool. We also discuss whether this migration is associated with an overall increase in trading volume. Second, we study what factors are important for determining the extent to which the dark platform attracts order flow away from the lit market. This topic is the focus of existing empirical research on dark pools, and our model can help researchers better design future empirical studies. Finally, we tackle the concerns expressed by regulators about welfare and fair access to dark venues by studying how the introduction of a dark pool affects the quality of the lit market as well as the distribution of welfare between retail and institutional traders, and how an increase in dark pool transparency affects

[1] This would happen in particular if informed traders chose to trade in the dark, thus also raising adverse selection costs for those liquidity traders who may decide to make use of these pools of liquidity (toxicity).

trading costs of different types of market participants.[2]

There is to date very limited academic research on dark trading venues that are competing with lit venues. Existing models focus on the comparison between a dealer market (DM) and a periodic crossing network (e.g., Degryse, Van Achter and Wuyts, DVW 2009), thus over-looking the fundamental interaction between liquidity suppliers and liquidity demanders that governs today's stock exchanges which are organized as LOBs. Moreover, extant models do not consider dark markets with a continuous execution system which allow traders to simultaneously access lit and dark venues. We review the extensive literature related to our model in Section I.

We aim to model a realistic dark venue that captures the most important features of how real life dark pools interact with a transparent limit order market. The most active types of dark pools in the U.S., Europe and Canada are Independent/Agency and Bank/Broker pools (Figure 1). The Independent/Agency pools, like ITG POSIT, are run by independent agency brokers and offer periodic executions at the midpoint of the primary market inside spread, which in the U.S. generally coincides with the National Best Bid and Offer (NBBO). The Bank/Broker pools are instead operated by banks and are used both for agency and proprietary trading. These pools generally offer continuous rather than periodic execution at the midpoint of the NBBO and sometimes also at other price points within the NBBO.[3]

[2]We do not discuss the effects of dark trading on the informational efficiency of prices and on toxicity, for two reasons: technically, because our model does not include asymmetric information; and in terms of contribution to the existing literature because this issue has already been investigated by Ye (2011) and Zhu (2013). While in case of order migration the price discovery process on the lit market would be affected by the introduction of a dark pool, we are less concerned about the possible toxicity of dark pools, as most banks are aware of the risk they run by allowing potentially informed traders to access their dark pool and as a consequence implement strict anti-gaming procedures that seem to be rather widespread.

[3]Within the Bank/Broker category of dark pools, the Market Maker pools are characterized by the fact that liquidity can only be provided by the manager of the pool, whereas the Consortium-Sponsored pools are actually owned by several banks which already own their dark pool and use the Consortium-Sponsored pools as trading venues of last resort. Finally, Exchange-Based dark pools are owned by exchanges and offer continuous execution Bank/Broker pools.

Our theoretical model builds on Parlour (1998), but we extend her to include a price grid, a dark pool and additional order types. We differentiate between retail and institutional traders and only allow the latter to access the dark pool. We need a LOB with a price grid to distinguish among books which differ in spread and depth. We also need additional order types because when institutional traders use both the lit and the dark venue at the same time, they rely on orders that are more sophisticated than simple market and limit orders. Therefore, we introduce immediate-or-cancel orders (IOC) which are first sent to the dark and, if not immediately executed, automatically routed to the LOB as market orders. We also introduce a combination of dark and limit orders which rest on both markets until execution.

We start by modeling a LOB competing with a dark pool which executes periodically at the prevailing LOB midpoint and which gathers orders from institutional traders. This protocol allows us to identify factors which determine dark pool market share and also allows us to show the effects of the introduction of a typical Independent/Agency pool on market quality and traders' welfare. We then model the same LOB but this time competing with a dark pool that offers continuous execution like the Bank/Broker and Exchange-Based pools discussed above. This protocol allows market participants not only to demand liquidity by sending orders to the dark venue, but also to supply and demand liquidity simultaneously on both trading platforms. This very rich set of strategies enables us to provide policy prescriptions not only for the group of Bank/Broker dark pools that executes 57%, 67% and 87% of dark volumes in the U.S., Europe and Canada respectively, but by extension also for the Exchange-Based dark pools for which official data is not available.[4]

[4] Admittedly, our dark pools do not allow for execution at prices within the NBBO, but this feature would probably be more relevant to investigate the effects of dark trading on price discovery, an issue which we do not address in this paper. Our framework does not include competition among different dark venues and it is therefore inadequate to model the Consortium-Sponsored dark pools that are sometimes used by banks to

By comparing results from the benchmark LOB model without a dark pool to the results from the model with a LOB competing with a dark pool, we are able to address the concerns raised by exchange officials and regulators discussed above. We show that the effects of the introduction of a dark pool crucially depend on the initial state of the LOB. Our results imply that regulation applied universally to all stocks may have negative consequences for market quality and for retail as well as institutional traders' welfare. We provide a brief discussion of our main results and of the main mechanisms that underpin our theoretical predictions below.

In our model, traders optimally trade-off the potential price improvement (midquote price) in the dark pool against the trading opportunities on the LOB. For stocks with greater depth at the inside and/or narrower spread, there is more competition for the provision of liquidity. This implies that a limit order submitted to the LOB has to be more aggressive to gain priority over the orders already on the book. As a result, the possibility of obtaining a midquote execution in the dark pool becomes relatively more attractive. Moreover, as liquidity in the lit market increases, more orders migrate to the dark venue and the execution probability of dark orders increases thus making these orders more profitable. Consequently, our model predicts that order migration and dark pool market share increase in liquidity. This prediction is confirmed in recent empirical work on dark pool data by Ready (2013) and Buti, Rindi, and Werner (2011).

While our results show that order migration and dark pool volume increase with liquidity, the number of trades and share volume in excess of the benchmark LOB only framework decrease as liquidity increases. When market orders move to the dark venue in our setting, fewer trades take place; whereas when limit orders move to the dark, more trades take place.

look for the execution of orders that do not find any matching interest in their main dark pool. This group of dark pools, however, execute only a minimal part of the dark volumes (Figure 1).

The reason is that market orders have certainty of execution in the LOB, but when they move to the dark to get a better price their execution probability declines and fewer trades occur. By contrast, limit orders only move to the dark venue if they expect a higher execution probability since they actually get a worse price in the dark venue than in the lit market. As a result more trades take place when limit orders go dark. Because traders tend to make a greater use of market orders at the expense of limit orders in deeper books, the increase in trades and share volume in excess of the benchmark framework is lower for liquid stocks. Similarly, because traders are more likely to use limit orders in shallower books, trades and share volume in excess of the benchmark LOB only framework is higher for illiquid stocks.

We next consider the consequences of traders' optimal use of dark pools for displayed LOB spread and depth. A dark pool always attracts orders away from the LOB, but the consequences for LOB market quality depends whether it is predominately limit or market orders that leave the book. When limit orders leave the LOB, the provision of liquidity decreases and this leads to a reduction in market depth and to a widening of the inside spread. By contrast, a reduction in market orders has a positive effect on both depth and inside spread as market orders subtract liquidity from the book. When a dark pool is introduced, it is always a mixture of market and limit orders that migrate away from the lit market. As explained above, for liquid stocks it is predominately market orders that traders use, and as a result the spread of the lit market improves. However, enough limit orders also migrate to the dark venue to cause inside order book depth to decline. By contrast, for illiquid stocks it is predominately limit orders that traders use, and market quality of the lit market therefore deteriorates.

While previous models of dark pools have only considered venues that cross orders periodically, we extend our model to address dark pools that trade continuously in parallel to the lit market. To make this setup realistic, together with continuous execution we introduce additional order types like immediate-or-cancel orders (IOC) and a combination of dark and limit orders which rest on both markets until execution. Several dark pools offer this type of functionality, for example Sigma X in the U.S. and Match Now in Canada.

With a continuous dark pool more orders and volume migrate to the dark both because executions take place at each trading round and because traders use the new orders that allow them to demand and supply liquidity simultaneously to the LOB and the dark pool. For liquid stocks the migration is so intense that it overcomes the fact that market orders have a lower execution probability in the dark pool than in the LOB. As a result, more trades take place and share volume increases even for liquid stocks following the introduction of a dark pool. And it is precisely the intense migration of market orders that preserves liquidity of the lit market and explains why both spread and depth improve in liquid stocks following the introduction of a continuous dark pool.

Because of the effects of dark pools on market quality discussed above, all traders benefit from the existence of a periodic dark pool for liquid stocks, whereas only institutional traders are better off when trading illiquid stocks. Retail traders are worse off when trading illiquid stocks as they are constrained to use only the LOB, and the market quality of the book deteriorates when a dark pool is introduced. Further, the results show that these effects are all amplified when the dark pool has continuous as opposed to periodic executions. This means that a continuous dark pool is more beneficial than a periodic dark pool for the welfare of retail traders in liquid stocks, the opposite being true for illiquid stocks.

Finally, we use our model to examine how an increase in the visibility of the liquidity residing in the dark pool (higher pre-trade transparency) affects the equilibrium. Here, our results are again very different for the framework with the periodic compared to the continuous dark pool. We show that when the dark pool has periodic execution, if large traders are allowed to have a preview of the state of the dark market, the execution uncertainty is resolved and more orders migrate from the LOB to the dark pool. This means that the execution probability of dark pool orders increases, which reinforces the already existing liquidity externality. As a consequence, an increase in transparency enhances the effects on market quality and traders' welfare which were previously discussed. When instead the dark pool is characterized by continuous executions, the effect of pre-trade transparency is negligible. If traders can access the dark market continuously and can use orders which bounce back to the LOB if unexecuted, they have little need to observe the imbalance directly.

Overall the results of our model emphasize that regulators have to be extremely careful when considering regulation of dark pools, as the consequences of any rules are likely to be very different for liquid compared to illiquid stocks and for dark pools with different market structures. There is clearly no such thing as one-size-fits-all when it comes to regulating dark venues. In fact, our results suggest that regulators should mainly worry about regulating dark venues that focus on illiquid stocks, as it is only for illiquid stocks that dark pool trading is likely to be associated with deteriorating market quality and welfare facing traders who primarily use the lit market.

Note that our predictions are very different from what would be obtained if the lit market was modeled as a DM as for instance in DVW (2009). In their model, traders who are patient and unwilling to pay the spread cannot submit limit orders and hence either stay out of the market or move to the dark pool to execute at the midquote. By contrast, patient traders in our model do not need to move to the dark as they can post their limit orders on the LOB. As a result, we find less order migration to the dark venue than what is predicted by DVW. Our model also generates very different predictions about the factors that drive orders to go dark. DVW find that the smaller the spread, the fewer orders go dark because the price improvement offered by the dark pool is small. When instead the spread is large, traders are more likely to route their orders to the dark venue since it offers a larger price improvement compared to dealer quotes. Our model predicts the opposite, i.e., that dark pools are more actively used for liquid stocks. DVW also conclude that dark trading is beneficial for stocks with larger spread, i.e., illiquid stocks. We find instead that it is precisely when trading illiquid stocks that retail traders loose the most. In fact, total welfare can deteriorate even though institutional traders are better off.

The paper is organized as follows. In Section I we review the related literature. In Section II we present both the benchmark framework and the framework with a dark pool, be it periodic or continuous. In Section III we report the results on factors that affect order flows and dark pool market share, and in Section IV on the effects on market quality and welfare. Section V is dedicated to the model's empirical implications and Section VI to the conclusions and policy implications. All proofs are in the Appendix.

I Literature on Dark Pools

The literature on multimarket competition is extensive.[5] As we model competition between a LOB and a dark pool, our paper is related in particular to the branch of the literature which deals with competition between trading venues with different pre-trade transparency and focuses on the interaction between crossing networks (CN) and DM. The paper which is closest to ours is DVW (2009), who investigate the interaction of a CN and a DM and show that the composition and dynamics of the order flow on both systems depend on the

[5] Works on competition among trading venues include: Barclay, Hendershott, and McCormick (2003), Baruch, Karolyi, and Lemmon (2007), Bessembinder and Kaufman (1997), Easley, Kiefer, and O'Hara (1996), Karolyi (2006), Lee (1993), Pagano (1989), Reiss and Werner (2004) and Subrahmanyam (1997).

level of transparency.[6] However, as we discuss in depth later in the paper, our contribution differs substantially from DVW (2009). First of all, we consider the interaction between a LOB -rather than a DM- and a dark venue, so that in our model traders can both demand liquidity (via market orders) and compete for the provision of liquidity (via limit orders). Second, in addition to a dark CN, we consider a dark pool with a continuous execution system where traders have simultaneous access both to the LOB and to the dark pool.

Another related paper is Zhu (2013) who uses the Glosten and Milgrom (1985) model to show that when the dark market is introduced to a DM, price discovery on the lit venue improves. The reason is that informed traders choose to send their market orders to the DM and not to the CN because they would all submit orders on the same side in the dark venue and no executions would take place. By contrast, in our LOB model traders can act as liquidity suppliers and earn the spread. We conjecture that if we were to extend our model to include asymmetric information, there would be no reason for informed traders to avoid the dark pool and go to the LOB. On the LOB they would not find an infinite supply of liquidity as in a DM. We also conjecture that dark pool trading would not cause a wider spread even if asymmetric information were introduced since informed traders can use limit orders in our model. This is especially likely to be the case in shallow books in which there is limited supply of liquidity at the inside LOB spread.[7] Also note that Ye (2011) finds opposite results on price discovery by modeling competition between a Kyle (1985) auction market and a dark pool. Ye assumes that only informed traders but not noise traders can

[6]Hendershott and Mendelson (2000) model the interaction between a CN and a DM and show costs and benefits of order flow fragmentation. Donges and Heinemann (2004) model intermarket competition as a coordination game among traders and investigate when a DM and a CN can coexist; Foster, Gervais and Ramaswamy (2007) show that a volume-conditional order-crossing mechanism next to a DM Pareto improves the welfare of additional traders.

[7]It should be mentioned, however, that it is technically very challenging to introduce asymmetric information in a model of this type.

strategically opt to trade in the dark pool, and finds that dark pools harm price discovery. In our model it would be unlikely for informed traders to concentrate in the dark pool because rational uninformed traders would exit the pool.

Our model is also closely related to Foucault and Menkveld (2008) who focus on the competition between two transparent LOBs. They show that when brokers can apply Smart Order Routing Technology (SORT), the execution probability of limit orders (i.e., the liquidity provision) in the incumbent LOB increases. In our model traders can use IOC instructions to route orders and we suggest that this routing technology enhances the competition from the new trading venue. The routing technology has a positive effect on liquidity when the book is deep, but a negative effect on liquidity when the book is shallow.

To our knowledge, there is still only limited empirical academic analysis on dark pools. Ready (2013) studies monthly volume by stock in two dark pools for the period June 2005 to September 2007: Liquidnet and ITG POSIT**.** The data suggests that these two dark pools executed roughly 2% of consolidated volume (third quarter 2007) in stocks where they were active, but this still only made up for less than 1% of total market consolidated volume. He finds that dark pools execute most of their volume in liquid stocks (low spreads, high share volume), but they execute the smallest fraction of share of volume in those same stocks. Buti, Rindi and Werner (2011) examine a unique dataset on dark pool activity for a large cross section of U.S. securities and find that liquid stocks are those characterized by more intense dark pool activity. They also find that dark pool volumes increase for stocks with narrow quoted spreads and high inside bid depths, suggesting that a higher degree of competition in the limit order book enhances dark pool activity. Nimalendran and Ray (2012) study detailed data from one dark pool and they find evidence suggesting that price discovery may take place in the dark venue, particularly for less liquid stocks. Finally, Degryse, de Jong and van Kervel (2011) consider a sample of 52 Dutch stocks and analyze both internalized trades and trades sent to dark pools. They find that when these two sources of dark liquidity are combined, the overall effect on global liquidity is detrimental.

Because dark pools are characterized by limited or no pre-trade transparency, our model is also related to the vast literature on anonymity and transparency.[8] In particular the recent paper by Boulatov and George (2012) shows that in a Kyle (1989) setting the quality of the market in a dark regime is better than in a transparent one. The reason is that more informed agents are drawn into providing liquidity, and they trade more aggressively as liquidity providers than as liquidity demanders. Our model differs from Boulatov and George's because it focuses on the trader's endogenous choice between a dark and a visible market, rather than on trading in a setting that can be either dark or transparent.

Finally, dark pools are currently competing with other dark options offered by exchanges to market participants and this provides a link to the recent literature on hidden orders. In Buti and Rindi (2013) and Moinas (2010) traders active in a LOB can choose between disclosed and undisclosed orders, whereas in our model they can choose between lit and dark trading venues. On the empirical side, Bessembinder, Panayides and Venkataraman (2009) study the costs and benefits of iceberg orders at Euronext and find that these orders are associated with smaller implementation shortfall costs, thus suggesting that similarly to dark pools, they provide a protection from price impact.[9]

[8] See for example the theoretical works by Admati and Pfleiderer (1991), Baruch (2005), Fishman and Longstaff (1992), Forster and George (1992), Madhavan (1995), Pagano and Röell (1996), Rindi (2008), and Röell (1991). Several empirical papers have recently explored the significance of anonymity and transparency in experimental settings and real data: Bloomfield and O'Hara (1999, 2000), Boehmer, Saar, and Yu (2005), Flood, Huisman, Koedijk and Mahieu (1999) and Foucault, Moinas and Theissen (2007).

[9] Other empirical contributions are De Winne and D'Hondt (2007), Frey and Sandas (2008), Hasbrouck and Saar (2004), and Tuttle (2006).

II The Model

Existing dark pools can be classified into two broad categories, Periodic Dark Pools ($PDPs$) and Continuous Dark Pools ($CDPs$). $PDPs$ cross orders periodically, for example every hour on the hour. Clients can submit orders directly to the dark pool, but have to wait until the next cross to see if their orders are executed. $CDPs$ cross orders continuously and allow investors to use more sophisticated trading strategies. Liquidity demanders may submit orders that can immediately bounce back to the lit market in case of non-fill or partial execution; liquidity suppliers may send orders simultaneously both to the CDP and to the lit market so that they can exploit trading opportunities on both platforms.

In this Section we present a model of a LOB with both retail and institutional traders and use it as a benchmark protocol. We then add a dark pool and investigate the competition between the LOB and either a PDP or a CDP.

A Benchmark Model (B)

We consider a three-period ($t = t_1, t_2, t_3$) trading protocol that features a LOB for a security which pays v at the end of the trading game. The LOB is characterized by a set of four prices and associated quantities, denoted by $\{p_i^z \& q_i^z\}$, where $z = \{A, B\}$ indicates the ask or bid side of the market, and $i = \{1, 2\}$ the level on the price grid. Therefore, prices are defined relative to the common value of the asset, v:

$$\begin{aligned} p_i^A &= v + i\,\frac{\tau}{2} \qquad (1) \\ p_i^B &= v - i\,\frac{\tau}{2}\ , \end{aligned}$$

where τ is the minimum price increment that traders are allowed to quote over the existing price, and hence it is the minimum spread that can prevail on the LOB. The associated quantities denote the number of shares that are available at each price level. Following Parlour (1998) and Seppi (1997), we assume that a trading crowd absorbs whatever amount of the asset is demanded or offered at the highest ask and lowest bid on the price grid, which in our model are p_2^A and p_2^B. Therefore the book depth is unlimited at the second level, whereas the number of shares available at p_1^A (p_1^B) forms the state of the book at each time t and is defined as $b_t = [q_1^A q_1^B]$.

In each period t a new risk neutral trader who can be with equal probability either a large trader (LT) or a small trader (ST) joins the market. Large traders can trade $j = [0, 2]$ shares, whereas small traders can only trade 1 share or refrain from trading. Upon arrival the trader selects an order type and his optimal trading strategy cannot be modified thereafter. The trader's personal valuation of the asset is represented by a multiplicative parameter, β_t, drawn from a uniform distribution with support $[0, 2]$: traders with a high value of β_t are impatient to buy the asset, while traders with a low β_t are impatient to sell it; traders with a β_t next to 1 are patient as their valuation of the asset is close to the common value.[10]

Traders observe the state of the LOB but not the identity of market participants. To select the optimal order type the incoming trader compares the expected profits from each of the different orders strategies, $\varphi(.)$. Both large and small traders can submit market orders to the first two levels of the price grid, $\varphi_M(j, p_i^z)$; they can post limit orders to the first level, $\varphi_L(j, p_1^z)$, and they can choose not to trade, $\varphi(0)$.[11] Marketable orders are large market

[10] Differently from Parlour (1998), we do not assume that traders arrive at the market with an exogenous probability of being a buyer or a seller, but let the individual βs determine their trade direction. This way, agents are not forced to refrain from trading when they have a high or a low evaluation of the asset and nature selects them as sellers or buyers respectively.

