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Working Paper n. 433

This Version: April, 2012

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Choquet Integration on Riesz Spaces and Dual Comonotonicity*

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Draft of April, 2012. First Draft, September, 2010

Abstract

We give a general integral representation theorem (Theorem 6) for nonadditive functionals defined on an Archimedean Riesz space X with order unit. Additivity is replaced by a weak form of modularity, or equivalently dual comonotonic additivity, and integrals are Choquet integrals. Those integrals are defined through the Kakutani [8] isometric identification of X with a $C(K)$ space. We further show that our novel notion of dual comonotonicity naturally generalizes and characterizes the notions of comonotonicity found in the literature when X is assumed to be a space of functions.

1 Introduction

Consider the following classical integral representation theorem in the space of real valued, bounded, and $\mathcal{F}$-measurable functions $B(\Omega, \mathcal{F})$ where $\mathcal{F}$ is a σ-algebra of sets.[1]

Theorem 1 *Let $X = B(\Omega, \mathcal{F})$ and V a functional from X to $\mathbb{R}$. The following statements are equivalent:*

(i) V is monotone and additive;

(ii) there exists a unique finitely additive measure $\mu : \mathcal{F} \to [0, \infty)$ such that

$$V(x) = \int_{\Omega} x(\omega)\, d\mu(\omega) \qquad \forall x \in X. \tag{1}$$

This paper aims at extending the above Riesz representation result along the following lines:

1. the domain X of the functional will be a general Archimedean Riesz space with (order) unit;
2. the functional $V : X \to \mathbb{R}$ will neither be assumed monotone nor additive.

In turn, X is identified with a $C(K)$ space and the integral in (1) is a Choquet integral (see Choquet [5]) with μ being a suitable nonadditive set function defined on a lattice of sets. Several partial extensions in this direction have already been established. For instance, when $X = B(\Omega, \mathcal{F})$, Schmeidler

*2010 Mathematics Subject Classification: 28A12, 28A25, 28C05, 46B40, 46B42, 46G12.

The financial support of ERC (Advanced Grant BRSCDP-TEA) is gratefully acknowledged.

[1] Theorem 1 applies also in the case $\mathcal{F}$ is an algebra. In such a case, $B(\Omega, \mathcal{F})$ might not be a vector space.

[14] introduced comonotonic additivity, for a functional, in place of additivity: a property actually shared by Choquet integrals.[2] Along the same lines Zhou [16] generalized X to be a Stone vector lattice while Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [4], among the others, also removed the monotonicity assumption and considered Stone lattices.[3]

Theorem 2 (Schmeidler 1986) *Let $X = B(\Omega, \mathcal{F})$ and V a functional from X to $\mathbb{R}$. The following statements are equivalent:*

(i) V is monotone and comonotonic additive;

(ii) there exists a unique capacity $\mu : \mathcal{F} \to [0, \infty)$ such that

$$V(x) = \int_\Omega x(\omega)\, d\mu(\omega) \qquad \forall x \in X. \tag{2}$$

In the above result there are two new elements: comonotonicity and the Choquet integral in (2). Recall that two functions f and g from Ω to $\mathbb{R}$ are said to be *comonotonic* if and only if

$$(f(\omega) - f(\omega'))(g(\omega) - g(\omega')) \geq 0 \qquad \forall \omega, \omega' \in \Omega. \tag{3}$$

Comonotonic additivity, in Theorem 2, means that the functional V is additive when restricted to pairs of comonotonic functions. On the other hand, when μ is a capacity, that is a monotone set function, the Choquet integral (2) is defined as the sum of two improper Riemann integrals:

$$\int_\Omega x(\omega)\, d\mu(\omega) = \int_0^\infty \mu(x \geq t)\, dt + \int_{-\infty}^0 [\mu(x \geq t) - \mu(\Omega)]\, dt \qquad \forall x \in X \tag{4}$$

where $(x \geq t) = \{\omega \in \Omega : x(\omega) \geq t\}$. It is immediate to check that this definition can be easily extended from capacities μ to set functions of bounded variation.[4] This was done in [11], [10], and [4]. At the same time, also the representation Theorem 2 has been extended to the nonmonotone case but always retaining the assumption that X is a space of functions. To state our main result, we recall some definitions regarding Riesz spaces.

Definition 3 *An Archimedean Riesz space with unit is a real vector space X with a partial order $\geq$ such that:*

(i) (Ordered vector space) $x \geq y$ implies $\alpha x + z \geq \alpha y + z$ for all $\alpha \geq 0$ and all $z \in X$;

(ii) (Archimedean property) $0 \leq nx \leq y$ for all $n \in \mathbb{N}$ implies $x = 0$;

(iii) (Riesz property) X is a lattice with respect to $\geq$;

(iv) (Existence of a unit) there exists an element $e \in X \setminus \{0\}$ such that

$$X = \bigcup_{n \in \mathbb{N}} \{x \in X : |x| \leq ne\}.$$

An element e satisfying (iv) is said to be a unit. We define $X_+ = \{x \in X : x \geq 0\}$. It is then well known that $\|\cdot\| : X \to [0, \infty)$, defined by

$$\|x\| = \min\{\lambda \geq 0 : |x| \leq \lambda e\} \qquad \forall x \in X, \tag{5}$$

[2] Comonotonicity was already used under different names by many authors. See Denneberg [6] for more details as well as for different characterizations of comonotonicity.

[3] See the Appendix for a definition of Stone vector lattice.

[4] See Section 1.1 for definitions, notation, and terminology.

is an M-norm over X.[5] Given an Archimedean Riesz space with unit X, we endow X with the norm above and we denote by X^* the norm dual. We endow X^* and any of its subsets with the w*-topology. The map $(x,\xi) \longmapsto \langle x,\xi\rangle$ for all $(x,\xi) \in X \times X^*$ denotes the dual pairing. The positive unit sphere is

$$\Delta = \{\xi \in X^* : \langle e,\xi\rangle = 1 \text{ and } \langle x,\xi\rangle \geq 0 \text{ for all } x \geq 0\}. \tag{6}$$

We denote by $\mathcal{E}$ the set of extreme points of the compact and convex set Δ. The set $\mathcal{E}$ is compact. By Krein-Milman's Theorem, Δ is the closed convex hull of $\mathcal{E}$. We also endow $\mathcal{E}$ with the Baire σ-algebra: $Baire(\mathcal{E})$. Next, we list few special properties for functionals $V : X \to \mathbb{R}$.

Definition 4 *Let X be an Archimedean Riesz space with unit. The functional $V : X \to \mathbb{R}$ is:*

(i) monotone if $V(x) \geq V(y)$ whenever $x \geq y$;

(ii) of bounded variation if

$$Var_V(0,x) = \sup\left\{\sum_{i=1}^{n} |V(x_i) - V(x_{i-1})| : 0 = x_0 \leq .... \leq x_n = x \text{ and } n \in \mathbb{N}\right\} < \infty \qquad \forall x \in X_+;$$

(iii) unit-additive if $V(x+\lambda e) = V(x) + V(\lambda e)$ for all $x \in X$ and all $\lambda \geq 0$;

(iv) unit-modular if $V(x \wedge \lambda e) + V(x \vee \lambda e) = V(x) + V(\lambda e)$ for all $x \in X$ and all $\lambda \in \mathbb{R}$;

(v) dual comonotonic additive if $V(x+y) = V(x) + V(y)$ for all $x, y \in X$ such that

$$(\langle x,\xi\rangle - \langle x,\xi'\rangle)(\langle y,\xi\rangle - \langle y,\xi'\rangle) \geq 0 \qquad \forall \xi,\xi' \in \mathcal{E};$$

(vi) supermodular if $V(x \wedge y) + V(x \vee y) \geq V(x) + V(y)$ for all $x, y \in X$;

(vii) superadditive if $V(x+y) \geq V(x) + V(y)$ for all $x, y \in X$.

Next statement is a Choquet-Bishop-DeLeeuw variant of Krein-Milman's theorem.

Theorem 5 *Let X be an Archimedean Riesz space with unit and V a functional from X to $\mathbb{R}$. The following statements are equivalent:*

(i) V is monotone and additive;

(ii) there exists a unique measure μ on $Baire(\mathcal{E})$ such that

$$V(x) = \int_{\mathcal{E}} \langle x,\xi\rangle \, d\mu(\xi) \qquad \forall x \in X. \tag{7}$$

The above result can be viewed as an abstract Gelfand's integral representation theorem.[6] The goal of this paper is to provide a nonadditive and, not necessarily, monotone version of the above representation result:

Theorem 6 *Let X be an Archimedean Riesz space with unit and V a functional from X to $\mathbb{R}$. The following statements are equivalent:*

[5] That is, $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for all $x, y \in X_+$.

[6] Since $\mathcal{E}$ is compact, it is worth noticing that the existence of a measure μ, in Theorem 5, when $V \in \Delta$, is merely a consequence of [12, Proposition 1.2]. By [12, Corollary 10.9] and since Δ is a Choquet simplex and $\mathcal{E}$ is an F_σ set, also the uniqueness of μ follows.

