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A note on comparative ambiguity aversion and justifiability*

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Abstract

We consider a decision maker who ranks actions according to the smooth ambiguity criterion of Klibanoff et al. (2005). An action is justifiable if it is a best reply to some belief over probabilistic models. We show that higher ambiguity aversion expands the set of justifiable actions. In turn, this implies that higher ambiguity aversion expands the set of rationalizable actions of a game. Our results follow from a generalization of the duality lemma of Wald (1949) and Pearce (1984).

1 Introduction

In this paper we consider a decision maker (DM) who ranks alternatives under uncertainty. The DM holds subjective beliefs over a set of probabilistic models $\Sigma \subseteq \Delta(S)$, where S is a set of states of nature, or actions of an opponent in a game. We assume that the DM ranks choices according to the smooth ambiguity criterion of Klibanoff et al. (2005). With this, we show that higher ambiguity aversion expands the set of actions that are best replies to at least one belief; for brevity, we call such actions "justifiable." Empirically, they are the actions that an outside observer can infer as possible from the knowledge of the DM attitudes toward uncertainty. Our result shows that such inference becomes rougher as ambiguity aversion increases. In turn, this implies that higher ambiguity aversion expands the set of rationalizable actions of a game, where the rationalizability concept is modified

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to take into account ambiguity attitudes. We derive our result from a generalization of the duality lemma of Wald (1949) and Pearce (1984) that should be of independent interest.

Another consequence of the same duality lemma is that, under ambiguity neutrality, higher risk aversion expands the set of justifiable actions, and hence the set of rationalizable actions in a game. This risk version of our result was independently obtained by Weinstein (2013) for subjective expected utility maximizers in finite games.[1] For expositional purposes and to exploit economies of scope, we present the results about comparative risk aversion and comparative ambiguity aversion jointly.

The result is not intuitively obvious. Indeed, if the DM deems possible very different probabilistic models, then higher ambiguity aversion increases the attractiveness of "safe" actions whose objective expected utility is somewhat low for each model, but does not change much with the model. Given the *same* belief over probabilistic models, actions that give high expected utility for some models and low expected utility for other models become instead *less* attractive. Yet, an increase in ambiguity aversion cannot make such actions unjustifiable, because —regardless of ambiguity attitudes— they can always be justified by extreme beliefs assigning high probability to models under which they yield high objective expected utility.

This comparative statics result is analogous to another result of ours, which also relies on the smooth ambiguity criterion: higher ambiguity aversion expands the set of self-confirming equilibria (Battigalli et al., 2015). However, as argued in the discussion (Section 5), the similarity between these results is only superficial, because they rely on different assumptions about the decision or game problem and have very different explanations.

The rest of the paper is structured as follows. Section 2 presents the decision criterion we use. Section 3 contains our main results, whose implications for rationalizability are explored in Section 4. Our findings are discussed in Section 5 where we also briefly discuss alternative decision models (e.g., preferences representable by quasiconcave functionals). All proofs are relegated in Section 6, where we state and prove the abstract version of the duality lemma of Wald (1949) and Pearce (1984) that underlies our analysis.

2 Criterion

We consider a standard decision problem under uncertainty with action space A, state space S and payoff function $r : A \times S \to \mathbb{R}$. We assume that A and S are separable metric spaces and r is continuous in each component and bounded. The payoff function may be interpreted as the composition of a consequence function, or game form, $g : A \times S \to C$, where C is the consequence space, and a von Neumann-Morgenstern utility function $u : C \to \mathbb{R}$, that is, $r = u \circ g$. For interpretational and expositional purposes, we assume that consequences are

[1] We thank Amanda Fridenberg for letting us know about this work.

monetary, i.e., $C \subseteq \mathbb{R}$.

Let Σ be a nonempty closed subset of the collection $\Delta(S)$ of all Borel probability measures σ on the state space, each σ being interpreted as a possible stochastic model for states.[2] Actions are ranked by the smooth ambiguity criterion $V_{\phi,r} : A \times \Delta(\Sigma) \to \mathbb{R}$ given by

$$V_{\phi,r}(a,\mu) = \phi^{-1}\left(\int_\Sigma \phi\left(\int_S r(a,s)\,\sigma(\mathrm{d}s)\right)\mu(\mathrm{d}\sigma)\right)$$

where $\phi : \overline{\mathrm{co}}\,\mathrm{Im}\,r \to \mathbb{R}$ is strictly increasing and continuous, and μ is a subjective probability measure on the posited set of stochastic models Σ.[3] Function ϕ is also known as the *second-order utility* because it can be interpreted as the "utility of objective expected utility." When ϕ is the identity, the criterion reduces to standard subjective expected utility, i.e.,

$$V_{\mathrm{Id},r}(a,\mu) = \int_\Sigma \int_S r(a,s)\,\sigma(\mathrm{d}s)\mu(\mathrm{d}\sigma) = \int_S r(a,s)\,\sigma_\mu(\mathrm{d}s),$$

where, for any (measurable) event $E \subseteq S$, $\sigma_\mu(E) = \int_\Sigma \sigma(E)\mu(\mathrm{d}\sigma)$ is the predictive probability of E induced by μ. Function ϕ captures the DM's attitudes toward ambiguity, whereas $r = u \circ g$ captures attitudes toward risk.

As a matter of interpretation, we emphasize that A represents the set of all feasible choices, possibly including some choices that yield an objectively random outcome for at least one state. In Subsection 3.5, we demonstrate how our framework can formally encompass this possibility.[4] Here, we only clarify our methodological position: it may be the case that not all randomizations are either feasible or credibly implementable. In particular, choosing an action according to the realization of a random variable is a credible "randomization" only if the actions with positive probability are optimal in A.

3 Main results

3.1 Justifiability

Definition 1 *The collection of* justifiable actions *for ambiguity attitudes ϕ and risk attitudes r given Σ is*

$$\mathcal{J}_{\phi,r}(\Sigma) = \{a \in A : \exists \mu \in \Delta(\Sigma), \forall a' \in A,\ V_{\phi,r}(a,\mu) \geq V_{\phi,r}(a',\mu)\}.$$

In words, $\mathcal{J}_{\phi,r}(\Sigma)$ is the collection of all actions that are best replies, according to $V_{\phi,r}$, to some belief μ over Σ.[5]

[2] In this presentation of the results, to fix ideas the reader can think of all sets defining the decision problem as finite. In Section 6, we prove our results in the general setting.

[3] Here $\overline{\mathrm{co}}\,\mathrm{Im}\,r$ is the smallest closed interval that contains the image $\mathrm{Im}\,r$ of the payoff function r.

[4] See also Section 3.3 in Marinacci (2015).

[5] The terminology is inspired by Milgrom and Roberts (1991). Lehrer and Teper (2011) have introduced

3.2 Risk attitudes

We first consider higher risk aversion in the subjective expected utility case. In our monetary setup, $r' = \psi \circ r = (\psi \circ u) \circ g$, with ψ concave, continuous, and strictly increasing, is the payoff function of a more risk averse DM. The following proposition says that, assuming ambiguity neutrality ($\phi = \mathrm{Id}$), a more risk averse DM has more justifiable actions. We denote by δ_s the Dirac probability measure supported by state s.