[11] For simplicity, we do not allow large traders to split their orders between a 1-unit market and a 1-unit limit order. The inclusion of such a strategy would not change the results qualitatively.

orders that walk up or down the book in search of execution, and we label them $\varphi_M(j, p^z)$.[12] The profitability of the orders depends on the state of the book, b_t, and on the personal evaluation of the trader, β_t.

[**Insert Figure 2 here**]

Figure 2 shows an example of the extensive form of the trading game, in which the market opens at t_1 with two units on the best bid and offer, $b_{t_1} = [22]$. We will use this extensive form to illustrate the strategies available to both large and small traders, their payoffs and the effects of different types of order on the state of the LOB. Assume first that an impatient large trader arrives and opts to submit a 2-share market sell order that hits the 2 shares on the first level of the bid side, $\varphi_M(2, p_1^B)$. This order pays the spread and executes with certainty with the following payoff:

$$\pi_{t_1}[\varphi_M(2, p_1^B)] = (p_1^B - \beta_{t_1} v)\ 2\ . \qquad (2)$$

After this order is executed, at t_2 the book opens with no shares on the first level of the bid side of the market, $b_{t_2} = [20]$.

If instead a small trader arrives at t_1 and opts for a market sell order, $\varphi_M(1, p_1^B)$, the book opens at t_2 as $b_{t_2} = [21]$. Further, suppose now that a large impatient seller arrives at t_2, observes $b_{t_2} = [21]$, and decides to submit a market sell order of 2 shares. In this case, because there is only 1 share standing at the best bid price, he effectively submits a marketable order, $\varphi_M(2, p^B)$, that walks down the LOB hitting p_1^B and p_2^B to complete execution, with the following payoff:

[12] We omit the subscript i for the level of the book since the order will be executed at different prices.

$$\pi_{t_2}[\varphi_M(2, p^B)] = (p_1^B + p_2^B) - 2\beta_{t_2} v \ . \tag{3}$$

Hence, at t_3 the book opens empty on the bid side. In this last period, traders do not submit limit orders because the market closes and their execution probability is zero. Therefore at t_3 both large and small traders submit either market buy orders to p_1^A, or market sell orders to p_2^B, or refrain from trade and gets no profits, $\pi_{t_3}[\varphi(0)] = 0$.

Starting again from t_1, assume that a more patient large trader arrives who wishes to sell at a higher price and hence chooses to submit a limit sell order to p_1^A, $\varphi_L(2, p_1^A)$, thus forgoing execution certainty. His profits depend on the probability of the order being executed in the following trading rounds, t_2 and t_3:

$$\begin{aligned} \pi_{t_1}^e[\varphi_L(2, p_1^A)] &= (p_1^A - \beta_{t_1} v)[\sum_{w_{t_2}=1,2} w_{t_2} \Pr_{w_{t_2}}(p_1^A | b_{t_2}) \\ &\quad + \sum_{w_{t_2}=0,1} \Pr_{w_{t_2}}(p_1^A | b_{t_2}) \sum_{w_{t_3}=1}^{2-w_{t_2}} w_{t_3} \Pr_{w_{t_3}}(p_1^A | b_{t_3})] \ , \end{aligned} \tag{4}$$

where $\Pr_{w_t}(p_1^A | b_t)$ is the probability that w_t shares get executed at t. It follows that the book opens at t_2 with 4 shares on the best ask price, $b_{t_2} = [42]$. The strategies on the buy side are symmetric and are left out for brevity.

To summarize, at each trading round, the arriving risk-neutral trader selects the optimal order submission strategy which maximizes his expected profits, conditional on the state of the LOB, b_t, and on his type captured by his personal evaluation of the asset, β_t. The large trader chooses:

$$\max_{\varphi} \ \pi_t^e[\varphi_M(j, p_i^z), \varphi_M(2, p^z), \varphi_L(j, p_1^z), \varphi(0) \ |\beta_t, b_t] \ , \tag{5}$$

and the small trader:

$$\max_{\varphi} \pi_t^e[\varphi_M(1, p_i^z), \varphi(0), \varphi_L(1, p_1^z)|\beta_t, b_t] \ . \tag{6}$$

Note that in this model the standard trade-off between execution costs and price opportunity costs applies. Impatient traders generally minimize execution costs by choosing market orders, whereas patient traders minimize the costs of trading at an unfavorable price by choosing limit orders.

We find the solution of this game by backward induction, and from now on we assume without loss of generality that $\tau = 0.1$ and $v = 1$ for simplicity of exposition. We start from the end-nodes at time t_3 and for all the possible states of the book we compare trading profits from both the large and the small traders' optimal strategies. This allows us to determine the probability of the equilibrium trading strategies at t_3, which can be market orders on the buy or sell side, as well as no trading. We can hence calculate the execution probabilities of limit orders placed at t_2, which in turn allows us to compute the equilibrium order submission strategies for period t_2. Given the probability of market orders submitted at t_2, we can finally compute the equilibrium order submission strategies at t_1.

In this model, traders are indifferent between orders with zero execution probability and therefore a unique equilibrium always exists due to the recursive structure of the game:

Definition 1 *An equilibrium of the trading game is a set of $n \in N_t$ order submission decisions, $\{\varphi_a^n\}$, where $a = \{LT, ST\}$, such that at each period the large and the small trader maximize the expected payoff π_t^e according to their Bayesian updated beliefs over the execution probabilities, $\Pr_{w_t}(p_1^z|b_t)$.*

B Limit Order Book and Dark Pools

We now extend the model to include a dark pool that operates alongside our benchmark LOB. As mentioned, we consider two different types of dark pools: the PDP and the CDP. The PDP has periodic execution and resembles the historical Independent/Agency dark pools. The CDP captures the most relevant microstructure features of the Bank/Broker and Exchange-Based dark pools.

In the dark pool modeled herein traders are unable to observe the orders previously submitted by the other market participants. It follows that they can only infer the state of the dark pool by monitoring the LOB and by Bayesian updating their expectations. We assume that at t_1 the dark pool opens with equal probability either empty, or full on one or the other side of the market:[13]

$$\widetilde{PDP}_{t_1} = \widetilde{CDP}_{t_1} = \begin{cases} +6 \;\; with\ prob = \frac{1}{3} \\ \;\;0 \;\; with\ prob = \frac{1}{3} \\ -6 \;\; with\ prob = \frac{1}{3} \end{cases} . \tag{7}$$

B.1 Periodic Dark Pool Framework - $L\&P$

A PDP is organized like an opaque crossing network where time priority is enforced. In this trading venue, orders are crossed at the end of the trading game only if enough orders on the opposite side have been submitted to the dark pool prior to the cross. The execution price is the spread midquote prevailing on the LOB at the end of period t_3 which we indicate with $\widetilde{p}_{Mid}$. Hence, in a PDP not only the execution probability is uncertain but also the

[13] At t_1 there are three periods left in the trading game. So, if for example six shares to sell are already standing on the ask side of the dark pool, $PDP_{t_1} = [-6]$, then the execution probability of any other share posted to the ask side is zero. The reason is that at the most two shares can be executed in each trading round.

execution price.

In this framework the large traders' action space includes the possibility to submit orders to buy or to sell the asset directly on the PDP, as shown in Figure 3.[14] Therefore, each trader decides not only his optimal order type, as in the benchmark framework, but also his preferred trading venue. He still compares the expected profits from the different order types but now the feasibility and profitability of these orders depend also on the expected state of the PDP at the time of the order submission, $\widetilde{PDP}_t$.

[**Insert Figure 3 here**]

Consider for example the sell side. A large incoming trader at t_1 may now in addition to the previously considered strategies submit a 2-share order to the PDP, $\varphi(-2, \widetilde{p}_{Mid})$. Submitting a sell order to the PDP has the following expected payoff:

$$\pi_1^e[\varphi(-2, \widetilde{p}_{Mid})] = E[(\widetilde{p}_{Mid} - \beta_{t_1} v) \Pr_{-2}(\widetilde{p}_{Mid} \,|\Omega_{t_1})]\ 2\ , \tag{8}$$

where $\Omega_{t_1} = \{b_{t_1}, \widetilde{PDP}_{t_1}\}$ is the information set of the trader and $\Pr_{-2}(\widetilde{p}_{Mid} \,|\Omega_{t_1})$ is the probability that 2 shares to sell will be executed in the PDP at the end of the game. Clearly, this additional order submitted to the dark venue adds a new element of uncertainty to the investors' updating process. At t_2 the book opens unchanged and traders are uncertain on whether a large order to buy or to sell was submitted to the PDP. Traders update their expectations on the state of the dark pool as follows:

[14] In this model dark pools are designed to trade large blocks. For this reason, we do not allow either small traders to post their orders to the dark pool, or large traders to split their orders between the dark pool and the LOB. We relax the last assumption when we study the competition between a LOB and a CDP. With continuous executions we have to allow traders to use the two platforms simultaneously.

$$\widetilde{PDP}_{t_2} = \begin{cases} \pm 8 \;\; with\ prob = \frac{1}{3}\frac{\Pr_{t_1}\ \varphi(\pm 2,\widetilde{p}_{Mid})}{\Pr_{t_1}\ \varphi(+2,\widetilde{p}_{Mid})+\Pr_{t_1}\ \varphi(-2,\widetilde{p}_{Mid})} \\ \pm 4 \;\; with\ prob = \frac{1}{3}\frac{\Pr_{t_1}\ \varphi(\mp 2,\widetilde{p}_{Mid})}{\Pr_{t_1}\ \varphi(+2,\widetilde{p}_{Mid})+\Pr_{t_1}\ \varphi(-2,\widetilde{p}_{Mid})} \\ \pm 2 \;\; with\ prob = \frac{1}{3}\frac{\Pr_{t_1}\ \varphi(\pm 2,\widetilde{p}_{Mid})}{\Pr_{t_1}\ \varphi(+2,\widetilde{p}_{Mid})+\Pr_{t_1}\ \varphi(-2,\widetilde{p}_{Mid})} \end{cases} . \tag{9}$$

If in the following period, t_2, a small trader arrives and decides not to trade, traders at t_3 will face a double uncertainty when Bayesian updating the state of the PDP: not only they have to assess whether a buy or a sell dark pool order was submitted at t_1 but they also have to figure out the trading strategies at t_2 that would result in the visible book $b_{t_3} = [22]$. To summarize, the fact that a dark order may have been submitted influences market participants' estimate of the state of the dark pool. Therefore, it also influences the estimated execution probability of future dark orders and the order submission decisions of incoming traders.

More generally, at each trading round the risk-neutral large trader takes all these effects into account and chooses the optimal order submission strategy which maximizes his expected profits, conditional on his valuation of the asset, β_t, and his information set, Ω_t, respectively:

$$\max_{\varphi}\ \pi_t^e[\varphi_M(j,p_i^z),\varphi_M(2,p^z),\varphi(\pm j,\widetilde{p}_{Mid}),\varphi_L(j,p_1^z),\varphi(0)\ |\beta_t,\Omega_t]\ . \tag{10}$$

Small traders still solve problem (6), however they now condition their strategies not only on their own β and on the state of the LOB but also on the inferred state of the PDP. The game is solved as before by backward induction starting from t_3.

B.2 Continuous Dark Pool Framework - *L&C*

We now consider the dark pool that offers continuous execution. In our discrete model, this means that the CDP crosses orders at each trading round at the spread midquote prevailing on the LOB in that period, $\widetilde{p}_{Mid,t}$. In addition to the orders discussed so far, the CDP offers traders a set of more sophisticated strategies that allow large investors to simultaneously send orders to the LOB and to the dark venue.

Consider again the sell side. A large impatient trader may send a IOC sell order to the CDP, $\varphi_M(-j, p_{Mid,t}, p_i^B)$. If the order does not execute immediately, it is automatically routed to the LOB as a market order. This strategy provides the following payoff:

$$\pi_t^e[\varphi_M(-j, p_{Mid,t}, p_i^B)] = \Pr_{-j,t}(p_{Mid,t} \mid \Omega_t)(p_{Mid,t} - \beta_t v)\ j + [1 - \Pr_{-j,t}(p_{Mid,t} \mid \Omega_t)](p_i^B - \beta_t v)\ j\ , \quad (11)$$

where $\Pr_{-j,t}(p_{Mid,t} \mid \Omega_t)$ is the probability that j shares to sell are executed in the CDP at t.[15]

Analogously, a large patient trader may simultaneously send a sell order to the CDP and a limit sell order to the LOB, $\varphi_L(-j, \widetilde{p}_{Mid,t}, p_i^A)$. In this case, after submission, the unexecuted part of the order rests both in the dark and in the lit market until a buy order arrives and executes against it. As soon as the order is executed on one of the platforms, it is immediately cancelled from the other one. Investors' expected payoff from this strategy is equal to:

$$\pi_{t_q}^e[\varphi_L(-j, \widetilde{p}_{Mid,t}, p_1^A)] = \Pr_{-j,t_q}(p_{Mid,t_q} \mid \Omega_{t_q})(p_{Mid,t} - \beta_{t_q} v)\ j + [1 - \Pr_{-j,t_q}(p_{Mid,t_q} \mid \Omega_{t_q})] \quad (12)$$
$$\{[(p_{Mid,t_{q+1}} - \beta_{t_q} v)\ j \Pr_{-j,t_{q+1}}(p_{Mid,t_{q+1}} \mid \Omega_{t_{q+1}}) + (p_1^A - \beta_{t_q} v) \sum_{w_{t_{q+1}}=1,j} w_{t_{q+1}} \Pr_{w_{t_{q+1}}}(p_1^A | \Omega_{t_{q+1}})]$$

[15] When the opposite side of the LOB has only one share at the first level, the IOC bounces back as a marketable order, see Eq. (3).

$$+[1-\Pr_{-j,t_{q+1}}(p_{Mid,t_{q+1}}\,|\Omega_{t_{q+1}})-\sum_{w_{t_{q+1}}=1,j}w_{t_{q+1}}\Pr_{w_{t_{q+1}}}(p_1^A|\Omega_{t_{q+1}})]$$

$$[(p_{Mid,t_{q+2}}-\beta_{t_q}v)\;j\Pr_{-j,t_{q+2}}(p_{Mid,t_{q+2}}\,|\Omega_{t_{q+2}})+(p_1^A-\beta_{t_q}v)\sum_{w_{t_{q+3}}=1,j}^{j-w_{t_{q+1}}}w_{t_{q+2}}\Pr_{w_{t_{q+2}}}(p_1^A|\Omega_{t_{q+2}})\}\;.$$

Therefore, the CDP allows both liquidity demanders and liquidity suppliers to access the dark venue in search of trading opportunities. Because the dark pool crosses at each trading round, the Bayesian updating on the state of the CDP is faster and more effective than in the previous framework.

Figure 4 illustrates this point. It shows that after the simultaneous submission of a sell order to the CDP and a limit sell order to the LOB, incoming traders may observe different possible states of the LOB. With probability $\frac{2}{3}$ they observe one additional share at p_1^A, and infer that a 1-unit limit order was submitted at the first level of the book. However, they are uncertain because this order could come not only from the large trader but also from a small one. If instead they observe the book unchanged after a fast cancellation of a 1-unit limit order (probability $\frac{1}{3}$), they infer that a combined limit and dark pool sell order was submitted and executed immediately on the CDP, so that $CDP_{t_2}=+4$.[16] Therefore the trader arriving at t_2, whether small or large, is aware that now the execution probability of a dark pool order to sell or to buy is equal to 1 or 0 respectively, and trades accordingly.

[**Insert Figure 4 here**]

Under this new trading protocol, at each trading round the risk-neutral large trader

[16] When traders observe a limit sell order that is immediately cancelled, they realize that this order is different from the combined market and dark pool sell or buy order (immediately executed), even if for all these trading strategies the resulting visible LOB is $b_{t_2}=[22]$.

chooses the optimal order submission strategy which maximizes his expected profits, depending on his evaluation of the asset, β_t and on his information set, Ω_t:

$$\max_{\varphi} \pi_t^e[\varphi_M(j, p_i^z), \varphi_M(2, p^z), \varphi_L(j, p_1^z), \varphi(\pm j, \widetilde{p}_{Mid,t}), \\ \varphi_M(\pm j, \widetilde{p}_{Mid,t}, p_i^z), \varphi_L(\pm j, \widetilde{p}_{Mid,t}, p_1^z), \varphi(0) \; |\beta_t, \Omega_t] \; . \tag{13}$$

As before, small traders solve problem (6), and shape their strategies depending on the expected state of the CDP.

III What's Driving Volume into the Dark?

Having solved numerically both the benchmark model (B) and the two models with a periodic ($L\&P$) and a continuous ($L\&C$) dark pool alongside a LOB, we can now compare the results which are derived from the agents' equilibrium strategies. This allows us to answer a number of questions, related respectively to order migration, trade creation and volume creation, which we believe are of particular interest both to market participants and even more so to exchange officials.

When a dark pool is added alongside a LOB, should we expect orders to migrate to the dark venue? And if the dark pool generates order migration, should we expect orders simply to move from one trading platform to the other, or the aggregate execution rate of orders submitted to either platform to increase, thus generating more trades? Also, considering that orders may differ in size, should we expect this variation in the fill rate to lead to volume creation? Our model allows us to discuss these issues and also to investigate which factors attract order flow away from the lit market and into the dark pool. Finally, by comparing the $L\&P$ with the $L\&C$, we can discuss how the design of the dark markets affects the dynamics of such order flow.

We define order migration (OM) as the average probability that in equilibrium an order migrates to the dark pool. The average is computed over the three periods of the game and over all of the equilibrium states of the book and of the dark pool:

$$OM = \tfrac{1}{3}\sum_{t=t_1}^{t_3} \Pr(LT)\ E_{\Omega_t}\left[\int_0^2 \varphi_{LT}^n \cdot f\left(\beta_t\right) d\beta_t\right] , \tag{14}$$

where for the $L\&P$ framework $\varphi_{LT}^n = \varphi(\pm j, \widetilde{p}_{Mid})$, whereas for the $L\&C$ framework $\varphi_{LT}^n = \{\varphi(\pm j, \widetilde{p}_{Mid,t}), \varphi_M(\pm j, \widetilde{p}_{Mid,t}, p_i^z), \varphi_L(\pm j, \widetilde{p}_{Mid,t}, p_1^z)\}$.

We define trade creation (TC) as the difference between the sum of the fill rates on the LOB and the dark pool, and the fill rate in the benchmark model:[17]

$$TC = \sum_{t=t_1}^{t_3}(FR_t^Z - FR_t^B) , \tag{15}$$

where $Z = \{L\&P, L\&C\}$ and

$$FR_t^{Z,B} = \sum_{a=ST,LT} \Pr(a)\ E_{\Omega_t}\left[\int_0^2 \varphi_a^n \cdot f\left(\beta_t\right) d\beta_t\right] . \tag{16}$$

The equilibrium strategies (φ_a^n) considered in Eq. (15) include all -large and small- market orders for the B framework, and both market orders and executed dark pool orders for the $L\&P$ and $L\&C$ frameworks.

