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Mean-Risk Analysis with Enhanced Behavioral Content

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Abstract

We study a Mean-Risk model derived from a behavioral theory of Disappointment with multiple reference points. One distinguishing feature of the risk measure is that it is based on mutual deviations of outcomes, not deviations from a specific target. We prove necessary and sufficient conditions for strict first and second order stochastic dominance, and show that the model is, in addition, a Convex Risk Measure. The model allows for richer, and behaviorally more plausible, risk preference patterns than competing models with equal degrees of freedom, including Expected Utility (EU), Mean-Variance (MV), Mean-Gini (MG), and models based on non-additive probability weighting, such as Dual Theory (DT). For example, in asset allocation, the decision-maker can abstain from diversifying in a risky asset unless it meets a threshold performance, and gradually invest beyond this threshold, which appears more acceptable than the extreme solutions provided by either EU and MV (always diversify) or DT and MG (always plunge). In asset trading, the model allows no-trade intervals, like DT and MG, in some, but not all, situations. An illustrative application to portfolio selection is presented. The model can provide an improved criterion for Mean-Risk analysis by injecting a new level of behavioral realism and flexibility, while maintaining key normative properties.

Key words: Risk analysis; Uncertainty modeling; Utility theory; Stochastic dominance; Convex risk measures.

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1. Introduction

Mean-Risk analysis is an appealing approach to decision under risk that has sprung abundant literature and applications. This is because measuring the value of gambles as a function of their rewards and risks goes to the heart of decision makers' concerns in a direct, transparent manner (Jia and Dyer 1996). There seems to be general agreement—even compelling arguments (de Giorgi 2005)—that the potential reward of a gamble should be captured by its expected value, i.e., its mean. There is less accord about what constitutes an acceptable measure of risk. The challenge is to balance desirable normative properties with intuitively or behaviorally appealing considerations. This tension ultimately lies at the heart of any prescriptive theory of choice under risk.

Here, we propose a Mean-Risk model that results from a reformulation of Disappointment without Prior Expectation (Delquié and Cillo 2006), a theory of Disappointment in which every outcome of a prospect can act as a reference point for any other outcome. We show that this Mean-Risk model presents advantages over the standard competing models because it is able to produce solutions to mean-risk optimization problems that are behaviorally more realistic, and at the same time it retains key normative properties required for use in a wide range of applications. Due to this flexibility, our model may provide an attractive criterion to capture decision makers' risk-return preference patterns in Mean-Risk analysis.

The paper proceeds as follows. Section 2 introduces the model. We show how it relates to other models of risk, and that it defines a class of risk measure distinct from the classic families widely considered throughout the literature. In Section 3, we provide necessary and sufficient conditions for monotonicity with respect to first and, more importantly, second order stochastic dominance, two essential normative criteria for ordering risky prospects. This generalizes previous results concerning the Mean-Gini model (Yitzhaki 1982, Ogryczak and Ruszczyński 2002). In Section 4, we show that the model yields a Convex risk measure, which is highly desirable for use in risk management because it rewards diversification. Next, the model's implications for asset trading and optimal allocation are examined in Section 5. There, we show that the model allows for a richer pattern of risk taking behaviors than other standard models, and we specify the conditions under which qualitatively different types of behaviors occur. The risk taking behaviors produced by the model appear more realistic than those of other classic models with comparable degrees of freedom. These results are closely tied to the model's ability to bridge first order and second order risk aversion. Section 6 addresses practical issues in calibrating and using the model for applications, and provides a numerical example in stock portfolio selection. By way of summary, Section 7 concludes that the model provides a tractable, sound analysis of choice under risk, offering a wider range of available solutions in Mean-Risk analysis. All proofs appear in the Appendices.

2. The Proposed Mean-Risk Model

The literature on risk emanates from several intellectual traditions, notably Statistics (measures of dispersion and the moments approach), Economics (the EU approach, but also the inequality measurement approach), Finance (the portfolio efficient set approach), and Psychology (the behavioral/cognitive approach). Sarin and Weber (1993) present an overview of the Risk-Value models literature at the time of their writing; Pederson and Satchell (1998) provide a fairly detailed review of risk measures.

Expected Utility stands as *the* ultimately rational approach to choice under risk, however, there is no explicit construction of a risk index as a primitive in EU. For an individual with utility function u, the risk of a gamble X can be measured as its risk premium, defined as $\pi(X) = \mathrm{E}[X] - u^{-1}(\mathrm{E}u[X])$ (Pratt 1964), but the valuation of a gamble, i.e., its certainty equivalent, cannot in general be calculated directly from its expected value and its risk premium in a Risk-Value spirit, because the estimation of $\pi(X)$ usually requires calculating the certainty equivalent, leading to a circularity. Under particular conditions on the u function and/or the distribution of X, EU can take a Risk-Value form. For example, if the utility function is exponential and gambles have a normal distribution, or if the utility function is quadratic, then EU is equivalent to a Mean-Variance model. Further ways to cast EU as a function of risk and return have been explored in some depth by Bell (1995) and Jia and Dyer (1996): the possibilities seem confined to a limited set. Because the notion of risk in EU is entirely driven by the concavity of the utility function, it is completely intertwined with the concept of diminishing marginal utility of money. To require that the valuation of each and every risk be entirely and only determined by the pattern of utility for wealth may be too rigid for some decision makers. That is, EU may leave out some aspects of risk that legitimately matter to the decision maker.

The so-called Risk-Value framework may offer more flexibility in dealing with risk (Dyer and Jia 1997) by allowing to define a risk measure "from scratch," that is, unconstrained by whether it is consistent with the maximization of a particular EU function. Because risk is associated with the presence of uncertainty in the payoffs, that is, the extent to which their distribution departs from a sure outcome, risk measures are germane with measures of dispersion. Risk is traditionally measured as the propensity of a random outcome to deviate from some reference level. Stone (1973) proposes that three basic ingredients are relevant to devising a risk measure: (i) a reference level, from which deviations are measured; (ii) the range of deviations taken into account; and (iii) how deviations are weighed. He shows that this defines a general family that includes the standard risk measured used in Finance: variance, semi-variance, mean absolute deviation, and the probability of a loss worse than some specified level.

A wide variety of risk measures has been proposed, some of which have received special attention. For example, Mean Absolute Deviation, Semi-lower Deviation, Conditional Value-at-Risk, and the Gini Absolute Difference, among others, have been studied in depth as regards their normative properties (e.g. compliance with stochastic orderings, consistency) and computational performance in optimization (see the work of Ogryczak and Ruszczyński (1999, 2002); Mansini et al. (2003, 2007); Krzemienowski and Ogryczak (2005)). The risk measure we propose here was motivated by a desire to account for risk preferences that deviate systematically from EU, such as the widely observed Allais (1953) paradox and certainty effect (Kahneman and Tversky 1979), the common ratio effect, and reference-dependence in valuing outcomes.

2.1 A Behaviorally Motivated Mean-Risk Model

Delquié and Cillo (2006) developed the Disappointment without Prior Expectation model of choice under risk based on the postulate that individuals are liable to experience a *mixture* of disappointment and contentment from comparing the outcome received from a gamble to all the other possible outcomes, worse and better, rather than a single prior expectation. This extends the notion of reference dependence by allowing each and every outcome in the gamble to play the role of a reference point, that is, the value of an outcome is relative to the *entire* context in which it is embedded. In all previous formulations of Disappointment, including Bell (1985) and Loomes and Sugden (1986), the gamble is summarized into a single reference point. Kőszegi and Rabin (2007) proposed a model of reference-dependent risk taking behavior in which the reference level is stochastic, consisting of the expectations the decision maker held in the recent past.

It was also shown in Delquié and Cillo (2006) that Disappointment without Prior Expectation could be reformulated as a Risk-Value model, taking the following form:

$$V(X) = \sum_{i=1}^{n} p_i v(x_i) - \sum_{i=1}^{n} \sum_{j \geq i} p_i p_j H\big(v(x_i) - v(x_j)\big),$$

where X is a gamble that yields payoff x_i with probability p_i, $i = 1, \ldots, n$, $\sum p_i = 1$ and $x_1 \geq x_2 \geq \ldots \geq x_n$; $v(\cdot)$ is an increasing function that describes the subjective value of outcomes; and the function H describes how an individual values discrepancies between achieved and missed outcomes, that is, the losses associated with less than desired outcomes. The immutable properties of H, that stem from its very definition, are: (i) $H(0) = 0$, and (ii) H is defined on the non-negative domain, that is, it takes non-negative deviations as argument, i.e., differences between ordered outcomes.