Proposition 1 *Let S be compact and $\{\delta_s\}_{s \in S} \subseteq \Sigma$. If $r' = \psi \circ r$ for some concave, continuous, and strictly increasing function $\psi : \overline{\mathrm{co}}\, \mathrm{Im}\, r \to \mathbb{R}$, then $\mathcal{J}_{\mathrm{Id},r}(\Sigma) \subseteq \mathcal{J}_{\mathrm{Id},r'}(\Sigma)$.*

Example 1 Consider the following game form with monetary consequences:

$g:$	s'	s''
t	0	1
m	$\frac{1}{3}$	$\frac{1}{3}$
b	1	0

(1)

Suppose the DM is a subjective expected utility maximizer ($\phi = \mathrm{Id}$). If the DM is risk neutral ($r = g$), action m is unjustifiable: for every belief $\mu \in \Delta(\Sigma)$, indeed

$$V_{\mathrm{Id},g}(m,\mu) = \frac{1}{3} < \frac{1}{2} \leq \max\{V_{\mathrm{Id},g}(b,\mu), V_{\mathrm{Id},g}(t,\mu)\} = \max\{\sigma_\mu(s'), 1 - \sigma_\mu(s')\}.$$

If Σ contains the two Dirac measures $\delta_{s'}$ and $\delta_{s''}$, i.e., S is embedded in Σ, then $\mathcal{J}_{\mathrm{Id},g}(\Sigma) = \{t, b\}$. In particular b (resp., t) is a best reply to μ if and only if $\sigma_\mu(s') \geq 1/2$ (resp., $\sigma_\mu(s') \leq 1/2$). Now suppose that the DM is risk averse, with a power utility function $u_\theta(c) = c^{1/\theta}$ (where $\theta \geq 1$ parametrizes risk aversion). Then, the payoff function is $r_\theta = u_\theta \circ g$ and

$$\mathcal{J}_{\mathrm{Id},r_\theta}(\Sigma) = \begin{cases} \{t,b\}, & \theta < \bar{\theta}, \\ \{t,m,b\}, & \theta \geq \bar{\theta}, \end{cases}$$

where $\bar{\theta} = \log_2 3$ solves $u_\theta(g(m)) = 1/2$. The collection of justifiable actions thus expands as θ increases. Note, however, that the sets of beliefs justifying the risky actions b and t *shrink* as soon as θ increases above the threshold.[6] Also note that the assumption $\{\delta_s\}_{s \in S} \subseteq \Sigma$ is needed. Otherwise, assuming that Σ is compact, either $V_{\mathrm{Id},r_\theta}(b,\mu) = \sigma_\mu(s')$ or $V_{\mathrm{Id},r_\theta}(t,\mu) = \sigma_\mu(s'')$ would be bounded below 1 whereas $V_{\mathrm{Id},r_\theta}(m,\mu) = (1/3)^{1/\theta} \to 1$ as $\theta \to +\infty$. ▲

a new class of "justifiable preferences" under uncertainty. The connection with our notion of justifiability is, however, limited: ours is just the old best-reply-to-some-belief concept, applied here to the smooth ambiguity model.

[6] We comment in more detail on this for the analogous case of increasing ambiguity aversion.

3.3 Ambiguity attitudes

Next we consider a change in ambiguity attitudes. The following proposition says that a more ambiguity averse DM has a larger set of justifiable actions. As argued in the Introduction, the result is not intuitively obvious. Note that the hypothesis of Σ being compact is weaker than the hypothesis, made in the previous proposition, of S being compact.

Proposition 2 *Let Σ be compact. If $\phi' = \varphi \circ \phi$ for some concave, continuous, and strictly increasing function $\varphi : \overline{\mathrm{co}}\,\mathrm{Im}\,\phi \to \mathbb{R}$, then $\mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\phi',r}(\Sigma)$.*

Example 2 Consider again the game form (1) and suppose, just for simplicity, that the DM is risk-neutral, i.e., $r = g$, and $\Sigma = \{\delta_{s'}, \delta_{s''}\}$. Let $\phi_\theta(x) = x^{1/\theta}$, where $\theta \geq 1$ parametrizes ambiguity aversion. Then, it can be shown that the belief μ that maximizes $V_{\phi_\theta,g}(m,\mu) - \max\{V_{\phi_\theta,g}(t,\mu), V_{\phi_\theta,g}(b,\mu)\}$ satisfies $\mu(\delta_{s'}) = \mu(\delta_{s''}) = 1/2$ (cf. Battigalli et al. 2015, Lemma 6). With this, calculations similar to those of Example 1 yield

$$\mathcal{J}_{\phi_\theta,r}(\Sigma) = \begin{cases} \{t,b\}, & \theta < \bar{\theta}, \\ \{t,m,b\}, & \theta \geq \bar{\theta}, \end{cases}$$

where $\bar{\theta} = \log_2 3$ solves $\phi_\theta(g(m)) = 1/2$. The collection of justifiable actions thus expands as θ increases. Note, however, that the sets of beliefs justifying ambiguous actions b and t *shrink*: In fact, $V_{\phi_\theta,g}(m,\mu) = 1/3$ regardless of θ, whereas $V_{\phi_\theta,g}(b,\mu) = \mu^\theta(\delta_{s'})$ and $V_{\phi_\theta,g}(t,\mu) = \mu^\theta(\delta_{s''})$ is strictly decreasing in θ; as θ increases above the threshold $\bar{\theta}$, the probability $\mu(\delta_{s'})$ must increase to make b a best reply, and similarly for t (see Figure 1). On the horizontal (resp., vertical) axis we report the second-order utility of objective expected utility given model $\sigma^1 = \delta_{s'}$ (resp., $\sigma^2 = \delta_{s''}$). As ambiguity aversion increases, the expected utility vector corresponding to action m shifts North-Eastward. ▲

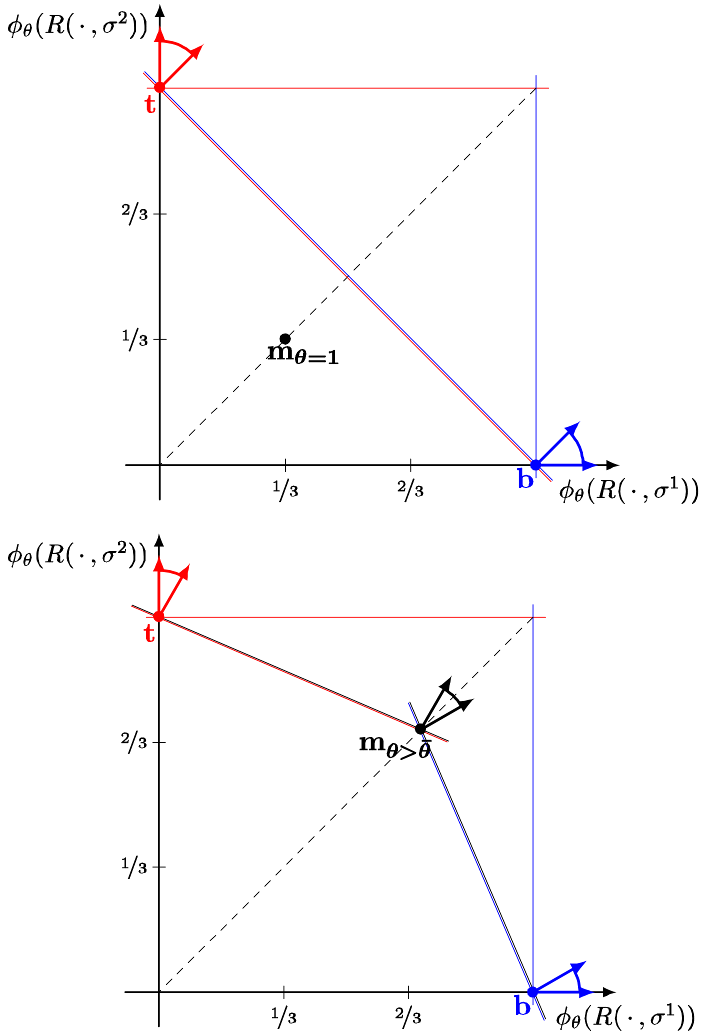

Figure 1. As θ increases, the sets of beliefs justifying b and t shrink.

In sum, higher aversion to either ambiguity or risk (under ambiguity neutrality) expands the collection of justifiable actions. As for the set of beliefs justifying any action, we can only say that, if it is not empty, an increase in risk or ambiguity aversion cannot make it empty. Propositions 1 and 2 are purely comparative results that do not require either risk or ambiguity aversion (i.e., the functions r and ϕ are not assumed to be concave).

The proof is based on an abstract version of the duality lemma of Pearce (cf. Pearce 1984, Lemma 3) presented in Section 6, which is a version of the classic Complete Class Theorem of Wald (see, e.g., Wald, 1949, Theorem 2.2).