Finally, we define volume creation (VC) as the total LOB plus dark pool volume, V_t^Z, in excess of the total LOB volume in the benchmark framework, V_t^B:

[17] We compute the total change over all periods rather than the average across periods because the PDP executes only at the end of the trading game. Considering per-period changes would imply an arbitrarily allocation to a particular period of the trade creation and volume creation that take place in the PDP.

$$VC = \sum_{t=t_1}^{t_3} (V_t^Z - V_t^B) \ , \tag{17}$$

where we measure volume in each period t by weighting the fill rate, FR_t, by the order size, $q_t = \{1, 2\}$:

$$V_t^{Z,B} = \sum_{a=ST,LT} \Pr(a) E_{\Omega_t} \left[\int_0^2 q_t \cdot \varphi_a^n \cdot f(\beta_t) \, d\beta_t \right] \ . \tag{18}$$

Proposition 1 *In equilibrium, when a dark pool is introduced in a market with a LOB we observe:*

- *OM, which is positively related to the liquidity of the book, measured either by the spread and/or by market depth.*
- *TC, which is inversely related to the liquidity of the book. TC is positive in the L&P only when the book is shallow, while in the L&C it is always positive.*
- *VC, which is also inversely related to the liquidity of the book. VC is positive both in the L&P and in the L&C.*
- *OM, TC and VC, which are greater for the L&C than for the L&P and are positively related to the magnitude of the tick size.*

When institutional traders who are active on a LOB are offered the additional option to trade in the dark at a better price but with execution uncertainty, orders migrate to the dark pool (Figure 5). Migration is more intense when the book becomes more liquid in terms of spread and depth: as liquidity increases, some traders find dark pool orders more attractive than limit and market orders (Table I). The reason is that when competition for the provision of liquidity increases, the queue becomes longer due to time priority and there is less room on the LOB; so, dark orders become attractive for the more aggressive of the patient traders. At the same time, as the liquidity of the book increases the execution uncertainty of the dark pool decreases and dark orders become more attractive for the impatient traders. In the $L\&C$ framework, dark pool orders become even more attractive because they may be executed at each trading round (lower execution uncertainty). Hence, both effects are stronger and the migration is more intense than in the $L\&P$ one.

[**Insert Table I and Figure 5 here**]

This finding is consistent with Ready (2013), who shows that the size of the spread on the primary market influences volumes on Liquidnet and ITG POSIT, two Independent/Agency dark pools with periodic crossing. According to Ready, the larger the percentage spread of a stock, the lower is the share of institutional volumes traded on these two dark pools. Our result is also consistent with Buti, Rindi and Werner (2011) who find that stocks with narrower quoted spreads have greater dark pool volumes, suggesting that dark pools are more attractive when the degree of competition on the LOB is high.

Having discussed the migration of orders away from the lit market into the dark pool, we now consider the model's results on trade creation. TC measures the overall increase in the execution rate following the introduction of the dark pool and hence it is the sum of the orders executed on the dark and on the LOB in excess of the benchmark framework. TC decreases with the liquidity of the book for both the $L\&P$ and the $L\&C$ (Figures 5) and this result is driven by the different effect that the migration of limit and market orders has on executions. When limit orders migrate from the LOB into the dark, executions overall increase, whereas when market orders migrate to the dark pool executions decrease as the execution probability of dark orders is larger than that of limit orders and smaller than that of market orders. When the book becomes deeper, traders use more market than limit orders and the second effect is stronger, so that total executions in excess of the benchmark model decrease. This effect is present both in the $L\&P$ and in the $L\&C$ frameworks. However, because with the CDP traders can post orders simultaneously to the LOB and to the dark pool, the execution uncertainty of dark orders is substantially smaller and traders opt for these orders more extensively. As a result, in the $L\&C$ framework TC is greater than in the $L\&P$ one, and it is positive even when the book is deep.

Because we measure volume by weighing the fill rates by the size of the orders executed, a similar pattern characterizes the dynamics of total volumes and VC. However, the average size of the orders executed on the dark pool is larger than the average size of the orders executed on the LOB. Therefore, the total effect on volumes can be positive even if TC is negative, like in the $L\&P$ framework when the book is deep.

Finally, we show that when the tick size is smaller, e.g., 0.05 instead of 0.1, the effect of the introduction of a dark pool on OM, TC and VC is smaller (Figure 6). In equilibrium the smaller the tick size, the smaller the proportion of limit to market orders (as limit orders are less profitable), so that when the dark market is introduced, fewer limit orders switch to dark orders. Furthermore, the smaller the tick size, the smaller the inside spread and the less expensive market orders are compared to dark pool orders. Hence overall OM is smaller and the effects previously described diminish when the lit market tick size is smaller.

In DVW (2009), the introduction of a CN alongside a DM leads to the creation of new orders, as the CN attracts investors who would previously refrain from trading, and it generates order migration only as a secondary effect. By contrast, in our model order migration is the main driver of the results. The creation of new orders takes place almost exclusively in the last period of the trading game, when our model resembles a DM because limit orders have zero execution probability.

However, our model shares with DVW (2009) a feedback effect generated from traders' perception of dark pool liquidity which influences traders' estimate of the execution probability of dark pool orders and hence their use. This result is summarized in Proposition 2:

Proposition 2 *Dark pools generate a liquidity-externality effect as existing dark liquidity begets future liquidity.*

When traders perceive that liquidity is building in the dark pool, they update their estimate of the dark pool depth and assign a higher probability of execution to dark orders, the result being that they are more likely to opt for dark trading. This positive liquidity-externality effect intensifies when traders can observe the dark pool and perceive that dark volume is growing. This prediction is consistent with the empirical results by Buti, Rindi and Werner (2011) that show the existence of a positive auto-correlation between contemporaneous and lagged dark activity.

[**Insert Figure 6 here**]

IV Who Benefits from a Dark Pool?

Even though dark trading has existed for several decades, it is only recently that its share of consolidated equity volume has increased to more than 14% in the U.S. and almost 5% in Europe (Figure 1). It is therefore understandable that regulators are concerned about the effects on market quality and traders' welfare of the widespread use of dark pools. Is market quality affected by the overall reduction in transparency that the growing use of dark trading entails? Should regulators be concerned about the welfare implications of dark trading, and about the degree of dark pools' transparency? We address these issues by first investigating how the introduction of a dark pool affects the quality of the primary market. Because changes in market quality influence agents' gains from trade, we then study how total welfare and the distribution of welfare across market participants change after the introduction of a dark pool. Finally, we extend the model to increase the visibility of the dark pool and investigate whether market participants benefit from enhanced pre-trade transparency.

A Market Quality

To evaluate the effect of dark trading on the quality of the LOB, we consider two standard measures of market quality, i.e., inside spread (S) and market depth (D). We compute expected spread and depth in period t_{i+1} by weighing the realized values in the equilibrium states of the book with the corresponding order submission probabilities in the previous periods:

$$y_{t_{i+1}} = \sum_{a=ST,LT} \Pr(a)\ E_{\Omega_{t_i}} \left[\int_0^2 y_{t_{i+1}} \cdot \varphi_a^n \cdot f\left(\beta_{t_i}\right) d\beta_{t_i} \right] , \tag{19}$$

where $y_{t_{i+1}} = \left\{S_{t_{i+1}}, D_{t_{i+1}}\right\}$. We then compute the percentage difference between these indicators of market quality for the $L\&P$ (and $L\&C$) and the B framework, and average them across periods:[18]

$$\Delta y = \tfrac{1}{2} \sum_{t=t_2}^{t_3} (y_t^Z - y_t^B)/y_t^B \ , \tag{20}$$

where $y = \{S, D\}$. The following Proposition summarizes our results.

Proposition 3 *When a dark pool is added alongside a LOB, changes in market quality depend on the state of the book:*

[18] As in period t_1 spread and depth are exogenous, we only consider the following two periods.

- *when the book is shallow, both the inside spread and depth at the best bid-offer worsen;*

- *when the book is deep, the inside spread improves; depth at the best bid-offer improves for the L&C framework while it worsens in the L&P framework, even though to a less extent compared to the regime with a shallow book.*

- *The effects on depth and spread are stronger when traders use the dark pool more intensively, i.e., in the L&C framework.*

Our results show that the introduction of a dark pool has a negative effect on market quality when the book is shallow. By contrast, overall market quality generally improves when the book is deep. However, note that in the $L\&P$ framework, depth at the best bid-offer still slightly worsens even when the book is deep but not as much as when the book is shallow.

These overall effects can be explained by considering that a dark pool attracts orders away from the LOB and that the effects of OM on the liquidity of the LOB depends on whether it is limit orders or market orders that leave the book. When limit orders leave the LOB, the provision of liquidity decreases and this leads to a reduction in market depth and to a widening of the inside spread. By contrast, a reduction in market orders may have a positive effect on both depth and inside spread as market orders subtract liquidity from the book. Because when the book becomes deeper, traders, all else equal, switch from limit orders to market orders, the positive effects of market orders leaving the LOB dominates. The effect on depth and spread is stronger in the $L\&C$ model because of the higher OM. When the book is deep, the migration of market orders is so intense that liquidity in the LOB is preserved and both spread and depth improve.

[**Insert Figure 7 here**]

Our results are related to the recent empirical studies on the effect of fragmentation on market quality. O'Hara and Ye (2011) find that during 2008 fragmentation improved market quality for NYSE and NASDAQ stocks. As a proxy of volume on off-exchange venues they use trades reported to the Trade Reporting Facilities (TRFs). Unfortunately, the TRF data does not distinguish between dark markets, internalization by broker-dealers and fully transparent limit order books like BATS or Direct Edge. Therefore, they cannot focus on the specific effects of dark pools on the quality of lit markets. Degryse, de Jong and van Kervel (2011) investigate the effect of fragmentation in Europe and find that dark trading has a detrimental effect on the liquidity of Dutch stocks. Yet, their results cannot be interpreted as a test of our empirical predictions as their definition of dark trading includes not only orders executed in dark pools but also internalized trades.

B Welfare Analysis

Traders in our model have a private motive to trade. Hence we can fully characterize welfare and further differentiate between the effects of introducing a dark venue on retail and institutional traders' welfare. In light of our results on OM, TC and VC, and on market quality, we can assess to what extent dark pools enable traders to realize welfare gains. Finally, we can address the policy question of whether in a competitive setting the dark trading option enhances total welfare.

Following Goettler, Parlour and Rajan (2005) and DVW (2009), we measure welfare for a large or a small trader as:

$$W_{a,t} = \int_0^2 \pi_t^e(\varphi_a^n) d\beta_t \ . \tag{21}$$

Total welfare at period t is equal to the sum of the gains from trade for both large and small traders:

$$W_t = \sum_{a=ST,LT} \Pr(a) W_{a,t} \ . \tag{22}$$

We then compute the percentage difference between the $L\&P$ (and $L\&C$) and the B framework for each trader's type and in total, and average them out across the three periods. The following Proposition summarizes our results.

Proposition 4 *The introduction of a dark pool changes traders' welfare as follows.*

- *Small traders are worse off when the book is shallow and better off when it is deep.*
- *Large traders are always better off; the positive change in the gains from trading by large traders increases with the liquidity of the book for the L&P, whereas it is not monotonic for the L&C framework.*
- *In the L&P framework total welfare increases only when the book is deep, whereas in the L&C framework it always improves. The change of total welfare following an increase in the liquidity of the LOB has the same pattern observed for the change in welfare of large traders.*
- *Changes in welfare are larger in the L&C than in the L&P framework.*

Welfare of small traders. The effect of the introduction of a dark pool on the welfare of small traders is mainly driven by the variation in the spread: because these agents only trade one unit, depth only marginally affects their profits (Figure 8). When the book is shallow, the spread deteriorates and their gains from trades decrease. When the book is deep, the spread improves and their welfare increases.

[**Insert Figure 8 here**]

Welfare of large traders. Large traders always benefit from the introduction of a dark pool, be it periodic or continuous. When they are patient, they switch from limit to dark pool orders. The deeper the book, the stronger the competition for the provision of liquidity, and the greater are the benefits from the alternative trading venue. When they are impatient, however, and they switch from market to dark pool orders, the marginal benefit from dark trading decreases with the liquidity of the book. Even though greater liquidity enhances the execution probability of dark orders, the attractiveness of trading at the midquote is greater when the book is empty because large traders have to walk up the LOB in search of execution.

In the $L\&P$ framework, in which dark pool trades have to wait for execution until the end of the game, dark orders are mainly used by patient traders so that overall the positive effect on welfare increases with the liquidity of the book. In the $L\&C$, in which both patient and impatient traders use the dark pool together with the LOB, the final effect on the welfare of large trader depends on the relative advantage that they gain by switching from market or limit orders to the dark pool. As the market becomes more liquid and traders use more market than limit orders, the positive effect of dark trading on the welfare of large patient traders is somewhat attenuated by the reduced marginal benefit that large impatient traders gain by switching from market to dark pool orders.

Total welfare. The introduction of a dark pool increases total welfare, the only exception being a PDP with a shallow book.[19] In this case the limited OM by large traders does not compensate the lower welfare that small traders achieve due to the deterioration of the spread. For the $L\&P$ the variation in total welfare, as well as the variation in the welfare

[19] Interestingly, because DVW (2009) consider a DM - rather than a LOB - competing with an opaque crossing network, they find that overall welfare improves only for assets with a high relative spread: the wider the spread, the greater order creation. Traders' welfare instead always improves because the introduction of a crossing network widens traders' opportunity sets.

of small and large traders, increases with the liquidity of the book. For the $L\&C$, in which there is a greater migration of large orders to the dark venue, the change in overall welfare following a change in the liquidity of the book is driven by that of large traders previously discussed.

Consistently with our findings on volume and market quality, the effects on welfare are magnified with continuous (rather than periodic) dark trading.

C Transparency

In light of the growing volume of dark trading, regulators are concerned about the effects that the lack of pre-trade transparency may have on the quality of the lit markets and on the distribution of welfare among market participants. Because dark markets may allow some investors to receive privileged information on the state of the dark pool, the SEC is also concerned about the effects of unfair access to undisplayed liquidity and has recently proposed various changes to the regulation of non-public trading interest that have been grouped under the SEC releases No. 34-60997 and No. 34-61658. These proposals aim at enhancing dark pool transparency and thus leveling the playing field.

In this Section, we extend our model to consider a framework in which the state of the dark pool, whether periodic or continuous, is visible to large traders.[20] Our aim is to illustrate the effects on market quality and traders' welfare of a stylized two-tiered market in which large traders get a preview of the dark pool liquidity. The following Proposition

[20] As a robustness check, we have solved the model also for the case in which the dark pool is visible to both retail and institutional traders. We do not observe any substantial change in the effects on OM, volume, market quality and welfare that only marginally increase compared to the case with the dark pool only visible to large traders. The divergence between the two frameworks with different levels of transparency decreases in the liquidity of the LOB: small traders' profits are influenced by the state of the dark pool only when they submit limit orders, which are used more extensively when the LOB is empty (Tables I and II). Results are available from the authors upon request.

summarizes the results shown in Figures 9 and 10 and Table II:

Proposition 5 *When large traders are allowed to observe the state of the dark pool,*

- *if the dark pool has periodic execution (L&P) we observe that:*
  - *OM, TC and VC increase across all states of the book;*
  - *spread and depth deteriorate when the book is shallow and improve when it is liquid;*
  - *for illiquid books, small traders are worse of, whereas for liquid ones all traders are better off. Total welfare increases across all books.*
- *If instead the dark pool has continuous execution (L&C), pre-trade transparency does not generate substantial effects, with the exception of OM which decreases.*

In the $L\&P$ framework, traders have to wait until the end of the trading game to resolve the execution uncertainty on the PDP. When large traders can observe the state of the dark pool, this uncertainty is significantly reduced and traders switch from market to dark pool orders (Table II). As a result, volumes move to the dark venue more intensively, and the effects on market quality and welfare are magnified: illiquid books deteriorate further and liquid ones become even more liquid. All investors are better off when trading liquid stocks but trading illiquid stocks is now even more detrimental for small traders. However, because large investors have greater gains from trading across both venues (than losses experienced by small traders), total welfare increases.

[**Insert Table II here**]

Within the $L\&C$ framework, in a transparent CDP traders do not need to send tentative orders to the dark market in search of liquidity because they can use it exclusively when they observe that some liquidity is available. Therefore, OM decreases substantially as traders now have no need to assess the state of the dark pool, and there are effectively no differences in terms of TC, VC, market quality or welfare. Transparency does not significantly matter when the dark pool has a continuous execution.

Interestingly, our results show that the overall effect of pre-trade transparency is to make the difference between the periodic and the continuous dark pool less relevant. The convergence is somewhat weaker for liquid books for which immediacy is more important. Because our model does not include an exogenous time discount factor, the only reason why traders worry about time to execution is the uncertainty on the state of the dark pool. When such an uncertainty is resolved, time to execution no longer matters and the two market structures converge.

An interesting extension of our model would be to add an appropriate time discount parameter to show that even if the endogenous discount factor based on the dark pool uncertainty is resolved with transparency, traders still request greater immediacy for liquid stocks so that the convergence might be stronger for illiquid rather than liquid stocks.

[**Insert Figures 9 and 10 here**]

V Empirical Implications

Our model generates several empirical predictions pertaining to dark pool order flow, to the effects of the introduction of a dark pool both on the quality of the LOB and on the welfare of market participants, and to pre-trade transparency.

Predictions on order flow. Our results show that when a dark pool is added to a LOB, the effects of intermarket competition crucially depend on the liquidity of the lit market order book. We expect OM to be more intense and volume on the dark pool to increase more for liquid stocks. Therefore a smaller spread and a greater inside depth should drive greater volumes into the dark. This prediction is in line both with Buti, Rindi and Werner (2011) and with Ready (2013). Besides depth and spread, our model also suggests there is a third factor influencing dark pool trading, as it shows that dark pool volume increases with the tick size.[21] This empirical prediction should be tested with caution, as our model does not include sub-penny trading which may take place in some dark markets and is particularly sensitive to tick size variations (Buti, Rindi, Wen, and Werner 2011). To study the effect that a tick size change can have on dark trading, empiricists should control for the average order size, which is generally smaller in dark venues that engage in sub-penny trading.

Our model also predicts that when the liquidity of the stock increases, LOB volume and the overall executions and volume created by the introduction of the dark pool decrease. In addition, order migration, executions and volumes increase significantly across all stocks if the dark pool is run continuously.

To our knowledge no attempt has been made in the literature so far to test predictions regarding how fill rates and volumes are affected by dark pool trading. There is also no work on testing how the tick size affects dark pool trading. Further, no attempt has been made to test predictions on the effects of a periodic vs. a continuous dark pool on patterns of trading, market quality, and welfare.

Predictions on market quality and welfare. For illiquid stocks, dark pools have a detri-

[21] Because in the U.S. the tick size is one penny for all stocks priced above 1 USD, empiricists could test this prediction by considering changes in the price of the stock which affect the tick-to-price ratio. Rather than the absolute tick size, it is in fact the size of the minimum price change relative to the price of the stock that affects traders' order submission strategies.

mental effect on the quality of the LOB measured by spread and depth, whereas for liquid stocks dark pools improve market quality. We therefore expect retail traders to be worse off when trading illiquid stocks and better off when trading liquid ones. We also expect that institutional traders overall benefit from dark trading and that all these effects are magnified with continuous dark pools. O'Hara and Ye (2010) show that the overall effect of fragmentation on NASDAQ and NYSE stocks is positive, and Buti, Rindi and Werner (2011) show that more dark pool activity is associated with better market quality. Yet no empirical attempt has been made to measure the gains from dark trading accruing to retail compared to institutional traders, in liquid vs. illiquid stocks, as well as in periodic vs. continuous dark pools.

Predictions on pre-trade transparency. Our results show that the increase in pre-trade transparency that comes with the enhanced dark pool visibility offered to institutional traders has different effects depending on the execution system that governs the dark pool. With periodic execution, pre-trade transparency amplifies the effects of the introduction of the dark pool. In other words, forcing a dark pool that crosses periodically to show orders as they cumulate prior to the cross would further enhance liquidity for liquid stocks and would be associated with a further deterioration of liquidity for illiquid stocks. Moreover, welfare for retail traders would generally deteriorate furthermore as a result of forcing pre-trade transparency. By contrast, with continuous execution pre-trade transparency does not substantially affect the quality of the market and the welfare of traders. This implies that if regulators were to force pre-trade transparency, it would be innocuous for the majority of the dark pools. Note, however, that our model does not allow for asymmetric information about the fundamental value of the security. In a setting where traders gather costly information about the future value of stocks, transparency may reduce the welfare gains to informed institutional traders to the extent that it invites front-running and/or imitation.

VI Conclusions and Policy Implications

We model a multi-period market where institutional traders can access a limit order book as well as a dark pool to satisfy their trading needs. The dark pool can either be a periodic dark pool which gathers orders and executes buy orders against sell orders at the end of the trading game, or a continuous dark pool which traders can use as a complement to the LOB to demand and supply liquidity at each trading round.