(i) V is dual comonotonic additive and of bounded variation;

(ii) V is unit-additive, unit-modular, and of bounded variation;

(iii) there exists an outer continuous set function $\gamma : \mathcal{U}(\mathcal{E}) \to \mathbb{R}$ of bounded variation such that

$$V(x) = \int_{\mathcal{E}} \langle x, \xi \rangle \, d\gamma(\xi) \qquad \forall x \in X.$$

Moreover, γ is unique and V is monotone if and only if γ is.

Here $\mathcal{U}(\mathcal{E})$ denotes the lattice of upper level sets generating $Baire(\mathcal{E})$, that is,

$$\mathcal{U}(\mathcal{E}) = \{(f \geq t) : f \in C(\mathcal{E}),\ t \in \mathbb{R}\}.$$

It is apparent the close relation of Theorem 6 with Theorem 5. Actually, Theorem 6 collapses into Theorem 5 whenever $V : X \to \mathbb{R}$ is monotone and additive (see also Section 4) and γ also admits a unique extension to $Baire(\mathcal{E})$. It is also worth emphasizing that the equivalence between (i) and (ii) of Theorem 6 is novel also in the standard case $X = B(\Omega, \mathcal{F})$ where dual comonotonic additivity and comonotonic additivity coincide. This equivalence is obtained in Theorem 19 in the Appendix. This latter theorem extends all the existing representation results in the literature when X is assumed to be a space of functions.

Section 2 is entirely devoted to the proof of Theorem 6. Behind the definition of dual comonotonic additivity established in Definition 4, there is the property of two elements x and y in X to be *dually comonotonic*, that is, such that

$$(\langle x, \xi \rangle - \langle x, \xi' \rangle)(\langle y, \xi \rangle - \langle y, \xi' \rangle) \geq 0 \qquad \forall \xi, \xi' \in \mathcal{E}. \tag{8}$$

Section 3 provides a careful analysis of this notion and, among the others, shows how dual comonotonicity naturally generalizes and characterizes the notions of comonotonicity found in the literature when X is assumed to be a space of functions.

Finally, Section 4 is dedicated to the special case in which the functional V has the further property of being either supermodular or superadditive. We study the close relation of these two properties in view of their integral representation. Moreover, we show that superaddivity is crucial in guaranteeing the uniqueness of γ in Theorem 6 also when γ is extended to $Baire(\mathcal{E})$.

1.1 Notation

We close this introductory section by adding few further definitions that were and will be used in the paper.

If Σ is a lattice of subsets of a set Ω such that $\emptyset, \Omega \in \Sigma$, a function $\gamma : \Sigma \to \mathbb{R}$ is a set function if $\gamma(\emptyset) = 0$. In particular, a set function is:

(i) *monotone* or a *capacity* if $\gamma(A) \leq \gamma(B)$ whenever $A \subseteq B$;

(ii) *supermodular* if $\gamma(A \cap B) + \gamma(A \cup B) \geq \gamma(A) + \gamma(B)$ for all $A, B \in \Sigma$;

(iii) γ is *outer continuous at* $A \in \Sigma$ if $\lim_n \gamma(A_n) = \gamma(A)$ whenever $A_n \downarrow A$;

(iv) γ is *outer continuous* if γ is outer continuous at each $A \in \Sigma$;

(v) γ is of *bounded variation* if

$$\sup \left\{ \sum_{i=1}^{n} |\gamma(A_i) - \gamma(A_{i-1})| : \varnothing = A_0 \subseteq .... \subseteq A_n = \Omega \text{ and } n \in \mathbb{N} \right\} < \infty.$$

Submodularity and inner continuity are defined similarly. We say that a set function is continuous (resp., modular) if and only if it is inner and outer continuous (resp., submodular and supermodular). The notion of set function of bounded variation goes back to Aumann and Shapley [3]. The space of set functions of bounded variation on Σ is denoted by $bv(\Sigma)$.[7] It is immediate to see that capacities are set functions of bounded variation. Finally, if F is a Stone vector lattice, we denote by Σ_F the lattice of upper level sets, that is, $\Sigma_F = \{(f \geq t) : f \in F \text{ and } t \in \mathbb{R}\}$. Given a subset $A \subseteq \Omega$, we denote by χ_A the indicator function of A.

2 Proof of the Main Theorem

Let X be an Archimedean Riesz space with unit endowed with the supnorm (5). Let $\mathcal{E}$ be the set $\{\xi \in \Delta : \xi \text{ is an extreme point of } \Delta\}$. By [2, Theorems 3.14 and 4.28], we have that

$$\emptyset \neq \mathcal{E} = \{\xi \in \Delta : \xi \text{ is a lattice homomorphism and } \langle e, \xi \rangle = 1\}.$$

For this reason, $\mathcal{E}$, endowed with the w*-topology, is Hausdorff and compact. We define by $C(\mathcal{E})$ the space of all continuous functions over $\mathcal{E}$ and we endow it with the supnorm. Define $T : X \to C(\mathcal{E})$ by

$$x \mapsto T(x) = \hat{x} \text{ where } \hat{x}(\xi) = \langle x, \xi \rangle \qquad \forall \xi \in \mathcal{E}, \forall x \in X. \tag{9}$$

By the classical Kakutani-Bohnenblust-M. Krein-S. Krein theorem (see for instance [2, Theorem 4.29]), T turns out to be an isometric lattice homomorphism such that $T(e) = \chi_{\mathcal{E}}$. Thus, X is lattice isometric to $C = T(X)$.[8,9] By [1, Theorem 9.12], the latter is uniformly dense in $C(\mathcal{E})$. Then, we consider $T : X \to C$. In this way, T is an isometry, a lattice isomorphism, and the same applies for its inverse T^{-1}. Both maps are positive.

Proof of Theorem 6. (i) implies (ii). Let $x \in X$ and $\lambda \in \mathbb{R}$. Since $(\langle x, \xi \rangle - \langle x, \xi' \rangle)(\langle \lambda e, \xi \rangle - \langle \lambda e, \xi' \rangle) = 0$ for all $\xi, \xi' \in \mathcal{E}$, x and λe are dually comonotonic. Since V is dual comonotonic additive, we have that

$$V(x + \lambda e) = V(x) + V(\lambda e), \tag{10}$$

proving, in particular, that V is unit-additive. Since T is a lattice isomorphism such that $T(e) = \chi_{\mathcal{E}}$, we have that $T(x \wedge \lambda e) = \hat{x} \wedge \lambda \chi_{\mathcal{E}}$ and $T(x \vee \lambda e) = \hat{x} \vee \lambda \chi_{\mathcal{E}}$. By [10, Lemma 4.6], $\hat{x} \wedge \lambda \chi_{\mathcal{E}}$ and $\hat{x} \vee \lambda \chi_{\mathcal{E}}$ are comonotonic in C. This implies that $x \wedge \lambda \chi_{\mathcal{E}}$ and $x \vee \lambda \chi_{\mathcal{E}}$ are dually comonotonic. By (10) and since V is dual comonotonic additive, it follows that

$$V(x \wedge \lambda e) + V(x \vee \lambda e) = V(x \wedge \lambda e + x \vee \lambda e) = V(x + \lambda e) = V(x) + V(\lambda e),$$

proving that V is unit-modular.

(ii) implies (iii). Suppose V is unit-additive, unit-modular, and of bounded variation. Given T as defined in (9), define $\hat{V} : C \to \mathbb{R}$ by $\hat{V} = V \circ T^{-1}$.

The set $C \subseteq C(\mathcal{E})$ is a Stone vector lattice. Since T^{-1} is a positive operator such that $T^{-1}(\chi_{\mathcal{E}}) = e$ and V is unit-additive, we have that

$$\begin{aligned}\hat{V}(f + \lambda \chi_{\mathcal{E}}) &= V\left(T^{-1}(f + \lambda \chi_{\mathcal{E}})\right) = V\left(T^{-1}(f) + \lambda e\right) = V\left(T^{-1}(f)\right) + V(\lambda e) \\ &= \hat{V}(f) + \hat{V}(\lambda \chi_{\mathcal{E}}) \qquad \forall f \in C, \forall \lambda \geq 0,\end{aligned}$$

[7] A detailed study of the space of set functions of bounded variation can be found in [4, Section 3].

[8] From here on, all compact topological spaces are understood to be Hausdorff.