3.4 Extreme ambiguity attitudes: a discontinuity

As ambiguity aversion becomes higher and higher, i.e., as $-\phi''/\phi' \uparrow +\infty$, we have larger and larger collections $\mathcal{J}_{\phi,r}(\Sigma)$ of justifiable actions. It is natural to wonder how this property relates with the well known fact (Klibanoff et al., 2005, p. 1867) that in this case the criterion $V_{\phi,r}(a,\mu)$ tends to the maxmin criterion $V_{\infty,r}(a,\mu)=\min_{\sigma\in\operatorname{supp}\mu}\int_S r(a,s)\,\sigma(\mathrm{d}s)$, a version of the classic criterion of Gilboa and Schmeidler (1989). Denote by

$$\mathcal{J}_{\infty,r}(\Sigma)=\{a\in A:\exists\mu\in\Delta(\Sigma),\forall a'\in A,\,V_{\infty,r}(a,\mu)\geq V_{\infty,r}(a',\mu)\}$$

the collection of actions that are maxmin justifiable. Let $\bigcup_{\phi\in\Phi}\mathcal{J}_{\phi,r}(\Sigma)$ be the collection of all actions that are justifiable with the smooth ambiguity criterion, for some $\phi\in\Phi$, where Φ is the collection of continuous and strictly increasing real-valued functions on $\overline{\mathrm{co}}\,\mathrm{Im}\,r$. Interestingly, despite the continuity in value, the next two examples show that none of the inclusions $\bigcup_{\phi\in\Phi}\mathcal{J}_{\phi,r}(\Sigma)\subseteq\mathcal{J}_{\infty,r}(\Sigma)$ and $\mathcal{J}_{\infty,r}(\Sigma)\subseteq\bigcup_{\phi\in\Phi}\mathcal{J}_{\phi,r}(\Sigma)$ holds in general.[7] We begin by showing that the latter inclusion may fail.

Example 3 Given $0\leq\varepsilon<1$, consider the payoff function:

$r:$	s'	s''
t	0	1
m	ε	ε
b	1	0

Suppose $\{\delta_s\}_{s\in S}\subseteq\Sigma$ where Σ is finite. We have $m\in\mathcal{J}_{\infty,r}(\Sigma)$ for each $\varepsilon\geq0$. Moreover, if $\varepsilon>0$ there exists $\phi\in\Phi$ such that $m\in\mathcal{J}_{\phi,r}(\Sigma)$. However, if $\varepsilon=0$ there is no such ϕ. This shows that the inclusion $\mathcal{J}_{\infty,r}(\Sigma)\subseteq\bigcup_{\phi\in\Phi}\mathcal{J}_{\phi,r}(\Sigma)$ may fail. ▲

Next we show that, quite surprisingly, also the converse inclusion may fail.

Example 4 Consider the payoff function:

$r:$	s'	s''
t	0	1
h	c'	c''
m	ε	ε
b	1	0

Assume $\Sigma=\{\delta_{s'},\delta_{s''}\}$ and, wlog, $c'\leq c''$. If $c'=\varepsilon=1/3<c''<1$, it is easy to check that $m\in\mathcal{J}_{\infty,r}(\Sigma)$ but $m\notin\bigcup_{\phi\in\Phi}\mathcal{J}_{\phi,r}(\Sigma)$. This confirms the finding of the previous, simpler, example. More interesting, assume $0\leq c'<\varepsilon<1/2\leq c''<1\leq c'+c''$. For concreteness,

[7] For a characterization of the set $\bigcup_{\phi\in\Phi}\mathcal{J}_{\phi,r}(\Sigma)$ see Subsection 5.2.

set $c' = 2/7$, $\varepsilon = 1/3$ and $c'' = 6/7$. If the DM is ambiguity neutral, $\phi = \mathrm{Id}$, action h is justifiable. For, if $\mu \in \Delta(\Sigma)$ is the uniform belief $\mu(\{\delta_{s'}\}) = 1/2 = \mu(\{\delta_{s''}\})$, then

$$V_{\mathrm{Id},r}(h,\mu) = \frac{4}{7} > \frac{1}{2} = V_{\mathrm{Id},r}(t,\mu) = V_{\mathrm{Id},r}(b,\mu) > \frac{1}{3} = V_{\mathrm{Id},r}(m,\mu).$$

Hence, $h \in \mathcal{J}_{\mathrm{Id},r}(\Sigma)$ and so, by Proposition 2, $h \in \mathcal{J}_{\phi,r}(\Sigma)$ for all concave $\phi \in \Phi$. On the other hand, $h \notin \mathcal{J}_{\infty,r}(\Sigma)$. For, any $\mu \in \Delta(\Sigma)$ has to be in one of the following three cases:

(i) $\operatorname{supp}\mu = \{\delta_{s'}\}$: we have

$$V_{\infty,r}(b,\mu) = 1 > \frac{1}{3} = V_{\infty,r}(m,\mu) > \frac{2}{7} = V_{\infty,r}(h,\mu) > 0 = V_{\infty,r}(t,\mu).$$

(ii) $\operatorname{supp}\mu = \{\delta_{s''}\}$: we have

$$V_{\infty,r}(t,\mu) = 1 > \frac{6}{7} = V_{\infty,r}(h,\mu) > \frac{1}{3} = V_{\infty,r}(m,\mu) > 0 = V_{\infty,r}(b,\mu).$$

(iii) $\operatorname{supp}\mu = \{\delta_{s'}, \delta_{s''}\}$: we have

$$V_{\infty,r}(m,\mu) = \frac{1}{3} > \frac{2}{7} = V_{\infty,r}(h,\mu) > 0 = V_{\infty,r}(t,\mu) = V_{\infty,r}(b,\mu).$$

Points (i)-(iii) show that $h \notin \mathcal{J}_{\infty,r}(\Sigma)$, i.e., for each $\mu \in \Delta(\Sigma)$ action h is never optimal (hence, justifiable) for the maxmin criterion $V_{\infty,r}$. We conclude that the inclusion $\bigcup_{\phi\in\Phi} \mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\infty,r}(\Sigma)$ fails. ▲

Summing up, the previous two examples show that there are actions that are justifiable under the maxmin criterion $V_{\infty,r}$ but not under any smooth ambiguity criterion $V_{\phi,r}$ with $\phi \in \Phi$, as well as actions that are justifiable under all smooth ambiguity criteria with concave $\phi \in \Phi$ but not under the maxmin criterion. The continuity in value, as ambiguity aversion becomes higher and higher, does not translate at all in a continuity of the associated collections of justifiable actions. As is well known, best reply correspondences need not be lower hemicontinuous. Here we have a more dramatic continuity failure, with neither upper nor lower hemicontinuity.[8]

We close by establishing some sufficient conditions that restore continuity. Some terminology: $\phi' \succeq \phi$ means that ϕ' is more concave than ϕ, i.e., $\phi' = f \circ \phi$ for some strictly increasing and concave f; moreover, for each $a \in A$ we denote by $\mathcal{B}_{\phi,r}(a,\Sigma) = \{\mu \in \Delta(\Sigma) : \forall a' \in A, V_{\phi,r}(a,\mu) \geq V_{\phi,r}(a',\mu)\}$ the collection of beliefs that make a a best reply, given ϕ and r.

Proposition 3 *Let Σ be compact. Then*

[8] A similar discontinuity holds under risk, though we omit details for brevity.

(i) $\mathcal{J}_{\infty,r}(\Sigma) \subseteq \bigcup_{\phi\in\Phi} \mathcal{J}_{\phi,r}(\Sigma)$ *if A is finite and, for each $\mu \in \Delta(\Sigma)$, the set* $\arg\min_{a\in A} V_{\infty,r}(a,\mu)$ *is a singleton.*

(ii) $\bigcup_{\phi\in\Phi} \mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\infty,r}(\Sigma)$ *if, for each finite chain $\phi_m \succeq \cdots \succeq \phi_1$ in Φ and for each $a \in A$, we have* $\bigcap_{i=1}^{m} \mathcal{B}_{\phi_i,r}(a,\Sigma) \neq \emptyset$ *whenever* $\mathcal{B}_{\phi_1,r}(a,\Sigma) \neq \emptyset$.

3.5 Risk and ambiguity

If we enrich the setup with randomized consequences, we can consider cases where risk attitudes are immaterial and only ambiguity aversion is relevant. In this richer setup, the payoff function may be interpreted as $r(a,s) = \sum_{c\in C} u(c)\, g(a,s)(c)$, where $\Delta_0(C)$ is the set of simple lotteries and $g : A \times S \to \Delta_0(C)$ is a stochastic consequence function that associates lotteries to action-state pairs. Thus, each action a corresponds to the Anscombe-Aumann act $g(a,\cdot) : S \to \Delta_0(C)$. The set of actions A allows for all the randomizations that are feasible under the commitment technology of the DM.

This richer setup helps disentangle, conceptually and formally, the effects of risk aversion and of ambiguity aversion. In particular, when the consequence space is binary, i.e., $C = \{c^0, c^1\}$, risk attitudes are mute, but ambiguity attitudes may still be important.[9] The following example shows a case where only ambiguity attitudes matter for justifiability.