Our results show that the consequences of introducing dark pools depend crucially on the liquidity of the initial limit order book. Following the introduction of a dark pool, orders migrate to the dark market. When the initial book is liquid, trades and share volume increase and the quality of the LOB, measured by spread and depth, improves. As a result, all traders are better off, i.e., there is a Pareto improvement of welfare.

When instead the initial book is illiquid, trades and share volume increase but the quality of the LOB deteriorates. The result is that retail investors are harmed, and even though institutional investors are better off, total welfare can deteriorate. This is more likely to occur when the dark pool is completely dark and based on a periodic execution system. When the dark pool instead has continuous executions and/or is more transparent, total welfare increases following the introduction of a dark pool. In this market setup, the welfare-gains for institutional traders out-weigh the welfare-losses facing retail traders.

Our results suggest that the regulatory objective to preserve retail traders' welfare could clash with the objective of dark pool operators to maximize trade and volume-related revenues. The reason is that when institutional traders have access to a dark pool for illiquid stocks the lit market spread tends to widen, and retail traders face higher trading costs as a result. Note also that since fill rates and share volume in excess of the public LOB increase when a dark pool is available for illiquid stocks, the operator of the dark pool has an incentive to boost dark trading even when the operator is the exchange which also runs the lit market. Our model also shows that managers of dark pools seeking to maximize revenue would prefer continuous executions to periodic crossings as this further enhances executions and share volume, but this comes at an even higher cost to retail investors in terms of a wider lit market spread.

Rule 301 (b) of Regulation ATS defines the threshold above which dark pools are obliged to display their best-priced orders in the consolidated quotation data. The SEC (SEC, N.34-60997) recently proposed to substantially lower the trading volume threshold from the current 5% to 0.25%, aiming to reduce dark volume.[22] Similar rule changes are on the table as part of Markets in Financial Instruments Directive (MiFID) II in Europe.[23]

The SEC proposal, aimed at increasing dark pool pre-trade transparency and at leveling the playing field when indications of interest (IOIs) are sent to large traders, can also be evaluated based on our model.[24] We show that pre-trade transparency does not affect market quality significantly when the dark pool is run with continuous execution. Hence in this case IOIs should not significantly affect the welfare of market participants. By contrast, when IOIs are permitted for a dark pool characterized by period executions, all the effects previously discussed are amplified. Specifically, retail traders who are not allowed to access the dark

[22] Currently the display requirement applies if the average daily trading volume that a dark pool has in a stock during at least 4 of the 6 preceding months is 5% of the aggregate average daily share volume for that stock in that period.

[23] MiFID II, if approved, will implement a 4% volume cap on the amount of trading that can be conducted in a single security on a single venue using a reference price waiver, as well as an 8% upper limit on trading in a single name across all such venues. Beyond these levels, orders must be redirected to lit markets.

[24] IOIs are sales messages reflecting an indication of interest to either buy or sell securities. They can contain security names, prices and order size.

pool might be harmed. Our model therefore suggests that it would be beneficial to allow retail traders to access dark markets as was recently discussed by the SEC. We leave this extension for future work.

Our model allows us to discuss a wide range of policy issues which are currently on the agenda of financial regulators. However, there are several caveats that should be kept in mind when deriving policy conclusions from our results. First, the model does not include asymmetric information, so we cannot say anything about whether dark markets are likely to affect price discovery. However, this topic is addressed in complementary theoretical work by Ye (2011) and Zhu (2013). Unfortunately, their models reach opposite conclusions: Ye (2011) finds that informed traders are attracted to the dark pool while Zhu (2013) finds that informed traders avoid the dark pool.

Second, we do not discuss price manipulation. While smart traders could in principle trade on the lit market in advance to manipulate the execution price in the dark, we conjecture that this would primarily be an issue for illiquid stocks. Therefore, the possibility of manipulation provides a further incentive for the regulator to limit dark pool volumes for illiquid stocks.

Third, our model does not embed sub-penny trading as our dark pool trades execute at the midpoint of the lit market spread. Buti, Rindi, Wen, and Werner (2011) show, however, that sub-penny trading also harms illiquid rather than liquid stocks. Therefore, our main policy implications are supported even for market structures where dark pools offer sub-penny trading.

Finally, our model focuses on the competition between a transparent LOB and a dark market. However, some exchanges also allow traders to use hidden orders, thus offering an alternative to dark pool trading. Among the wide range of existing undisclosed orders, the closest competitors to dark pool orders are Hidden Mid-Point Peg orders which are totally invisible and are submitted at the spread mid-point. Compared to dark pool orders, Hidden Mid-Point Peg execute against the LOB order flow and therefore have a higher execution probability than dark pool orders. Tackling the issue of competition for the provision of dark venues between exchanges and ATSs is therefore an extremely interesting issue that we leave for future research.

Appendix

Proof of Proposition 1

Consider first the benchmark case. The model is solved by backward induction, starting from $t = t_3$. The t_3-trader solves a simplified version of Eq. (5), if large, or (6), if small:

$$\max_{\varphi} \pi^e_{t_3} \left\{ \varphi_M(j, p^B_i), \varphi_M(2, p^B), \varphi(0), \varphi_M(2, p^A), \varphi_M(j, p^A_i) | \beta_{t_3}, b_{t_3} \right\} \tag{5'}$$

$$\max_{\varphi} \pi^e_{t_3} \left\{ \varphi_M(1, p^B_i), \varphi(0), \varphi_M(1, p^A_i) \,|\, \beta_{t_3}, b_{t_3} \right\} \ . \tag{6'}$$

Without loss of generality, assume that depending on β_{t_3} and the state of the book b_{t_3} the trader selects one of the equilibrium strategy φ^n_a, with $a = \{ST, LT\}$ and $n \in N_{t_3}$, being N_{t_3} the number of the equilibrium strategies at t_3. The β-thresholds between two different strategies are determined as follows:

$$\beta^{\varphi^{n-1}_a, \varphi^n_a}_{t_3} : \pi^e_{t_3}(\varphi^{n-1}_a \,|\, b_{t_3}) - \pi^e_{t_3}(\varphi^n_a \,|\, b_{t_3}) = 0 \ .$$

These strategies are ordered in such a way that the β-thresholds are increasing, $\beta^{\varphi^{n-1}_a, \varphi^n_a}_{t_3} < \beta^{\varphi^n_a, \varphi^{n+1}_a}_{t_3}$. Hence, the ex-ante probability that a trader submits a certain order type at t_3 is determined as follows:

$$\Pr_{t_3}(\varphi^n_a \,|\, b_{t_3}) = F(\beta^{\varphi^n_a, \varphi^{n+1}_a}_{t_3} \,|\, b_{t_3}) - F(\beta^{\varphi^{n-1}_a, \varphi^n_a}_{t_3} \,|\, b_{t_3}) \ .$$

Consider now period t_2. The incoming trader solves Eq. (5) or (6) if large or small respectively, and uses $\Pr_{t_3}(\varphi^n_a \,|\, b_{t_3})$ to compute the execution probabilities of his limit orders. Given the optimal strategies at t_3, the β-thresholds and the order type probabilities at t_2 are derived using the same procedure as for period t_3, which is then reiterated for period t_1. When traders are indifferent between strategies φ^{n-1}_a and φ^n_a, i.e., $\beta_t = \beta^{\varphi^{n-1}_a, \varphi^n_a}_t$, we assume without loss of generality that they choose φ^{n-1}_a.

The solution of the $L\&P$ and $L\&C$ frameworks follows the same methodology, but now the large trader solves Eq. (10) or (13) respectively.

L&P . We provide examples for the three trading periods analyzed when the book opens at t_1 as $b_{t_1} = [22]$. From now onwards we assume that for large traders the optimal order size is $j^* = \max_j \, [\varphi \,|\, \Omega_t]$, since $\partial \pi^e_t(\varphi)/\partial j \geq 0$ due to agents' risk neutrality. To ensure the uniqueness of the equilibrium, we also assume that when traders are indifferent between trading on the LOB or on the dark pool, they will stay on the LOB.

Consider the following information sets available to traders at t_3, $\Omega_{t_3} = [b_{t_3}, y_{t_1}, y_{t_2}]$ where

$y_t \in \{vis_t, inv_t\}$: (I) $\Omega^{I}_{t_3} = [20, vis_{t_1}, vis_{t_2}]$, (II) $\Omega^{II}_{t_3} = [20, inv_{t_1}, vis_{t_2}]$. The book opens as $b_{t_3} = [20]$ but in (I) a visible (vis) change in the LOB is observed by traders in both periods (i.e., a market order hitting the book or a limit order posted on the book), and in (II) no change in the LOB (inv) is observed at t_1 and a visible change in the LOB (vis) is observed at t_2. We focus on the large trader's profits that for (I) are:

$$\begin{aligned}
\pi^e_{t_3}[\varphi_M(2, p^B_2) \,|\, \Omega^I_{t_3}] &= 2(p^B_2 - \beta_{t_3} v) = 2(1 - \tfrac{3\tau}{2} - \beta_{t_3}) \\
\pi^e_{t_3}[\varphi(-2, p_{Mid}) \,|\, \Omega^I_{t_3}] &= 2(\tfrac{p^A_1 + p^B_2}{2} - \beta_{t_3} v) \Pr_{-2}(\tfrac{p^A_1 + p^B_2}{2} \,|\, \Omega_{t_3}) = 2(1 - \tfrac{\tau}{2} - \beta_{t_3}) \times \tfrac{1}{3} \\
\pi^e_{t_3}[\varphi(+2, p_{Mid}) \,|\, \Omega^I_{t_3}] &= 2(\beta_{t_3} v - \tfrac{p^A_1 + p^B_2}{2}) \Pr_{+2}(\tfrac{p^A_1 + p^B_2}{2} \,|\, \Omega_{t_3}) = 2(\beta_{t_3} - 1 + \tfrac{\tau}{2}) \times \tfrac{1}{3} \\
\pi^e_{t_3}[\varphi_M(2, p^A_1) \,|\, \Omega^I_{t_3}] &= 2(\beta_{t_3} v - p^A_1) = 2(\beta_{t_3} - 1 - \tfrac{\tau}{2}) \,.
\end{aligned}$$

By solving Eq. (10) for this case, it is straightforward to show that all strategies are optimal in equilibrium ($N_{t_3} = 4$) and that for the LT: $\varphi^1_{LT,\Omega^I_{t_3}} = \varphi_M(2, p^B_2)$, $\varphi^2_{LT,\Omega^I_{t_3}} = \varphi(-2, p_{Mid})$, $\varphi^3_{LT,\Omega^I_{t_3}} = \varphi(+2, p_{Mid})$ and $\varphi^4_{LT,\Omega^I_{t_3}} = \varphi_M(2, p^A_1)$. As an example we compute the probability of $\varphi^1_{LT,\Omega^I_{t_3}}$ and to ease the notation in the following formula we omit the subscript "$LT, \Omega^I_{t_3}$":

$$\begin{aligned}
\beta^{\varphi^1,\varphi^2}_{t_3} &: \quad \pi^e_{t_3}[\varphi^1] - \pi^e_{t_3}[\varphi^2] = 0, \text{ and so } \beta^{\varphi^1,\varphi^2}_{t_3} = 1 - 2\tau \\
\Pr_{t_3} \varphi^1 &= F(\beta^{\varphi^1,\varphi^2}_{t_3}) = \tfrac{1}{2}(1 - 2\tau) \,.
\end{aligned}$$

In case (II), profits for DP orders differ:

$$\begin{aligned}
\pi^e_{t_3}[\varphi(-2, p_{Mid}) \,|\, \Omega^{II}_{t_3}] &= 2(\tfrac{p^A_1 + p^B_2}{2} - \beta_{t_3} v)[\tfrac{1}{3} \times 1 + \tfrac{1}{3} \tfrac{\Pr_{t_1} \varphi^n(+2, \widetilde{p}_{Mid})}{\Pr_{t_1} \varphi^n(+2, \widetilde{p}_{Mid}) + \Pr_{t_1} \varphi^n(-2, \widetilde{p}_{Mid}) + \Pr_{T-2} \varphi^n(0)}] \\
\pi^e_{t_3}[\varphi(+2, p_{Mid}) \,|\, \Omega^{II}_{t_3}] &= 2(\beta_{t_3} v - \tfrac{p^A_1 + p^B_2}{2})[\tfrac{1}{3} \times 1 + \tfrac{1}{3} \tfrac{\Pr_{t_1} \varphi^n(-2, \widetilde{p}_{Mid})}{\Pr_{t_1} \varphi^n(+2, \widetilde{p}_{Mid}) + \Pr_{t_1} \varphi^n(-2, \widetilde{p}_{Mid}) + \Pr_{t_1} \varphi^n(0)}] \,,
\end{aligned}$$

where $\varphi^n(..)$ are equilibrium strategies, and we omit that all probabilities at t_1 are conditional to Ω_{t_1}. In this case both the β-thresholds and the order probabilities depend on the equilibrium strategies at t_1, that are rationally computed by the t_3-trader. For example, if the equilibrium strategies are such that $\varphi^1_{\Omega^{II}_{t_3}} = \varphi_M(2, p^B_2)$ and $\varphi^2_{\Omega^{II}_{t_3}} = \varphi(-2, p_{Mid})$, we obtain (subscript "$LT, \Omega^{II}_{t_3}$" is omitted):

$$\begin{aligned}
\beta^{\varphi^1,\varphi^2}_{t_3} &: \quad \pi^e_{t_3}[\varphi^1] - \pi^e_{t_3}[\varphi^2] = 0, \text{ and so } \beta^{\varphi^1,\varphi^2}_{t_3} = \tfrac{(2-7\tau)\Pr_{t_1} \varphi^n(+2, \widetilde{p}_{Mid}) + 4(1-2\tau)[\Pr_{t_1} \varphi^n(-2, \widetilde{p}_{Mid}) + \Pr_{t_1} \varphi^n(0)]}{2\Pr_{t_1} \varphi^n(+2, \widetilde{p}_{Mid}) + 4[\Pr_{t_1} \varphi^n(-2, \widetilde{p}_{Mid}) + \Pr_{t_1} \varphi^n(0)]} \\
\Pr_{t_3} \varphi^1 &= F(\beta^{\varphi^1,\varphi^2}_{t_3}) = \tfrac{1}{2} \beta^{\varphi^1,\varphi^2}_{t_3} \,.
\end{aligned}$$

To determine the equilibrium strategies $\varphi^n_{\Omega^{II}_{t_3}}$ at t_3 for $n \in N_{t_3}$, the model has to be solved up to period t_1. We anticipate that $\varphi_M(2, p_2^B)$ is indeed an equilibrium strategy, and that the corresponding probability is: $\Pr_{t_3} \varphi^1_{\Omega^{II}_{t_3}} = \frac{(2-5\tau)}{4}$.

For t_2 and t_1 we only specify the profit formulas, as the derivation of the β-thresholds and order probabilities follows the same steps presented for period t_3. Consider the case $\Omega_{t_2} = [20, vis_{t_1}]$ as an example, small traders' profits are as follows:

$$\begin{aligned}
\pi_{t_2}[\varphi_M(1, p_2^B) \,|\, \Omega_{t_2}] &= (p_2^B - \beta_{t_2} v) \\
\pi^e_{t_2}[\varphi_L(1, p_1^A) \,|\, \Omega_{t_2}] &= \pi_{t_2}[\varphi(0)] = 0 \\
\pi^e_{t_2}[\varphi_L(1, p_1^B) \,|\, \Omega_{t_2}] &= (\beta_{t_2} v - p_1^B)\tfrac{1}{2}[\Pr_{t_3}(\varphi_M(1, p_1^B) \,|\, \Omega_{t_3} = [21, vis_{t_1}, vis_{t_2}]) + \Pr_{t_3}(\varphi_M(2, p^B) \,|\, \Omega_{t_3} = [21, vis_{t_1}, vis_{t_2}])] \\
\pi_{t_2}[\varphi_M(1, p_1^A) \,|\, \Omega_{t_2}] &= (\beta_{t_2} v - p_1^A) \ .
\end{aligned}$$

Large traders' strategies are similar, the only difference being that $j = 2$, and that they can submit dark pool orders:

$$\begin{aligned}
\pi^e_{t_2}[\varphi(-2, \widetilde{p}_{Mid})] &= E[(\widetilde{p}_{Mid} - \beta_{t_2} v) \Pr_{-2}(\widetilde{p}_{Mid} | \Omega_{t_2})] \\
\pi^e_{t_2}[\varphi(+2, \widetilde{p}_{Mid})] &= E[(\beta_{t_2} v - \widetilde{p}_{Mid}) \Pr_{+2}(\widetilde{p}_{Mid} | \Omega_{t_2})] \ .
\end{aligned}$$

We specify the first profit formula:

$$\begin{aligned}
\pi^e_{t_2}[\varphi(-2, \widetilde{p}_{Mid})] = {} & \tfrac{1}{3} \times 2 \times \tfrac{1}{2}(\tfrac{p_1^A + p_2^B}{2} - \beta_{t_2} v) \Pr_{t_3} \varphi(+2, p_{Mid}) \\
& + \tfrac{1}{3} \times 2 \times \tfrac{1}{2}\{(\tfrac{p_2^A + p_2^B}{2} - \beta_{t_2} v) \Pr_{t_3} \varphi_M(2, p_1^A) + (\tfrac{p_1^A + p_2^B}{2} - \beta_{t_2} v) \\
& [1 + \Pr_{t_3} \varphi(+2, p_{Mid}) + \Pr_{t_3} \varphi(-2, p_{Mid}) + \Pr_{t_3} \varphi_M(2, p_2^B)]\} \ ,
\end{aligned}$$

where we omit that all strategies at t_3 are conditional to $\Omega_{t_3} = [20, vis_{t_1}, inv_{t_2}]$.

At t_1 we consider the book $b_{t_1} = [22]$ and present profit formulas only for the sell side of the market, the buy side being symmetric:

$$\begin{aligned}
\pi_{t_1}[\varphi_M(2, p_1^B)] &= 2(p_1^B - \beta_{t_1} v) \\
\pi_{t_1}[\varphi(0)] &= 0 \\
\pi^e_{t_1}[\varphi_L(2, p_1^A)] &= (p_1^A - \beta_{t_1} v) \Big\{ \tfrac{1}{2}\Pr_{t_2}(\varphi_M(1, p_1^A) \,|\, \Omega_{t_2} = [42, vis_{t_1}]) \tfrac{1}{2}\Pr_{t_3}(\varphi_M(2, p_1^A) \,|\, \Omega_{t_3} = [32, vis_{t_1}, vis_{t_2}]) \\
& + \tfrac{1}{2}\Pr_{t_2}(\varphi_M(2, p_1^A) \,|\, \Omega_{t_2} = [42, vis_{t_1}])[\tfrac{1}{2}\Pr_{t_3}(\varphi_M(1, p_1^A) \,|\, \Omega_{t_3} = [22, vis_{t_1}, vis_{t_2}]) \\
& + \tfrac{1}{2} 2 \Pr_{t_3}(\varphi_M(2, p_1^A) \,|\, \Omega_{t_3} = [22, vis_{t_1}, vis_{t_2}])] \Big\}
\end{aligned}$$

$$\pi^e_{t_1}[\varphi(-2,\widetilde{p}_{Mid})] = E[(\widetilde{p}_{Mid}-\beta_{t_1}v)\Pr_{-2}(\widetilde{p}_{Mid}\,|\,\Omega_{t_1})]\ ,$$

where to economize space we do not specify the formula for $\pi^e_{t_1}[\varphi(-2,\widetilde{p}_{Mid})]$.