Here, for parsimony and for the sake of having a Risk-Value representation comparable to those that have appeared before, we focus on a special case of the above model: we will assume v linear throughout this paper. This assumption does not play a role in the essential results and claims developed in the paper, and it will enable to us to concentrate on what can be accomplished with the simplest form. Thus, the model we are interested in here is:

$$V(X) = \mathrm{E}[X] - \Delta(X)$$

$$\text{with} \quad \Delta(X) = \sum_{i=1}^{n} \sum_{j \geq i} p_i p_j H(x_i - x_j), \tag{1}$$

where E[X] is the mean of X, a measure of its potential reward, and Δ(X) defines a risk-premium, that is, the amount by which the reward will be discounted to account for the presence of risk in X. For example, for a binary gamble X with outcomes x, y with probabilities p, $1–p$ respectively, and $x \geq y$, we have: $V(X) = px + (1–p)y – p(1–p)H(x–y)$. Note that if the outcomes are not ordered, we can just enter their absolute difference in the H function. If F denotes the cumulative distribution of X, the continuous form of Δ(X) is:

$$\Delta(X) = \int_{-\infty}^{+\infty} \int_{-\infty}^{x} H(x-y)dF(y)dF(x) = \mathrm{E}\left[\int_{-\infty}^{X} H(X-y)dF(y)\right]. \tag{2}$$

From now on, we will refer to the model expressed in (1), either its discrete or continuous form (2), as the M-Δ model. Note that the M-Δ model produces a valuation of a gamble directly in the form of a certainty equivalent. That is, $V(X)$ in (1) is in the same units as the gamble's payoffs, and a gamble is acceptable if and only if $V(X) > 0$.

The function H weighs the relative impact of large and small deviations. Although H could be an increasing, decreasing, or even non-monotonic function, given the pervasiveness of risk averse behavior, it is sensible to focus on the case of H (strictly) increasing: this will imply $H(y) > 0$ for all $y > 0$ (since $H(0) = 0$), leading to a positive risk premium Δ(X) for any gamble. Under this assumption, a sure payoff equal to the expected value of the gamble will always be preferred to the gamble itself, and zero mean gambles will always be rejected.

Δ(X) measures the riskiness of a gamble with no regard to its location, that is, irrespective of how good or bad the outcomes are (this, of course, is captured by E[X]). This may be regarded as a desirable feature for a risk measure, because a risk judgment itself should be distinct from the overall desirability of a gamble. Equation (1) specifies how risk should be traded off against the reward, that is, a decision rule. This rule emanates naturally from the behavioral hypothesis from which the model was derived.

M-Δ in (1) has constant risk aversion (CRA) for any form of H. That is, if a constant is added to a gamble, the valuation of the gamble increases by the same constant. The property of decreasing risk aversion is often regarded as behaviorally more compelling than CRA. However, CRA presents a great practical advantage for applications, because it allows analyzing problems in terms of gains and losses rather than in terms of total wealth, which is often impossible. This convenience is the reason why exponential utility is used so routinely in applications of EU. The essence of a Risk-Value representation is to accept risk as a primitive construct, not necessarily tied to the valuation of sure, final wealth: in this mind-set, the property that $\Delta(X)$ is independent of wealth is not shocking (Mitchell and Gelles 2003). Also, if the random variables considered are returns, i.e. relative payoffs, the M-Δ model would imply constant *relative* risk aversion, that is, *diminishing* risk aversion in the absolute payoffs.

2.2 Relationship to Some Risk Measures

Fishburn (1977) considers a family of risk measures in which risk is measured as a probability weighted function of the deviations below a specified target return, defined as follows:

$$\rho_t(F) = \int_{-\infty}^{t} \varphi(t-x)dF(x), \tag{3}$$

where F is the cumulative distribution of the random payoff, t the target level, and φ measures how deviations below the target are weighed. Fishburn (1977) examines the special case $\varphi(t-x)=(t-x)^{\alpha}$, for $x \leq t$, so-called the 'α-t' model. The α-t model belongs to a general family considered by Stone (1973).

$\Delta(X)$ is neither a particular case of the general measure considered by Fishburn (1977), Equation (3), nor part of the family proposed by Stone (1973). Indeed, one essential difference is that these traditional risk measures are sprung from the outcomes' deviations from a fixed reference level, whereas $\Delta(X)$ is built on the mutual deviations of outcomes among one another. Nevertheless, Equation (2) makes a relationship to Equation (3) apparent: for $X = x$, the expression $\int_{-\infty}^{x} H(x-y)dF(y)$ within the expectation in (2) is nothing but the Fishburn (1977) measure of risk, $\rho_x(X)$, representing the risk of failing to achieve at least outcome x in gamble X. Thus, $\Delta(X)$ can be thought of as the mathematical expectation of the collection of Fishburn's risk measures generated by taking as target level each and every value of X in turn. In $\Delta(X)$, each outcome of X can be viewed as playing the role of a target and contributing its own 'à la Fishburn' risk to the gamble: the total risk $\Delta(X)$ of the gamble is just the average of the risks associated with individual outcomes. Thus (1) can be written as:

$$V(X) = \mathrm{E}[X] - \mathrm{E}[\rho_X(X)] = \mathrm{E}[X-\rho_X(X)] = \mathrm{E}[u_X(X)]$$

with $u_X(x) = x-\rho_x(X)$,

where $u_X(x)$ can be interpreted as the risk-adjusted utility of outcome x in gamble X. In other words, $\rho_x(X)$ is the "risk premium" associated with just outcome x in gamble X, and $\mathrm{E}[\rho_X(X)] = \Delta(X)$ is the risk premium of the whole gamble.

Notice from (1) that every pairwise difference between two outcomes enters exactly once in the makeup of Δ(*X*). For *H* non-decreasing, Δ(*X*) constitutes a general measure of dispersion, which includes an important case. Indeed, for *H* linear, Δ(*X*) yields the Gini Mean Difference (up to a positive multiplicative constant), also known in Statistics as the 'Absolute Mean Difference' measure of dispersion. Gini's Mean Difference is defined as the mean of the absolute difference between two observations of a random variable.[1] The Gini measure is most prominently used as a measure of inequality of income or wealth among a population, but it has also been used as a risk measure (Yitzhaki 1982, Ogryczak and Ruszczyński 2002). Observing that Δ(*X*) in (1) can also be written as:

$$\Delta(X) = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j H(|x_i - x_j|),$$

we see that for $H(y) = y$, $\Delta(X) = \frac{1}{2}G(X)$, where G(*X*) is the Gini Mean Difference of *X*.

Various generalizations of the Gini measure have been proposed before (e.g. Donaldson and Weymark 1980, Yitzhaki 1983). Some generalizations introduce parameters that, in effect, transform the decumulative distribution function of *X*. As another type of extension, Krzemienowski and Ogryczak (2005) consider the Gini measure computed over below-mean outcomes in order to capture downside risk only. Our extension of Gini differs from previous generalizations by introducing a weighting function over the deviations. However, our approach did not start with the Gini measure seek to extend it; instead our purpose was to account for widely observed non-EU preference patterns, and this produced a risk measure that happens to include Mean-Gini as a particular case. [2]

3. Efficiency of the M-Δ Model

To avoid the difficulties connected with knowing decision makers' utility functions, several authors have examined the merits of ordering prospects in terms of dominance rules (Hadar and Russell 1969; Wong 2007, Egozcue and Wong 2010). Let *X* and *Y* be two random variables, and *F* and *G*, respectively, their cumulative distribution functions. Let us recall the definitions of first and second order stochastic dominance (FSD and SSD):

DEFINITION 1. X is said to be as large as Y in the sense of FSD, denoted as $X \succcurlyeq_{FSD} Y$ if and only if

$$F(x) - G(x) \leq 0 \text{ for all } x.$$

[1] The Gini *Index*, also called Gini Coefficient of Concentration, is a normalized, unit-free measure obtained by dividing the Gini Mean Difference by twice the mean of the distribution.

[2] Maccheroni et al. (2006) show that the Gini concentration index also arises in the so-called divergence preferences model they introduce to represent ambiguity preferences, and which generalizes the multiple priors model of Gilboa and Schmeidler (1989). The index of ambiguity aversion in Maccheroni et al.'s (2006) model includes the relative Gini concentration index as a particular case. They also show that their model with Gini as index of ambiguity aversion is equivalent to mean-variance preferences, when restricted to the domain of monotonicity of mean-variance.

DEFINITION 2. X is said to be as large as Y in the sense of SSD, denoted as $X \succcurlyeq_{SSD} Y$, if and only if

$$\int_{-\infty}^{x}(F(t)-G(t))dt \leq 0 \text{ for all } x.$$

The above define weak ordering relations. In both Definition 1 and 2, the strict stochastic dominance (SD) relation is defined as:

$$X \succ_{SD} Y \text{ if and only if } X \succcurlyeq_{SD} Y \text{ and } Y \not\succcurlyeq_{SD} X,$$

i.e., the inequality is strict for at least one x in the above definitions; that is to say, F and G are not identical.

Stochastic dominance establishes a partial ordering of probability distributions, and it can be shown that distribution F dominates distribution G in the sense of n^{th}-order stochastic dominance if and only if all EU maximizing individuals with utility functions whose derivatives to order n alternate in sign (that is, u such that $\operatorname{sign} u^{(j)} = (-1)^{j+1}$ for $j = 1,\ldots,n$) prefer F to G (Levy 1992).

In selecting a risk measure ad hoc, we expose ourselves to—and presumably tolerate— violating some normative principles of EU, but we would like to maintain others. In particular, we do not want to give up monotonicity with respect to larger payoffs and decreasing risk, that is, respectively, FSD and SSD orders. SSD is critical because it lies at the heart of fundamental notions of risk and risk aversion, also it can rank more prospects than FSD. However, working directly with stochastic dominance orders, e.g., as constraints in portfolio optimization (Dentcheva and Ruszczyński 2006), is computationally challenging and not always tractable. Thus, a key issue in using any Risk-Value model is: does it rank prospects consistently with FSD and SSD? Propositions 1 and 2 below address this question for the M-Δ model.

