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Niveloids and Their Extensions:
Risk Measures on Small Domains*

S. Cerreia-Vioglio,[a] F. Maccheroni,[a] M. Marinacci,[a] and A. Rustichini[a,b]
[a]Università Bocconi and [b]University of Minnesota

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Abstract

Given a functional defined on a nonempty subset of an Archimedean Riesz space with unit, necessary and sufficient conditions are obtained for the existence of a (convex or concave) niveloid that extends the functional to the entire space. In the language of mathematical finance, this problem is equivalent to the one of verifying if the policy adopted by a regulator is consistent with monetary risk measurement, when only partial information is available.

Keywords: extension theorems, Daniell-Stone theorem, risk measures, variational preferences

1 Introduction

Let S be a nonempty set and L a vector sublattice of $B(S)$ containing the constant functions.[1] Dolecki and Greco [11] call *niveloid* a functional $\varphi : L \to \mathbb{R}$ which is

- *monotone*: $x \geqslant y$ implies $\varphi(x) \geq \varphi(y)$;
- *translation invariant*: $\varphi(x + c) = \varphi(x) + c$ for all $x \in L$ and $c \in \mathbb{R}$.

In mathematics, an important example of niveloid is the *Choquet integral* (introduced by Choquet [8], see also Dellacherie [9]). In this case, S is a Polish space, $L = C_b(S)$, and

$$\varphi(x) = \int_0^\infty \phi(x \geq t)\, dt + \int_{-\infty}^0 [\phi(x \geq t) - \phi(S)]\, dt$$

where ϕ is an increasing set function, such that $\phi(S) = 1$, defined on the closed subsets of S.

In economics, niveloids play a central role in the modelling of *decisions under (Knightian) uncertainty* (see Gilboa and Marinacci [17] for a recent survey on this topic). For example, in the case of Gilboa and Schmeidler [19]'s multiple priors preferences, (S, Σ) is a measurable space and an action a generating utility $u_a(s)$ in each state s is evaluated by

$$\varphi(u_a) = \inf_{p \in \Gamma} \int_S u_a(s)\, dp(s) \tag{1}$$

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[1] As usual, $B(S)$ denotes the Banach lattice of real-valued bounded functions on S endowed with the pointwise order and the supnorm. A vector sublattice of $B(S)$ containing the constant functions is called *Stone vector lattice*.

where Γ is a set of probabilities on Σ. The interpretation in terms of robust statistical decision theory is clear. Maccheroni, Marinacci, and Rustichini [20] axiomatically characterize the more general criterion

$$\varphi(u_a) = \inf_{p \in \Delta(S,\Sigma)} \left\{ \int_S u_a(s)\, dp(s) + \gamma(p) \right\} \tag{2}$$

where the hard constraint $p \in \Gamma$ is relaxed to a soft constraint $\gamma : \Delta(S,\Sigma) \to [0,\infty]$.[2]

In mathematical finance, niveloids are called *monetary measures of risk* and they represent capital requirements. In this perspective, $\varphi(x)$ represents the minimal reserve amount that should be invested in a risk-free manner at date 0 to cover date 1 state-contingent loss x. Monotonicity means that higher losses require higher reserves, while translation invariance requires that if a loss is augmented by c dollars in every state the capital requirement have to be augmented by the same amount.[3] The active research in this topic started with the seminal paper of Artzner, Delbaen, Eber, and Heath [3] and is strongly motivated by banking regulation and supervision problems. A complete and updated treatment of monetary measures of risk can be found in Föllmer and Schied [14]. As they argue (see also [5]), a central principle of risk management requires that diversification cannot increase risk. For niveloids this principle translates into convexity. In particular, convex niveloids are called *convex risk measures*. They were introduced by Föllmer and Schied [15] and Frittelli and Rosazza Gianin [16], and are the economically most relevant class of monetary measures of risk.

This paper studies the conditions under which a functional defined only on a nonempty subset X of L can be extended to a (convex or concave) niveloid on the entire space L. This question is natural from a mathematical viewpoint in light of the extension results of measure theory and functional analysis. But, here it is also compelling from the viewpoint of applications. For example, in the theory of choice under uncertainty the natural domain of φ only consists of the set $\{u_a : a \in A\}$, where A is the set of available actions and concavity corresponds to uncertainty aversion. Analogously, in a banking context, the European Central Bank can observe the capital requirements $\varphi(x_1), ..., \varphi(x_N)$ imposed by the Bank of Italy on the portfolios $x_1, x_2, ..., x_N$ of the N Italian banks and try to gauge if the policy is consistent with a monetary measure of risk. Plausibly, the European Central Bank might also want to verify if diversification is favoured by the Bank of Italy, that is, if there exists a convex niveloid that extends φ.

Section 2 contains the mathematical framework of the paper plus some basic definitions and properties. The, mostly introductory, Section 3 adapts the extension results of Dolecki and Greco [11] to the different setup of this paper and investigates some basic representations of niveloids. Section 4 characterizes the conditions for the existence of a convex niveloidal extension, while a dual representation of the maximal of such extensions is obtained in the subsequent Section 5, where the Fenchel-Moreau duality of niveloids is studied in some detail. A version of the Daniell-Stone Theorem for niveloids is obtained in Section 6. The concluding Section 7 studies niveloids on domains of the form

$$\left\{ x \in L : \inf_{s \in S} x(s), \sup_{s \in S} x(s) \in I \right\}$$

where I is an interval of the real line. This class of sets contains the open and closed unit balls of L, as well as the positive cone, the negative cone, and the entire space.

[2] As usual, $\Delta(S,\Sigma)$ denotes the set of all probabilities on Σ.

[3] While monotonicity is uncontroversial, translation invariance (called *cash additivity* in this context) is non-problematic only if it is possible to transfer cash from date 0 to date 1 in a risk-free manner without frictions. In this case the assumption of zero interest rate is an innocuous normalization. See the discussion in El Karoui and Ravanelli [12] and Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [5].

2 Setup and preliminaries

Stone vector lattices form a special case of Archimedean Riesz spaces with unit, but the latter class is larger, including for example all $L^{\infty}(\Omega, \mathcal{F}, \mathcal{N}, \mathbb{R}^n)$ spaces where $\mathcal{N}$ is the σ-ideal of null subsets of $\mathcal{F}$ (see [4]). For this reason, with the exception of Section 6, functionals on Archimedean Riesz spaces with unit rather than on Stone vector lattices are considered in the rest of the paper.

Let E be a vector space and $\geqslant$ a partial order on E such that

$$x \geqslant y \Longrightarrow cx + z \geqslant cy + z$$

for all $c \in \mathbb{R}_+$ and $z \in E$. The pair $(E, \geqslant)$ is called an *ordered vector space*. An element x of E is *positive* if $x \geqslant 0$, E_+ denotes the set of all positive elements of E, $E_- = -E_+$. For each $y, z \in E$, $[y, z]$ is the (possibly empty) set $\{x \in E : y \leqslant x \leqslant z\}$.

An ordered vector space is a *Riesz space* if each pair of elements $x, y \in E$ admits a supremum and an infimum in E, respectively denoted by $x \vee y$ and $x \wedge y$. A Riesz space is *Archimedean* if whenever $0 \leqslant nx \leqslant y$ for all $n \in \mathbb{N}$ and some $y \in E$, then $x = 0$. If E is a Riesz space and $x \in E$, the *absolute value* of x is defined by $|x| = (-x) \vee x$. An element $e \neq 0$ in a Riesz space E is a *unit* if for each $x \in E$ there exists $n \in \mathbb{N}$ such that $|x| \leqslant ne$. The vector subspace spanned by e is denoted by $\mathbb{R}e$.

Throughout this paper $(E, \geqslant, e)$ is an Archimedean Riesz space with unit e, endowed with the *supnorm*

$$\|x\|_e = \inf\{c \in \mathbb{R}_+ : |x| \leqslant ce\} \quad \forall x \in E.$$

It is important to recall that E_+ is closed in the induced topology.[4]

Let X be any *nonempty* subset of E. A functional $\varphi : X \to \mathbb{R}$ is:

- *monotone* if $x \geqslant y$ implies $\varphi(x) \geq \varphi(y)$ for all $x, y \in X$;
- *translation invariant* if $\varphi(x + ce) = \varphi(x) + c$ for all $x \in X$ and $c \in \mathbb{R}$ such that $x + ce \in X$.[5]

The natural domains for translation invariant functionals are sets X that contain the coset $x + \mathbb{R}e$ whenever they contain x: these sets are called *tubes*. If X is not a tube, then $X + \mathbb{R}e$ is the smallest tube that contains X. The next proposition, inspired by the results on risk measures of [3] and [14], characterizes monotone and translation invariant functionals on tubes.

Proposition 1 *Let X be a tube. A function $\varphi : X \to [-\infty, \infty]$ is real valued, monotone, and translation invariant if and only if there exists $\varnothing \subset Y \subset X$, such that $x \in X$, $y \in Y$ and $x \leqslant y$ imply $x \in Y$, for which*

$$\varphi(x) = \inf\{c \in \mathbb{R} : x - ce \in Y\} \quad \forall x \in X. \tag{3}$$

Moreover, if X and Y are convex so is φ.

Proof. If φ is real valued, monotone, and translation invariant, take $Y = \{\varphi < 0\}$ or $Y = \{\varphi \leq 0\}$.

Conversely, for each $x \in X$, set $C_x = \{c \in \mathbb{R} : x - ce \in Y\}$ and notice that:

- C_x is not empty, since for each $y \in Y$ eventually $x - y \leqslant ne$ so that $x - ne \leqslant y \in Y$;

[4] See Chapters 8 and 9 of Aliprantis and Border [1] for the properties of Riesz spaces. For example, see Theorem 9.28 for the properties of $\|\cdot\|_e$ and Theorem 8.43 for the closure of E_+.

[5] This is equivalent to require $\varphi(x + ce) = \varphi(x) + c$ for all $x \in X$ and $c > 0$. In fact, assume this is the case; let $x \in X$ and $c < 0$ be such that $x + ce \in X$. Set $y = x + ce \in X$ and $d = -c$. This yields $y, y + de = x \in X$ and $d > 0$, then $\varphi(y + de) = \varphi(y) + d$, that is, $\varphi(x) = \varphi(x + ce) - c$. Obviously, $\varphi(x + ce) = \varphi(x) + c$ for all $x \in X$ if $c = 0$.

- C_x is an unbounded above half-line ($c \in C_x$ and $d \geq c$ imply $d \in C_x$);
- C_x is bounded below, otherwise $x + ne \in Y$ for all $n \in \mathbb{N}$, but for each $z \in X$ eventually $z \leqslant x + ne \in Y$, and this would contradict $Y \subset X$.