[9] Some of the results cited in [2] apply to the case when X is an AM-space and, in particular, it is norm complete. Nevertheless, it is routine to check that they apply to an Archimedean Riesz space X with unit when endowed with the supnorm.

proving that $\hat{V}$ is unit-additive. Since T^{-1} is a lattice isomorphism such that $T^{-1}(\chi_{\mathcal{E}}) = e$ and V is unit-modular, we have that

$$\begin{aligned}\hat{V}(f \wedge \lambda\chi_{\mathcal{E}}) + \hat{V}(f \vee \lambda\chi_{\mathcal{E}}) &= V\left(T^{-1}(f \wedge \lambda\chi_{\mathcal{E}})\right) + V\left(T^{-1}(f \vee \lambda\chi_{\mathcal{E}})\right) \\ &= V\left(T^{-1}(f) \wedge \lambda e\right) + V\left(T^{-1}(f) \vee \lambda e\right) \\ &= V\left(T^{-1}(f)\right) + V(\lambda e) = \hat{V}(f) + \hat{V}(\lambda\chi_{\mathcal{E}}) \qquad \forall f \in C, \forall \lambda \in \mathbb{R},\end{aligned}$$

proving that $\hat{V}$ is unit-modular. Finally, consider $0 \leq f \in C$ and a chain $\{f_i\}_{i=0}^n$ such that $0 = f_0 \leq f_1 \leq \cdots \leq f_n = f$. Define x and $\{x_i\}_{i=0}^n$ by $x = T^{-1}(f)$ and $x_i = T^{-1}(f_i)$ for $i \in \{0, ..., n\}$. Since T^{-1} is a positive operator, it follows that $0 = x_0 \leq x_1 \leq \cdots \leq x_n = x$ and

$$\sum_{i=1}^{n} \left|\hat{V}(f_i) - \hat{V}(f_{i-1})\right| = \sum_{i=1}^{n} |V(x_i) - V(x_{i-1})| \leq Var_V(0, x).$$

Since V is of bounded variation, $\hat{V}$ is of bounded variation. By Lemma 18, we have that $\hat{V} : C \to \mathbb{R}$ is Lipschitz continuous. Since C is uniformly dense in $C(\mathcal{E})$, it follows that $\hat{V} : C \to \mathbb{R}$ admits a unique Lipschitz continuous extension to $C(\mathcal{E})$. We denote the extension by $\bar{V}$. Since $\hat{V}$ is unit-additive, unit-modular, and of bounded variation, it follows that $\bar{V}$ shares the same properties. Finally, consider $f \in C(\mathcal{E})$ and a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq C(\mathcal{E})$ such that $f_n \downarrow f$ pointwise. By Dini's Theorem, we have that $\{f_n\}_{n\in\mathbb{N}}$ converges in supnorm to f. Since $\bar{V}$ is Lipschitz continuous, it follows that $\lim_n \bar{V}(f_n) = \bar{V}(f)$. By Theorem 19, there exists an outer continuous $\gamma \in bv\left(\Sigma_{C(\mathcal{E})}\right) = bv(\mathcal{U}(\mathcal{E}))$ such that

$$\bar{V}(f) = \int_{\mathcal{E}} f(\xi)\, d\gamma(\xi) \qquad \forall f \in C(\mathcal{E}).$$

In particular, we have that

$$V(x) = \int_{\mathcal{E}} \langle x, \xi \rangle\, d\gamma(\xi) \qquad \forall x \in X.$$

(iii) implies (i). Assume there exists an outer continuous $\gamma \in bv(\mathcal{U}(\mathcal{E}))$ such that

$$V(x) = \int_{\mathcal{E}} \langle x, \xi \rangle\, d\gamma(\xi) \qquad \forall x \in X. \tag{11}$$

We start by considering the case when γ is a capacity. Consider x and y in X such that $x \geq y$. Since T is positive, it follows that $\hat{x} \geq \hat{y}$. By (11) and the monotonicity of the Choquet integral when γ is a capacity, we have that

$$V(x) = \int_{\mathcal{E}} \hat{x}(\xi)\, d\gamma(\xi) \geq \int_{\mathcal{E}} \hat{y}(\xi)\, d\gamma(\xi) = V(y),$$

proving that V is monotone. Let now x and y be dually comonotonic, that is,

$$(\langle x, \xi \rangle - \langle x, \xi' \rangle)(\langle y, \xi \rangle - \langle y, \xi' \rangle) \geq 0 \qquad \forall \xi, \xi' \in \mathcal{E}.$$

It follows that the two functions, $\hat{x}$ and $\hat{y}$, are comonotonic. Since Choquet integrals are comonotonic additive, we have that

$$V(x+y) = \int_{\mathcal{E}} (\hat{x}(\xi) + \hat{y}(\xi))\, d\gamma(\xi) = \int_{\mathcal{E}} \hat{x}(\xi)\, d\gamma(\xi) + \int_{\mathcal{E}} \hat{y}(\xi)\, d\gamma(\xi) = V(x) + V(y),$$

proving that V is dual comonotonic additive according to (v) of Definition 4. Next, consider the case when γ is of bounded variation. By [4, Proposition 7] and since $\mathcal{U}(\mathcal{E})$ is a lattice of sets, there exist two capacities $\gamma_1, \gamma_2 : \mathcal{U}(\mathcal{E}) \to [0, \infty)$ such that $\gamma = \gamma_1 - \gamma_2$. By (11), this implies that $V = V_1 - V_2$ where

$$V_i(x) = \int_{\mathcal{E}} \langle x, \xi \rangle\, d\gamma_i(\xi) \qquad \forall x \in X, i = 1, 2.$$

By the previous part of the proof, V_1 and V_2 are monotone and dual comonotonic additive. Consequently, V is dual comonotonic additive and of bounded variation.

We are left to prove the uniqueness of γ and the characterization of monotonicity. We start by the uniqueness of γ. Consider two outer continuous set functions $\gamma_1, \gamma_2 \in bv(\mathcal{U}(\mathcal{E}))$ such that

$$V(x) = \int_{\mathcal{E}} \langle x, \xi \rangle \, d\gamma_i(\xi) \qquad \forall x \in X, i \in \{1, 2\}.$$

For $i \in \{1,2\}$ define $\hat{V}_i : C \to \mathbb{R}$ by $\hat{V}_i(f) = \int_{\mathcal{E}} f(\xi) \, d\gamma_i(\xi)$ for all $f \in C$. By Theorem 19, it follows that $\gamma_1(A) = \gamma_2(A)$ for all $A \in \Sigma_C \subseteq \mathcal{U}(\mathcal{E})$. On the other hand, if $A \in \mathcal{U}(\mathcal{E})$ then there exist $f \in C(\mathcal{E})$ and $t \in \mathbb{R}$ such that $A = (f \geq t)$. Since C is uniformly dense in $C(\mathcal{E})$ and $\chi_{\mathcal{E}} \in C$, there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq C$ such that $f_n \geq f_{n+1} \geq f$ for all $n \in \mathbb{N}$ and $\|f_n - f\| \to 0$. It follows that $(f_n \geq t) \downarrow (f \geq t)$. Since γ_1 and γ_2 are outer continuous and they coincide on Σ_C, we have that

$$\gamma_1(A) = \gamma_1(f \geq t) = \lim_n \gamma_1(f_n \geq t) = \lim_n \gamma_2(f_n \geq t) = \gamma_2(f \geq t) = \gamma_2(A),$$

proving uniqueness.

Finally, we prove the characterization of monotonicity. Consider $\hat{V} = V \circ T^{-1} : C \to \mathbb{R}$ and $\bar{V}$ the Lipschitz continuous extension of $\hat{V}$ from C to $\mathcal{C}(\mathcal{E})$, as in (ii) implies (iii). Since T^{-1} is a positive operator, if V is further monotone then $\hat{V}$ is also monotone and so is the extension $\bar{V}$. By Theorem 19, there exists a unique outer continuous capacity $\gamma : \mathcal{U}(\mathcal{E}) \to [0, \infty)$ such that $\bar{V}(f) = \int_{\mathcal{E}} f(\xi) \, d\gamma(\xi)$ for all $f \in C(\mathcal{E})$. In particular, we have that $V(x) = \int_{\mathcal{E}} \langle x, \xi \rangle \, d\gamma(\xi)$ for all $x \in X$. On the other hand, if γ is a capacity that satisfies (11) then V is monotone, as proved in the initial part of (iii) implies (i). ■

3 Dual Comonotonicity

In this section we study the dual comonotonicity relation contained in (8). Notice that two elements x and y in X are dually comonotonic if and only if the functions $\hat{x}(\cdot)$ and $\hat{y}(\cdot)$ in $C(\mathcal{E})$ are comonotonic in the usual sense. At the same time, to check that two elements are dually comonotonic, it suffices that the relation in (8) holds for all $\xi, \xi' \in \Gamma \subseteq \mathcal{E}$ where Γ is dense in $\mathcal{E}$.

When X is a space of functions we thus have two notions of comonotonicity: the traditional one and the dual one. We show that these notions coincide. This confirms that the new notion of dual comonotonicity we propose is a legitimate generalization, of the standard notion of comonotonicity, to Archimedean Riesz spaces with unit.

We start by considering a σ-algebra $\mathcal{F}$ of subsets of a nonempty set Ω. $B(\Omega, \mathcal{F})$ is a Banach lattice with unit χ_Ω. The norm dual of $B(\Omega, \mathcal{F})$ is lattice isometric to the set of bounded and finitely additive set functions $ba(\Omega, \mathcal{F})$. In this case, the dual pairing is such that $(f, \mu) \longmapsto \int_\Omega f d\mu$. Δ is the set of finitely additive probabilities on $\mathcal{F}$ and $\mathcal{E}$ is the set of $\{0,1\}$-valued probabilities. Next proposition shows the coincidence of comonotonicity and dual comonotonicity in this setting.