L&C . We now provide examples that also refer to the case with the book opening as $b_{t_1} = [22]$. By comparing Eq. (2) and (11), we observe that market orders are always dominated by IOC dark pool orders, unless the probability that the order executes on the CDP is zero:

$$\pi^e_t[\varphi_M(\pm j, p_{Mid,t}, p^z_i)] \geq \pi_t[\varphi_M(\pm j, p^z_i)]\ .$$

At t_3 traders can only choose between dark pool orders and IOC orders. Consider $b_{t_3} = [20]$ as an example. The large trader's profits are:

$$\begin{aligned}
\pi^e_{t_3}[\varphi_M(2,p_{Mid,t_3},p^B_2)\,|\,\Omega_{t_3}] &= 2(\tfrac{p^A_1+p^B_2}{2}-\beta_{t_3}v)\Pr_{-2,t_3}(\tfrac{p^A_1+p^B_2}{2}\,|\,\Omega_{t_3}) + 2(p^B_2-\beta_{t_3}v)[1-\Pr_{-2,t_3}(\tfrac{p^A_1+p^B_2}{2}\,|\,\Omega_{t_3})]\\
\pi^e_{t_3}[\varphi(-2,p_{Mid,t_3})\,|\,\Omega_{t_3}] &= 2(\tfrac{p^A_1+p^B_2}{2}-\beta_{t_3}v)\Pr_{-2,t_3}(\tfrac{p^A_1+p^B_2}{2}\,|\,\Omega_{t_3})\\
\pi^e_{t_3}[\varphi(+2,p_{Mid,t_3})\,|\,\Omega_{t_3}] &= 2(\beta_{t_3}v-\tfrac{p^A_1+p^B_2}{2})\Pr_{+2,t_3}(\tfrac{p^A_1+p^B_2}{2}\,|\,\Omega_{t_3})\\
\pi^e_{t_3}[\varphi_M(2,p_{Mid,t_3},p^A_1)\,|\,\Omega_{t_3}] &= 2(\beta_{t_3}v-\tfrac{p^A_1+p^B_2}{2})\Pr_{+2,t_3}(\tfrac{p^A_1+p^B_2}{2}\,|\,\Omega_{t_3}) + 2(\beta_{t_3}v-p^A_1)[1-\Pr_{+2,t_3}(\tfrac{p^A_1+p^B_2}{2}\,|\,\Omega_{t_3})]\ .
\end{aligned}$$

We compute the β-threshold between $\varphi_M(2,p_{Mid,t_3},p^B_2)$ and $\varphi(-2,p_{Mid,t_3})$:

$$\begin{aligned}
\beta^{\varphi_M(2,p_{Mid,t_3},p^B_2),\varphi(-2,p_{Mid,t_3})}_{t_3} &: \quad \pi^e_{t_3}[\varphi_M(2,p_{Mid,t_3},p^B_2)\,|\,\Omega_{t_3}] - \pi^e_{t_3}[\varphi(-2,p_{Mid,t_3})\,|\,\Omega_{t_3}] = 0\\
\beta^{\varphi_M(2,p_{Mid,t_3},p^B_2),\varphi(-2,p_{Mid,t_3})}_{t_3} &= 1-\tfrac{3}{2}\tau\ .
\end{aligned}$$

Note that the threshold is independent of $\widetilde{CDP}_{t_3}$ that influences only the execution probability $\Pr_{-2,t_3}(\frac{p^A_1+p^B_2}{2}\,|\,\Omega_{t_3})$. The same results hold at t_3 when comparing the other strategies and for other possible states of the book. By solving Eq. (13) for this case, it is straightforward to show that all the strategies are optimal in equilibrium ($N_{t_3} = 4$) and that for the LT: $\varphi^1_{LT,\Omega_{t_3}} = \varphi_M(2,p_{Mid,t_3},p^B_2)$, $\varphi^2_{LT,\Omega_{t_3}} = \varphi(-2,p_{Mid,t_3})$, $\varphi^3_{LT,\Omega_{t_3}} = \varphi(+2,p_{Mid,t_3})$ and $\varphi^4_{LT,\Omega_{t_3}} = \varphi_M(2,p_{Mid,t_3},p^A_1)$.

Consider now period t_2. In the $L\&C$ framework, traders update the state of the CDP not only by distinguishing the cases in which a visible change of the LOB is observed or not, but also by extracting information from the visible orders. Assume that at t_1 a 1-unit limit order to sell immediately cancelled is observed, so that $b_{t_2} = [22]$. Holding this information, at t_2 traders infer that a LT arrived at t_1 who submitted $\varphi_L(-1,p_{Mid,t_1},p^A_1)$; the order was then immediately executed on the CDP and the portion on the LOB cancelled. Therefore,

$CDP_{t_2} = +4$ and Ω_{t_2} =[22, $\varphi_L(-1, p_{Mid,t_1}, p_1^A)$]. The feasible strategies for a LT are:

$$\begin{aligned}
\pi_{t_2}[\varphi(-2, \widetilde{p}_{Mid,t_2}) \,|\, \Omega_{t_2}] &= 2(\tfrac{p_1^A+p_1^B}{2} - \beta_{t_2} v) \\
\pi_{t_2}[\varphi(0) \,|\, \Omega_{t_2}] &= 0 \\
\pi_{t_2}[\varphi_M(2, p_1^A) \,|\, \Omega_{t_2}] &= 2(\beta_{t_2} v - p_1^A) \ .
\end{aligned}$$

Assume now instead that at t_1 a 2-unit market order to sell is observed, so that $b_{t_2} = [20]$. Traders infer that a LT arrived at t_1 and submitted $\varphi_M(2, p_{Mid,t_1}, p_1^B)$, the order was not executed on the CDP and was re-routed to the LOB. Therefore $\widetilde{CDP}_{t_2} = 0$ or -6 with equal probability, and Ω_{t_2} =[20, $\varphi_M(2, p_{Mid,t_1}, p_1^B)$]. The feasible strategies for a large trader become:

$$\begin{aligned}
\pi_{t_2}[\varphi_M(2, p_2^B) \,|\, \Omega_{t_2}] &= 2(p_2^B - \beta_{t_2} v) \\
\pi^e_{t_2}[\varphi(-2, p_{Mid,t_2}) \,|\, \Omega_{t_2}] &= \tfrac{1}{2} 2(\tfrac{p_1^A+p_2^B}{2} - \beta_{t_2} v)\tfrac{1}{2}[\Pr_{t_3} \varphi(+2, \widetilde{p}_{Mid,t_3}) + \Pr_{t_3} \varphi_M(2, \widetilde{p}_{Mid,t_3}, p_1^A)] \\
\pi^e_{t_2}[\varphi(+2, p_{Mid,t_2}) \,|\, \Omega_{t_2}] &= 2(\beta_{t_2} v - \tfrac{p_1^A+p_2^B}{2})\{\tfrac{1}{2} + \tfrac{1}{4}[\Pr_{t_3} \varphi(-2, \widetilde{p}_{Mid,t_3}) + \Pr_{t_3} \varphi_M(2, \widetilde{p}_{Mid,t_3}, p_2^B)]\} \\
\pi^e_{t_2}[\varphi_L(2, p_1^B) \,|\, \Omega_{t_2}] &= 2(\beta_{t_2} v - p_1^B)\tfrac{1}{2}[\tfrac{1}{2} \Pr_{t_3} \varphi_M(1, p_1^B) + \tfrac{1}{2} \Pr_{t_3} \varphi_M(2, p_1^B)] \\
\pi^e_{t_2}[\varphi_L(1, p_{Mid,t_2}, p_1^B) \,|\, \Omega_{t_2}] &= \tfrac{1}{2} 2(\beta_{t_2} v - \tfrac{p_1^A+p_2^B}{2}) + \tfrac{1}{2}\{(\beta_{t_2} v - p_1^B)\tfrac{1}{2} \Pr_{t_3} \varphi_M(1, p_1^B) + (\beta_{t_2} v - v) \\
&\quad \tfrac{1}{2} 2[\Pr_{t_3} \varphi_M(1, \widetilde{p}_{Mid,t_3}, p_1^B) + \Pr_{t_3} \varphi_M(2, \widetilde{p}_{Mid,t_3}, p^B) + \Pr_{t_3} \varphi(-2, \widetilde{p}_{Mid,t_3})]\} \\
\pi^e_{t_2}[\varphi_M(2, p_{Mid,t_2}, p_1^A) \,|\, \Omega_{t_2}] &= \tfrac{1}{2} 2(\beta_{t_2} v - \tfrac{p_1^A+p_2^B}{2}) + \tfrac{1}{2} 2(\beta_{t_2} v - p_1^A) \ .
\end{aligned}$$

We do not specify profit formulas for t_1 given that they are similar to the ones presented for the $L\&P$ framework, the main difference being that now traders have the additional opportunity to submit IOC dark pool orders and combined limit and dark pool orders, as already shown for periods t_2 and t_3.

OM . Results on OM presented in Figures 5 and 6 are derived by straightforward comparison of the equilibrium strategies for the three frameworks: B, $L\&P$ and $L\&C$. In Figures A1-A6 we provide plots at t_1 for the large trader's profits as a function of β, for both the $L\&P$ and $L\&C$ frameworks. We consider only selling strategies, the plots being symmetric for the buy side. Each figure provides a graphical representation of the traders' optimization problem. Figure A1 shows how the introduction of a PDP changes the optimal order submission strategies of large traders by crowding out both market and limit orders, and generating OM. Consider first OM in the $L\&P$: compare Figures A1 and A3 for the effect of market depth, A1 and A5 for the effect of spread, and A1 and A7 for the tick size (to economize space we only report results for the book b_{t_1} = [22]). For the $L\&C$, compare instead Figures A2 and A4, A2 and A6, and A2 and A8 respectively.

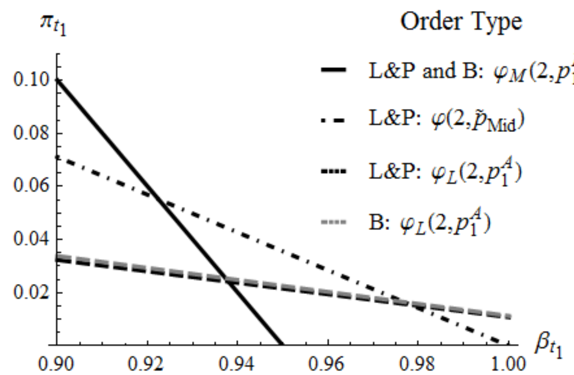


Figure A1. Order Migration on the L&P - b_{t_1}=[22]

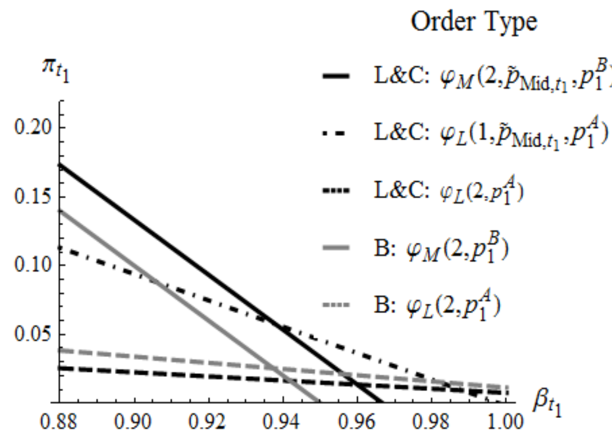


Figure A2. Order Migration on the L&C - b_{t_1}=[22]

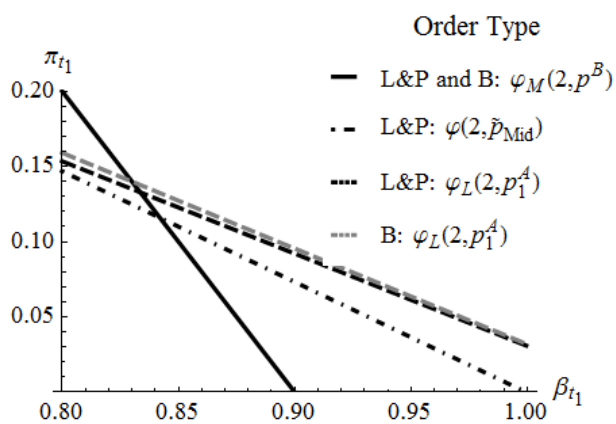


Figure A3. Order Migration on the L&P - b_{t_1}=[11]

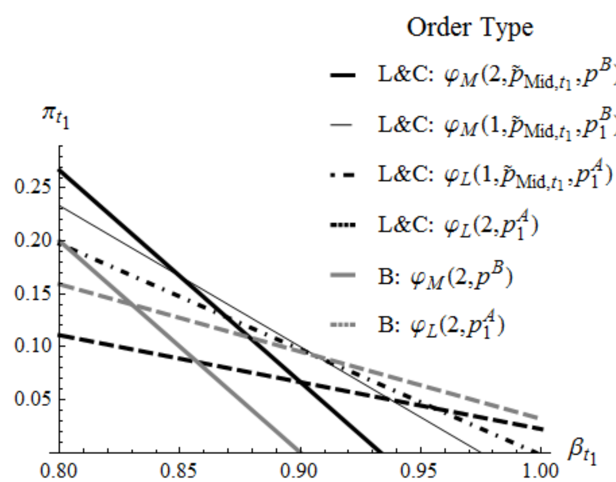


Figure A4. Order Migration on the L&C - b_{t_1}=[11]

TC and VC. Results for TC and VC are obtained by comparing fill rates and volumes for the B, $L\&P$ and $L\&C$ frameworks, as shown in Eq. (15) and (17) respectively. As an example, we consider the $L\&P$ model with an opening book equal to $b_{t_1} = [22]$ -hence omitted in subscript for φ- and specify formulas for the estimated fill rate and volume at

t_1. Equilibrium strategies at t_1 for a LT are as follows: $\varphi^1_{LT}= \varphi_M(2,p^B_1)$, $\varphi^2_{LT}= \varphi(-2,\widetilde{p}_{Mid})$, $\varphi^3_{LT}= \varphi_L(2,p^A_1)$, $\varphi^4_{LT}= \varphi_L(2,p^B_1)$, $\varphi^5_{LT}= \varphi(+2,\widetilde{p}_{Mid})$ and $\varphi^6_{LT}= \varphi_M(2,p^A_1)$. The ones for a ST are: $\varphi^1_{ST}= \varphi_M(1,p^B_1)$, $\varphi^2_{ST}= \varphi_L(1,p^A_1)$, $\varphi^3_{ST}= \varphi_L(1,p^B_1)$ and $\varphi^4_{ST}= \varphi_M(1,p^A_1)$.

$$
\begin{aligned}
FR^{L\&P}_{t_1,[22]} &= \tfrac{1}{2}(\Pr_{t_1}\varphi^1_{ST} + \Pr_{t_1}\varphi^4_{ST})+\tfrac{1}{2}(\Pr_{t_1}\varphi^1_{LT} + \Pr_{t_1}\varphi^6_{LT}) \\
V^{L\&P}_{t_1,[22]} &= \tfrac{1}{2}(\Pr_{t_1}\varphi^1_{ST} + \Pr_{t_1}\varphi^4_{ST})+\tfrac{1}{2}2(\Pr_{t_1}\varphi^1_{LT} + \Pr_{t_1}\varphi^6_{LT}) \ .
\end{aligned}
$$

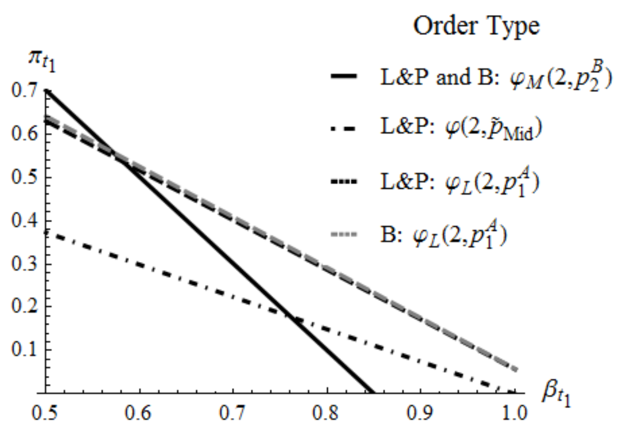


Figure A5. Order Migration on the L&P - b_{t_1}=[00]

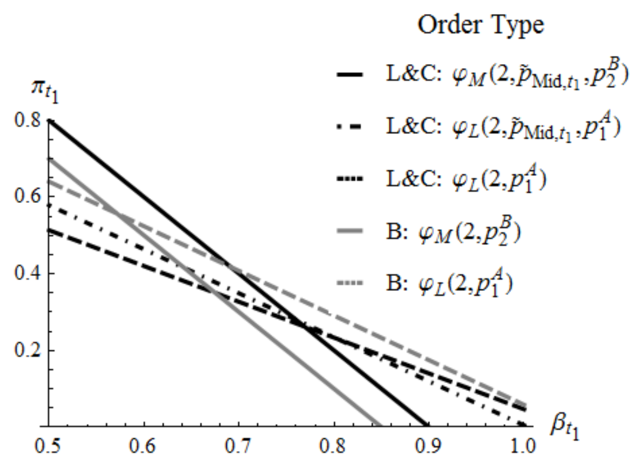


Figure A6. Order Migration on the L&C - b_{t_1}=[00]

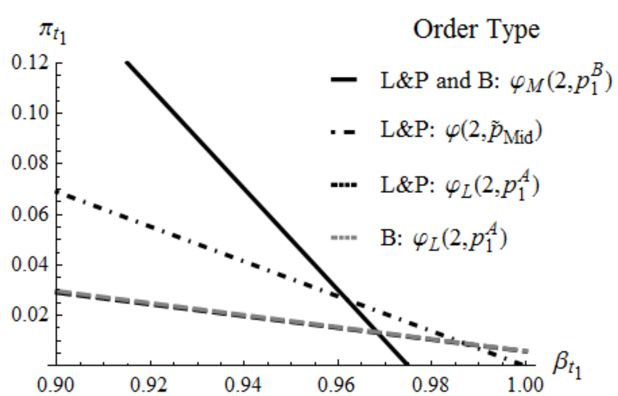


Figure A7. Order Migration on the L&P - b_{t_1}=[22]
Small Tick

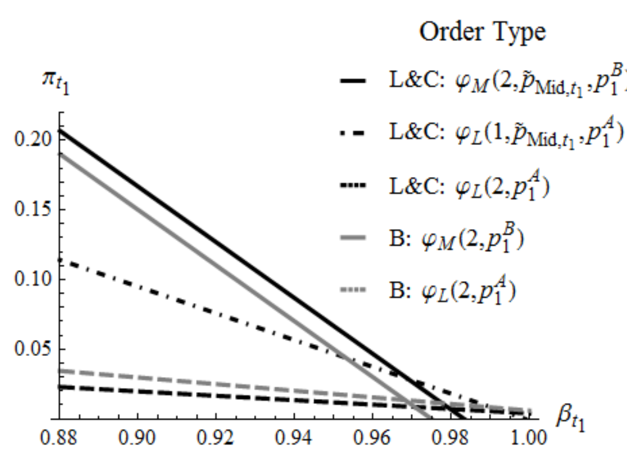


Figure A8. Order Migration on the L&C - b_{t_1}=[22]
Small Tick

Proof of Proposition 2

As for Proposition 1, the result is derived by straightforward comparison of the equilibrium strategies at t_2 and t_3 for the three frameworks: B, $L\&P$ and $L\&C$. We present as an example the book that opens at t_2 as $b_{t_2} = [22]$ for both the $L\&P$ and $L\&C$ frameworks. We observe that, everything equal,[25] profits for dark pool orders increase when no change is observed in the LOB (inv_{t_1}) and traders rationally assume that dark orders were submitted, compared to the case in which traders observe an order submitted to the LOB (vis_{t_1}). Therefore dark pool orders are used more extensively.

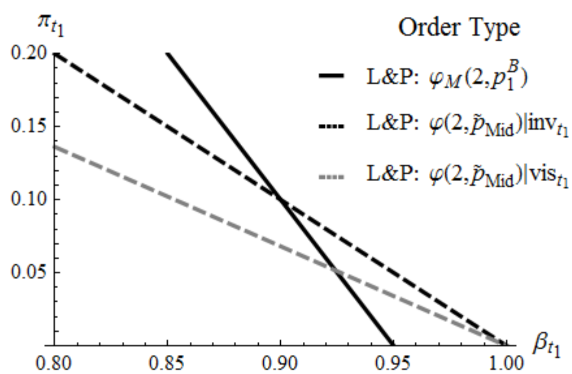


Figure A9. Liquidity Externality Effect on PDP - b_{t_2}=[22]

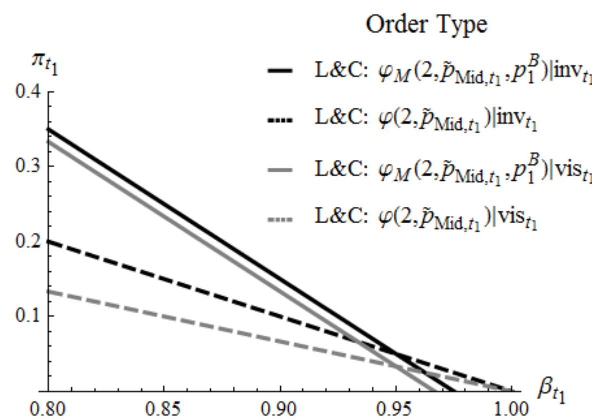


Figure A10. Liquidity Externality Effect on CDP - b_{t_2}=[22]

Proof of Proposition 3

Results presented in Figure 7, are obtained by comparing the two market quality measures for the B, $L\&P$ and $L\&C$ protocol. As an example, we consider again the $L\&P$ model with an opening book equal to $b_{t_1} = [22]$ and specify formulas for the estimated spread and depth at t_2. We refer to the proof of Proposition 1 for a list of the equilibrium strategies in this case.