Example 5 Consider the following game form with binary consequence space $C = \{0,1\}$ and stochastic monetary consequences:

$g:$	s'	s''
t	δ_0	δ_1
m	$\frac{2}{3}\delta_0 + \frac{1}{3}\delta_1$	$\frac{2}{3}\delta_0 + \frac{1}{3}\delta_1$
b	δ_1	δ_0

Here risk attitudes do not matter. In particular, set $u(0) = 0$ and $u(1) = 1$, so that r is given by the game form (1), and let $\phi_\theta(x) = x^{1/\theta}$, where $\theta \geq 1$ parametrizes ambiguity aversion. Also, assume that S is embedded in Σ. As in Example 2, the belief μ that maximizes

$$V_{\phi_\theta,g}(m,\mu) - \max\{V_{\phi_\theta,g}(b,\mu), V_{\phi_\theta,g}(t,\mu)\}$$

satisfies $\mu(\delta_{s'}) = \mu(\delta_{s''}) = 1/2$ and

$$\mathcal{J}_{\phi_\theta,r}(\Sigma) = \begin{cases} \{t,b\}, & \theta < \bar{\theta}, \\ \{t,m,b\}, & \theta \geq \bar{\theta}, \end{cases}$$

[9] Game forms with stochastic consequences and only two monetaty prizes have been implemented in some laboratory experiments precisely for this reason. See Roth and Malouf (1979), the survey by Roth (1995), and references therein.

where $\bar{\theta} = \log_2 3$. ▲

By allowing for a stochastic consequence function we can encompass within our framework the possibility that the DM can commit to objective randomizations. Such feasible randomizations can be represented as elements of A.

Example 6 Suppose we enrich game form (1) by allowing the DM to irreversibly delegate the final choice to a random device, which selects t or b with objective probability $1/2$. Then the stochastic consequence function is

$g:$	s'	s''
t	δ_0	δ_1
m	$\delta_{\frac{1}{3}}$	$\delta_{\frac{1}{3}}$
b	δ_1	δ_0
$t\frac{1}{2}b$	$\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$	$\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$

(2)

where $t\frac{1}{2}b$ denotes the 50:50 randomization. (In this new decision problem the degree of ambiguity aversion does not affect the set of justifiable actions. In particular, the set is $J_\phi(\Sigma) = \left\{t, b, t\frac{1}{2}b\right\}$ for each concave ϕ if Σ is symmetric and the DM is risk neutral.) ▲

Furthermore, we note that any decision problem with a stochastic consequence function $g : A \times S \to \Delta_0(C)$ and a set of possible stochastic models $\Sigma \subseteq \Delta(S)$ can be represented by an equivalent decision problem with modified state space $\bar{S}$ and model space $\bar{\Sigma} \subseteq \Delta\left(\bar{S}\right)$, and deterministic consequence function $\bar{g} : A \times \bar{S} \to C$.

Example 7 Consider game form (2) and fix $\Sigma \subseteq \Delta\left(\{s', s''\}\right)$. Then, in the equivalent decision problem we let

$$\bar{S} = \{s', s''\} \times \{00, 01, 10, 11\},$$

$\bar{g}$	$s', 00$	$s', 01$	$s', 10$	$s', 11$	$s'', 00$	$s'', 01$	$s'', 10$	$s'', 11$
t	0	0	0	0	1	1	1	1
m	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
b	1	1	1	1	0	0	0	0
$t\frac{1}{2}b$	0	0	1	1	0	1	0	1

and

$$\bar{\Sigma} = \left\{\bar{\sigma} \in \Delta\left(\bar{S}\right) : \exists \sigma \in \Sigma, \forall (s, x) \in \bar{S},\ \bar{\sigma}(s, x) = \frac{1}{4}\sigma(s)\right\}.$$

With this, for every model $\bar{\sigma} \in \bar{\Sigma}$, the objective expected utility of actions is the same as with (2) and $\sigma = marg_S \bar{\sigma} \in \Sigma$. ▲

Finally, it is known that when all randomizations are feasible ambiguity attitudes do not affect the set of justifiable actions as noted, for example, by Kuzmics (2015) in a decision theoretic setting.[10]

4 Rationalizability

The previous analysis has implications for rationalizability in games. Specifically, in games with complete information, higher risk or ambiguity aversion expands the set of rationalizable outcomes. In fact, consider a game with ambiguity attitudes $G(\phi, r) = \langle I, (A_i, \phi_i, r_i)_{i \in I} \rangle$, where, for each $i \in I$, A_i is a compact metric space, $r_i : A_i \times A_{-i} \to \mathbb{R}$ is (jointly) continuous, $\phi_i : \overline{\mathrm{co}}\, \mathrm{Im}\, r_i \to \mathbb{R}$ is strictly increasing and continuous, $r = (r_i)_{i \in I}$, and $\phi = (\phi_i)_{i \in I}$. To provide an appropriate notion of rationalizability with general attitudes toward ambiguity we have to specify a scenario for this game. For illustration purposes, we assume that each i is a population with a continuum of identical agents, who may – however – hold different beliefs.[11] Agents from each population are drawn and matched at random to play game $G(\phi, r)$, which is *commonly known*. As we allow for ambiguity aversion, we also assume that feasible randomizations, if any, are already accounted for as elements of each A_i, possibly reinterpreting $r_i(a_i, a_{-i})$ as the (objectively) expected utility of i given (a_i, a_{-i}) (see Subsection 3.5). With this, rationalizable actions can be defined as follows: for each $i \in I$, $\mathcal{A}_i^0(\phi, r) = A_i$ and

$$\mathcal{A}_i^n(\phi, r) = \mathcal{J}_{\phi_i, r_i}\left(\bigotimes_{j \neq i} \Delta(\mathcal{A}_j^{n-1}(\phi, r))\right),$$

for each $n \in \mathbb{N}$.[12] To ease interpretation, consider the case of finite action sets. Rationality implies that an action a_i can be played by a positive fraction of agents in population i if and only if $a_i \in \mathcal{J}_{\phi_i, r_i}\left(\bigotimes_{j \neq i} \Delta(A_j)\right)$. If, on top of this, everyone believes in rationality, then a_i

[10] For informal consideration of the same sort see Raiffa (1961). Instead, for similar results and considerations involving game theoretic solution concepts, see Proposition 3 of Battigalli et al. (2013) and the discussions in Battigalli et al. (2015), Klibanoff (1996), and Lo (1996).

[11] Alternatively, we could have considered the opposite scenario whereby I is the fixed set of agents/players. In this case, from the point of view of each $i \in I$, the set of possible stochastic models is isomorphic to a subset of A_{-i}. The analysis is simpler, ambiguity aversion is still relevant, and results are similar. We refer to the population game scenario in the text because we find it more effective to illustrate the application of the comparative statics result.

[12] $\bigotimes_{j \neq i} \Delta(\mathcal{A}_j)$ denotes the set of product measures on $\times_{j \neq i} \mathcal{A}_j$, which contains all the Dirac measures on $\times_{j \neq i} \mathcal{A}_j$.

can be played by a positive fraction of agents if and only if

$$a_i \in \mathcal{J}_{\phi_i,r_i}\left(\bigotimes_{j\neq i}\Delta\left(\mathcal{J}_{\phi_j,r_j}\left(\bigotimes_{k\neq j}\Delta(A_k)\right)\right)\right) = \mathcal{J}_{\phi_i,r_i}\left(\bigotimes_{j\neq i}\Delta(\mathcal{A}^1_j(\phi,r))\right),$$

and so on.

This can be expressed with the justification-operator language of Milgrom and Roberts (1991): For every closed (hence compact) Cartesian subset $C = \times_{i\in I}C_i \subseteq \times_{i\in I}A_i$, let

$$J_{\phi,r}(C) = \times_{i\in I}\mathcal{J}_{\phi_i,r_i}\left(\bigotimes_{j\neq i}\Delta(C_j)\right).$$

Standard arguments show that, under the compactness and continuity assumptions of this subsection, $J_{\phi,r}$ is a monotone self-map on the collection of the non-empty and closed (hence compact) Cartesian subsets of $\times_{i\in I}A_i$. Then $\mathcal{A}^n(\phi,r) = J^n_{\phi,r}(\times_{i\in I}A_i)$ is non-empty and compact for each $n\in\mathbb{N}$. The set of *rationalizable* action profiles is $\mathcal{A}^*(\phi,r) = \bigcap_{n\in\mathbb{N}} J^n_{\phi,r}(\times_{i\in I}A_i)$. Again by standard arguments, $\mathcal{A}^*(\phi,r)$ is also non-empty, and it is the largest compact Cartesian subset C such that $C\subseteq J_{\phi,r}(C)$.