This proves that (3) defines a real valued functional. If $x \leqslant z \in X$, then $C_z \subseteq C_x$ and hence φ is monotone. While translation invariance follows from $C_{x+de} = C_x + d$ for all $d \in \mathbb{R}$.

Finally, if X and Y are convex, take $x, y \in X$ and set $z = y + (\varphi(x) - \varphi(y))\, e$. By translation invariance, $\varphi(z) = \varphi(x)$, then C_x and C_z are half-lines with the same infimum and $\varphi(x) = \varphi(z) = \inf(C_x \cap C_z)$. For each $\lambda \in (0,1)$ and each $c \in C_x \cap C_z$, $x - ce, z - ce \in Y$, and the convexity of Y implies $\lambda x + (1-\lambda) z - ce \in Y$, then $c \in C_{\lambda x + (1-\lambda) z}$, so that

$$\varphi(x) = \inf(C_x \cap C_z) \geq \inf C_{\lambda x+(1-\lambda)z} = \varphi(\lambda x + (1-\lambda) z) = \varphi(\lambda x + (1-\lambda) y) + (1-\lambda)(\varphi(x) - \varphi(y))$$

which proves convexity of φ. ■

For $X = E$ and $Y = E_-$, Proposition 1 shows monotonicity, translation invariance and convexity of the *essential supremum*

$$\operatorname{esup}(x) = \inf\{c \in \mathbb{R} : x \leqslant ce\} \quad \forall x \in E. \tag{4}$$

Notice that $\|x\|_e = \operatorname{esup}(|x|)$ for all $x \in E$, and so $\operatorname{esup}(\cdot)$ is positively homogeneous. Moreover, since E_+ is closed, the infimum in (4) is attained. Therefore, given $x \in E$ and $c \in \mathbb{R}$,

$$x \leqslant ce \iff \operatorname{esup}(x) \leq c$$

and hence

$$|x| \leqslant ce \iff \|x\|_e \leq c.$$

In turn, this implies that the closed ball $U_\varepsilon(x_o)$ in E of radius ε centered in x_o coincides with the interval $[x_o - \varepsilon e, x_o + \varepsilon e]$.

The essential supremum can also be used to characterize the properties of monotonicity and translation invariance of functionals defined on tubes. As usual in convex analysis set $\infty - \infty = \infty$.[6]

Proposition 2 *A function $\varphi : X \to [-\infty, \infty]$ is real valued, monotone, and translation invariant if*

$$\varphi(x) - \varphi(y) \leq \operatorname{esup}(x - y) \quad \forall x, y \in X. \tag{5}$$

The converse is true if X is a tube.

Proof. Assume (5) holds, since the extended difference takes value ∞ whenever the usual one is undefined, φ must be real valued. If $x, y \in X$ and $x \leqslant y$, then $\varphi(x) - \varphi(y) \leq \operatorname{esup}(x - y) \leq 0$, so that φ is monotone. Moreover, if $x \in X$, $c \in \mathbb{R}$, and $x + ce \in X$, then $\varphi(x + ce) - \varphi(x) \leq \operatorname{esup}(x + ce - x) = c$ and $\varphi(x) - \varphi(x + ce) \leq \operatorname{esup}(x - x - ce) = -c$ so that φ is translation invariant.

Conversely, assume X is a tube and $\varphi : X \to \mathbb{R}$ is monotone and translation invariant. Since X is a tube, for all $x, y \in X$, $x - y \leqslant (\operatorname{esup}(x - y))\, e$, thus $x \leqslant y + (\operatorname{esup}(x - y))\, e \in X$. By monotonicity $\varphi(x) \leq \varphi(y + (\operatorname{esup}(x - y))\, e)$, by translation invariance $\varphi(x) \leq \varphi(y) + \operatorname{esup}(x - y)$. ■

Next example shows that the equivalence cannot be true for a generic subset X of E.

[6] See, e.g., [21].

Example 1 *Consider the anti-diagonal X of the real plane $\mathbb{R}^2$ endowed with the usual order and unit $(1,1)$. Set $\varphi(t,-t)=t^2$ for all $t\in\mathbb{R}$.*[7] *Clearly, φ is monotone and translation invariant on X, but*

$$\varphi(2,-2)-\varphi(0,0)=4>2=\operatorname{esup}((2,-2)-(0,0)).$$

Proposition 2 implies that φ cannot be extended to a monotone and translation invariant functional on the entire $\mathbb{R}^2$.

In particular, Proposition 2 implies that if a functional $\varphi: X\to\mathbb{R}$ admits a monotone and translation invariant extension to E, then φ satisfies (5). This observation suggests the following:

Definition 1 *A functional $\varphi: X\to\mathbb{R}$ is a* niveloid *if and only if*

$$\varphi(x)-\varphi(y)\leq\operatorname{esup}(x-y)\quad\forall x,y\in X. \tag{6}$$

A few remarks are in order.

- Although this is not the original definition of Dolecki and Greco [11] reported in the introduction, it is equivalent because of Proposition 2; in fact, Proposition 2 guarantees that a functional $\varphi: X\to\mathbb{R}$ on a tube is monotonic and translation invariant if and only if it satisfies (6).
- Given $\varphi: X\to\mathbb{R}$, define $\overline{\varphi}:-X\to\mathbb{R}$ by $\overline{\varphi}(y)=-\varphi(-y)$ for all $y\in -X$. It is easy to check that: $\overline{(\overline{\varphi})}=\varphi$; φ is monotone if and only if $\overline{\varphi}$ is monotone; φ is translation invariant if and only if $\overline{\varphi}$ is translation invariant; φ is a niveloid if and only if $\overline{\varphi}$ is a niveloid; finally, provided X is convex, φ is convex if and only if $\overline{\varphi}$ is concave. For example,

  $$\operatorname{einf}(x)=-\operatorname{esup}(-x)=\sup\{c\in\mathbb{R}: ce\leqslant x\}\quad\forall x\in E$$

  is a superlinear niveloid.
- φ is a niveloid if and only if $\varphi(y)-\varphi(x)\geq-\operatorname{esup}(x-y)=\operatorname{einf}(y-x)$ for all $x,y\in X$ if and only if $\operatorname{einf}(y-x)\leq\varphi(y)-\varphi(x)\leq\operatorname{esup}(y-x)$ for all $y,x\in X$.
- Niveloids are Lipschitz continuous of order 1. In fact

  $$\varphi(x)-\varphi(y)\leq\operatorname{esup}(x-y)\leq\operatorname{esup}(|x-y|)=\|x-y\|_e\quad\forall x,y\in X.$$
- In the risk measurement perspective of mathematical finance, (6) has a very natural interpretation: the additional reserve $\varphi(x)-\varphi(y)$ required to move from position y to position x cannot exceed the maximal additional loss $\operatorname{esup}(x-y)$.

3 Extensions of niveloids

In this section it is shown that $\varphi: X\to\mathbb{R}$ admits a monotone and translation invariant extension to E if and only if it is a niveloid. Actually, the next result is stronger since it explicitly describes the minimal and maximal niveloidal extensions of φ.

Theorem 1 *Let $\varphi: X\to\mathbb{R}$ be a niveloid. The functional defined on E by*

$$\hat{\varphi}(y)=\sup\{\varphi(x)+b: x\in X,\ b\in\mathbb{R},\ \text{and } x+be\leqslant y\}\quad\forall y\in E \tag{7}$$

[7] The elements of X can be seen as Bernoullian risks and φ is their variance relative to the uniform probability.

is the minimal niveloid on E that extends φ. Moreover, for each $y \in E$,

$$\hat{\varphi}(y) = \sup_{x \in X} (\varphi(x) + \operatorname{einf}(y - x)). \tag{8}$$

Analogously, the functional defined on E by

$$\check{\varphi}(y) = \inf \{\varphi(x) + b : x \in X,\ b \in \mathbb{R},\ \text{and } x + be \geqslant y\} \quad \forall y \in E \tag{9}$$

is the maximal niveloid on E that extends φ. Moreover, for each $y \in E$,

$$\check{\varphi}(y) = \inf_{x \in X} (\varphi(x) + \operatorname{esup}(y - x)). \tag{10}$$

Before entering the details of the proof notice that (8) and (10) imply $\hat{\varphi} = \overline{\check{\overline{\varphi}}}$ and $\check{\varphi} = \overline{\hat{\overline{\varphi}}}$.

Lemma 1 *If $\varphi : X \to \mathbb{R}$ is a translation invariant functional, then there exists a unique translation invariant extension $\tilde{\varphi}$ of φ to $\widetilde{X} = X + \mathbb{R}e$.*

Proof. If such an extension $\tilde{\varphi} : \widetilde{X} \to \mathbb{R}$ exists, for all $x \in X$ and $c \in \mathbb{R}$, it satisfies

$$\tilde{\varphi}(x + ce) = \tilde{\varphi}(x) + c = \varphi(x) + c. \tag{11}$$

In particular it is unique. It remains to be shown that (11) defines a translation invariant functional (that obviously extends φ) on $\widetilde{X}$.

If $x, y \in X$, $c, d \in \mathbb{R}$, and $x + ce = y + de$, then $x = y + (d - c)e$. In particular, $y \in X$ and $d - c \in \mathbb{R}$ are such that $y + (d - c)e \in X$, therefore

$$\varphi(x) + c = \varphi(y + (d - c)e) + c = \varphi(y) + d - c + c = \varphi(y) + d.$$

This proves that $\tilde{\varphi}$ is well defined by (11) on $\widetilde{X} = X + \mathbb{R}e$. If $x + ce \in \widetilde{X}$ (with $x \in X$ and $c \in \mathbb{R}$) and $d \in \mathbb{R}$, then $\tilde{\varphi}((x + ce) + de) = \tilde{\varphi}(x + ce + de) = \varphi(x) + c + d = \tilde{\varphi}(x + ce) + d$, that is, $\tilde{\varphi}$ is translation invariant. ■

Lemma 2 *The following statements are equivalent for a functional $\varphi : X \to \mathbb{R}$:*

(i) φ is translation invariant and its unique translation invariant extension $\tilde{\varphi}$ to $\widetilde{X}$ is monotone.

(ii) φ is translation invariant and $\tilde{\varphi}$ is a niveloid.

(iii) φ is a niveloid.