Proposition 7 *The functions $x, y \in B(\Omega, \mathcal{F})$ are comonotonic if and only if are dually comonotonic.*

Proof. For each $\omega \in \Omega$ define the Dirac measure δ_ω on $(\Omega, \mathcal{F})$ by

$$\delta_\omega(A) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} \quad \text{for all } A \in \mathcal{F}.$$

The set $\Gamma = \{\delta_\omega : \omega \in \Omega\}$ is dense in $\mathcal{E}$. If x and y are dually comonotonic, in particular, we have that

$$(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) = (\langle x, \delta_\omega\rangle - \langle x, \delta_{\omega'}\rangle)(\langle y, \delta_\omega\rangle - \langle y, \delta_{\omega'}\rangle) \geq 0 \qquad \forall \omega, \omega' \in \Omega,$$

proving that x and y are comonotonic. Conversely, if x and y are comonotonic then

$$0 \leq (x(\omega) - x(\omega'))(y(\omega) - y(\omega')) = (\langle x, \delta_\omega\rangle - \langle x, \delta_{\omega'}\rangle)(\langle y, \delta_\omega\rangle - \langle y, \delta_{\omega'}\rangle) \qquad \forall \omega, \omega' \in \Omega,$$

proving that the relation in (8) is true on the dense subset Γ of $\mathcal{E}$. This implies that x and y are dually comonotonic. ■

In light of Proposition 7, given a comonotonic additive functional $V : B(\Omega, \mathcal{F}) \to \mathbb{R}$ of bounded variation, we have two representations . The first one is a specification of Theorem 6 if X is assumed to be $B(\Omega, \mathcal{F})$. The second one is the direct representation according to Schmeidler's result as generalized to the nonmonotone case by [11] and [10]. It is interesting to realize their relation. More specifically, if $V : B(\Omega, \mathcal{F}) \to \mathbb{R}$ is comonotonic additive and of bounded variation then there exists a unique set function $\eta \in bv(\mathcal{F})$ such that

$$V(x) = \int_\Omega x(\omega)\, d\eta(\omega) \qquad \forall x \in B(\Omega, \mathcal{F}). \tag{12}$$

On the other hand, Theorem 6 implies that there exists a unique outer continuous set function $\gamma \in bv(\mathcal{U}(\mathcal{E}))$ such that

$$V(x) = \int_\mathcal{E} \hat{x}(\xi)\, d\gamma(\xi) \qquad \forall x \in B(\Omega, \mathcal{F}). \tag{13}$$

Next proposition clarifies the relationship among the objects $\mathcal{E}$, $C(\mathcal{E})$, η, and γ. By definition, $\hat{x}(\xi) = \langle x, \xi\rangle$ for all $\xi \in \mathcal{E}$ and $x \in X$. To this purpose, if A is an element of the σ-algebra $\mathcal{F}$, define the following subset of $\mathcal{E}$:

$$\hat{A} = \{\mu \in \mathcal{E} : \mu(A) = 1\}.$$

That is, $\hat{A}$ is the collection of $\{0,1\}$-valued probabilities having the set A as carrier. Set $\hat{\mathcal{F}} = \left\{\hat{A} : A \in \mathcal{F}\right\}$.

Proposition 8 *For the representations (12) and (13), the following properties hold:*

(i) the space $\mathcal{E}$ can be identified with the compact, Hausdorff, and totally disconnected Stone space of the Boolean algebra $\mathcal{F}$;

(ii) $\hat{\mathcal{F}}$ is the σ-algebra of clopen sets of $\mathcal{E}$ which is a base of the topology in $\mathcal{E}$;

(iii) $C(\mathcal{E}) = B\left(\hat{\mathcal{F}}\right)$ and $T(\chi_A) = \chi_{\hat{A}}$;

(iv) $\gamma\left(\hat{A}\right) = \eta(A)$ for all $A \in \mathcal{F}$.

Proof. Before starting we denote by τ_1 the topology on $\mathcal{E}$ induced by the w*-topology and by τ_2 the topology generated by $\hat{\mathcal{F}}$.

(i) and (ii). Define the map $h : \mathcal{F} \to \hat{\mathcal{F}}$ by $A \longmapsto \hat{A}$ for all $A \in \mathcal{F}$. It is routine to check that it is a lattice isomorphism as in [13, Definition 1.4.4]. This implies that $\hat{\mathcal{F}}$ is an algebra. Next, consider $\left\{\hat{A}_n\right\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{F}}$, it follows that there exists $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ such that $\hat{A}_n = \{\mu \in \mathcal{E} : \mu(A_n) = 1\}$ for all $n \in \mathbb{N}$. Since $\mathcal{F}$ is a σ-algebra, it is immediate to see that $\cap_{n\in\mathbb{N}}\hat{A}_n = \{\mu \in \mathcal{E} : \mu(\cap_{n\in\mathbb{N}}A_n) = 1\} \in \hat{\mathcal{F}}$, proving that $\hat{\mathcal{F}}$ is a σ-algebra. It is routine to check that each set in $\hat{\mathcal{F}}$ is a τ_1-clopen set. Since $\hat{\mathcal{F}}$ is closed under intersection and complementation, $\hat{\mathcal{F}}$ is a base for τ_2 and each τ_2-closed set is intersection of sets in $\hat{\mathcal{F}}$. It follows that $\tau_2 \subseteq \tau_1$. Consider the identity $id_\mathcal{E} : (\mathcal{E}, \tau_1) \to (\mathcal{E}, \tau_2)$. Since $\tau_2 \subseteq \tau_1$, $id_\mathcal{E}$ is a continuous bijection. $(\mathcal{E},\tau_1)$ is compact and Hausdorff while $(\mathcal{E},\tau_2)$ is Hausdorff. In fact, consider μ_1 and μ_2 in $\mathcal{E}$. By contradiction, assume that $\mu_1 \neq \mu_2$ and that for each $C \in \hat{\mathcal{F}}$

$$\mu_1 \in C \Longrightarrow \mu_2 \in C.$$

Since μ_1 and μ_2 are $\{0,1\}$-valued, this implies that $\mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{F}$, that is, $\mu_1 = \mu_2$, a contradiction. It follows that if $\mu_1, \mu_2 \in \mathcal{E}$ and $\mu_1 \neq \mu_2$ then there exists $C \in \hat{\mathcal{F}}$ such that $\mu_1 \in C$ and $\mu_2 \notin C$. Since $\hat{\mathcal{F}}$ is closed by complementation, we have that $\mu_1 \in C$ and $\mu_2 \in C^c$ and τ_2 is Hausdorff. We can conclude that $id_{\mathcal{E}}$ is an homeomorphism. Thus, τ_1 and τ_2 coincide, proving that $(\mathcal{E},\tau)$ is compact, Hausdorff, totally disconnected, and $\hat{\mathcal{F}}$ is a base for τ. It is routine to check that any clopen set of $(\mathcal{E},\tau)$ belongs to $\hat{\mathcal{F}}$.

(iii) Clearly, for each $\hat{A} \in \hat{\mathcal{F}}$ we have that $T(\chi_A) = \chi_{\hat{A}} \in C(\mathcal{E})$. Since T is linear, this implies that $B_0\left(\mathcal{E},\hat{\mathcal{F}}\right) \subseteq C(\mathcal{E}) \subseteq B\left(\mathcal{E},\hat{\mathcal{F}}\right)$. Since $C(\mathcal{E})$ is a Banach space, the statement follows.

(iv) By (12) and (13), we have that for each $x \in X$

$$\int_\Omega x(\omega)\,d\eta(\omega) = \int_{\mathcal{E}} \hat{x}(\xi)\,d\gamma(\xi). \tag{14}$$

Setting $x = \chi_A$ in (14), we get the relation $\eta(A) = \gamma\left(\hat{A}\right)$. ■

Next result though interesting in itself will be useful in the sequel.

Proposition 9 *Let X and Y be two AM-spaces with units e_X and e_Y and let $J : X \to Y$ be a lattice homomorphism such that $J(e_X) = e_Y$. The following statements are true:*

(i) If x and y in X are dually comonotonic then $J(x)$ and $J(y)$ in Y are dually comonotonic.

(ii) If J is further injective, x and y are dually comonotonic if and only if $J(x)$ and $J(y)$ are.

Proof. Since X and Y are two AM-spaces, Kakutani's maps $T_X : X \to C(\mathcal{E}_X)$ and $T_Y : Y \to C(\mathcal{E}_Y)$ are isomorphisms. Define the lattice homomorphism $\hat{J} : C(\mathcal{E}_X) \to C(\mathcal{E}_Y)$ by $\hat{J} = T_Y \circ J \circ T_X^{-1}$. $\hat{J}$ is such that $\hat{J}(\chi_{\mathcal{E}_X}) = \chi_{\mathcal{E}_Y}$. It follows that $\hat{J} \circ T_X = T_Y \circ J$. From a general result on homomorphisms between spaces of continuous functions (see for instance [1, Theorem 14.23]), it follows that there exists a continuous map $h : \mathcal{E}_Y \to \mathcal{E}_X$ such that

$$\hat{J}(f) = f \circ h \qquad \forall f \in C(\mathcal{E}_X).$$

(i). By contradiction, assume x and y in X are dually comonotonic and that $J(x)$ and $J(y)$ are not. It follows that there exist ξ_1 and ξ_2 in $\mathcal{E}_Y$ such that

$$\begin{aligned}
0 &> \left[T_Y(J(x))(\xi_1) - T_Y(J(x))(\xi_2)\right]\left[T_Y(J(y))(\xi_1) - T_Y(J(y))(\xi_2)\right] \\
&= \left[\hat{J}(T_X(x))(\xi_1) - \hat{J}(T_X(x))(\xi_2)\right]\left[\hat{J}(T_X(y))(\xi_1) - \hat{J}(T_X(y))(\xi_2)\right] \\
&= \left[\hat{J}(\hat{x})(\xi_1) - \hat{J}(\hat{x})(\xi_2)\right]\left[\hat{J}(\hat{y})(\xi_1) - \hat{J}(\hat{y})(\xi_2)\right] = \left[\hat{x}(h(\xi_1)) - \hat{x}(h(\xi_2))\right]\left[\hat{y}(h(\xi_1)) - \hat{y}(h(\xi_2))\right],
\end{aligned}$$

a contradiction with the hypothesis that x and y are dually comonotonic.