[25] In Figure A10 we presents strategies for the case in which the visible order does not provide any information on the state of the CDP, i.e., an order clearly identifiable as submitted by a ST. If the visible order was submitted by a LT, traders would update their expectation on the state of the CDP and this would influence dark pool trading even if liquidity on the CDP was not affected (think for example of a IOC dark order that ends up being executed on the LOB, so that traders anticipate that the CDP is empty on that side).

$$
\begin{aligned}
S^{L\&P}_{t_2,[22]} \quad = \quad & \tfrac{1}{2}[\left(p_1^A - p_2^B\right) \Pr_{t_1} \varphi^1_{LT} + \left(p_1^A - p_1^B\right) (\Pr_{t_1} \varphi^2_{LT} + \Pr_{t_1} \varphi^3_{LT} + \Pr_{t_1} \varphi^4_{LT} + \Pr_{t_1} \varphi^5_{LT}) \\
& + \left(p_2^A - p_1^B\right) \Pr_{t_1} \varphi^6_{LT}] + \tfrac{1}{2} \left(p_1^A - p_1^B\right) (\Pr_{t_1} \varphi^1_{ST} + \Pr_{t_1} \varphi^2_{ST} + \Pr_{t_1} \varphi^3_{ST} + \Pr_{t_1} \varphi^4_{ST})
\end{aligned}
$$

$$
\begin{aligned}
D^{L\&P}_{t_2,[22]} = & \tfrac{1}{2}[2(\Pr_{t_1} \varphi^1_{LT} + \Pr_{t_1} \varphi^6_{LT}) + 6(\Pr_{t_1} \varphi^3_{LT} + \Pr_{t_1} \varphi^4_{LT}) + 4(\Pr_{t_1} \varphi^2_{LT} + \Pr_{t_1} \varphi^5_{LT})] \\
& + \tfrac{1}{2}[3(\Pr_{t_1} \varphi^1_{ST} + \Pr_{t_1} \varphi^4_{ST}) + 5(\Pr_{t_1} \varphi^2_{ST} + \Pr_{t_1} \varphi^3_{ST})]
\end{aligned}
$$

Similar computations make it possible to derive the market quality measures for all the other cases.

Proof of Proposition 4

Results presented in Figure 8, are obtained by comparing welfare values for the LT, the ST and on average in the B, $L\&P$ and $L\&C$ protocol. To provide an example, we consider again the $L\&P$ model with an opening book equal to $b_{t_1} = [22]$ and specify the welfare formula at t_1 for the LT. We refer again to the proof of Proposition 1 for a list of the equilibrium strategies in this case.

$$
\begin{aligned}
W^{L\&P}_{LT,t_1,[22]} = & \int_0^{\beta_{t_1}^{\varphi^1_{LT},\varphi^2_{LT}}} \pi_{t_1}(\varphi^1_{LT}) d\beta_{t_1} + \int_{\beta_{t_1}^{\varphi^1_{LT},\varphi^2_{LT}}}^{\beta_{t_1}^{\varphi^2_{LT},\varphi^3_{LT}}} \pi_{t_1}(\varphi^2_{LT}) d\beta_{t_1} + \int_{\beta_{t_1}^{\varphi^2_{LT},\varphi^3_{LT}}}^{\beta_{t_1}^{\varphi^3_{LT},\varphi^4_{LT}}} \pi_{t_1}(\varphi^3_{LT}) d\beta_{t_1} \\
& + \int_{\beta_{t_1}^{\varphi^3_{LT},\varphi^4_{LT}}}^{\beta_{t_1}^{\varphi^4_{LT},\varphi^5_{LT}}} \pi_{t_1}(\varphi^4_{LT}) d\beta_{t_1} + \int_{\beta_{t_1}^{\varphi^4_{LT},\varphi^5_{LT}}}^{\beta_{t_1}^{\varphi^5_{LT},\varphi^6_{LT}}} \pi_{t_1}(\varphi^4_{LT}) d\beta_{t_1} + \int_{\beta_{t_1}^{\varphi^5_{LT},\varphi^6_{LT}}}^{2} \pi_{t_1}(\varphi^6_{LT}) d\beta_{t_1}
\end{aligned}
$$

Similarly, we can derive welfare values for the ST, and for the other protocols.

Proof of Proposition 5

The model with transparency is a simplified case of the one presented in the proof of Proposition 1, the only difference being that now LTs observe the state of the dark pool and do not need to Bayesian update their belief. Therefore we refer to that proof for the solution of the $L\&P$ and $L\&C$ frameworks. Compare Figures A11 and A13 with Figures A12 and A14 respectively for the convergence of the $L\&P$ framework to the $L\&C$ one. We obtain similar results for $b_{t_1} = [00]$ and $b_{t_1} = [11]$ but to economize space we do not present figures for these cases. We cannot show graphically that transparency does not affect substantially the $L\&C$ framework, because we would need "average profits" comparable to the ones in

Figures A2, A4 and A6. The result can be easily shown though by comparing equilibrium strategies for the non-transparent case (Table I) and average equilibrium strategies for the transparent case (Table II).

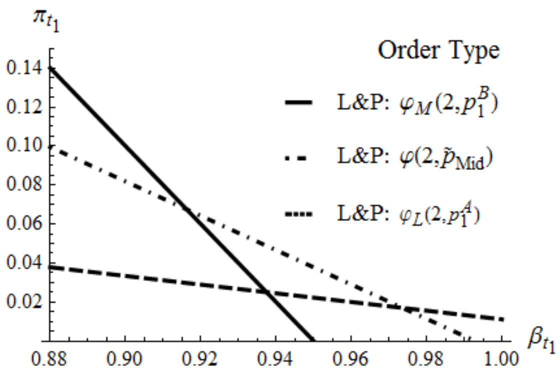


Figure A11. Order Migration on the L&P - b_{t_1}=[22] Transparency, PDP_{t_1}=[0]

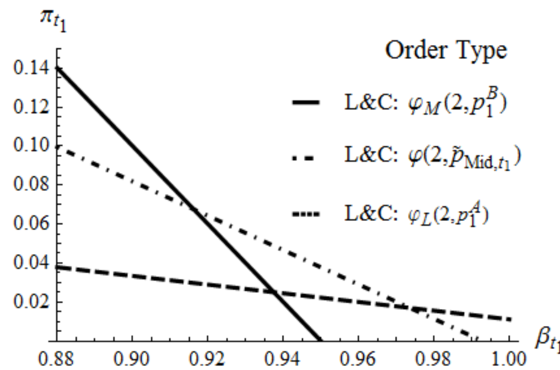


Figure A12. Order Migration on the L&C - b_{t_1}=[22] Transparency, CDP_{t_1}=[0]

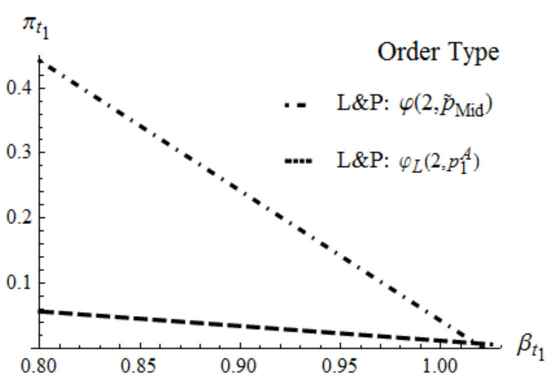


Figure A13. Order Migration on the L&P - b_{t_1}=[22] Transparency, PDP_{t_1}=[+6]

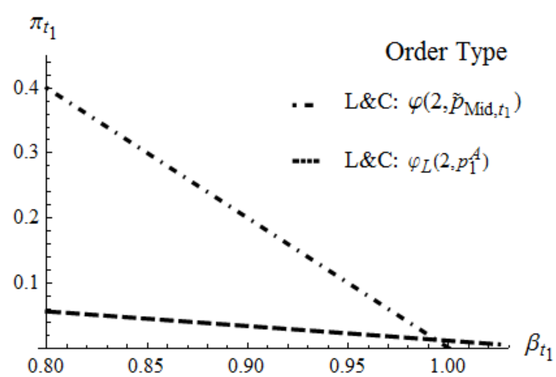


Figure A14. Order Migration on the L&C - b_{t_1}=[22] Transparency, CDP_{t_1}=[+6]

References

Admati, A. and P. Pfleiderer, 1991, Sunshine trading and financial market equilibrium, *Review of Financial Studies* 4, 443-481.

Barclay, M. J., T. Hendershott, and D. T. McCormick, 2003, Competition Among Trading Venues: Information and Trading on Electronic Communication Networks, *Journal of Finance* 58, 2637-2665.

Baruch, S., 2005, Who Benefits from and Open Limit-Order Book?, *Journal of Business* 78, 1267-1306.

Baruch, S., A. Karolyi, and M. Lemmon, 2007, Multi-Market Trading and Liquidity: Theory and Evidence, *Journal of Finance* 62, 2169-2200.

Bessembinder, H. and H. Kaufman, 1997, A cross-exchange comparison of execution costs and information flow for NYSE-listed stocks, *Journal of Financial Economics* 46, 293-319.

Bessembinder, H., M. Panayides, and K. Venkataraman, 2009, Hidden Liquidity: An Analysis of Order Submission Strategies in Automated Markets, *Journal of Financial Economics* 94, 361-383.

Bloomfield, R. and M. O'Hara, 1999, Market Transparency: Who Wins and Who Loses?, *Review of Financial Studies* 12, 5-35.

Bloomfield, R. and M. O'Hara, 2000, Can transparent markets survive?, *Journal of Financial Economics* 55, 425-459.

Boehmer E., G. Saar and L. Yu, 2005, Lifting the Veil: An Analysis of Pre-trade Transparency at the NYSE, *Journal of Finance* 60, 783-815.

Boulatov, A. and T. George, 2012, Hidden and Displayed Liquidity in Securities Markets with Informed Liquidity Providers, *Review of Financial Studies*, forthcoming.

Buti, S. and B. Rindi, 2013, Undisclosed Orders and Optimal Submission Strategies in a Limit Order Market, *Journal of Financial Economics*, forthcoming.

Buti, S., B. Rindi and I. M. Werner, 2011, Diving into Dark Pools, Working paper, Fisher College of Business.

Buti, S., B. Rindi, Y. Wen and I. M. Werner, 2011, Tick Size Regulation and Sub-Penny Trading, Working paper.

Degryse, H., F. de Jong and V. van Kervel, 2011, The impact of dark and visible fragmentation on market quality, TILEC Discussion Paper.

Degryse, H., M. Van Achter and G. Wuyts, 2009, Dynamic order submission strategies with competition between a dealer market and crossing network, *Journal of Financial Economics* 91, 319-338.

De Winne, R. and C. D'Hondt, 2007, Hide-and-Seek in the Market: Placing and Detecting Hidden Orders, *Review of Finance* 11, 663-692.

Donges, J. and F. Heinemann, 2004, Competition for Order Flow as a Coordination Game, Working paper.

Easley, D., N. M. Kiefer and M. O'Hara, 1996, Cream-skimming or profit-sharing? The curious role of purchased order flow, *Journal of Finance* 51, 811-833.

Fishman, M. and F. Longstaff, 1992, Dual Trading in Futures Markets, *Journal of Finance* 47, 643-672.

Flood, M. D., R. Huisman, K. C.G. Koedijk and R. Mahieu, 1999, Quote disclosure and price discovery in multiple-dealer markets, *Review of Financial Studies* 12, 37-59.

Forster, M. and T. George, 1992, Anonymity in Securities Markets, *Journal of Financial Intermediation* 2, 168-206.

Foster, D. P., S. Gervais and K. Ramaswamy, 2007, The Benefits of Volume-Conditional Order-Crossing, Working paper.

Foucault, T. and A. J. Menkveld, 2008, Competition for Order Flow and Smart Order Routing Systems, *Journal of Finance* 63, 119-158.

Foucault, T., S. Moinas and E. Theissen, 2007, Does Anonymity Matter in Electronic Limit Order Markets?, *Review of Financial Studies* 20, 1707-1747.

Frey, S. and P. Sandas, 2008, The Impact of Hidden Liquidity in Limit Order Books, Working paper, University of Tubingen.

Goettler, R. L., C.A. Parlour and U. Rajan, 2005, Equilibrium in a dynamic limit order market, *Journal of Finance* 60, 2149–2192.

Glosten, L. and P. Milgrom, 1985, Bid, Ask, and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders, *Journal of Financial Economics* 14, 71-100.

Hasbrouck, J. and G. Saar, 2004, Limit Orders and Volatility in a Hybrid Market: The Island ECN, Working paper, NYU.

Hendershott, T. and H. Mendelson, 2000, Crossing Networks and Dealer Markets: Competition and Performance, *Journal of Finance* 55, 2071-2115.

Karolyi, A. G., 2006, The World of Cross-Listings and Cross-Listings of the World: Challenging Conventional Wisdom, *Review of Finance* 10, 73-115.

Kyle, A., 1985, Continuous Auctions and Insider Trading, *Econometrica* 53, 1315-1335.

Kyle, A., 1989, Informed speculation with imperfect competition, *Review of Economic Studies*, 56, 317-56.

Lee, C. M., 1993, Market Integration and Price Execution for NYSE-Listed Securities, *Journal of Finance* 48, 3, 1009-1038.

Madhavan, A., 1995, Consolidation, Fragmentation, and the Disclosure of Trading Information, *Review of Financial Studies* 8, 579-603.

Moinas, S., 2010, Hidden Limit Orders and Liquidity in Limit Order Markets, Working paper, Toulouse Business School.

Nimalendran, M. and S. Ray, 2012, Informational Linkages Between Dark and Lit Trading Venues, Working paper, University of Florida.

O'Hara, M. and M. Ye, 2011, Is market fragmentation harming market quality?, *Journal of Financial Economics* 100, 459-474.

Pagano, M., 1989, Trading volume and asset liquidity, *Quarterly Journal of Economics* 104, 255-274.
Pagano, M. and A. Röell, 1996, Transparency and Liquidity: A Comparison of Auction and Dealer Markets with Informed Trading, *Journal of Finance* 51, 579-611.
Parlour, C. A., 1998, Price Dynamics in Limit Order Markets, *Review of Financial Studies* 11, 789-816.
Ready, M., 2013, Determinants of Volume in Dark Pools, Working paper, University of Wisconsin.
Reiss, P. C., and I. M. Werner, 2004, Anonymity, Adverse Selection, and the Sorting of Interdealer Trades, *Review of Financial Studies* 17, 599-636.
Rindi, B., 2008, Informed traders as liquidity providers: anonymity, liquidity and price formation. *Review of Finance* 12, 497-532.
Röell, A., 1991, Dual capacity trading and the quality of the market, *Journal of Financial Intermediation* 1, 105-124.
Securities and Exchange Commission, 2009, Regulation of Non-Public Trading Interest, Release No. 34-60997, File No. S7-27-09.
Securities and Exchange Commission, 2010, Concept Release on Equity Market Structure, Release No. 34-61358, File No. S7-02-10.
Seppi, D., 1997, Liquidity Provision with Limit Orders and a Strategic Specialist, *Review of Financial Studies* 10, 103-150.
Subrahmanyam, A., 1997, Multi-market trading and the informativeness of stock trades: An empirical intraday analysis, *Journal of Economics and Business* 49, 515-531.
Tuttle, L., 2006, Hidden orders, trading costs and information, Working paper, American University of Sharjah.
Ye, M., 2011, A Glimpse into the Dark: Price Formation, Transaction Cost and Market Share of the Crossing Network, Working paper, Cornell University.
Zhu, H., 2013, Do Dark Pools Harm Price Discovery?, Working paper.

Figure 1 - Dark Pools Volume: Percentage of Consolidated U.S., European and Canadian Equity Volume, December 2012. Data source: Rosenblatt Securities Inc.

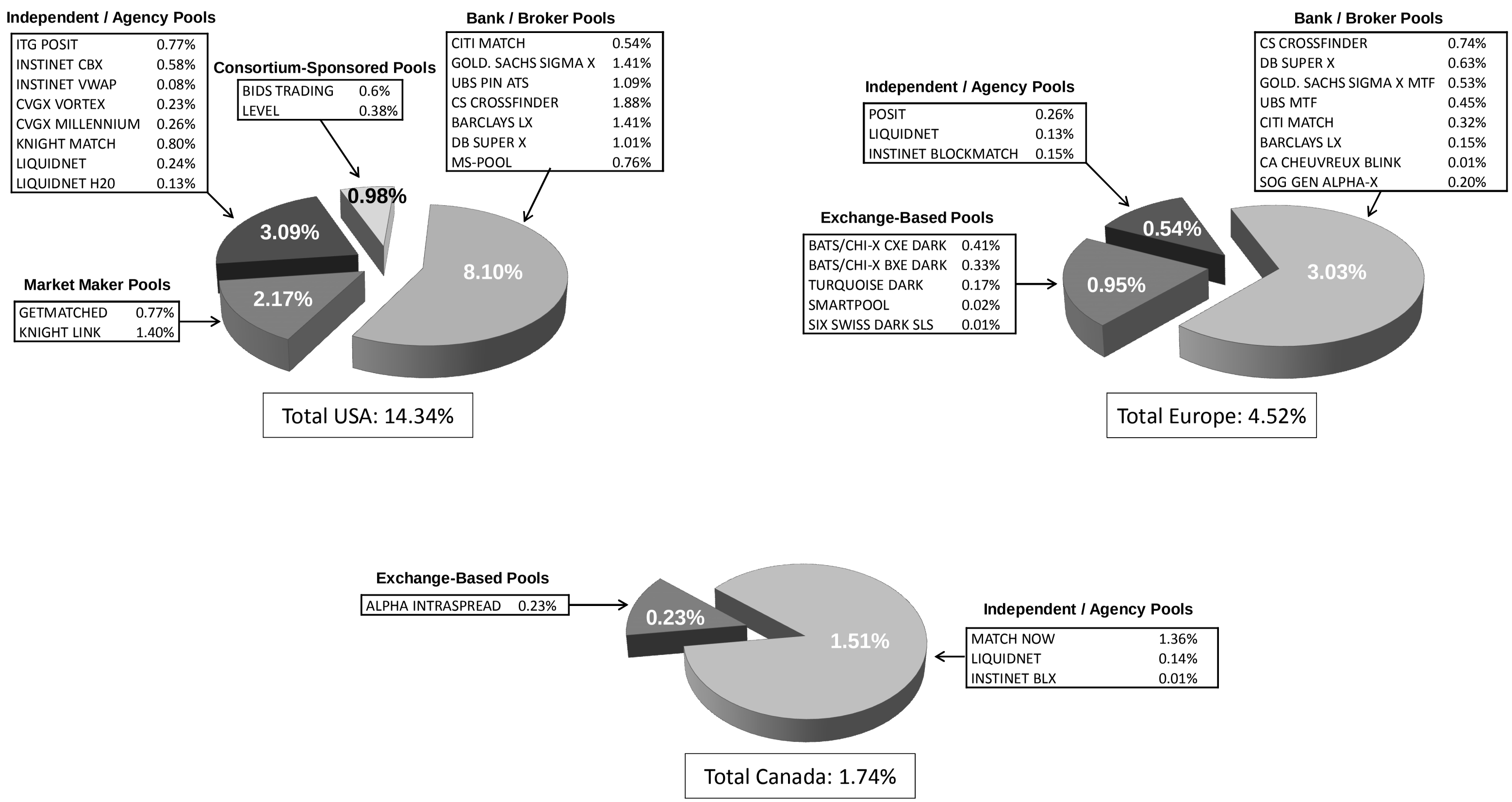


Figure 2 - Benchmark Model of Limit Order Book (B). Example of the extensive form of the game when the opening book at t_1 is $b_{t_1} = [22]$, where j indicates the number of shares traded by the large trader. Only equilibrium strategies are presented.