Proof. By Proposition 2, (i) implies (ii). (ii) trivially implies (iii). Finally, if φ is a niveloid, by Proposition 2 again, φ is translation invariant. Moreover, if $x, y \in X$ and $c, d \in \mathbb{R}$ are such that $x + de \leqslant y + de$, then $\varphi(x) - \varphi(y) \leq \operatorname{esup}(x - y)$ implies

$$\tilde{\varphi}(x + ce) - \tilde{\varphi}(y + de) = \varphi(x) - \varphi(y) + c - d \leq \operatorname{esup}(x - y) + c - d = \operatorname{esup}((x + ce) - (y + de)) \leq 0$$

that is $\tilde{\varphi}$ is monotone. ■

Proof of Theorem 1. Observe that

$$\{\varphi(x) + b : x \in X,\ b \in \mathbb{R}, \text{ and } x + be \leqslant y\} = \left\{\tilde{\varphi}(z) : z \in \widetilde{X} \text{ and } z \leqslant y\right\}.$$

Therefore

$$\hat{\varphi}(y) = \sup \left\{\tilde{\varphi}(z) : z \in \widetilde{X} \text{ and } z \leqslant y\right\} \quad \forall y \in E$$

from E to $[-\infty,\infty]$ is the minimal monotone extension of $\tilde{\varphi}$ to E ($\tilde{\varphi}$ is monotone by Lemma 2).

Moreover, choose $x \in X$, for each $y \in E$ there exists $c \in \mathbb{R}$ such that $ce \leqslant y - x$ (since e is a unit), thus $y \geqslant x + ce \in \widetilde{X}$ and $\left\{\tilde{\varphi}(z) : z \in \widetilde{X} \text{ and } z \leqslant y\right\} \neq \varnothing$, so that $\hat{\varphi}$ never takes value $-\infty$. Choose $x \in X$ and $n \in \mathbb{N}$ such that $x + ne \geqslant 0$ (eventually $x \geqslant -ne$), then, for each $y \in E$, $z \in \widetilde{X}$ and $z \leqslant y$ imply $z \leqslant \operatorname{esup}(y)\, e \leqslant x + ne + \operatorname{esup}(y)\, e$, so that $\tilde{\varphi}(z) \leq \varphi(x) + n + \operatorname{esup}(y)$. That is $\hat{\varphi}$ is real valued.

For each $y \in E$, $\left\{\tilde{\varphi}(z) : z \in \widetilde{X} \text{ and } z \leqslant y + ce\right\} = \left\{\tilde{\varphi}(z) : z \in \widetilde{X} \text{ and } z \leqslant y\right\} + c$ for all $c \in \mathbb{R}$, therefore $\hat{\varphi}$ is translation invariant, hence a niveloid.

Clearly, $\hat{\varphi}$ extends φ. If ψ is a niveloid that extends φ, $x \in X$, $b \in \mathbb{R}$, and $x + be \leqslant y$, then

$$\psi(y) \geq \psi(x + be) = \psi(x) + b = \varphi(x) + b$$

thus $\hat{\varphi}$ is minimal.

Moreover, for each $y \in E$,

$$\begin{aligned}
\hat{\varphi}(y) &= \sup\{\varphi(x) + b : x \in X,\ b \in \mathbb{R}, \text{ and } x + be \leqslant y\} \\
&= \sup\{\varphi(x) + b : x \in X,\ b \in \mathbb{R}, \text{ and } \operatorname{einf}(y - x) \geq b\} \\
&= \sup_{x \in X}(\varphi(x) + \operatorname{einf}(y - x)).
\end{aligned}$$

This proves the first part of Theorem 1. The second follows from analogous arguments. ■

Theorem 1 extends Theorem 5.2 of Dolecki and Greco to Archimedean Riesz spaces with unit. The merit of representations (8) and (10) is that they allow to explicitly express extensions $\hat{\varphi}$ and $\check{\varphi}$ in terms of φ. Next proposition provides some alternative explicit representations featuring the strict epigraph $\operatorname{epi}(\varphi)$ and ipograph $\operatorname{ipo}(\varphi)$ of φ.[8]

Proposition 3 *Let $\varphi : X \to \mathbb{R}$ be a niveloid. Then, for each $y \in E$,*

$$\begin{aligned}
\hat{\varphi}(y) &= \sup\{c \in \mathbb{R} : y - ce \in A_{\varphi}\} = \sup_{z \in A_{\varphi}}(\operatorname{einf}(y - z)) \\
\check{\varphi}(y) &= \inf\{c \in \mathbb{R} : y - ce \in A^{\varphi}\} = \inf_{z \in A^{\varphi}}(\operatorname{esup}(y - z))
\end{aligned}$$

where

$$\begin{aligned}
A_{\varphi} &= \{x - te : (x,t) \in \operatorname{ipo}(\varphi)\} + E_{+} \\
A^{\varphi} &= \{x - te : (x,t) \in \operatorname{epi}(\varphi)\} + E_{-}
\end{aligned}$$

in particular, $A_{\varphi} = \{\hat{\varphi} > 0\}$ and $A^{\varphi} = \{\check{\varphi} < 0\}$.

Proof. Let $\varphi : X \to \mathbb{R}$ be a niveloid. By (9)

$$\{y \in E : \check{\varphi}(y) < d\} = \left\{z \in \widetilde{X} : \tilde{\varphi}(z) < d\right\} + E_{-} \quad \forall d \in \mathbb{R}.$$

In particular,

$$\begin{aligned}
\{\check{\varphi} < 0\} &= \left\{z \in \widetilde{X} : \tilde{\varphi}(z) < 0\right\} + E_{-} && (12) \\
&= \{x - te : x \in X,\ t \in \mathbb{R}, \text{ and } \varphi(x) < t\} + E_{-} && (13) \\
&= \{x - te : (x,t) \in \operatorname{epi}(\varphi)\} + E_{-} = A^{\varphi}. && (14)
\end{aligned}$$

[8] Recall that $\operatorname{epi}(\varphi) = \{(x,t) \in X \times \mathbb{R} : t > \varphi(x)\}$ and $\operatorname{ipo}(\varphi) = \{(x,t) \in X \times \mathbb{R} : t < \varphi(x)\}$.

Therefore, for each $y \in E$,

$$\breve{\varphi}(y) = \inf\{c \in \mathbb{R} : c > \breve{\varphi}(y)\} = \inf\{c \in \mathbb{R} : y - ce \in \{\breve{\varphi} < 0\}\} \tag{15}$$
$$= \inf\{c \in \mathbb{R} : y - ce \leqslant z \text{ for some } z \in \{\breve{\varphi} < 0\}\} \tag{16}$$
$$= \inf\{c \in \mathbb{R} : \operatorname{esup}(y - z) \leq c \text{ for some } z \in \{\breve{\varphi} < 0\}\} \tag{17}$$
$$= \inf_{z \in \{\breve{\varphi} < 0\}} (\operatorname{esup}(y - z)). \tag{18}$$

which together with $\{\breve{\varphi} < 0\} = A^{\varphi}$ delivers the second part of the statement.[9] The first is proved with analogous arguments. ■

Notice that the chain of equalities (15)-(18) holds for any niveloid on any tube.

4 Convexity

As discussed in the introduction, convex niveloids play an important role in mathematical finance.[10] The next example shows that there exist niveloids satisfying the usual convexity condition

$$\varphi(x_0) \leq \textstyle\sum_1^n \lambda_i \varphi(x_i) \tag{19}$$

for all $x_0, x_1, ..., x_n \in X$ and all $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$ such that $\sum_1^n \lambda_i = 1$ and $x_0 = \sum_1^n \lambda_i x_i$, but that *do not* admit a convex niveloidal extension.

Example 2 *Consider the set $X = \{(0,0), (2,0), (0,2)\}$ in $\mathbb{R}^2$ and the niveloid $\varphi \equiv 0$ on X. Clearly φ satisfies (19). But, if there existed a convex niveloid ψ extending φ to $\mathbb{R}^2$, it would follow*

$$1 = \psi(1,1) - \psi(0,0) = \psi(1,1) - \varphi(0,0) = \psi(1,1) \leq \frac{1}{2}\psi(2,0) + \frac{1}{2}\psi(0,2) = \frac{1}{2}\varphi(2,0) + \frac{1}{2}\varphi(0,2) = 0.$$

As the next theorem shows, a strengthening of the defining condition for niveloids delivers the desired extension property.

Theorem 2 *Let $\varphi : X \to \mathbb{R}$. There exists a convex niveloid that extends φ to E if and only if*

$$\varphi(x_0) - \textstyle\sum_1^n \lambda_i \varphi(x_i) \leq \operatorname{esup}(x_0 - \sum_1^n \lambda_i x_i) \tag{20}$$

for all $x_0, x_1, ..., x_n \in X$ and all $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$ such that $\sum_1^n \lambda_i = 1$.

In this case,

$$\dot{\varphi}(y) = \inf\{c \in \mathbb{R} : y - ce \in \operatorname{co}(A^{\varphi})\} \quad \forall y \in E$$

is the maximal convex niveloid that extends φ.

Before entering the details of the proof, notice that:

- For $n = 1$, condition (20) reduces to the definition (6) of niveloid.
- For each $x_0, x_1, ..., x_n \in X$, the function

$$(\lambda_1, ..., \lambda_n) \mapsto \operatorname{esup}(x_0 - \textstyle\sum_1^n \lambda_i x_i) - (\varphi(x_0) - \sum_1^n \lambda_i \varphi(x_i))$$
$$= \operatorname{esup}(\textstyle\sum_1^n \lambda_i (x_0 - x_i)) - \sum_1^n \lambda_i (\varphi(x_0) - \varphi(x_i))$$

  is continuous and convex on the compact n-dimensional simplex and (20) amounts to require that its minimum is not negative. This allows to check the existence of a convex niveloidal extension by solving some finite dimensional optimization problems.

[9] The last equality follows from $\inf R = \inf(R + \mathbb{R}_+)$ for all $R \subseteq \mathbb{R}$.

[10] Analogously, in the theory of choice under uncertainty, concavity of the niveloid φ representing the preferences of a decision maker corresponds to uncertainty aversion (see [19] and [20]).

- Also (20), like (6), has a simple interpretation in terms of capital requirements: the additional reserve $\varphi(x_0) - \sum_1^n \lambda_i \varphi(x_i)$ required to make x_0 acceptable, once $\sum_1^n \lambda_i \varphi(x_i)$ has already been invested in a risk-free manner, cannot exceed the maximal additional loss $\operatorname{esup}(x_0 - \sum_1^n \lambda_i x_i)$ of x_0 relative to the diversified loss portfolio $\sum_1^n \lambda_i x_i$. In fact, since diversification cannot increase risk, $\sum_1^n \lambda_i \varphi(x_i)$ already provides a sufficient deposit for $\sum_1^n \lambda_i x_i$.