(ii). If J is further injective then the map $h : \mathcal{E}_Y \to \mathcal{E}_X$ is onto. Necessity follows from point (i). By contradiction, suppose that $J(x)$ and $J(y)$ are dually comonotonic but x and y are not. It follows that there exist ξ_1 and ξ_2 in $\mathcal{E}_X$ such that $\left[\hat{x}(\xi_1) - \hat{x}(\xi_2)\right]\left[\hat{y}(\xi_1) - \hat{y}(\xi_2)\right] < 0$. Since h is onto, there exist ξ_1' and ξ_2' in $\mathcal{E}_Y$ such that $\xi_1 = h(\xi_1')$ and $\xi_2 = h(\xi_2')$. Since $\hat{J}(\hat{x}) = T_Y(J(x))$ and $\hat{J}(\hat{y}) = T_Y(J(y))$, this implies that

$$\begin{aligned}
0 &> \left[\hat{x}(h(\xi_1')) - \hat{x}(h(\xi_2'))\right]\left[\hat{y}(h(\xi_1')) - \hat{y}(h(\xi_2'))\right] = \left[\hat{J}(\hat{x})(\xi_1') - \hat{J}(\hat{x})(\xi_2')\right]\left[\hat{J}(\hat{y})(\xi_1') - \hat{J}(\hat{y})(\xi_2')\right] \\
&= \left[T_Y(J(x))(\xi_1') - T_Y(J(x))(\xi_2')\right]\left[T_Y(J(y))(\xi_1') - T_Y(J(y))(\xi_2')\right],
\end{aligned}$$

a contradiction with $J(x)$ and $J(y)$ being dually comonotonic. ■

The AM**-spaces** $C_b(X)$ **and** c We give some immediate applications of Proposition 9. Let $C_b(S)$ be the space of all bounded and continuous functions on a topological space S. $C_b(S)$, endowed with the supnorm and the pointwise order, is a Banach lattice with unit χ_S.

Proposition 10 *Two functions f and g in the AM-space $C_b(S)$ are dually comonotonic if and only if they are comonotonic.*

Proof. Let $\mathcal{B}_S$ be the Borel σ-algebra of S and consider the space $B(S, \mathcal{B}_S)$. Every continuous function is $\mathcal{B}_S$-measurable. Thus, the inclusion map $J : C_b(S) \to B(S, \mathcal{B}_S)$ is an injective lattice homomorphism. By Propositions 7 and 9, the statement follows. ■

Consider the subspace c of l^∞ where the latter is the space of all bounded sequences and the former is the space of all convergent sequences

$$c = \left\{ x \in l^\infty : x_\infty = \lim_{i \to \infty} x_i \text{ exists in } \mathbb{R} \right\}.$$

Notice that $l^\infty = B(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$. We endow c with the supnorm $\|x\| = \sup_{i \in \mathbb{N}} |x_i|$. Clearly, c is an AM-space with unit and its norm dual can be identified with the AL-space $l^1 \oplus \mathbb{R}$ via the lattice isometry $\xi \longmapsto (y, \lambda)$ with $y \in l^1$ and $\lambda \in \mathbb{R}$ such that (see also [1, Theorem 16.14])

$$\langle x, \xi \rangle = \lambda x_\infty + \sum_{i=1}^{\infty} y_i x_i \qquad \forall x \in c.$$

We have that $\mathcal{E} = \{e_i\}_{i \in \mathbb{N}} \cup \{e_\infty\}$ where $\langle e_i, x \rangle = x_i$ for all $i \in \mathbb{N}$ and $\langle e_\infty, x \rangle = x_\infty$. The elements x and y in c are functions defined over $\mathbb{N}$ thus comonotonicity means

$$[x_i - x_j][y_i - y_j] \geq 0 \qquad \forall i, j \in \mathbb{N}. \tag{15}$$

By Propositions 7 and 9 and since the inclusion map $J : c \to l^\infty$ is a lattice homomorphism, we have that x and y in c are dually comonotonic if and only if they are comonotonic.

The AM**-space** $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathcal{N})$ Let $(\Omega, \mathcal{F})$ be a measurable space where $\mathcal{F}$ is a σ-algebra. Let $\mathcal{N} \subseteq \mathcal{F}$ be a proper σ-ideal.[10] Define by $L^\infty(\Omega, \mathcal{F}, \mathcal{N})$ the space of all real valued $\mathcal{F}$-measurable functions that are essentially bounded. That is, for each $f \in L^\infty(\Omega, \mathcal{F}, \mathcal{N})$ there exists a scalar k such that $(|f| > k) \in \mathcal{N}$. The function $\|\cdot\|_\infty : L^\infty(\Omega, \mathcal{F}, \mathcal{N}) \to [0, \infty)$ defined by

$$\|f\|_\infty = \inf \{k \geq 0 : (|f| > k) \in \mathcal{N}\} \qquad \forall f \in L^\infty(\Omega, \mathcal{F}, \mathcal{N})$$

is a seminorm. By introducing the equivalence relation

$$f \sim g \text{ if and only if } (|f - g| > 0) \in \mathcal{N} \text{ if and only if } \|f - g\|_\infty = 0$$

the quotient space $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathcal{N}) = L^\infty(\Omega, \mathcal{F}, \mathcal{N}) / \sim$ is an AM-space with unit.[11]

By $\mathbf{f}$ we will denote an equivalence class in $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathcal{N})$ and by f a representative element of the class $\mathbf{f}$. The norm dual of $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathcal{N})$ is lattice isomorphic to $ba(\Omega, \mathcal{F}, \mathcal{N})$ and the extreme points $\mathcal{E}$ of Δ are the set of all finitely additive probabilities on $(\Omega, \mathcal{F})$ that are $\{0, 1\}$-valued and take value zero on sets belonging to $\mathcal{N}$. If we consider a σ-additive probability measure $P : \mathcal{F} \to [0, 1]$ and we define $\mathcal{N} = \{E \in \mathcal{F} : P(E) = 0\}$ then $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathcal{N})$ becomes the standard space $\mathcal{L}^\infty(\Omega, \mathcal{F}, P)$. On the other hand, when $\mathcal{N} = \{\emptyset\}$ we have that $B(\Omega, \mathcal{F}) = L^\infty(\Omega, \mathcal{F}, \mathcal{N}) = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathcal{N})$.

[10] That is, $\Omega \notin \mathcal{N}$; $\mathcal{F} \ni A \subseteq B \in \mathcal{N}$ implies $A \in \mathcal{N}$; $\bigcup_{n=1}^\infty A_n \in \mathcal{N}$ if $\{A_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathcal{N}$.

[11] For a thorough analysis of the space $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathcal{N})$ and its norm dual see [13, pp. 137-140].

If $\mathbf{f} \in \mathcal{L}^{\infty}(\Omega, \mathcal{F}, \mathcal{N})$ then we denote by $\hat{f} \in C(\mathcal{E})$ the image of $\mathbf{f}$ under Kakutani's map T. That is, $\hat{f} : \mathcal{E} \to \mathbb{R}$ is such that $\hat{f}(\mu) = \int_{\Omega} f d\mu$ for all $\mu \in \mathcal{E}$.[12] Given two functions $f, g \in L^{\infty}(\Omega, \mathcal{F}, \mathcal{N})$, $f \sim g$ if and only if $\int_{\Omega} f d\mu = \int_{\Omega} g d\mu$ for all $\mu \in \mathcal{E}$ if and only if $\mathbf{f} = \mathbf{g}$. Therefore, we can also define $\mathcal{L}^{\infty}(\Omega, \mathcal{F}, \mathcal{N}) = L^{\infty}(\Omega, \mathcal{F}, \mathcal{N}) / \ker T$.