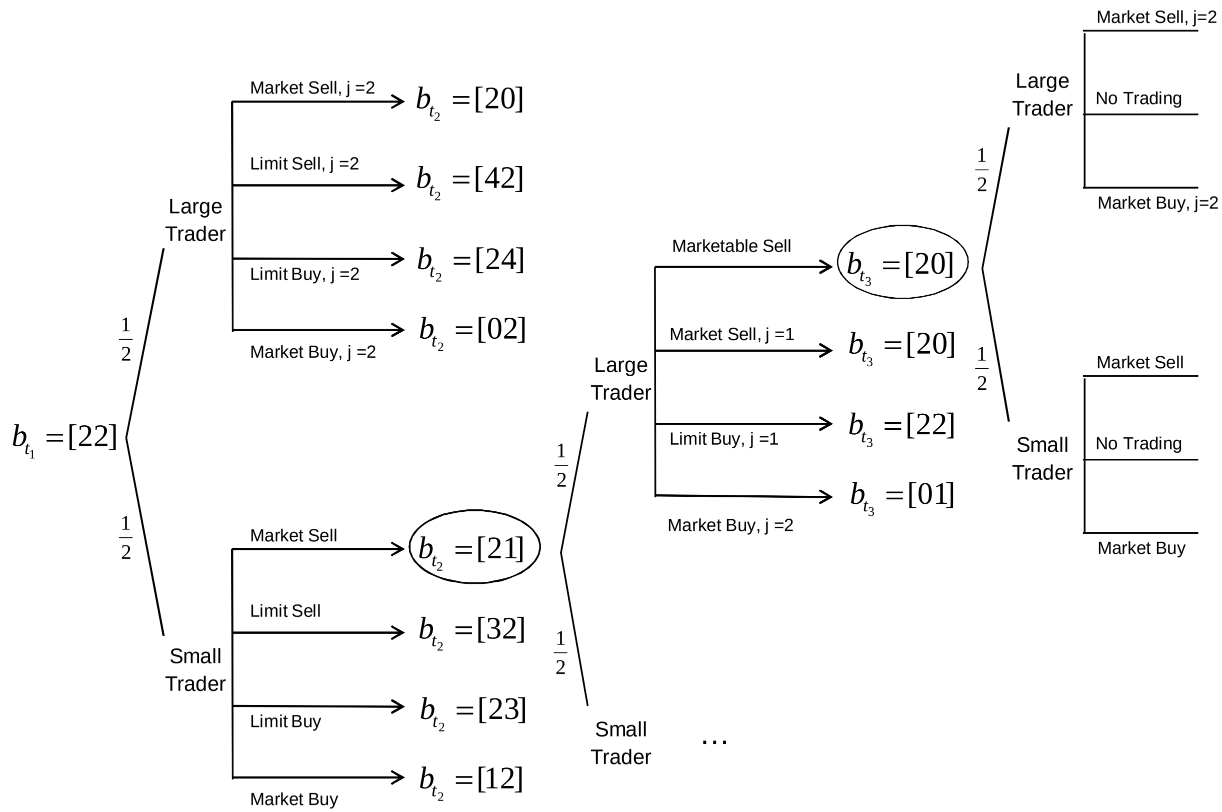


Figure 3 - Limit Order Book and Periodic Dark Pool (L&P). Example of the extensive form of the game when the opening book at t_1 is $b_{t_1} = [22]$, where j indicates the number of shares traded by the large trader. Books that belong to the same information set, and hence are undistinguishable, are inside a squared box. Only equilibrium strategies are presented.

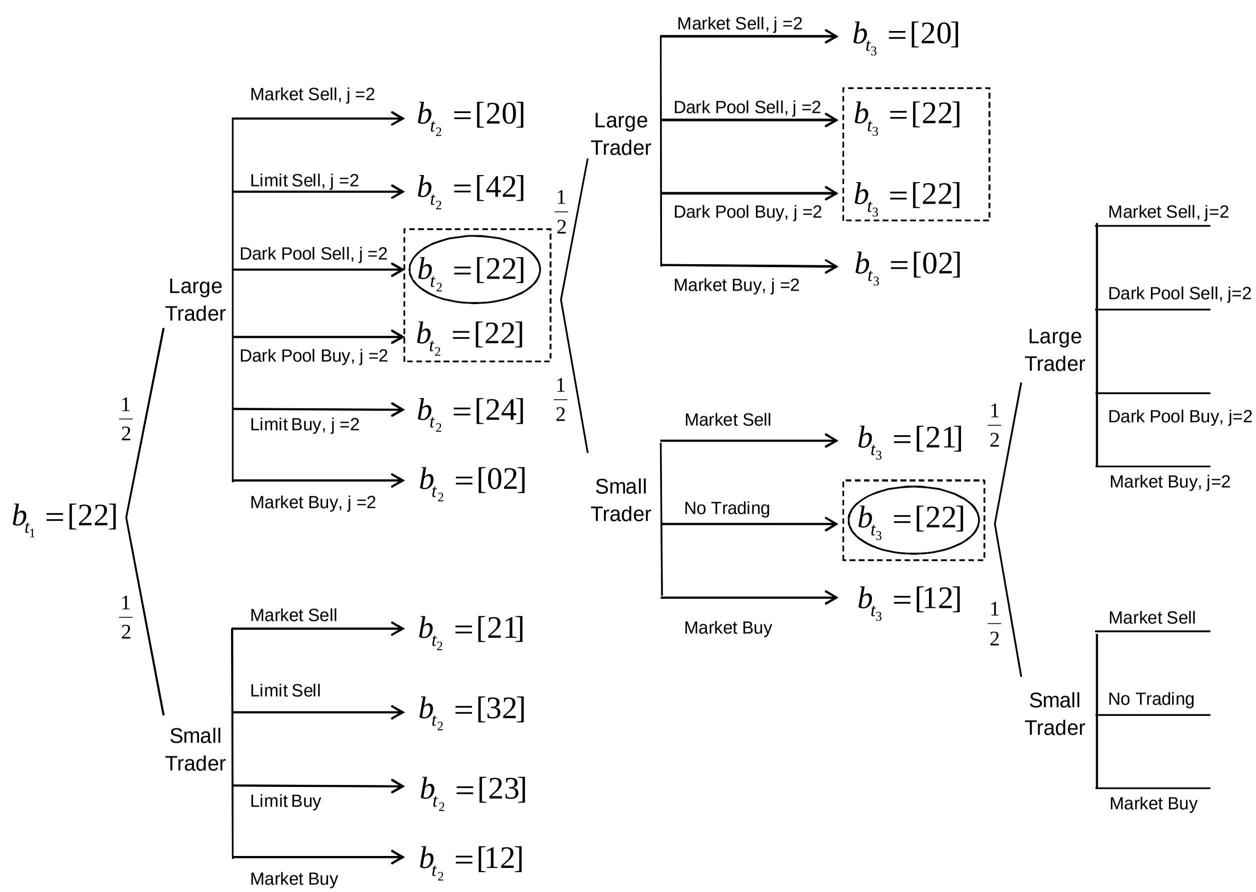


Figure 4 - Limit Order Book and Continuous Dark Pool (L&C). Example of the extensive form of the game when the opening book at t_1 is $b_{t_1} = [22]$ where j indicates the number of shares traded by the large trader. Books that belong to the same information set, and hence are undistinguishable, are inside a squared box and have the same line format. For example, $b_{t_2} = [23]$ can be observed either when a small trader arrives and submits a limit buy, or when a large trader arrives and submits a combined limit and dark order to buy that is not executed immediately. Only equilibrium strategies are presented.

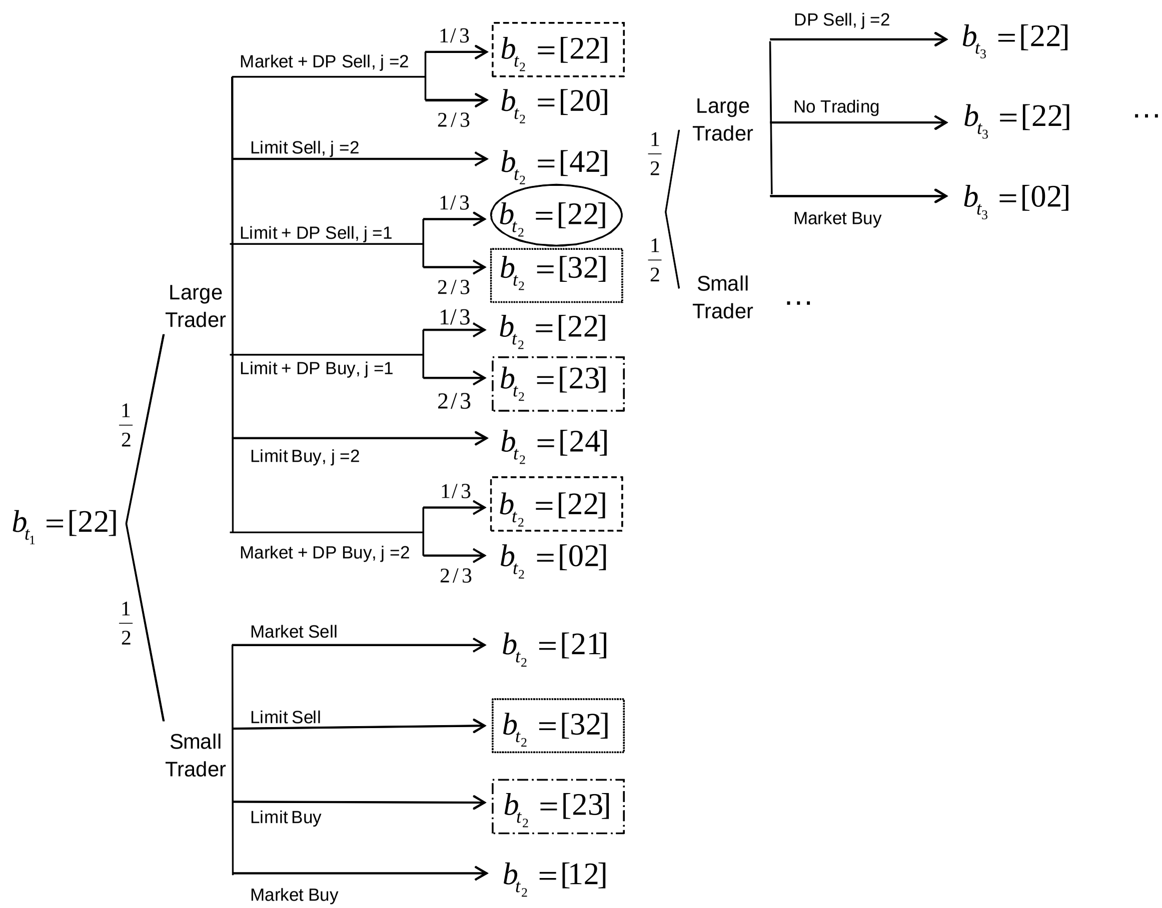
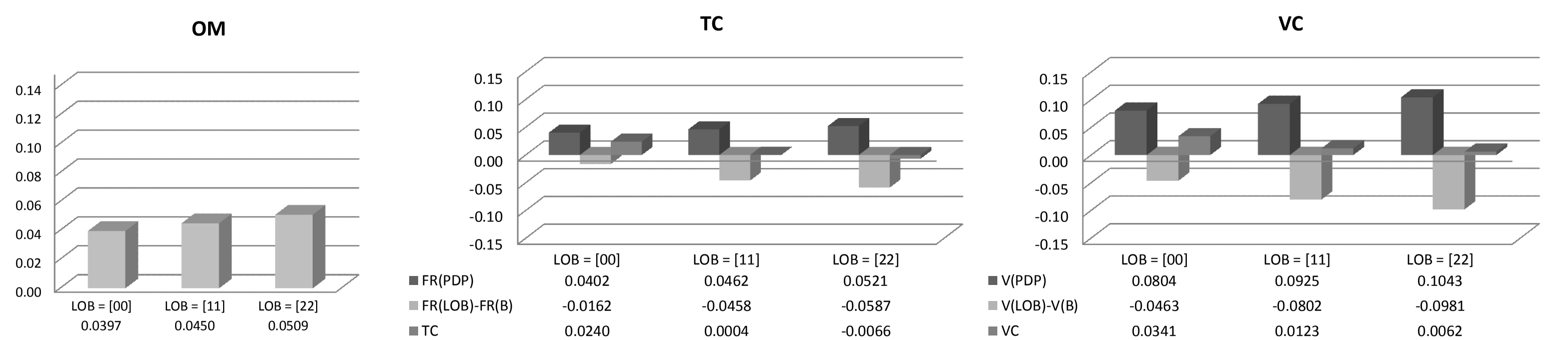


Figure 5 - Order Migration, Trade Creation and Volume Creation. This Figure presents results for two frameworks, $L\&P$ (Panel A) and $L\&C$ (Panel B). The first one combines a limit order book (LOB) and a periodic dark pool (PDP); the second a LOB and a continuous dark pool (CDP). For each framework we report order migration (OM), trade creation (TC), and volume creation (VC). OM is the average probability over the three periods that an order migrates to the dark pool. TC is the sum of two components. The first one is the total fill rate in the PDP, $FR(PDP)$, or in the CDP, $FR(CDP)$. The second one is the difference between the LOB fill rate in the $L\&P$ or $L\&C$, $FR(LOB)$, and the LOB fill rate in the benchmark, $FR(B)$. Three initial states of the LOB are considered that differs for the number of shares at the first level of the book: [00], [11], and [22]. Results are computed assuming that the tick size is equal to $\tau = 0.1$.

PANEL A: L&P



PANEL B: L&C

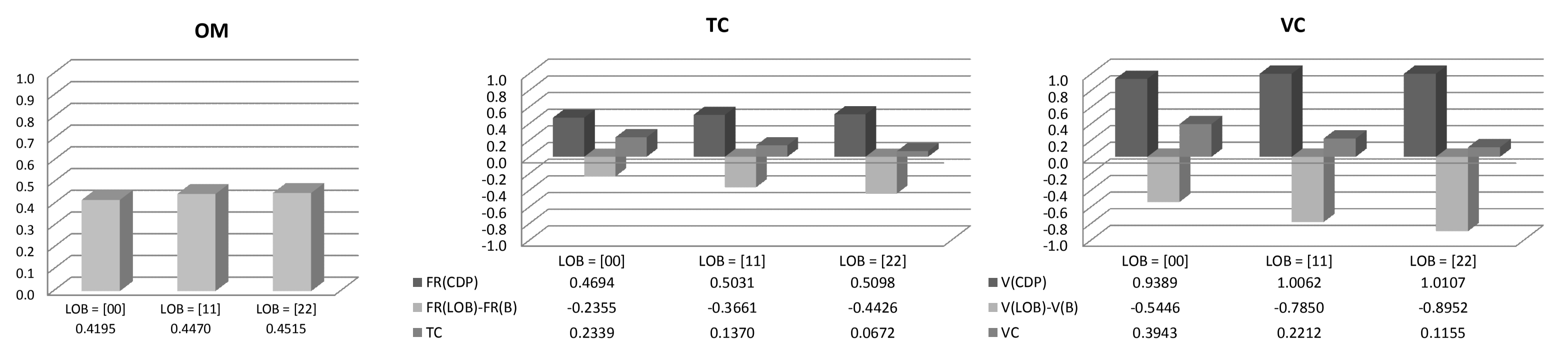
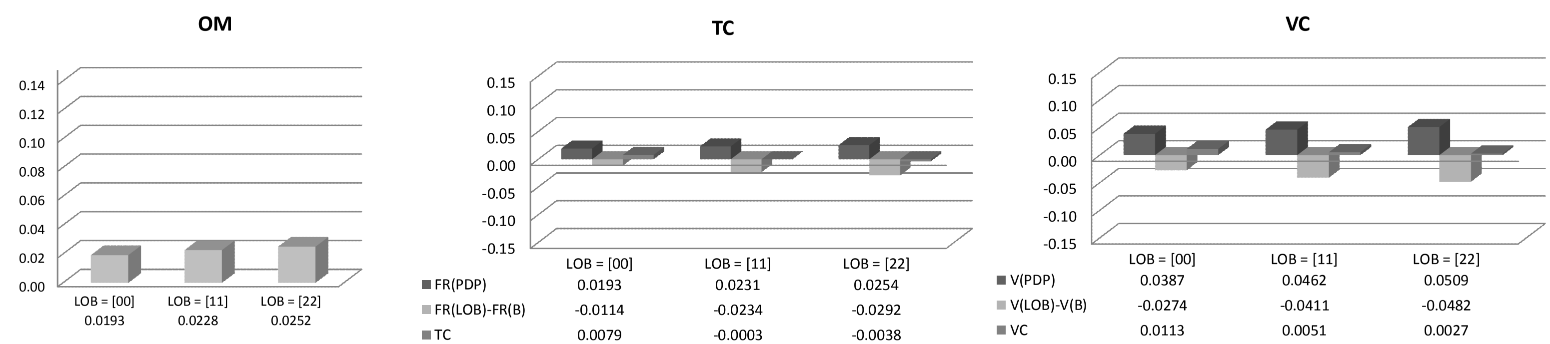


Figure 6 - Order Migration, Trade Creation and Volume Creation - Small Tick Size . This Figure presents results for two frameworks, $L\&P$ (Panel A) and $L\&C$ (Panel B). The first one combines a limit order book (LOB) and a periodic dark pool (PDP); the second a LOB and a continuous dark pool (CDP). For each framework we report order migration (OM), trade creation (TC), and volume creation (VC). OM is the average probability over the three periods that an order migrates to the dark pool. TC is the sum of two components. The first one is the total fill rate in the PDP, $FR(PDP)$, or in the CDP, $FR(CDP)$. The second one is the difference between the LOB fill rate in the $L\&P$ or $L\&C$, $FR(LOB)$, and the LOB fill rate in the benchmark, $FR(B)$. Three initial states of the LOB are considered that differs for the number of shares at the first level of the book: [00], [11], and [22]. Results are computed assuming that the tick size is equal to $\tau = 0.05$.

PANEL A: L&P – small tick



PANEL B: L&C – small tick

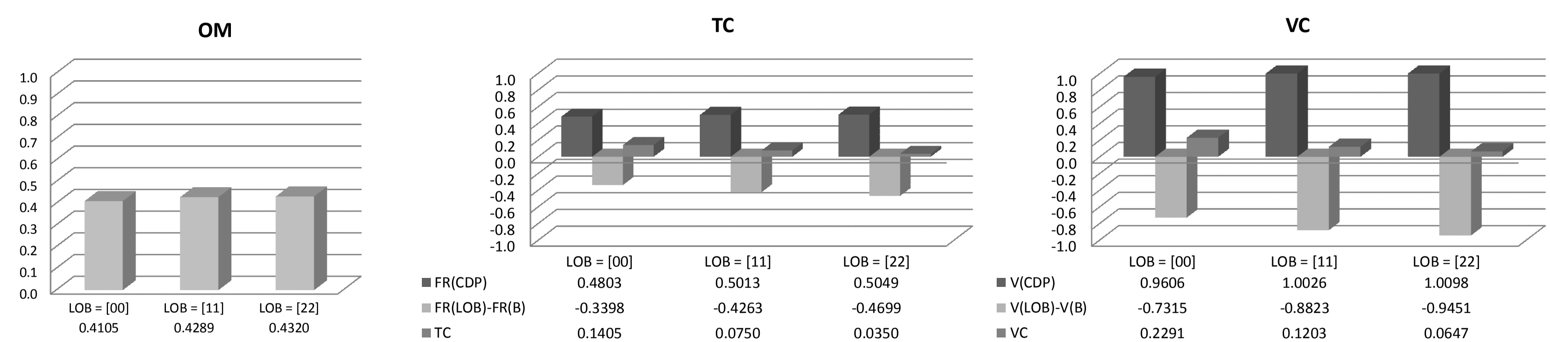
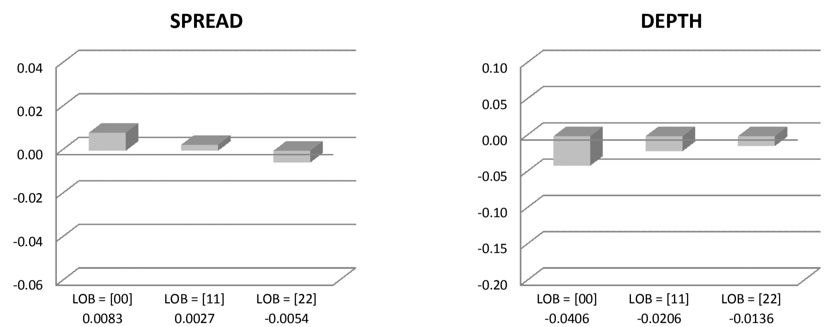


Figure 7 - Market Quality: Spread and Depth. This Figure presents results for spread and depth in two frameworks, $L\&P$ (Panel A) and $L\&C$ (Panel B). $L\&P$ combines a limit order book (LOB) and a periodic dark pool (PDP); $L\&C$ combines a LOB and a continuous dark pool (CDP). The market quality measures are computed as the average percentage difference between their value for the $L\&P$ or $L\&C$ framework and the benchmark framework. The average is computed over periods t_2 and t_3 only, because at t_1 spread and depth are exogenous. Three initial states of the LOB are considered that differ for the number of shares at the first level of the book: [00], [11], and [22]. Results are computed assuming that the tick size is equal to $\tau = 0.1$.