Lemma 3 *For a functional $\varphi : X \to \mathbb{R}$, condition (20) is equivalent to*

$$\varphi(x_0) \leq \sum_1^n \lambda_i \varphi(x_i) + \lambda_0 \tag{21}$$

for all $x_0, x_1, ..., x_n \in X$ and all $\lambda_0, \lambda_1, ..., \lambda_n \in \mathbb{R}$ such that $\lambda_1, ..., \lambda_n \geq 0$, $\sum_1^n \lambda_i = 1$, and $x_0 \leqslant \sum_1^n \lambda_i x_i + \lambda_0 e$.

Proof. For all $x_0, x_1, ..., x_n \in X$ and all $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$ such that $\sum_1^n \lambda_i = 1$,

$$x_0 \leqslant \sum_1^n \lambda_i x_i + \operatorname{esup}(x_0 - \sum_1^n \lambda_i x_i)\, e \tag{22}$$

so that (21) implies (20).

Conversely, for all $x_0, x_1, ..., x_n \in X$ and all $\lambda_0, \lambda_1, ..., \lambda_n \in \mathbb{R}$ such that $\lambda_1, ..., \lambda_n \geq 0$, $\sum_1^n \lambda_i = 1$, and $x_0 \leqslant \sum_1^n \lambda_i x_i + \lambda_0 e$, the relation $\operatorname{esup}(x_0 - \sum_1^n \lambda_i x_i) \leq \lambda_0$ holds. Then (20) delivers

$$\varphi(x_0) \leq \sum_1^n \lambda_i \varphi(x_i) + \operatorname{esup}(x_0 - \sum_1^n \lambda_i x_i) \leq \sum_1^n \lambda_i \varphi(x_i) + \lambda_0$$

that is, (21). ■

Proof of Theorem 2. If there exists a convex niveloid ψ extending φ to E, then (22) implies

$$\begin{aligned} \varphi(x_0) = \psi(x_0) &\leq \psi\left(\sum_1^n \lambda_i x_i + \operatorname{esup}(x_0 - \sum_1^n \lambda_i x_i)\, e\right) = \psi\left(\sum_1^n \lambda_i x_i\right) + \operatorname{esup}(x_0 - \sum_1^n \lambda_i x_i) \\ &\leq \sum_1^n \lambda_i \psi(x_i) + \operatorname{esup}(x_0 - \sum_1^n \lambda_i x_i) = \sum_1^n \lambda_i \varphi(x_i) + \operatorname{esup}(x_0 - \sum_1^n \lambda_i x_i) \end{aligned}$$

for all $x_0, x_1, ..., x_n \in X$ and all $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$ such that $\sum_1^n \lambda_i = 1$. Therefore (20) is necessary for the existence of a convex niveloidal extension.

Conversely, assume condition (20) is satisfied. By Lemma 2, $\tilde{\varphi}$ is a niveloid. By the chain of equalities (12)-(14), $A^\varphi = \{\tilde{\varphi} < 0\} + E_-$. Then, convexity of E_- implies

$$\operatorname{co}(A^\varphi) = \operatorname{co}(\{\tilde{\varphi} < 0\}) + E_-. \tag{23}$$

Moreover, $z \in \widetilde{X}$ and $z \leqslant y$ for some $y \in \operatorname{co}(\{\tilde{\varphi} < 0\})$ implies $z \in \{\tilde{\varphi} < 0\}$, that is

$$\widetilde{X} \cap \operatorname{co}(A^\varphi) = \{\tilde{\varphi} < 0\}. \tag{24}$$

In fact, $z = x_0 + c_0 e$ ($x_0 \in X$ and $c_0 \in \mathbb{R}$), and there exist $z_i = x_i + c_i e \in \{\tilde{\varphi} < 0\}$ ($x_i \in X$, $c_i \in \mathbb{R}$, $i = 1, ..., n$) and $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$ with $\sum_1^n \lambda_i = 1$ such that $z \leqslant \lambda_1 z_1 + \lambda_2 z_2 + ... + \lambda_n z_n$, that is, $x_0 \leqslant \sum_1^n \lambda_i x_i + \sum_1^n \lambda_i c_i e - c_0 e$. By (21), $\varphi(x_0) \leq \sum_1^n \lambda_i \varphi(x_i) + \sum_1^n \lambda_i c_i - c_0$, that is,

$$\tilde{\varphi}(z) = \varphi(x_0) + c_0 \leq \sum_1^n \lambda_i \varphi(x_i) + \sum_1^n \lambda_i c_i = \sum_1^n \lambda_i \tilde{\varphi}(z_i) < 0.$$

The first implication of (24) is that $\varnothing \neq \{\tilde{\varphi} \geq 0\} \subseteq E \setminus \operatorname{co}(A^\varphi)$, so that:

- $\varnothing \subset \operatorname{co}(A^\varphi) \subset E$;

- $\operatorname{co}(A^\varphi) = \operatorname{co}(A^\varphi) + E_-$, because of (23);

- $\operatorname{co}(A^{\varphi})$ is convex.

By Proposition 1, $\dot{\varphi}$ is a convex niveloid. If $x \in X$, then for each $c \in \mathbb{R}$ such that $x - ce \in \operatorname{co}(A^{\varphi})$, (24) implies $x - ce \in \{\tilde{\varphi} < 0\}$, then

$$\{c \in \mathbb{R} : x - ce \in \operatorname{co}(A^{\varphi})\} = \{c \in \mathbb{R} : x - ce \in \{\tilde{\varphi} < 0\}\}$$

and $\dot{\varphi}(x) = \tilde{\varphi}(x) = \varphi(x)$.[11] Thus $\dot{\varphi}$ extends φ, and (20) is sufficient for the existence of a convex niveloidal extension.

Finally, if ψ is a convex niveloid that extends φ, by Lemma 1, it extends $\tilde{\varphi}$, then $\{\psi < 0\} \supseteq \{\tilde{\varphi} < 0\}$, by convexity $\{\psi < 0\} \supseteq \operatorname{co}(\{\tilde{\varphi} < 0\})$, by monotonicity $\{\psi < 0\} \supseteq \operatorname{co}(A^{\varphi})$. Therefore

$$\dot{\varphi}(y) = \inf\{c \in \mathbb{R} : y - ce \in \operatorname{co}(A^{\varphi})\} \geq \inf\{c \in \mathbb{R} : y - ce \in \{\psi < 0\}\} = \psi(y)$$

for all $y \in E$. ■

In the very special and very important case in which X is convex, (20) is equivalent to the convexity of φ.

Proposition 4 *If X is convex and $\varphi : X \to \mathbb{R}$ is a niveloid, then (20) is equivalent to the convexity of φ in the usual sense, and $\dot{\varphi} = \check{\varphi}$.*

Proof. Let X and φ be convex. In this case $\widetilde{X} = X + \mathbb{R}e$ is convex, and

$$\begin{aligned}\tilde{\varphi}(\lambda z_1 + (1-\lambda) z_2) &= \varphi(\lambda x_1 + (1-\lambda) x_2) + \lambda c_1 + (1-\lambda) c_2 \\ &\leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) + \lambda c_1 + (1-\lambda) c_2 = \lambda\tilde{\varphi}(z_1) + (1-\lambda)\tilde{\varphi}(z_2)\end{aligned}$$

for all $z_i = x_i + c_i e$ ($x_i \in X$, $c_i \in \mathbb{R}$, $i = 1, 2$) and all $\lambda \in [0, 1]$. So that $\tilde{\varphi}$ is convex too. In turn, this implies that $\{\check{\varphi} < 0\} = \{\tilde{\varphi} < 0\} + E_-$ is convex and so is $\check{\varphi}$. Thus φ admits a convex extension to E and it satisfies (20).[12] Maximality of $\check{\varphi}$ and $\dot{\varphi}$ delivers $\dot{\varphi} = \check{\varphi}$. ■

Finally, if X is convex and it has nonempty interior, a convex niveloid $\varphi : X \to \mathbb{R}$ also admits a minimal niveloidal extension; this can be shown adapting the proof of Theorem 5 below.

5 Fenchel-Moreau duality

A *Radon measure* is a positive linear functional on E.[13] A *Radon probability measure* is a Radon measure p such that $\langle e, p\rangle = 1$. Denote by Δ the set of Radon probability measures. Clearly, Δ is convex and, being weak* closed and norm bounded, it is also weak* compact.

Theorem 3 *Let $\varphi : X \to \mathbb{R}$ satisfy (20). Then:*

(i) $\varphi^*_{|\Delta}$ is the minimal function $\alpha : \Delta \to (-\infty, \infty]$ such that

$$\varphi(x) = \sup_{p \in \Delta}(\langle x, p\rangle - \alpha(p)) \quad \forall x \in X. \tag{25}$$

Moreover, $\partial\varphi(x) \cap \Delta = \arg\max_{p\in\Delta}(\langle x, p\rangle - \varphi^*(p))$ for all $x \in X$.

[11] Remember the observation at the end of Section 3.

[12] Conversely, Theorem 2 guarantees that if φ satisfies (20), then it is convex on X (where it coincides with $\dot{\varphi}$).

[13] Notice that if f is a Radon measure, monotonicity implies $\sup_{x\in U_1(0)}\langle x, f\rangle = \langle e, f\rangle$ since $U_1(0) = [-e, e]$, in particular $f \in E^*$ and $\|f\|_e^* = f(e)$.

(ii) $\dot{\varphi}^*(p) = \varphi^*(p)$ *for all* $p \in \Delta$ *and* $\dot{\varphi}^*(f) = \infty$ *if* $f \in E^* \setminus \Delta$. *In particular,*

$$\dot{\varphi}(y) = \max_{p \in \Delta} \left(\langle y, p \rangle - \varphi^*(p) \right) \quad \forall y \in E. \tag{26}$$

(iii) *If* $X + \mathbb{R}e = E$ *and* $\alpha : \Delta \to (-\infty, \infty]$ *is such that (25) holds, then*

$$\varphi^*(p) = \sup \{ \langle y, p \rangle : y \in E \text{ and } \langle y, \cdot \rangle \leq \alpha(\cdot) \} \quad \forall p \in \Delta. \tag{27}$$

In particular, $\varphi^*_{|\Delta}$ *is the unique convex and weakly* lower semicontinuous* $\alpha : \Delta \to (-\infty, \infty]$ *such that (25) holds.*

(iv) *If* $\alpha : \Delta \to (-\infty, \infty]$ *is such that (25) holds, then* $\varphi(x) = \sup_{\{p \in \Delta : \alpha(p) \leq c - \dot{\varphi}(0)\}} (\langle x, p \rangle - \alpha(p))$ *for all* $x \in X$ *such that* $\operatorname{esup}(x) - \operatorname{einf}(x) < c$.