Say that $G(\mathrm{Id},r')$ *exhibits more risk aversion than* $G(\mathrm{Id},r)$ if there are profiles of concave, continuous and strictly increasing transformations $(\psi_i : \overline{\mathrm{co}}\,\mathrm{Im}\,r_i \to \mathbb{R})_{i\in I}$ such that $r'_i = \psi_i\circ r_i$ for each $i\in I$. Similarly, $G(\phi',r)$ *exhibits more ambiguity aversion than* $G(\phi,r)$ if there are profiles of concave, continuous and strictly increasing transformations $(\varphi_i : \overline{\mathrm{co}}\,\mathrm{Im}\,\phi_i \to \mathbb{R})_{i\in I}$ such that $\phi'_i = \varphi_i\circ\phi_i$ for each $i\in I$. Given the properties of the justifications operators $J_{\phi,r}$ (with continuous ϕ and r), Propositions 1 and 2 imply the sought-after property that higher risk or ambiguity aversion expands the set of rationalizable outcomes.

Corollary 1 *If $G(\mathrm{Id},r')$ exhibits more risk aversion than $G(\mathrm{Id},r)$, then $\mathcal{A}^*(\mathrm{Id},r)\subseteq\mathcal{A}^*(\mathrm{Id},r')$. If $G(\phi',r)$ exhibits more ambiguity aversion than $G(\phi,r)$, then $\mathcal{A}^*(\phi,r)\subseteq\mathcal{A}^*(\phi',r)$.*

Example 8 Consider the following two-person game form with monetary consequences:

$g_1,g_2:$	b'	b''
a'	0, 1	1, 0
a''	$\frac{1}{3}$,0	$\frac{1}{3}$,1
a'''	1, 1	0, 0

Let $r_{\theta,1} = g_1^{\frac{1}{\theta}}$ and $\phi_{\theta,1}(r) = r^{1/\theta}$, ϕ_2 be continuous and strictly increasing, $r_2 = u_2\circ g_2$ for any continuous and strictly increasing u_2 (the risk and ambiguity attitudes of player 2 are immaterial). We specifically consider the rationalizability correspondences $\theta\mapsto\mathcal{A}^*(\phi_{\theta,1},\phi_2,g_1,r_2)$ and $\theta\mapsto\mathcal{A}^*(\mathrm{Id},\phi_2,r_{\theta,1},r_2)$ for $\theta\geq 1$. Note that every action of player 2 is a best response to some belief, and the set of rationalizable actions of player 2 is $\{b',b''\}$ if a'' is justifiable for player 1, and $\{b'\}$ if a'' is unjustifiable. In the latter case, the only rationalizable action of player 1 is a''', the best reply to b'. With this, the calculations of Examples 1 and 2 imply

$$\mathcal{A}^*(\phi_1^\theta, \phi_2, g_1, r_2) = \mathcal{A}^*(\mathrm{Id}, \phi_2, r_{\theta,1}, r_2) = \begin{cases} \{(a''', b')\}, & \theta < \bar{\theta}, \\ A_1 \times A_2, & \theta \geq \bar{\theta}. \end{cases}$$

▲

5 Discussion

We discuss the related literature and briefly consider the case of incomplete information about ambiguity attitudes.

5.1 Related literature

A superficial analogy First we compare with Battigalli et al. (2015) and explain the difference. In that paper we proved that higher ambiguity aversion expands the set of self-confirming equilibria, a steady-state phenomenon resulting from the strong discipline on beliefs that the notion of self-confirming equilibrium imposes by requiring their consistency with the long-run data that agents observe in recurrent interaction. Specifically, assuming that each agent observes at least his realized payoff in each play, self-confirming equilibrium actions are perceived as unambiguous best replies by players, whereas unused alternatives are typically perceived as ambiguous. Therefore, holding beliefs fixed, an increase in ambiguity aversion leaves the value of self-confirming actions unaltered but decreases the value of unused alternatives. This implies that, for each equilibrium action, the set of confirmed beliefs justifying it *expands* as ambiguity aversion increases. All this stands in sharp contrast with the justifiability result of the present paper: Here we are not trying to characterize steady-state actions; hence, feedback is irrelevant and beliefs are not restricted by experience (in games, rationalizable beliefs are restricted by strategic thinking). Therefore, a justifiable action may well be perceived as ambiguous. In this case, as ambiguity aversion increases, the set of beliefs justifying this action typically *shrinks*, as demonstrated by our examples.[13]

Criterion As explained above, our work builds on the choice model of Klibanoff et al. (2005) and the duality lemma of Wald (1949) introduced back into game theory by Pearce (1984). We use the smooth ambiguity model for two reasons. (i) It is portable, i.e., it parametrizes personality traits that agents are supposed to exhibit in any decision problem:

[13] The failure of the inclusion $\bigcup_{\phi\in\Phi} \mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\infty,r}(\Sigma)$ further exemplifies how feable is the connection with Battigalli et al. (2015), where the set of smooth self-confirming equilibria is included in the set of maxmin ones.

risk attitudes given by the von Neumann-Morgenstern utility function u and ambiguity attitudes given by the second-order utility function ϕ. Such personality traits can be assumed to be constant across decision, or game situations; state spaces and beliefs, on the other hand, change according to the situation. (ii) Under this model an increase in ambiguity aversion is represented by a concave strictly increasing transformation of ϕ, which by a fortuitous coincidence (see Remark 1 in the appendix) allows to rely on the general version of the duality lemma.

As for (i), we can think of alternative models sharing the same properties of portability (see, for example, eq. (13) of Battigalli et al., 2015). But we feel that the smooth ambiguity model —among the known models of decision making under ambiguity— is the one where these features are most evident. As for the possibility to extend our comparative statics result (point ii), it may be natural to consider the class of preferences that can be represented by quasiconcave utility functionals on $\mathbb{R}^S$. But such extension does not hold. In fact, the criterion $V_{\infty,r}$ of Subsection 3.4 belongs to this class and is more ambiguity averse than $V_{\phi,r}$ for any concave ϕ. Yet, action h in Example 4 is justifiable for criterion $V_{\phi,r}$, but not for the more ambiguity averse criterion $V_{\infty,r}$.

Weinstein (2013) The paper most related to ours is Weinstein (2013). Our paper can be seen as a generalization and extension of his, independent, comparative statics result (his Proposition 1). It is a generalization because he considers finite games, while we allow for a continuum of actions and states. It is also an extension because, by assuming ambiguity neutral players, he can only study the effect of increasing risk aversion, whereas we allow for non-neutral ambiguity attitudes and study the effect of increasing ambiguity aversion.

Other papers Other papers in the literature analyzed notions of justifiability, or rationalizability with non-neutral attitudes toward ambiguity. Ghirardato and Le Breton (2000) characterize actions that are best replies to some possibly non-additive belief under the Choquet expected utility criterion of Schmeidler (1989). Epstein (1997) analyzes rationalizability under several criteria, including the Choquet criterion of Schmeidler (1989) and the maxmin criterion of Gilboa and Schmeidler (1989). In any case, to the best of our knowledge, ours is the first work reporting a result on comparative ambiguity and justifiability (or rationalizability).

5.2 Incomplete information about ambiguity attitudes

The set of justifiable actions can be determined if the ambiguity attitudes of the DM are known. Therefore, in Section 4 we are able to apply our comparative statics result to rationalizability in games with complete information, i.e., games where the game form and players personality traits are commonly known. What if ambiguity attitudes (of others) are unknown? Our comparative statics result suggests answers for the case where only an upper bound on ambiguity attitudes is known (in games, commonly known). To focus on ambiguity aversion, we assume for simplicity that there is (common knowledge of) risk neutrality; thus, r is the monetary payoff function. Recall that Φ denotes the set of "second-order utility functions," that is, the continuous and strictly increasing real-valued functions with domain $[\min r, \max r]$. Fix a concave function $\bar{\phi} \in \Phi$, and let $\bar{\Phi} \subseteq \Phi$ denote the set containing $\bar{\phi}$ and every other second-order utility function ϕ such that $\bar{\phi}$ is a concave and strictly increasing transformation of ϕ. Under our assumptions on A, S and Σ, Proposition 2 implies $\mathcal{J}_{\bar{\phi},r}(\Sigma) = \bigcup_{\phi \in \bar{\Phi}} \mathcal{J}_{\phi,r}(\Sigma)$. Of course, the same result holds when we consider any class of second-order utility functions parametrized by a measure of ambiguity aversion less than or equal to an upper bound. Thus, when only an "upper bound" $\bar{\phi}$ on ambiguity aversion is known (in games, commonly known) our results apply to changes in $\bar{\phi}$.