Lemma 11 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous. If $\mathbf{f} \in \mathcal{L}^{\infty}(\Omega, \mathcal{F}, \mathcal{N})$ then for each $f \in \mathbf{f}$*

$$\int_{\Omega} (\varphi \circ f)\, d\mu = \varphi\left(\int_{\Omega} f d\mu\right) \qquad \forall \mu \in \mathcal{E}. \tag{16}$$

Proof. If $\mathbf{f} \in \mathcal{L}^{\infty}(\Omega, \mathcal{F}, \mathcal{N})$ then there exists a function $\bar{f} \in B(\Omega, \mathcal{F})$ such that $\bar{f} \sim f$. Since φ is continuous, $\varphi \circ \bar{f} \sim \varphi \circ f$. If $\bar{f}$ is a simple function, that is $\bar{f} \in B_0(\Omega, \mathcal{F})$, then it is immediate to see that (16) holds for $\bar{f}$. On the other hand, if $\bar{f} \in B(\Omega, \mathcal{F})$ then there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq B_0(\Omega, \mathcal{F})$ that converges in supnorm to $\bar{f}$. It follows that

$$\begin{aligned}\varphi\left(\int_{\Omega} f d\mu\right) &= \varphi\left(\int_{\Omega} \bar{f} d\mu\right) = \varphi\left(\lim_n \int_{\Omega} f_n d\mu\right) = \lim_n \varphi\left(\int_{\Omega} f_n d\mu\right) \\ &= \lim_n \int_{\Omega} (\varphi \circ f_n)\, d\mu = \int_{\Omega} (\varphi \circ \bar{f})\, d\mu = \int_{\Omega} (\varphi \circ f)\, d\mu \qquad \forall \mu \in \mathcal{E},\end{aligned}$$

proving the statement. ■

Proposition 12 *Let $\mathbf{f}, \mathbf{g} \in \mathcal{L}^{\infty}(\Omega, \mathcal{F}, \mathcal{N})$. The following statements are equivalent:*

(i) $\mathbf{f}$ *and* $\mathbf{g}$ *are dually comonotonic;*

(ii) there exist $f \in \mathbf{f}$, $g \in \mathbf{g}$, and $N \in \mathcal{N}$ such that

$$[f(\omega) - f(\omega')]\,[g(\omega) - g(\omega')] \geq 0 \qquad \forall \omega, \omega' \in N^c; \tag{17}$$

(iii) for each $f \in \mathbf{f}$ and $g \in \mathbf{g}$ there exists $N \in \mathcal{N}$ such that

$$[f(\omega) - f(\omega')]\,[g(\omega) - g(\omega')] \geq 0 \qquad \forall \omega, \omega' \in N^c.$$

Proof. The canonical mapping $J : B(\Omega, \mathcal{F}) \to \mathcal{L}^{\infty}(\Omega, \mathcal{F}, \mathcal{N})$, such that $f \longmapsto \mathbf{f}$, is a lattice homomorphism.

(i) implies (ii). Let $\mathbf{f}$ and $\mathbf{g}$ be dually comonotonic. This implies that $\hat{f}$ and $\hat{g}$ are comonotonic. By [6, Proposition 4.5], there exist two continuous and monotone functions $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ such that $\hat{f} = \varphi_1\left(\hat{f} + \hat{g}\right)$ and $\hat{g} = \varphi_2\left(\hat{f} + \hat{g}\right)$. Set $h = f + g$. By previous lemma, we have that $f \sim \varphi_1 \circ h$ and $g \sim \varphi_2 \circ h$. On the other hand, by [6, Proposition 4.5], the functions $\varphi_1 \circ h$ and $\varphi_2 \circ h$ are comonotonic. This implies that $\varphi_1 \circ h$ and $\varphi_2 \circ h$ satisfy (17) for $N = \emptyset$.

(ii) implies (iii). If $f' \in \mathbf{f}$ and $g' \in \mathbf{g}$ then there exists $M \in \mathcal{N}$ such that $f'_{|M^c} = f_{|M^c}$ and $g'_{|M^c} = g_{|M^c}$. It follows that for each $\omega, \omega' \in (N \cup M)^c = N^c \cap M^c$

$$[f'(\omega) - f'(\omega')]\,[g'(\omega) - g'(\omega')] = [f(\omega) - f(\omega')]\,[g(\omega) - g(\omega')] \geq 0$$

and $N \cup M \in \mathcal{N}$.

(iii) implies (i). Let $f \in \mathbf{f}$, $g \in \mathbf{g}$, and $N \in \mathcal{N}$ be as in (iii). There exist $k \in \mathbb{R}$ and two functions $\bar{f}$ and $\bar{g}$ in $B_+(\Omega, \mathcal{F})$ that are further comonotonic in the standard sense and such that $\bar{f} \sim f + k\chi_{\Omega}$ and $\bar{g} \sim g + k\chi_{\Omega}$. By Propositions 7 and 9, $J\left(\bar{f}\right) = \mathbf{f} + \mathbf{k}$ and $J(\bar{g}) = \mathbf{g} + \mathbf{k}$ are dually comonotonic and so are $\mathbf{f}$ and $\mathbf{g}$. ■

[12] Notice that if $f_1, f_2 \in \mathbf{f}$ then $\int_{\Omega} f_1 d\mu = \int_{\Omega} f_2 d\mu$ for all $\mu \in ba(\Omega, \mathcal{F}, \mathcal{N})$.

4 Superadditivity

When $X = B(\Omega, \mathcal{F})$, the relation between superadditivity and supermodularity is well known and the results are simple and nice (see, e.g., [10, Corollary 4.2]). We shall extend this result to our general setting. Moreover, superadditivity will be a key property in allowing us to provide a representation where the set function γ is defined and unique not just over the class $\mathcal{U}(\mathcal{E})$, but on the entire Baire σ-algebra, $Baire(\mathcal{E})$, providing a complete generalization of Theorem 5.

Proposition 13 *Let X be an Archimedean Riesz space with unit and V a functional from X to $\mathbb{R}$. The following statements are equivalent:*

(i) V is monotone, dual comonotonic additive, and supermodular;

(ii) V is monotone, dual comonotonic additive, and superadditive;

(iii) there exists a unique supermodular and outer continuous capacity, $\gamma : \mathcal{U}(\mathcal{E}) \to [0, \infty)$, such that

$$V(x) = \int_{\mathcal{E}} \langle x, \xi \rangle \, d\gamma(\xi) \qquad \forall x \in X.$$

Proof. Consider $\hat{V} : C \to \mathbb{R}$ and $\bar{V} : C(\mathcal{E}) \to \mathbb{R}$ as in the proof of Theorem 6. Recall that $\hat{V} = V \circ T^{-1}$ where T is Kakutani's map and $\bar{V}$ is $\hat{V}$ Lipschitz continuous extension to $C(\mathcal{E})$.

(i) implies (iii) (resp., (ii) implies (iii)). Since V is monotone, dual comonotonic additive, and supermodular (resp., superadditive), and T is a lattice isomorphism, we have that $\hat{V}$ is monotone, comonotonic additive, and supermodular (resp., superadditive). By Lipschitz continuity, the extension $\bar{V}$ inherits the same properties. By Dini's Theorem, $\bar{V}$ is outer continuous. By Theorem 19, it follows that there exists an outer continuous supermodular capacity, $\gamma : \mathcal{U}(\mathcal{E}) \to [0, \infty)$, such that $\bar{V}(f) = \int_{\mathcal{E}} f(\xi) \, d\gamma(\xi)$ for all $f \in C(\mathcal{E})$. It follows that $V(x) = \int_{\mathcal{E}} \langle x, \xi \rangle \, d\gamma(\xi)$ for all $x \in X$. Uniqueness follows from Theorem 6.

(iii) implies (i) (resp., (iii) implies (ii)). Consider a supermodular and outer continuous capacity $\gamma : \mathcal{U}(\mathcal{E}) \to [0, \infty)$. Define $\gamma^* : Baire(\mathcal{E}) \to [0, \infty)$ by

$$\gamma^*(A) = \sup \{ \gamma(B) : \mathcal{U}(\mathcal{E}) \ni B \subseteq A \} \qquad \forall A \in Baire(\mathcal{E}).$$

It is immediate to see that γ^* is a supermodular capacity which further coincides to γ on $\mathcal{U}(\mathcal{E})$. Next, define $W : B(\mathcal{E}, Baire(\mathcal{E})) \to \mathbb{R}$ by $f \longmapsto \int_{\mathcal{E}} f(\xi) \, d\gamma^*(\xi)$. By [10, Corollary 4.2] and since γ^* is supermodular, W is monotone, comonotonic additive, supermodular, and superadditive. Since $V = W \circ T$ and T is a lattice isomorphism, it follows that V is monotone, dual comonotonic additive, supermodular, and superadditive. ■

In the next result we assume that V satisfies an extra property of continuity but is not necessarily monotone. Its proof is heavily dependent on the Daniel-Stone extension theorem given by [4].

Theorem 14 *Let X be an Archimedean Riesz space with unit and V a functional from X to $\mathbb{R}$. The following statements are equivalent:*

(i) V is dual comonotonic additive, superadditive, sequentially weakly continuous, and of bounded variation;

(ii) V is dual comonotonic additive, superadditive, sequentially weakly continuous at $x = 0$, and of bounded variation;

(iii) there exists a unique supermodular and continuous $\gamma \in bv\,(Baire\,(\mathcal{E}))$ such that

$$V(x) = \int_{\mathcal{E}} \langle x, \xi \rangle \, d\gamma(s) \qquad \forall x \in X.$$

Moreover, V is monotone if and only if γ is a capacity.

Proof. Again consider $\hat{V} : C \to \mathbb{R}$ as in the proof of Theorem 6. Recall that $\hat{V} = V \circ T^{-1}$ where T is Kakutani's map.

(i) implies (ii). It is obvious.