PROPOSITION 1. *Assume that H is differentiable and all expectations exist. The M-Δ model satisfies strict first order stochastic dominance if and only if* $0 \leq H'(y) \leq 1$ *for all* $y \geq 0$.

The intuitive interpretation of the condition is that the sensitivity to outcomes differences should not exceed the sensitivity to the outcomes themselves. Indeed, if the weight placed on deviations in payoffs should ever exceed the weight put on the payoffs themselves, it would be possible to have a situation in which a strict increase in a payoff (making the gamble strictly better) would increase the risk $\Delta(X)$ of the gamble more than its expected reward E[X]. Delquié and Cillo (2006) showed the result of Proposition 1 for weak FSD using Machina's (1982) concept of "local utility function". The new proof we provide here uses a different approach and shows the result for strict FSD.

PROPOSITION 2. *Assume that H is twice differentiable and all expectations exist. The M-Δ model satisfies strict second order stochastic dominance if and only if H is such that, for all* $y > 0$:

$0 < H'(y) \leq 1$ *and* $H''(y) \geq 0$, *that is, H is strictly increasing, convex, and never grows faster than the identity function.*

The convexity of *H* essentially guarantees that adding an independent, zero-mean risk to *X* will never cause $\Delta(X)$ to decrease. Behaviorally, it reflects increasing sensitivity to larger deviations. Yitzhaki (1982) showed that the Mean-½Gini model satisfies FSD and SSD. While Yitzhaki (1982) showed only weak SSD consistency, Ogryczak and Ruszczyński (2002) showed that the Mean-Gini model meets strict SSD. Therefore, Proposition 2 extends this result to the M-Δ model.

4. Convexity of the M-Δ Model

Several axiomatic approaches to constructing risk measures have been proposed. Some approaches have a prescriptive orientation, that is, they attempt to outline general properties deemed desirable or necessary for adequate management of risk (Ma and Wong 2010). In this section, we examine how our risk model relates to two classes that have received a lot of attention: the so-called "Coherent" and "Convex" risk measures.

Artzner et al. (1999) measure risk as the amount of cash that should be added to a risk position, i.e., a gamble, to make it acceptable (their formulation also incorporates the interest rate earned on the cash provisioned). They argue that the following axioms, P1-P4, are necessary for the proper management and regulation of risk, and they call measures satisfying them *Coherent risk measures*. Let ρ denote the risk measure as defined by Artzner et al. (1999), then for all *X*, *Y*:

P1. Translation invariance: $\rho(X+\delta) = \rho(X) - \delta$, for all δ.

P2. Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

P3. Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$, for all $\lambda \geq 0$.

P4. Monotonicity: If $X \leq Y$, $\rho(X) \geq \rho(Y)$.

The way Artzner et al. (1999) define risk is different from just variability or dispersion: it is essentially the negative of a mean-risk measure (or 0 if the mean risk measure is positive). Therefore, it corresponds to the M-Δ model by taking: $\rho(X) = \max(-V(X),0)$;[3] and the above axioms may be recast in terms of Δ(.) as defined in (1) and (2). The correspondence between deviation risk measures and the measures considered by Artzner et al. (1990) was studied by Mansini et al. (2003). Also see Rockafellar et al. (2006) on the one-to-one correspondence between deviation measures and the type of risk measures defined by Artzner et al. (1999).

[3] Indeed, if position *X* is unacceptable under the M-Δ model, i.e., $V(X) < 0$, then the amount by which the position needs to be augmented to make it, at the limit, acceptable is just $-V(X)$.

Coherent risk measures are generally not consistent with SSD (de Giorgi 2005), but some are. For example, Ogryczak and Ruszczyński (1999) show that certain coherent risk measures based on semi-deviations (standard or absolute) preserve SSD. Also, Mansini et al. (2003) showed that a SSD efficient measure that is LP decomposable is a coherent risk measure.

The property of positive homogeneity (P3) may be considered rather restrictive. One could argue that doubling the position in a gamble will at least double the risk incurred, that is:

P3'. $\rho(\lambda X) \geq \lambda\rho(X)$ for all $\lambda \geq 1$ (this implies that $\rho(\lambda X) \leq \lambda\rho(X)$ for $\lambda \leq 1$).

The property P3' may be deemed more compelling, and more flexible, than P3. In this spirit, Föllmer and Schied (2002) propose to replace P2 and P3 by the weaker property:

P2'. Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, for all $0 \leq \lambda \leq 1$.

Property P2' just provides that diversification should not increase risk, a cornerstone principle of risk management. Föllmer and Schied (2002) define risk measures satisfying Translation Invariance (P1), Convexity (P2') and Monotonicity (P4) as *Convex risk measures*. They show a representation theorem for Convex risk measures parallel to that obtained by Artzner et al. (1999) for Coherent risk measures. See de Giorgi (2005) and Brown and Sim (2009) for further characterization of Convex risk measures.

It is not difficult to show that the M-Δ model will comply with the axioms of Convex risk measures under the following conditions:

- *Translation invariance*: fulfilled for any H. Note that this, of course, implies constant risk aversion, which we discussed previously.
- *Convexity*: fulfilled whenever H is convex.
- *Monotonicity*: holds if and only if $H' \leq 1$, as seen in Proposition 1.

Thus, it turns out that the conditions for SSD (Proposition 2) ensure that M-Δ is, in addition, a Convex risk measure.

For those who would like to use a Coherent risk measure, the Subadditivity axiom (P2) will be fulfilled if H is subadditive, and Positive homogeneity will hold if and only if H is linear. Thus, to satisfy the four axioms of Coherent risk measures simultaneously, H has to be a seminorm, that is, we have to take H linear: $H(y) = \beta y$, with $0 \leq \beta \leq 1$. In that case, we have the Gini measure: $\Delta(X) = \beta/2\, G(X)$, where the parameter β reflects the decision maker's trade-off between risk and reward. This shows incidentally that the Gini measure is an example of a Coherent risk measure satisfying SSD.

In sum, the M-Δ model can accommodate the general axioms reviewed above, either individually or collectively, for convexity or coherence, and convexity of the risk measure is guaranteed whenever H fulfills the conditions of Propositions 1 and 2.

5. Asset Trading and Allocation under the M-Δ Model

In this section, we examine two central problems in the economic analysis of investment behavior: the trading of an asset, and the optimal allocation of wealth to a risky asset. Before analyzing these problems, we begin with a general result on the ability of M-Δ to produce first order or second order risk aversion, because this distinction has been shown to drive risk taking behavior in a general fashion. In each case, we show that the M-Δ model has the flexibility to encompass the predictions made by EU and by models based on non-additive probability weighting.

5.1 First Order vs. Second Order Risk Aversion

The notions of first and second order risk aversion were presented by Segal and Spivak (1990). Consider the gamble αX, corresponding to taking a fraction α of the random variable X. Under first order risk aversion, the risk premium for a small risk αX, i.e. for α sufficiently small, is proportional to α, that is, linear in the size of the risk taken. Under second order risk aversion, the risk premium is proportional to α^2, and thus approaches 0 faster than α. Thus, an individual with second order risk aversion becomes nearly risk neutral for small risks.

First and second order risk aversion imply qualitatively different behavior when risks are scalable, such as when it is possible to buy variable quantities of an asset, or partially insure against a risk.

Let X be a non-constant random variable, and denote by $\pi(\alpha)$ the risk premium of the gamble αX.

DEFINITION 3. An individual's risk aversion is of

- first order if for every non-constant X such that $E[X] = 0$, $d\pi/d\alpha|_{\alpha=0+} > 0$;
- second order if for every non-constant X such that $E[X] = 0$, $d\pi/d\alpha|_{\alpha=0} = 0$ and $d^2\pi/d\alpha^2|_{\alpha=0+} > 0$.

Note that the above definition, from Segal and Spivak (1990), is stated for the case of a risk-averse attitude, not risk-seeking. The inequalities in Definition 3 are reversed for a risk-seeking attitude. Also, we do not specify that the definition holds at a particular wealth level, because the individual's wealth level is immaterial in the M-Δ model, which has constant risk aversion.

All models based on rank-dependent probability weighting embody first order risk aversion, whereas EU and models with smooth Fréchet differentiability have second order risk aversion. The M-Δ model can embody either first or second order risk aversion, depending on a simple condition on the first derivative of H at 0.

PROPOSITION 3. *Consider an individual behaving according to the M-Δ model with a twice differentiable H function. The individual's risk aversion is:*

(i) *first order if and only if* $H'(0) > 0$*;*

(ii) *second order if and only if* $H'(0) = 0$ *and* $H''(0) > 0$.

The proof of Proposition 3 also shows that in the M-Δ model the risk premium for a small risk is proportional to the Gini Mean Difference. Under EU, the risk premium is a function of the variance, (Pratt 1964).

The order of risk aversion determines behavior in taking a position on an asset, as we study next.