(v) *If there exists* $k \in \mathbb{R}$ *such that* $ke \in X$ *and* $\varphi(ke) = k$, *then* $\dot{\varphi}(0) = 0$ *and* $\varphi^*(p) \geq 0$ *for all* $p \in \Delta$. *Moreover,* $\partial \varphi(ke) \cap \Delta = \left\{ \varphi^*_{|\Delta} = 0 \right\} = \arg\min_{p \in \Delta} \varphi^*(p)$.

Notice that to evaluate the Fenchel conjugate $\varphi^*(p) = \sup_{x \in X} \{\langle x, p \rangle - \varphi(x)\}$ it is not necessary to know $\dot{\varphi}$, therefore (26) describes an alternative (dual) way to construct $\dot{\varphi}$.

The proof of Theorem 3 starts with a simple application of the generalized Moreau duality. Let F be a nonempty subset of $\mathbb{R}^X$ and $\varphi : X \to \mathbb{R}$ be a function. The F-*subdifferential* at $x \in X$ of φ is

$$\partial^F \varphi(x) = \{ f \in F : \varphi(y) - \varphi(x) \geq \langle y, f \rangle - \langle x, f \rangle \text{ for all } y \in X \}$$

where $\langle \cdot, f \rangle = f(\cdot)$. The (lower) F-*conjugate* of φ is the extended real valued function

$$\varphi^F(f) = \sup_{x \in X} \{ \langle x, f \rangle - \varphi(x) \} \quad \forall f \in F.$$

Notice that when $F = E^*$, then $\partial^{E^*} \varphi = \partial \varphi$ and $\varphi^{E^*} = \varphi^*$ are the usual *subdifferential* and *conjugate* of convex analysis.

Lemma 4 *If* $\partial^F \varphi(x) \neq \varnothing$ *for all* $x \in X$, *then*

$$\varphi(x) = \max_{f \in F} \left(\langle x, f \rangle - \varphi^F(f) \right)$$
$$\partial^F \varphi(x) = \arg\max_{f \in F} \left(\langle x, f \rangle - \varphi^F(f) \right)$$

for all $x \in X$. *Moreover,* φ^F *is the minimal* $\alpha : F \to (-\infty, \infty]$ *such that*

$$\varphi(x) = \sup_{f \in F} \left(\langle x, f \rangle - \alpha(f) \right) \quad \forall x \in X.$$

Proof. By definition, $\varphi(x) \geq \langle x, f \rangle - \varphi^F(f)$ for all $x \in X$ and $f \in F$. Moreover,

$$f \in \partial^F \varphi(x) \Longleftrightarrow \varphi^F(f) = \langle x, f \rangle - \varphi(x) \Longleftrightarrow \varphi(x) = \langle x, f \rangle - \varphi^F(f)$$

which proves the first part of the statement. Moreover, if $\varphi(x) = \sup_{f \in F} (\langle x, f \rangle - \alpha(f))$ for all $x \in X$, then $\alpha(f) \geq \langle x, f \rangle - \varphi(x)$ for all $x \in X$ and $f \in F$, that is $\alpha(f) \geq \varphi^F(f)$ for all $f \in F$. ∎

Proof of Theorem 3. (i) Since $\varphi : X \to \mathbb{R}$ satisfies (20), $\dot{\varphi}$ is a convex niveloid that extends φ to E. Since $\dot{\varphi}$ is continuous, at each $y \in E$ the subdifferential $\partial \dot{\varphi}(y)$ is not empty, and it is contained in Δ because $\dot{\varphi}$ is monotone and translation invariant. A fortiori, for each $x \in X$, $\partial^\Delta \varphi(x) = \partial \varphi(x) \cap \Delta \supseteq \partial \dot{\varphi}(x)$ is not empty. Lemma 4 concludes.

(ii) For all $y \in E$, set $\psi(y) = \sup_{p\in\Delta}(\langle y,p\rangle - \varphi^*(p))$, ψ is a convex niveloid on E (that extends φ). Point (i) applied to ψ guarantees that $\psi^*(p) \le \varphi^*(p)$ for all $p \in \Delta$. Maximality of $\dot{\varphi}$ as a convex niveloidal extension guarantees that $\psi \le \dot{\varphi}$. Therefore

$$\psi^*(p) \le \varphi^*(p) = \sup_{x\in X}(\langle x,p\rangle - \varphi(x)) \le \sup_{y\in E}(\langle y,p\rangle - \dot{\varphi}(y)) \le \sup_{y\in E}(\langle y,p\rangle - \psi(y)) = \psi^*(p)$$

that is $\psi^*(p) = \varphi^*(p) = \dot{\varphi}^*(p)$ for all $p \in \Delta$. Point (ii) follows because $\dot{\varphi}$ is a niveloid on E.

(iii) Since $E = X + \mathbb{R}e = \widetilde{X}$, then $\tilde{\varphi} = \dot{\varphi}$ and, by the previous point, $\tilde{\varphi}^*(p) = \varphi^*(p)$ for all $p \in \Delta$. Let $\alpha : \Delta \to (-\infty,\infty]$ be such that $\varphi(x) = \sup_{p\in\Delta}(\langle x,p\rangle - \alpha(p))$ for all $x \in X$. Then, for all $y \in E$,

$$\tilde{\varphi}(y) = \sup_{p\in\Delta}(\langle y,p\rangle - \alpha(p)) = \alpha^*(y)$$

when the dual pair (E, E^*) is considered and α is extended to ∞ outside Δ. Point (iii) follows from the implied equality $\tilde{\varphi}^* = \alpha^{**}$ and the properties of conjugation (see, e.g., [13, p.102] and [21, p.77]).

(iv) If $\mathrm{esup}(x) - \mathrm{einf}(x) < c$, there exists $\varepsilon > 0$ such that $\mathrm{esup}(x) - \mathrm{einf}(x) + \varepsilon < c$. For all $p \in \Delta$ such that $\alpha(p) > c - \dot{\varphi}(0)$,

$$\begin{aligned}\varphi(x) - \varepsilon &\ge \dot{\varphi}(\mathrm{einf}(x)e) - \varepsilon = \dot{\varphi}(0) + \mathrm{einf}(x) - \varepsilon > \dot{\varphi}(0) - c + \mathrm{esup}(x) > -\alpha(p) + \langle \mathrm{esup}(x)e, p\rangle \\ &\ge \langle x,p\rangle - \alpha(p).\end{aligned}$$

On the other hand,

$$\varphi(x) = \sup_{p\in\Delta}(\langle x,p\rangle - \alpha(p)) = \max\left(\sup_{p\in\{\alpha > c-\dot{\varphi}(0)\}}(\langle x,p\rangle - \alpha(p)), \sup_{p\in\{\alpha \le c-\dot{\varphi}(0)\}}(\langle x,p\rangle - \alpha(p))\right)$$

which concludes the proof of this point, since $\sup_{p\in\{\alpha > c-\dot{\varphi}(0)\}}(\langle x,p\rangle - \alpha(p)) \le \varphi(x) - \varepsilon$.

(v) The assumption $\varphi(ke) = k$ implies $\dot{\varphi}(0) = \dot{\varphi}(0) + k - k = \dot{\varphi}(ke) - k = \varphi(ke) - k = 0$. Moreover,

$$\varphi^*(p) = \sup_{x\in X}\{\langle x,p\rangle - \varphi(x)\} \ge \langle ke,p\rangle - \varphi(ke) = 0 \quad \forall p \in \Delta.$$

By point (i),

$$\partial\varphi(ke) \cap \Delta = \arg\max_{p\in\Delta}(\langle ke,p\rangle - \varphi^*(p)) = \arg\max_{p\in\Delta}(k - \varphi^*(p)) = \arg\min_{p\in\Delta}\varphi^*(p).$$

Finally, $\max_{p\in\Delta}(\langle ke,p\rangle - \varphi^*(p)) = \varphi(ke) = k$ implies $\min_{p\in\Delta}\varphi^*(p) = 0$, (v) is proved. ■

6 Daniell-Stone extensions

In this section, $X = L$ is a Stone vector lattice in $B(S)$ and $E = B(S, \sigma(L))$ is the space of bounded functions on S that are measurable with respect to the σ-algebra $\sigma(L)$ generated by L.

A functional $\varphi : L \to \mathbb{R}$ has the *Lebesgue property* if $\varphi(x_n) \to \varphi(x)$ whenever x_n is a norm bounded sequence in L which converges pointwise to x in L. The set L^σ of linear functionals on L that have the Lebesgue property is a vector sublattice of L^* and the map

$$\begin{array}{rccc} T: & ca(S,\sigma(L)) & \to & L^\sigma \\ & \mu & \mapsto & \int_S d\mu \end{array}$$

that associates to each $\mu \in ca(S,\sigma(L))$ the restriction to L of the integral with respect to μ is a linear lattice isomorphism, and an isometry when $ca(S,\sigma(L))$ is endowed with the total variation norm.[14]

[14] See, e.g., [7, p.909-910] and notice that *a fortiori* T is a homeomorphism when both spaces are endowed with their weak topologies.

The elements of $\Delta^\sigma = \Delta \cap L^\sigma$ are called *Daniell-Stone integrals*. By the eponymous theorem, for each $q \in \Delta^\sigma$ there exists a unique probability measure Q on $\sigma(L)$ such that $\langle x, q \rangle = E_Q[x]$ for all $x \in L$, that is, $Q = T^{-1}(q)$. Denote by $\mathcal{Q}$ the set of all probability measures on $\sigma(L)$ and by q in Δ^σ the image $T(Q)$ of Q in $\mathcal{Q}$. Observe that $Q \mapsto q = T(Q)$ is a bijection between $\mathcal{Q}$ and Δ^σ.

Proposition 5 *Let L be a Stone vector lattice in $B(S)$ and $\varphi : L \to \mathbb{R}$ be a convex niveloid. The following conditions are equivalent:*

(i) φ has the Lebesgue property.

(ii) $\{p \in \Delta : \varphi^*(p) \leq c\}$ is a weakly compact subset of Δ^σ for all $c \in \mathbb{R}$.

In this case, the only convex niveloid with the Lebesgue property that extends φ to $B(S, \sigma(L))$ is

$$\varphi^\sigma(y) = \max_{Q \in \mathcal{Q}} (E_Q[y] - \varphi^*(q)) \quad \forall y \in B(S, \sigma(L)).$$

Proof. First notice that, by point (ii) of Theorem 3 with $E = L = X$, $\varphi^*(f) = \infty$ if $f \in L^* \setminus \Delta$. Therefore $\{p \in \Delta : \varphi^*(p) \leq c\} = \{\varphi^* \leq c\}$.