(ii) implies (iii). Since V is dual comonotonic additive, superadditive, and of bounded variation (resp., monotone), we have that $\hat{V}$ is comonotonic additive, superadditive, and of bounded variation (resp., monotone). Next, consider a uniformly bounded sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq C$ such that $f_n \longrightarrow 0$ pointwise. It follows that the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, defined by $x_n = T^{-1}(f_n)$ for all $n \in \mathbb{N}$, is bounded and it weakly converges to 0. Since V is sequentially weakly continuous at $x = 0$, we have that $\hat{V}(0) = V(0) = \lim_n V(x_n) = \lim_n \hat{V}(\hat{x}_n)$, proving that $\hat{V}$ is bounded pointwise continuous at 0. By [4, Theorem 22], it follows that there exists a continuous and supermodular $\gamma \in bv\,(\sigma(\Sigma_C))$ (resp., capacity) such that

$$\hat{V}(f) = \int_{\mathcal{E}} f(\xi)\, d\gamma(\xi) \qquad \forall f \in C.$$

Since C is uniformly dense in $C(\mathcal{E})$, the σ-algebra generated by Σ_C coincides to $Baire\,(\mathcal{E})$, that is, $\gamma \in bv\,(Baire\,(\mathcal{E}))$. Finally, since $V = \hat{V} \circ T$ and given uniqueness of γ in [4, Theorem 22], the statement follows.

(iii) implies (i). By Theorem 6, it follows that V is dual comonotonic additive and of bounded variation (resp., monotone if γ is a capacity). Define $W : B(\mathcal{E}, Baire\,(\mathcal{E})) \to \mathbb{R}$ by $W(f) = \int_{\mathcal{E}} f(\xi)\, d\gamma(\xi)$ for all $f \in B(\mathcal{E}, Baire\,(\mathcal{E}))$. Notice that $V = W \circ T$. By [10, Corollary 4.2] and since γ is supermodular, we have that W is superadditive. Since T is a positive operator, V is superadditive. Finally, consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $x_n \rightharpoonup x$. It follows that the sequence $\{\hat{x}_n\}_{n\in\mathbb{N}} \subseteq B(\mathcal{E}, Baire\,(\mathcal{E}))$ is bounded and it converges pointwise to $\hat{x}$. By [4, Theorem 22] and since γ is continuous, it follows that

$$V(x) = W(\hat{x}) = \lim_n W(\hat{x}_n) = \lim_n V(x_n),$$

proving that V is sequentially weakly continuous. ■

Remark 15 *Notice that we could have provided a version of Theorem 14 where superadditivity was replaced by subadditivity and supermodularity by submodularity. If V is both monotone and additive then it is immediate to see that $V \in X^*$. Thus, V is sequentially weakly continuous. By previous observations, it further follows that Theorem 14 guarantees the existence of a unique continuous and modular capacity γ on $Baire\,(\mathcal{E})$. This delivers that γ is a σ-additive measure and that Theorem 5 is a corollary of Theorem 14.*

5 Appendix

In this appendix, we provide a nonadditive integral representation theorem (Theorem 19) for functionals defined over a Stone vector lattice L.[13] This result is ancillary in proving Theorem 6. Theorem

[13] A Stone vector lattice is a vector lattice, wrt the pointwise order, of real valued and bounded functions on a set Ω which further contains χ_Ω. L is endowed with the supnorm. The main statement in Theorem 19 could be proven when L is a subset of $B(\Omega, \mathcal{P}(\Omega))$ such that if $f \in L$ then αf, $f \wedge \beta$, $f \vee \beta$, $f + \beta \in L$ for all $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$.

19 is closely related to Sipos's approach [15] (see also Greco [7]). We denote by $\mathcal{F}$ a σ-algebra of sets of Ω such that $L \subseteq B(\Omega, \mathcal{F})$. In the sequel, with a small abuse of notation, given $k \in \mathbb{R}$, we will denote by k both the real number and the constant function on Ω that takes value k. Moreover, we set $L_+ = \{f \in L : f \geq 0\}$. In view of Definition 4, given a functional $V : L \to \mathbb{R}$ and two points $f, g \in L$ such that $f \leq g$, we define

$$Var_V(f, g) = \sup \sum_{i=1}^{n} |V(f_i) - V(f_{i-1})| \in [0, \infty],$$

where the supremum is taken over all finite chains $f = f_0 \leq f_1 \leq \cdots \leq f_n = g$ contained into L. The notions of monotonicity, unit-additivity, unit-modularity, bounded variation are the ones of Definition 4 applied to this setting where the unit is the function χ_Ω. Moreover, we say that V is outer continuous if $\lim_n V(f_n) = V(f)$ whenever $f_n \downarrow f$ and the convergence is pointwise. It is immediate to check that if V is the difference of monotone functionals then V is of bounded variation. Unit-modularity is very close to the following property introduced by Sipos [15] (see also [7]):

$$V(f) = V(f \wedge \beta) + V(f \vee \beta - \beta) \qquad \forall f \in L, \forall \beta \in \mathbb{R}. \tag{18}$$

We next report some simple relations among the above properties. Their proof follows from standard arguments.

Proposition 16 *Let V be a functional from L to $\mathbb{R}$. The following statements are true.*

(i) If V is unit-additive then $V(0) = 0$.

(ii) If V is unit-additive then $V(\alpha) = \alpha V(1)$ for all $\alpha \in \mathbb{Q}$.

(iii) If V is monotone and unit-additive then $V(\alpha) = \alpha V(1)$ for all $\alpha \in \mathbb{R}$ and V is Lipschitz continuous.

(iv) If V is unit-additive then $V(f + \alpha) = V(f) + V(\alpha)$ holds for all $f \in L$ and all $\alpha \in \mathbb{R}$.

(v) Given V unit-additive, V is unit-modular if and only if it satisfies (18).

Next proposition delivers a nonadditive integral representation theorem when the functional V is further monotone and it is fundamentally due to Greco [7]. It is ancillary to prove Theorem 19. The proof of Proposition 17 consists in checking that all our assumptions on V allow us to apply Greco's result. We omit it.

Proposition 17 *Let V be a functional from L to $\mathbb{R}$. The following statements are equivalent:*

(i) V is monotone, unit-additive, and unit-modular;

(ii) there exists a capacity $\gamma : \Sigma_L \to [0, \infty)$ such that

$$V(f) = \int_\Omega f(\omega)\, d\gamma(\omega) = \int_0^\infty \gamma(f \geq t)\, dt + \int_{-\infty}^0 [\gamma(f \geq t) - \gamma(\Omega)]\, dt \qquad \forall f \in L. \tag{19}$$

Next lemma is crucial to extend Proposition 17 to nonmonotone functionals.

Lemma 18 *Let V be a functional from L to $\mathbb{R}$. The following statements are equivalent:*

(i) $V : L \to \mathbb{R}$ is unit-additive, unit-modular, and of bounded variation;

(ii) V is the difference, that is $V = V_1 - V_2$, of two monotone, unit-additive, and unit-modular functionals $V_1, V_2 : L \to \mathbb{R}$. In particular, V is Lipschitz continuous.

Proof. (ii) implies (i). The proof of this implication is trivial and details are omitted.

(i) implies (ii). We will proceed by Steps. We start by defining $V_1 : L_+ \to \mathbb{R}$ by

$$V_1(f) = Var_V(0, f) \qquad \forall f \in L_+.$$

Step 1. V_1 is monotone and unit-additive on L_+.

Proof of the Step.

The proof follows from the arguments contained in the proof of [4, Lemma 14]. □

Step 2. V_1 is unit-modular on L_+.

Proof of the Step.

Fix $f \in L_+$ and $\alpha \in \mathbb{R}_+$. It is immediate to see that $0 \leq \alpha \leq f \vee \alpha$, $0 \leq f \wedge \alpha \leq f$, and

$$Var_V(0, f \vee \alpha) \geq Var_V(0, \alpha) + Var_V(\alpha, f \vee \alpha) \tag{20}$$

and

$$Var_V(0, f) \geq Var_V(0, f \wedge \alpha) + Var_V(f \wedge \alpha, f). \tag{21}$$

Next, we show that also the converse inequalities hold implying that the above relations are indeed equalities. Let $\xi < Var_V(0, f \vee \alpha)$. It follows that there exists a chain $0 = f_0 \leq f_1 \leq \cdots \leq f_n = f \vee \alpha$ such that

$$\xi < \sum_{i=1}^{n} |V(f_i) - V(f_{i-1})|.$$

Since V is unit-modular, we have that

$$\begin{aligned}
&\xi < \sum_{i=1}^{n} |V(f_i) - V(f_{i-1})| = \sum_{i=1}^{n} |[V(f_i) + V(\alpha)] - [V(f_{i-1}) + V(\alpha)]| \\
&= \sum_{i=1}^{n} |[V(f_i \vee \alpha) - V(f_{i-1} \vee \alpha)] + [V(f_i \wedge \alpha) - V(f_{i-1} \wedge \alpha)]| \\
&\leq \sum_{i=1}^{n} |V(f_i \vee \alpha) - V(f_{i-1} \vee \alpha)| + \sum_{i=1}^{n} |V(f_i \wedge \alpha) - V(f_{i-1} \wedge \alpha)| \\
&\leq Var_V(\alpha, f \vee \alpha) + Var_V(0, \alpha)
\end{aligned}$$

where the last step follows from the fact that $\{f_i \vee \alpha\}_{i=0}^{n}$ is a chain from α to $f \vee \alpha$, while $\{f_i \wedge \alpha\}_{i=0}^{n}$ is a chain from 0 to α. This implies that

$$Var_V(0, f \vee \alpha) \leq Var_V(0, \alpha) + Var_V(\alpha, f \vee \alpha).$$