5.2 Trading of an Asset and the No-trade Interval

Consider an asset whose present monetary value can be described by the random variable *X*. Assume that it is possible to trade the asset in any (small) quantity. Under EU, a risk-averse individual with differentiable utility function will invest a positive quantity of his/her money in the asset if and only if the expected value E[*X*] of the asset exceeds its price, and he/she will short sell (some of) the asset if and only if the asset price exceeds E[*X*]. This result, shown in Arrow (1974) and also discussed in Segal and Spivak (1990), is due to the fact that risk aversion is of the second order under EU: as mentioned in Section 5.1, this implies that for sufficiently small risks the individual is locally risk-neutral and decides according to expected value. Therefore, the individual will neither buy nor short sell the asset if, and only if, the price is exactly equal to the expected value of the asset. Dow and Werlang (1992) showed a contrasting result that, for an individual maximizing expected utility with non-additive subjective probabilities, an interval exists such that, for any price in this interval, the individual abstains from trading the asset. If the price is lower than the lower bound of the interval, the individual buys; if the price is higher than the upper bound, he/she short sells; for prices within the interval, he/she declines holding a position. This result, intuitively plausible and compatible with observed investment behavior, is further extended by Chateauneuf and Ventura (2010).[4]

COROLLARY 4. *Consider an individual behaving according to the M-Δ model with a convex, differentiable H function and constant initial wealth. For any random asset X, the individual will:*

(i) *hold no position when the asset price is in the interval*
$\left[\mathrm{E}[X] - \frac{1}{2}\mathrm{G}(X)H'(0), \mathrm{E}[X] + \frac{1}{2}\mathrm{G}(X)H'(0)\right]$ *if he/she has first order risk aversion;*

(ii) *take a position as an EU maximizer if he/she has second order risk aversion.*

The no-trade interval is always centered on E[*X*] and it has a strictly positive length if and only if the individual has first order risk aversion. In the case of first order risk aversion, the size of the no-trade

[4] Under EU, no-trade intervals may be explained by the presence of trading costs, but it is unclear whether this accounts for empirical observations.

interval depends on both the individual's attitude towards infinitesimal risks and the riskiness of the asset as measured by the Gini measure: the riskier the asset, the wider the range of prices within which the individual is unwilling to take a position. For $H'(0) = 0$, the individual is (nearly) risk-neutral for infinitesimal risks, just like an EU maximizer.

Remark. When initial wealth *W* is random (includes background risks), if *X* and *W* are *comonotonic*, i.e., *X* does not provide a hedge against *W*, it can be shown that the individual will be willing to buy an amount of *X* only if the asset price π is $< \mathrm{E}[X] - \frac{1}{2}\mathrm{G}(X)H'(0)$, that is, the no-trade interval includes (is wider than) the interval given in Corollary 4. Because in this case *X* tends to compound the risk in *W*, the individual will be more conservative in buying it than he/she would be in the absence of background risk, which conforms to intuition. If, on the other hand, *X* and *W* are *countermonotonic* (*W* and –*X* are comonotonic), *X* provides a hedge against the risk in *W*, then the individual (even with first order risk aversion) will want to buy a strictly positive amount of *X* whenever π < E[*X*]. Thus, for such *X*, we do not have a no-trade interval. Indeed, there is always an advantage to buying (some of) *X* whenever it is priced at anything less than its expected value, because it will dampen the background risk while providing an increase in expected total wealth. This again matches common sense.

The class of models considered by Dow and Werlang (1992) always produce a no-trade interval of non-zero length, while EU never produces a no-trade interval. The M-Δ model bridges these two situations. Chateauneuf and Ventura (2010) show that Dow and Werlang's (1992) result holds for non-positive assets. Note that, owing to translation invariance, no assumptions were necessary about the sign of *X* in the M-Δ analysis above. If $X < 0$, the price at which the individual would be willing to buy the asset is, of course, negative, which is tantamount to selling insurance against *X*.

5.3 The Asset Allocation Problem

A question of interest for any model of choice under risk is the kind of solutions it provides to the optimal asset allocation problem. This is especially relevant if M-Δ is to be used as a criterion for building optimal portfolios of risky assets.

Under EU, a risk-averse individual with differentiable utility should always invest a strictly positive amount of money in a risky asset that has a positive expected value, no matter how risky the asset, or how risk-averse the individual. This is because a risk-averse EU maximizer behaves arbitrarily close to risk-neutral for risks sufficiently small. Other models, such as Yaari's (1987) Dual Theory (DT) model, predict "plunging," that is, for any risky asset, invest either nothing or the full capital available in the risky asset.[5] Yaari (1987) argues that the two classes of solutions produced by EU and DT (always an interior solution, or always a corner solution, respectively) are extreme, and that an intermediate between these two situations would be more satisfactory. The M-Δ model is able to bridge these two extremes and, more interestingly, also produce anew intermediate solution between them: it allows an investor to hold back or diversify depending on whether the performance of the risky asset meets a certain threshold. That is, the M-Δ model does not prescribe diversification in all cases, but it does not have the problem of a bang-bang solution (as DT does).

To show this, let us consider a simple asset allocation problem, involving a safe asset with 0 rate of return and a risky asset with a random rate of return θ distributed in the interval $[-1, a]$ with cumulative distribution F. Assume that the risky asset has a positive expected return $E[\theta] > 0$. Let K be the total amount available to invest, and x the amount to be invested in the risky asset, $0 \le x \le K$. Thus, the net payoff is described by the random variable $X = K + \theta x$. According to the M-Δ model, the investor's valuation of this portfolio is given by:

$$V(K + \theta x) = K + E[\theta]x - R(\theta x) = \Psi(x). \quad (4)$$

To examine how $V(K + \theta x)$ varies with the investment level, let us take its derivative with respect to x:

$$\Psi'(x) = E[\theta] - E\left[\int_{-1}^{\theta} H'((\theta - t)x)(\theta - t)dF(t)\right]. \quad (5)$$

To analyze the sign of $\Psi'(x)$ and state results below, it will be helpful to define the quantity:

$$S(\theta) = \frac{E[\theta]}{\frac{1}{2}G(\theta)}, \quad (6)$$

where G(.) is the Gini measure of risk. S(θ) defines a measure of performance of the risky asset: its return per "unit of risk," or its reward-to-risk ratio, akin to the Sharpe ratio. It depends solely on the characteristics of the risky asset, not on the investor's risk preferences, which are captured by H. S(.) so defined also happens to be the inverse of the Gini coefficient. Proposition 5 states that the solution to maximizing (4) can be either a corner solution (as in DT) or an interior solution (as in EU) all depending on S(θ) and the individual's pattern of risk aversion.

PROPOSITION 5. *Consider an individual behaving according to the M-Δ model with a differentiable, convex H function. For this individual, the optimal allocation to the risky asset may be to invest none, some, or all of the capital available in a risky asset depending on the performance of the risky asset* S(θ). *Specifically, if x^* denotes the optimal allocation:*

(i) *for* $S(\theta) \le H'(0)$, *the optimal allocation is the corner solution* $x^* = 0$;

[5] "Dual Theory" (Yaari 1987) is an axiomatic model of Rank-Dependent Utility (RDU) (Quiggin 1982) with linear utility. The Mean-½Gini model considered by Yitzhaki (1982) also predicts plunging, because it can be shown to be equivalent to RDU with linear utility, i.e., DT, and probability weighting function $w(p) = p^2$.

(ii) *for* $H'(0) < \mathrm{S}(\theta) < H'((a+1)K)$, *there exists a unique solution* $0 < x^{*} \leq K$;

(iii) *for* $\mathrm{S}(\theta) \geq H'((a+1)K)$, *the optimal allocation is* $x^{*} = K$.

If the individual has second order risk aversion ($H'(0) = 0$), we have $\mathrm{S}(\theta) > H'(0)$ whenever $\mathrm{E}[\theta] > 0$. In such case, the solution necessarily involves a strictly positive investment, $x^{*} > 0$, as under EU. An individual with first order risk aversion will hold back if the reward-to-risk performance of the risky asset remains below a certain threshold, and begin to diversify if the performance is beyond the threshold. The optimal diversification x^{*} will gradually augment with S(θ). That is, M-Δ provides more flexibility not only by allowing different individuals to behave differently as we saw in asset trading, but also by allowing the *same* individual to adopt qualitatively different behaviors in different situations.

Remark. Suppose the individual has $H' \leq 1$ (still with $H'' > 0$), that is, complies with the conditions of Proposition 2. Then, if $\mathrm{S}(\theta) > 1$, $\Psi'(x) > 0$ for all x. Thus for any security with $\mathrm{S}(\theta) > 1$, the maximum amount should be invested in the risky asset.[6]

In sum, the value of the portfolio, Eq. (4), can be monotone decreasing, monotone increasing, or non-monotone single peaked over the range of possible investment levels. Thus, the M-Δ model is able to produce a richer pattern of optimal solutions to the asset allocation problem, depending on the features of the risky asset relative to the investor's pattern of risk aversion over the range of the portfolio's outcome. To illustrate the point, Figure 1 provides an example of how the pattern of asset allocation under M-Δ is intermediate between those produced by EU and DT. The discontinuity in the DT pattern is the so-called plunging phenomenon.

[6] For H linear: $H'(y) = c$, with $0 \leq c \leq 1$, we have plunging, i.e., corner solutions, for *any* risky asset. If the risky asset is such that $\mathrm{S}(\theta) > c$, the optimal solution is $x^* = K$; if $\mathrm{S}(\theta) < c$, the optimal solution is $x^* = 0$; if $\mathrm{S}(\theta) = c$, the investor is indifferent toward any level of investment between 0 and K. This, of course, concords with Yaari (1987), since the case H linear yields a Mean-Gini model equivalent to Yaari's Dual Theory with quadratic, convex probability weighting.