What if it is only known that ambiguity aversion is finite, but it can be arbitrarily high? How can we characterize the set of actions that are justifiable for at least one second-order utility function $\phi \in \Phi$? Though outside the scope of the present paper, we expect that it is possible to adapt results from Boergers (1993) and Weinstein (2013) to provide a kind of dominance characterization. In particular, suppose for simplicity that A and Σ are finite. Adapting a result proved for the case of risk by Boergers (1993), one can show that an action a is justifiable for some second-order utility function —$a \in \bigcup_{\phi \in \Phi} \mathcal{J}_{\phi,r}(\Sigma)$— if and only if a is not purely weakly dominated on any subset of Σ, that is, for each $\emptyset \neq \hat{\Sigma} \subseteq \Sigma$ there is no $\hat{a} \in A$ with

$$\forall \sigma \in \hat{\Sigma},\ R(\hat{a}, \sigma) \geq R(a, \sigma) \quad \text{and} \quad \exists \sigma \in \hat{\Sigma},\ R(\hat{a}, \sigma) > R(a, \sigma),$$

where $R(\hat{a}, \sigma)$ is the expected payoff of $\hat{a}$ under stochastic model σ. Adapting a result of Weinstein (2013, Proposition 3), such set of undominated actions is the limit of the justifiable set $\mathcal{J}_{\phi,r}(\Sigma)$ as ambiguity aversion goes to infinity. In the parametrized decision problem of Example 3, this set is $\{t, b\}$ if $\varepsilon = 0$, because in this case m is weakly dominated by t and b on Σ, strictly dominated by b on each nonempty subset $\hat{\Sigma} \subseteq \Sigma \backslash \{\delta_{s''}\}$, and strictly dominated by t on each nonempty subset $\hat{\Sigma} \subseteq \Sigma \backslash \{\delta_{s'}\}$.

6 Proofs and related material

6.1 Abstract Pearce-Wald lemma

Fix two nonempty subsets A_1 and A_2 of a Hausdorff locally convex topological vector space. Let $B_i = \operatorname{co} A_i$ and $\bar{B}_i = \overline{\operatorname{co}} A_i$ denote respectively the convex hull of A_i and its closure, for $i = 1, 2$.

Given $F : B_1 \times \bar{B}_2 \to \mathbb{R}$, we say that $a_1^* \in A_1$ is *dominated* if and only if

$$\exists a_1 \in A_1, \forall a_2 \in A_2, \qquad F(a_1^*, a_2) < F(a_1, a_2),$$

otherwise we say that a_1^* is *undominated;* we say that a_1^* is *co-dominated* if and only if

$$\exists b_1 \in B_1, \forall a_2 \in A_2, \qquad F(a_1^*, a_2) < F(b_1, a_2),$$

otherwise we say that a_1^* is *co-undominated.*

Lemma 1 *Suppose that:*

(i) A_2 *is closed and* $\bar{B}_2$ *is compact;*

(ii) F *is quasiconcave and upper semicontinuous on* B_1*;*

(iii) F *is affine and continuous on* $\bar{B}_2$*.*

An element $a_1^* \in A_1$ *is co-undominated only if there exists some* $b_2 \in \bar{B}_2$ *such that* $a_1^* \in \arg\max_{a_1 \in A_1} F(a_1, b_2)$*. The converse is true if* F *is affine on* B_1*.*

Of course, condition (i) implies that A_2 (a subset of $\bar{B}_2$) is compact. In many examples, condition (i) is equivalent to the compactness of A_2.[14] Condition (ii) is satisfied if F is *concave* and upper semicontinuous on B_1.

Proof. First note that, since A_2 is compact, there exists a function $\bar{\iota}_2 : \bar{B}_2 \to \Delta(A_2)$ (the set $\Delta(A_2)$ here denotes the set of all regular Borel probability measures) such that

$$\phi(b_2) = \int_{A_2} \phi(a_2)\, \bar{\iota}_2(b_2)(\mathrm{d}a_2)$$

for all continuous and affine $\phi : \bar{B}_2 \to \mathbb{R}$.[15] Moreover, by definition of convex hull, there is a function $\iota_1 : B_1 \to \Delta(A_1)$ such that

$$b_1 = \sum_{a_1 \in A_1} a_1 \iota_1(b_1)(a_1).$$

(Only if) Suppose that a_1^* is co-undominated. We must show that there exists $b_2^* \in \bar{B}_2$ such that $F(a_1^*, b_2^*) \geq F(a_1, b_2^*)$ for all $a_1 \in A_1$. Define the function $h : B_1 \times \bar{B}_2 \to \mathbb{R}$ by

$$h(b_1, b_2) = F(a_1^*, b_2) - F(b_1, b_2).$$

Since a_1^* is co-undominated, for each $b_1 \in B_1$ there exists $a_2^{b_1} \in A_2$ such that $F(a_1^*, a_2^{b_1}) \geq F(b_1, a_2^{b_1})$, that is, $h(b_1, a_2^{b_1}) \geq 0$. We can conclude that

$$\forall b_1 \in B_1, \ \max_{b_2 \in \bar{B}_2} h(b_1, b_2) \geq h(b_1, a_2^{b_1}) \geq 0.$$

[14] Consider, e.g., quasi-complete locally convex topological spaces (p. 61, Holmes, 1975).

[15] See Propositions 1.2 and 4.5 of Phelps (1966). For later use, we find it convenient to denote $\bar{\iota}_2$ this map.

In turn, this yields $\inf_{b_1 \in B_1} \max_{b_2 \in \bar{B}_2} h(b_1, b_2) \geq 0$. Given the properties of F, the function h satisfies all the assumptions of the Sion Minimax Theorem (Corollary 3.3 of Sion, 1958), namely, h is quasiconvex and lower semicontinuous on B_1, as well as affine and continuous on $\bar{B}_2$. This implies that

$$\max_{b_2 \in \bar{B}_2} \inf_{b_1 \in B_1} h(b_1, b_2) = \inf_{b_1 \in B_1} \max_{b_2 \in \bar{B}_2} h(b_1, b_2) \geq 0.$$

By choosing $b_2^* \in \arg\max_{b_2 \in \bar{B}_2} \left(\inf_{b_1 \in B_1} h(b_1, b_2)\right)$, we have that

$$0 \leq \inf_{b_1 \in B_1} h\left(b_1, b_2^*\right) = \inf_{b_1 \in B_1} \left(F(a_1^*, b_2^*) - F(b_1, b_2^*)\right),$$

thus, $F(a_1^*, b_2^*) \geq F(a_1, b_2^*)$ for each $a_1 \in A_1$.

(If) Suppose that F is also affine on B_1. By way of contraposition, suppose that a_1^* is co-dominated, that is, there exists $b_1 \in B_1$ such that

$$\forall a_2 \in A_2, \quad F(a_1^*, a_2) < F\left(b_1, a_2\right) = \sum_{a_1 \in A_1} F(a_1, a_2)\iota_1\left(b_1\right)\left(a_1\right).$$

We must show that for each $b_2 \in \bar{B}_2$ there exists some $a_1^{b_2} \in A_1$ such that $F\left(a_1^{b_2}, b_2\right) > F\left(a_1^*, b_2\right)$. Fix $b_2 \in \bar{B}_2$ arbitrarily. Since a_1^* is co-dominated by b_1, integrating over A_2 and by using the maps ι_1 and $\bar{\iota}_2$, we obtain

$$\begin{aligned} F(a_1^*, b_2) &= \int_{A_2} F(a_1^*, a_2)\bar{\iota}_2(b_2)\left(\mathrm{d}a_2\right) < \int_{A_2} \left(\sum_{a_1 \in A_1} F(a_1, a_2)\iota_1\left(b_1\right)\left(a_1\right)\right) \bar{\iota}_2\left(b_2\right)\left(\mathrm{d}a_2\right) \\ &= \sum_{a_1 \in A_1} \left(\int_{A_2} F(a_1, a_2)\bar{\iota}_2(b_2)(\mathrm{d}a_2)\right) \iota_1(b_1)(a_1) = \sum_{a_1 \in A_1} F(a_1, b_2)\iota_1(b_1)(a_1). \end{aligned}$$