(i) $\Rightarrow$ (ii) If x_n is a norm bounded sequence in L which converges pointwise to x in L, then $z_n = x_n - x$ is a norm bounded sequence in L which converges pointwise to 0. Arbitrarily choose $c \in \mathbb{R}$. For all $p \in \{\varphi^* \leq c\}$, $n \in \mathbb{N}$, and $\lambda > 0$,

$$\langle \lambda |z_n|, p \rangle - \varphi(\lambda |z_n|) \leq \varphi^*(p) \leq c \Longrightarrow |\langle z_n, p \rangle| \leq \langle |z_n|, p \rangle \leq \frac{c}{\lambda} + \frac{\varphi(\lambda |z_n|)}{\lambda}$$

so that

$$0 \leq \sup_{p \in \{\varphi^* \leq c\}} |\langle z_n, p \rangle| \leq \frac{c + \varphi(\lambda |z_n|)}{\lambda}.$$

Passing to the limit delivers $0 \leq \overline{\lim}_n \sup_{p \in \{\varphi^* \leq c\}} |\langle z_n, p \rangle| \leq \lambda^{-1}(c + \varphi(0))$, since φ has the Lebesgue property. But this is true for all $\lambda > 0$, therefore

$$\sup_{p \in \{\varphi^* \leq c\}} |\langle z_n, p \rangle| \to 0 \quad \text{as} \quad n \to \infty. \tag{28}$$

This implies that the elements of $\{\varphi^* \leq c\}$ have the Lebesgue property, that is, $\{\varphi^* \leq c\} \subseteq \Delta^\sigma$. Moreover, if x_n is a norm bounded disjoint sequence in L,[15] then it is pointwise convergent to 0, and (28) implies that $\{\varphi^* \leq c\}$ is weakly relatively compact by the Grothendieck Theorem.[16] Since φ^* is weakly* lower semicontinuous on L^*, then $\{\varphi^* \leq c\}$ is weakly closed in L^* and hence weakly compact.

(ii) $\Rightarrow$ (i) Consider the function defined by

$$\alpha(\mu) = \begin{cases} \varphi^*(T(\mu)) & \mu \in \mathcal{Q} \\ \infty & \mu \notin \mathcal{Q} \end{cases} \tag{29}$$

from $ba(S, \sigma(L))$ to $(-\infty, \infty]$. For each $c \in \mathbb{R}$, $\{\alpha \leq c\} = T^{-1}\{\varphi^* \leq c\}$ is a weakly compact subset of $\mathcal{Q}$ (in $ca(S, \sigma(L))$ and so in $ba(S, \sigma(L))$), in particular, α is weakly lower semicontinuous and convex.

Let y_n be a norm bounded sequence in $B(S, \sigma(L))$ which converges pointwise to y_0. By point (iv) of Theorem 3, there exists $c \in \mathbb{R}$ such that

$$\varphi^\sigma(y_m) = \max_{Q \in \{\alpha \leq c\}} (E_Q[y_m] - \alpha(Q)) \quad \forall m \in \mathbb{N}_0.$$

[15] *Disjoint* means that $|x_l| \wedge |x_m| = 0$ for all $l \neq m$ in $\mathbb{N}$.

[16] In Theorem 4.41 of [2] the assumption of norm completeness is redundant.

If $y_n \downarrow y_0$, $Q \mapsto E_Q[y_m] - \alpha(Q)$ is weakly upper semicontinuous for all $m \in \mathbb{N}_0$, and by the Dominated Convergence Theorem $E_Q[y_n] - \alpha(Q) \downarrow E_Q[y_0] - \alpha(Q)$ for all $Q \in \{\alpha \leq c\}$. By the Dini-Cartan Lemma,[17] $\sup_{Q\in\{\alpha\leq c\}}(E_Q[y_n] - \alpha(Q)) \downarrow \sup_{Q\in\{\alpha\leq c\}}(E_Q[y_0] - \alpha(Q))$, that is $\varphi^\sigma(y_n) \downarrow \varphi^\sigma(y_0)$. If $y_n \uparrow y_0$, then $\varphi^\sigma(y_n) \uparrow k \leq \varphi^\sigma(y_0)$, but $\varphi^\sigma(y_0) = E_{Q_0}[y_0] - \alpha(Q_0)$ for some $Q_0 \in \{\alpha \leq c\}$, and by the Dominated Convergence Theorem again $\varphi^\sigma(y_0) = \lim_n (E_{Q_0}[y_n] - \alpha(Q_0)) \leq \lim_n \max_{Q\in\{\alpha\leq c\}}(E_Q[y_n] - \alpha(Q)) = k$. Monotonicity of φ^σ and σ-Dedekind completeness of $B(S, \sigma(L))$ imply that φ^σ has the Lebesgue property, and so does $\varphi = \varphi^\sigma_{|L}$.

Finally, if ψ is a convex niveloid with the Lebesgue property that extends φ to $B(S, \sigma(L))$ and $\beta = \psi^*$, then

$$\varphi(x) = \max_{Q\in\mathcal{Q}}(E_Q[x] - \beta(Q)) = \max_{q\in\Delta}(\langle x, q\rangle - \gamma(q)) \quad \forall x \in L$$

where γ is defined by

$$\gamma(f) = \begin{cases} \beta\left(T^{-1}(f)\right) & f \in \Delta^\sigma \\ \infty & f \notin \Delta^\sigma \end{cases} \tag{30}$$

from L^* to $(-\infty, \infty]$. By (ii) $\{\beta \leq c\}$ is weakly compact in $ba(S, \sigma(L))$ for all $c \in \mathbb{R}$, and this implies that γ is weakly* lower semicontinuous since $\{\gamma \leq c\} = T\{\beta \leq c\}$. By point (iii) of Theorem 3 with $E = L = X$, $\gamma = \varphi^*$. Positions (29) and (30) deliver $\alpha = \beta$ whence $\varphi^\sigma = \psi$. ■

7 Niveloids on boxes

If I is an interval in $\mathbb{R}$, the *box* with diagonal I is the set

$$E(I) = \{x \in E : \operatorname{einf}(x), \operatorname{esup}(x) \in I\}.$$

The closed unit ball, the open unit ball, and the positive cone E_+ are boxes.[18] The financial intuition is again immediate; $E\left(\left[\underline{\ell}, \overline{\ell}\right]\right)$, for example, is the set of losses that are bounded below by $\underline{\ell}$ and above by $\overline{\ell}$.

Notation *Throughout this section, I is a nonsingleton interval in $\mathbb{R}$ and $X = E(I)$.*[19]

Now consider the following decision theoretic setup *à la* Anscombe-Aumann: (S, Σ) is a measurable space, there is a finite set of n consequences, and each action a determines in each state s an objective probability distribution $a(s) = (a_1(s), ..., a_n(s))$ over consequences; that is, the set A of actions can be identified with the set of measurable functions from S to the n dimensional simplex. Denote by u_i the utility of consequence $i = 1, ..., n$ and set

$$u_a(s) = \sum_{i=1}^n u_i a_i(s) \quad \forall a \in A, s \in S.$$

Taking $E = B(S, \Sigma)$ and $I = [\min(u_1, ..., u_n), \max(u_1, ..., u_n)]$, it is easy to check that $E(I) = \{u_a\}_{a\in A}$ and that

$$u_{\lambda a+(1-\lambda)b} = \lambda u_a + (1-\lambda) u_b \quad \forall a, b \in A, \forall \lambda \in [0, 1]. \tag{31}$$

This is the motivating example for the last exercise of this paper that consists in giving conditions on $\varphi : E(I) \to \mathbb{R}$ that guarantee the existence of a concave niveloidal extension and *only* rely on convex

[17] See, e.g., [10, p. 98].

[18] For $I = [-1, 1]$, $(-1, 1)$, and $[0, \infty)$, respectively.

[19] If I were a singleton, then X would be a singleton, and distinctions should be introduced in the proofs to discuss functionals defined on a single point.

combinations of elements of $E(I)$. In view of (31), these conditions can be directly translated into conditions on actions,[20] while conditions such as (20) cannot: convex combinations of elements of $E(I)$ correspond to randomizations of actions, while the interpretation of general linear combinations is less natural in this decision theoretic framework.

Definition 2 *A functional $\varphi : X \to \mathbb{R}$ is* translation quasinvariant *if and only if whenever*

$$\varphi(\lambda x + (1-\lambda) ke) \geq \varphi(\lambda y + (1-\lambda) ke) \tag{32}$$

holds for some $\lambda \in (0,1)$, $x, y \in X$, and $k \in \mathbb{R}$ such that $ke \in X$, then it holds for all $k \in \mathbb{R}$ such that $ke \in X$.

Translation quasinvariance is implied by translation invariance (like quasiconcavity is implied by concavity) and preserved by increasing transformations. But, clearly, it is weaker and – differently from translation invariance – it only involves convex combinations of elements of X.

Theorem 4 *Let I be a nonsingleton interval in $\mathbb{R}$. The following statements are equivalent for a functional $\varphi : E(I) \to \mathbb{R}$.*

(i) φ is a niveloid.

(ii) φ is monotone, translation quasinvariant, and $\varphi(ke) - \varphi(he) = k - h$ for all $h, k \in I$.

(iii) φ is monotone and translation invariant.

In this case, φ is concave if and only if

$$\varphi(\lambda y + (1-\lambda) x) \geq \varphi(x)$$

for all $x, y \in E(I)$ such that $\varphi(y) = \varphi(x)$ and all $\lambda \in (0,1)$; moreover, also $\hat{\varphi}$ is concave.

First observe that $E(I)$ is a convex sublattice of E.

Proposition 6 *$E(I)$ is convex, and $[x \wedge y, x \vee y] \subseteq E(I)$ for all $x, y \in E(I)$.*

Proof. Since $\operatorname{esup}(\cdot)$ is convex and $\operatorname{einf}(\cdot)$ is concave, for each $x, y \in E(I)$ and $\lambda \in [0,1]$,

$$\begin{aligned} \lambda \operatorname{einf}(x) + (1-\lambda)\operatorname{einf}(y) &\leq \operatorname{einf}(\lambda x + (1-\lambda) y) \\ &\leq \operatorname{esup}(\lambda x + (1-\lambda) y) \leq \lambda \operatorname{esup}(x) + (1-\lambda)\operatorname{esup}(y) \end{aligned}$$

and the first and the last term of the chain of inequalities belong to I, thus all terms do, because I is an interval. Next observe that $\operatorname{esup}(\cdot)$ is maxitive and $\operatorname{einf}(\cdot)$ is minitive,[21] therefore for each $x, y \in X$ and all $z \in [x \wedge y, x \vee y]$

$$\operatorname{einf}(x) \wedge \operatorname{einf}(y) = \operatorname{einf}(x \wedge y) \leq \operatorname{einf}(z) \leq \operatorname{esup}(z) \leq \operatorname{esup}(x \vee y) = \operatorname{esup}(x) \vee \operatorname{esup}(y)$$

again the first and the last term of the chain of inequalities belong to I, thus all terms do. ■

Lemma 5 *Let $0 \in I^o$. Then $\varphi : E(I) \to \mathbb{R}$ is translation invariant if and only if*

$$\varphi(\lambda x + (1-\lambda) ke) = \varphi(\lambda x) + (1-\lambda) k \tag{33}$$

for all $x, ke \in E(I)$ and all $\lambda \in (0,1)$.