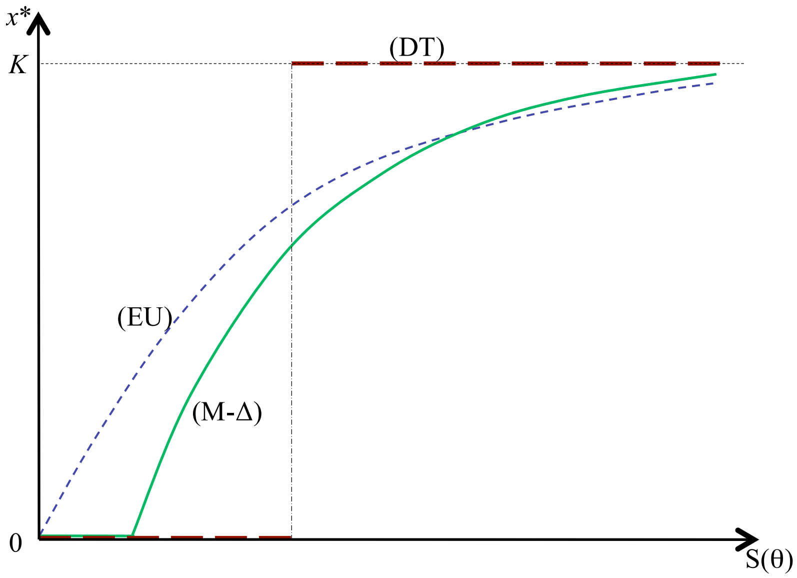

Figure 1. Optimal diversification in the risky asset under EU, DT, and M-Δ, as a function of the asset's performance ratio S(θ)

To wrap up this section, the M-Δ model can yield thresholds in diversification and no-trade intervals, without resorting to non-linear probability weighting. The decision maker's sensitivity to small risks (i.e., the derivative of *H* at 0) and the reward-to-risk ratio of the asset, as defined by S(.) in (6), both play special roles in these results. Figure 2 illustrates three main patterns of weighting of deviations in the M-Δ model. An individual with a linear *H* function with slope less than 1, as shown in (a), is a DT maximizer. An individual with pattern (c) will behave qualitatively as an EU maximizer. An individual with pattern (b) will behave as an EU maximizer in some cases, although he/she is not acting according to any specific utility function, and as a RDU maximizer in other cases, although he/she is not acting according to any non-additive probability weighting, all depending on the reward-to-risk performance of the gambles faced relative to the obtuseness of the kink in *H* at 0. Patterns (a) and (b) have first order risk aversion, while (c) has second order risk aversion. The strength of first order risk aversion depends on the size of $H'(0)$, thus M-Δ offers a continuum of some sort between first order and second order risk aversion.

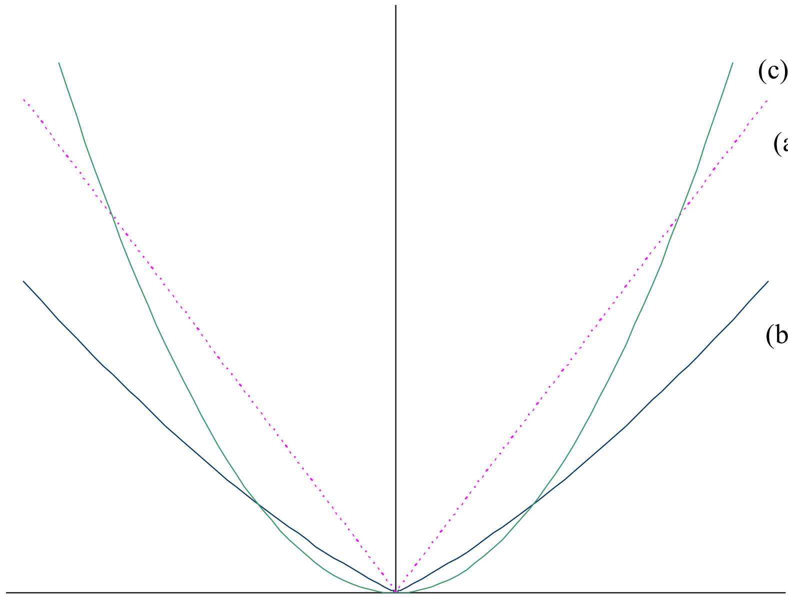

Figure 2. Three types of H functions: (a) Linear: Mean-Gini; (b) Nonlinear with a kink at 0; (c) Nonlinear with no kink at 0, e.g. Mean-Variance

6. Practical Aspects of Using the M-Δ Model

6.1 Assessment and Encoding of the H function

Using the M-Δ model requires obtaining an estimate of the deviations weighting function, H. This function can be assessed by eliciting the DM's preferences for simple gambles, much like in utility assessment. For example, the well-known methods of Certainty Equivalence (CE) and Probability Equivalence (PE) (Hershey and Schoemaker 1985) can be used to obtain a set of indifference statements, from which non-parametric estimates of values of H can be directly calculated. The CE and PE methods obtain indifference statements between a binary gamble $X = \{x, p\ ; 0, 1–p\}$ and a sure payoff s by varying s or p, respectively. This readily yields a point value estimate of H as follows:

$$V(s) = V(X)$$

$$s = px + (1–p)0 – p(1–p)H(x–0) = px – p(1–p)H(x)$$

Hence: $H(x) = (px – s)/(p(1–p))$.

Other elicitation methods can be used, of course. The elicitation questions will produce a system of linear equations in the unknowns, and these can be designed to have as many unknowns as (independent) equations so as to yield an exact, unique solution. For example, three outcome lotteries with equally spaced outcomes will result in only two unknowns on H. Consider $X = \{x+d, p_1; x, p_2; x–d, 1–p_1–p_2\}$ and $Y = \{y+d, q; y, 1–q\}$. A preference relation between X and Y involves up to 5 different outcomes, but only 2 unknowns, $H(2d)$ and $H(d)$. One other equation involving either or both of these unknowns would be sufficient to solve. Realizing this can provide great flexibility in designing easily solvable assessment questionnaires.

An arbitrary set of indifference statements may result in equations involving a set of unknown *H* values that cannot be solved exactly (as may also arise in assessing utility functions). In this case, a numerical method may be used to find a set of *H* values providing a best fit to the preference data, such as minimizing least-square error, or other appropriate criterion.

Selecting a parametric form for *H* can further simplify the assessment, for in that case, it reduces to the estimation of just the parameter(s) of the functional form. Examples of one-parameter *H* functions that satisfy the conditions of Propositions 1 and 2 are: $H(y) = y^2/(y+\alpha)$, $\alpha \geq 0$; or $H(y) = y + e^{-\frac{y}{\alpha}} - 1$ with $0 \leq \alpha \leq 1$. The latter features first order risk aversion for $0 \leq \alpha < 1$, and second order risk aversion for $\alpha = 1$ (see Proposition 3). A somewhat simple possibility could be to assume a piecewise linear function: $H(y) = 0$ for $0 \leq y \leq \delta$, $H(y) = y - \delta$ for $y > \delta$, that is, $H(y) = \max(0, y - \delta)$. This function lets risk aversion kick in when the spread of gambles exceeds δ, that is, deviations up to a certain level are just ignored, while deviations beyond this range are weighed linearly. The value of the parameter δ could be readily determined by asking: "what is the largest range of deviations for which risk would not be a concern at all?" Of course, this simple function satisfies SSD in the weak sense only for small gambles (whose spread does not exceed δ), because it is risk-neutral for such gambles.

The above functions are proposed just as illustrative examples, not to suggest that they are more desirable than other possible forms. The choice of an appropriate *H* function should be based on how well it accounts for the decision maker's risk preference patterns, and other considerations such as computational tractability.

6.2 Computational Issues

The computational tractability of a risk measure is an important consideration for use in large scale optimization problems. Risk measures that enable linear programming (LP) formulations are of special interest, due to the great computational efficiency of LP optimization (Mansini et al. 2003, Krzemienowski and Ogryczak 2005). The M-Δ model with non-linear weighting of deviations will, of course, not allow an LP formulation of mean-risk optimization, and thus sacrifice computational power. The Gini measure, which weights deviations linearly, does give rise to an LP specification, although it produces larger size optimization models than linear risk measures based on deviations from a target, such as, e.g., Mean Absolute Deviation. Indeed, for an optimization problem in which the data set consists of random variables (e.g. stock returns) with *n* discrete realizations, linear single target deviation measures will require *n* additional decision variables and associated constraints, while the Gini measure will require n^2 additional variables and associated constraints, specifically one for each deviation between any two realizations. The use of piecewise linear *H* functions in M-Δ would permit LP formulations, although this would produce LP problems larger than Mean-Gini, because each piecewise linear segment of the *H* function would necessitate its own set of decision variables in the LP formulation. Nonetheless, very large LP problems can routinely be solved efficiently nowadays. Therefore, piecewise linear H functions could offer a good compromise between solvability, by allowing LP formulations, and descriptive flexibility, by allowing a wide diversity of risk preference patterns.[7] See Mansini et al. (2007) for a study of this issue in the case of using Conditional Value-at-Risk (C-VaR) as a risk measure involving an LP formulation.