If $a_1^{b_2} \in \arg\max_{a_1 \in \operatorname{supp} \iota_1(b_1)} F(a_1, b_2)$, then $F(a_1^{b_2}, b_2) \geq \sum_{a_1 \in A_1} F(a_1, b_2)\iota_1(b_1)(a_1) > F(a_1^*, b_2)$. ■

6.2 Randomization

Now let A_1 and A_2 be two separable metric spaces. Denote by $\Delta_0\left(A_i\right)$ the collection of all simple Borel probability measures, and by $\Delta\left(A_i\right)$ the collection of all Borel probability measures on A_i. Given a function $f : A_1 \times A_2 \to \mathbb{R}$, say that $a_1^* \in A_1$ is *dominated under randomization* if and only if

$$\exists \beta_1 \in \Delta_0\left(A_1\right), \forall a_2 \in A_2, \qquad f(a_1^*, a_2) < \sum_{a_1 \in A_1} f(a_1, a_2)\beta_1(a_1).$$

Otherwise, we say that a_1^* is *undominated under randomization*.

At this level of generality, the separable metric space spaces A_1 and A_2 are *not* required to be subsets of some Hausdorff locally convex topological vector space. Denote by $\mathcal{B}_i$ the

Borel σ-algebra of A_i. In this framework, we can identify each element a of A_i with the Dirac δ_a at a. The set of Dirac probability measures is a subset of the space of all Borel countably additive measures of bounded variation $ca(A_i, \mathcal{B}_i)$ which, when endowed with the w^*-topology $\sigma(ca(A_i, \mathcal{B}_i), C_b(A_i))$, is a Hausdorff locally convex topological vector space. Since A_i is a separable metric space, the set of corresponding Dirac probabilities is also closed. Under this identification, B_i corresponds to the set $\Delta_0(A_i)$ of all probability measures on A_i with finite support, while $\bar{B}_i$ corresponds to the set $\Delta(A_i)$ of all Borel probability measures on A_i. The set $\bar{B}_i$ is compact if and only if A_i compact.[16] Finally, note that if any of the two sets A_1 and A_2 is finite, then it is a separable metric space once endowed with the discrete metric; moreover, if both are finite, then f is continuous in each component and bounded.

In what follows, with a small abuse of notation, we will denote by A_i both the original set A_i and the set of corresponding Dirac probability measures. Also, we will denote the elements of B_1 and $\bar{B}_2$ with the letter β rather than b to stress that we interpret them as probability measures.

Corollary 2 *Let A_1 and A_2 be two separable metric spaces. If*

(i) A_2 is compact;

(ii) f is continuous on A_1 and A_2 and bounded;

then, an element $a_1^* \in A_1$ is undominated under randomization if and only if there exists $\beta_2 \in \bar{B}_2 = \Delta(A_2)$ such that $a_1^* \in \arg\max_{a_1 \in A_1} \int_{A_2} f(a_1, a_2)\beta_2(\mathrm{d}a_2)$.

Proof. Define $F : B_1 \times \bar{B}_2 \to \mathbb{R}$ by

$$F(\beta_1, \beta_2) = \int_{A_2} \left(\sum_{a_1 \in A_1} f(a_1, a_2)\, \beta_1(a_1) \right) \beta_2(\mathrm{d}a_2).$$

It is routine to check that the function F is affine and continuous in each component. Given our identifications, $a_1^* \in A_1$ is undominated under randomization if and only if $\delta_{a_1^*}$ is co-undominated. By Lemma 1, the statement follows. ■

Set

$$\mathcal{J}_f = \left\{ a_1 \in A_1 : \exists \beta_2 \in \Delta(A_2), \forall a_1' \in A_1, \int_{A_2} f(a_1, a_2)\beta_2(\mathrm{d}a_2) \geq \int_{A_2} f(a_1', a_2)\beta_2(\mathrm{d}a_2) \right\}.$$

Say that a function $f : A_1 \times A_2 \to \mathbb{R}$ is *nice* (resp., *semi-nice*) when any $a_1 \in A_1$ is undominated under randomization if and only if (resp., only if) $a_1 \in \mathcal{J}_f$. Corollary 2 establishes conditions for niceness.

[16] See Chapter 15 of Aliprantis and Border (2006) for all the above notions.

Corollary 3 *Let $f, h : A_1 \times A_2 \to \mathbb{R}$ be, respectively, nice and semi-nice. If $h = \varphi \circ f$, with $\varphi : \mathrm{co}(\mathrm{Im}\, f) \to \mathbb{R}$ concave and strictly increasing, then $\mathcal{J}_f \subseteq \mathcal{J}_h$.*

Proof. Let $\bar{a}_1 \in \mathcal{J}_f$. Since f is nice, $\bar{a}_1$ is undominated under randomization. Hence, since φ is concave and strictly increasing, this implies that

$$\forall \beta_1 \in \Delta_0(A_1), \exists a_2 \in A_2,\ f(\bar{a}_1, a_2) \geq \sum_{a_1 \in A_1} f(a_1, a_2)\beta_1(a_1) \geq \varphi^{-1}\left(\sum_{a_1 \in A_1} (\varphi \circ f)(a_1, a_2)\beta_1(a_1)\right)$$

that is,

$$\forall \beta_1 \in \Delta_0(A_1), \exists a_2 \in A_2,\ (\varphi \circ f)(\bar{a}_1, a_2) \geq \sum_{a_1 \in A_1} (\varphi \circ f)(a_1, a_2)\beta_1(a_1).$$

Since $h = \varphi \circ f$ is semi-nice, this implies $\bar{a}_1 \in \mathcal{J}_h$. ■

6.3 Proofs of Propositions 1 and 2

First observe that, since S is a separable metric space, also $\Delta(S)$ is a separable metric space (once endowed with the Prohorov metric). We denote by $\mathcal{B}$ its Borel sigma-algebra. Given a set $\Sigma \in \mathcal{B}$, we denote by $\mathcal{B}_{|\Sigma}$ the relative Borel sigma-algebra and by $\Delta(\Sigma)$ the collection of all Borel probability measures $\mu : \mathcal{B}_{|\Sigma} \to [0,1]$. We endow $\Delta(\Sigma)$ with the w*-topology. $\Delta(\Sigma)$ is compact if and only if Σ is w*-compact in $\Delta(S)$.

Proof of Proposition 1. Let $A_1 = A$, $A_2 = S$, $f = r$, and $h = r' = \psi \circ r$. By Corollary 2 and given the properties of A, S, r, and ψ, it is immediate to see that f and h are nice. By Corollary 3 and since ψ is concave and strictly increasing, we have that $\mathcal{J}_f \subseteq \mathcal{J}_h$. Finally, since $\Sigma \supseteq \{\delta_s\}_{s \in S}$, we can conclude that $\mathcal{J}_{\mathrm{Id},r}(\Sigma) = \mathcal{J}_f$ and $\mathcal{J}_h = \mathcal{J}_{\mathrm{Id},r'}(\Sigma)$. Therefore, $\mathcal{J}_{\mathrm{Id},r}(\Sigma) \subseteq \mathcal{J}_{\mathrm{Id},r'}(\Sigma)$. ■

Proof of Proposition 2. Define $R(a, \sigma) = \int_S r(a,s)\,\sigma(\mathrm{d}s)$ for all $a \in A$ and $\sigma \in \Sigma$. Note that R is continuous in each component and bounded. Let $A_1 = A$, $A_2 = \Sigma$, $f = \phi \circ R$, and $h = \phi' \circ R = (\varphi \circ \phi) \circ R = \varphi \circ f$. By Corollary 2 and given the properties of A, Σ, R, and ϕ and φ, it is immediate to see that f and h are nice. By Corollary 3 and since φ is concave and strictly increasing, we have that $\mathcal{J}_f \subseteq \mathcal{J}_h$. By construction, we have that $\mathcal{J}_{\phi,r}(\Sigma) = \mathcal{J}_f$ and $\mathcal{J}_h = \mathcal{J}_{\phi',r}(\Sigma)$. Therefore, $\mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\phi',r}(\Sigma)$. ■

Remark 1 To invoke the abstract Pearce-Wald lemma, in the proof of Proposition 2, from a decision theoretic viewpoint, we consider randomized actions β_1 on A as ex-ante randomizations rather than the more customary ex-post randomizations a la Anscombe and Aumann. Such ex-ante randomizations are merely ancillary analytical objects. Formally, in our proofs the value of a randomized action β_1 is

$$\phi^{-1}\left(\sum_{a \in A} \int_\Sigma \phi\left(\int_S r(a,s)\,\sigma(\mathrm{d}s)\right) \mu(\mathrm{d}\sigma)\,\beta_1(a)\right)$$

rather than

$$\phi^{-1}\left(\int_{\Sigma}\phi\left(\sum_{a\in A}\int_{S}r\left(a,s\right)\sigma(\mathrm{d}s)\beta_{1}\left(a\right)\right)\mu\left(\mathrm{d}\sigma\right)\right).$$

Conceptually, we reiterate that DM can only choose in A, which encompasses any feasible randomization. It is a fortuitous coincidence that the smooth model permits this treatment of randomized actions that allows to exploit the abstract Pearce-Wald lemma. This is in contrast to what happens with the maxmin criterion (see Subsection 3.4).