[20] E.g., quasiconcavity of φ on $E(I)$ amounts to uncertainty aversion on A.

[21] That is, for each $x, y \in E(I)$, $\operatorname{esup}(x \vee y) = \operatorname{esup}(x) \vee \operatorname{esup}(y)$ and $\operatorname{einf}(x \wedge y) = \operatorname{einf}(x) \wedge \operatorname{einf}(y)$.

Proof. Assume (33) holds (and notice that $0_{\mathbb{R}} \in I^o \iff 0_E \in E(I)^o$).

Step 1. If $I = (a, b)$, then φ is translation invariant.

Proof of Step 1. Let $x \in E(I)$ and $c > 0$ be such that $x + ce \in E(I)$. Then $a < \operatorname{einf}(x), \operatorname{esup}(x + ce) < b$, and there exist $\lambda \in (0,1)$ such that $\dfrac{x}{\lambda}, \dfrac{x + ce}{\lambda} \in E(I)$. Fix such a λ and fix $\varepsilon > 0$ such that $[-\varepsilon e, \varepsilon e] \in E(I)$. Choose $n \geq 2$ such that $\dfrac{\frac{1}{n}c}{1-\lambda} < \varepsilon$, thus $\dfrac{\frac{1}{n}c}{1-\lambda} e \in E(I)$. By convexity,

$$\frac{x + \frac{m}{n}ce}{\lambda} = \frac{m}{n}\frac{x + ce}{\lambda} + \left(1 - \frac{m}{n}\right)\frac{x}{\lambda} \in E(I) \quad \forall m = 0, 1, ..., n-1.$$

Then

$$\varphi\left(x + \frac{m+1}{n}ce\right) = \varphi\left(x + \frac{m}{n}ce + \frac{1}{n}ce\right) = \varphi\left(\lambda\left(\frac{x + \frac{m}{n}ce}{\lambda}\right) + (1-\lambda)\frac{\frac{1}{n}c}{1-\lambda}e\right)$$

but $\dfrac{x + \frac{m}{n}ce}{\lambda}$ and $\dfrac{\frac{1}{n}c}{1-\lambda}e \in E(I)$. By (33),

$$\varphi\left(x + \frac{m+1}{n}ce\right) = \varphi\left(\lambda\left(\frac{x + \frac{m}{n}ce}{\lambda}\right)\right) + (1-\lambda)\frac{\frac{1}{n}c}{1-\lambda} = \varphi\left(x + \frac{m}{n}ce\right) + \frac{c}{n} \quad \forall m = 0, 1, ..., n-1.$$

and

$$\varphi(x + ce) - \varphi(x) = \sum_{m=0}^{n-1}\left(\varphi\left(x + \frac{m+1}{n}ce\right) - \varphi\left(x + \frac{m}{n}ce\right)\right) = \sum_{m=0}^{n-1}\frac{c}{n} = c$$

as wanted.[22] □

Step 2. If $I = (a, b]$, then φ is translation invariant.

Proof of Step 2. Let $x \in E(I)$ and $c > 0$ be such that $x + ce \in E(I)$. Then $a < \operatorname{einf}(x)$ and $\operatorname{esup}(x + ce) \leq b$. Choose $n \geq 2$ such that $b - \dfrac{c}{n} > 0$ and $\operatorname{einf}\left(\dfrac{b}{b - \frac{c}{n}}x\right) > a$. Set $\lambda = \dfrac{b - \frac{c}{n}}{b} \in (0,1)$ so that $\dfrac{\frac{1}{n}c}{1-\lambda} = b \in I$. For each $m = 0, 1, ..., n-1$,

$$a < \operatorname{einf}(x) \leq \operatorname{einf}\left(x + \frac{m}{n}ce\right) \leq \operatorname{esup}\left(x + \frac{m}{n}ce\right) \leq b - \frac{c}{n} < b. \tag{34}$$

Divide all the terms by λ to obtain

$$a < \operatorname{einf}\left(\frac{x}{\lambda}\right) \leq \operatorname{einf}\left(\frac{x + \frac{m}{n}ce}{\lambda}\right) \leq \operatorname{esup}\left(\frac{x + \frac{m}{n}ce}{\lambda}\right) \leq \frac{b - \frac{c}{n}}{\lambda} = b$$

and hence $\dfrac{x + \frac{m}{n}ce}{\lambda} \in E(I)$ for all $m = 0, 1, ..., n-1$. Then, since $\dfrac{\frac{1}{n}c}{1-\lambda}e = be \in E(I)$,

$$\begin{aligned}\varphi(x + ce) &= \varphi\left(x + \frac{n-1}{n}ce + \frac{1}{n}ce\right) = \varphi\left(\lambda\left(\frac{x + \frac{n-1}{n}ce}{\lambda}\right) + (1-\lambda)\frac{\frac{1}{n}c}{1-\lambda}e\right) \\ &= \varphi\left(\lambda\left(\frac{x + \frac{n-1}{n}ce}{\lambda}\right)\right) + (1-\lambda)\frac{\frac{1}{n}c}{1-\lambda} = \varphi\left(x + \frac{n-1}{n}ce\right) + \frac{c}{n}.\end{aligned}$$

Also, by (34), $x, x + \frac{n-1}{n}e \in E((a, b)) \subseteq E((a, b])$, and by Step 1,

$$\varphi(x + ce) = \varphi\left(x + \frac{n-1}{n}ce\right) + \frac{c}{n} = \varphi(x) + \frac{n-1}{n}c + \frac{c}{n}$$

as wanted. □

[22] Actually, this proves that the lemma is true for any open convex subset X of E containing 0.

Step 3. If $I = [a, b)$, then φ is translation invariant.

Proof of Step 3. Notice that $-I = (-b, -a]$ contains 0 in its interior. Moreover, $-E(I) = E(-I)$ and $\overline{\varphi} : E((-b, -a]) \to \mathbb{R}$ satisfies (33). In fact, for each $x, ke \in -E(I)$ and each $\lambda \in (0,1)$, $-x, -ke \in E(I)$, then

$$\begin{aligned}\overline{\varphi}(\lambda x + (1-\lambda) ke) &= -\varphi(-(\lambda x + (1-\lambda) ke)) = -\varphi(\lambda(-x) + (1-\lambda)(-k) e) \\ &= -\varphi(\lambda(-x)) - (1-\lambda)(-k) = \overline{\varphi}(\lambda x) + (1-\lambda) k.\end{aligned}$$

Step 2 implies that $\overline{\varphi}$ is translation invariant, and so is φ. □

Step 4. If $I = [a, b]$, then φ is translation invariant.

Proof of Step 4. If $I = [a, b]$, $c > 0$, and $x, x + ce \in E(I)$, then $x, x + \frac{c}{2}e \in E([a, b))$ and $x + \frac{c}{2}e, x + ce \in E((a, b])$. By the previous steps, φ is translation invariant both on $E([a, b))$ and on $E((a, b])$. Therefore

$$\varphi(x + ce) = \varphi\left(\left(x + \frac{c}{2}e\right) + \frac{c}{2}e\right) = \varphi\left(x + \frac{c}{2}e\right) + \frac{c}{2} = \varphi(x) + \frac{c}{2} + \frac{c}{2}.$$

as wanted. □

Steps 1-4 prove that if (33) holds, then φ is translation invariant, the converse is trivial. ■

Lemma 6 *$\varphi : E(I) \to \mathbb{R}$ is translation invariant if and only if*

$$\varphi(\lambda x + (1-\lambda) ke) = \varphi(\lambda x + (1-\lambda) he) + (1-\lambda)(k-h) \tag{35}$$

for all $x, ke, he \in E(I)$ and $\lambda \in (0,1)$.

Proof. Fix $h \in I^o$. For all $x, ke \in E(I)$ and $\lambda \in (0,1)$, (35) implies

$$\varphi(he + \lambda(x - he) + (1-\lambda)(k-h) e) = \varphi(he + \lambda(x - he)) + (1-\lambda)(k-h).$$

Thus

$$\psi(y) = \varphi(y + he) \quad \forall y \in E(I - h) \tag{36}$$

satisfies (33) and, by Lemma 5 it is translation invariant. In turn,

$$\varphi(x) = \psi(x - he) \quad \forall x \in E(I)$$

is translation invariant too.

Conversely, translation invariance of φ clearly implies (35). ■

Lemma 7 *Let $\varphi : E(I) \to \mathbb{R}$ be monotone. Then φ is translation invariant if and only if it is translation quasinvariant, and $\varphi(ke) - \varphi(he) = k - h$ for all $h, k \in I$.*

Proof. Assume φ is translation quasinvariant, and $\varphi(ke) - \varphi(he) = k - h$ for all $h, k \in I$. Let $x, ke, he \in E(I)$ and $\lambda \in (0,1)$. Monotonicity and the fact that there exists $c \in \mathbb{R}$ such that

$$\varphi(te) = t + c \quad \forall t \in I \tag{37}$$

imply that

$$\varphi(\lambda x + (1-\lambda) ke) = \varphi(\lambda \varkappa e + (1-\lambda) ke) \tag{38}$$

for some $\varkappa \in [\operatorname{einf}(x), \operatorname{esup}(x)] \subseteq I$. Translation quasinvariance implies

$$\varphi(\lambda x + (1-\lambda) he) = \varphi(\lambda \varkappa e + (1-\lambda) he) \tag{39}$$

which together with (37) delivers (35) so that φ is translation invariant. The converse is trivial.[23] ■

[23] Notice that the assumption of translation quasinvariance can be replaced with the weaker requirement that (38) implies (39) provided $x, ke, he, \varkappa e \in E(I)$ and $\lambda \in (0,1)$.