6.3 Numerical Illustration

We developed a set of compact computational formulas for calculating $\Delta(X)$ of any discrete distribution, for a number of parametric H functions, in spreadsheet applications. The functions accept data arrays as arguments, which can be either a set of observations of the random variable X, $\{x_1, x_2, ..., x_n\}$ (a one-dimensional array), or a frequency distribution, $\{p_i, x_i;\ i = 1, \ldots, n\}$ (a two-dimensional array). The parameter(s) of the H function can also be specified as arguments. These functions (available from the authors) can be loaded in the function library of the spreadsheet program, and used like other standard spreadsheet functions.

For numerical illustration purposes, we built the M-Δ efficient frontier for a basket of 15 stocks of large companies, selected to cover a diversified range of industries and geographical origins (North America, Europe, and Asia). For each stock, monthly prices adjusted for dividends and stock splits were obtained for the period from January 1999 to January 2010, allowing calculation of monthly returns for 11 years, that is, 132 observations. The efficient frontier was computed by minimizing the portfolio risk, Δ, for different levels of expected return set as a constraint, assuming no short sales (i.e., non-negativity constraints on stock weights). The decision variables in the optimization model are the weights on the stocks, with the constraint that they sum to 1. The H function used in the Δ risk measure was linear plus exponential form with parameter $\alpha = 0.2$.

Because Mean-Variance (M-V) plays a central role in modern finance and, despite shortcomings, is still the most widely used criterion to select portfolios of securities, it is relevant to compare the portfolios generated by M-Δ to those of M-V. The optimization model formulation for M-V is identical to that described above, except that the objective function is to minimize the portfolio variance instead of the Δ measure of risk. The portfolios produced by M-Δ and M-V have generally similar profiles, but with differences that appear to be systematic.

First, M-Δ appears to produce more diversified portfolios than M-V over the range of achievable returns, except at high returns. Everywhere except toward the northeast extremity of the efficient

[7] Care should be taken to verify that the results shown in Sections 3 and 5 for differentiable functions hold for piecewise linear functions meeting the required monotonicity and convexity conditions, which we believe to be the case.

frontier, the M-Δ portfolio includes more stocks than the M-V portfolio, also the mean absolute deviation of portfolio weights from equal weights (the so-called "naïve diversification" portfolio) is lower for M-Δ than for M-V. When high returns are required, the M-Δ and M-V portfolios tend, as expected, to become more concentrated on a smaller number of stocks, those capable of producing high expected returns. In those cases, M-Δ selects the same number or one or two fewer stocks than M-V. This phenomenon was again observed by replicating the analysis on a different set of 10 stocks, selected arbitrarily by ticker symbol alphabetical order from the CRSP data base of Wharton Research Data Services, with monthly return history from 1999 to 2008. Throughout the range of the efficient frontier, the M-Δ portfolios include a greater or equal number of stocks than M-V. Also, the mean absolute deviation of weights from equal weights is lower for M-Δ than M-V with one exception; again, at high returns (the same holds if the standard deviation is used as a measure of how spread out the weights are).

Second, M-Δ portfolios have return distributions with more pronounced skewness. This is reported in Table 1, showing a summary of comparative features of optimal portfolios obtained by M-Δ and M-V at different levels of target expected return covering the efficient frontier.

Table 1. Comparative features of M-Δ and M-V portfolios at different return levels, showing min, max and skewness of returns, and number of stocks selected out of 15 in the optimal portfolio

Mean portfolio (monthly) return		**M-Δ portfolio**	**M-V portfolio**
0.56%	*Min return*	-9.54%	-9.39%
	Max return	13.96%	11.25%
	Skewness	0.238	0.069
	Nbr. of stocks	9	8
0.65%	*Min return*	-9.02%	-8.90%
	Max return	17.19%	12.68%
	Skewness	0.527	0.200
	Nbr. of stocks	9	7
0.75%	*Min return*	-8.32%	-8.40%
	Max return	21.93%	15.25%
	Skewness	1.068	0.408
	Nbr. of stocks	7	5
0.85%	*Min return*	-8.30%	-8.45%
	Max return	27.73%	18.34%
	Skewness	1.803	0.671
	Nbr. of stocks	5	7
0.95%	*Min return*	-11.81%	-8.49%
	Max return	35.01%	23.90%
	Skewness	2.683	1.148
	Nbr. of stocks	6	6

As can be seen in Table 1, the M-Δ optimal portfolios have systematically and sizably higher positive skewness than the M-V optimal portfolios. Of course, positive skew in the portfolios is due to the presence of positive skew in the distributions of individual stock returns. As it turns out, the other data set of 10 stocks contained stocks with mostly negative skewness. For those stocks, M-Δ portfolios have stronger negative skewness than M-V portfolios. The point is that M-Δ efficient portfolios seem to retain more of the skewness of the component stocks than M-V portfolios. Empirical evidence indicates that investors often prefer positive skewness. To the extent that investors are able to screen stocks for positive skewness, M-Δ may help construct portfolios that preserve this desirable feature. Alternatively, because higher skewness is associated with higher risk-return combinations (as evidenced in Table 1), M-Δ may better allow investors to satisfy their desire for upside potential without having to sacrifice efficiency or take excessive risk exposure (see Mitton and Vorkink 2007 for a study of this issue under M-V).

Finally, the data set and optimization model at hand for this illustration gave us the opportunity to verify the predictions of Proposition 5. For this, we construct an efficient portfolio (either by M-Δ or M-V) and calculate the reward-to-risk ratio, S defined in (6), of this portfolio. By solving the optimal allocation between cash and the portfolio for investors with different risk aversion levels, we verify that investors with $H'(0) \leq$ S (which is equal to $1 - \alpha$ for the H function we used) do not wish to invest and prefer to keep all cash, while investors with $H'(0) >$ S allocate a positive proportion of their money to the portfolio, and the higher $H'(0)$, the larger this proportion.

7. Conclusion

The issue in selecting a risk measure for Mean-Risk analysis is that riskiness of a gamble, like intelligence of a person, is a complex, multifaceted concept: reducing it to a single index will necessarily leave out some aspects of it. The question is how much relevance and flexibility can be captured by a single index. The M-Δ model appears to increase behavioral realism without sacrificing normative compliance and parsimony. It owes its flexibility to its very mathematical structure, which is based on mutual deviations among outcomes instead of deviations from a given benchmark.

First, M-Δ combines normative properties that are highly desirable for the practice of risk management. With H convex increasing (but less steep than 1), M-Δ satisfies stochastic dominance properties and provides convexity in the risk measure. Second, it derives entirely from one of the most robust findings of behavioral research: that people's appraisal of something depends on the context in which it is embedded. Third, M-Δ is parsimonious: as EU or DT it will require the elicitation of only one function, which is less onerous and complex than the concurrent assessment of utility and probability weighting of RDU. Furthermore, despite having no more degrees of freedom than EU and DT, M-Δ produces a richer —and more empirically plausible— range of risk taking behaviors than either of these two models. For example, in specific circumstances, it can variously produce no-trade zones, declining to invest, or thresholds in diversification decisions, and allows an individual to adopt qualitatively different behaviors in different situations, thus producing commonsensical solutions that EU or DT cannot generate. Thus, M-Δ can help popularize the use of Mean-Risk analysis in areas of decision under risk where this approach has not been considered, or provide a more flexible criterion in the wide range of situations where it is already used, such as finance, project selection, or energy, to name a few. For large scale applications, consideration needs to be given to computational performance, and the M-Δ model in its general form (with non-linear *H*) may be less efficient. The trade-off between computational performance and risk behavior flexibility has to be balanced by the analyst based on the purposes at hand.

Further work can be pursued along several lines. We have considered only differentiable *H* functions for ease of deriving mathematical results and characterizing risk-taking behavior. For computational applications, it may be advantageous to use piecewise linear functions, which are continuous but not differentiable, having different right and left derivatives at a number of points. Further work could seek to derive our main theoretical results (particularly on SD) without assuming differentiability. Another area of interest would be to derive the predictions of the M-Δ model for preferences toward different kinds of insurance contracts, as Doherty and Eeckhoudt (1995) do for RDU. Also, the possibilities of using the M-Δ model as a basis for pricing assets, that is, deriving a CAPM, should be explored because this might incorporate more behavioral relevance in asset pricing.

Appendix A. Proof of Proposition 1

Let *X* and *Y* be two random variables, and *F* (*f*) and *G* (*g*), respectively, their cumulative distribution (probability density) functions.

Sufficiency. Suppose that $X \succ_{FSD} Y$. Let us show that this implies $V(X) > V(Y)$. We want to show:

$$V(X) = \int_{-\infty}^{+\infty} \left(x - \int_{-\infty}^{x} H(x-t)dF(t) \right) dF(x) > \int_{-\infty}^{+\infty} \left(x - \int_{-\infty}^{x} H(x-t)dG(t) \right) dG(x) = V(Y).$$

Define $u_F(x) = x - \int_{-\infty}^{x} H(x-t)dF(t)$. The first derivative of $u_F(.)$ is (being mindful that the variable of differentiation, *x*, appears both as a bound of the integral and inside the integral):

$$u'_F(x) = 1 - \int_{-\infty}^{x} H'(x-t)dF(t) - H(0)f(x) = 1 - \int_{-\infty}^{x} H'(x-t)f(t)dt.$$

Because $H'(y) \le 1$ for all y, $\int_{-\infty}^{x} H'(x-t)f(t)dt \le \int_{-\infty}^{x} f(t)dt \le 1$, and therefore $u'_F(x) \ge 0$, that is, u_F is increasing. Because $X \succ_{FSD} Y$ and u_F is increasing, we have: E[u_F(X)] ≥ E[u_F(Y)] (Hadar and Russell 1969), that is: $\int_{-\infty}^{+\infty} u_F(x)dF(x) \ge \int_{-\infty}^{+\infty} u_F(x)dG(x)$. Let us now show that $\int_{-\infty}^{+\infty} u_F(x)dG(x) > V(Y)$.