By (20), it follows that

$$Var_V(0, f \vee \alpha) = Var_V(0, \alpha) + Var_V(\alpha, f \vee \alpha). \tag{22}$$

Next, we show that $Var_V(\alpha, f \vee \alpha) = Var_V(f \wedge \alpha, f)$. First, consider a chain $\{f_i\}_{i=0}^{n}$ such that $f \wedge \alpha = f_0 \leq f_1 \leq \cdots \leq f_n = f$. Since V is unit-modular, it follows that

$$\sum_{i=1}^{n} |V(f_i) - V(f_{i-1})| = \sum_{i=1}^{n} |V(f_i \vee \alpha) - V(f_{i-1} \vee \alpha) + V(f_i \wedge \alpha) - V(f_{i-1} \wedge \alpha)|.$$

At the same time, we have that $f_i \wedge \alpha = f \wedge \alpha$ for all $i \in \{0, ..., n\}$ and $(f_i \vee \alpha)_{i=0}^n$ is a chain from α to $f \vee \alpha$. This implies that

$$\sum_{i=1}^{n} |V(f_i) - V(f_{i-1})| = \sum_{i=1}^{n} |V(f_i \vee \alpha) - V(f_{i-1} \vee \alpha)| \leq Var_V(\alpha, f \vee \alpha),$$

proving that $Var_V(f \wedge \alpha, f) \leq Var_V(\alpha, f \vee \alpha)$. The converse inequality follows from a similar argument. By the same methods, we obtain that

$$Var_V(0, f) = Var_V(0, f \wedge \alpha) + Var_V(f \wedge \alpha, f). \tag{23}$$

Finally, given (22) and (23), we have that

$$\begin{aligned} V_1(f \vee \alpha) + V_1(f \wedge \alpha) &= Var_V(0, f \vee \alpha) + Var_V(0, f \wedge \alpha) \\ &= Var_V(0, \alpha) + Var_V(\alpha, f \vee \alpha) + Var_V(0, f) - Var_V(f \wedge \alpha, f) \\ &= V_1(\alpha) + V_1(f) \end{aligned}$$

and V_1 is unit-modular over L_+. □

Step 3. The functional $V_2 : L_+ \to \mathbb{R}$, defined by $V_2 = V_1 - V$, is monotone, unit-additive, and unit-modular. Moreover, $V(f) = V_1(f) - V_2(f)$ for all $f \in L_+$.

Proof of the Step.

Consider $V_2 = V_1 - V$. It is immediate to see that $V = V_1 - V_2$. If $0 \leq f \leq g$ then we have that

$$V(g) - V(f) \leq |V(g) - V(f)| \leq Var_V(f, g) \leq V_1(g) - V_1(f),$$

proving that V_2 is monotone. Consequently, by Step 1 and Step 2 and since $V_2 = V_1 - V$, the functional V_2 is also unit-additive and unit-modular over L_+. □

Step 4. V is the difference of two monotone, unit-additive, and unit-modular functionals $V_1, V_2 : L \to \mathbb{R}$. In particular, V is Lipschitz continuous.

Proof of the Step.

Given a monotone and unit-additive functional on L_+, it is a routine argument to show that such a functional admits a unique monotone and unit-additive extension to L. Moreover, if such a functional is unit-modular then the extension is also unit-modular. In light of this fact, we consider $V_1, V_2 : L_+ \to \mathbb{R}$, as defined in the previous part of the proof, and, without loss of generality, we denote their unique monotone, unit-additive, and unit-modular extensions by the same symbols. It is routine to check that $V = V_1 - V_2$ over the entire space L and that V is Lipschitz continuous in light of Proposition 16. □

Step 4 concludes the proof. ■

We can state the main result of the appendix:

Theorem 19 *Let $V : L \to \mathbb{R}$ be a functional over a Stone vector lattice L. The following statements are equivalent:*

(i) V is comonotonic additive and of bounded variation;

(ii) V is unit-additive, unit-modular, and of bounded variation;

(iii) there exists $\gamma \in bv(\Sigma_L)$ such that

$$V(f) = \int_\Omega f(\omega)\, d\gamma(\omega) \qquad \forall f \in L. \tag{24}$$

Moreover, V is outer continuous if and only if γ can be chosen to be outer continuous. Given γ outer continuous then

1. *γ is unique.*
2. *V is monotone if and only if γ is a capacity.*
3. *V is supermodular if and only if γ is supermodular.*
4. *V is superadditive only if γ is supermodular.*

Proof. (i) implies (ii). Consider $f \in L$, $\alpha \in \mathbb{R}$, and $\beta \in \mathbb{R}$. It is immediate to check that f and α are comonotonic and also $f \wedge \beta$ and $f \vee \beta$ are comonotonic (see also [10, Lemma 4.6]). Since V is comonotonic additive, this implies that

$$V(f+\alpha) = V(f) + V(\alpha) \text{ and } V(f \wedge \beta) + V(f \vee \beta) = V(f+\beta) = V(f) + V(\beta),$$

proving that V is unit-additive and unit-modular.

(ii) implies (iii). By Lemma 18, there exist $V_1, V_2 : L \to \mathbb{R}$ where V_1 and V_2 are monotone, unit-additive, and unit-modular and such that $V = V_1 - V_2$. By Proposition 17, there exist $\gamma_1, \gamma_2 : \Sigma_L \to [0,\infty)$ that represent, respectively, V_1 and V_2 as in (19). The set function $\gamma = \gamma_1 - \gamma_2$ represents V as in (24).

(iii) implies (i). By [4, Proposition 7], γ is the difference of two capacities γ_1 and γ_2 on Σ_L. Define $\gamma_i^* : \mathcal{F} \to [0,\infty)$ by

$$\gamma_i^*(A) = \sup\{\gamma_i(B) : \mathcal{F} \ni B \subseteq A\} \qquad \forall A \in \mathcal{F}.$$

Since γ_i is a capacity, γ_i^* is a capacity for $i \in \{1,2\}$ and $\gamma_i^*(A) = \gamma_i(A)$ for all $A \in \Sigma_L$. It follows that $\gamma^* = \gamma_1^* - \gamma_2^*$ belongs to $bv(\mathcal{F})$ and extends γ. Define $W : B(\Omega,\mathcal{F}) \to \mathbb{R}$ by $W(f) = \int_\Omega f(\omega)\, d\gamma^*(\omega)$. By [11] or [10], it follows that $V(f) = W(f)$ for all $f \in L$ and the latter functional is comonotonic additive and of bounded variation, proving that also V is.

Moreover, by [4, Theorem 13], V is outer continuous if and only if γ can be chosen to be outer continuous. Similarly, provided γ is outer continuous, points 1., 2., and 3., follow from the same result. We just need to prove 4. Assume V is superadditive and consider $A, B \in \Sigma_L$. By [4, Lemma 16], there exists $\{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}} \subseteq L_+$ such that $f_n \downarrow 1_A$ and $g_n \downarrow 1_B$ pointwise. It follows that $\{f_n + g_n\}_{n\in\mathbb{N}} \subseteq L_+$ and $f_n + g_n \downarrow 1_A + 1_B$. Moreover, we have that $\lim_n V(f_n) = \gamma(A)$ and $\lim_n V(g_n) = \gamma(B)$. Define $k = \|f_1\| + \|g_1\| + 2$. Observe that $(f_n + g_n \geq t) \downarrow (1_A + 1_B \geq t)$ for all $t \in [0,k]$. Since Σ_L is a lattice, for each $t \in [0,k]$

$$(1_A + 1_B \geq t) \in \{\emptyset, A\cap B, A \cup B, \Omega\} \subseteq \Sigma_L.$$

Define $h_n : [0,k] \to \mathbb{R}$ by

$$h_n(t) = \gamma(f_n + g_n \geq t) \qquad \forall t \in [0,k], \forall n \in \mathbb{N}.$$

Since γ is outer continuous and of bounded variation, we have that for each $t \in [0,k]$

$$|h_n(t)| = |\gamma(f_n + g_n \geq t)| < \infty \text{ and } h_n(t) \to \gamma(1_A + 1_B \geq t).$$

By (24) and Arzelà's Theorem (see [9]), it follows that

$$\lim_n V(f_n + g_n) = \lim_n \int_0^k h_n(t)\,dt = \int_0^k \left[\lim_n h_n(t)\right] dt = \int_0^k \gamma(1_A + 1_B \geq t)\,dt = \gamma(A\cap B) + \gamma(A \cup B).$$

Since V is superadditive, it follows that $V(f_n + g_n) \geq V(f_n) + V(g_n)$ for all $n \in \mathbb{N}$. Passing to the limit, we obtain that

$$\gamma(A \cup B) + \gamma(A \cap B) = \lim_n V(f_n + g_n) \geq \lim_n V(f_n) + \lim_n V(g_n) = \gamma(A) + \gamma(B).$$

■

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