Lemma 8 *A translation invariant and monotone functional* $\varphi : E(I) \to \mathbb{R}$ *is a niveloid.*

Proof. In view of Lemma 2, it is sufficient to show that $\tilde{\varphi}$ is monotone; i.e., that $\varphi(x) + c \leq \varphi(y) + d$ if $x, y \in E(I)$, $c, d \in \mathbb{R}$, and $x + ce \leqslant y + de$. Setting $k = d - c$, it will be shown that $y + ke \geqslant x$ implies $\varphi(y) + k \geq \varphi(x)$.

Let $b = \operatorname{esup}(x) \vee \operatorname{esup}(y) \in I$.

- If $k \leq b - \operatorname{esup}(y)$, then $\operatorname{einf}(x) \leq \operatorname{einf}(y + ke)$ and $\operatorname{esup}(y + ke) = \operatorname{esup}(y) + k \leq b$, so that $y + ke \in E(I)$ and $\varphi(x) \leq \varphi(y + ke) = \varphi(y) + k$.
- Else $k > b - \operatorname{esup}(y) \geq 0$ and $x \leqslant be$. There are two subcases:
  - ▶ if $k \geq b - \operatorname{einf}(y)$, then

$$\begin{aligned}\varphi(y) + k &\geq \varphi(\operatorname{einf}(y)\, e) + k = \tilde{\varphi}(0) + \operatorname{einf}(y) + k \\ &\geq \tilde{\varphi}(0) + b = \varphi(be) \geq \varphi(x);\end{aligned}$$

  - ▶ if $k < b - \operatorname{einf}(y)$ and $\operatorname{einf}(y) < b - k < \operatorname{esup}(y)$, then $y + ke \geqslant x$ implies $(y + ke) \wedge be \geqslant x$, but $(y + ke) \wedge be \in E(I)$, in fact $x \leqslant (y + ke) \wedge be \leq be$, and $(y + ke) \wedge be = (y \wedge (b - k)\, e) + ke$. Notice that also $y \wedge (b - k)\, e \in E(I)$ since $(b - k) \in (\operatorname{einf}(y), \operatorname{esup}(y)) \subseteq I$. Therefore

$$\begin{aligned}\varphi(y) + k &\geq \varphi(y \wedge (b - k)\, e) + k = \varphi((y \wedge (b - k)\, e) + ke) \\ &= \varphi((y + ke) \wedge be) \geq \varphi(x).\end{aligned}$$

As wanted. ■

Lemma 9 *Let* $X = E(I)$ *or* X *be a convex tube, and* $\varphi : X \to \mathbb{R}$ *be a niveloid. Then*

$$\varphi(\lambda y + (1 - \lambda)\, x) \geq \varphi(x) \tag{40}$$

for all $x, y \in X$ *such that* $\varphi(y) = \varphi(x)$ *and all* $\lambda \in (0, 1)$ *if and only if* φ *is concave.*

Proof. Clearly concavity of φ imply (40) for all $x, y \in X$ such that $\varphi(y) = \varphi(x)$ and all $\lambda \in (0, 1)$.

Conversely, assume first $X = E(I)$. Let $x_o \in X^o$, there exists $\varepsilon > 0$ such that

$$X^o \supseteq U_\varepsilon(x_o) = [x_o - \varepsilon e, x_o + \varepsilon e].$$

By Lipschitz continuity of order 1, $\|x - x_o\| \leq \varepsilon/3$ implies $|\varphi(x) - \varphi(x_o)| \leq \varepsilon/3$. Then, if $x, y \in U_{\varepsilon/3}(x_o)$,

$$|\varphi(x) - \varphi(y)| \leq |\varphi(x) - \varphi(x_o)| + |\varphi(x_o) - \varphi(y)| \leq \frac{2}{3}\varepsilon$$

and $-\frac{2}{3}\varepsilon \leq \varphi(x) - \varphi(y) \leq \frac{2}{3}\varepsilon$. That is, $-\frac{2}{3}\varepsilon e \leqslant te \leqslant \frac{2}{3}\varepsilon e$ where $t = \varphi(x) - \varphi(y)$. Moreover, $x_o - \frac{1}{3}\varepsilon e \leqslant y \leqslant x_o + \frac{1}{3}\varepsilon e$ and

$$x_o - \varepsilon e \leqslant y + te \leqslant x_o + \varepsilon e$$

so that $y + te \in X^o$. Since $y \in X^o$ too, then $\varphi(y + te) = \varphi(y) + t = \varphi(x)$, and (40) implies

$$\varphi(\lambda(y + te) + (1 - \lambda)\, x) \geq \varphi(x) \quad \forall \lambda \in (0, 1). \tag{41}$$

Hence

$$\begin{aligned}\varphi(x) &\leq \varphi(\lambda(y + te) + (1 - \lambda)\, x) = \varphi(\lambda y + (1 - \lambda)\, x + \lambda te) \\ &= \varphi(\lambda y + (1 - \lambda)\, x) + \lambda t = \varphi(\lambda y + (1 - \lambda)\, x) + \lambda(\varphi(x) - \varphi(y))\end{aligned}$$

that is

$$\varphi(\lambda y + (1-\lambda) x) \geq \lambda \varphi(y) + (1-\lambda) \varphi(x) \quad \forall \lambda \in (0,1) \tag{42}$$

and φ is concave in $U_{\varepsilon/3}(x_o)$. Conclude, by arbitrarity of the choice of x_o, that φ is locally concave on X^o. A standard result from convex analysis yields concavity on X^o. Finally, the continuity of φ implies its concavity on the whole X. This proves the first case. To prove the second, for all $x, y \in X$ and $\lambda \in (0,1)$, set $t = \varphi(x) - \varphi(y)$. Since X is a tube, $y + te \in X$, and $\varphi(y + te) = \varphi(y) + t = \varphi(x)$. Repeat the argument leading from (41) to (42). ■

Proof of Theorem 4. (i) $\Longrightarrow$ (ii) is trivial. (ii) $\Longrightarrow$ (iii) is Lemma 7. (iii) $\Longrightarrow$ (i) is Lemma 8. The concavity properties descend from Lemma 9, Proposition 4, and $\hat{\varphi} = \overline{\check{\bar{\varphi}}}$. □

By Theorem 4, if $\varphi : E(I) \to \mathbb{R}$ is a concave niveloid, then its minimal niveloidal extension $\hat{\varphi}$ is concave. A fortiori, $\hat{\varphi}$ is the minimal concave niveloidal extension of φ. Clearly, the maximal niveloidal extension $\check{\varphi}$ may fail to be concave. Nonetheless, the next and final theorem of this paper shows that φ admits a maximal concave niveloidal extension, denoted by $\breve{\varphi}$. Such extension is the pointwise supremum of all concave niveloidal extensions of φ. This result is somehow surprising since the supremum of concave (resp. quasiconcave) functions is not concave (resp. quasiconcave) in general.

Theorem 5 *Let $\varphi : E(I) \to \mathbb{R}$ be a concave niveloid. Then*

$$\hat{\varphi}(y) = \inf \left\{ \psi(y) : \psi \textit{ is a concave niveloid on } E \textit{ such that } \psi_{|E(I)} = \varphi \right\} \quad \forall y \in E \tag{43}$$

is the minimal concave niveloid on E that extends φ.
Analogously, the functional defined on E by

$$\breve{\varphi}(y) = \sup \left\{ \psi(y) : \psi \textit{ is a concave niveloid on } E \textit{ such that } \psi_{|E(I)} = \varphi \right\} \quad \forall y \in E \tag{44}$$

is the maximal concave niveloid on E that extends φ.

In particular, if ψ is a concave niveloid on E such that $\psi_{|E(I)} = \varphi$, then

$$\hat{\varphi} \leq \psi \leq \breve{\varphi}.$$

Before entering the proof's details, notice that, in general $\breve{\varphi} < \check{\varphi}$.

Example 3 *Consider the positive unit box $[0,1]^2$ in the real plane $\mathbb{R}^2$, and $\varphi(t,r) = \min\{t,r\}$. Then $\hat{\varphi} = \breve{\varphi}$ and by (10) it follows that*

$$\breve{\varphi}(2,0) = 0 < 1 = \inf_{0 \leq t,r \leq 1} (2 + \min\{0, r-t\}) = \inf_{0 \leq t,r \leq 1} (\min\{t,r\} + \max\{2-t, -r\}) = \check{\varphi}(2,0).$$

Proof of Theorem 5. As already observed, the first part of the statement follows from Theorem 4. Set $X = E(I)$ and

$$\begin{aligned} \Psi &= \left\{ \psi : E \to \mathbb{R} \mid \psi \text{ is a concave niveloid on } E \text{ and } \psi_{|X} = \varphi \right\} \\ &= \left\{ \psi : E \to \mathbb{R} \mid \psi \text{ is a concave niveloid on } E \text{ and } \psi_{|X^o} = \varphi \right\}. \end{aligned}$$

For each $\psi \in \Psi$ denote by ψ^* and $\partial\psi$ the *concave conjugate* and *superdifferential* of ψ.[24] Set

$$\eta(f) = \inf_{\psi \in \Psi} \psi^*(f) \quad \forall f \in E^*.$$

[24] If $f \in E^*$ and $y \in E$, $\psi^*(f) = \inf_{z \in E} (\langle z, f \rangle - \psi(z))$ and $\partial\psi(y) = \{g \in E^* : \psi(z) \leq \psi(y) + \langle z - y, g \rangle \ \forall z \in E\}$. Recall that $\partial\psi(y) = \{g \in E^* : \langle y, g \rangle - \psi(y) = \psi^*(g)\} = \left\{g \in E^* : g \geq d^+\psi(y)\right\}$, see [1], [13], and [21].

The effective domain of all ψ^*, and hence that of $\eta : E^* \to [-\infty, \infty)$, is contained in Δ. Moreover, since all ψ^* are weakly* upper semicontinuous, so is η. For each $x \in X^o$ and $q \in \partial\varphi(x)$ ($= \partial\psi(x)$ for all $\psi \in \Psi$)

$$\langle x, q\rangle - \varphi(x) = \langle x, q\rangle - \psi(x) = \psi^*(q) = \eta(q) \tag{45}$$

so that η is proper. Therefore

$$\eta^*(y) = \min_{p\in\Delta}(\langle y, p\rangle - \eta(p)) \quad \forall y \in E$$

is a concave niveloid, and $\eta^* \geq \psi$ for all $\psi \in \Psi$. Finally, for each $x \in X^o$ and $q \in \partial\varphi(x)$ it follows $\eta^*(x) \geq \varphi(x) = \langle x, q\rangle - \eta(q) \geq \eta^*(x)$, that is, $\eta^* \in \Psi$ and $\eta^* = \breve{\varphi}$. ■

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