For this, we first show that $\int_{-\infty}^{x} -H(x-t)dF(t) \ge \int_{-\infty}^{x} -H(x-t)dG(t)$ for all x with a strict inequality for at least one x.

Integration by parts yields:

$$\int_{-\infty}^{x} -H(x-t)dF(t) = -H(x-t)F(t)\Big|_{-\infty}^{x} - \int_{-\infty}^{x} H'(x-t)F(t)dt = -\int_{-\infty}^{x} H'(x-t)F(t)dt.$$

Likewise $\int_{-\infty}^{x} -H(x-t)dG(t) = -\int_{-\infty}^{x} H'(x-t)G(t)dt$.

Therefore, $\int_{-\infty}^{x} -H(x-t)dF(t) - \int_{-\infty}^{x} -H(x-t)dG(t) = -\int_{-\infty}^{x} H'(x-t)\big(F(t)-G(t)\big)dt$. (7)

Because $X \succ_{FSD} Y$, $F(x) - G(x) \le 0$ for all x, and the inequality is strict for at least one x. Also $H' \ge 0$, therefore we have:

$-\int_{-\infty}^{x} H'(x-t)\big(F(t)-G(t)\big)dt \ge 0$ for all x, and there exists x such that the inequality is strict, unless $H' = 0$ everywhere (we will deal with that case separately). Hence, if there exists y such that $H'(y) > 0$, we have:

$\int_{-\infty}^{+\infty}\left(-\int_{-\infty}^{x} H'(x-t)\big(F(t)-G(t)\big)dt\right)dG(x) > 0$, that is, from (7):

$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{x} -H(x-t)dF(t)\right)dG(x) > \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{x} -H(x-t)dG(t)\right)dG(x)$, adding $\int_{-\infty}^{+\infty} xdG(x)$ on both sides:

$\int_{-\infty}^{+\infty} xdG(x) + \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{x} -H(x-t)dF(t)\right)dG(x) > \int_{-\infty}^{+\infty} xdG(x) + \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{x} -H(x-t)dG(t)\right)dG(x)$, i.e.

$$\int_{-\infty}^{+\infty} u_F(x)dG(x) > V(Y)$$

In sum, we have shown: $V(X) = \int_{-\infty}^{+\infty} u_F(x)dF(x) \ge \int_{-\infty}^{+\infty} u_F(x)dG(x) > V(Y)$.

Now, suppose $H'(y) = 0$ for all y. Then the Δ measure of risk is constant equal to 0 and $V(X) = \mathrm{E}[X]$ for all X. Because $X \succ_{FSD} Y$, we have $\mathrm{E}[X] > \mathrm{E}[Y]$, that is: $V(X) > V(Y)$.

Necessity. Suppose there exists x such that $H'(x) > 1$, that is, $H'(x) = 1+h$ with $h > 0$. We construct a pair of gambles X, Y such that $X \succ_{FSD} Y$ and $V(X) < V(Y)$. Going back to the definition of the derivative of a function, we have:

$$H'(x) = \lim_{\varepsilon \to 0}\left(\frac{H(x+\varepsilon) - H(x)}{\varepsilon}\right) = 1 + h.$$

Now going back to the definition of a limit, we know that there exists δ > 0 such that: for $0 < \varepsilon < \delta$

$\left|\frac{H(x+\varepsilon) - H(x)}{\varepsilon} - (1+h)\right| < h/2$. That is, there exists $\varepsilon > 0$ such that:

$$H(x+\varepsilon) > H(x) + (1 + h/2)\varepsilon. \qquad (8)$$

Let us take such ε, and consider the following binary gambles: $Y = \{x, p; 0, 1-p\}$, $X = \{x+\varepsilon, p; 0, 1-p\}$, with $p = h/(h+2)$. Clearly $X \succ_{FSD} Y$, and for these gambles we have:

$$\begin{aligned} V(X) &= p(x+\varepsilon) - p(1-p)H(x+\varepsilon) \\ &< p(x+\varepsilon) - p(1-p)(H(x)+(1+h/2)\varepsilon) && \text{by (8)} \\ &= V(Y) + p\varepsilon(1 - (1-p)(1+h/2)) \\ &= V(Y) && \text{because } (1-p)(1+h/2) = 1. \end{aligned}$$

This completes the proof.

Appendix B. Proof of Proposition 2

Let X and Y be two random variables, and F (f) and G (g), respectively, their cumulative distribution (probability density) functions.

Sufficiency. Suppose that $X \succ_{SSD} Y$. Let us show that this implies $V(X) > V(Y)$. We want to show:

$$V(X) = \int_{-\infty}^{+\infty}\left(x - \int_{-\infty}^{x} H(x-t)dF(t)\right)dF(x) > \int_{-\infty}^{+\infty}\left(x - \int_{-\infty}^{x} H(x-t)dG(t)\right)dG(x) = V(Y).$$

Define $u_F(.)$ as in Appendix A. The second derivative of u_F is (taking notice again that the variable of differentiation, x, appears both as a bound of the integral and inside the integral):

$$u_F''(x) = -\int_{-\infty}^{x} H''(x-t)dF(t) - H'(0)f(x).$$

Because $H''(y) \geq 0$ and $H'(0) \geq 0$, $u_F''(x) \leq 0$, thus u_F is concave. Also u_F is increasing, as we saw in Appendix A.

Because $X \succ_{SSD} Y$ and u_F is increasing, concave, we have $\int_{-\infty}^{+\infty} u_F(x)dF(x) \geq \int_{-\infty}^{+\infty} u_F(x)dG(x)$ (Hadar and Russell 1969).

Let us now show that $\int_{-\infty}^{+\infty} u_F(x)dG(x) > V(Y)$.

Integration by parts on the right side of Equation (7) yields:

$$-\int_{-\infty}^{x} H'(x-t)\big(F(t)-G(t)\big)dt = -H'(x-t)\int_{-\infty}^{t}\big(F(y)-G(y)\big)dy\Bigg|_{-\infty}^{x} - \int_{-\infty}^{x} H''(x-t)\int_{-\infty}^{t}\big(F(y)-G(y)\big)dydt$$

$$= -H'(0)\int_{-\infty}^{x}\big(F(y)-G(y)\big)dy - \int_{-\infty}^{x} H''(x-t)\int_{-\infty}^{t}\big(F(y)-G(y)\big)dydt \qquad (9)$$

Because $X \succ_{SSD} Y$, $\int_{-\infty}^{x}(F(t)-G(t))dt \leq 0$ for all x, and the inequality is strict for at least one x. Thus, because $H'(0) \geq 0$ and $H'' \geq 0$, the expression in (9) is positive for all x and, unless both $H'(0) = 0$ and $H'' = 0$, this expression is strictly positive for some x (because $\int_{-\infty}^{x}(F(t)-G(t))dt < 0$ for some x, due to $X \succ_{SSD} Y$). The case $H'(0) = 0$ and $H'' = 0$ simultaneously is excluded because it corresponds to H constant equal to 0, which is not strictly increasing.

Thus, from (7) we have for all x: $-\int_{-\infty}^{x} H(x-t)dF(t) \geq -\int_{-\infty}^{x} H(x-t)dG(t)$ with a strict inequality for some x. Therefore:

$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{x} -H(x-t)dF(t)\right)dG(x) > \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{x} -H(x-t)dG(t)\right)dG(x).$$

Adding $\int_{-\infty}^{+\infty} xdG(x)$ on both sides of the preceding inequality, we get: $\int_{-\infty}^{+\infty} u_F(x)dG(x) > V(Y)$.

In sum, we have shown: $V(X) = \int_{-\infty}^{+\infty} u_F(x)dF(x) \geq \int_{-\infty}^{+\infty} u_F(x)dG(x) > V(Y)$.

Necessity. Suppose there exists x_0 such that $H''(x_0) < 0$. Then, there exists $\varepsilon > 0$ such that: $H(x_0+\varepsilon) - H(x_0) < H(x_0) - H(x_0-\varepsilon)$. Using such ε, we can construct a gamble involving x_0 and a mean preserving spread of it (Rothschild and Stiglitz 1970) that would cause a *decrease* in Δ.