PANEL A: L&P



PANEL B: L&C

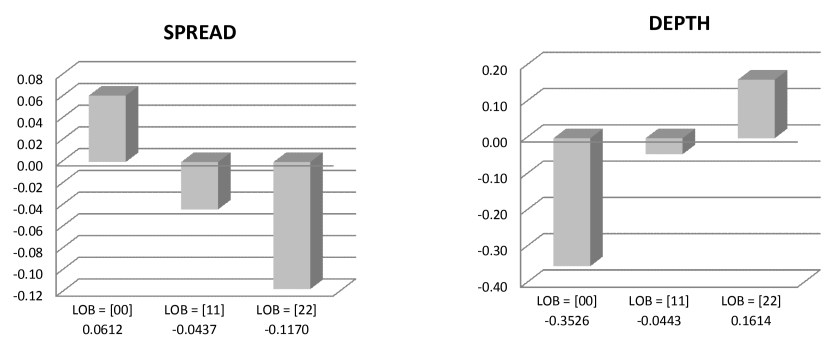


Figure 8 - Welfare. This Figure presents results for welfare in the two frameworks, $L\&P$ (Panel A) and $L\&C$ (Panel B). The first one combines a limit order book (LOB) and a periodic dark pool (PDP); the second a LOB and a continuous dark pool (CDP). For each framework three measures of welfare are computed as the average percentage difference across the three periods between their value in the $L\&P$ or $L\&C$ framework and in the benchmark framework. The three measures are: the welfare of a small trader (ST), the welfare of a large trader (LT), and total welfare. Three initial states of the LOB are considered that differs for the number of shares at the first level of the book: [00], [11], and [22]. Results are computed assuming that the tick size is equal to $\tau = 0.1$.

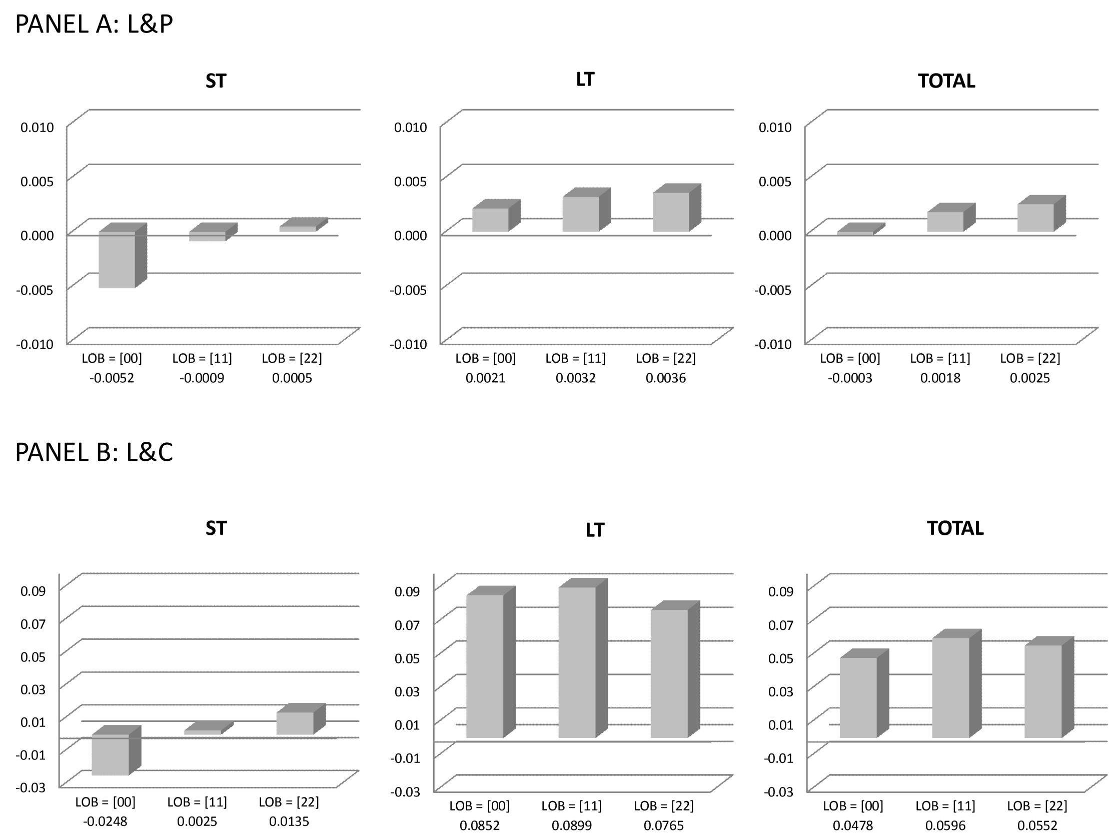
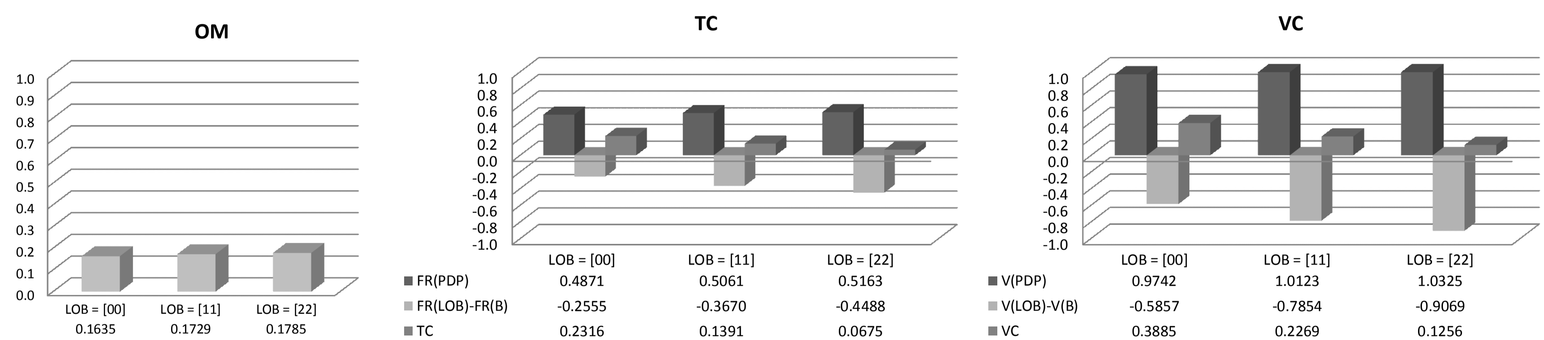


Figure 9 - Order Migration, Trade Creation and Volume Creation - Transparency. This Figure presents results for two frameworks, $L\&P$ (Panel A) and $L\&C$ (Panel B). The first one combines a limit order book (LOB) and a periodic dark pool (PDP); the second a LOB and a continuous dark pool (CDP). Under transparency large traders can observe the state of the dark pool. For each framework we report order migration (OM), trade creation (TC), and volume creation (VC). OM is the average probability over the three periods that an order migrates to the dark pool. TC is the sum of two components. The first one is the total fill rate in the PDP, $FR(PDP)$, or in the CDP, $FR(CDP)$. The second one is the difference between the LOB fill rate in the $L\&P$ or $L\&C$, $FR(LOB)$, and the LOB fill rate in the benchmark, $FR(B)$. Three initial states of the LOB are considered that differs for the number of shares at the first level of the book: [00], [11], and [22]. Results are computed assuming a value for the tick size equal to $\tau = 0.1$.

PANEL A: L&P - Transparency



PANEL B: L&C - Transparency

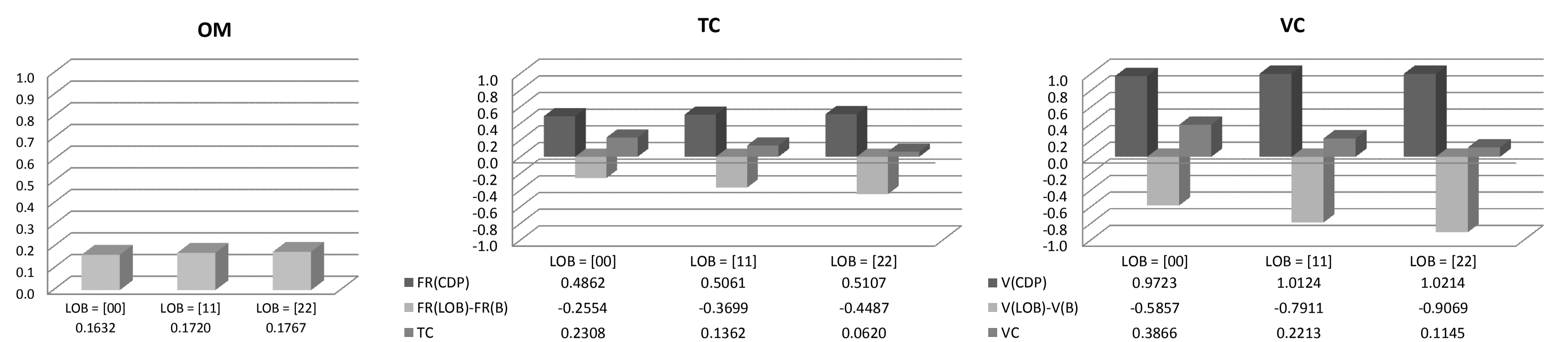


Figure 10 - Market Quality and Welfare - Transparency. This Figure presents results for spread, depth and three measures of welfare in two frameworks, $L\&P$ (Panel A) and $L\&C$ (Panel B). $L\&P$ combines a limit order book (LOB) and a periodic dark pool (PDP); $L\&C$ combines a LOB and a continuous dark pool (CDP). Under transparency large traders can observe the state of the dark pool. All the measures are computed as the average percentage difference between their value for the $L\&P$ or $L\&C$ framework and the benchmark framework. For the market quality measures the average is computed over periods t_2 and t_3 only, because at t_1 spread and depth are exogenous. The three measures of welfare are: the welfare of a small trader (ST), the welfare of a large trader (LT), and total welfare. Three initial states of the LOB are considered that differ for the number of shares at the first level of the book: [00], [11], and [22]. Results are computed assuming that the tick size is equal to $\tau = 0.1$.

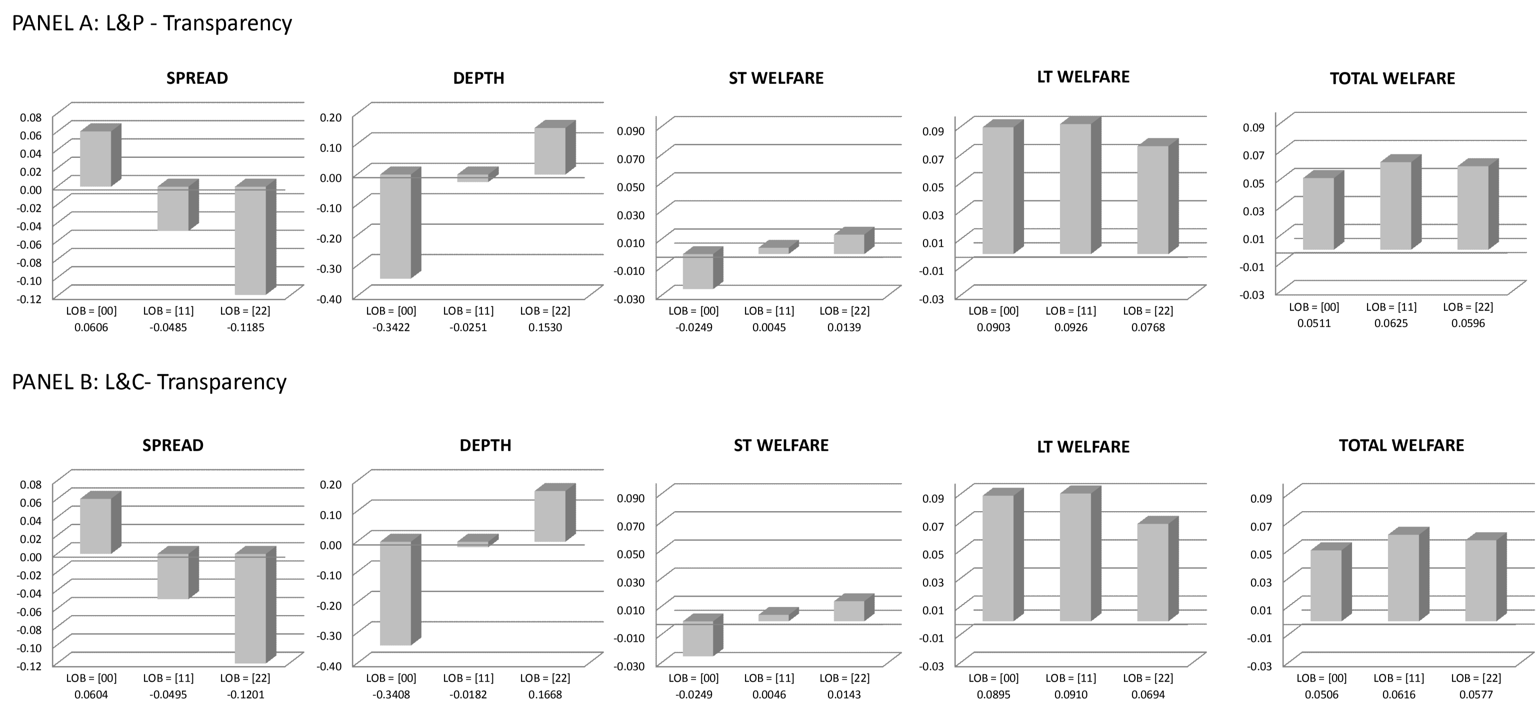


Table I: Order Submission Probabilities at t_1. This Table reports the submission probabilities of large (LT) and small traders (ST) for the orders listed in column 1 for the benchmark framework (B), for the model with a limit order book and a periodic dark pool ($L\&P$) and for the model with a limit order book and a continuous dark pool ($L\&C$). Three opening books at t_1 are considered that differ for the number of shares at the first level of the book: $b_{t_1} = [00]$, $b_{t_1} = [11]$, and $b_{t_1} = [22]$. Since the model is symmetric, we present equilibrium strategies only for the ask side. Results are computed assuming that the tick size is equal to $\tau = 0.1$.

PANEL A - ST	$b_{t_1} = [00]$			$b_{t_1} = [11]$			$b_{t_1} = [22]$		
Trading Strategy	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$
$\varphi_M\left(1, p_2^B\right)$	0.1971	0.2142	0.2721						
$\varphi_M\left(1, p_1^B\right)$				0.4306	0.4333	0.4501	0.4648	0.4656	0.4687
$\varphi_L\left(1, p_1^A\right)$	0.3029	0.2858	0.2279	0.0694	0.0667	0.0499	0.0352	0.0344	0.0313

PANEL B - LT	$b_{t_1} = [00]$			$b_{t_1} = [11]$			$b_{t_1} = [22]$		
Trading Strategy	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$
$\varphi_M(2, p_2^B)$	0.2857	0.2912							
$\varphi_M(2, \widetilde{p}_{Mid,t_1}, p_2^B)$			0.3800						
$\varphi_M(2, p^B)$				0.4150	0.4168				
$\varphi_M(2, \widetilde{p}_{Mid,t_1}, p^B)$						0.4250			
$\varphi_M(1, \widetilde{p}_{Mid,t_1}, p_1^B)$						0.0288			
$\varphi_M(1, p_1^B)$									
$\varphi_M(2, p_1^B)$							0.4686	0.4612	
$\varphi_M(2, \widetilde{p}_{Mid,t_1}, p_1^B)$									0.4689
$\varphi(-2, \widetilde{p}_{Mid})$								0.0279	
$\varphi_L(1, \widetilde{p}_{Mid,t_1}, p_1^A)$						0.0241			0.0252
$\varphi_L(1, p_1^A)$									
$\varphi_L(2, \widetilde{p}_{Mid,t_1}, p_1^A)$			0.0220						
$\varphi_L(2, p_1^A)$	0.2143	0.2088	0.0980	0.0850	0.0832	0.0221	0.0314	0.0109	0.0059

Table II: Order Submission Probabilities at t_1 - Transparency. This Table reports the average submission probabilities over the three possible initial states of the dark pool, $\{0, +6, -6\}$, of large (LT) and small traders (ST) for the orders listed in column 1 for the benchmark framework (B), for the model with a limit order book and a periodic dark pool ($L\&P$) and for the model with a limit order book and a continuous dark pool ($L\&C$) when large traders can observe the imbalance in the dark pool. Three opening books at t_1 are considered that differs for the number of shares at the first level of the book: $b_{t_1} = [00]$, $b_{t_1} = [11]$, and $b_{t_1} = [22]$. Since the model is symmetric, we present equilibrium strategies only for the ask side. Results are computed assuming that the tick size is equal to $\tau = 0.1$.

PANEL A - ST	$b_{t_1} = [00]$			$b_{t_1} = [11]$			$b_{t_1} = [22]$		
Trading Strategy	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$
$\varphi_M\left(1, p_2^B\right)$	0.1971	0.2723	0.2723						
$\varphi_M\left(1, p_1^B\right)$				0.4306	0.4502	0.4502	0.4648	0.4687	0.4687
$\varphi_L\left(1, p_1^A\right)$	0.3029	0.2277	0.2277	0.0694	0.0498	0.0498	0.0352	0.0313	0.0313

PANEL B - LT	$b_{t_1} = [00]$			$b_{t_1} = [11]$			$b_{t_1} = [22]$		
Trading Strategy	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$	B	$L\&P$	$L\&C$
$\varphi_M(2, p_2^B)$	0.2857	0.2284	0.2284						
$\varphi_M(2, \widetilde{p}_{Mid,t_1}, p_2^B)$									
$\varphi_M(2, p^B)$				0.4150	0.2796	0.2784			
$\varphi_M(2, \widetilde{p}_{Mid,t_1}, p^B)$									
$\varphi_M(1, \widetilde{p}_{Mid,t_1}, p_1^B)$									
$\varphi_M(1, p_1^B)$					0.0157	0.0157			
$\varphi_M(2, p_1^B)$							0.4686	0.3110	0.3110
$\varphi_M(2, \widetilde{p}_{Mid,t_1}, p_1^B)$									
$\varphi(-2, \widetilde{p}_{Mid})$		0.1554	0.1547		0.1692	0.1627		0.1791	0.1752
$\varphi_L(1, \widetilde{p}_{Mid,t_1}, p_1^A)$						0.0075			
$\varphi_L(1, p_1^A)$					0.0023	0.0023			
$\varphi_L(2, \widetilde{p}_{Mid,t_1}, p_1^A)$									
$\varphi_L(2, p_1^A)$	0.2143	0.1162	0.1170	0.0850	0.0332	0.0334	0.0314	0.0099	0.0138