6.4 Proof of Corollary 1

We give the proof for the case of comparative ambiguity aversion. Recall that we are assuming that, for each $i\in I$, A_i is a compact metric space and r_i is (jointly) continuous on $A_i\times A_{-i}$. Suppose that $G(\phi',r)$ exhibits more ambiguity aversion than $G(\phi,r)$. For each closed (hence compact) Cartesian subset $C\subseteq\times_{j\in I}A_j$ and each $i\in I$, the assumptions of Proposition 2 apply with $A=C_i$ and $\Sigma=\bigotimes_{j\neq i}\Delta(C_j)$, which are both compact; therefore $J_{\phi,r}(C)\subseteq J_{\phi',r}(C)$. In particular,

$$J^{1}_{\phi,r}(\times_{i\in I}A_i)=J_{\phi,r}(\times_{i\in I}A_i)\subseteq J_{\phi',r}(\times_{i\in I}A_i)=J^{1}_{\phi',r}(\times_{i\in I}A_i).$$

Suppose, by way of induction, that $J^{n-1}_{\phi,r}(\times_{i\in I}A_i)\subseteq J^{n-1}_{\phi',r}(\times_{i\in I}A_i)$. By monotonicity of $J_{\phi,r}\left(\cdot\right)$ and Proposition 2

$$J^{n}_{\phi,r}\left(\times_{i\in I}A_i\right)=J_{\phi,r}\left(J^{n-1}_{\phi,r}(\times_{i\in I}A_i)\right)\subseteq J_{\phi,r}\left(J^{n-1}_{\phi',r}(\times_{i\in I}A_i)\right)\subseteq J_{\phi',r}\left(J^{n-1}_{\phi',r}(\times_{i\in I}A_i)\right)=J^{n}_{\phi',r}\left(\times_{i\in I}A_i\right).$$

Therefore, for each $n\in\mathbb{N}$, $\mathcal{A}^n(\phi,r)\subseteq\mathcal{A}^n(\phi',r)$, and

$$\mathcal{A}^*(\phi,r)=\bigcap_{n\in\mathbb{N}}\mathcal{A}^n(\phi,r)\subseteq\bigcap_{n\in\mathbb{N}}\mathcal{A}^n(\phi',r)=\mathcal{A}^*(\phi',r).$$

Taking into account that $\Sigma=\bigotimes_{j\neq i}\Delta(C_j)$ contains all the Dirac probability measures on C_{-i}, the proof for the case of comparative risk aversion is analogous, with Proposition 1 in the role of Proposition 2. ■

6.5 Proof of Proposition 3

(i) Consider $a^*\in\mathcal{J}_{\infty,r}\left(\Sigma\right)$. There exists $\mu\in\Delta\left(\Sigma\right)$ such that for each $a\in A\backslash\left\{a^*\right\}$

$$\min_{\sigma\in\operatorname{supp}\mu}\int_S r\left(a^*,s\right)\sigma\left(\mathrm{d}s\right)>\min_{\sigma\in\operatorname{supp}\mu}\int_S r\left(a,s\right)\sigma\left(\mathrm{d}s\right). \tag{3}$$

Consider the sequence of functions $\left(\phi_n\right)_{n\in\mathbb{N}}$ defined as $\phi_n\left(t\right)=-e^{-nt}$ for all $n\geq 1$ and for all $t\in\mathbb{R}$. It is immediate to see that

$$\forall n\in\mathbb{N},\ \phi_n^{-1}\left(\int_{\Sigma}\phi_n\left(\int_S r\left(a^*,s\right)\sigma\left(\mathrm{d}s\right)\right)\mu\left(\mathrm{d}\sigma\right)\right)\geq\min_{\sigma\in\operatorname{supp}\mu}\int_S r\left(a^*,s\right)\sigma\left(\mathrm{d}s\right).$$

On the other hand, by (3) for each $a \in A\backslash\{a^*\}$ there exists $n_a \geq 1$ such that, for each $n \geq n_a$,

$$\min_{\sigma\in\operatorname{supp}\mu}\int_S r(a^*,s)\,\sigma(\mathrm{d}s) > \phi_n^{-1}\left(\int_\Sigma \phi_n\left(\int_S r(a,s)\,\sigma(\mathrm{d}s)\right)\mu(\mathrm{d}\sigma)\right) \geq \min_{\sigma\in\operatorname{supp}\mu}\int_S r(a,s)\,\sigma(\mathrm{d}s).$$

Since A is finite, let $\bar{n} = \max_{a\in A\backslash\{a^*\}} n_a$. It follows that for each $a \in A\backslash\{a^*\}$

$$\begin{aligned}\phi_{\bar{n}}^{-1}\left(\int_\Sigma \phi_{\bar{n}}\left(\int_S r(a^*,s)\,\sigma(\mathrm{d}s)\right)\mu(\mathrm{d}\sigma)\right) &\geq \min_{\sigma\in\operatorname{supp}\mu}\int_S r(a^*,s)\,\sigma(\mathrm{d}s)\\ &> \phi_{\bar{n}}^{-1}\left(\int_\Sigma \phi_{\bar{n}}\left(\int_S r(a,s)\,\sigma(\mathrm{d}s)\right)\mu(\mathrm{d}\sigma)\right),\end{aligned}$$

proving that $a^* \in \mathcal{J}_{\phi_{\bar{n}},r}(\Sigma)$ and, thus, the statement. (ii) Consider $a^* \in \bigcup_{\phi\in\Phi}\mathcal{J}_{\phi,r}(\Sigma)$. It follows that there exists $\bar{\phi} \in \Phi$ and $\mu \in \Delta(\Sigma)$ such that $V_{\bar{\phi},r}(a^*,\mu) \geq V_{\bar{\phi},r}(a,\mu)$ for all $a \in A$. Define $\phi_n = \varphi_n \circ \bar{\phi}$ such that $\varphi_n(t) = -e^{-nt}$ for all $n \geq 1$ and for all $t \in \mathbb{R}$. By Proposition 2, $\mathcal{B}_{\phi_n,r}(a^*,\Sigma) \neq \emptyset$ for all $n \geq 1$. By the finite intersection property of compact sets, we can conclude that there is $\bar{\mu} \in \bigcap_{i=1}^{\infty}\mathcal{B}_{\phi_i,r}(a^*,\Sigma)$. It follows that $V_{\phi_n,r}(a^*,\bar{\mu}) \geq V_{\phi_n,r}(a,\bar{\mu})$ for all $n \geq 1$ and for all $a \in A$. By passing to the limit, for each $a \in A$

$$\begin{aligned}V_{\infty,r}(a^*,\bar{\mu}) &= \min_{\sigma\in\operatorname{supp}\bar{\mu}}\int_S r(a^*,s)\,\sigma(\mathrm{d}s) = \lim_n V_{\phi_n,r}(a^*,\bar{\mu})\\ &\geq \lim_n V_{\phi_n,r}(a,\bar{\mu}) = \min_{\sigma\in\operatorname{supp}\bar{\mu}}\int_S r(a,s)\,\sigma(\mathrm{d}s) = V_{\infty,r}(a,\bar{\mu}),\end{aligned}$$

proving that $a^* \in \mathcal{J}_{\infty,r}(\Sigma)$. ■

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