Define: $H'_+(x_0) = \big(H(x_0+\varepsilon) - H(x_0)\big)/\varepsilon$ and $H'_-(x_0) = \big(H(x_0) - H(x_0-\varepsilon)\big)/\varepsilon$. Thus $H'_+(x_0) < H'_-(x_0)$. Take the binary gamble $X = \{x_0, p; 0, 1-p\}$, with $0 < p/(1-p) < H'_-(x_0) - H'_+(x_0)$. Now consider the three-outcome gamble $Y = \{x_0+\varepsilon, p/2; x_0-\varepsilon, p/2; 0, 1-p\}$. Y is a mean preserving spread of X, that is, $X \succ_{SSD} Y$. However, we have:

$$\begin{aligned}
\Delta(Y) &= p(1-p)H(x_0+\varepsilon)/2 + p(1-p)H(x_0-\varepsilon)/2 + (p/2)^2 H(2\varepsilon) \\
&= \frac{p}{2}(1-p)\left(H(x_0+\varepsilon) + H(x_0-\varepsilon) + \frac{p}{2(1-p)} H(2\varepsilon) \right) \\
&\le \frac{p}{2}(1-p)\left(H(x_0+\varepsilon) + H(x_0-\varepsilon) + \frac{p}{(1-p)} \varepsilon \right) \qquad \text{because } H(2\varepsilon) \le 2\varepsilon \\
&< \frac{p}{2}(1-p)\left(H(x_0+\varepsilon) + H(x_0-\varepsilon) + (H'_-(x_0) - H'_+(x_0))\varepsilon \right) \quad \text{by choice of } p \\
&= p(1-p)H(x_0) \\
&= \Delta(X)
\end{aligned}$$

that is, $\Delta(Y) < \Delta(X)$, hence, $V(Y) > V(X)$. This completes the proof.

Appendix C. Proof of Proposition 3

Consider a non-constant random variable X and denote by F its cumulative distribution. Under the M-Δ model, the risk premium associated with the gamble αX is: $\pi(\alpha) = \Delta(\alpha X)$. For first order risk aversion we need to show that $d\pi/d\alpha|_{\alpha=0+} > 0$; for second order risk aversion, we need to show that $d\pi/d\alpha|_{\alpha=0} = 0$ and $d^2\pi/d\alpha^2|_{\alpha=0+} > 0$.

We have: $\pi(\alpha) = \Delta(\alpha X) = \mathrm{E}\left[\int_{-\infty}^{X} H(\alpha(X-y))dF(y)\right]$,

thus: $\dfrac{d\pi}{d\alpha} = \mathrm{E}\left[\int_{-\infty}^{X} H'(\alpha(X-y))(X-y)dF(y)\right]$

therefore, $\left.\dfrac{d\pi}{d\alpha}\right|_{\alpha=0} = \mathrm{E}\left[\int_{-\infty}^{X} H'(0)(X-y)dF(y)\right] = H'(0)\,\mathrm{E}\left[\int_{-\infty}^{X} (X-y)dF(y)\right] = H'(0) \times \tfrac{1}{2}\mathrm{G}(X)$,

where $\mathrm{G}(X)$ is the Gini Mean Difference measure.

Because X is non-constant, $\mathrm{G}(X) > 0$. Therefore, $d\pi/d\alpha|_{\alpha=0+} > 0$ if and only if $H'(0) > 0$, which shows Part (i).

The above also shows that $d\pi/d\alpha|_{\alpha=0} = 0$ if and only if $H'(0) = 0$. Besides:

$$\frac{d^2\pi}{d\alpha^2} = \mathrm{E}\left[\int_{-\infty}^{X} H''(\alpha(X-y))(X-y)^2 dF(y)\right],$$

thus, $\left.\dfrac{d^2\pi}{d\alpha^2}\right|_{\alpha=0^+} = H''(0) \times \mathrm{E}\left[\int_{-\infty}^{X} (X-y)^2 dF(y)\right] = H''(0) \times \sigma^2(X)$.

Because X is non-constant, $\sigma^2(X) > 0$, hence $d^2\pi/d\alpha^2|_{\alpha=0+} > 0$ if and only if $H''(0) > 0$, that is, H is strictly convex at 0, which shows (ii). Note that the convexity of H at 0 is only there to ensure that the risk attitude is negative (averse), which is the case of interest, as opposed to positive (seeking) or neutral. Indeed, if we had $H''(0) \le 0$ when $H'(0) = 0$, then H would be decreasing negative (or constant), implying risk-seeking (or neutral) behavior.

Appendix D. Proof of Corollary 4

Let π be the price of one unit of the asset. Under the M-Δ model, the individual's utility increase for buying a quantity $\alpha \geq 0$ of the asset is:

$$V(\alpha(X-\pi)) = \mathrm{E}[\alpha(X-\pi)] - \Delta(\alpha(X-\pi)) = \alpha(\mathrm{E}[X] - \pi) - \Delta(\alpha X).$$

The analysis of interest here is how $V(\alpha(X-\pi))$ varies with α: in particular, starting from no position ($\alpha = 0$), is there a positive increase in utility for moving to $\alpha > 0$? To understand the variations in $V(\alpha(X-\pi)) = \Psi(\alpha)$ as a function of α, let us study its derivative:

$$\Psi'(\alpha) = \mathrm{E}[X] - \pi - \mathrm{E}\left[\int_{-\infty}^{X} H'(\alpha(X-y))(X-y)dF(y)\right].$$

Because H is convex and differentiable, H' is continuous increasing, therefore Ψ' is continuous decreasing in α. Because $\Psi(0) = 0$, $\Psi(\alpha)$ can take > 0 values if and only if there exists a range where Ψ' is positive. Such a range exists if and only if $\Psi'(0) > 0$, that is:

$$\Psi'(0) = \mathrm{E}[X] - \pi - \tfrac{1}{2}\mathrm{G}(X)H'(0) > 0.$$

Thus the individual will be willing to buy (some of) the asset if and only if:

$$\pi < \mathrm{E}[X] - \tfrac{1}{2}\mathrm{G}(X)H'(0).$$

The optimal amount to buy will be determined by where Ψ' becomes 0, if at all. Note that if the individual complies with the conditions of Proposition 2, that is, $H' \leq 1$, we have $\Psi'(\alpha) \geq \mathrm{E}[X] - \pi - \tfrac{1}{2}\mathrm{G}(X)$ for all $\alpha \geq 0$. Then, if the price of the asset is such that $\pi < \mathrm{E}[X] - \tfrac{1}{2}\mathrm{G}(X)$, $V(\alpha(X-\pi))$ increases indefinitely with α. For such a favorably priced asset, the individual would buy the maximum quantity possible of the asset, subject to budget or other constraints.

Now consider short selling the asset. The individual's utility resulting from selling a quantity $\alpha \geq 0$ of the asset is:

$$V(\alpha(\pi - X)) = \alpha(\pi - \mathrm{E}[X]) - \Delta(-\alpha X) = \alpha(\pi - \mathrm{E}[X]) - \Delta(\alpha X).$$

Reasoning as before, we find that this utility can be positive if and only if: $\pi - \mathrm{E}[X] - \tfrac{1}{2}\mathrm{G}(X)H'(0) > 0$. That is, the individual will sell the asset if and only if its price is such that:

$$\pi > \mathrm{E}[X] + \tfrac{1}{2}\mathrm{G}(X)H'(0).$$

Therefore, for prices within the interval $\left[\mathrm{E}[X] - \tfrac{1}{2}\mathrm{G}(X)H'(0), \mathrm{E}[X] + \tfrac{1}{2}\mathrm{G}(X)H'(0)\right]$, the individual is not willing to hold a position on the asset. If the individual has second order risk aversion, $H'(0) = 0$, the no-trade interval reduces to the single point $\{\mathrm{E}[X]\}$, as under EU.

Appendix E. Proof of Proposition 5

Because H is convex, H' is increasing, therefore $\Psi'(x)$ in (5) is decreasing in x and the question is whether it changes sign in the interval [0, K]. A strictly positive solution, $x^* > 0$, can occur if, and only if, $\Psi'(0) > 0$. From (5), we have:

$$\Psi'(0) = \mathrm{E}[\theta] - \mathrm{E}\left[\int_{-1}^{\theta} H'(0)(\theta - t)dF(t)\right] = \mathrm{E}[\theta] - \frac{1}{2}H'(0)\,\mathrm{G}(\theta).$$

Thus $\Psi'(0) > 0$ if and only if $\mathrm{S}(\theta) > H'(0)$, which shows (i).

Working from (5) again, we can derive the following inequalities:

$$\begin{aligned}\Psi'(x) &= \mathrm{E}[\theta] - \mathrm{E}\left[\int_{-1}^{\theta} H'((\theta - t)x)(\theta - t)dF(t)\right]\\ &\geq \mathrm{E}[\theta] - \mathrm{E}\left[\int_{-1}^{\theta} H'((a+1)x)(\theta - t)dF(t)\right]\\ &= \mathrm{E}[\theta] - \tfrac{1}{2}H'((a+1)x)\,\mathrm{G}(\theta)\\ &\geq \mathrm{E}[\theta] - \tfrac{1}{2}H'((a+1)K)\,\mathrm{G}(\theta)\end{aligned}$$

Thus, if $\mathrm{S}(\theta) \geq H'((a+1)K)$, then $\Psi'(x) \geq 0$ for $0 \leq x \leq K$, and the optimal solution is at $x^* = K$. An interior, unique solution away from the bound K will occur if $\Psi'(K) < 0$. As an example of this, if θ has a two point distribution with all probability mass at b and a ($a > b$), it is easy to verify that the solution is such that $0 < x^* < K$ when $H'(0) < \mathrm{S}(\theta) < H'((a-b)K)$.

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